ADVANCED DIFFERENTIAL EQUATIONS

ADVANCED DIFFERENTIAL EQUATIONS

[CONTAINING, COMPLETE UNIFIED U.G.C. SYLLABUS FOR B.A./B.Sc. (GENERAL AND HONOURS) PART II AND BOUNDARY VALUE PROBLEMS]

[For B.A., B.Sc. and Honours (Mathematics and Physics), M.A., M.Sc. (Mathematics and Physics), B.E. Students of Various Universities and for P.C.S., A.M.I.E. GATE, C.S.I.R. U.G.C. NET and Various Competitive Examinations]

Dr. M.D. RAISINGHANIA
M.Sc., Ph.D
Formerly Reader and Head,
Department of Mathematics,
S.D. College, Muzaffarnagar,
U.P.

S.CHAND & COMPANY PVT. LTD.
(AN ISO 9001 : 2008 COMPANY)
RAMNAGAR, NEW DELHI - 110 055

8.3	Cauchy's problem of second order partial differential equations	8.3
8.4	Solved examples based on Art. 8.3	8.4
8.5	Laplace transformation. Reduction to canonical (or normal) form.	8.4
8.6	Working rule for reducing a hyperbolic equation to its canonical form	8.7
8.7	Solved examples based on Art. 8.6	8.7
8.8	Working rule for reducing a parabolic equations to its canonical form	8.23
8.9	Solved examples based on Art. 8.8	8.24
8.10	Working rule for reducing an elliptic equation to its canonical form	8.31
8.11	Solved examples based on Art. 8.10	8.31
8.12	The solution of linear hyperbolic equations	8.36
8.13	Riemann method of solution of general linear hyperbolic equation of second order	8.36
8.14	Solved examples based on Art. 8.13	8.39
8.15	Riemann-Volterra method of solving Cauchy problem for the one-dimensional wave-equations	8.44
9. Monge's method		**9.1 – 9.51**
9.1	Introduction	9.1
9.2	Monge's method of integrating $Rr + Ss + Tt = V$	9.1
9.3	Type-I: When the given equation $Rr + Ss + Tt = V$ leads to two distinct intermediate intergrals and both of them are used to get the desired solution	9.3
9.4	Solved examples based on Art. 9.3	9.3
9.5	Type 2: When the given equation $Rr + Ss + Tt = V$ leads to two distinct intermediate integrals and only one is employed to get the desired solution	8.11
9.6	Solved examples based on Art. 9.5	9.12
9.7	Type 3: When the given equation $Rr + Ss + Tt = V$ leads to two identitcal intermediate integrals	9.22
9.8	Solved examples based on Art. 9.7	9.22
9.9	Type 4: When the given equation $Rr + Ss + Tt = V$ fails to yield an intermediate integral as in types 1, 2 and 3	9.30
9.10	Solved examples based on Art. 9.9	9.30
9.11	Monge's method of integrating $Rr + Ss + t + U(rt - s^2) = V$	9.33
9.12	Type 1: When the roots of λ-quadratic are identical	9.34
9.13	Type 2: When the roots of λ-quadratic are distinct	9.40
9.14	Miscellaneous examples on $Rr + Ss + Tt + U(rt - s^2) = V$	9.47
10. Transport equation		**10.1 – 10.5**
10.1	Introduction	10.1
10.2	An important theorem	10.1
10.3	Generalised or weak solution	10.2
10.4	Transport equation for a linear-hyperbolic system	10.3
	Miscellaneous problems on this part of the book	M.1 – M.10

S. CHAND & COMPANY PVT. LTD.

(An ISO 9001 : 2008 Company)

Head Office: 7361, RAM NAGAR, NEW DELHI - 110 055

Phone: 23672080-81-82, 9899107446, 9911310888 Fax: 91-11-23677446

Shop at: **schandgroup.com;** e-mail: **info@schandgroup.com**

Branches :

AHMEDABAD	:	1st Floor, Heritage, Near Gujarat Vidhyapeeth, Ashram Road, **Ahmedabad** - 380 014, Ph: 27541965, 27542369, ahmedabad@schandgroup.com
BENGALURU	:	No. 6, Ahuja Chambers, 1st Cross, Kumara Krupa Road, **Bengaluru** - 560 001, Ph: 22268048, 22354008, bangalore@schandgroup.com
BHOPAL	:	Bajaj Tower, Plot No. 2&3, Lala Lajpat Rai Colony, Raisen Road, **Bhopal** - 462 011, Ph: 4274723, 4209587. bhopal@schandgroup.com
CHANDIGARH	:	S.C.O. 2419-20, First Floor, Sector - 22-C (Near Aroma Hotel), **Chandigarh** -160 022, Ph: 2725443, 2725446, chandigarh@schandgroup.com
CHENNAI	:	No.1, Whites Road, Opposite Express Avenue, Royapettah, **Chennai** - 600014 Ph. 28410027, 28410058, chennai@schandgroup.com
COIMBATORE	:	1790, Trichy Road, LGB Colony, Ramanathapuram, **Coimbatore** -6410045, Ph: 2323620, 4217136 coimbatore@schandgroup.com **(Marketing Office)**
CUTTACK	:	1st Floor, Bhartia Tower, Badambadi, **Cuttack** - 753 009, Ph: 2332580; 2332581, cuttack@schandgroup.com
DEHRADUN	:	1st Floor, 20, New Road, Near Dwarka Store, **Dehradun** - 248 001, Ph: 2711101, 2710861, dehradun@schandgroup.com
GUWAHATI	:	Dilip Commercial (Ist floor), M.N. Road, Pan Bazar, **Guwahati** - 781 001, Ph: 2738811, 2735640 guwahati@schandgroup.com
HALDWANI	:	Bhatt Colony, Talli Bamori, Mukhani, **Haldwani** -263139 **(Marketing Office)** Mob. 09452294584
HYDERABAD	:	Padma Plaza, H.No. 3-4-630, Opp. Ratna College, Narayanaguda, **Hyderabad** - 500 029, Ph: 27550194, 27550195, hyderabad@schandgroup.com
JAIPUR	:	1st Floor, Nand Plaza, Hawa Sadak, Ajmer Road, **Jaipur** - 302 006, Ph: 2219175, 2219176, jaipur@schandgroup.com
JALANDHAR	:	Mai Hiran Gate, **Jalandhar** - 144 008, Ph: 2401630, 5000630, jalandhar@schandgroup.com
KOCHI	:	Kachapilly Square, Mullassery Canal Road, Ernakulam, **Kochi** - 682 011, Ph: 2378740, 2378207-08, cochin@schandgroup.com
KOLKATA	:	285/J, Bipin Bihari Ganguli Street, **Kolkata** - 700 012, Ph: 22367459, 22373914, kolkata@schandgroup.com
LUCKNOW	:	1st Floor, Arya Pratinidhi Sabha, 5-Mirabai Marg, Near Narhi Sabzi Mandi, Hazratganj, **Lucknow** - 226 001, Ph: 4065646, 4026791, 4027188, 4022948 lucknow@schandgroup.com
MUMBAI	:	Blackie House, IInd Floor, 103/5, Walchand Hirachand Marg, Opp. G.P.O., **Mumbai** - 400 001, Ph: 22690881, 22610885, mumbai@schandgroup.com
NAGPUR	:	Karnal Bagh, Near Model Mill Chowk**, Nagpur** - 440 032, Ph: 2720523, 2777666 nagpur@schandgroup.com
PATNA	:	104, Citicentre Ashok, Mahima Palace , Govind Mitra Road, **Patna** - 800 004, Ph: 2300489, 2302100, patna@schandgroup.com
PUNE	:	Sadguru Enclave, Ground floor, Survey No. 114/3, Plot no. 8 Alandi Road , Vishrantwadi **Pune** – 411015 Ph: 64017298 pune@schandgroup.com
RAIPUR	:	Kailash Residency, Plot No. 4B, Bottle House Road, Shankar Nagar, **Raipur -** 492 007, Ph: 2443142,Mb. : 09981200834, raipur@schandgroup.com **(Marketing Office)**
RANCHI	:	Shanti Deep Tower, Opp.Hotel Maharaja, Radium Road, **Ranchi**-834001 Mob. 09430246440 ranchi@schandgroup.com
SILIGURI	:	122, Raja Ram Mohan Roy Road, East Vivekanandapally, P.O., Siliguri, **Siliguri-**734001, Dist., Jalpaiguri, (W.B.) Ph. 0353-2520750 **(Marketing Office)** siliguri@schandgroup.com
VISAKHAPATNAM	:	No. 49-54-15/53/8, Plot No. 7, 1st Floor, Opp. Radhakrishna Towers, Seethammadhara North Extn., **Visakhapatnam** - 530 013, Ph-2782609 (M) 09440100555, visakhapatnam@schandgroup.com **(Marketing Office)**

© 1988, M.D. Raisinghania

All rights reserved. No part of this publication may be reproduced or copied in any material form (including photocopying or storing it in any medium in form of graphics, electronic or mechanical means and whether or not transient or incidental to some other use of this publication) without written permission of the copyright owner. Any breach of this will entail legal action and prosecution without further notice.

***Jurisdiction :** All disputes with respect to this publication shall be subject to the jurisdiction of the Courts, Tribunals and Forums of New Delhi, India only.*

First Edition 1988

Subsequent Editions and elarged Edition 1991, 95, 97, 2000, 2001, 2003, 2004, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014

Eighteenth Revised Editon 2015

ISBN: 978-81-219-0893-1 **Code :** 1014F 271

PRINTED IN INDIA

By Nirja Publishers & Printers Pvt. Ltd., 54/3/2, Jindal Paddy Compound, Kashipur Road, Rudrapur-263153, Uttarakhand and published by S. Chand & Company Pvt. Ltd., 7361, Ram Nagar, New Delhi -110 055.

Dedicated to the memory of my parents

PREFACE TO THE SIXTEENTH EDITION

Questions asked in recent papers of GATE and various university examinations have been inserted at appropriate places. Number of new solved problems have been added.

I hope that these changes will enhence the utlity of the book.

All valuable suggestions for further improvement of the book will be highly appreciated.

AUTHOR

PREFACE TO THE TWELFTH EDITION

Due to recent changes in syllabi of various universities, a thorough revision of the book was overdue. Accordingly, almost each chapter of this book has been enlarged and rearranged. References to the latest papers of various universities, I.A.S. and GATE etc. have been inserted at proper places. A set of objective problems (including those asked in various universities, I.A.S., Gate etc.) has been provided at the end of each chapter.

Four new chapters, namely, Chebyshev polynomials, Beta and Gamma functions, Power series and Transport equation, have been added. In part IV-B of this book, I have added ''Fourier Transforms and their applications''.

I hope that these changes will make the material more accessible and more attractive to the reader.

All valuable suggestions for further improvement of the book will be highly appreciated.

AUTHOR

PREFACE TO THE SEVENTH EDITION

It gives me great pleasure to inform the reader that the present edition of the book has been improved, well-organised, enlarged and made up-to-date in the light of latest syllabi. The following major changes have been made in the present edition:

- Almost all the chapters have been rewritten so that in the present form, the reader will not find any difficulty in understanding the subject matter.
- The matter of the previous edition has been re-organised so that now each topic gets its proper place in the book.
- More solved examples have been added so that the reader may gain confidence in the techniques of solving problems.
- References to the latest papers of various universities and I.A.S. examination have been made at proper places.
- Errors and omissions of the previous edition have been corrected.

In view of the above mentioned features it is expected that this new edition will prove more useful to the reader.

I am extremely thankful to the Managing Director, Shri Rajendra Kumar Gupta and the Director, Shri Ravindra Gupta for showing keen interest throughout the publication of the book.

Suggestions for further improvement of the book will be greatefully received.

AUTHOR

PREFACE TO THE FIRST EDITION

This book has been designed for the use of honours and postgraduate students of various Indian universities. It will also be found useful by the students preparing for various competitive examinations. During my long teaching experience I have fully understood the need of the students and hence I have taken great care to present the subject matter in the most clear, interesting and complete form from the student's point of view.

Do not start this book with an unreasonable fear. There are no mysteries in Mathematics. It is all simple and honest reasoning explained step by step which anybody can follow with a little effort and concentration. Often a student has difficulty in following a mathematical explanation only because the author skips steps which he assumes the students to be familiar with. If the student fails to recount the missing steps, he may be faced with a gap in the reasoning and the author's conclusion may become mysterious to him. I have avoided such gaps by giving necessary references throughout the book. I have been influenced by the following wise-saying.

"My passion is for lucidity. I don't mean simple mindedness. If people can't understand it, why write it."

AUTHOR

New unified U.G.C. syllabus of Differential Equations for B.A./B.Sc. (General and Honours) Part II

TOPICS	Chapters covering the topics
Unit I. Series solutions of differential equations, Power series method, Bessel, Legendre and hypergeometric equations. Bessel, Legendre and hypergeometric functions and their properties, convergence, recurrence and generating relations. Orthogonality of functions. Strum-Liouville problems. Orthogonality of eigen functions. Reality of eigen values. Orthogonality of Bessel functions and Legendre polynomials.	Refer chapter 8, 9, 11, 15 of Part I of this book.
Unit II. Partial differential equations of the first order. Lagrange's solution. Some special types of equations which can be solved easily by other than the general method. Charpit's general method of solution. Partial differential equations of second and higher orders. Classification of linear partial differential equations of second order. Homogeneous and non-homogeneous equations with constant coefficients. Partial differential equations reducible to equations with constant coefficients. Monge's methods.	Refer Chapters 1 to 9 of Part II of this book.
Unit III. Laplace transformation. Linearity of Laplace transformation. Existence theorem for Laplace transforms. Laplace transform of derivatives and integrals. Shifting theorems. Differentiation and integration of transforms. Convolution theorem. Solution of integral equations and system of differential equations using the Laplace transformation.	Refer Part IV-A of this book
Unit IV. Calculus of variations. Variational problems with fixed boundaries. Euler's equation for functionals containing first order derivative and one independent variable. Extremals. Functionals dependent on higher order derivatives. Functions dependent on more than one independent variable. Variational problem in parametric form. Invariance of Euler's equation under coordinate transformation. Variational problems with moving boundaries. Functionals dependent on one or two functions. One sided variations. Sufficient conditions for an extremum. Jacobi and Legendre conditions. Second variation Variational principle of least action.	Refer Part V of this book

CONTENTS

PART-I

ADVANCED ORDINARY DIFFERNTIAL EQUATIONS AND SPECIAL FUNCTIONS

CHAPTERS		PAGES
1.	**Picard's iterative method. Uniqueness and existence theorem**	**1.3–1.25**
1.1	Introduction	1.3
1.2A.	Picard's method of successive approximation (or Picard's iteration method)	1.3
1.2B.	Solved examples based on Art. 1.2A	1.4
1.3A.	Working rule for Picard's method of solving simulataneous differential equations with initial conditions	1.10
1.3B.	Solved examples based on Art. 1.3A	1.10
1.4	Problems of existence and uniqueness	1.14
1.5	Lipschitz condition	1.14
1.6	Picard's theorem. Existence and uniqueness theorem	1.15
1.7	An important theorem	1.18
1.8	Solved examples based on Articles 1.4 to 1.7	1.18
2.	**Simultaneous differential equations of the form $(dx)/P = (dy)/Q = (dz)/R$**	**2.1–2.24**
2.1	Introduction	2.1
2.2	The nature of solution of $(dx)/P = (dy)/Q = (dz)/R$	2.1
2.3	Geometrical interpretation of $(dx)/P = (dy)/Q = (dz)/R$	2.1
2.4	Rule I for solving $(dx)/P = (dy)/Q = (dz)/R$	2.1
2.5	Solved examples based on Art. 2.4	2.1
2.6	Rule II for solving $(dx)/P = (dy)/Q = (dz)/R$	2.3
2.7	Solved examples based on Art. 2.6	2.3
2.8	Rule III for solving $(dx)/P = (dy)/Q = (dz)/R$	2.5
2.9	Solved examples based on Art. 2.8	2.5
2.10	Rule IV for solving $(dx)/P = (dy)/Q = (dz)/R$	2.12
2.11	Solved examples based on Art. 2.10	2.13
2.12	Orthogonal trajectories of a system of curves on a surface	2.23
2.12 A.	Solved examples based on Art. 2.12	2.23
3.	**Total (or Pfaffian) differential equations**	**3.1–3.32**
3.1	Introduction	3.1
3.2	Total differential equation or Pfaffian differential equation	3.1
3.3	Necessary and sufficient conditions for integability of a single differential equation $Pdx + Qdy + Rdz = 0$	3.1
3.4	The conditions for exactness of $Pdx + Qdy + Rdz = 0$	3.3
3.5	Method of solving $Pdx + Qdy + Rdz = 0$	3.4
3.6	Special method I. Solution by inspection	3.4
3.7	Solved examples based on Art. 3.6	3.4
3.8	Special method II. Solution of homogeneous equation	3.12

3.9	Solved examples based on Art. 3.8	3.13
3.10	Special method III. Use of auxiliary equations	3.17
3.11	Solved examples based on Art. 3.10	3.17
3.12	General method of solving $Pdx + Qdy + Rdz = 0$ by taking one variable as constant	3.19
3.13	Solved examples based on Art. 3.12	3.20
3.14	Solution of $Pdx + Qdy + Rdz = 0$ when it is exact and homogeneous of degree $n \neq -1$.	2.24
3.15	The non-integrable single equation	3.25
3.16	Working rule for finding the curves represented by the solution of non-integrable total differential equation	3.25
3.17	Solved examples bsed on working rule 3.16	2.25
3.18	Geometrical interpretation of $Pdx + Qdy + Rdz = 0$	3.27
3.19	To show that the locus of $Pdx + Qdy + Rdz = 0$ is orthogonal to the locus of $(dx)/P = (dy)/Q = (dz)/R$	3.27
3.20	Total differential equation containing more than three variables	3.27
3.21	Solved examples based on Art. 3.20	3.28
3.22.	Working rule based on Art. 3.3. for solving $Pdx + Qdy + Rdz = 0$	3.31
4. Riccati's equation		**4.1 – 4.5**
4.1	Introduction	4.1
4.2	General solution of Riccati's equation	4.1
4.3	The cross-ratio of any four particular integrals of a Riccati's equation is independent of x	4.2
4.4	Method sof solving Riccati's equation when three particular integrals are known	4.2
4.5	Method of solving Riccati's equation when two particular integrals are known	4.3
4.6	Method of solving Riccati's equation when one particular integral is known	4.4
4.7	Solved examples	4.4
5. Chebyshev polynomials		**5.1 – 5.9**
5.1	Chebyshev polynomials	5.1
5.2	$T_n(x)$ and $U_n(x)$ are independent solutions of Chebyshev equation	5.1
5.3	Orthogonal properties of Chebyshev polynomials	5.2
5.4	Recurrence relations (formulas)	5.3
5.5	Some theorems on Chebyshev polynomials	5.3
5.6	First few Chebyshev polynomials	5.5
5.7	Generating functions for Chebyshev polynomials	5.6
5.8	Specal values of Chebyshev polynomials	5.7
5.9	Illustrative solved examples	5.8
6. Beta and Gamma functions		**6.1 – 6.22**
6.1	Introduction	6.1
6.2	Euler's integrals. Beta and Gamma functions	6.1
6.3	Properties of Gamma function	6.1
6.4	Extension of definition of Gamma function	6.2
6.5	To show that $\Gamma(1/2) = \sqrt{\pi}$	6.3
6.6	Transformation of Gamma function	6.3
6.7	Solved examles based on Gamma function	6.4

6.8	Symmetrical property of Beta function	6.8
6.9	Evaluation of Beta function $B(m, n)$ in an explicit form when m or n is a positive integer	6.8
6.10	Transformation of Beta function	6.9
6.11	Relation between Beta and Gamma functions	6.12
6.12	Solved examples	6.15
6.13	Legendre duplication formula	6.20
6.14	Solved examlpes	6.21
7. Power series		**7.1 – 7.7**
7.1	Introduction	7.1
7.2	Summary of useful results	7.1
7.3	Power series	7.2
7.4	Some important facts about the power series	7.2
7.5	Radius of convergence and interval of convergence	7.2
7.6	Formulas for determining the radius of convergence	7.3
7.7	Solved examples based on Art. 7.6	7.4
7.8	Some theorems about power series	7.6
8. Integration in series		**8.1 – 8.60**
8.1	Introduction	8.1
8.2	Some basic definitions	8.1
8.3	Ordinary and singular points	8.2
8.4	Solved examples based on Art. 8.3	8.2
8.5	Power series solution in powers of $(x - x_0)$	8.4
8.6	Solved examples based on Art. 8.5	8.4
8.7	Series solution about regular singular point $x = 0$. Frobenius method	8.15
8.8	Working rule for solution by Frobenius method	8.17
8.9	Examples of type-1 based on Frobenius method	8.18
8.10	Examples of type-2 based on Frobenius method	8.29
8.11	Examples of type-3 based on Frobenius method	8.35
8.12	Examples of type-4 based on Fronenius method	8.44
8.13	Series solution about regular singular point at infinity	8.51
8.14	Solved examples based on Art. 8.13	8.51
8.15	Series solution in descending powers of independent variable	8.55
8.16	Solved examples based on Art. 8.15	8.56
8.17	Method of differentiation	8.57
	Objective problems on chapter 8	8.58
9. Legendre polynomials		**9.1 – 9.50**
PART I: Legendre function of the first kind		**9.1 – 9.43**
9.1	Legendre's equation and its solution	9.1
9.2	Legendre function of the first kind or Legendre polynomial of degree n	9.3
9.3	Generating function for Legendre polynomials	9.4
9.4	Solved examples based on Art. 9.2 and Art. 9.3	9.5
9.5	Trigonometric series for $P_n(x)$	9.10
9.6	Laplace's definite integrals for $P_n(x)$	9.12

9.7	Some bounds on $P_n(x)$	9.13
9.8	Orthogonal properties of Legendre's polynomials	9.14
9.9	Recurrence relations (formulas)	9.15
9.10	Beltrami's result	9.17
9.11	Christoffel's summation formula	9.17
9.12	Christoffel's expansion	9.18
9.13	Solved examples based on Art. 9.8 and Art. 9.9	9.18
9.14	Rodigue's formula	9.26
9.15	Solved examples based on Art. 9.14	9.27
9.16	Legendre's series for $f(x)$, where $f(x)$ is a polynomial	9.34
9.17	Solved examples based on Art. 9.16	9.35
9.18	Expansion of function $f(x)$ in a series of Legendre polynomials	9.36
9.19	Even and odd functions	9.37
9.20	Expansion of x^n is Legendre polynomials	9.38
9.21	Solved examples based on Art. 9.20	9.41
	Objective problems on chapter 9	9.43
PART II: Associated Legendre functions of the first kind		**9.43 – 9.50**
9.22	Associated Legendre functions	9.43
9.23	Properties of associated Legendre functions	9.45
9.24	Orthogonality relations for associated Legendre functions	9.46
9.25	Recurrence relations for associated Legendre functions	9.48
10. Legendre functions of the second kind		**10.1 – 10.12**
10.1	Some useful results	10.1
10.2	Recurrence relations	10.2
10.3	Theorem	10.5
10.4	Complete solution of Legendre's equation	10.5
10.5	Christoffel's second summation formula	10.6
10.6	A relation connecting $P_n(x)$ and $Q_n(x)$	10.7
10.8	Solved examples on chapter 8	10.8
11. Bessel functions		**11.1 – 11.45**
11.1	Bessel's equations and its solution	11.1
11.2	Bessel's function of the first kind of order n	11.2
11.3	List of important results of Gamma and Beta functions	11.3
11.4	Relation between $J_n(x)$ and $J_{-n}(x)$, n being an integer	11.3
11.5	Bessel's function of the second kind of order n	11.5
11.6	Integration of Bessel equation in series for $n = 0$	11.5
	Bessel's function of zeroth order, *i.e.*, $J_0(x)$	11.5
11.6A.	Solved examples based on Articles 11.1 to 11.6	11.6
11.7	Recurrence relations for $J_n(x)$	11.7
11.7A.	Solved examples based on Art. 11.7	11.19
11.7B.	Solved examles involving integration and recurrence relations	11.27
11.8	Generating function for the Bessel's function $J_n(x)$	11.31

11.9	Trigonometric expansions involving Bessel functions	11.33
11.9 A.	Solved examples based on Art. 11.8 and Art. 11.9	11.33
11.10	Orthogonality of Bessel functions	11.40
11.11	Bessel-series or Fourier-Bessel expansion for $f(x)$	11.42
11.11 A.	Solved examples based on Art. 11.11	11.42
	Objective problems on chapter 11	11.44
12 Hermite polynomials		**12.1 – 12.12**
12.1	Hermite equation and its solution	12.1
12.2	Hermite polynomial of order n	12.3
12.3	Generating function for Hermite polynomials	12.3
12.4	Alternative expressions for the Hermite polynomials	12.3
	Rodrigue's formula for Hermite polynomials	12.3
12.5	Hermite polynomials for some special values of n	10.4
12.6	Evaluation of values of $H_{2n}(0)$ and $H_{2n+1}(0)$	12.5
12.7	Orthogonality properties of the Hermite polynomials	12.5
12.8	Recurrence relations (or formulas)	12.6.
12.9	Solved examples	12.7
13. Laguerre polynomial		**13.1 – 13.11**
13.1	Laguerre equation and its solution	13.1
13.2A.	Laguerre polynomial of order (or degree) n	13.2
13.2B.	Alternative definition of Leguerre polynomial of order (or degree) n	13.2
13.3	Generating function for Laguerre polynomials	13.2
13.4	Alternating expression for the Laguerre polynomials	13.3
13.5	First few Laguerre polynomials	13.3
13.6	Orthogonal properties of Laguerres polynomials	13.4
13.7	Expansion of a polynomial in a series of Laguerre polynomials	13.5
13.8	Relation between Laguerre polynomial and their derivatives	13.6
13.9	Solved examples	13.7
14. Hypergeometric function		**14.1 – 14.18**
14.1	Pochhammer symbol	14.1
14.2	General hypergeometric function	14.1
14.3	Confluent hypergeometric (or Kummer) function	14.1
14.4	Hypergeometric function	14.1
14.5	Gauss's hypergeometric equation	14.2
14.6	Solution of hypergeometric equation	14.2
14.7	Symmetric property of hypergeometric function	14.3
14.8	Differentiation of hypergeometric function	14.3
14.9	Integral representation for hypergeometric function	14.4
14.10	Gauss theorem	14.5
14.11	Vandermonde's theorem	14.5
14.12	Kummer's theorem	14.6
14.13	More about confluent hypergeometric function	14.6
14.14	Differentiation of confluent hypergeometric function	14.8
14.15	Integral representation for confluent hypergeometric function	14.8

14.16	Kummer's relation	14.9
14.17	Contiguous hypergeometric functions	14.9
14.18	Contiguous relationship	14.9
14.19	Contiguous relationship for confluent hypergeometric function	14.9
14.20	Solved examples	14.10
15. Orthogonal set of functions and Strum-Liouville problem		**15.1 – 15.25**
15.1	Orthogonality	15.1
15.2	Orthogonal set of function	15.1
15.3	Orthonormal set of functions	15.1
15.4	Orthogonality with respect to a weight function	15.1
15.5	Orthogonal set of functions with respect to a weight function	15.1
15.6	Orthogonal set of functions with respect to a weight function	15.1
15.7	Working rule for getting orthonormal set	15.2
15.8	Gram-Schmidth process of orthonormalization	15.2
15.9	Illustrative solved examples	15.3
15.10	Strum-Liouville problem	15.10
	Eigen (or characteristic) functions and eigen (or characteristic) values	15.10
15.11	Orthogonality of eigenfunctions	15.10
15.12	Reality of eigenvalues	15.12
15.13	Solved examples	15.13
15.14	Orthogonality of Legendre polynomials	15.21
15.15	Orthogonality of Bessel functions	15.21
15.16	Orthogonality on an infinite interval	15.22
15.17	Orthogonal expansion or generalised Fourier series	15.22
	Objective problems on chapter 15	15.24
	Miscellaneous problems on this part of the book	M.1 – M.4

PART-II

PARTIAL DIFFERENTIAL EQUATIONS

CHAPTERS		PAGES
1. Origin of partial differential equations		**1.3 – 1.19**
1.1	Introduction	1.3
1.2	Partial differential equation. Definition	1.3
1.3	Order of a partial differential equation	1.3
1.4	Degree of a partial differential equation	1.3
1.5	Linear and non-linear partial differential equation	1.3
1.6	Notations	1.3
1.7	Classification of first order partial differential equations	1.4
1.8	Origin of partial differential equations	1.4
1.9	Rule I. Derivation of partial differential equations by the elimination of arbitrary constants	1.4
1.10	Solved examples based on rule I of Art. 1.9	1.5
1.11	Rule II. Derivation of partial differential equations by elimination of arbitrary functions ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z	1.11

1.12	Solved examples based on rule II of Art. 1.11	1.11
1.13	Cauchy's problem for first order equations	1.17
	Objective problems on chapter 1	1.18
2. Linear partial differential equations of order one		**2.1 – 2.40**
2.1	Lagrange's equations	2.1
2.2	Lagrange's method of solving $Pp + Qq = R$	2.1
2.3	Working rule for solving $Pp + Qq = R$ by Lagrange's method	2.2
2.4	Examples based on working rule of Art. 2.3	2.2
2.5	Type 1 based on rule I for solving $(dx)/P = (dy)/Q = (dz)/R$	2.2
2.6	Solved examples based on Art. 2.5	2.2
2.7	Type 2 based on rule II for solving $(dx)/P = (dy)/Q = (dz)/R$	2.4
2.8	Solved examples based on Art. 2.7	2.5
2.9	Type 3 based on rule III for solving $(dx)/P = (dy)/Q = (dz)/R$	2.8
2.10	Solved examples based on Art. 2.9	2.8
2.11	Type 4 based on rule IV for solving $(dx)/P = (dy)/Q = (dz)/R$	2.16
2.12	Solved examples based on Art. 2.11	2.16
2.13	Miscellaneous examples based on $Pp + Qq = R$	2.26
2.14	Integral surfaces passing through a given curve	2.27
2.15	Solved examples based on Art. 2.14	2.27
2.16	Surfaces orthogonal to a gvien system of surfaces	2.31
2.17	Solved examples based on Art. 2.16	2.31
2.18 (*a*)	Geometrical description of the solutions of $Pp + Qq = R$ and the system of equations $(dx)/P = (dy)/Q = (dz)/R$ and relationship between the two	2.33
2.18 (*b*)	Another geometrical interpretation of Lagrange's equation	2.34
2.19	Solved examples based on Art. 2.18(*a*) and 2.18(*b*)	2.34
2.20	The linear partial differentil equation with n independent variables and its solutions	2.35
2.21	Solved examples based on Art. 2.20	2.35
3. Non-linear partial differential equations of order one		**3.1 – 3.83**
3.1	Complete integral (or complete solution), particular integral, singular integral (or singular solution) and general integral (or general solution)	3.1
3.2	Geometrical interpretation of integrals of $f(x, y, z, p, q) = 0$	3.2
3.3	Method of getting singular integral directly from the partial differential equation of first order	3.3
3.4	Compatible system of first-order equations	3.3
3.5 A	particular case of Art. 3.4	3.4
3.6	Solved examples based on Art. 3.4 and Art. 3.5	3.5
3.7	Charpit's method	3.11
3.8 A.	Working rule while using Charpit's method	3.12
3.8 B.	Solved examples based on Art. 3.8A	3.12
3.9	Special methods of solutions applicable to certain standard forms	3.33
3.10	Standard form I. Only p and q present	3.33
3.11	Solved examples based on Art. 3.10	3.34
3.12	Standard form II. Clairaut's equations	3.44
3.13	Solved examles based on Art. 3.12	3.44

3.14	Standard form III. Only p, q and z present	3.49
3.15	Working rule for solving equations of the form $f(p, q, z) = 0$	3.49
3.16	Solved examples based on Art. 3.15	3.50
3.17	Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$	3.57
3.18	Solved examples based on Art. 3.17	3.57
3.19	Jacobi's method	3.63
3.20	Working rule for solving partial differential equations with three or more independent variables. Jacobi's method	3.64
3.21	Solved examples based on Art. 3.20	3.65
3.22	Jacobi's method for solving a non-linear first order partial differential equation in two independent variables	3.74
3.23	Cauchy's method of characteristics for solving non-linear partial differential equations	3.76
3.24	Some theorems	3.79
3.25	Solved examples based on Art. 3.23 and Art. 3.24	3.79

4. Homogeneous linear partial differential equations with constant coefficients 4.1 – 4.34

4.1	Homogeneous and non-homogeneous linear partial differential equations with constant coefficients	4.1
4.2	Solution of a homogeneous linar partial differential equation with constant coefficients	4.1
4.3	Method of finding the complementary function (C.F.) of linear homogeneous partial differential equation with constant coefficients	4.2
4.4 A.	Working rule for finding C.F. of linear partial differential equations with constant coefficients	4.4
4.4 B.	Alternative working rule for finding C.F.	4.4
4.5	Solved examples based on Art. 4.4A and 4.4B	4.5
4.6	Particular integral (P.I.) of homogeneous partial differential equations	4.6
4.7	Short methods of finding P.I. in certain cases	4.7
4.8	Short method I. When $f(x, y)$ is of the form $\phi(ax + by)$	4.7
4.9	Solved examples based on Art. 4.8	4.9
4.10	Short method II. When $f(x, y)$ is of the form $x^m y^n$ or a rational integral algebraic function of x and y	4.18
4.11	Solved examples based on Art. 4.10	4.19
4.12	A general method of finding the P.I. of linear homogeneous parial differential equation with constant coefficients	4.25
4.13	Solved examples based on Art. 4.12	4.26
4.14	Solutions under given geometrical conditions	4.32
4.15	Solved examples based on Art. 4.14	4.32
	Objective problems on chapter 4	4.34

5. Non-homogeneous linear partial differential equations with constant coefficients 5.1 – 5.30

5.1	Non-homogeneous linear partial differential equation with constant coefficients	5.1
5.2	Reducible and irreducible linear differential operators	5.1
5.3	Reducible and irreducible linear partial differential equations with constant coefficients	5.1
5.4	Theorem. If the operator $F(D, D')$ is reducible, then the order in which the linear factors occur is unimportant	5.1

5.5	Determination of complementary function (C.F) of a reducible non-homogeneous linear partial differential equation with constant coefficients	5.2
5.6	Working rule for finding C.F. of reducible non-homogeneous linear partial differential equations with constant coefficients	5.3
5.7	Solved examples based on Art. 5.6	5.4
5.8	Method of finding C.F. of irreducible linear partial differential equation with constant coefficients	5.5
5.9	Solved examples based on Art. 5.8	5.6
5.10	General solution of non-homogeneous linear partial differential equations with constant coefficients	5.9
5.11	Particulars integral (P.I.) of non-homogeneous linear partical differetial equations	5.9
5.12	Determination of P.I. of non-homogeneous linear partial differential equations (reducible or irreducible)	5.9
5.13	Solved examples based on Art. 5.6, Art. 5.8 and Art. 5.12.	5.10
5.14	General method of finding P.I. for only reducible non-homogeneous linear partial differential equations with constant coefficients	5.26
5.15	Working rule for finding P.I. of any reducible linear partial differential equations (homogeneous or non-homogeneous)	5.27
5.16	Solved examples based on Art. 5.15	5.28
5.17	Solutions under given geometrical conditions	5.30
6.	**Partial differential equations reducible to equations with constant coefficients**	**6.1–6.11**
6.1	Introduction	6.1
6.2	Method of reducible Euler-Cauchy type equation to linear partial differential equation with constant coefficients	6.1
6.3	Working rule for solving Euler-Cauchy type equations	6.2
6.4	Solved examples based on Art. 6.3	6.2
6.5	Solutions under given geometical conditions	6.10
7.	**Partial differential equations of order two with variables coefficients**	**7.1 – 7.14**
7.1	Introduction	7.1
7.2	Type I	7.1
7.3	Solved examles based on Art 7.2	7.2
7.4	Type II	7.4
7.5	Solved examples based on Art. 7.4	7.4
7.6	Type III	7.7
7.7	Solved examples based on Art. 7.6	7.7
7.8	Type IV	7.11
7.9	Solved examples based on Art. 7.8	7.12
7.10	Solutions of equations under given geometrical conditions	7.12
7.11	Solved examles based on Art. 7.10	7.13
8.	**Classifiation of partial differential equations Reduction to cononial or normal form. Riemann method**	**8.1 – 8.50**
8.1	Classification of partial differential equation of second order	8.1
8.2	Classificationof partial differential equations in three independent variables	8.2
8.2A.	Solved examples based on Art. 8.2	8.2

PART-III

BOUNDARY VALUE PROBLEMS AND THEIR SOLUTIONS BY SEPARATION OF VARIABLES

CHAPTERS **PAGES**

1. Heat, wave and telegraph equation. Method of separation of variables **1.3 – 1.18**

1.1 Introduction 1.3

1.2 Derivation of one dimensional wave equation 1.3

1.3 Derivation of two-dimensional wave equation 1.4

1.4 Derivation of one dimensional heat equation 1.5

1.5 Derivation of Fourier equation of heat conduction 1.6

1.6 A. Laplace's equation 1.7

1.6 B. Laplace's equation in plane polar coordinates 1.8

1.6 C. Laplace's equation in cylindrical coordinates 1.9

1.6 D. Laplace's equation in spherical coordinates 1.10

1.7 Derivation of telegraph (or transmission) line equations 1.11

1.8 Boundary value problms (B.V.P.) 1.12

1.9 Method of separation of variables or product method 1.12

1.10 A. The principle of superposition 1.13

1.10 B. Notations 1.13

1.11 Solutions of boundary value problems 1.13

2. Boundary value problems in cartesian coordinates **2.1 – 2.98**

2.1 Introduction 2.1

PART I: Problems based on one-dimensional heat (or diffusion) equations **2.3 – 2.27**

2.2 A. General solution of one dimensional heat flow equation by the method of separation of variables 2.3

2.2 B. Solved examples based on Art. 2.2A 2.4

2.3 A. General solution of heat equation when both the ends of a bar are kept at temperature zero and the initial temperature is prescribed 2.5

2.3 B. Working rule for solving heat equation when both the ends of a bar are kept at temperature zero and the initial temperature is prescribed 2.6

3.3 C. Solved examples based on Art. 2.3A and 2.3B 2.7

2.4 A. General solution of heat equation when both the ends of a bar are insulated and the initial temperature is prescribed 2.15

2.4. B. Working rule for solving heat equation when both the ends of bar are insulated and the initial temperature is prescribed 2.17

4.4 C. Solved examples based on Art. 2.4 A and Art. 2.4B 2.18

2.5 Solution of heat equation when only one end is inslated 2.20

2.6 Miscellaneous examples based on heat (or diffusion) equation 2.23

PART II: Problems based on two-dimensional heat (or diffusion) equation **2.27 – 2.33**

2.7 A. General solution of two-dimensional heat (or diffusion) equation 2.27

2.7 B. General solution of two-dimensional heat (or diffusion) equation satisfying the given boundary and initial conditions 2.28
2.7 C. Working rule for solving two-dimensional heat (or diffusion) equation 2.29
2.7 D. Solved examples based on Art. 2.7A, Art 2.7B and Art 2.7C 2.30
PART III: Problems based on three-dimensional heat (or diffusion) equation 2.33 – 2.35
2.8 A. General solution of three-dimensional heat (or diffusion) equation 2.33
2.8 B. Solution of three-dimensional heat (or diffusion) equation satisfying the given boundary and initial conditons 2.34
PART IV: Problems based on one-dimensional wave equations 2.35 – 2.64
2.9 A. General solution of one-dimensional wave equation 2.35
2.9 B. Examples based on Art 2.9A 3.36
2.10 A. General solution of one-dimensional wave equation satifying the given boundary and initial conditions 2.37
2.10 B. Working rule for solving one-dimensional wave equation when both the ends of a string are kept fixed and initial deflection (or shape) and velocity are prescribed 2.39
2.10 C. Working rule for solving one-dimensional wave equation when both the ends of the string are fixed and initial velocity of the string is zero, *i.e.*, the string starts from the position of rest 2.39
2.10 D Working rule for solving one-dimensional wave equation when the ends of the string are kept fixed and the initial deflection of the string is zero 2.39
2.10 E. Solved examples based on Articles 2.10 A to 2.10D 2.40
2.10 F. Miscellaneous solved examples based on one dimensional wave equation 2.54
2.10 G. Derivation of one-dimensional wave equation in another form and its solution for a hanging chain suspended vertically from one end 2.61
PART V: Problems based on two-dimensional wave equation 2.64 – 2.68
2.11 A General solution of two-dimensional wave equation 2.64
2.11 B Solved examples based on Art. 2.11A 2.67
PARTVI: Problems based on three dimensional wave equation 2.68 – 2.70
2.12 A. General soluton of three-dimensional wave equation 2.69
2.12 B. Solved examples based on Art. 2.12 A 2.69
PART VII: Problems based on two-dimensional Laplace's equation 2.70 – 2.90
2.13 A. General solution of two-dimensional Laplace's equation 2.71
2.13 B. Dirichlet problem in a rectangle 2.71
2.13 C. Solved examlpes based on Art. 2.13B or problems similar to the problems based on Art. 2.13A and Art. 2.13B 2.73
2.13 D. The Neumann problem in a rectangle 2.81
2.13 E. ''Mixed problems'' in a rectangle 2.83
PART VIII: Problem based on three-dimensional Laplace's equation 2.91 – 2.93
2.14 A. General solution of three dimensional Laplace's equation 2.91
2.14 B. Solved examples based on Art. 2.14A 2.91
PART IX: Prolems based on telegraph (or transimision) line equation 2.93 – 2.95
3. Boundary value problems in polar coordinates 3.1 – 3.24
3.1. Introduction 3.1
3.2 Solution of Laplace's equation in polar coordinates 3.2

3.3 Temperature inside a circular plate 3.22
3.4 The viberation of a circular plate 3.23
4. Boundary value problems in cylindrial coordinates 4.1 – 4.20
4.1 Introduction 4.1
4.2 Solution of Laplace's equation 4.2
4.3 Circular harmonics 4.7
4.4 Solution of heat (or diffusion) equation 4.9
4.5 Solution of wave equation 4.13
5. Boundary value problems in spherical coordinates 5.1 – 5.22
5.1 Introduction 5.1
5.2 Solution of Laplace's equation 5.2
5.3 Solution of heat (or diffusion) equation 5.12
5.4 Solution of wave equation 5.17

PART-IV-A

LAPLACE TRANFORMS WITH APPLICATIONS

CHAPTERS **PAGES**

1. The Laplace transform 1.3 – 1.54
1.1 Introduction 1.3
1.2 Laplace transform. Definition 1.3
1.3 Working rule to find Laplace transform 1.3
1.4 Piecewise (or sectinally) continuous function 1.4
1.5 Functions of exponential order 1.4
1.5 A. Solved examples based on functions of exponential order 1.4
1.6 Functions of class A 1.5
1.7 Sufficient condition for the existence of Laplace transform 1.5
1.8 Linearity property of Laplace transform 1.6
1.9 Laplace transforms of some elementary functions 1.6
1.10 Table of Laplace transforms 1.9
1.10 A. Solved examples based on Art. 1.8 and Art. 1.9 1.9
1.11 First shifting (or first translation) theorem 1.16
1.11 A. Solved examles based on Art. 1.11 1.17
1.12 The unit step function or Heaviside's unit function 1.21
1.13 Second shifting (or second translation) theorem 1.21
1.13 A. Solved examples based on Art. 1.13 1.22
1.14 Change of scale property 1.23
1.14 A. Solved examples based on Art. 1.14 1.23
1.15 Laplace transform of derivatives 1.24
1.15 A. Solved examples based on Art. 1.15 1.27
1.16 Multiplication by positive integral powers of t. Derivatives of Laplace transform. 1.29
1.16 A. Solved examples based on Art. 1.16 1.30
1.17 Division by t. Theorem 1.34

1.17 A.	Solved examples based on Art. 1.17	1.35
1.18	Laplace transforms of integrals	1.38
1.18 A.	Solved examples based on Art. 1.18	1.39
1.19	Initial value theorem and final value theorem	1.40
1.20	Laplace transform of periodic functions	1.41
1.20 A.	Solved examples based on Art. 1.20	1.41
1.21	Evaluation of integrals	1.43
1.21 A.	Solved examples based on Art. 1.21	1.43
1.22	Laplace transforms of some special functions	1.45
	The error function	1.45
1.22 A.	Solved examples based on error functions	1.46
	The sine integral	1.46
	The cosine integral	1.47
	The exponential integral	1.48
	The Laguerre polynomial	1.48
	The Dirac' delta function	1.49
	The Bessel function	1.49
1.22 B.	Solved examples based on Bessel function	1.50
1.23	Table of some Laplace transform theorems for ready reference	1.52
	Objective problems on Chapter 1	1.52
2. The inverse Laplace transform		**2.1 – 2.58**
2.1	Introduction	2.1
2.2	Inverse Laplace transform. Definition	2.1
2.3	Null function. Definition	2.1
2.3 A.	Uniqueness of inverse Laplace transforms	2.1
	Lerch's theorem	2.1
2.4	Inverse Laplace transform of some elementary functions	2.2
2.5	Linearity propety of inverse Laplace transform	2.3
2.5 A.	Solved examples based on Art. 2.4 and Art. 2.5	2.3
2.6	Method of partial fractions	2.9
2.6 A.	Solved examples based on Art. 2.6	2.11
2.7	Heaviside's expansion theorem (or formula)	2.15
2.7 A.	Solved examples based on Art. 2.7	2.15
2.8	First translation or first shifting theorem	2.17
2.8 A.	Solved examples based on Art. 2.8	2.18
2.9	Second shifting theorem or second translation theorem	2.28
2.9 A.	Solved examples based on Art. 2.9	2.29
2.10	Change of scale property	2.32
2.10 A.	Solved examples based on Art. 2.10	2.32
2.11	Inverse Laplace transform of derivatives	2.33
2.11 A.	Solved examples based on Art. 2.11	2.33
2.12	Inverse Laplace transform of integrals	2.35
2.13	Multiplication by powers of s	2.36
2.14 A.	Division by powers of s	2.37

2.14 B.	An important theorem	2.37
2.14 C.	Solved examples based on Art. 2.14A and 2.14B	2.38
2.15	Convolution (or Falting or resultant)	2.42
2.16	The convolution theorem (or convolution property)	2.43
2.16 A.	Solved examples based on Art. 2.16	2.44
2.17	Table of some inverse Laplace transform for ready reference	2.56
	Objective problems on chapter 2	2.56
3. Applications to ordinary differential equations		**3.1 – 3.38**
3.1	Solution of ordinary differential equation with constant coefficients	3.1
3.1 A.	Solved examples based on Art 3.1	3.1
3.2	Solution of ordinary differential equations with variables coefficients	3.24
3.2 A.	Solved examples based on Art. 3.2	3.24
3.3	Soluion of simultaneous ordinary differential equations	3.28
3.3 A.	Solved examples based on Art. 3.3	3.28
4. Application to Integral equations		**4.1 – 4.13**
4.1	Integral equation. Definition	4.1
4.2	Some important results to be committed to memory	4.1
4.3	Application of Laplace transform in solving Volterra integral equation type with convolution working rule	4.2
4.4	Abel's integral equation	4.2
4.5	Integro-differential equation	4.2
4.6	Solved examples based on integral equations	4.2
5. Application to boundary value problems		**5.1 – 5.26**
5.1 A.	Initial and boundary conditions. Definitions	5.1
5.1 B.	Boundary value problems involving partial differential equations	5.1
5.1 C.	Working rule for solving boundary value problems involving partial differential equations	5.1
5.2	Solved examples based on Art. 5.1C	5.2
5.3	Summary of some important partial differential equations	5.13
	One dimensional heat conduction (or diffusion) equation	5.13
	One dimensional wave equation	5.13
5.4	Solution of one-dimensional heat conduction (or diffusion) equation	5.13
5.5	Solution of one-dimensional wave equation	5.22

PART-IV-B

FOURIER TRANSFORMS AND THEIR APPLICATIONS

CHAPTERS		PAGES
1. Fourier integrals and Fourier Transforms		**1.3 – 1.49**
PART I: Fourier integrals and their applications		**1.3 – 1.10**
1.1	Introduction	1.3
1.2	Dirichlet's conditions	1.3
1.3	Fourier integral theorem (or formula). Fourier integral representation	1.3
1.4	Different forms of Fourier integral theorem (or formula)	1.3
1.5	Solved examples based on Art. 1.4	1.6

PART II: (Infinite) Fourier transforms **1.10 – 1.35**

1.6 (Infinite) Fourier tranform or complex Fourier transform 1.10
1.7 Fourier sine transform (or infinite Fourier sine transform) 1.12
1.8 Fourier cosine tranform (or infinite Fourier cosine transform) 1.12
1.9 Relationship between Fourier and Laplace transforms 1.13
1.10 Some useful results of Integral calculus for direct applications 1.14
1.11 Linearity property of Fourier transforms 1.14
1.12 Change of scale property 1.15
1.13 Shifting property of Fourier transforms 1.16
1.14 Modulation theorem 1.16
1.15 Solved examples based on infinite Fourier transform 1.16
1.16 Solved examples based on infinite Fourier sine transform and infinite Fourier cosine transform 1.20
1.17 Solution of integral equations with help of infinite Fourier sine and cosine transforms 1.30
1.18 Solved examples based on Art. 1.17 1.31
1.19 Convolution or falting 1.33
1.20 The convolution (or falting) theorem for Fourier transforms 1.33
1.21 Parseval's identity for Fourier transform Rayleigh's theorem or Plancherel's theorem 1.33
1.21 A Examples based on Art. 1.21 1.34
1.22 Fourier transforms of the derivatives of a funcion. Derivative theorem 1.35
1.23 Multiple Fourier transforms 1.35

PART III: Applications of infinite Fourier transforms to boundary value problems **1.35 – 1.49**

1.24 Some useful results for direct applications 1.35
1.25 Solved examples based on applications of infinite Fourier transforms to boundary value problems 1.37

2.Finite Fourier transforms **2.1 – 2.30**

PART I: Finite Fourier transforms **2.1 – 2.13**

2.1 Introduction 2.1
2.2 The finite Fourier sine transform 2.1
2.3 The finite Fourier cosine transform 2.2
2.4 Some useful results for direct applications 2.3
2.5 Solved examples based on Finite sine and cosine transforms 2.4
2.6 Relation between the finite Fourier transforms of the derivatives of a function 2.12
2.7 Multiple finite Fourier transform 2.12

PART II: Application of Finite Fourier sine and cosine transforms to boundary value problems **2.13 – 2.30**

2.8 Some useful results for direct applications 2.13
2.9 Solved examples based on applications of finite Fourier sine and cosine transforms to boundary value problems 2.14

PART-V

CALCULUS OF VARIATIONS

CHAPTERS **PAGES**

1. Variational problems with fixed boundaries **1.3 – 1.83**

1.1 Introduction 1.3

1.2 Variation and its properties 1.4

1.2 A. Solved examples based on Art. 1.1 and Art. 1.2 1.6

1.3 The fundamental lemma of the calculus of variations 1.8

1.4 Euler's equation 1.9

1.4 A An alternative proof of Euler's equation 1.10

1.5 Alternative forms of Euler's equation 1.12

1.5 A. Illustrative solved examples based on Art. 1.5 1.13

1.5 B. Case I: If F is independent of y' 1.19

1.5 C. Case II: The function F is linearly dependent of y' such that

$F(x, y, y') = M(x, y) + N(x, y)y'$ 1.20

1.5 D. Case III: The function F is independent of x and y 1.21

1.5 E. Case IV : F is independent of y 1.22

1.5 F. Case V : F is independent of x 1.24

1.6 Invariance of Euler's equation under co-ordinate transformation 1.28

1.6 A. Solved examples based on Art 1.6 1.29

1.7 Geoderics 1.32

1.7 A. Illustrative solved examples based on Art. 1.7 1.32

1.8 Variational problems for functional involving several dependent varaibles 1.37

1.8 A. Alternative proof of getting the system of Euler's equations 1.39

1.8 B. Solved example based on Art. 1.8 and 1.8A 1.39

1.9 Functional dependent on higher order derivatives 1.43

Particular cases 1.43

1.9 A. General case. Functional dependent on higher order derivatives 1.46

1.9 B. Solved examples based on Art. 1.9 and 1.9A 1.47

1.10 Functional dependent on the functions of several independent variables 1.53

1.10 A. Illustrative solved examples based on Art. 1.10 1.55

1.11 Variational problems in parametric form 1.58

1.11 A. Euler's equations in parametric form (an alternative form) 1.60

1.11 B. Solved examples based on Art. 1.11 and 1.11A 1.61

1.12 Isoperimetric problems 1.65

1.13 Working rule for solving isoperimetric problems 1.67

1.13 A. Illustrative solved examples based on Art. 1.13 1.68

1.14 Reciprocity principle 1.78

1.15 Hamilton principle. Variational principle of least action 1.81

2. Variational problems with moving (or free) boundaries. One sided variations 2.1 – 2.35

2.1 Variational problems with moving (or free) boundaries 2.1

2.1 A. Some important results 2.5

2.2	Illustrative solved examples based on Art. 2.1 and Art. 2.1 A	
2.3	Variational problem with a moving boundary for a functional dependent on two functions	2.23
2.3 A.	Working rule for solving variational problems with a moving boundary for a functional dependent on two functions	2.25
2.4	Illustrative solved examples based on Art. 2.3 and Art. 2.3A	2.26
2.5	One sided variations	2.31
2.6	Illustrative solved examples based on Art. 2.5	2.33
3. Sufficient conditions for an extremum		**3.1 – 3.28**
3.1	Proper field, central field and field of extremals	3.1
3.1 A	Solved examples based on Art. 3.1	3.2
3.2	Jacobi Condition	3.4
3.2 A.	solved examples based on Art. 3.1 and 3.2.	3.6
3.3	Weierstrass function	3.10
3.3 A.	Solved examples based on Art. 3.3	3.12
3.4	Legendre condition	3.14
3.4 A.	Working rule for testing a weak minimum and weak maximum	3.15
3.4 B.	Solved exampes based on Art. 3.4 and 3.4 A	3.15
3.4 C.	An alternative form of Jacobi equation. Jacobi's conditions	3.20
3.4 D.	Solved examples based on Art. 3.4C	3.21
3.4 E.	Second variation and condition for week minimum	3.22
3.5	Determination of second variation of a function	3.25
	Miscellanceous Problems on Part V of this book	**M.1–M.4**
	Additional problems on the entire book	**A.1–A.18**

LIST OF SOME USEFUL RESULTS FOR DIRECT APPLICATIONS

I Table of elementary integrals

$$\int \sin x\, dx = -\cos x \qquad \int \sinh x\, dx = \cosh x$$

$$\int \cos x\, dx = \sin x \qquad \int \cosh x\, dx = \sinh x$$

$$\int \sec^2 x\, dx = \tan x \qquad \int \operatorname{sech}^2 x\, dx = \tanh x$$

$$\int \operatorname{cosec}^2 x\, dx = -\cot x \qquad \int \operatorname{cosech}^2 x\, dx = -\coth x$$

$$\int \sec x \tan x\, dx = \sec x \qquad \int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x$$

$$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x \qquad \int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x$$

$$\int e^x dx = e^x \qquad \int a^x dx = (a^x)/\log_e a$$

$$\int x^n dx = \frac{x^{n+1}}{n+1},\ n \neq -1 \qquad \int \{f(x)\}^n f'(x) dx = \frac{\{f(x)\}^{n+1}}{n+1},\ n \neq -1$$

$$\int \frac{1}{x} dx = \log x \qquad \int \frac{f'(x)}{f(x)} dx = \log x$$

$$\int \tan x\, dx = \log \sec x = -\log \cos x \qquad \int \cot x\, dx = \log \sin x$$

$$\int \operatorname{cosec} x\, dx = \log \tan(x/2) = \log(\operatorname{cosec} x - \cot x)$$

$$\int \sec x\, dx = \log \tan(\pi/4 + x/2) = \log(\sec x + \tan x)$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a};\quad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\frac{x-a}{x+a},\ x > a;\quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\frac{a+x}{a-x},\ x < a$$

$$\int \frac{dx}{x(x^2 - a^2)^{1/2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} \quad \text{or} \quad -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}$$

$$\int \frac{dx}{(a^2 - x^2)^{1/2}} = \sin^{-1}\frac{x}{a} \quad \text{or} \quad -\cos^{-1}\frac{x}{a}$$

$$\int \frac{dx}{(a^2 + x^2)^{1/2}} = \sinh^{-1}\frac{x}{a} \quad \text{or} \quad \log\{x + (x^2 + a^2)^{1/2}\}$$

$$\int \frac{dx}{(x^2 - a^2)^{1/2}} = \cosh^{-1}\frac{x}{a} \quad \text{or} \quad \log\{x + (x^2 - a^2)^{1/2}\}$$

$$\int (a^2 - x^2)^{1/2}\,dx = (x/2) \times (a^2 - x^2)^{1/2} + (a^2/2) \times \sin^{-1}(x/a)$$

$$\int (a^2 + x^2)^{1/2}\,dx = (x/2) \times (a^2 + x^2)^{1/2} + (a^2/2) \times \log\{x + (a^2 + x^2)^{1/2}\}$$

$$\int (x^2 - a^2)^{1/2}\,dx = (x/2) \times (x^2 - a^2)^{1/2} - (a^2/2) \times \log\{x + (x^2 - a^2)^{1/2}\}$$

Note. If x is replaced by $ax + b$ (a and b being constants) on both sides of any formula of the above table, then the standard form remains true, provided the result on R.H.S. is divided by a, the coefficient of x. For examples,

$$\int \cos(ax+b)dx = \frac{\sin(ax+b)}{a}; \qquad \int e^{ax+b}dx = \frac{e^{ax+b}}{a}$$

II. $$\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}; \qquad \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}; \qquad \int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$$

III. Integration by parts

$$\int f_1(x) f_2(x)dx = f_1(x)\left\{\int f_2(x)\,dx\right\} - \int\left[\left\{\frac{d}{dx} f_1(x)\right\} \times \left\{\int f_2(x)\,dx\right\}\right] dx$$

In words, this formula states

The integral of the product of two functions

= Ist function × integral of 2nd – integral of (diff. coeff. of 1st × integral of 2nd)

The success of this method depends upon the choosing the first and second functions in such a way that the second term on the R.H.S. is easily integrable.

Note. While choosing the first and second function, note carefully the following facts:

(*i*) The second function must be chosen in such a way that its integral is known

(*ii*) If the integrals of both the functions in the product to be integrated are known, then the second function must be chosen in much a way that the new integral on the R.H.S. should be integrable directly or it should the simpler than the original integral.

(*iii*) If the integrals of both the functions are known and if one them be of the form x^n or $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ (where n is a positive integers and a_0, a_1, ..., a_n are constants), then that function must be chosen as the first function. For example, in $\int x^3 e^x dx$, x^3 must be chosen as the first function.

(*iv*) If in the product of two functions the integral of one of the functions is not known, then that function must be taken as the first function. For example, in $\int x \tan^{-1} x\, dx$ and $\int x \log x\, dx$ etc we do not know the integrals of $\tan^{-1} x$ and $\log x$ and hence we must choose $\tan^{-1}x$ and $\log x$ etc as first function.

(*v*) Sometimes we are to evaluate the integral of a single function by the method of integration by parts. In such cases, unity (*i.e.*, 1) must be taken as the second function. For example, to find $\int \tan^{-1} x\, dx$, $\int \log x\, dx$ etc, we always take 1 as the function. Thus, we write

$$\int \tan^{-1} x\, dx = \int (\tan^{-1} x)\cdot 1\, dx, \qquad \int \log x\, dx = \int (\log x)\cdot 1\, dx \text{ etc.}$$

(*vi*) The formula of integration by parts can be applied more than once, if necessary.

BERNOULLI'S FORMULA OR GENERALISED RULE OF INTEGRATION BY PARTS OR CHAIN RULE OF INTEGRATION BY PARTS.

Let u and v be two functions of x. Let dashes denote differentiation and suffixes integration with respect to x. Thus, we have

$$u' = \frac{du}{dx}, \qquad u'' = \frac{du'}{dx^2} = \frac{d^2u}{dx^2}, ..., \qquad v_1 = \int v\,dx, \qquad v_2 = \int v_1\,dx = \iint v(dx)^2, \quad \text{and so on.}$$

Then $$\int u\,v\,dx = u\,v_1 - u'v_2 + u''v_3 - u'''v_4 + ...$$

The above rule is applied when u is of the form x^n or $a_0x^n + a_1\,x^{n-1} + ... + a_{n-1}\,x + a_n$ (where n is a positive integer) and v is a function of the forms e^{ax}, a^x, sin ax or cos ax.

While applying the above rule, simplification should be done only when the whole process of integration is over. Study the solutions of the following problems carefully.

Example 1. $\int x^4e^x dx = (x^4)(e^x) - (4x^3)\,(e^x) + (12x^2)(e^x) - (24x)\,(e^x) + (24)(e^x)$

$= e^x\,(x^4 - 4x^3 + 12x^2 - 24x + 24)$

Example 2. $\int_0^{\pi} x^5 \sin x\,dx = \Big[(x^5)(-\cos x) - (5x^4)\,(-\sin x) + (20x^3)\,(\cos x)$

$$-(60x^2)\,(\sin x) + (120x)\,(-\cos x) - (120)\,(-\sin x)\Big]_0^{\pi}$$

$$= \Big[(-x^5 + 20x^3 - 120x)\cos x + (5x^4 - 60x^2 + 120)\sin x\Big]_0^{\pi}$$

$$= (-\pi^5 + 20\pi^3 - 120\pi)\cos\pi + (-\pi^5 + 20\pi^3 - 120\pi) \times (-1) = \pi^5 - 20\pi^3 + 120\pi$$

Some useful direct results based on integration by parts

$\int e^{ax}\{a\,f(x) + f'(x)\}dx = e^{ax}f(x)$. Its particulars case are

$\int e^x\{f(x) + f'(x)\}dx = e^x f(x);$ $\qquad\qquad$ $\int e^{-x}\{-f(x) + f'(x)\}dx = e^{-x}f(x)$

IV Properties of definite integrals

(*i*) $\int_a^b f(x)dx = \int_a^b f(t)\,dt$ $\qquad\qquad$ (*ii*) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(*iii*) $\int_a^b f(x)\,dx = \int_a^b f(x)dx + \int_c^b f(x)\,dx,$ where $a < c < b$

(*iv*) $\int_a^a f(x)dx = 2\int_0^a f(x)\,dx,$ if $f(-x) = f(x);$ $\qquad\qquad$ $\int_{-a}^a f(x)dx = 0,$ if $f(-x) = -f(x)$

(*v*) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $f(2a - x) = f(x);$ $\int_0^{2a} f(x)\,dx = 0,$ if $f(2a - x) = -f(x)$

V Walli's formulas

(*i*) If n is an even positive integer, then

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

For example $$\int_0^{\pi/2} \sin^{10} x\,dx = \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

Note carefully that answer is written down very easily by beginning with the denominator. We then have the ordinary sequence of natural numbers written down backwards. Thus, in the above example, we write (10 under 9) × (8 under 7) × (6 under 5) .... etc. stopping at (2 under 1), and writing a factor $\pi/2$ in the end.

(*ii*) If n is an odd positive integer, then

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}$$

For example. $$\int_0^{\pi/2} \sin^9 x\,dx = \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$$

Thus, as above, we begin with the denominator. We then have the ordinary sequence of natural numbers written down backwards. Thus, in the above example, we write (9 under 8) × (7 under 6) × .... etc stopping at (3 under 2) and additional factor $\pi/2$ is not written in the end.

(*iii*) If m and n are positive integers, then

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{(m-1)\,(m-3)\,(m-5)...(n-1)\,(n-3)(n-5)...}{(m+n)\,(m+n-2)\,(m+n-4)...} \times k,$$

where k is $\pi/2$ if m and n are both both positive even integers otherwise $k = 1$. The last factor in each of the three products (namely, $(m-1)\,(m-3)\,(m-5)$ ..., $(n-1)\,(n-3)\,(n-5)$ .... and $(m+n)\,(m+n-2)\,(m+n-1)$ ...) is either 1 or 2. In case any of m or n is 1, we simply write 1 as the only factor to replace its product.

Example 1 $$\int_0^{\pi/2} \sin^4 x \cos^2 x\,dx = \frac{3\cdot1\cdot1}{6\cdot4\cdot2}\times\frac{\pi}{2} = \frac{\pi}{32}$$

Example 2 $$\int_0^{\pi/2} \sin^4 x \cos^3 x\,dx = \frac{3\cdot1\cdot2}{7\cdot5\cdot3\cdot1}\times1 = \frac{2}{35}$$

Example 3 $$\int_0^{\pi/2} \sin^4 x \cos x\,dx = \frac{3\cdot1\cdot1}{5\cdot3\cdot1} = \frac{1}{5}$$

THE GREEK ALPHABET

alpha	α	A	nu	ν	N
beta	β	B	xi	ξ	Ξ
gamma	γ	Γ	omicron	o	O
delta	δ	Δ	pi	π	∏
epsilon	ε	E	rho	ρ	P
zeta	ζ	Z	sigma	σ	Σ
eta	η	H	tau	τ	T
theta	θ	Θ	upsilon	υ	ϒ
iota	ι	I	phi	ϕ	Φ
kappa	κ	K	chi	χ	X
lambda	λ	Λ	psi	ψ	Ψ
mu	μ	M	omega	ω	Ω

ADVANCED ORDINARY DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTIONS

WHERE IS WHAT

Chapters		*Pages*
1.	Picard's iterative method. Uniqueness and existence theorems	1.3-1.25
2.	Simultaneous equations of the form (dx)/P = (dy)Q = (dz)/R	2.1-2.24
3.	Total (or Pfaffian) differential equations	3.1-3.32
4.	Riccati's equation	4.1-4.5
5.	Chebyshev polynomials	5.1-5.9
6.	Beta and Gamma functions	6.1-6.22
7.	Power Series	7.1-7.7
8.	Integration in series	8.1-8.60
9.	Legendre polynomials	9.1-9.50
10.	Legendre functions of the second order	10.1-10.12
11.	Bessel functions	11.1-11.45
12.	Hermite polynomials	12.1-12.12
13.	Laguerre polynomials	13.1-13.11
14.	Hypergeometric function	14.1-14.18
15.	Orthogonal sets of functions and Strum-Liouville problem	15.1-15.25

1

Picard's Iterative Method Uniqueness And Existence Theorems

1.1. Introduction. In many practical problems we come across with a differential equation which cannot be solved by one of the standard methods known so far. Various methods have been formulated for getting to any desired degree of accuracy the numerical solution of the above mentioned type of differential equation with numerical coefficients and given conditions. In this chapter we propose to discuss Picard's iteration method for finding an approximate solution of the initial value problem of the form

$$dy/dx = f(x, y), \qquad y(x_0) = y_0.$$

The condition $y(x_0) = y_0$ is called the *initial condition*. Here $y(x_0)$ denotes the value of y at $x = x_0$. Sometimes $y(x_0) = y_0$ is also expressed by saying that $y = y_0$ when $x = x_0$.

An *iteration method* is a method which consists of a repeatred application of exactly the same type of steps where in each step we use the result of the previous step (or steps).

1.2A. Picard's method of successive approximations (or Picard's iteration method) [Himanchal 2004, Bangalore 2002, 06; Allahabad 2001; Meerut 2000, 10; Ujjan 2003]

Consider an initial value problem of the form

$$dy/dx = f(x, y), \qquad y(x_0) = y_0. \qquad ...(1)$$

By integrating over the interval (x_0, x), (1) gives

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx \qquad \text{or} \qquad y(x) - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

or

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx. \qquad ...(2)$$

Thus, the solving of initial value problem (1) is equivalent to finding a function $y(x)$ which satisfies the equation (2), since by differentiating (2) we get $dy/dx = f(x, y)$ and putting $x = x_0$ in (2) yields $y(x_0) = y_0 + 0$ *i.e.*, $y(x_0) = y_0$. Conversely, (2) has been obtained from (1) by integration over the interval (x_0, x) and employing the initial condition $y(x_0) = y_0$.

Since the information concerning the expression of y in terms of x is absent, the integral on the R.H.S. of (2) cannot be evaluated. Hence the exact value of y cannot be obtained. Therefore we determine a sequence of approximations to the solution (2) as follows. As a crude approximation, we put $y = y_0$ in the integral on the right of (2) and obtain

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx, \qquad ...(3)$$

where $y_1(x)$ is the corresponding value of $y(x)$ and is called *first approximation* and is better approximation of $y(x)$ at any x. To determine still better approximation we replace y by y_1 in the integral on R.H.S. in (2) and obtain the *second approximation* y_2 as

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx. \qquad ...(4)$$

Proceeding in this way, the *nth approximation* y_n is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx. \qquad ...(5)$$

Thus, we arrive at a sequence of approximate solutions $y_1(x), y_2(x), y_3(x), \ldots\ldots\ldots y_n(x), \ldots\ldots$

1.2B. Solved examples based on Art. 1.2A

Ex. 1. *Apply Picard's method to solve the following initial value problem upto third approximation : $dy/dx = 2y - 2x^2 - 3$ given that $y = 2$ when $x = 0$.*

[Agra 2005; Gwalior 2003; Delhi Maths (Hons.) 1998, 2005; Meerut 2000, 04, 05, 11]

Sol. Given problem is $dy/dx = 2y - 2x^2 - 3$, where $y = 2$, $x = 0$. ... (1)

We know that the nth approximation y_n of the initial value problem

$dy/dx = f(x, y)$, where $y = y_0$ when $x = x_0$ ...(2)

is given by $$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = 2y - 2x^2 - 3$, $x_0 = 0$ and $y_0 = 2$. ...(4)

$\therefore$ from (3), $$y_n = 2 + \int_0^x (2y_{n-1} - 2x^2 - 3)\, dx. \qquad ...(5)$$

First approximation. Putting $n = 1$ in (5), we have

$$y_1 = 2 + \int_0^x (2y_0 - 2x^2 - 3)\, dx = 2 + \int_0^x (4 - 2x^2 - 3)\, dx, \text{ using (4)}$$

or $$y_1 = 2 + \int_0^x (1 - 2x^2)\, dx = 2 + \left[x - 2\left(\frac{x^3}{3} \right) \right]_0^x = 2 + x - \frac{2x^3}{3}. \qquad ...(6)$$

Second approximation. Putting $n = 2$ in (5), we have

$$y_2 = 2 + \int_0^x (2y_1 - 2x^2 - 3)\, dx = 2 + \int_0^x \left[2\left(2 + x - \frac{2x^3}{3} \right) - 2x^2 - 3 \right] dx, \text{ using (6)}$$

or $$y = 2 + \int_0^x \left(1 + 2x - 2x^2 - \frac{4x^3}{3} \right) dx = 2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3}. \qquad ...(7)$$

Third approximation : Putting $n = 3$ in (5), we have

$$y_3 = 2 + \int_0^x \left(2y_2 - 2x^2 - 3 \right) dx = 2 + \int_0^x \left[2\left(2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3} \right) - 2x^2 - 3 \right] dx, \text{ using (7)}$$

or $$y = 2 + \int_0^x \left[1 + 2x - \frac{4x^3}{3} - \frac{2}{3}x^4 \right] dx = 2 + x + x^2 - \frac{x^4}{3} - \frac{2x^5}{15}.$$

Ex. 2. *Using the Picard's method of successive approximations, find the third approximation of the solution of the equation : $dy/dx = x + y^2$, where $y = 0$ when $x = 0$.*

[Delhi Maths (Hons) 1996; Meerut 2007; Ravishankar 2001
Indore 2001; Jabalpur 2000, 02, 05; Gwalior 2006; Rohilkhand 2004]

Sol. Given problem is $dy/dx = x + y^2$, where $y = 0$ when $x = 0$. ...(1)

We know that the nth approximation y_n of the initial problem

$dy/dx = f(x, y)$ where $y = y_0$ when $x = x_0$. ...(2)

is given by $$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = x + y^2$, $x_0 = 0$ and $y_0 = 0$. ...(4)

$\therefore$ from (3) , $$y_n = \int_0^x \left[x + y_{n-1}^2\right] dx. \qquad ...(5)$$

First approximation. Putting $n = 1$ in (5) and using (4) we have

$$y_1 = \int_0^x \left(x + y_0^2\right) dx = \int_0^x x\, dx = \frac{1}{2}x^2. \qquad ...(6)$$

Second approximation. Putting $x = 2$ in (5) and using (6), we have

$$y_2 = \int_0^x \left(x + y_1^2\right) dx = \int_0^x \left[x + \frac{x^4}{4}\right] dx = \frac{1}{2}x^2 + \frac{1}{20}x^5. \qquad ...(7)$$

Third approximation. Putting $n = 3$ in (5), we get

$$y_3 = \int_0^x \left(x + y_2^2\right) dx = \int_0^x \left[x + \left(\frac{1}{2}x^2 + \frac{1}{20}x^5\right)^2\right] dx, \text{ using (7)}$$

$$= \int_0^x \left[x + \frac{1}{4}x^4 + \frac{1}{400}x^{10} + \frac{1}{20}x^7\right] dx = \frac{1}{2}x^2 + \frac{1}{20}x^5 + \frac{1}{4400}x^{11} + \frac{1}{160}x^8.$$

Ex. 3. *Find the third approximation of the solution of the equation* $dy/dx = 2 - (y/x)$ *by Picard's method, where* $y = 2$ *when* $x = 1$.

[Delhi Maths (Hons.) 1997, 99, 2008, 2009; Gwalior 2004; Meerut 2002, 11; Rohilkhand 2000]

Sol. Given problem is $dy/dx = 2 - (y/x)$, where $y = 2$ when $x = 1$. ...(1)

We know that the nth approximation y_n of the initial value problem

$dy/dx = f(x, y)$, where $y = y_0$ when $x = x_0$. ...(2)

is given by $$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\, dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = 2 - (y/x)$, $x_0 = 1$ and $y_0 = 2$. ...(4)

$\therefore$ from (3), $$y_n = 2 + \int_1^x [2 - (1/x) y_{n-1}]\, dx. \qquad ...(5)$$

First Approximation. Putting $n = 1$ in (5), we get

$$y_1 = 2 + \int_1^x \left[2 - (1/x) y_0\right] dx = 2 + \int_1^x \left[2 - (2/x)\right] dx, \text{ using (4)}$$

$$= 2 + \left[2x - 2\log x\right]_1^x = 2 + 2x - 2\log x - 2 = 2x - 2\log x. \qquad ...(6)$$

Second Approximation. Putting $n = 2$ in (5), we get

$$y_2 = 2 + \int_1^x \left(2 - \frac{y_1}{x}\right) dx = 2 + \int_1^x \left[2 - \frac{1}{x}(2x - 2\log x)\right] dx, \text{ by (6)}$$

$$= 2 + 2\int_1^x \log x \cdot \frac{1}{x}\, dx = 2 + \left[(\log x)^2\right]_1^x = 2 + (\log x)^2. \qquad ...(7)$$

Third Approximation. Putting $n = 3$ in (5), we get

$$y_3 = 2 + \int_1^x \left[2 - \left(\frac{1}{x}\right) y_2\right] dx = 2 + \int_1^x \left[2 - \left(\frac{1}{x}\right)\left\{2 + (\log x)^2\right\}\right] dx, \text{ by (7)}$$

$$= 2 + \int_1^x \left[2 - \frac{2}{x} - (\log x)^2 \frac{1}{x}\right] dx = 2 + \left[2x - 2\log x - \frac{(\log x)^3}{3}\right]_1^x$$

$$= 2 + 2x - 2\log x - (1/3) \times (\log x)^3 - 2 = 2x - 2\log x - (1/3) \times (\log x)^3.$$

Ex. 4. *(a) Using Picard's method of successive approximation, find a sequence of two functions which approach solution of the initial value problem* $dy/dx = e^x + y^2$, $y(0) = 1$.

[Delhi Maths (Hons.) 1994, 2002]

Sol. Given problem is $dy/dx = e^x + y^2$, where $y = 1$ when $x = 0$. ...(1)

We know that the nth approximation y_n of the initial value problem

$dy/dx = f(x, y)$, where $y = y_0$ when $x = x_0$ ...(2)

is given by $$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx.$$ ...(3)

Comparing (1) and (2), $f(x, y) = e^x + y^2$, $x_0 = 0$ and $y_0 = 1$. ...(4)

$\therefore$ from (3), $$y_n = 1 + \int_0^x (e^x + y_{n-1}^2)\, dx.$$ ...(5)

First Approximation. Putting $n = 1$ in (5), and using (4), we get

$$y_1 = 1 + \int_0^x (e^x + y_0^2)\, dx = 1 + \int_0^x (e^x + 1)\, dx = 1 + \Big[e^x + x\Big]_0^x = 1 + e^x + x - 1 = e^x + x.$$ ...(6)

Second Approximation. Putting $n = 2$ in (5), we have

$$y_2 = 1 + \int_0^x (e^x + y_1^2)\, dx = 1 + \int_0^x \Big[e^x + (e^x + x)^2\Big] dx, \text{ by (6)}$$

$$= 1 + \int_0^x (e^x + e^{2x} + x^2 + 2x\, e^x)\, dx = 1 + \left[e^x + \frac{1}{2}e^{2x} + \frac{x^3}{3}\right]_0^x + 2\int_0^x x\, e^x\, dx$$

$$= 1 + e^x + \frac{1}{2}e^{2x} + \frac{x^3}{3} - 1 - \frac{1}{2} + 2\left[\Big[x\, e^x\Big]_0^x - \int_0^x (1.e^x)\, dx\right] = e^x + \frac{1}{2}e^{2x} + \frac{1}{3}x^3 - \frac{1}{2} + 2\Big[xe^x - [e^x]_0^x\Big]$$

$= e^x + (1/2) \times e^{2x} + x^3/3 - (1/2) + 2x\, e^x - 2\,(e^x - 1) = (1/2) \times e^{2x} + x^3/3 + 3/2 + (2x - 1)\, e^x$.

Ex. 4(b). *Find three successive approximations of the solution of* $dy/dx = e^x + y^2$, $y(0) = 0$.

[Delhi Maths (Hons.) 2007]

Sol. Given problem in $dy/dx = e^x + y^2$, $y(0) = 0$ ... (1)

We know that the nth approximation y_n of the initial value problem.

$dy/dx = f(x, y)$, where $y = y_0$ when $x = x_0$ ... (2)

in given by $$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$ ... (3)

Comparing (1) and (2), here $f(x, y) = e^x + y^2$, $x_0 = 0$, $y_0 = 0$ ... (4)

$\therefore$ (3) reduces to $$y_n = \int_0^x (e^x + y_{n-1}^2)\, dx$$ ... (5)

First approximation : Putting $n = 1$ in (5) and using (4), we get

$$y_1 = \int_0^x (e^x + y_0^2) dx = \int_0^x e^x dx = [e^x]_0^x = e^x - 1$$ ... (6)

Second approximation : Putting $n = 2$ in (5) and using (4) and (6), we get

$$y_2 = \int_0^x (e^x + y_1^2) dx = \int_0^x \{e^x + (e^x - 1)^2\} dx = \int_0^x (e^{2x} - e^x + 1)\, dx$$

$$= [(1/2) \times e^{2x} - e^x + x]_0^x = (1/2) \times (e^{2x} - 2e^x + 2x + 1)$$ ... (7)

Third approximation : Putting $n = 3$ in (5) and using (4) and (7), we get

$$y_3 = \int_0^x (e^x + y_2^2)dx = \int_0^x \{e^x + (1/4)\times(e^{2x} - 2e^x + 2x + 1)^2\}dx$$

$$= \frac{1}{4}\int_0^x (4e^x + e^{4x} + 4e^{2x} + 4x^2 + 1 - 4e^{3x} + 4xe^{2x} + 2e^{2x} - 8xe^x - 4e^x + 4x)\,dx$$

$$= \frac{1}{4}\int_0^x (e^{4x} - 4e^{3x} + 2e^{2x} + 4x^2 + 4x + 1 + 4xe^{2x} - 8xe^x)\,dx$$

$$= \frac{1}{16}\Big[e^{4x}\Big]_0^x - \frac{1}{3}\Big[e^{3x}\Big]_0^x + \frac{1}{4}\Big[e^{2x}\Big]_0^x + \frac{1}{4}\left[\frac{4x^3}{3} + 2x^2 + x\right]_0^x + \left[x\times\left(\frac{1}{2}e^{2x}\right) - (1)\times\left(\frac{1}{4}e^{2x}\right)\right]_0^x$$

$$-2\Big[(x)(e^x) - (1)\times(e^x)\Big]_0^x, \text{ on integrating by parts the last two terms}$$

$$= (1/16)\times(e^{4x} - 1) - (1/3)\times(e^{3x} - 1) + (1/4)\times(e^{2x} - 1) + (1/12)\times(4x^3 - 6x^2 + 3x)$$

$$+ (x/2)\times e^{2x} - (1/4)\times e^{2x} + 1/4 - 2(xe^x - e^x + 1)$$

$$= (1/16)\times e^{4x} - (1/3)\times e^{3x} + (1/4)\times e^{2x} + (1/12)\times(4x^3 - 6x^2 + 3x) + (1/4)\times(2x-1)e^{2x} - 2\,(x-1)e^x - (83/48).$$

Ex. 5. *Use Picard's method to obtain a solution of the differential equation:* $dy/dx = x^2 - y$, $y(0) = 0$. *Find at least the fourth approximation to each solution.* **[Meerut 1996]**

Sol. Given $dy/dx = x^2 - y$, where $y = 0$ when $x = 0$. ...(1)

We know that the nth approximation y_n of the initial value problem

$dy/dx = f(x, y)$, when $y = y_0$ when $x = x_0$ ...(2)

is given by $$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\,dx.$$ ...(3)

Comparing (1) and (2), $f(x, y) = x^2 - y$, $x_0 = 0$ and $y_0 = 0$. ...(4)

$\therefore$ from (3), $$y_n = \int_0^x (x^2 - y_{n-1})\,dx.$$ ...(5)

First approximation. Putting $n = 1$ in (5) and using (4), we get

$$y_1 = \int_0^x (x^2 - y_0)\,dx = \int_0^x x^2\,dx = \left[\frac{1}{3}x^3\right]_0^x = \frac{1}{3}x^3.$$ ...(6)

Second approximation. Putting $n = 2$ in (5) and using (6), we get

$$y_2 = \int_0^x (x^2 - y_1)\,dx = \int_0^x \left(x^2 - \frac{1}{3}x^3\right)dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4\right]_0^x = \frac{1}{3}x^3 - \frac{1}{12}x^4.$$ ...(7)

Third approximation. Putting $n = 3$ in (5) and using (7), we get

$$y_3 = \int_0^x \left(x^2 - y_2\right)dx = \int_0^x \left[x^2 - \left(\frac{1}{3}x^3 - \frac{1}{12}x^4\right)\right]dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right]_0^x$$

or $$y_3 = x^3/3 - x^4/12 + x^5/60.$$ ...(8)

Fourth approximation. Putting $n = 4$ in (5) and using (7), we get

$$y_4 = \int_0^x (x^2 - y_3)\,dx = \int_0^x \left[x^2 - \left(\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right)\right]dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5 - \frac{1}{360}x^6\right]_0^x$$

or $y = x^3/3 - x^4/12 + x^5/60 - x^6/360.$

Ex. 6. (a) *Apply Picard's method to find the solution of the problem* $dy/dx = y - x$, $y(0) = 2$. *Show that the iterative solution approaches the exact solution.* **[Meerut 1995]**

Sol. Given $dy/dx = y - x$, where $y = 2$ when $x = 0$. ...(1)

We know that the nth approximation of the initial value problem

$dy/dx = f(x, y)$ where $y = y_0$ when $x = x_0$ ...(2)

is given by $y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\,dx.$ ...(3)

Comparing (1) and (2), $f(x, y) = y - x$, $x_0 = 1$ and $y_0 = 2$. ...(4)

$\therefore$ from (3), $y_n = 2 + \int_0^x (y_{n-1} - x)\,dx.$ ...(5)

First approximation. Putting $n = 1$ in (5) and using (4), we get

$$y_1 = 2 + \int_0^x (y_0 - x)\,dx = 2 + \int_0^x (2 - x)\,dx = 2 + 2x - \frac{1}{2}x^2. \quad ...(6)$$

Second approximation. Putting $n = 2$ in (5) and using (6) we get

$$y_2 = 2 + \int_0^x (y_1 - x)\,dx = 2 + \int_0^x \left(2 + 2x - \frac{1}{2}x^2 - x\right)dx = 2 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \quad ...(7)$$

Third approximation. Putting n = 3 in (5) and using (7), we get

$$y_3 = 2 + \int_0^x (y_2 - x)\,dx = 2 + \int_0^x [2 + 2x + x^2/2 - x^3/6 - x]\,dx$$

$$= 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} = 1 + x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} \quad ...(8)$$

To find the exact solution of (1). Rewriting (1), we have

$(dy/dx) - y = -x$, which is a linear differential equation ...(9)

Its I.F. $= e^{\int (-1)\,dx} = e^{-x}$ and hence its solution is

$$ye^{-x} = \int (-x)(e^{-x})\,dx + c = -\left[x(-e^{-x}) - \int 1.(-e^{-x})\,dx\right] + c,\ c \text{ being an arbitrary constant}$$

or $ye^{-x} = xe^{-x} + e^{-x} + c$ or $y = x + 1 + c\,e^x.$ ...(10)

Given that $y = 2$, when $x = 0$, so (10) gives $2 = 1 + c$ or $c = 1$.

Hence, from (10), the exact solution is $y = x + 1 + e^x.$ ...(11)

We know that $e^x = 1 + x + (x^2/2!) + (x^3/3!) + \ldots\ldots$ ad inf. ...(12)

Keeping (12) and (8) in view, we find that the approximate solution tends to

$y = 1 + x + 1 + x/1! + x^2/2! + x^3/3! + \ldots. = 1 + x + e^x$ *i.e.*, which is exact solution of (4).

Ex. 6 (b) *Apply Picard's iteration method to the initial value problem* $dy/dx = y$, $y(0) = 1$ *and show that the successive approximations tend to the limit* $y = e^x$, *the exact solution.*

[Allahabad 2001, 05; Gwalior 2005; Indore 2002, 03; Kurukshetra 2003; Pune 2002]

Sol. Proceeding as in Ext. 6 (*a*), we have

$y_1 = 1 + x$, $y_2 = 1 + \frac{x}{1!} + \frac{x^2}{2!}$, $y_3 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$.

In general, $y_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!}$. ...(1)

Also, solving $dy/dx = y$, we have $(1/y)dy = dx$ so that $y = c\,e^x$. ... (2)

Given that $y = 1$ when $x = 0$, so (2) given $1 = c$ and hence

the exact solution (2) becomes $y = e^x$. ...(3)

From (1), we see that the successive approximations tend to the limit $y = e^x$ as $n \to \infty$, which is the exact solution.

Ex. 6. (*c*). *Show that successive approximation* $\phi_n(x)$ *for the equation* $y'(x) = y(x)$ $(-\infty < x < \infty)$ *with initial condition* $y(0) = 1$ *are given by* $\phi_n(x) = 1 + x + x^2/2! + x^3/3! + \ldots + x^n/n!$ **[Kolkata 2002, 04, 07]**

EXERCISE 1 (A)

1. *Apply Picard's method to the following initial value problems and find the first three successive approximations.*

(*i*) $dy/dx = 2xy$, $y(0) = 1$. **[Allahabad 1994, Delhi Maths (Hons.) 95, 2001, 04, 06]**

Ans. $y_1 = 1 + x^2$, $y_2 = 1 + x^2 + (x^4/2)$, $y_3 = 1 + x^2 + (x^4/2) + (x^6/6)$

(*ii*) $dy/dx = 3e^x + 2y$, $y(0) = 0$.

[Himanchal 2002; Meerut 1993, 94, M.K.U. (Tamil Nadu), 2002, 02]

Ans. $y_1 = 3(e^x - 1)$, $y_2 = 9e^x - 6x - 9$, $y_3 = 21\,e^x - 6x^2 - 18x - 21$.

(*iii*) $dy/dx = x + y$, $y(0) = 1$. **[Allahabad 1999; Calicut 2004; Meerut 1993, 95]**

Ans. $y_1 = 1 + x + (x^2/2)$, $y_2 = 1 + x + x^2 + (x^3/6)$, $y_3 = 1 + x + x^2 + (x^3/3) + (x^4/24)$

(*iv*) $dy/dx = 1 + xy$, $y(0) = 2$.

[Agra 2001, 04, 05 Himanchal 2004, 05; Lucknow 2003, Meerut 2001, 06]

Ans. $y_1 = 2 + x + x^2$, $y_2 = 2 + x + x^2 + (x^3/3) + (x^4/4)$,

$y_3 = 2 + x + x^2 + (x^3/3) + (x^4/4) + (x^5/15) + (x^6/24)$

(*v*) $dy/dx = 2x - y^2$, where $y = 0$ at $x = 0$.

Ans. $y_1 = x^2$, $y_2 = x^2 - (x^5/5)$, $y_3 = x^2 - (x^5/5) + (x^8/120) - (x^{11}/275)$.

(*vi*) $dy/dx = e^x + y^2$, $y(0) = 0$. **Ans.** $y_1 = e^x - 1$, $y_2 = (1/2)e^{2x} - e^x + x + (1/2)$, $y_3 = (1/16)e^{4x} - (1/3)e^{3x} + (1/2)xe^{2x} + (1/2)e^{2x} - 2xe^x + 2e^x + (1/3)x^3 + (1/2)x^2 + (1/4)x - (107/48)$.

2. *Solve the differential equation* $dy/dx = x - y$ *with the condition* $y = 1$ *when* $x = 0$ *and show that the sequence of approximations given by Picard's method tend to the exact solution as a limit.* **[Agra 2001; Meerut 2003]**

Ans. $y_1 = 1 - x + \frac{x^2}{2!}$, $y_2 = 1 - x + \frac{2x^2}{2!} - \frac{x^3}{3!}$, $y_3 = 1 - x + \frac{2x^2}{2!} - \frac{2x^3}{3!} + \frac{x^4}{4!}$;

$y_4 = 1 - x + \frac{2x^2}{2!} - \frac{2x^3}{3!} + \frac{2x^4}{4!} - \frac{x^5}{5!}$ tending to $y = -1 + x + 2e^{-x}$, which is exact solution.

3. *Use Picard's method to approximate the solution of the equation* $dy/dx + 2xy^2 = 0$ *with* $y = 1$ *when* $x = 0$ *and hence show that* $y = 1/(1 + x^2)$. **[Rohtak 2001, Meerut 1995]**

Ans. $y_1 = 1 - x^2, y_2 = 1 - x^2 + x^4 - (1/3)x^6, y_3 = 1 - x^2 + x^4 - x^6 + (2/3)x^8 - (1/3)x^{10} + (1/9)x^{12} - (1/63)x^{14}$.

4. *State the conditions under which the initial value problem* $dy/dx = x^2$, $x_0 = 2$, $y_0 = 1$ *has unique solution by Picard's method of successive approximation. Obtain the solution of the initial value problem* $dy/dx = x^2 + y^2$, $y(0) = 1$, *by Picards method as far as the term, involving* x^4.

[Himanchal 2003, 05; Jabalpur 2004, 06; Kurukshetra 2006]

Ans. $y_1(x) = 1 + x + x^3/3, y_2(x) = 1 + x + x^2 + (2x^3/3) + (2x^4)/12 + (2x^5)/15 + x^6/63$

5. *Using Picard's method find the third approximation of the solution of the initial value problem* $dy/dx = 1 + y^2$, $y(0) = 0$. **[Kolkata 2004, 07, Pune 2002]**

Ans. $y_1(x) = x$, $y_2(x) = x + x^3/3, y_3(x) = x + x^3/3 + (2x^5)/15 + x^7/63$

6. *Find the exact solution of initial value problem* $dy/dx = x + y$, $y(0) = 0$. *Next apply Picard's iterative method to obtain three successive approximate solutions* $y_1(x)$, $y_2(x)$, $y_3(x)$.

[Allahabad 2003, 07]

7. *Solve the initial value problem* $dy/dx = y^2$, $y(0) = 1$ *by method of successive approximation.*

[Jabalpur 2004; Osmania 2006]

Ans. $y_1(x) = 1 + x$; $y_2(x) = 1 + x + x^2 + x^3/3$; $y_3(x) = 1 + x + x^2 + x^3 + (2x^4)/3 + x^5/3 + x^6/9 + x^7/63$

8. (*a*) *Under what condition does the initial value problem of the form* $y' = f(x, y)$, $y(x_0) = y_0$ *has unique solution ? Give statement only.*

(b) Consider the initial value problem $y' = x - y^2$, $y(0) = 1/2$. (*i*) *Does this have a unique solution ? Justify your answer* (*ii*) *Applying Picard method find an approximate solution of the above initial value problem containing at least four no-zero terms (iii) Give your comments regarding the utility of Picard's method in view of the above initial value problem.*

[Lucknow 2004, 05]

9. *Consider the initial value problem* $dy/dx = f(x, y)$, $y(a) = b$ *and discuss Picard's method of successive approximation to solve it.* **[Bangalore 2002, 06; Himanchal 2004]**

10. *Let* $\{y_n\}$ *be a sequence of successive approximation to the solution of* $dy/dx = f(x, y)$, $y(x_0) = y_0$ *such that* $y_{10} = y_9$. *Show* y_{10} *is exact solution.* **[Himanchal 2002, 03, 05, 07]**

11. *Explain initial value problem and equivalent integral equation.*

[Kanpur 2002, 07; Rohilkhand 2001, Ujjain 2000, 01, 06]

1.3 A. Working rule for Picard's method of solving simultaneous differntial equations with initial conditions, namely

$dy/dx = f(x, y, z)$, $dz/dx = g(x, y, z)$ where $y = y_0$, $z = z_0$ when $x = x_0$. ...(1)

The nth approximation (y_n, z_n) to the initial value problem (1) is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\, dx \qquad ...(2)$$

and

$$z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\, dx\,. \qquad ...(3)$$

1.3B. Solved examples based on Art 1.3A.

Ex. 1(a) *Find the third approximation of the solution of the equation* $dy/dx = z$, $dz/dx = x^3(y + z)$ *by Picard's method where* $y = 1$, $z = 1/2$ *where* $x = 0$.

[Meerut 2006,07; Rohilkhand 2002; Gwalior 2003]

Sol. Given $dy/dx = z$, $dz/dx = x^3(y + z)$, $y = 1$, $z = 1/2$ when $x = 0$. ...(1)

We know that the nth approximation (y_n, z_n) to the initial value problem

$dy/dx = f(x, y, z)$, $dz/dx = g(x, y, z)$, when $y = y_0$, $z = z_0$ when $x = x_0$ ...(2)

is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\, dx \qquad ...(3)$$

and $$z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\, dx . \qquad ...(4)$$

Comparing (1) and (2), we have

$f(x, y, z) = z$, $g(x, y, z) = x^3 (y + z)$, $y_0 = 1$, $z_0 = 1/2$, $x_0 = 0$. ...(5)

$\therefore$ From (3), $$y_n = 1 + \int_0^x z_{n-1}\, dx \qquad ...(6)$$

and from (4), $$z_n = \frac{1}{2} + \int_0^x x^3 \left(y_{n-1} + z_{n-1}\right) dx . \qquad ...(7)$$

First approximation. Putting $n = 1$ in (6) and using (5), we get

$$y_1 = 1 + \int_0^x z_0\, dx = 1 + \int_0^x \frac{1}{2}\, dx = 1 + \frac{1}{2} x . \qquad ...(8)$$

Next, putting $n = 1$ in (7) and using (5), we gets

$$z_1 = \frac{1}{2} + \int_0^x x^3 (y_0 + z_0)\, dx = \frac{1}{2} + \int_0^x x^3 \left(1 + \frac{1}{2}\right) dx = \frac{1}{2} + \frac{3}{8} x^4 . \qquad ...(9)$$

Second approximation. Putting $n = 2$ in (6) and using (9), we get

$$y_2 = 1 + \int_0^x z_1\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8} x^4\right) dx = 1 + \frac{1}{2} x + \frac{3x^5}{40} . \qquad ...(10)$$

Next, putting $n = 1$ in (7) and using (8) and (9), we get

$$z_2 = \frac{1}{2} + \int_0^x x^3 (y_1 + z_1)\, dx = \frac{1}{2} + \int_0^x x^3 \left(1 + \frac{1}{2} x + \frac{1}{2} + \frac{3}{8} x^4\right) dx$$

$$= \frac{1}{2} + \int_0^x \left(\frac{3}{2} x^3 + \frac{1}{2} x^4 + \frac{3}{8} x^7\right) dx = \frac{1}{2} + \frac{3}{8} x^4 + \frac{1}{10} x^5 + \frac{3}{64} x^8 . \qquad ...(11)$$

Third approximation. Putting $n = 3$ in (6) and (11), we get

$$y_3 = 1 + \int_0^x z_2\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8} x^4 + \frac{1}{10} x^5 + \frac{3}{64} x^8\right) dx = 1 + \frac{1}{2} x + \frac{3}{40} x^5 + \frac{1}{60} x^6 + \frac{1}{192} x^9 .$$

Next, putting $n = 3$ in (7) and using (10) and (11), we get

$$z_3 = \frac{1}{2} + \int_0^x x^3 (y_2 + z_2)\, dx = \frac{1}{2} + \int_0^x x^3 \left[1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right] dx$$

$$= \frac{1}{2} + \int_0^x \left(\frac{3}{2} x^3 + \frac{1}{2} x^4 + \frac{3}{8} x^7 + \frac{7}{40} x^8 + \frac{3}{64} x^{11}\right) dx = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^9}{360} + \frac{x^{12}}{256}$$

Ex. 1 (b) *Find the third approximation of the solution of the equation* $d^2y/dx^2 = x^3(y + dy/dx)$, *where* $y = 1$ *and* $dy/dx = 1/2$ *when* $x = 0$

(*Agra 2000, 02; Himanchal 2004, 05; Meerut 2000; Rohilkhand 2002*)

Sol. Given that $d^2y/dx^2 = x^3(y + dy/dx)$, where $y = 1$ and $dy/dx = 1/2$ when $x = 0$. ...(1)

Let $dy/dx = z$ so that $d^2y/dx^2 = dz/dx$. Then, we have

$dy/dx = z$, $dz/dx = x^3(y + z)$, where $y = 1$ and $z = 1/2$ when $x = 0$. ...(2)

which, is the same as given in solved Ex. 1 (a) Proceed now as before and obtain the required third approximation: $y_3 = 1 + x/2 + 3x^5/40 + x^6/60 + x^9/192$.

Note : For complete solution of Ex. 1(b), you need not find z_3.

Ex. 2 (a) *Find the third approximation of the solution of the equation* $dy/dx = z$, $dz/dx = x^2z + x^4y$ *by Picard's method,* $y = 5$ *and* $z = 1$ *when* $x = 0$. **[Bangalore 2001, Meerut 2006, 07]**

Sol. Given $dy/dx = z$, $dz/dx = x^2z + x^4y$, $y = 5$, $z = 1$ when $x = 0$. ...(1)

We know that the nth approximation (y_n, z_n) to initial value problem

$dy/dx = f(x, y, z)$, $dz/dx = g(x, y, z)$ where $y = y_0$, $z = z_0$, when $x = x_0$ ...(2)

is given by $$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\,dx \qquad ...(3)$$

and $$z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\,dx\,. \qquad ...(4)$$

Comparing (1) and (2), we have

$f(x, y, z) = z$, $g(x, y, z) = x^2z + x^4y$, $y_0 = 5$, $z_0 = 1$, $x_0 = 0$. ...(5)

$\therefore$ from (3), $$y_n = 5 + \int_0^x z_{n-1}\,dx \qquad ...(6)$$

and from (4), $$z_n = 1 + \int_0^x (x^2 z_{n-1} + x^4 y_{n-1})\,dx. \qquad ...(7)$$

First approximation. Putting $n = 1$ in (6) and using (5), we get

$$y_1 = 5 + \int_0^x z_0\,dx = 5 + \int_0^x dx = 5 + x. \qquad ...(8)$$

Next putting $n = 1$ in (7) and using (5), we get

$$z_1 = 1 + \int_0^x (x^2 z_0 + x^4 y_0)\,dx = 1 + \int_0^x (x^2 + 5x^4)\,dx = 1 + \frac{x^3}{3} + x^5. \qquad ...(9)$$

Second approximation. Putting $n = 2$ in (6) and using (9), we get

$$y_2 = 5 + \int_0^x z_1\,dx = 5 + \int_0^x \left(1 + \frac{x^3}{3} + x^5\right)dx = 5 + x + \frac{x^4}{12} + \frac{x^6}{6}. \qquad ...(10)$$

Next, putting $n = 2$ in (7) and using (8) and (9), we get

$$z_2 = 1 + \int_0^x \left(x^2 z_1 + x^4 y_1\right)dx = 1 + \int_0^x \left[x^2\left(1 + \frac{x^3}{3} + x^5\right) + x^4(5 + x)\right]dx$$

$$= 1 + \int_0^x \left(x^2 + 5x^4 + \frac{4}{3}x^5 + x^7\right)dx = 1 + \frac{1}{3}x^3 + x^5 + \frac{2}{9}x^6 + \frac{1}{8}x^8. \qquad ...(11)$$

Third approximation. Putting $n = 3$ in (6) and using (11), we get

$$y_3 = 5 + \int_0^x z_2\,dx = 5 + \int_0^x \left(1 + \frac{1}{3}x^3 + x^5 + \frac{2}{9}x^6 + \frac{1}{8}x^8\right)dx = 5 + x + \frac{x^4}{12} + \frac{x^6}{6} + \frac{2x^7}{63} + \frac{x^9}{72}$$

Next, putting $n = 3$ in (7) and using (10) and (11), we have

$$z_3 = 1 + \int_0^x \left(x^2 z_2 + x^4 y_2\right)dx = 1 + \int_0^x \left[x^4\left(5 + x + \frac{x^4}{12} + \frac{x^6}{6}\right) + x^2\left(1 + \frac{x^3}{3} + x^5 + \frac{2x^6}{9} + \frac{x^8}{8}\right)\right]dx$$

$$= 1 + (1/3)x^3 + x^5 + (2/9)x^6 + (1/8)x^8 + (11/224)x^9 + (7/264)x^{11}.$$

Ex. 2 (b) *Find the third approximation of the solution of the equation* $d^2y/dx^2 = x^2(dy/dx) + x^4y$ *where* $y = 5$ and $dy/dx = 1$ *when* $x = 0$.

Sol. Let $dy/dx = z$ so that $dz/dx = d^2y/dx^2 = x^2z + x^4y$, where $y = 5$ and $z = 1$ when $x = 0$. ($\because$ $dy/dx = 1 \Rightarrow z = 1$)

This is exactly the same problem as given in Ex. 2.(a) Proceed as above and obtain the required value of y_3.

In the solution of Ex. 2 (b), you need not compute z_3 because we want third approximation of the solution of the original equation in which y is dependent variable.

$\therefore$ The required third approximation $= y_3(x) = 5 + x + (1/12)x^4 + (1/6)x^6 + (2/63)x^7 + (1/72)x^9$.

Ex. 3. *Find the third approximation of the initial value problem* $d^2y/dx^2 = xy + 1$, *when* $(y)_0 = 1$ and $(dy/dx)_0 = 0$. **[Meerut 1994]**

Sol. Let $dy/dx = z$ so that $dz/dx = d^2y/dx^2 = xy + 1$.

Again, re–writing the initial conditions, $y_0 = 1$, $z_0 = 0$ when $x = 0$.

In order to solve the given problem, we shall solve the following initial value problem involving differential equations : $dy/dx = z$, $dz/dx = 1 + xy$, $y = 1$, z = 0 when $x = 0$ ...(1)

We know that the nth approximation (y_n, z_n) to the initial value problem

$dy/dx = f(x, y, z)$, $dz/dx = g(x, y, z)$, when $y = y_0$, $z = z_0$ when $x = x_0$ ...(2)

is given by $$y_n = y_0 + \int_{x_0}^{x} f\left(x, y_{n-1}, z_{n-1}\right) dx \qquad ...(3)$$

and $$z_n = z_0 + \int_{x_0}^{x} g\left(x, y_{n-1}, z_{n-1}\right) dx. \qquad ...(4)$$

Comparing (1) and (2), we have

$f(x, y, z) = z$, $g(x, y, z) = 1 + xy$, $y_0 = 1$, $z_0 = 0$ and $x_0 = 0$. ...(5)

$\therefore$ From (3), $$y_n = 1 + \int_0^x z_{n-1}\, dx \qquad ...(6)$$

and from (4), $$z_n = \int_0^x (1 + xy_{n-1})\, dx. \qquad ...(7)$$

First approximation. Putting $n = 1$ in (6) and using (5), we get

$$y_1 = 1 + \int_0^x z_0\, dx = 1 + \int_0^x (0)\, dx = 1. \qquad ...(8)$$

Next, putting $n = 1$ in (7) and using (5), we get

$$z_1 = \int_0^x (1 + xy_0)\, dx = \int_0^x (1 + x)\, dx = x + \frac{x^2}{2}. \qquad ...(9)$$

Second approximation. Putting $n = 2$ in (6) and using (9), we get

$$y_2 = 1 + \int_0^x z_1\, dx = 1 + \int_0^x \left(x + \frac{x^2}{2}\right) dx = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3. \qquad ...(10)$$

Next, putting $n = 2$ in (7) and using (8), we get

$$z_2 = \int_0^x (1 + xy_1)\, dx = 1 + \int_0^x (1 + x)\, dx = x + \frac{1}{2}x^2. \qquad ...(11)$$

Third approximation. Putting $n = 3$ in (6) and using (11), we get

$$y_3 = 1 + \int_0^x z_2\, dx = 1 + \int_0^x \left(x + \frac{1}{2}x^2\right) dx = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

Next, putting $n = 3$ in (7) and using (10), we get

$$z_3 = \int_0^x (1 + xy_2)\, dx = \int_0^x \left[1 + x\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)\right] dx = x + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^5}{30}$$

EXERCISE 1 (B)

1. Apply Picard's method upto third approximation to solve the equations:
$dy/dx = x + z$, $dz/dx = x - y^2$, given that $y = 2$, $z = 1$ when $x = 0$

[Bhopal, 2002; Meerut 2000; Agra 2003, 04; Gwalior 2004]

[**Ans.** $y_1 = 2 + x + (1/2)x^2$, $z_1 = 1 - 4x + (1/2)x^2$; $y_2 = 2 + x - (3/2)x^2 + (1/6)x^3$; $z_2 = 1 - 4x - (3/2)x^2 - x^3 - (1/4)x^4 - (1/20)x^5$; $y_3 = 2 + x - (3/2)x^2 - (1/2)x^3 - (1/4)x^4 - (1/20)x^5 - (1/120)x^6$; $z_3 = 1 - 4x - (3/2)x^2 + (5/3)x^3 + (7/12)x^4 - (31/60)x^5 + (1/12)x^6 - (1/252)x^8$.]

2. Find the third approximation of the solution of the following equations:
$dy/dx = 2x + z$, $dz/dx = 3xy + x^2z$ where $y = 2$ and $z = 0$ when $x = 0$.

[Meerut 2000, 05; I.A.S. 2000]

[**Ans.** $y_3 = 2 + x^2 + x^3 + (3/20)x^5 + (1/10)x^6$, $z_3 = 3x^2 + (3/4)x^4 + (6/5)x^5 + (3/28)x^7 + (3/40)x^8$.]

1.4. Problems of existence and uniqueness : An Introduction.

Consider the initial value problem

$$| dy/dx | + | y | = 0, \qquad y(0) = 1. \qquad ...(1)$$

If possible, let $y \neq 0$. Then division by $| y |$ and integration leads to an absurd result. Hence $y = 0$ is the only solution of the differential equation. Clearly this solution does not satisfy the initial condition $y(0) = 1$. Thus, we see that the initial value problem (1) has no solution at all.

Now let us consider the initial value problem

$$dy/dx = x, \qquad y(0) = 1. \qquad ...(2)$$

Separating the variables, we get $dy = x\,dx$.

Integrating, $y = (x^2/2) + c$, where c is an arbitrary constant.

Using the initial condition $y(0) = 1$ *i.e.*, $x = 0$, $y = 1$, we get $c = 1$. Hence, the initial value problem (2) has only one solution, namely, $y = (x^2/2) + 1$.

Finally consider the following initial value problem

$$dy/dx = (y - 1)/x, \qquad y(0) = 1. \qquad ...(3)$$

Separating the variables, $(dy)/(y - 1) = (dx)/x$.

Integrating, $\log (y - 1) = \log x + \log c$ or $y - 1 = xc$.

Using the given initial condition, *i.e.*, $x = 0$, $y = 1$, we see that c cannot be dertermined. Thus, the given initial value problem (3) has infinite solutions given by $y - 1 = xc$, where c is an arbitrary constant.

From the above examples, we conclude that an initial value problem

$$dy/dx = f(x, y), \qquad y(x_0) = y_0 \qquad ...(4)$$

may have none, exactly one, or more than one solution. This leads us to the following two fundamental questions.

Problem of existence. *Under what conditions does an initial value problem of the form (4) has at least one solution ?*

Problem of uniqueness. *Under what conditions does that problem has a unique solution, that is, only one solution ?*

Theorems which state such conditions are called *existence theorem* and *uniqueness theorem*, respectively.

It may be noted that the above three examples are very simple and investigation about their existence and uniqueness is evident by mere inspection (or by actually solving), without using any theorem. However, when the equation cannot be solved by standard methods, existence and uniqueness theorem will play an important role.

1.5. Lipschitz condition. A function $f(x, y)$ is said to satisfy a *Lipschitz condition* in a region D in xy–plane if there exists a positive constant k such that

$$|f(x, y_2) - f(x, y_1)| \le K |y_2 - y_1|$$

whenever the points (x, y_1) and (x, y_2) both lie in D. The constant K is called a *Lipschitz constant* for the function $f(x, y)$.

1.6. Picard's Theorem. Existence and uniqueness theorem.

[Himanchal 2008, 09; Agra 2003, Calicut 2003, I.A.S. 1985 ; Meerut 2001, 02, 07, 11; Gwalior 2004, 05; G.N.D.U. Amritsar 2000; Jiwaji 2002 Ravishankar 2002, Rajasthan 2004, Rohilkhand 2007; Kolkata 2003; Ujjain 2003, 06]

Statement. Let $f(x, y)$ *be continuous in a domain D of the* (x, y) *plane and let M be a constant such that* $|f(x, y)| \le M$ in D. ...(1)

Let $f(x, y)$ *satisfy in D the Lipschitz condition in y namely*

$$|f(x, y_1) - f(x, y_2)| \le K |y_1 - y_2|, \quad ...(2)$$

where the constant K is independent of x, y_1, y_2.

Let the rectangle R, defined by $|x - x_0| \le h, \quad |y - y_0| \le k,$ ...(3)

lie in D, where $Mh < k$. *Then, for* $|x - x_0| \le h$, *the differential equation* $dy/dx = f(x, y)$ *has a unique solution* $y = y(x)$ *for which* $y(x_0) = y_0$.

Proof. [Read article 1.2A also for this proof carefully.]

We shall prove this theorem by the method of successive approximations. Let x be such that $|x - x_0| \le h$. We now define a sequence of functions $y_1(x), y_2(x), \ldots y_n, (x) \ldots$, called the successive approximations (or Picard Iterants) as follows :

$$\left.\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(x, y_0)\, dx \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(x, y_1)\, dx \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ y_{n-1}(x) &= y_0 + \int_{x_0}^{x} f(x, y_{n-2})\, dx \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \end{aligned}\right\} \quad ...(4)$$

We shall divide the proof into five main steps.

First Step. *We prove that, for* $x_0 - h \le x \le x_0 + h$ *the curve* $y = y_n(x)$ *lies in the rectangle R, that is to say* $y_0 - k < y < y_0 + k$.

Now, $$|y_1 - y_0| = \left| \int_{x_0}^{x} f(x, y_0)\, dx \right| \le \int_{x_0}^{x} |f(x, y_0)|.|dx|, \text{ by (4)}$$

or $|y_1 - y_0| \le M |x - x_0| \le Mh < k$, using (1), (3) and the given result viz $Mh < k$.

This proves the desired result for $n = 1$. Assume that $y = y_{n-1}(x)$ lies in R and so $f(x, y_{n-1})$ is defined and continuous and satisfies

$$|f(x, y_{n-1})| \le M \quad \text{on} \quad [x_0 - h, x_0 + h].$$

From (4), we have $$|y_n - y_0| = \left| \int_{x_0}^{x} f(x, y_{n-1})\, dx \right| \le \int_{x_0}^{x} |f(x, y_{n-1})|.|dx| \le M|x - x_0| \le Mh < k,$$

as before which shows that $y_n(x)$ lies in R and hence $f(x, y_n)$ is defined and continuous on $[x_0 - h, x_0 + h]$. The above arguments show that the desired result holds for all n by induction.

Second Step. *We prove again by induction, that* $$|y_n - y_{n-1}| \le \frac{MK^{n-1}}{n!} |x - x_0|^n. \quad ...(5)$$

We have already verified (5) for $n = 1$ in first step where we have shown that $| y_1 - y_0 | \le M | x - x_0 |$. Assume that this inequality (5) holds for $n - 1$ in place of n, that is, let

$$\left| y_{n-1} - y_{n-2} \right| \le \frac{MK^{n-2}}{(n-1)!} \left| x - x_0 \right|^{n-1}. \qquad ...(6)$$

Then, we have $\left| y_n - y_{n-1} \right| = \left| \int_{x_0}^{x} \left\{ f(x, y_{n-1}) - f(x, y_{n-2}) \right\} dx \right|$, by (4)

or $$\left| y_n - y_{n-1} \right| \le \int_{x_0}^{x} \left| f(x, y_{n-1}) - f(x, y_{n-2}) \right| . \left| dx \right|. \qquad ...(7)$$

Lipschitz condition (2) gives $$\left| f(x, y_{n-1}) - f(x, y_{n-2}) \right| \le K \left| y_{n-1} - y_{n-2} \right| \qquad ...(8)$$

From (7) and (8), we get

$$\left| y_n - y_{n-1} \right| \le \int_{x_0}^{x} K \left| y_{n-1} - y_{n-2} \right| . \left| dx \right| \le \text{K} \cdot \frac{MK^{n-1}}{(n-1)!} . \frac{|x - x_0|^n}{n}, \text{ by (6)}$$

Hence by mathematical induction, we conclude that (5) is true for each natural number n.

Third Step. We shall now prove that the sequence y_n converges uniformly to a limit for $x_0 - h \le x \le x_0 + h$. For the interval under consideration, $\left| x - x_0 \right| \le h$.

Hence from second step, we get $$\left| y_n - y_{n-1} \right| \le \frac{MK^{n-1} h^n}{n!} \text{ is true for all } n.$$

Using this, the infinite series

$$y_0 + (y_1 - y_0) + (y_2 - y_1) - \ldots + (y_n - y_{n-1}) + ... \qquad ...(9)$$

$$\le y_0 + Mh + \frac{1}{2!} MKh^2 + \ldots + \frac{1}{n!} MK^{n-1} h^n + \ldots \le y_0 + \frac{M}{K} \left[e^{Kh} - 1 \right],$$

which is known to be convergent for all values of K, h and M. Consequently, the series (9) is surely convergent. Thus, by the Weirstrass M–test, the series (9) converges uniformly on $[x_0 - h, x_0 + h]$. Now since the terms of (9) are continuous functions of x, therefore,

$$\text{its sum} = \lim_{n \to \infty} y_n (x) = y (x), \text{ say,} \qquad \text{as} \qquad y_n = y_0 + \sum_{n=1}^{n} (y_n - y_{n-1}) \qquad ...(10)$$

must be continuous.

Fourth Step. *We now show that $y = y(x)$ satisfies the differential equation $dy/dx = f(x, y)$.*

Since $y_n (x)$ tends uniformly to $y (x)$ in $[x_0 - h, x_0 + h]$ and by Lipschitz condition,

$$\left| f(x, y) - f(x, y_n) \right| \le K \left| y - y_n \right|,$$

it follows that $f [x, y_n (x)]$ tends uniformly to $f [x, y (x)]$. Again from (4) we have

$$y_n (x) = y_0 + \int_{x_0}^{x} f[x, y_{n-1} (x)] dx$$

or $$\lim_{n \to \infty} y_n (x) = y_0 + \lim_{n \to \infty} \int_{x_0}^{x} f[x, y_{n-1} (x)] dx, \text{ letting } n \to \infty.$$

Since the sequence $f [x, y_n (x)]$, consisting of continuous functions on the given interval, converge uniformly to $f [x, y(x)]$ on the same interval, the interchanges of limiting operations given below are valid. Thus using (10), we have

$$y(x) = y_0 + \int_{x_0}^{x} \lim_{n \to \infty} f[x, y_{n-1} (x)] dx \qquad \text{or} \qquad y(x) = y_0 + \int_{x_0}^{x} f[x, y(x)] dx \quad ....(11)$$

The integrand on the right–hand side of (11) being a continuous function of x, we conclude that the integral has the derivative. Thus, the limit function $y(x)$ satisfies the differential equation $dy/dx = f(x, y)$ on $[x_0 - h, x_0 + h]$ and is such that $y(x_0) = y_0$.

[**In the above four steps we have thus proved the existence of a solution of the given initial value problem. The next step will show that the solution $y(x)$ is unique.**]

Fifth Step. Uniqueness of the solution : We now prove that the solution $y = y(x)$ just found is the only solution for which $y(x_0) = y_0$.

Assume if possible $y = Y(x)$, say, is another solution of the given initial value problem.

Let $\left| Y(x) - y(x) \right| \le B$, where $x_0 - h \le x \le x_0 + h$. ...(12)

It may be noted here that we can surely take $B = 2K$. From (11), we get

$$\left| Y(x) - y(x) \right| = \left| \int_{x_0}^{x} \left[f\{x, Y(x)\} - f\{x, y(x)\} \right] \right| \text{ or } \left| Y(x) - y(x) \right| \le \int_{x_0}^{x} \left| f\{x, Y(x)\} - f\{x, y(x)\} \right| . |dx|$$

or $$\left| Y(x) - y(x) \right| \le K \int_{x_0}^{x} |Y(x) - y(x)| . |dx|. \quad ...(13)$$

$$\left[\because \quad \left| f\{x, Y(x)\} - f\{x, y(x)\} \right| \le K \left| Y(x) - y(x) \right|, \text{ by Lipschitz condition} \right.$$

or $$\left| Y(x) - y(x) \right| \le K.B \left| x - x_0 \right|, \text{ using (12)} \quad ...(14)$$

Now substituting (14) for integrand in (13), we get

$$| Y(x) - y(x) | \le K^2 B \int_{x_0}^{x} | x - x_0 | dx \le \frac{K^2 B | x - x_0 |^2}{2!}. \quad ...(15)$$

Again, substituting (15) for the integrand in (13), we get

$$\left| Y(x) - y(x) \right| \le \frac{K^3 B}{2!} \int_{x_0}^{x} | x - x_0 |^2 \, dx \le \frac{K^3 B | x - x_0 |^3}{3!}$$

Continuing in this way, we shall surely get

$$\left| Y(x) - y(x) \right| \le \frac{K^n B | x - x_0 |^n}{n!} \le B \frac{(Kh)^n}{n!}, \quad \text{as} \quad | x - x_0 | \le h \quad ...(16)$$

Now the series $\sum_{n=0}^{n} B \frac{(Kh)^n}{n!}$ converges, and so $\lim_{n \to \infty} B \frac{(Kh)^n}{n!} = 0$.

Thus $\left| Y(x) - y(x) \right|$ can be made less than any number however small and consequently we conclude that $Y(x) - y(x) = 0$ *i.e.*, $Y(x) = y(x)$

This shows that the solution $y = y(x)$ is always unique, and the proof of the theorem is complete.

Important note : The above theorem is an existence theorem because it says that the initial value problem does have a solution. It is also a uniqueness theorem, because it says that there is only one solution.

Remark 1. If the existence theorem is asked then you need not mention uniqueness in the statement of theorem and finish up the proof just after the fourth step. Again if you are asked to state and prove the uniqueness theorem, then give the complete proof of first four steps and fifth step.

Remark 2. If $f(x, y)$ satisfies the condition $\left| \partial f / \partial y \right| \le K$ ...(*i*)

for all values of x, y in the given range then for the same constant K the Lipschitz's condition is also satisfied.

By the mean value theorem of differential calculus, we get

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\left(\frac{\partial f}{\partial y}\right)_{y=\bar{y}}, \qquad \text{where } y_1 < \bar{y} < y_2, \qquad ...(ii)$$

where (x, y_1) and (x, y_2) are assumed in the given range.

Now (i) and (ii) $\Rightarrow$ $|f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1|$, ...(*iii*)

which is Lipschitz condition. It follows that the Lipschitz condition (*iii*) can be replaced by the stronger condition (*i*).

1.7. An Important Theorem. *If S in either a rectangle* $|x - x_0| \le h, |y - y_0| \le k$, $(h, k > 0)$ *or a strip* $|x - x_0| \le h, |y| < \infty$ $(h > 0)$, *and if* $f(x, y)$ *is a real valued function defined on S such that* $\partial f/\partial y$ *exists, is continuous on S, and* $\left|\frac{\partial}{\partial y} f(x,y)\right| \le K$, $(x, y) \in S$ *for a positive constant K, then* $f(x, y)$ *satisfies a Lipschitz condition on S with Lipschitz constant K.* **[Meerut 1994]**

Proof. Now, $|f(x,y_1) - f(x,y_2)| = \left|\int_{y_1}^{y_2} \frac{\partial}{\partial y} f(x,y)\, dy\right| = \int_{y_1}^{y_2} \left|\frac{\partial}{\partial y} f(x,y)\right| |dx| \le K \int_{y_1}^{y_2} |dy|$.

Thus, $|f(x,y_1) - f(x,y_2)| = K|y_1 - y_2|$ for $(x, y_1), (x, y_2) \in S$, showing that $f(x, y)$ satisfies Lipschitz condition on S with Lipschitz constant K.

1.8. Solved examples based on Articles 1.4 to 1.7

Ex. 1. *Show that* $f(x, y) = xy^2$ *satisfies the Lipschitz condition on the rectangle R:* $|x| \le 1, |y| \le 1$ *but does not satisfy a Lipschitz condition on the strip S* $|x| \le 1, |y| < \infty$.

[Meerut 2005, 07, Kanpur 2002, Bilaspur 2004]

Sol. Method I. We have $|f(x,y_2) - f(x,y_1)| = |xy_2^2 - xy_1^2|$, as $f(x, y) = xy^2$

or $$|f(x,y_2) - f(x,y_1)| = |x|\,|y_2 + y_1|\,|y_2 - y_1|. \qquad ...(1)$$

Hence in the rectangle $|x| \le 1, |y| \le 1$, (1) $\Rightarrow |f(x,y_2) - f(x,y_1)| \le$ (1) × (2) × $|y_2 - y_1|$, showing that Lipschitz condition is satisfied.

Next, $\left|\frac{f(x,y_2) - f(x,0)}{y_2 - 0}\right| = |x||y_2| \to \infty$ when $|y_2| \to \infty$ if $|x| \ne 0$,

showing that the Lipschitz condition is not satisfied on the strip $|x| \le 1, |y| < \infty$.

Method II. We have $$|\partial f/\partial y| = 2|xy| = 2|x||y| \qquad ...(1)$$

$\therefore$ In the rectangle $|x| \le 1, |y| \le 1$, $|\partial f/\partial y| \le 2$, for eah $(x, y) \in R$,

showing that $f(x, y)$ satisfies Lipschitz condition in R, with Lipschitz constant 2.

On the other hand, on the strip $S: |x| \le 1, |y| < \infty$, (1) shows that $|\partial f/\partial y|$ is unbounded on the strip S as $|y| \to \infty$. Hence Lipschitz condition is not satisfied on the strip S.

Ex. 2. *If S is defined by the rectangle* $|x| \le a, |y| \le b$, *show that the* $f(x, y) = x^2 + y^2$, *satisfies the Lipschitz condition. Find the Lipschitz constant.*

[Meerut 2000, 05, 07; Himanchal 2003; Rohilkhand 2007]

Sol. Let (x, y_1) and (x, y_2) be two arbitrary points in the rectangle S. Then, we have

$$|f(x,y_2) - f(x,y_1)| = \left|\left(x^2 + y_2^2\right) - \left(x^2 + y_1^2\right)\right|, \text{ as } f(x, y) = x^2 + y^2$$

$$= \left| y_2^2 - y_1^2 \right| = \left| y_2 + y_1 \right| \left| y_2 - y_1 \right|.$$

Thus, $\left| f(x, y_2) - f(x, y_1) \right| \le 2b \left| y_2 - y_1 \right|$, since $|y| < b$ in S.

showing that the Lipschitz condition is satisfied. Here the Lipschitz constant $K = 2b$.

Ex. 3. *Prove that the continuity of $f(x, y)$ is not enough to guarantee the uniqueness of the solution of the initial value problem* : $dy/dx = f(x, y) = \sqrt{|y|}$, $y(0) = 0$.

[Himanchal 2003; Lucknow 2006; Meerut 1998]

OR Show that the solution of the initial value problem $dy/dx = f(x, y)$, $y(x_0) = y_0$ may not be unique although $f(x, y)$ is continuous.

Sol. Consider the initial value problem $dy/dx = \sqrt{|y|}$, $y(0) = 0$. ...(1)

Clearly $f(x, y) = \sqrt{|y|}$ is continuous for all y and (1) has the following two solutions :

$y \equiv 0$ and $y = \begin{cases} x^2/4, & \text{when } x \ge 0 \\ -x^2/4, & \text{when } x < 0. \end{cases}$

We now show that Lipschitz condition $\dfrac{\left| f(x, y_2) - f(x, y_1) \right|}{\left| y_2 - y_1 \right|} \le K$ ...(2)

does not hold good in any region which includes the line $y = 0$. For example, when $y_1 = 0$ and y_2 is positive, then $\dfrac{\left| f(x, y_2) - f(x, y_1) \right|}{\left| y_2 - y_1 \right|} = \dfrac{\sqrt{y_2}}{y_2} = \dfrac{1}{\sqrt{y_2}}$, as $\sqrt{y_2} > 0$

showing that quantity on L.H.S. of (2) can be made as large as we please by choosing y_2 sufficiently small. This violates (2) because the quantity on L.H.S. of (2) should not exceed a fixed constant K.

Ex. 4. *Illustrate by an example that a continuous function may not satisfy a Lipschitz condition on a rectangle.* **[Meerut 1995]**

Sol. As an example, consider the function $f(x, y) = y^{2/3}$ on the rectangle $S : |x| \le 1, |y| \le 1$. ..(1)

Clearly, $f(x, y)$ is continuous in the rectangle S.

Here $\left| \dfrac{\partial}{\partial y} f(x, y) \right| = \left| \dfrac{2}{3y^{1/3}} \right| \to \infty$, when, $y \to 0$. ...(2)

Since $y = 0$ is a point of the rectangle S, (2) shows that the Lipschitz condition is not satisfied by the function $f(x, y) = y^{2/3}$ on the rectangle S.

Ex. 5. *(a) For the initial value problem $dy/dx = e^y$, $y(0) = 0$. Find the largest interval $|x| \le a$ in which the Picard's theorem guarantees existence of a unique solution.*

[Meerut 2007, 11 Gwalior 2000, 02, 05]

(b) For the initial value problem $dy/dx = e^y$, $y(0) = 0$; find the largest interval $|x| \le a$ in which the Picard's theorem holds. **[Meerut 2007]**

Sol. Here the condition of boundedness of $f(x, y)$, namely, $\left| f(x, y) \right| \le M$ for $\left| y - y_0 \right| \le Ma$,

reduces to $e^y \le M$ for $\left| y - 0 \right| \le Ma$, as $f(x, y) = e^y$

Let y_1, y_2 lie in the range $|y| \le Ma$ and $y_1 < y_2$. Then, using the mean value theorem, we have

$$e^{y_2} - e^{y_1} = (y_2 - y_1) \left(\frac{\partial e^y}{\partial y} \right)_{y = \bar{y}}, \quad \text{where} \quad y_1 < \bar{y} < y_2$$

or $e^{y_2} - e^{y_1} \le (y_2 - y_1)\, M$, since $e^{\bar{y}} \le M$,

showing that the function e^y satisfies the Lipschitz condition.

Again, the inequality $e^y \le M$ will be satisfied for all values of y such that $|y| \le Ma$ provided it is satisfied for $y = Ma$. Hence $e^{Ma} \le M$ or $a \le (\log M)/M$.

Using the well known methods of finding maximum and minimum values of a function, we can easily show that the function $(\log M)/M$ is maximum when $M = e = 2.718$. Hence uniqueness theorem of the given initial value problem gives $|x| \le a$ where $a = 1/e = 0.308$. Thus, the required largest interval is $|x| \le 0.308$.

Ex. 6. *Show that for the problem* $dy/dx = y$, $y(0) = 1$, *the constant a in Picard's theorem must be smaller than unity.* **[Agra 1992; Meerut 1993]**

Sol. The condition of bounded of $f(x, y)$, namely $|f(x,y)| \le M$ for $|y - y_0| \le Ma$

reduces to $|y| \le M$ for $|y - 1| \le Ma$.

Choose $M \ge 1$. Then the Lipschitz condition is also satisfied because in this situation, the Lipschitz condition takes the form

$$|f(x,y_2) - f(x,y_1)| = |y_2 - y_1| \le M\,|y_2 - y_1|, \quad \text{as} \quad M \ge 1.$$

Again, $|y - 1| \le Ma \quad \Rightarrow \quad |y| - 1 \le Ma$.

It follows that the condition $|y| \le M$ will be satisfied for all values of $|y - 1| \le Ma$ provided it is satisfied for $|y| = 1 + Ma$.

$\therefore \quad 1 + Ma \le M$ or $a < (\mathrm{M} - 1)/\mathrm{M} = 1 - (1/\mathrm{M}) < 1$, as $1 \le M < \infty$.

Ex. 7. *For the initial value problem statement* (*I.V.P.*) $y' = f(x, y)$ *with* $y(0) = 0$ *which of the following statement is true:*

(*a*) $f(x, y) = (xy)^{1/2}$ *satisfies Lipschitz's condition and so I.V.P. has unique solution.*

(*b*) $f(x, y) = (xy)^{1/2}$ *does not satisfy Lipschitz's condition and so, I.V.P. has no solution.*

(*c*) $f(x, y) = |y|$ *satisfies Lipschitz's condition and so I.V.P. has unique solution.*

(*d*) $f(x, y) = |y|$ *does not satisfy Lipschitz's condition still I.V.P. has unique solution.*

Sol. Ans. (a) Proceed as in Ex. 3 of Art. 1.8. **(GATE 2003)**

Ex. 8. (*i*) *Define Lipschitz conditions and Lipschitz constant.*

(*ii*) *Give an example to show that there exist functions which do not satisfy the Lipschitz condition.*

(*iii*) *Give an example to show that the existence of partial derivative of* $f(x, y)$ *is not necessary for* $f(x, y)$ *to be a Lipschitz function.*

Sol. (*i*) **Lipschitz condition and Lipschitz constant**

Definitions. Let $f(x, y)$ be a function defined for all (x, y) in a domain D. We say, (x, y) satisfies a *Lipschitz condition* on D, if there exists a constant $K > 0$, such that

$$|\, f(x,y_1) - f(x,y_2)\,| \le K\,|\, y_1 - y_2\,|$$

for all (x, y_1), (x, y_2) in D. The constant K is known as *Lipschitz constant.*

As a consequence of the definition, a function $f(x, y)$ satisfies Lispschitz condition if and only if there exists a constant $K > 0$ such that

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} \le K, \quad y_1 \ne y_2\,, \text{ whenever } (x, y_1), (x, y_2) \text{ belongs to } D.$$

We now state a general criterion which would ensure the Lipschitz condition :

Let $f(x, y)$ be a continuous function defined over a rectangle $D = \{(x, y); |x - x_0| \le a, |y - y_0| \le b\}$.

Here a, b are some positive real numbers. Let $\partial f / \partial y$ be defined and continuous on D and

$$|\partial f / \partial y| \le K, \text{ for each } (x, y) \in D$$

for some $K > 0$. Then f satisfies a Lipschitz conditon on D with Lipschitz constant K.

(*ii*) Let $f(x, y) = y^{1/2}$ be defined on the rectangle $D = \{(x, y) : |x| \le 2, |y| \le 2\}$, Let $y_1 > 0$.

Then, $$\frac{|f(x, y_1) - f(x, 0)|}{|y_1 - 0|} = \frac{y_1^{1/2}}{y_1} = \frac{1}{y_1^{1/2}},$$ which is unbounded as $y_1 \to 0$.

Hence $f(x, y)$ does not satisfy the Lipschitz condition in D.

(*iii*) Let $f(x, y) = |y|$ be defined on the square $D = \{(x, y) : |x| \le 1, |y| \le 1\}$.

Recall that the partial derivatives of $f(x, y)$ w.r.t. 'y' at the point (x', y') is defined as

$$\left(\frac{\partial f}{\partial y}\right)_{(x', y')} = \lim_{k \to 0} \frac{f(x', y' + k) - f(x', y')}{k}$$

$\therefore$ Using the above definition, we get $$\left(\frac{\partial f}{\partial y}\right)_{(x, 0)} = \lim_{k \to 0} \frac{f(x, 0 + k) - f(x, 0)}{k} = \lim_{k \to 0} \frac{|k|}{k},$$

which does not exist. Thus $\partial f / \partial y$ fails to exist at $(x, 0)$.

Again, $$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{||y_1| - |y_2||}{|y_1 - y_2|} \le 1, \quad \text{as} \quad |y_1 - y_2| \ge ||y_1| - |y_2||$$

showing that $f(x, y)$ satisfies Lipschitz condition in x on D with Lipschitz constant $K = 1$.

Ex.9. *If S is defined by the rectangle $|x| \le a$, $|y| \le b$, show that the function $f(x, y) = x \sin y + y \cos x$, satisfy the Lipschitz condition. Find the Lipschitz constant.*

[Agra 2006; Jabalpur 2005; Kanpur 2002; Meerut 2003]

Sol. From the given function, $$\partial f / \partial y = x \cos y + \cos x \quad \ldots (1)$$

Since x, $\cos y$ and $\cos x$ are continuous functions, $\partial f / \partial y$ is also continuous on S. Also, we have $$|\partial f / \partial y| = |x \cos y + \cos x| \le |x \cos y| \le |\cos x|$$

or $$|\partial f / \partial y| = |x| |\cos y| + |\cos x| \le |x| + 1 \le a + 1 \text{ for each } (x, y) \in S, \quad [\because |x| < a]$$

showing that $f(x, y)$ satisfies Lipschitz condition and Lipschitz constant is $a + 1$.

Ex. 10. *Show that the function $f(x, y) = y^{2/3}$ does not satisfy the Lipschitz condition on the rectangle $R : |x| \le 1, |y| \le 1$.*

Sol. We have $|\partial f / \partial y| = |2 / 3y^{1/3}|$, which is unbounded in every neighbourhood of the origin.

Hence $f(x, y)$ does not satisfy the Lipschitz condition

Second method. We have $$\frac{|f(0, y) - f(0, 0)|}{|y - 0|} = \frac{|y^{2/3} - 0|}{y} = \frac{1}{|y|^{1/3}},$$

which is unbounded in every neighbourhood of the origin and so $f(x, y)$ does not satisfy the Lipschitz condition.

Ex. 11. *Give an example to prove that we cannot drop the Lipschitz condition in the statement of Picard's theorem.*

Sol. Consider the problem $dy/dx = 3\,y^{2/3}, \qquad y(0) = 0 \qquad \ldots (1)$

and let R be the rectangle $|x| \le 1,\ |y \le 1|$. Clearly $f(x, y) = 3\,y^{2/3}$, which is continuous on R.

We easily verify that $y_1(x) = x^3$ and $y_2(x) = 0$ are two distinct solutions of (1) valid for all x. Thus (1) has solution that is not unique. The reason for this non uniqueness lies in the fact that $f(x,\ y)$ does not satisfy a Lipschitz condition on the rectangle R, since

$$\frac{|f(0, y) - f(0,0)|}{|y - 0|} = \left|\frac{3y^{2/3}}{y}\right| = \frac{3}{|y|^{1/3}}$$

is unbounded in every neighbourhood of the origin.

Ex. 12. *Show that the* $f(x, y) = 4x^2 + y^2$ *on* $R : |x| \le 1, |y| \le 1$ *satisfies Lipschitz condition.*

Sol. Since $|\partial f / \partial y| = |2y| \le 2, \qquad$ for each $\qquad (x, y) \in R.$

Hence $f(x, y)$ satisfies the Lipschitz condition with Lipschitz constant 2.

Ex. 13. *Consider* $f(x,\ y) = x^3|\ y\ |$. *Prove that f satisfies a Lipschitz condition on* $R : |x| \le 2, |y| \le 2$ *even though* $\partial f / \partial y$ *does not exist at* $(x, 0)$ *if* $x \ne 0$.

Sol. By definition of partial derivative, we have

$$\left(\frac{\partial f}{\partial y}\right)_{(x,0)} = \lim_{k \to 0} \frac{f(x,k) - f(x,0)}{k} = \lim_{k \to 0} \frac{x^3\,|k|}{k},$$

which does not exist. However, we have

$$|f(x, y_1) - f(x, y_2)| = \left|x^3\,|y_1| - x^3\,|y_2|\right| = |x^3| \times \left||y_1| - |y_2|\right| \qquad \ldots (1)$$

Also, we have $\qquad \left||y_1| - |y_2|\right| \ \le\ |y_1 - y_2| \qquad \ldots (2)$

(1) and (2) $\qquad \Rightarrow \qquad |f(x, y_1) - f(x, y_2)| \ \le\ |x|^3\,|y_1 - y_2|$

$\therefore\ |f(x, y_1) - f(x, y_2)| \ \le\ 8|y_1 - y_2|,$ which is true for all (x, y_1), (x, y_2) in R. $\qquad \ldots (3)$

(3) shows that $f(x, y)$ satisfies Lipschitz condition with Lipschitz constant 8.

Ex. 14. *For the initial value problem* $dy/dx = y^2 + \cos^2 x$, $y(0) = 0$, *determine the interval of existence of its solution given that R is the rectangle containing origin,*

$$R : \{(x, y) : \qquad 0 \le x \le a, \qquad |y| \le b, \qquad a > 1/2, \qquad b > 0\}$$

Sol. Let $f(x, y) = y^2 + \cos^2 x$ Also, $|f(x, y)| = |y^2 + \cos^2 x| \le |y|^2 + |\cos x|^2 \le b^2 + 1$

Let $\qquad b^2 + 1 = M \qquad$ so that $\qquad |f(x, y)| \le M$

Again, since $|\partial f / \partial y| = |2y| = 2b = K$ (say), we see that $f(x, y)$ satisfies Lipschitz condition

We find that $y(x)$ exists for $\quad 0 \le x \le h = \min(a, b/M) = \min(a, b/(1 + b^2))$ $\qquad \ldots (1)$

Now, $\dfrac{b}{1+b^2} = \dfrac{1}{1/b + b} = \dfrac{1}{\left(1/\sqrt{b} - \sqrt{b}\right)^2 + 2}, \quad \Rightarrow$ The maximum value of $\dfrac{b}{1+b^2}$ is $\dfrac{1}{2}$

Hence (1) $\Rightarrow$ $h = 1/2$ and so $y(x)$ exists on the interval $0 \le x \le 1/2$.

Ex. 15. *Consider the initial value problem* $dy/dx = y^2$, $y(0) = 2$. *Let R be the rectangle* $R : \{(x, y) : |x| \le a, |y-2| \le b, a > 0, b > 0\}$. *Find the largest interval of existence of its solution. By explicitly solving the given initial value problem, show that there exists an interval of existence which is larger than that obtained by the application of Picards' method.*

Sol. Given $dy/dx = y^2$, $y(0) = 2$ ... (1)

To find the interval of existence by application of Picard's theorem: Let $f(x, y) = y^2$. Then, in R, we have $|f(x,y)| = |y^2| \le (b+2)^2 = M$, say and hence the interval of existence of solution is given by $|x| \le h$, where $h = \min\left(a, b/(b+2)^2\right) = 1/8$.

$\Rightarrow$ The interval of existence is $-(1/8) \le x \le (1/8)$.

***To find the interval of existence by solving* (1).** We have $dy/dx = y^2$ so that $y^{-2}dy = dx$.

Integrating, $-y^{-1} = x + c$, c being an arbitrary constant. ... (2)

Putting $x = 0$ and $y = 2$, (2) gives $-2^{-1} = c$. Then (2) gives $-\frac{1}{y} = x - \frac{1}{2}$ or $y(x) = \frac{2}{1-2x}$,

showing that $y(x)$ exists on $-\infty < x < 1/2$. This interval of existence is much larger than that obtained by the application of Picard's method.

Ex. 16. *The Picard's theorem assumes Lipschitz condition. Can we drop this condition. If answer is no, give an example to illustrate this point.*

Sol. In the following two examples, we shall show that if Lipschitz condition is not satisfied then we do not arrive at unique solution as stated in Picard's theorem.

Consider $dy/dx = 4y^{3/4}$, $y(0) = 0$, $x \ge 0$ ... (1)

Let $f(x, y) = 4y^{3/4}$. Then, we get $\frac{f(x,y) - f(x,0)}{y - 0} = \frac{4y^{3/4}}{y} = \frac{4}{y^{1/4}}$, $y \ne 0$

which is unfounded for $x \ge 0$, since it can be made as large as possible by choosing y close to zero. Thus $f(x, y)$ fails to satisfy Lipschitz condition.

Using Picard's method of successive approximation (refer Art. 1.2), we easily see that $y_n(x) = 0$ for $n = 0, 1, 2, 3.....$ Hence $y(x) = \lim_{n\to\infty} y_n(x) = 0$ on $[0, \infty[$. We also not that $y(x) = x^4$ is also solution of (1). Hence (1) does not possess unique solution.

EXERCISE 1(C)

Ex. 1 (*a*) Show that the function f given by $f(x, y) = y^{1/2}$ does not satisfy a Lipschitz condition on $R : |x| \le 1, 0 \le y \le 1$. Show that f satisfies a Lipschitz condition on any rectangle R of the form $R : |x| \le a, b \le y \le c, (a, b, c > 0)$

(*b*) Show that a function f given by $f(x, y) = x^2 |y|$ satisfies a Lipschitz condition on $R : |x| \le 1, |y| \le 1$

(*c*) Show that $f(x, y) = xy^2$ (*i*) satisfies a Lipschitz condition on any rectangle $a \le x \le b$, $c \le y \le d$. (*ii*) does not satisfy a Lipschitz condition on any strip $a \le x \le b$, $-\infty < y \le \infty$

(*d*) Show that $f(x, y) = xy$ (*i*) satisfies Lipschitz condition on any rectangle $a \le x \le b$, $c \le y \le d$ (*ii*) satisfies a Lilpschitz condition on any strip $a \le x \le b$, $-\infty < y < \infty$ (*iii*) does not satisfy a Lipschitz condition on the entire plane.

(*e*) For what points (x_0, y_0) does Picard's theorem imply that the initial value problem $dy/dx = y\,|\,y\,|$, $y(x_0) = y_0$ has a unique solution on some interval $|\,x - x_0\,| \le h$? [**Ans.** All point (x_0, y_0)]

(*f*) Consider the initial value problem : $dy/dx = 2y/x$, if $x > 0$ and $dy/dx = 0$, if $x = 0$; $y(0) = 0$. Show that $2y/x$ does not satisfy Lipschitz condition in any closed rectangle containing (0, 0) and the given initial value problem has solution which is not unique.

Sol. Left as an exercise for the reader.

Ex. 2. By computing appropriate Lipschitz constants, show that the following functions satisfy Lipschitz conditions on the domain D of xy–plane indicated.

(*i*) $f(x, y) = 4x^2 + y^2$, on $D : |\,x\,| \le 1, \quad |\,y\,| \le 1$

(*ii*) $f(x, y) = x^2\cos^2 y + y\sin^2 x$, on $D : |\,x\,| \le 1, \quad |\,y\,| < \infty$

(*iii*) $f(x, y) = x^3 e^{-xy^2}$, on $D : 0 \le x \le a, \quad |\,y\,| < \infty, \quad (a > 0)$

(*iv*) $f(x, y) = a(x)\,y^2 + b(x)\,y + c(x)$, on $D : |\,x\,| \le 1, \quad |\,y\,| \le 2,$

where $a(x)$, $b(x)$ and $c(x)$ are continuous functions on $|\,x\,| \le 1$.

(*v*) $f(x, y) = a(x)\,y + b(x)$, on $D : |\,x\,| \le 1, |\,y\,| < \infty$, where $a(x)$ and $b(x)$ are continuous on $|\,x\,| < 1$

Ans. (*i*) $K = 2$ (*ii*) $K = 3$ (*iii*) $K = \max\{2a^3, 2a^4\}$

(*iv*) $K = 4\,M_a + M_b$; where $M_a = \max_{|x| \le 1} |\,a(x)\,|$, $M_b = \max_{|x| \le 1} |\,b(x)\,|$ (*v*) $K = \max_{|x| \le 1} |\,a(x)\,|$

3. Let (x_0, y_0) be an interior point of a closed rectangle R, $a \le x \le b$, $c \le y \le d$ in which $f(x, y)$ is continuous. Let $f(x, y)$ satisfy the Lipschitz condition. $|\,f(x, y_1) - f(x, y_2)\,| \le K\,(y_1 - y_2)$, for all possible (x, y_1) and (x, y_2) in R and some fixed constant K. Prove that $dy/dx = f(x, y)$, $y(x_0) = y_0$ has a unique solution.

4. Write a note on Picard's existence theorem regarding the existence and uniquencess of the solution of the equation $dy/dx = f(x, y)$, where f satisfies Lipschitz condition.

5. (*a*) Show that (*i*) $|\,y'\,| + |\,y\,| = 0$, $y(0) = 1$ has no solution.
(*ii*) $y' = x$, $y(0) = 1$ has one solution (*iii*) $y' = (y - 1)/x$, $y(0) = 1$ has an infinity of solutions.

Explain what you mean by existence and uniqueness of a differential equation.

(*b*) State carefully any existence and uniqueness theorem for differential equation you know and apply it to the above three examples.

6. Show that the conditions for the exitence and uniqueness of a solution of the following initial value problem are not satisfied by the function $f(x, y) = (y - 1)/x$ in any rectangle R of xy–plane with (0, 1) as its centre : $y' = (y - 1)/x$, $y(0) = 1$; but a solution does exist of above problem. Give reason for your answer. Draw some possible solution curves.

7. Apply Picard's iteration process to get the solution of $y' = (y - 1)/x$, $y(1) = 1$. Give your arguments, why Picard's iteration process is applicable to the above initial value problem.

8. If $f(x, y) = y^{2/3}$, show that Lipschitz condition in not satisfied in any region containing the origin and that the solution of the differential equation $dy/dx = f(x, y)$ satisfying the initial condition $y = 0$ when $x = 0$ is not unique. **[Meerut 1999]**

9. By giving suitable example prove that a function does not satisfy the Lipschitz condition on the prescribed domain. **[Madurai 2001, 05]**

10. Examine existence and uniqueness of the solution of the initial value problem $dy/dx = y^2$, $y(1) = -1$. **[Meerut 2011]**

11. Show that the Picard's theorem ensures a unique solution in the interval $|\,x\,| \le 1/2$ for the initial value problem $dy/dx = x + y^2$, $y(0) = 0$.

12. State Picard's theorem on existence of solutions of differential equations. Show that $f(x, y) = y^{1/2}$ does not satisfy Lipschtiz's condition on $|x| \le 1$, $0 \le y \le 1$ while $f(x, y) = x^2 \cos^2 y + y \sin^2 x$ satisfies Lipschitz's condition on $|x| \le 1, |y| < \infty$. Find the Lipschitz's constant. **[I.A.S. 2000]**

13. Examine the exitence and uniqueness of solution of the initial value problem $dy/dx = y^{1/3}$, $y(0) = 0$ **[Himanchal 2003; Kanpur 2000; Kolkata 2000, Osmania 2005, 07; G.N.D.U. Amritsar 2003, 05; Meerut 2000]**

14. Show that for the problem $dy/dx = y$, $y(0) = 1$, the constant h in Picard's theorem must be smaller than unity. **[Rohilkhand 2007; Meerut 1995]**

15. Discuss the existence and uniqueness of a solution of the initial value problem $dy/dx = y^{4/3}$, $y(x_0) = y_0$. **[Bhopal 2010; Rajasthan 2010; Himachal 2010]**

16. Examine existence and uniqueness of the solution of the initial value problem $dy/dx = y^2$, $y(1) = -1$. **[Gwalior 2000, 01, 03; Ujjain 2000, 02, 04, Rajasthan 2000; Agra 1997; Kanpur 2000; Meerut 1994, 96]**

17. State and prove the uniqueness theorem for the initial value problem $dy/dx = f(x, y)$, $y(x_0) = y_0$. **[Meerut 1998]**

18. Examine whether the following differential equation possesses unique solution. Justify your answer. $\frac{dy}{dx} = \begin{cases} y(1-2x), x > 0 \\ y(2x-1), x < 0 \end{cases}$ subject to the condition: $y = 1$ at $x = 1$ **[Jabalpur 2003]**

19. Verify that the initial value problem $xy' - y = 0, y(0) = 0$ has two solutions $y_1(x) = 0$ and $y_2(x) = x$. Does it contradict the Picard's theorem. **[Himanchal 2002, 03, 05]**

20. Let $f(x, y) = (\cos y)/(1 - x^2)$, $(|x| < 1)$. Show that f satisfied a Lipschitz condition on every strip $S_a : |x| \le a,$, where $0 < a < 1$. **[Himanchal 2009]**

21. Define Lipschitz condition (with respect to y) of the function $f(x, y)$ and investigate its geometrical significance. Show that the function $f(x, y) = x^2 e^{x+y}$ satisfies Lipschitz condition in the rectangle defined by $|x| \le a, |y| \le b$. **[Himanchal 2009]**

22. State and prove Picard's existence theorem for the solution of differential equation $dy/dx = f(x, y)$, $y(x_0) = y_0$. Show that $dy/dx = x^2 + y^2$, $y(0) = 0$ has a unique solution. **[Himanchal 2009]**

2

Simultaneous Equations of the Form $(dx)/P = (dy)/Q = (dz)/R$

2.1. Introduction.

In this chapter we shall study simultaneous equations of the first order and of the first degree in the derivatives. Equations containing only three variables will be studied. It will be noted that the method of solution presented here can be applied to equations involving any number of variables.

The general type of a set of simultaneous equations of the first order having three variables is

$P_1dx + Q_1dy + R_1dz = 0$ and $P_2dx + Q_2dy + R_2dz = 0$, ...(1)

where the coefficients are functions of x, y, z. Solving these equations simultaneously, we have

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1},$$

which is of the form $(dx)/P = (dy)/Q = (dz)/R,$ ...(2)

where P, Q, and R are functions of x, y, z. Thus we note that the simultaneous equations (1) can always be put in the form (2).

2.2. The nature of solution of (dx)/P = (dy)/Q = (dz)/R.

The given equations are said to be completely solved when we get a solution of the form $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$, where u_1 and u_2 are two independent integrals (solutions) of the given equations. u_1 and u_2 are said to be independent integrals if u_1/u_2 is not merely a constant. For example, $u_1 = x^2 + y^2 + z^2$ and $u_2 = x + y + z$ are independent integrals whereas $u_1 = 2x + 2y + 2z$ and $u_2 = 2(x + y + z)$ are not independent.

2.3. Geometrical Interpretation of (dx)/P = (dy)/Q = (dz)/R.

From three dimensional coordinate geometry, it is known that the direction cosines of the tangent to a curve are proportional to dx: dy: dz. The given differential equations, therefore, express the fact that the direction cosines of the tangent to the curve at that point are proportional to $P : Q : R$. Suppose that the solution of the given equations is given by $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$. Then we observe that the solution represents the curves of intersection of the surfaces $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$. Since c_1 and c_2 can take any values in infinite number of ways, we get a doubly infinite number of such curves.

2.4. Rule I for solving (dx)/P = (dy)/Q = (dz)/R. ...(1)

By equating two of the three fractions of (1), we may be able to get an equation in only two variables. Sometimes such an equation is obtained after cancellation of some factor from the chosen two fractions of (1). On integrating the differential equation in only two variables by well known methods, we shall obtain one of the relations in the general solution of (1). This method may be repeated to give another relation with help of two other fractions of (1).

2.5. Solved examples based on Art. 2.4.

Ex. 1. *Solve* (a) $\dfrac{xdx}{y^2z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$. **[Nagpur 1996; Bangalore 2005]**

(*b*) $\dfrac{xdx}{y^2z}=\dfrac{dy}{xz}=\dfrac{dz}{y^2z}$ **[Poona 2006; Vikram 1996]**

Sol. (*a*) Taking the first two fractions, we get $x^2dx = y^2dy$

or $3x^2dx - 3y^2dy = 0$ and so $x^3 - y^3 = c_1$. ...(1)

Next, taking the first and the third fractions, we get $xdx = zdz$

or $2xdx - 2zdz = 0$ and so $x^2 - z^2 = c_2$. ...(2)

Since $x^3 - y^3$ and $x^2 - z^2$ are independent, the required general solution is given by the relations (1) and (2), c_1 and c_2 being arbitrary constants.

(*b*) Proceed as in part (*a*). **Ans.** $x^3 - y^3 = c_1$, $x^2 - 2z = c_2$

Ex. 2. *Solve* $\dfrac{dx}{yz}=\dfrac{dy}{zx}=\dfrac{dz}{xy}$. **[Agra 1996; Delhi Maths (G) 1998]**

Sol. Taking the first two fractions, we have

$xdx = ydy$ or $2xdx - 2ydy = 0$ so that $x^2 - y^2 = c_1$. ...(1)

Again, taking the first and the third fractions, we have

$xdx = zdz$ or $2xdx - 2zdz = 0$ so that $x^2 - z^2 = c_2$. ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 3. $(dx)/x = (dy)/0 = (dz)/(-x)$

Sol. From the second fraction, we have $dy = 0$ so that $y = c_1$. ...(1)

Taking the first and the third fractions, $xdx + zdz = 0$ or $2xdx + 2zdz = 0$.

Integrating, $x^2 + z^2 = c_2$, c_2 being an arbitrary to constant. ...(2)

The required solution is given by the relations (1) and (2).

Ex. 4. *Solve* $\dfrac{dx}{y^2}=\dfrac{dy}{x^2}=\dfrac{dz}{x^2y^2z^2}$. **[Delhi Maths (G) 2000]**

Sol. Taking the first two fractions, $3x^2dx - 3y^2dy = 0$.

Integrating, $x^3 - y^3 = c_1$, being an arbitrary constant ...(1)

Taking the first and third fractions, $3x^2dx - 3z^{-2}dz = 0$.

Integrating, $x^3 + 3z^{-1} = c_2$, c_2 being an arbitrary constant ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 5. *Solve* $\dfrac{dx}{x^2+2y^2}=\dfrac{dy}{-xy}=\dfrac{dz}{xz}$. **[Delhi Maths (Hons.) 1994]**

Sol. Taking the last two fractions, $(1/y)dy + (1/z)dz = 0$.

Integrating, $\log y + \log z = \log c_1$ or $yz = c_1$. ...(1)

Again, taking the first two fractions, we have

$\dfrac{dx}{dy}=\dfrac{x^2+2y^2}{-xy}$ or $x\dfrac{dx}{dy}=-\dfrac{x^2}{y}-2y$ or $2x\dfrac{dx}{dy}+\dfrac{2}{y}x^2=-4y$ ... (2)

Putting $x^2 = v$ so that $2x(dx/dy) = dv/dx$, (2) reduces to

$(dv/dx) + (2/y)v = -4y$, which is linear differential equation ...(3)

Integrating factor of (3) $= e^{\int(2/y)dy} = e^{2\log y} = y^2$ and so its solution is

$vy^2 = \int\{(-4y)\times y^2\}\times dy + c_2$ or $x^2y^2 + y^4 = c_2$. ...(4)

The required general solution is given by the relations (1) and (4).

EXERCISE 2 (A)

Solve the following simultaneous differential equations :

1. $dx = dy = dz$ **Ans.** $x - y = c_1$, $x - z = c_2$

2. $(dx)/a = (dy)/a = dz$ **Ans.** $x - y = c_1$, $y - az = c_2$

3. $(dx)/x = (dy)/y = (dz)/z$ **Ans.** $x/y = c_1$, $x/z = c_2$

4. $(dx)/\tan x = (dy)/\tan y = (dz)/\tan z$ **Ans.** $(\sin x)/(\sin y) = c_1$, $(\sin x)/(\sin z) = c_2$

5. $dx = dy = (dz)/\sin x$ **Ans.** $x - y = c_1$, $z + \cos x = c_2$

2.6. Rule II for solving (dx)/P = (dy)/Q = (dz)/R. ...(1)

Suppose only one relation $u_1(x, y, z) = c_1$ can be found by using rule I of Art 2.4. Then, sometimes we try to use this relation in expressing one variable in terms of the others. This may help us to obtain an equation in two variables. The solution of this equation will give us second relation for the general solution of (1). Note that the second relation will involve the arbitrary constant c_1. To find the final form the second relation, the arbitrary constant c_1 must be removed with help of the first relation $u_1(x, y, z) = c_1$.

2.7. Solved examples based on Art. 2.6

Ex. 1. *Solve* $\dfrac{dx}{xz(z^2 + xy)} = \dfrac{dy}{-yz(z^2 + xy)} = \dfrac{dz}{x^4}$. **[Delhi Maths (H) 2001]**

Sol. Cancelling $z(z^2 + xy)$, the first two fractions give

$(1/x)dx = -(1/y)dy$ or $(1/x)dx + (1/y)dy = 0$,

Integrating, $\log x + \log y = \log c_1$ or $xy = c_1$. ...(1)

Using (1), the first and third fractions give

$x^4dx = xz(z^2 + c_1)dz$ or $x^3dx - (z^3 + c_1z)dz = 0$.

Integrating, $\frac{1}{4}x^4 - \left(\frac{1}{4}z^4 + \frac{1}{2}c_1z^2\right) = \frac{1}{4}c_2$ or $x^4 - z^4 - 2c_1z^2 = c_2$.

Using (1) to remove c_1, we get $x^4 + z^4 - 2xyz^2 = c_2$. ...(2)

The complete solution is given by the relations (1) and (2).

Ex. 2. *Solve* $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{zxy - 2x^2}$. **[Rohilkhand 1993]**

Sol. Taking the first two fractions, $(1/x)dx - (1/y)dy = 0$ so that $x/y = c_1$. ...(1)

From (1), $x = c_1y$. So the second and third fractions give

$$\frac{dy}{y^2} = \frac{dz}{c_1zy^2 - 2c_1^2y^2} \quad \text{or} \quad c_1dy = \frac{dz}{z - 2c_1^2}.$$

Integrating, $c_1y - \log(z - 2c^2_1) = c_2$, c_1 and c_2 being arbitrary constants. ...(2)

Using (1) to remove c_1, (2) gives $x - \log(z - 2x^2/y^2) = c_2$. ...(3)

The complete solution is given by the relations (1) and (3).

Ex. 3. *Solve* $\dfrac{dx}{1} = \dfrac{dy}{2} = \dfrac{dz}{5z + \tan(y - 2x)}$. **(Delhi Maths (H) 2005; Mumbai 2007)**

Sol. Taking the first two fractions, $dy - 2dx = 0$.

Integrating, $y - 2x = c_1$, c_1 being an arbitrary constant. ...(1)

Using (1), the first and the third fractions give

$dx = dz/(5z + \tan c_1)$ so that $x - (1/5) \times \log(5z + \tan c_1) = c_2/5$.

Using (1) to remove c_1, this gives on simplification

$$5x - \log [5z + \tan (y - 2x)] = c_2, \; c_2 \text{ being an arbitrary constant.} \qquad ...(2)$$

The complete solution is given by the relations (1) and (2).

Ex. 4. *Solve* $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xyz^2(x^2 - y^2)}$.

Sol. Taking the first two fractions, $2xdx - 2ydy = 0$.

Integrating, $x^2 - y^2 = c_1$, c_1 being an arbitrary constant ...(1)

Using relation (1), the first and third fractions give

$$\frac{dx}{y} = \frac{dz}{xyz^2 c_1} \quad \text{or} \quad 2c_1 x dx - 2z^{-2} dz = 0.$$

Integrating, $c_1 x^2 + 2z^{-1} = c_2$ or $(x^2 - y^2)x^2 + 2z^{-1} = c_2$, by (1) ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 5. *Solve* $(dx)/(xz) = (dy)/(yz) = (dz)/(xy)$.

Sol. Taking the first two fractions, $(1/x)dx - (1/y)dy = 0$.

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(1)

From the second and third fractions, $xdy = zdz$.

or $2c_1 ydy = 2zdz$ $[\because$ from (1), $x = c_1 y]$

Integrating, $c_1 y^2 - z^2 = c_2$ or $(x/y) \times y^2 - z^2 = c_2$, using (1).

Thus, $xy - z^2 = c_2$, c_2 being an arbitrary constant ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 6. *Solve* $\dfrac{dx}{-xy^2} = \dfrac{dy}{y^3} = \dfrac{dz}{axz}$.

Sol. Taking the first two fractions, $(1/x)dx + (1/y)dy = 0$.

Integrating, $\log x + \log y = \log c_1$ or $xy = c_1$. ...(1)

From (1), $x = c_1/y$. Hence the last two fractions give

$$\frac{dy}{y^3} = \frac{dz}{az \times (c_1 / y)} \quad \text{or} \quad \frac{dz}{z} - ac_1 \, y^{-4} dy = 0.$$

Integrating, $\log z - (ac_1) \times [(y^{-3}/(-3)] = c_2$, c_2 being an arbitrary constant.

Using (1), we get $\log z + (axy)/3y^3 = c_2$ or $\log z + (ax/3y^2) = c_2$. ...(2)

The required general solution is given by the relations (1) and (2).

EXERCISE 2 (B)

Solve the following simultaneous differential equations :

1. *Solve* $\dfrac{dx}{1} = \dfrac{dt}{-2} = \dfrac{dz}{3x^2 \sin(y + 2x)}$ **(Gulbarga 2005)** **Ans.** $y + 2x = c_1, x^3 \sin(y + 2x) - z = c_2$

2. *Solve* $\dfrac{dx}{z} = \dfrac{dy}{-z} = \dfrac{dz}{z^2 + (x + y)^2}$. **Ans.** $x + y = c_1$ and $z^2 + (x + y)^2 = c_1 e^{2x}$

3. *Solve* $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{dz}{z/(x + y)}$ **Ans.** $x + y = c_1$ and $x - (x + y) \log z = c_2$

4. $(dx)/(zx) = (dy)/(-zy) = (dz)/(z + xy)$ **[Delhi Maths (G) 2006]**

Ans. $xy = c_1$, $\log x + xy \log (z + xy) - z = c_2$

2.8. Rule III for solving (dx)/P = (dy)/Q = (dz)/R ...(1)

Let P_1, Q_1, R_1 be functions of x, y, z. Then, by a well–unknown principle of algebra, each fraction in (1) will be equal to

$$(P_1dx + Q_1dy + R_1dz) / (P_1P + Q_1Q + R_1R). \quad ...(2)$$

If $P_1P + Q_1Q + R_1R = 0$ in (2), then we know that numerator of (2) is also zero. This gives $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible with help of multipliers. In such cases second integral should be obtained by using Rule I of Art. 2.4 or Rule II of Art. 2.6 as the case may be.

2.9. Solved examples based on Art. 2.8

Ex. 1. *Solve the simultaneous equations* $\dfrac{adx}{(b-c)yz} = \dfrac{bdy}{(c-a)zx} = \dfrac{cdz}{(a-b)xy}$.

[Kolkata 2001; Kumaoun 2002; Guwahati 2001, 02; Nagpur 2003, 04; Delhi Maths 1995, 1996; Bangalore 2005; Lucknow 2006]

Sol. Choosing x, y, z as multipliers, each fraction of given equations

$$= \frac{axdx + bydy + czdz}{xyz[(b-c)+(c-a)+(a-b)]} = \frac{axdx + bydy + czdz}{0}.$$

$\therefore$ $ax\,dx + by\,dy + cz\,dz = 0$ or $2ax\,dx + 2by\,dy + 2cz\,dz = 0$.

Integrating, $ax^2 + by^2 + cz^2 = c_1$, c_1 being an arbitrary contant.

Again choosing ax, by, cz as multipliers, each fraction of the given equations

$$= \frac{a^2xdx + b^2ydy + c^2zdz}{xyz[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2xdx + b^2ydy + c^2zdz}{0}.$$

$\therefore$ $a^2x\,dx + b^2y\,dy + c^2z\,dz = 0$ or $2a^2x\,dx + 2b^2y\,dy + 2c^2z\,dz = 0$.

Integrating, $a^2x^2 + b^2y^2 + c^2z^2 = c_2$, c_2 being an arbitrary constant ...(2)

The complete solution is given by relations (1) and (2).

Ex. 2. *Solve :* $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$. **[Delhi Maths (H) 2009]**

[Agra 2005; Gauhati 1996, Meerut 2006, 11; Kanpur 2002; Rajasthan 2003]

Sol. Choosing, x, $-y$, $-z$ as multipliers, each fraction of the given equations

$$= \frac{xdx - ydy - zdz}{xz(x+y) - yz(x-y) - z(x^2+y^2)} = \frac{xdx - ydy - zdz}{0}.$$

$\therefore$ $x\,dx - y\,dy - z\,dz = 0$ or $2x\,dx - 2y\,dy - 2z\,dz = 0$.

Integrating, $x^2 - y^2 - z^2 = c_1$, c_1 being an arbitrary constants

Now choosing, y, x, $-z$ as multipliers, each fraction of the given equations

$$= \frac{ydx + xdy - zdz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{ydx + xdy - zdz}{0}.$$

$\therefore$ $2y\,dx + 2x\,dy - 2z\,dz = 0$ or $2d(xy) - d(z^2) = 0$.

Integrating, $2xy - z^2 = c_2$, c_2 being an arbitrary constant ...(2)

The complete solution is given by the relations (1) and (2).

Ex. 3. *Solve* $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$. **[Bangalore 1997, Delhi Maths (H) 2001; Lucknow 2002; Karnataka 2004; Mysore 2005; Rajasthan 2006]**

Sol. Choosing l, m, n as multipliers, each fraction of given equations

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + my + ndz}{0}.$$

Thus, $l\,dx + m\,dy + n\,dz = 0$ so that $lx + my + nz = c_1$. ...(1)

Similarly, choosing x, y, z as multipliers, each fraction of the given equations

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - ly) + z(ly - mx)} = \frac{xdx + ydy - zdz}{0}.$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$ or $2x\,dx + 2y\,dy + 2z\,dz = 0$.

Integrating, $x^2 + y^2 + z^2 = c_2$, c_2 being an arbitrary constant. ...(2)

The complete solution consists of (1) and (2).

Ex.4. *Solve* $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$. **[Bangalore1993,DelhiMaths(Hons.)2006]**

Sol. Choosing x, y, z as multipliers, each fraction of the given equations

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{xdx + ydy + zdz}{0}.$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$ or $2x\,dx + 2y\,dy + 2z\,dz = 0$.

Integrating, $x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(1)

Again choosing $1/x$, $1/y$, $1/z$ as multipliers, each fraction of the given equations

$$= \frac{dx/x + dy/y + dz/z}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)} = \frac{dx/x + dy/y + dz/z}{0}.$$

$\therefore$ $dx/x + dy/y + dz/z = 0$. so that $\log x + \log y + \log z = \log c_2$

or $\log xyz = \log c_2$ or $xyz = c_2$, c_2 being an arbitrary constant ...(2)

The complete solution is given by the relations (1) and (2).

Ex. 5. *Solve* $(dx)/y = (dy)/(-x) = (dz)/(bx - ay)$.

Sol. Taking a, b, 1 as multipliers, each fraction of the given equations.

$= (a\,dx + b\,dy + dz)/0$ so that $a\,dx + b\,dy + dz = 0$.

Integrating, $ax + by + z = c_1$, c_1 being an arbitrary constant. ... (1)

From first two fractions, $x\,dx + y\,dy = 0$ so that $x^2 + y^2 = c_2$ ... (2)

The required solution is given by (1) and (2).

Ex. 6. *Solve* $\frac{xdx}{z^2 - 2yz - y^2} = \frac{dy}{y + z} = \frac{dz}{y - z}$. **[Bangalore 2007, Rohilkhand 1997]**

Sol. Choose 1, y, z as multipliers, each fraction of the given equations

$= (xdx + ydy + zdz)/0$ so that $2x\,dx + 2y\,dy + 2z\,dz = 0$.

Integrating, $x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(1)

From the last two fractions, $(y - z)dy = (y + z)dz$

or $2(ydz + zdy) - 2ydy + 2zdz = 0$ or $2d(yz) - d(y^2) + d(z^2) = 0$.

Integrating, $2yz - y^2 + z^2 = c_2$, c_2 being an arbitrary constant. ...(2)

The required complete solution is given by (1) and (2).

Ex. 7. *Solve* $\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}$.

Sol. Choosing $1/x$, $1/y$, $1/3z$ as multipliers, each fraction of the given equations

$$= \frac{dx/x + dy/y + dz/3z}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)} = \frac{dx/x + dy/y + dz/3z}{0}.$$

$\therefore$ $(1/x)dx + (1/y)dy + (1/3z)\,dz = 0$ so that $\log x + \log y + (1/3) \times \log z = \log c_1$

or $xyz^{1/3} = c_1$, c_1 being an arbitrary constant. ...(1)

Now the first two fractions give $(2y^4 - x^3y)dx = (y^3x - 2x^4)dy$.

Dividing by x^3y^3, $\left(\frac{2y}{x^3} - \frac{1}{y^2}\right)dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right)dy$ or $\left(\frac{1}{x^2}dy - \frac{2y}{x^3}dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0$.

or $d(y/x^2) + d(x/y^2) = 0$. so that $y/x^2 + x/y^2 = c_2$. ...(2)

The required solution is given by the relations (1) and (2).

Ex. 8. *Solve* $(dx)/y^2 = (dy)/x^2 = (dz)/x^2y^2z^2$. **[Mysore 2004]**

Sol. First two fractions give $3x^2dx - 3y^2dy = 0$ so that $x^3 - y^3 = c_1$. ...(1)

Choosing x^2, y^2, $-2/z^2$ as multipliers, each fraction of the given equations

$$= \frac{x^2dx + y^2dy - (2/z^2)dz}{x^2y^2 + x^2y^2 - 2x^2y^2} = \frac{x^2dx + y^2dy - (2/z^2)dz}{0}.$$

$\therefore$ $x^2dx + y^2dy - (2/z^2)dz = 0$ or $3x^2dx + 3y^2dy - (6/z^2)dz = 0$.

Integrating, $x^3 + y^3 + 6/z = c_2$, c_2 being an arbitrary constant. ...(2)

(1) and (2) together give the complete solution.

Ex. 9. $\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}$. **[Meerut 2007]**

Sol. Choosing $1/x$, $1/y$, $1/z$ as multipliers, each fraction of the given equations

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2 + z - (x^2 + z) + x^2 - y^2} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}.$$

$\therefore$ $(1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $xyz = c_1$. ...(1)

Next choosing x, y, -1 as multipliers, each fraction of the given equations

$$= \frac{xdx + ydy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{xdx + ydy - dz}{0}$$

$\therefore$ $2xdx + 2ydy - 2dz = 0$ so that $x^2 + y^2 - 2z = c_2$. ...(2)

(1) and (2) together give the complete solution.

Ex. 10. (*a*) *Solve* $\frac{dx}{z - y} = \frac{dy}{x - z} = \frac{dz}{y - x}$. **[Delhi Maths (G) 1994]**

(*b*) *Show that* $u = x + y + z$, $v = x^2 + y^2 + z^2$ *are integrals of the linear system* $dx/dt = y - z$, $dy/dt = z - x$, dz/dt $x - y$. **[Amravati 2003]**

Sol. (*a*) Choosing 1, 1, 1 as multipliers, each fraction of the given equations

$$= \frac{dx + dy + dz}{z - y + x - z + y - x} = \frac{dx + dy + dz}{0}.$$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_1$. ...(1)

Again, choosing x, y, z as multipliers, each fraction

$$= \frac{xdx + ydy + zdz}{x(z - y) + y(x - z) + z(y - x)} = \frac{xdx + ydy + zdz}{0}.$$

$\therefore$ $xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0$.

Integrating, $x^2 + y^2 + z^2 = c_2$, c_2 being arbitrary constant. ...(2)

The required solution is given by the relations (1) and (2).

(*b*) Given system yields $(dx)/(y - z) = (dy)/(z - x) = (dz)/(x - y)$. Now proceed as in part to show that $x + y + z = c_1$ and $x^2 + y^2 + z^2 = c_2$. Hence $u = x + y + z$ and $v = x^2 + y^2 + z^2$, as required.

Ex. 11. *Solve* $\dfrac{yzdx}{y-z} = \dfrac{zxdy}{z-x} = \dfrac{xydz}{x-y}$. **[Delhi Maths (G) 1996]**

Sol. Choosing 1, 1, 1 as multipliers, each fraction of the given equations

$$= \frac{yzdx + zxdy + xydz}{y - z + z - x + x - y} = \frac{d(xyz)}{0}.$$

$\therefore$ $d(xyz) = 0$ so that $xyz = c_1$. ...(1)

Again choosing $1/yz$, $1/zx$, $1/xy$ as multipliers, each fraction of the given equations

$$= \frac{dx + dy + dz}{(y-z)/yz + (z-x)/zx + (x-y)/xy} = \frac{dx + dy + dz}{1/z - 1/y + 1/x - 1/z + 1/y - 1/x} = \frac{dx + dy + dz}{0}$$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_2$. ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 12. *Solve* $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$. **[Delhi Maths (H) 1993]**

Sol. Choosing x, y, z as multipliers, each fraction of the given equations

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore$ $xdx + ydy + zdz = 0$ so that $x^2 + y^2 + z^2 = c_1$. ...(1)

Again, choosing $1/x$, $-1/y$, $-1/z$ as multipliners, each fraction

$$= \frac{(1/x)dx - (1/y)dy - (1/z)dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = \frac{(1/x)dx - (1/y)dy - (1/z)dz}{0}.$$

$\therefore$ $(1/x)dx - (1/y)dy - (1/z)dz = 0$ so that $\log x - \log y - \log z = \log c_2$

or $\log (x/yz) = \log c_2$ or $x/yz = c_2$. ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 13. *Solve* $\dfrac{dx}{y - zx} = \dfrac{dy}{x + yz} = \dfrac{dz}{x^2 + y^2}$. **[Nagpur 1996, Delhi Maths (H) 1998]**

Sol. Choosing x, $-y$, z as multipliers, each fraction of the given equations

$$= \frac{xdx - ydy + zdz}{x(y - zx) - y(x + yz) + z(x^2 + y^2)} = \frac{xdx - ydy + zdz}{0}$$

$\therefore$ $xdx - ydy + zdz = 0$ so that $2xdx - 2ydy + 2zdz = 0$.

Integrating it, $x^2 - y^2 + z^2 = c_1$, c_1 being an arbitrary constants. ...(1)

Again, choosing y, x, -1 as multipliers, each fraction of the given equation

$$= \frac{y\,dx + x\,dy - dz}{y(y - zx) + x(x + yz) - (x^2 + y^2)} = \frac{y\,dx + x\,dy - dz}{0}.$$

$\therefore$ $y\,dx + x\,dy - dz = 0$ or $d(xy) - dz = 0$.

Integrating it, $xy - z = c_2$, c_2 being an arbitrary constant. ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 14. *Solve* $\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{nxy}$. **[Delhi Maths (G) 1995]**

Sol. Taking the first two fractions, $x^{-2}dx = y^{-2}dy$.

Integrating $-\frac{1}{x} = -\frac{1}{y} - c_1$ or $\frac{1}{x}-\frac{1}{y} = c_1$

or $y - x = c_1xy$ or $x - y + c_1xy = 0$. ...(1)

Choosing $1/x$, $-1/y$, c_1/n as multipliers, each fraction of the given equations

$$= \frac{(1/x)dx-(1/y)dy+(c_1/n)dz}{x-y+c_1xy}=\frac{(1/x)dx-(1/y)dy+(c_1/n)dz}{0}, \text{ using (1)}$$

$\therefore$ $(1/x)dx - (1/y)dy + (c_1/n)dz = 0$.

Integrating, $\log x - \log y + (c_1/n)z = (c_1/n)c_2$, c_2 being arbitrary constant

or $\frac{c_1}{n}z = \frac{c_1}{n}c_2 + \log\frac{y}{x}$ or $z = c_2 + \frac{n}{c_1}\log\frac{y}{x} = c_2 + \frac{nxy}{y-x}\log\frac{y}{x}$, using (1)

The required general solution is given by the relations (1) and (2).

Ex. 15. *Solve* $\frac{dx}{y^2+z^2}=\frac{dy}{-xy}=\frac{dz}{-xz}$. **[Delhi Maths. (H) 1995]**

Sol. Taking the last two fractions, $(1/y)dy - (1/z)dz = 0$.

Integrating, $\log y - \log z = \log c$, or $y/z = c_1$. ...(1)

Choosing x, y, z or multipliers, each given fraction $= \frac{xdx+ydy+zdz}{x(y^2+z^2)-xy^2-xz^2}=\frac{xdx+ydy+zdy}{0}$.

$\therefore$ $xdx + ydy + zdz = 0$ so that $x^2 + y^2 + z^2 = c_2$.

The required general solution is given by the relations (1) and (2).

Ex. 16. *Solve* $\frac{dx}{x(2y^4-z^4)}=\frac{dy}{y(z^4-2x^4)}=\frac{dz}{z(x^4-y^4)}$.

Sol. Choosing $1/x$, $1/y$, $2/z$ as multipliers, each fraction

$$= \frac{(1/x)dx+(1/y)dy+(2/z)dz}{(2y^4-z^4)+(z^4-2x^4)+2(x^4-y^4)}=\frac{(1/x)dx+(1/y)dy+2(1/z)dz}{0}$$

$\therefore$ $(1/x)dx + (1/y)dy + 2(1/z)dz = 0$ so that $\log x + \log y + 2\log z = \log c_2$

or $\log x + \log y + \log z^2 = \log c_2$ or $xyz^2 = c_2$. ...(1)

Again, choosing x^3, y^3, z^3 as multipliers, each fraction

$$= \frac{x^3dx+y^3dy+z^3dz}{x^4(2y^4-z^4)+y^4(z^4-2x^4)+z^4(x^4-y^4)}=\frac{x^3dx+y^3dy+z^3dz}{0}$$

$\therefore$ $x^3dx + y^3dy + z^3dz = 0$ so that $x^4 + y^4 + z^4 = c_2$. ...(2)

The required general solution is given by the relations (1) and (2).

Important Note: *Sometimes multipliers are chosen by using a trial method. The whole procedure is explained in the next solved Ex. 17.*

Ex. 17. Solve $\frac{dx}{4y-3z}=\frac{dy}{4x-2z}=\frac{dz}{2y-3x}$.

Sol. Choosing l, m, n as multipliers, each fraction of the given equations

$$= \frac{ldx+mdy+ndz}{l(4y-3z)+m(4x-2z)+n(2y-3x)} \quad ...(1)$$

Now, choose l, m, n such that $l(4y - 3z) + m(4x - 2z) + n(2y - 3x) = 0$

or $(4m - 3n)x + (4l + 2n)y + (-3l - 2m)z = 0$,

which is satisfied if $4m - 3n = 0$, $4l + 2n = 0$, $-3l - 2m = 0$ or $l : m : n = 2 : -3 : -4$.

$\therefore$ from (1), each given ratio $= (2dx - 3dy - 4dz)/0$.

$\therefore$ $2dx - 3dy - 4dz = 0$ so that $2x - 3y - 4z = c_1$.

Again, choose l, m, n such that $l(4y - 3z) + m(4x - 2z) + n(2y - 3x) = 0$

or $4(ly + mx) + 3(-lz - nx) + 2(ny - mz) = 0$,

which is satisfied if $ly + mx = 0$, $-lz - nx = 0$, $ny - mz = 0$ or $l : m : n = x : -y : -z$.

$\therefore$ from (1), each given ratio $= (xdx - ydy - zdz)/0$.

$\therefore$ $xdx - ydy - zdz = 0$ so that $x^2 - y^2 - z^2 = c_2$. ...(3)

The required general solution is given by the relations (1) and (2).

Ex. 18. *Solve* $\dfrac{dx}{z^2 - 2yz - y^2} = \dfrac{dy}{y + z} = \dfrac{dz}{y - z}$.

Sol. From the last two fractions, $(y - z)dy = (y + z)dy$

or $ydy - zdz - (zdy + ydz) = 0$ or $2ydy - 2zdz - 2d(yz) = 0$.

Integrating, $y^2 - z^2 - 2yz = c_1$, c_1 being an arbitrary constant ...(1)

Taking, 1, y, z as multipliers, each fraction

$$= \frac{dx + ydy + zdz}{z^2 - 2yz - y^2 + y(y + z) + z(y - z)} = \frac{dx + ydy + zdz}{0}.$$

$\therefore$ $dx + ydy + zdz = 0$ or $2dx + 2ydy + 2zdz = 0$.

Integrating, $2x + y^2 + z^2 = c_2$, c_2 being an arbitrary constant ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 19. *Solve* $\dfrac{dx}{y - xz} = \dfrac{dy}{yz + x} = \dfrac{dz}{x^2 + y^2}$.

Sol. Choosing y, x, -1 and x, $-y$, z as multipliers by turn each given fraction,

$$= \frac{ydx + xdy - dz}{0} = \frac{xdx - ydy + zdz}{0}$$

$\therefore$ $ydx + xdy - dz = 0$ and $xdx - ydy + zdz = 0$.

Integrating, $xy - z = c_1$ and $x^2 - y^2 + z^2 = c_2$. ... (3)

The required general solution is given by (3).

Ex. 20. *Solve* $\dfrac{dx}{x^2(y - z)} = \dfrac{dy}{y^2(z - x)} = \dfrac{dz}{z^2(x - y)}$.

Sol. Choosing $1/x$, $1/y$, $1/z$ and $1/x^2$, $1/y^2$, $1/z^2$ as multipliers by turn, each fraction

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0} = \frac{(1/x^2)dx + (1/y^2)dy + (1/z^2)dz}{0}$$

$\therefore$ $(1/x)dx + (1/y)dy + (1/z)dz = 0$ and $x^{-2}dx + y^{-2}dy + z^{-2}dz = 0$.

Integrating, $\log x + \log y + \log z = \log c_1$ and $-x^{-1} - y^{-1} - z^{-1} = -c_2$

or $xyz = c_1$ and $x^{-1} + y^{-1} + z^{-1} = c_2$, which give the desired solution.

Ex. 21. $\dfrac{dx}{x + 2z} = \dfrac{dy}{4zx - y} = \dfrac{dy}{2x^2 + y}$.

Sol. Choosing y, x, $-2z$ as multipliers, each fraction

$$= \frac{ydx + xdy - 2zdz}{y(x + 2z) + x(4zx - y) - 2z(2x^2 + y)} = \frac{d(xy) - 2zdz}{0}$$

$\therefore \quad d(xy) - 2zdz = 0$ so that $xy - z^2 = c_1$. ...(1)

Choosing $2x, -1, -1$ as multipliers, each fraction

$$= \frac{2xdx - dy - dz}{2x(x+2z) - (4zx - y) - (2x^2 + y)} = \frac{2xdx - dy - dz}{0}.$$

$\therefore \quad 2xdx - dy - dz = 0$ so that $x^2 - y - z = c_2$. ...(2)

The required general solution is given by relations (1) and (2).

Ex. 22. *Solve* $\dfrac{dx}{x-y} = \dfrac{dy}{x+y} = \dfrac{dz}{2xz}$.

Sol. Choosing $1, 1, -1/z$ as multipliers, each given fraction

$$= \frac{dx + dy - (1/z)dz}{(x-y) + (x+y) - (1/z) \times (2xz)} = \frac{dx + dy - (1/z)dz}{0}.$$

$\therefore \quad dx + dy - (1/z)dz = 0$ so that $x + y - \log z = c_1$. ...(1)

Taking the first two fractions $\dfrac{dy}{dx} = \dfrac{x+y}{x-y} = \dfrac{1 + (y/x)}{1 - (y/x)}$. ...(2)

Let $y/x = v$ so that $y = xv$. ...(3)

From (3), $dy/dx = v + x(dv/dx)$. ...(4)

Using (3) and (4), (2) reduces to

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v} \quad \text{or} \quad x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$$

or $\dfrac{1-v}{1+v^2}dv = \dfrac{dx}{x}$ or $\left(\dfrac{2}{1+v^2} - \dfrac{2v}{1+v^2}\right)dv = \dfrac{2dx}{x}$

Integrating, $2\tan^{-1} v - \log(1+v^2) = 2\log x - \log c_2$ or $\log x^2 - \log(1+v^2) - \log c_2 = 2\tan^{-1} v$

or $\log\{x^2(1+v^2)/c_2\} = 2\tan^{-1} v$ or $x^2(1+v^2) = c_2 e^{2\tan^{-1} v}$

or $x^2\{1 + (y^2/x^2)\} = c_2\ e^{2\tan^{-1}(y/x)}$, as $v = y/x$ by (3)

or $(x^2 + y^2)e^{-2\tan^{-1}(y/x)} = c_2$, c_2 being an arbitrary constant ...(5)

The required general solution is given by the relations (1) and (5).

Ex. 23. *Solve* $\dfrac{dx}{x(x^2+3y^2)} = \dfrac{dy}{-y(3x^2+y^2)} = \dfrac{dz}{2z(y^2-x^2)}$.

Sol. Choosing $1/x, 1/y, -1/z$ as multipliers, each fraction of the given equations.

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0} \quad \text{so that} \quad \frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0.$$

Integrating, $\log x + \log y - \log z = \log c_1$ so that $(xy)/z = c_1$. ...(1)

Taking the first two fractions, $\dfrac{dy}{dx} = -\dfrac{y(3x^2+y^2)}{x(x^2+3y^2)} = -\dfrac{y}{x} \times \dfrac{3 + (y/x)^2}{1 + 3(y/x)^2}$

Putting $y/x = v$ or $y = xv$ so that $dy/dx = v + x\,(dv/dx)$, we get

$$v + x\frac{dv}{dx} = -v\,\frac{3+v^2}{1+3v^2} \quad \text{or} \quad x\frac{dv}{dx} = -v\left[\frac{3+v^2}{1+3v^2} + 1\right]$$

or $x\frac{dv}{dx} = -\frac{4(1+v^2)v}{1+3v^2}$ or $4\frac{dx}{x}+\frac{1+3v^2}{v(1+v^2)}dv=0$

or $4\frac{dx}{x}+\left(\frac{1}{v}+\frac{2v}{1+v^2}\right)dv = 0$, on resolving into partial fractions

Integrating, $4 \log x + \log v + \log (1 + v^2) = \log c_2'$ or $x^4v(1 + v^2) = c_2'$

or $x^4(y/x)[1 + (y/x)^2)] = c_2'$ or $xy(x^2 + y^2) = c_2'$ or $c_1z(x^2 + y^2) = c_2'$, using (1)

or $z(x^2 + y^2) = c_2$, where $c_2 = c_2'/c_1$, c_2 being an arbitrary constant ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 24. *Solve* $\frac{dx}{y+z}=\frac{dy}{-(x+z)}=\frac{dz}{x-z}$. **[Mysore 2004]**

Sol. Choosing 1, 1, 1 as multipliers each fraction $=\frac{dx+dy+dz}{y+z-(x+z)+x-y}=\frac{dx+dy+dz}{0}$

so that $dx + dy + dz = 0$ and so $x + y + z = C_1$ ... (1)

From (1), $y + z = C_1 - x$ and $x + z = C_1 - y$

Hence the first two fractions of the given problem may be re-written as

$$\frac{dx}{C_1-x}=\frac{dy}{-(C_1-y)} \quad \text{or} \quad \frac{dx}{C_1-x}+\frac{dy}{C_1-y}=0$$

Integrating, $-\log (C_1 - x) - \log (C_1 - y) = -\log C_2$, C_2 being an arbitrary constant

or $(C_1 - x)(C_1 - y) = C_2$ or $(y + z)(x + z) = C_2$ ... (2)

The required solution is given by the relations (1) and (2).

EXERCISE 2 (C)

Solve the following simultaneous differential equations :

1. $\frac{dx}{x^2\left(y^3-z^3\right)}=\frac{dy}{y^2\left(z^3-x^3\right)}=\frac{dz}{z^2\left(x^3-y^3\right)}$ **[Delhi Maths (G) 2005]**

Hint. Do like Ex. 4. of Art. 2.9, **Ans.** $x^2 + y^2 + z^2 = c_1$, $1/x + 1/y + 1/z = c_2$

2. $\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$. **[Pune 2010; Nagpur 1996; Bangalore 1993]**

Sol. Try yourself **Ans.** $x + y + z = c_1$ and $xyz = c_2$

3. $\frac{ldx}{mn(y-z)}=\frac{mdy}{nl(z-x)}=\frac{ndz}{lm(x-y)}$. **[Delhi Maths (G) 2001]**

Sol. Try yourself as in Ex. 1 of Art. 2.9 **Ans.** $l^2x + m^2y + n^2z = c_1$, $l^2x^2 + m^2y^2 + n^2z^2 = c_2$

4. $(dx)/y = (dy)/(-x) = (dz)/(2x - 3y)$ **[Osmania 2003]** **Ans.** $x^2 + y^2 = C_1$, $3x + 2y + z = C_2$

5. $(dx)/(zx) = (dy)/(-zy) = (dz)/(y^2 - x^2)$ **Ans.** $xy = C_1$, $x^2 + y^2 + z^2 = C_2$

2.10. Rule IV for solving (dx)/P = (dy)/Q = (dz)/R. ...(1)

Let P_1, Q_1, R_1 be functions of x, y, z. Then, by a well known principle of algebra, each fraction in (1) will be equal to

$$(P_1dx + Q_1dy + R_1dz)/(P_1P + Q_1Q + R_1R). \quad ...(2)$$

Suppose the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$(P_2dx + Q_2dy + R_2dz)/(P_2P + Q_2Q + R_2R) \quad ...(3)$$

is such that its numerator is exact differential of denominator. Fractions (2) and (3) are then combined to give an integral. This method may also be repeated in some problems to get another integral.

Sometimes only one integral is possible with help of multipliers. In such cases second integral should be obtained by using rule I of Art. 2.4 or rule II of Art. 2.6 or rule III of Art 2.8 as the case may be.

2.11. Solved examples based on Art. 2.10

Ex. 1. *Solve* $\dfrac{dx}{y^2(x-y)} = \dfrac{dy}{-x^2(x-y)} = \dfrac{dz}{z(x^2+y^2)}$

Sol. Given $$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dx}{z(x^2+y^2)}. \quad ... (1)$$

Taking the first two fractions in (1), we get

$x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0.$

Integrating, $x^3 + y^3 = c_1$, c_1 being an arbitrary constant ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)}. \quad ...(3)$$

Combining the third fraction in (1) with fraction (3), we get

$$\frac{dz}{z(x^2+y^2)} = \frac{dx-dy}{(x-y)(x^2+y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx-dy}{x-y}$$

Integrating $\log(x-y) - \log z = \log c_2$ or $(x-y)/z = c_2.$ ...(4)

The required solution is given by (2) and (4).

Ex. 2. *Solve* $\dfrac{dx}{x^2-y^2-z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}.$ **[Guwahati 2007; Delhi Maths (H) 2000, Delhi Maths (Prog) 2009]**

or $\dfrac{dx}{y^2+z^2-x^2} = \dfrac{dy}{-2xy} = \dfrac{dz}{-2xz}.$ **[Delhi Maths (G) 2000; Nagpur 1995]**

Sol. Given $$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad ... (1)$$

Taking the last two fractions of (1), $(1/y)dy - (1/z)dz = 0$

Integrating, $\log y - \log z = \log c_1$ or $y/z = c_1.$ ...(2)

Choosing x, y, z as multipliers, each fraction in (1)

$$= \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}. \quad ...(3)$$

Combining the third fraction in (1) with fraction (3), we get

$$\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz} \quad \text{or} \quad \frac{2(xdx+ydy+zdz)}{x^2+y^2+z^2} = \frac{dz}{z}.$$

Integrating, $\log(x^2+y^2+z^2) - \log z = \log c_2$ or $(x^2+y^2+z^2)/z = c_2.$ ...(4)

The required solution is given by (2) and (4).

Ex. 3. *Solve* $\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$ **[Bangalore 2005, Delhi Maths (G) 1998, 2008]**

Sol. Given $$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \quad \text{... (1)}$$

Choosing 1, –1, 0 and 0, 1, –1 as multipliers by turn, each fraction of (1)

$$= \frac{dx-dy}{x^2-y^2+z(x-y)}=\frac{dy-dz}{y^2-z^2+x(y-z)}$$

$$\therefore \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)} \quad \text{or} \quad \frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}.$$

Integrating, $\log(x-y)-\log(y-z)=\log c_1$ so that $(x-y)/(y-z)=c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction in (1)

$$= \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{...(3)}$$

Again, choosing 1, 1, 1 as multipliers, each fraction in (1)

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{...(4)}$$

From (3) and (4), we get $(xdx+ydy+zdz)/(x+y+z)=dx+dy+dz$

or $2(x+y+z)(dx+dy+dz)-2(xdx+ydy+zdz)=0$

Integrating, $(x+y+z)^2-(x^2+y^2+z^2)=2c_2$

or $(x^2+y^2+z^2+2xy+2yz+2zx)-(x^2+y^2+z^2)=2c_2$

or $xy+yz+zx=c_2$, c_2 being an arbitrary constant. ...(5)

The required solution is given by (2) and (5).

Ex. 4. *Solve* $\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$. **[Delhi Maths 1999; 2002]**

Sol. Given $$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y} \quad \text{... (1)}$$

Choosing, 1, –1, 0 and 0, 1, –1 as multipliers each fraction of (1)

$$= \frac{dx-dy}{(y+z)-(z+x)} = \frac{dy-dz}{(z+x)-(x+y)}. \quad \text{...(2)}$$

So $\frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)}$ or $\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$.

Integrating, $\log(x-y)-\log(y-z)=\log c_1$ so that $(x-y)/(y-z)=c_1$. ...(3)

Choosing 1, 1, 1 as multiplers, each given fraction of (1) $= \frac{dx+dy+dz}{2(x+y+z)}$. ...(4)

Combining the first fraction in (2) which fraction (4), we have

$\frac{dx-dy}{-(x-y)} = \frac{dx+dy+dz}{2(x+y+z)}$ or $\frac{dx-dy}{x-y}+\frac{dx+dy+dz}{2(x+y+z)}=0$.

Integrating, $\log(x-y)+(1/2)\times\log(x+y+z)=\log c_2$ or $(x-y)(x+y+z)^{1/2}=c_2$. ...(5)

The required solution is given by (3) and (5).

Ex. 5. *Solve* $\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}$. **[Delhi Maths (Hons) 2007, 08; Meerut 2005]**

Sol. Given $\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}$ ...(1)

Taking the first two fractions in (1), we have

$2(1+x)dx - 2(1+y)dy = 0$ so that $(1+x)^2 - (1+y)^2 = c_1$. ...(2)

Taking 1, 1, 0 as multipliers, each fraction in (1) $= (dx+dy)/(2+x+y)$. ...(3)

Combining the last fraction in (1) with (3), we get

$\frac{dx+dy}{2+x+y} = \frac{dz}{z}$ so that $\log(2+x+y) - \log z = \log c_2$

or $(2+x+y)/z = c_2$, c_2 being an arbitrary constant. ...(4)

The required solution is given by (2) and (4).

Ex. 6. *Solve* $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a(x^2+y^2+z^2)^{1/2}}$.

Sol. Given $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a(x^2+y^2+z^2)^{1/2}}$ ... (1)

Taking the first two fractions in (1), $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction in (1)

$$= \frac{xdx+ydy+zdz}{x^2+y^2+z^2-az(x^2+y^2+z^2)^{1/2}} = \frac{tdt}{t^2-azt} = \frac{dt}{t-az}. \quad ...(3)$$

[Put $x^2+y^2+z^2 = t^2$ so that $xdx+ydy+zdz = tdt$]

Putting $x^2+y^2+z^2 = t^2$ in (1), we get $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-at}$ ... (4)

Then, (3) and (4) $\Rightarrow$ $\frac{dz}{z-at} = \frac{dt}{t-az} = \frac{dx}{x}$. ...(5)

Choosing 1, 1, 0 as multipliers, each fraction in (5)

$$= \frac{dz+dt}{z+t-a(t+z)} = \frac{dz+dt}{(z+t)(1-a)}. \quad ...(6)$$

Combining the last fraction in (5) with (6), we get

$\frac{dz+dt}{(z+t)(1-a)} = \frac{dx}{x}$ or $(1-a)\frac{dx}{x} - \frac{dz+dt}{z+t} = 0$.

Integrating, $(1-a)\log x - \log(z+t) = \log c_2$

or $\frac{x^{a-1}}{z+t} = c_2$ or $\frac{x^{a-1}}{z+(x^2+y^2+z^2)^{1/2}} = c_2$. ...(7)

The complete solution is given by (2) and (7).

Ex. 7. *Solve* $(dx)/\cos(x+y) = (dy)/\sin(x+y) = (dz)/z$.

Sol. Given $(dx)/\cos(x+y) = (dy)/\sin(x+y) = (dz)/z$. ... (1)

From (1), using 1, 1, 0 and 1, –1, 0 as multipliers by turn, we have

$$\frac{dz}{z} = \frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{dx - dy}{\cos(x+y) - \sin(x+y)}. \qquad ...(2)$$

Putting $x + y = t$ so that $dx + dy = dt$, the first two fractions give

$$\frac{dz}{z} = \frac{dt}{\cos t + \sin t} \quad \text{or} \quad \sqrt{2}\frac{dz}{z} = \operatorname{cosec}\left(t + \frac{\pi}{4}\right)dt$$

$$\left[\because \cos t + \sin t = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right) = \sqrt{2}\sin\left(t + \frac{\pi}{4}\right)\right]$$

Integrating, $\sqrt{2}\log z = \log\tan\frac{1}{2}\left(t + \frac{\pi}{4}\right) + \log c_1$ or $z^{\sqrt{2}} = c_1 \tan\frac{1}{2}\left(t + \frac{\pi}{4}\right)$

or $$z^{\sqrt{2}} \cot\frac{1}{2}\left(x + y + \frac{\pi}{4}\right) = c_1, \quad \text{as} \quad t = x + y. \qquad ...(3)$$

Now from the last two fractions in (2), we get

$$\frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)}(dx + dy) = dx - dy \quad \text{or} \quad \frac{\cos t - \sin t}{\cos t + \sin t}dt = dx - dy.$$

Integrating, $\log(\cos t + \sin t) - \log c_2 = x - y$ or $(\cos t + \sin t)/c_2 = e^{x-y}$.

or $$[\cos(x+y) + \sin(x+y)]e^{y-x} = c_2, \quad \text{as} \quad t = x + y. \qquad ...(4)$$

The complete solution is given by (3) and (4).

Ex. 8. *Solve* $\dfrac{dx}{x^2 - y^2 - yz} = \dfrac{dy}{x^2 - y^2 - zx} = \dfrac{dz}{z(x-y)}$. **[Garhwal 2010]**

Sol. Choosing 1, –1, 0 and x, $-y$, 0 as multipliers by turn, each fraction of the given equations.

$$= \frac{dx - dy}{z(x-y)} = \frac{xdx - ydy}{(x^2 - y^2)(x-y)}$$

In view of the last fraction of the given equations and the above fractions, we have

$$\frac{dz}{z(x-y)} = \frac{dx - dy}{z(x-y)} = \frac{xdx - ydy}{(x^2 - y^2)(x-y)} \qquad ... (1)$$

From the first two fractions of (1), $dz = dx - dy$.

Integrating, $z = x - y + c_1$ so that $z - x + y = c_1$. ...(2)

Now taking the first and the last fraction in (1), $\dfrac{2dz}{z} = \dfrac{2(xdx - ydy)}{x^2 - y^2}$.

Integrating, $2\log z = \log(x^2 - y^2) - \log c_2$. or $(x^2 - y^2)/z^2 = c_2$. ...(3)

The required general solution is given by the relations (2) and (3).

Ex. 9. *Solve* $(dx)/xz = (dy)/yz = (dz)/xy$.

Sol. Given $(dx)/xy = (dy)/yz = (dz)/xy$ ... (1)

Taking the first two fraction, $(1/x)dx - (1/y)dy = 0$.

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(2)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy}{(1/x)\times xz + (1/y)\times yz} = \frac{ydx + xdy}{2xyz} \qquad ...(3)$$

Combining the last fraction of (1) with fraction (3), we have

$$\frac{ydx + xdy}{2xyz} = \frac{dz}{xy} \quad \text{or} \quad ydx + xdy = 2zdz$$

or $d(xy) = 2zdz$ so that $xy - z^2 = c_2$. ...(4)

The required general solution is given by the relations (2) and (4).

Ex. 10. *Solve* $\frac{dx}{x^2 + y^2 + yz} = \frac{dy}{x^2 + y^2 - xz} = \frac{dz}{z(x + y)}$. **[Kanpur 2009]**

Sol. Given $$\frac{dx}{x^2 + y^2 + yz} = \frac{dy}{x^2 + y^2 - xz} = \frac{dz}{z(x + y)} \quad \text{... (1)}$$

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 + y^2 + yz) - (x^2 + y^2 - xz)} = \frac{dx - dy}{z(x + y)}. \quad ...(2)$$

Choosing x, y, 0 as multipliers, each fraction of (1)

$$= \frac{xdx + ydy}{x(x^2 + y^2 + yz) + y(x^2 + y^2 - xz)} = \frac{xdx + ydy}{(x + y)(x^2 + y^2)} \quad ...(3)$$

From (1), (2) and (3), $$\frac{dz}{z(x + y)} = \frac{dx - dy}{z(x + y)} = \frac{xdx + ydy}{(x + y)(x^2 + y^2)} \quad ...(4)$$

Taking the first two fractions of (4), $dz - dx + dy = 0$.

Integrating, $z - x + y = c_1$, c_1 being an arbitrary constant ...(5)

Taking the first and third fractions of (4), $\frac{d(x^2 + y^2)}{x^2 + y^2} - 2\frac{dz}{z} = 0$.

Integrating, $\log (x^2 + y^2) - 2 \log z = \log c_2$ or $(x^2 + y^2)/z^2 = c_2$. ...(6)

The required general solution is given by relations (1) and (2)

Ex. 11. *Solve* $\frac{dx}{x^3 + 3xy^2} = \frac{dy}{y^3 + 3x^2y} = \frac{dz}{2z(x^2 + y^2)}$.

Sol. Given $$\frac{dx}{x^3 + 3xy^2} = \frac{dy}{y^3 + 3x^2y} = \frac{dz}{2z(x^2 + y^2)} \quad \text{... (1)}$$

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \frac{dx + dy}{x^3 + 3xy^2 + 3x^2y + y^3} = \frac{d(x + y)}{(x + y)^3}$. ... (2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \frac{dx - dy}{x^3 + 3xy^2 - y^3 - 3x^2y} = \frac{d(x - y)}{(x - y)^3}$. ...(3)

From (2) and (3), $(x + y)^{-3}d(x + y) = (x - y)^{-3}d(x - y)$.

or $u^{-3}du - v^{-3}dv = 0$, [putting $u = x + y$ and $v = x - y$]

Integrating, $u^{-2}/(-2) - v^{-2}/(-2) = c_1/2$ or $v^{-2} - u^{-2} = c_1$

or $(x - y)^{-2} - (x + y)^{-2} = c_1$, as $u = x + y$, $v = x - y$. ...(4)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy}{(1/x)(x^3 + 3xy^2) + (1/y)(y^3 + 3x^2y)} = \frac{(1/x)dx + (1/y)dy}{4(x^2 + y^2)}. \quad ...(5)$$

Combining the last fraction of (1) with fraction (5), we have

$$\frac{dz}{2z(x^2+y^2)} = \frac{(1/x)dx + (1/y)dy}{4(x^2+y^2)} \quad \text{or} \quad \frac{dx}{x} + \frac{dy}{y} - 2\frac{dz}{z} = 0.$$

Integrating, $\log x + \log y - 2 \log z = \log c_2$ or $(xy)/z^2 = c_2$. ...(6)

The required general solution is given by the relations (4) and (6).

Ex. 12. *Solve* $dx = dy = (dz)/(x + y + z)$.

Sol. Given $$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x+y+z} \quad \text{... (1)}$$

Taking the first two fractions, $dx - dy = 0$ so that $x - y = c_1$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{1 + 1 + (x + y + z)} = \frac{d(2 + x + y + z)}{2 + x + y + z} \quad \text{...(3)}$$

Combining the first fraction of (1) with fraction (3), we get

$$\frac{d(2 + x + y + z)}{2 + x + y + z} = dx \quad \text{so that} \quad \log (2 + x + y + z) - \log c_2 = x.$$

or $[(2 + x + y + z)/c_2] = e^x$ or $e^{-x}(2 + x + y + z) = c_2$. ...(4)

The required general solution is given by (2) and (4).

Ex. 13. *Solve* $\frac{dx}{y^2+yz+z^2} = \frac{dy}{z^2+zx+x^2} = \frac{dz}{x^2+xy+y^2}$. **[Delhi Maths (H) 2004; Meerut 1996]**

Sol. Given $$\frac{dx}{y^2+yz+z^2} = \frac{dy}{z^2+zx+x^2} = \frac{dz}{x^2+xy+y^2} \quad \text{... (1)}$$

Choosing 1, –1, 0 as multipliers, each ratio of (1)

$$= \frac{dx - dy}{(y^2+yz+z^2) - (z^2+zx+x^2)} = \frac{dx - dy}{(y^2-x^2) + z(y-x)} = -\frac{dy - dx}{(y-x)(y+x+z)}. \quad \text{...(2)}$$

Again, choosing 0, 1, –1 as multipliers, each ratio of (1)

$$= \frac{dy - dz}{(z^2+zx+x^2) - (x^2+xy+y^2)} = \frac{dy - dz}{(z^2-y^2) + x(z-y)} = -\frac{dz - dy}{(z-y)(z+y+x)}. \quad \text{...(3)}$$

From (2) and (3), $$\frac{d(y-x)}{y-x} - \frac{d(z-y)}{z-y} = 0.$$

Integrating, $\log (y - x) - \log (z - y) = \log c_1$ or $(y - x)/(z - y) = c_1$. ...(4)

Choosing x, y, z as multipliers, each ratio of (1)

$$= \frac{xdx + ydy + zdz}{x(y^2+yz+z^2) + y(z^2+zx+x^2) + z(x^2+xy+y^2)} = \frac{xdx + ydy + zdz}{(x+y+z)(xy+yz+zx)} \quad \text{...(5)}$$

Again, choosing $y + z, z + x, x + y$ as multipliers, each ratio of (1)

$$= \frac{(y+z)dx + (z+x)dy + (x+y)dz}{(y+z)(y^2+yz+z^2) + (z+x)(z^2+zx+x^2) + (x+y)(x^2+xy+y^2)}$$

$$= \frac{(ydx + xdy) + (ydz + zdy) + (zdx + xdz)}{2(x^3+xy^2+xz^2+y^3+yx^2+yz^2+z^3+zx^2+zy^2)}$$

$$= \frac{d(xy) + d(yz) + d(zx)}{2(x+y+z)(x^2+y^2+z^2)} = \frac{d(xy+yz+zx)}{2(x+y+z)(x^2+y^2+z^2)}. \quad \text{...(6)}$$

From fractions (5) and (6), we have

$$\frac{xdx + ydy + zdz}{(x + y + z)(xy + yz + zx)} = \frac{d(xy + yz + zx)}{2(x + y + z)(x^2 + y^2 + z^2)}.$$

or $(xy + yz + zx)\, d(xy + yz + zx) - (x^2 + y^2 + z^2)\,(2xdx + 2ydy + 2zdz) = 0.$

Integrating, $(xy + yz + zx)^2 - (x^2 + y^2 + z^2)^2 = c_2.$ ...(7)

The required general solution is given by the relations (4) and (7).

Ex. 14. *Solve* $\dfrac{dx}{x(x + y)} = \dfrac{dy}{-y(x + y)} = \dfrac{dz}{-(x - y)(2x + 2y + z)}.$

Sol. Given $$\frac{dx}{x(x + y)} = \frac{dy}{-y(x + y)} = \frac{dz}{-(x - y)(2x + 2y + z)} \quad \text{... (1)}$$

Taking the first two fractions in (1), we get $(1/x)dx + (1/y)dy = 0.$

Integrating, $\log x + \log y = \log c_1$ or $xy = c_1.$ ...(2)

Choosing, 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx + dy}{x(x + y) - y(x + y)} = \frac{dx + dy}{(x + y)(x - y)}. \quad \text{...(3)}$$

Again, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{x(x + y) - y(x + y) - (x - y)(2x + 2y + z)} = \frac{dx + dy + dz}{(x - y)(x + y) - (x - y)(2x + 2y + z)}$$

$$= \frac{dx + dy + dz}{(x - y)\{x + y - (2x + 2y + z)\}} = -\frac{dx + dy + dz}{(x - y)(x + y + z)}. \quad \text{...(4)}$$

From fractions (3) and (4), $$\frac{dx + dy}{(x + y)(x - y)} = -\frac{dx + dy + dz}{(x - y)(x + y + z)}$$

or $$\frac{dx + dy}{x + y} + \frac{dx + dy + dz}{x + y + z} = 0.$$

Integrating, $\log (x + y) + \log (x + y + z) = \log c_2,$ so that $(x + y)(x + y + z) = c_2.$ ...(5)

The required general solution is given by the relations (2) and (5).

Ex. 15. *Solve* $(dx)/x^2 = (dy)/y^2 = (dz)/z(x + y)$

Sol. Given $$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x + y)} \quad \text{... (1)}$$

From the first two fractions in (1), $x^{-2}dx - y^{-2}dy = 0.$

Integrating, $-x^{-1} + y^{-1} = c_1$ or $(1/y) - (1/x) = c_1.$ ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx - dy}{x^2 - y^2} = \dfrac{dx - dy}{(x - y)(x + y)}.$

...(3)

Taking the last fraction of (1) and fraction (3), we have

$$\frac{dz}{z(x + y)} = \frac{dx - dy}{(x - y)(x + y)} \quad \text{or} \quad \frac{d(x - y)}{x - y} - \frac{dz}{z} = 0.$$

Integrating, $\log (x - y) - \log z = \log c_2$ or $(x - y)/z = c_2.$ ...(4)

The required general solution is given by the relations (2) and (4).

Ex. 16. *Solve* $(dx)/(x^2 + y^2) = (dy)/(2xy) = (dz)/z(x + y).$

Sol. Given $$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{(x + y)z} \quad \text{... (1)}$$

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx+dy}{x^2+y^2+2xy} = \dfrac{dx+dy}{(x+y)^2}$. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx-dy}{x^2+y^2-2xy} = \dfrac{dx-dy}{(x-y)^2}$. ...(3)

From fractions (2) and (3), $(x+y)^{-2}(dx+dy) = (x-y)^{-2}(dx-dy)$.

Integrating, $-(x+y)^{-1} = -(x-y)^{-1} + c_1$ or $(x-y)^{-1} - (x+y)^{-1} = c_1$....(4)

From the last fraction of (1) and fraction (2), we have

$$\frac{dx+dy}{(x+y)^2} = \frac{dz}{(x+y)z} \quad \text{or} \quad \frac{d(x+y)}{x+y} - \frac{dz}{z} = 0.$$

Integrating, $\log(x+y) - \log z = \log c_2$ or $(x+y)/z = c_2$. ...(5)

The required general solution is given by relations (4) and (5).

Ex. 17. *Solve* $(dx)/y = (dy)/x = (dz)/z$.

Sol. Given $(dx)/y = (dy)/x = (dx)/z$ ... (1)

From the first two fractions, $2xdx - 2ydy = 0$ so that $x^2 - y^2 = c_1$. ...(2)

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= (dx+dy)/(y+x)$.

Combining this fraction which the last fraction (1), we get

$$\frac{dx+dy}{x+y} = \frac{dz}{z} \quad \text{so that} \quad \log(x+y) - \log z = \log c_2.$$

or $\log[(x+y)/z] = \log c_2$ or $(x+y)/z = c_2$. ...(3)

The required general solution is given by the relations (2) and (3).

Ex. 18. *Solve* $(dx)/x = (dy)/z = (dz)/(-y)$.

Sol. Given $(dx)/x = (dy)/z = (dz)(-y)$ ... (1)

From the last two fractions, $2ydy + 2zdz = 0$. so that $y^2 + z^2 = c_1$. ...(2)

Choosing, 0, z, $-y$ as multipliers each fraction of (1)

$$= \frac{zdy - ydz}{z.z - y(-y)} = \frac{zdy - ydz}{z^2+y^2} = \frac{(1/z)dy - (y/z^2)dz}{1+(y/z)^2} = \frac{d(y/z)}{1+(y/z)^2} = \frac{dt}{1+t^2}, \text{ where } t = \frac{y}{z} \quad ...(3)$$

Combining the above fraction with the first fraction of (1), we get

$$\frac{dx}{x} = \frac{dt}{1+t^2} \quad \text{so that} \quad \log x - \log c_2 = \tan^{-1} t = \tan^{-1}(y/z).$$

or $\log(x/c_2) = \tan^{-1}(y/z)$ or $x = c_2\, e^{\tan^{-1}(y/z)}$ ...(4)

The required general solution is given by the relations (2) and (4).

Ex. 19. *Solve* $(dx)/(x^2+a^2) = (dy)/(xy-az) = (dz)/(xz+ay)$.

Sol. Given $$\frac{dx}{x^2+a^2} = \frac{dy}{xy-az} = \frac{dz}{xz+ay}$$ ... (1)

Taking 0, z, $-y$ as multipliers, each fraction of (1)

$$= \frac{zdy - ydz}{z(xy-az) - y(xz+ay)} = \frac{zdy - ydz}{-a(y^2+z^2)}. \quad ...(2)$$

Taking 0, y, z as multipliers, each fraction of (1)

$$= \frac{ydy + zdz}{y(xy-az) + z(xz+ay)} = \frac{ydy + zdz}{x(y^2+z^2)}. \quad ...(3)$$

Taking the first fraction of (1) and fraction (3), we have

$$\frac{dx}{x^2+a^2} = \frac{ydy+zdz}{x(y^2+z^2)} \quad \text{or} \quad \frac{2xdx}{x^2+a^2} - \frac{2ydy+2zdz}{y^2+z^2} = 0.$$

Integrating, $\log(x^2+a^2) - \log(y^2+z^2) = \log c_2$.

or $(x^2+a^2)/(y^2+z^2) = c_2$, c_2 being an arbitrary constant ...(4)

Taking the first fraction of (1) and fraction (2), we have

$$\frac{dx}{x^2+a^2} = \frac{zdy-ydz}{-a(y^2+z^2)} \quad \text{or} \quad \frac{adx}{x^2+a^2} + \frac{zdy-ydz}{z^2+y^2} = 0$$

or $$\frac{adx}{x^2+a^2} + \frac{(1/z)dy-(y/z^2)dz}{1+(y/z)^2} = 0 \quad \text{or} \quad \frac{adx}{x^2+a^2} + \frac{d(y/z)}{1+(y/z)^2} = 0$$

or $$\frac{adx}{x^2+a^2} + \frac{dt}{1+t^2} = 0, \quad \text{where} \quad t = \frac{y}{z}.$$

Integrating, $a\tan^{-1}(x/a) + \tan^{-1} t = c_2$ or $a\tan^{-1}(x/a) + \tan^{-1}(y/z) = c_2$. ...(5)

The required general solution is given by the relations (4) and (5).

Ex. 20. *Solve* $\dfrac{dx}{y(x+y)+az} = \dfrac{dy}{x(x+y)-az} = \dfrac{dz}{z(x+y)}$.

[Delhi Maths (Hons) 2005, Nagpur 2005, 10]

Sol. Given $$\frac{dx}{y(x+y)+az} = \frac{dy}{x(x+y)-az} = \frac{dz}{z(x+y)} \quad \text{... (1)}$$

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{y(x+y)+az+x(x+y)-az} = \frac{dx+dy}{(x+y)^2}. \quad \text{...(2)}$$

From the last fraction of (1) and the fraction (2), we have

$$\frac{dx+dy}{(x+y)^2} = \frac{dz}{z(x+y)} \quad \text{or} \quad \frac{d(x+y)}{x+y} - \frac{dz}{z} = 0.$$

Integrating, $\log(x+y) - \log z = \log c_1$ or $(x+y)/z = c_1$. ...(3)

Choosing x, $-y$, 0 as multipliers, each fraction of (1)

$$= \frac{xdx-ydy}{x[y(x+y)+az]-y[x(x+y)-az]} = \frac{xdx-ydy}{az(x+y)}. \quad \text{...(4)}$$

From the first fraction of (1) and fraction (4), we have

$$\frac{xdx-ydy}{az(x+y)} = \frac{dz}{z(x+y)} \quad \text{or} \quad 2xdx - 2ydy - 2adz = 0.$$

Integrating, $x^2 - y^2 - 2az = c_2$, c_2 being an arbitrary constant ...(5)

The required general solution is given by relations (3) and (5).

Ex. 21. *Solve* $\dfrac{dx}{x+y-xy^2} = \dfrac{dy}{x^2y-x-y} = \dfrac{dz}{z(y^2-x^2)}$ **[Delhi Maths (H) 1997, 2002, 07]**

Sol. Given $$\frac{dx}{x+y-xy^2} = \frac{dy}{x^2y-x-y} = \frac{dz}{z(y^2-x^2)} \quad \text{... (1)}$$

Each fraction of (1) $= \dfrac{y\,dx + x\,dy}{y(x + y - xy^2) + x(x^2y - x - y)} = \dfrac{y\,dx + x\,dy}{(y^2 - x^2)(1 - xy)}$

Combining the above fraction with last fraction of (1), we get

$$\frac{y\,dx + x\,dy}{(y^2 - x^2)(1 - xy)} = \frac{dz}{z(y^2 - x^2)} \qquad \text{or} \qquad \frac{-y\,dx - x\,dy}{1 - xy} + \frac{dz}{z} = 0.$$

Integrating, $\log(1 - xy) + \log z = \log c_1$ or $z(1 - xy) = c_1$ ... (2)

Again, each fraction of (1) = $[x\,dx + y\,dy + (1/z)\,dz]/0$

Hence $2x\,dx + 2y\,dy + 2(1/z)\,dz = 0$ so that $x^2 + y^2 + 2\log z = c_2'$

or $x^2 + y^2 + 2\log\{c_1/(1 - xy)\} = c_2'$, using (2)

or $x^2 + y^2 - 2\log(1 - xy) = c_2$, where $c_2 = c_2' - 2\log c_1$ ... (3)

The required general solution is given by (2) and (3)

EXERCISE 2 (D)

Solve the following simultaneous differential equations :

1. $(dx)/x = (dy)/z = (dz)/y$ **Ans.** $y^2 - z^2 = C_1$, $(y + z)/x = C_2$

2. $(dx)/\cos(x + y) = (dy)/\sin(x + y) = (dz)/(z + 1/z)$

Ans. $e^{y-x}\{\cos(x + y) + \sin(x + y)\} = C_1$, $(z^2 + 1)^{1/2}\tan\{3\pi/8 - (x + y)/2\} = C_2$

3. $(dx)/x = (dy)/(-y) = (dz)/(y^2 - x^2)$ **(Bangalore 2005)** **Ans.** $xy = C_1$, $x^2 + y^2 - 2z = C_2$

4. $(dx)/y^2 = (dy)/x^2 = (dz)/z^2(x^2 - y^2)$ **(Bangalore 2005)** **Ans.** $x^3 - y^3 = c_1, x + y + (1/z) = C_2$

5. Solve $(dx)/(xz - y) = (dy)/(yz - x) = (dz)/(1 - z^2)$ **(Pune 2010)**

Ans. $(x - y)(1 - z) = c_1, (x + y)(1 + z) = c_2$

2.12. Orthogonal trajectories of a system of curves on a surface

Let the given surface be $f(x, y, z) = 0$ ... (1)

and let the given system of surface be $\phi(x, y, z) = c$, c being a parameter. ... (2)

Then, the given system of curves lying on the surface (1) are the curves of intersection of (1) and (2).

Clearly the direction ratios dx, dy, dz of the tangent at any point (x, y, z) on the given curve lying on the surfaces (1) and (2) are given by

$(\partial f/\partial x)dx + (\partial f/\partial y)dy + (\partial f/\partial z)dz = 0$ and $(\partial\phi/\partial x)dx + (\partial\phi/\partial y)dy + (\partial\phi/\partial z)dz = 0$

Solving these for dx, dy, dz, we have

$$(dx)/P = (dy)/Q = (dz)/R \qquad \text{... (3)}$$

where $P = \dfrac{\partial f}{\partial y}\dfrac{\partial\phi}{\partial z} - \dfrac{\partial f}{\partial z}\dfrac{\partial\phi}{\partial y}$, $Q = \dfrac{\partial f}{\partial z}\dfrac{\partial\phi}{\partial x} - \dfrac{\partial f}{\partial x}\dfrac{\partial\phi}{\partial z}$, $R = \dfrac{\partial f}{\partial x}\dfrac{\partial\phi}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial\phi}{\partial x}$ ... (4)

Thus, P, Q, R are the direction ratio of tangents to the given curves of intersection of (1) and (2). By, definition, P, Q, R will be direction ratios of normal of the required orthogonal trajectories. Therefore, if dx, dy, dz be the direction ratios of the required orthogonal trajectorie, then the differential equations of the orthogonal trajectories are given by

$(\partial f/\partial x)dx + (\partial f/\partial y)dy + (\partial f/\partial z)\,dz = 0$ and $P\,dx + Q\,dy + R\,dz = 0$

Solving these for dx, dy, dz, we obtain

$$(dx)/P' = (dy)/Q' = (dz)/R', \qquad \text{... (5)}$$

where $P' = R(\partial f/\partial y) - Q(\partial f/\partial z)$, $Q' = P(\partial f/\partial z) - R(\partial f/\partial x)$, $R' = Q(\partial f/\partial x) - P(\partial f/\partial y)$ ... (6)

The solution of simultaneous differential equations (5) together with the given surface (1) gives the system of required orthogonal trajectories.

2.12A. Solved Example based on Art. 2.12.

Ex. 1. *Find the orthogonal trajectories on the cone* $x^2 + y^2 = z^2 \tan^2\alpha$ *of its intersection with the family of planes parallel to* $z = 0$. **[Madurai 2005; Pune 2010]**

Sol. Given surface is $f(x, y, z) = x^2 + y^2 - z^2\tan^2\alpha = 0$ ... (1)

and the family of planes parallel to $z = 0$ is $z = k$. ... (2)

where k is parameter. Then, the system of differential equations of the given curves of intersection of (1) and (2) is given by

$$2x\,dx + 2ydy - 2z\tan^2\alpha\,dz = 0; \qquad dz = 0$$

Solving these equation for dx, dy, dz, we get

$$(dx)/y = (dy)/(-x) = (dz)/0$$

Hence the system of differential equations of the required orthogonal trajectories of the given curves is $xdx + y\,dy - 2z\tan^2\alpha\,dz = 0$ and $y\,dx - x\,dy + 0\cdot dz = 0$ ... (3)

Solving these for dx, dy, dz, we have

$$\frac{dx}{xz\tan^2\alpha} = \frac{dy}{yz\tan^2\alpha} = \frac{dz}{x^2 + y^2} \qquad \text{... (4)}$$

Taking x, y, 0 as multipliers, each fraction of (4) $= \dfrac{xdx + ydy}{(x^2 + y^2)z\tan^2\alpha}$

Combining this fraction with last fraction in (4), we have

$$\frac{xdx + ydy}{(x^2 + y^2)z\tan^2\alpha} = \frac{dz}{x^2 + y^2} \qquad \text{so that} \qquad 2x\,dx + 2y\,dy - 2z\tan^2\alpha\,dz = 0$$

Integrating, $x^2 + y^2 - z^2\tan^2\alpha = c'$, c' being an arbitrary constant.

Choosing $c' = 0$, we obtain the given surface (1).

Taking the first and second fractions of (4), $(1/x)\,dx - (1/y)dy = 0$.

Integrating, $\log x - \log y = \log c$ or $x/y = c$, c being an arbitrary constant.

Hence the required family of the orthogonal trajectories is given by $x^2 + y^2 = z^2\tan^2\alpha$ and $x = cy$.

Ex. 2. *Find the orthogonal trajectories on the conicoid* $(x + y)z = 1$ of the conics in which it is cut by the system of planes $x - y + z = k$, where k is parameter. **[Kolkata 2001]**

Sol. Given surface is $f(x, y, z) = xz + yz - 1 = 0$ ... (1)

and the given system of planes is $x - y + z = k$ ... (2)

Then the system of differential equations of the given curves of intersection of (1) and (2) is given by

$zdx + z\,dy + (x + y)dz = 0$ and $dx - dy + dz = 0$

Solving these for dx, dy, dz we see $\dfrac{dx}{z + x + y} = \dfrac{dy}{x + y - z} = \dfrac{dz}{-2z}$

Hence the system of differential equations of the required orthogonal trajectories of the given curves is

$$\left.\begin{aligned} z\,dx + z\,dy + (x+y)\,dz = 0 \\ (z+x+y)\,dx + (x+y-z)\,dy - 2z\,dz = 0 \end{aligned}\right\} \quad \text{and} \quad \ldots(3)$$

Solving the above simultaneous system of equations for dx, dy, dz, we get

$$\frac{dx}{-2z^2-(x+y)^2+z(x+y)} = \frac{dy}{(x+y)^2+z(x+y)+2z^2} = \frac{dz}{-2z^2} \quad \ldots(4)$$

$$\therefore \quad (4) \quad \Rightarrow \quad \frac{dx+dy}{2z(x+y)} = \frac{dz}{-2z^2} \qquad \text{or} \qquad \frac{dx+dy}{x+y} + \frac{dz}{z} = 0$$

Integrating, $\log(x+y) + \log z = \log c$ or $(x+y)z = c, \ldots(5)$

where c is an arbitrary constant. Choosing $c = 1$, we get the given surface (1), namely, $(x+y)\,z = 1 \ldots(6)$

Now, choosing the first and the last fractions in (4) and using (6), we have

$$\frac{dx}{-2z^2-1/z^2+1} = \frac{dz}{-2z^2} \qquad \text{or} \qquad dx = \{1+(1/2)\times z^{-4} - (1/2)\times z^{-2}\}\,dz$$

Integrating, $x + c' = z - (1/6z^3) + (1/2z)$, c' being an arbitrary constant.

Hence the required family of the orthogonal trajectories is given by

$$(x+y)z = c \qquad \text{and} \qquad x + c' = z - (1/6z^3) + (1/2z).$$

EXERCISE 2(E)

1. Find the orthogonal trajectories on the surface $x^2 + y^2 + 2fyz + d = 0$ of its curves of intersection with planes parallel to xy-plane. **[Nagarajuna 1993]**

Ans. $x^2 + y^2 + 2fyz + d = 0$, $fyz + d = c'x$, c' being an arbitrary constant.

2. Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to intersection with the hyperboids $xy = z + c$. **Ans.** $2y = x^2$, $3z + 2(x - 1/x) = c'$.

3. Find the orthogonal trajectories on the sphere $x^2 + y^2 + z^2 = a^2$ of its intersections with the paraboloids $xy = cz$, c being parameter.

4. Show that the orthogonal trajectories on the hyperboloid $x^2 + y^2 - z^2 = 1$ of the conics in which it is cut by the system of planes $x + y = c$ are its curves of intersection with the surfaces $(x - y)\,z = k$, where k is parameter.

MISCELLANEOUS PROBLEMS ON CHAPTER 2

1. If $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are integral curves of $dx/P = dy/Q = dz/R$, then what is the geometrical meanings of P, Q, R. **[Pune 2010]**

3

Total (or Pfaffian) Differential Equations

3.1. Introduction. In this chapter, we propose to discuss differential equations with one independent variable and more than one dependent variables.

Pfaffian differential form. Pfaffian differential equation. Definitions

Let u_i, $i = 1, 2, .., n$ be n functions of some or all of n independent variables $x_1, x_2, .., x_n$.

Then, $\sum_{i=1}^{n} u_i\, dx_i$ is called a Pfaffian differential form in n variables and $\sum_{i=1}^{n} u_i\, dx_i = 0$ is called a Pfaffian differential equation in n variables $x_1, x_2, .., x_n$.

3.2. Total (or single) differential equation (or Pfaffian differential equation).

An equation of the form $$Pdx + Qdy + Rdz = 0, \qquad ...(1)$$ where P, Q, R are functions of x, y, z is called the *single or total differential equation* in three variables x, y, z.

Equation (1) can be directly integrated if there exists a function $u(x, y, z)$ whose total differential du is equal to the left hand number of (1). In other cases (1) may or may not be integrable. We now proceed to find the condition which P, Q, R must satisfy, so that (1) may be integrable. This will be called the *condition or criterian of integrability* of the single differential equation (1).

3.3. Necessary and sufficient conditions for integrability of total (or single) differential equation Pdx + Qdy + Rdz = 0. [Bangalore 2005; Patna 2003; Delhi Maths Hons. 1994; Agra 2002, 05, 07; Indore 2001, 02; Kanpur 1999; Lucknow 2003, 05; Meerut 2006, 07; Gwalior 2006; Karnataka 2003; Nagarajuna 2004; Sagar 2001, 04, Ujjain 2001, 02]

Necessary condition : Consider the total (or single) differential equation

$$Pdx + Qdy + Rdz = 0, \text{ where } P, Q, R \text{ are functions of } x, y, z. \qquad ...(1)$$

Let (1) have an integral $$u(x, y, z) = c. \qquad ...(2)$$

Then total differential du must be equal to $Pdx + Qdy + Rdz$, or to it multiplied by a factor. But, we know that $$du = (\partial u/\partial x)dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz. \qquad ...(3)$$

Since (2) is an integral of (1), P, Q, R must be proportional to $\partial u/\partial x, \partial u/\partial y, \partial u/\partial z$

Therefore $$\frac{\partial u/dx}{P} = \frac{\partial u/dy}{Q} = \frac{\partial u/dz}{R} = \lambda(x, y, z), \text{ say.}$$

$$\therefore \lambda P = \partial u/\partial x, \qquad \lambda Q = \partial u/\partial y \qquad \text{and} \qquad \lambda R = \partial u/\partial z. \qquad ...(4)$$

From the first two equations of (4), we get

$$\frac{\partial}{\partial y}(\lambda P) = \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\lambda Q)$$

or $$\lambda\frac{\partial P}{\partial y}+P\frac{\partial \lambda}{\partial y}=\lambda\frac{\partial Q}{\partial x}+Q\frac{\partial \lambda}{\partial x}$$

i.e. $$\lambda\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial \lambda}{\partial x}-P\frac{\partial \lambda}{\partial y}. \qquad ...(5)$$

Similarly, $$\lambda\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial \lambda}{\partial y}-Q\frac{\partial \lambda}{\partial z} \qquad ...(6)$$

and $$\lambda\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial \lambda}{\partial z}-R\frac{\partial \lambda}{\partial x}. \qquad ...(7)$$

Multiplying (5), (6) and (7) by R, P and Q respectively and adding, we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \qquad ...(8)$$

This is, therefore, the necessary condition for the integrability of the equation (1).

Sufficient condition : Suppose that the coefficients P, Q, R, of (1) satisfy the relation (8). It will now be proved that this relation gives the required sufficient condition for the existence of an integral of (1). For this we show that an integral of (1) can be found when relation (8) holds.

We first prove that if we take $P_1 = \mu P$, $Q_1 = \mu Q$, $R_1 = \mu R$, where μ is any function of x, y and z, the same condition is satisfied by P_1, Q_1, R_1 as by P, Q, R. We have

$$\frac{\partial Q_1}{\partial z}-\frac{\partial R_1}{\partial y}=\mu\frac{\partial Q}{\partial z}+Q\frac{\partial \mu}{\partial z}-\left(\mu\frac{\partial R}{\partial y}+R\frac{\partial \mu}{\partial y}\right), \quad \text{as} \quad Q_1=\mu Q, \quad \text{and} \quad R_1=\mu R$$

or $$\frac{\partial Q_1}{\partial z}-\frac{\partial R_1}{\partial y}=\mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\frac{\partial \mu}{\partial z}-R\frac{\partial \mu}{\partial y}. \qquad ...(9)$$

Similarly, $$\frac{\partial R_1}{\partial x}-\frac{\partial P_1}{\partial z}=\mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\frac{\partial \mu}{\partial x}-P\frac{\partial \mu}{\partial z} \qquad ...(10)$$

and $$\frac{\partial P_1}{\partial y}-\frac{\partial Q_1}{\partial x}=\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)+P\frac{\partial \mu}{\partial y}-Q\frac{\partial \mu}{\partial x}. \qquad ...(11)$$

Multiplying (9), (10) and (11) by P_1, Q_1, R_1 respectively, adding and replacing P_1, Q_1, R_1 by μP, μQ, μR respectively in resulting R.H.S., we obtain

$$P_1\left(\frac{\partial Q_1}{\partial z}-\frac{\partial R_1}{\partial y}\right)+Q_1\left(\frac{\partial R_1}{\partial x}-\frac{\partial P_1}{\partial z}\right)+R_1\left(\frac{\partial P_1}{\partial y}-\frac{\partial Q_1}{\partial x}\right)$$

$$=\mu\left\{P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)\right\}=0, \text{ by (8)} \qquad ...(12)$$

Now $Pdx + Qdy$ may be regarded as an exact differential. For if it is not so, then multiplying the equation (1) by the integrating factor $\mu(x, y, z)$, we can make it so. *Thus there is no loss of generality in regarding $Pdx + Qdy$ as an exact differential. For this the condition is

$$\partial P/\partial y = \partial Q/\partial x \qquad ...(13)$$

Let $$V = \int (Pdx + Qdy) \qquad ...(14)$$

then it follows that $P = \partial V/\partial x$ and $Q = \partial V/\partial y$. ...(15)

---

*By this we mean here that if $Pdx + Qdy$ is not exact differential, then as explained $\mu Pdx + \mu Qdy$, *i.e.*, $P_1dx + Q_1dy$ will be exact differential and in the whole discussion (12) may be used in place of (8).

From (15), $\dfrac{\partial P}{\partial z} = \dfrac{\partial^2 V}{\partial z\,\partial x}$ and $\dfrac{\partial Q}{\partial z} = \dfrac{\partial^2 V}{\partial z\,\partial y}$.

Using the above relations, (13) and (15), (8) gives

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z\,\partial x} - \frac{\partial R}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z\,\partial x}\right) = 0. \quad \text{or} \quad \frac{\partial V}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) - \frac{\partial V}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) = 0.$$

or
$$\begin{vmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial z} - R\right) \\ \dfrac{\partial V}{\partial y} & \dfrac{\partial}{\partial y}\left(\dfrac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0.$$

*This shows that a relation independent of x and y exists between V and $(\partial V/\partial z) - R$. Consequently $(\partial V/\partial z) - R$ can be expressed as a function of z and V alone. That is, we can take

$$(\partial V/\partial z) - R = \phi(z, V). \qquad \ldots (16)$$

Now, $Pdx + Qdy + Rdz = \dfrac{\partial V}{\partial x}dx + \dfrac{\partial V}{\partial y}dy + \left(\dfrac{\partial V}{\partial z} - \phi\right)dz$, using (14) and (16)

$$= \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) - \phi\, dz = dV - \phi\, dz.$$

Thus (1) may be written as $dV - \phi\, dz = 0$ which is an equation in two variables. Hence its integration will give an integral of the form $F(V, z) = 0$.

Hence the condition (8) is sufficient.

Thus (8) is both the necessary and sufficient condition that (1) has an integral.

Theorem. *Prove that the necessary condition for integrability of the total differential equation.* $\mathbf{A}\cdot d\mathbf{r} = Pdx + Qdy + Rdz = 0$ *is* $\mathbf{A}\cdot curl\ \mathbf{A} = 0$.

[Himanchal 2003, 04, 05, 06; Indore 2004; Lucknow 2001, 04]

Proof. Given $\mathbf{A}\cdot d\mathbf{r} = Pdx + Qdy + Rdz = 0.$ ...(1)

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ so that $d\mathbf{r} = dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k}$ ...(2)

and $\mathbf{A} = P\ \mathbf{i} + Q\ \mathbf{j} + R\ \mathbf{k}.$ ...(3)

Then we see that (1) is satisfied by usual rule of dot product of two vectors $\mathbf{A}$ and $d\mathbf{r}$.

Now show (as explained in Art 3.2) that the necessary condition for integrability of (1) is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \qquad \ldots(4)$$

From vector calculus, we know that

$$\text{Curl } \mathbf{A} = \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\mathbf{i} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathbf{k}. \qquad \ldots(5)$$

Hence, using (3) and (5) and applying the usual rule of dot product of two vectors, the necessary condition (4) may be rewritten as $\mathbf{A}\cdot$ Curl $\mathbf{A} = 0$ as desired.

3.4. The conditions for exactness of Pdx + Qdy + Rdz = 0.

The given total differential equation is said to be exact if the following three conditions are

---

*Refer a chapter on Jacobians in any text book on advanced Differential Calculus.

satisfied

$$\partial P/\partial y = \partial Q/\partial x, \qquad \partial Q/\partial z = \partial R/\partial y \qquad \text{and} \qquad \partial R/\partial x = \partial P/\partial z. \qquad ...(1)$$

Note that when conditions (1) are satisfied, the condition for integrability of $Pdx + Qdy + Rdz = 0$, namely,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad ... (2)$$

is also satisfied, for each term of (2), vanishes identically.

3.5. Methods of solving Pdx + Qdy + Rdz = 0. ...(1)

There are several methods of solving (1). We know that (1) is integrable when the following condition is satisfied

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \qquad ...(2)$$

Before applying any method of solving (1), the students are strongly advised not to forget to verify the condition (2) for integrability of (1). Since this verification is quite simple, we have verified the same in some problems only and have left the same for the students to do themselves.

3.6. Special Method I. Solution by inspection. Sometimes by the rearranging the terms of the given equation and/or by dividing by a suitable function of x, y, z, the equation thus obtained will contain several parts which are exact differentials. The following list will help to re–write the given equation. Students should note that each formula of the list is quite general, that is, we can replace x by y (or z) and so on as may be necessary in a particular problem:

(i) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$ (ii) $\dfrac{xdy - ydx}{xy} = d\left(\log \dfrac{y}{x}\right)$

(iii) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1} \dfrac{y}{x}\right)$ (iv) $\dfrac{xdy + ydx}{xy} = d[\log (xy)]$.

(v) $\dfrac{xdy + ydx}{x^2 + y^2} = d\left[\frac{1}{2} \log (x^2 + y^2)\right]$. (vi) $\dfrac{2xy\, dy - y^2\, dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$.

(vii) $x\, dy + y\, dx = d\,(xy)$. $(viii)$ $d(xyz) = xy\, dz + xz\, dy + yz\, dx$.

(ix) $\int [f(x, y, z)]^n\, df(x, y, z) = \dfrac{[f(x, y, z)]^{n+1}}{n+1}$, $(n \neq -1)$. [Power formula]

(x) $\int \dfrac{df(x, y, z)}{f(x, y, z)} = \log f(x, y, z)$. [Log formula] (xi) $\dfrac{2x^2y\, dy - 2xy^2\, dx}{x^4} = d\left(\dfrac{y^2}{x^2}\right)$

(xii) $-\dfrac{xdy + ydx}{x^2y^2} = d\left(\dfrac{1}{xy}\right)$ $(xiii)$ $\dfrac{ye^x dx - e^x dy}{y^2} = d\left(\dfrac{e^x}{y}\right)$

(xiv) $y^2dx + 2xy\, dy = d(y^2x)$ (xv) $2(xdx + ydy) = d(x^2 + y^2)$.

(xvi) $2(xdx + ydy + zdz) = d(x^2 + y^2 + z^2)$. $(xvii)$ $3x^2ydx + x^3dy = d(x^3y)$.

3.7. Solved examples based on Special Method I of Art. 3.6.

Ex. 1(a). *Verify the condition of integrability for* $z\, dx + z\, dy + 2(x + y + \sin z)dz = 0$. **[Bangalore 2004, 07]**

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, here

$$P = z, \qquad Q = z \qquad \text{and} \qquad R = 2(x + y + \sin z). \qquad ...(1)$$

Now, $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= z(1 - 2) + z(2 - 1) + 2(x + y + \sin z)\,(0 - 0) = 0$, using (1)

showing that the condition of integrability is satisfied.

Ex. 1(b). *Show that* $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = 0$ *is integrable.* **[Bangalore 1995]**

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, here

$$P = 2x + y^2 + 2xz, \qquad Q = 2xy \qquad \text{and} \qquad R = x^2. \qquad \ldots (1)$$

Now, $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= (2x + y^2 + 2xz)\ (0 - 0) + 2xy\ (2x - 2x) + x^2\ (2y - 2y) = 0$, using (1)

showing that the condition of integrability is satisfied and hence the given equation is integrable.

Ex. 1(c). *Test for the integrability of the equation* $(y^2 + yz)\ dx + (xz + z^2)\ dy + (y^2 - xy)\ dz = 0$ **[Gulberga 2005]**

Sol. Comparing the given equation with $Pdx + Qdy + Rdq = 0$, we get

$$P = y^2 + yz, \qquad Q = xz + z^2 \qquad \text{and} \qquad R = y^2 - xy \qquad \ldots (1)$$

Here, $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= (y^2 + yz)\ \{(x + 2z) - (2y - x)\} + (xz + z^2)\ (-y - y) + (y^2 - xy)\ \{(2y + z) - z\}$

$= 2(y^2 + yz)\ (x - y + z) - 2y(xz + z^2) + 2y(y^2 - xy)$

$= 2(xy^2 + xyz - y^3 - y^2z + y^2z + yz^2 - xyz - yz^2 + y^3 - xy^2) = 0,$

showing that the condition of integrability is satisfied and hence the given equation is integrable.

Ex. 2(a). *Solve* $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$.

[Delhi Maths (G) 2002; Agra 2002; Gujrat 2001, 05; Indore 1997; Karnataka 2001; Meerut 1998; Rajasthan 2004; Vikram 1999]

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$P = yz + xyz, \qquad Q = zx + xyz \qquad \text{and} \qquad R = xy + xyz.$$

$\therefore\ P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= yz(1 + x)\ \{(x + xy) - (x + xz)\} + zx(1 + y)\ \{(y + yz) - (y + xy) + xy(1 + z)\ \{(z + xz) - (z + yz)\}$

$= yz(1 + x)\ x(y - z) + zx(1 + y)\ y(z - x) + xy(1 + z)\ z(x - y)$

$= xyz\ \{(1 + x)\ (y - z) + (1 + y)\ (z - x) + (1 + z)\ (x - y)\}$

$= xyz\ [\{(y - z) + (z - x) + (x - y)\} + \{x(y - z) + y(z - x) + z(x - y)\}] = xyz\ [0 + 0] = 0,$

showing that the given total differential equation is integrable.

Dividing each term by xyz, the given equation becomes

$$(1/x + 1)dx + (1/y + 1)dy + (1/z + 1)dz = 0.$$

Integrating it, $\log x + x + \log y + y + \log z + z = c$ or $\log (xyz) + x + y + z = c$,

which is the required general solution, c being an arbitrary constant.

Ex. 2(b). *Solve* $(yz + 2x)dx + (zx - 2z)dy + (xy - 2y)dz = 0$.

[Delhi Maths (G) 1996; Garhwal 1993; Meerut 1997, 2007]

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$P = yz + 2x, \qquad Q = zx - 2z \qquad \text{and} \qquad R = xy - 2y. \qquad \ldots (1)$$

$\therefore\ P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= (yz + 2x)\ \{(x - 2) - (x - 2)\} + (zx - 2z)\ (y - y) + (xy - 2y)\ (z - z) = 0$, by (1)

showing that the given total differential equation is integrable.

On rearranging, the given equation can be rewritten as

$(yz\ dx + zx\ dy + xy\ dz) + 2xdx - 2(zdy + ydz) = 0.$ or $d(xyz) + d(x^2) - 2d(yz) = 0$

Integrating, $xyz + x^2 - 2yz = c,$

which is the required general solution, c being an arbitrary constant.

Ex. 3. *Solve* $(y^2 + z^2 - x^2)dx - 2xy\ dy - 2xz\ dz = 0$.

[Delhi B.A. (Prog) II 2010; Kumaun 2005; Meerut 2007]

Sol. As usual, verify that the given equation is integrable.

Adding and subtracting x^2dx, the given equation becomes

$$(x^2 + y^2 + z^2)dx - x(2x\,dx + 2y\,dy + 2z\,dz) = 0 \quad \text{or} \quad \frac{dx}{x} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}$$

Integrating, $\log x + \log c = \log (x^2 + y^2 + z^2)$ or $xc = x^2 + y^2 + z^2$, which is the required solution, c being an arbitrary constant.

Ex. 4. *Solve* $2yz\,dx + zx\,dy - xy(1 + z)dz = 0$. **[Meerut 1998; Garhwal 2010]**

Sol. As usual, verify that the given equation is integrable.

Dividing each term by xyz, the given equation becomes

$$(2/x)dx + (1/y)dy - (1 + 1/z)dz = 0.$$

Integrating, $2 \log x + \log y - z - \log z - \log c = 0$ or $\log [(x^2y)/(cz)] = z$ or $x^2y = cze^z$, which is the required general solution, c being an arbitrary constant.

Ex. 5(*a*). *Solve* $x\,dy - y\,dx - 2x^2z\,dz = 0$. **[Rajasthan 2001; Banglore 1997]**

Sol. As usual, verify that the given equation is integrable.

Dividing the given equation by x^2, we have

$$\frac{xdy - ydx}{x^2} - 2z\,dz = 0 \quad \text{or} \quad d\left(\frac{y}{x}\right) - 2z\,dz = 0.$$

Integrating, $(y/x) - z^2 = c$, c being an arbitrary constant.

Ex. 5(*b*). *Solve* $zy\,dx = zx\,dy + y^2\,dz$.

[Mysore 2004; Delhi B.Sc. (H) 2000, 04, 06, 08; Delhi BA/B.Sc (Prog.) Maths 2007]

Sol. As usual, verify that the given equation is integrable.

Dividing by zy^2, the given equation can be rewritten as

$$\frac{y\,dx - x\,dy}{y^2} = \frac{dz}{z} \quad \text{or} \quad d\left(\frac{x}{y}\right) = \frac{dz}{z}.$$

Integrating, $(x/y) = \log z - \log c$ or $\log (z/c) = x/y$ or $z = c\,e^{x/y}$.

Ex. 6. *Solve* $yz \log z\,dx - zx \log z\,dy + xy\,dz = 0$. **[Delhi Maths (G) 1997, 1999; Agra 1998; Kanpur 1999; Garhwal 1997; Sgar 1996; Delhi Math (Prog.) 2007]**

Sol. As usual, verify that the given equation is integrable.

Dividing by $xy \log z$, the given equation becomes $\frac{1}{x}dx - \frac{1}{y}dy + \frac{(1/z)dz}{\log z} = 0.$

Integrating, $\log x - \log y + \log \log z = \log c$ or $x \log z = cy$.

Ex. 7. *Solve* $(a - z)(ydx + xdy) + xy\,dz = 0$. **[Delhi Maths (G) 2002; Meerut 1999]**

Sol. As usual, verify that the given equation is integrable.

Dividing by $xy(a - z)$, the given equation becomes

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{a - z} = 0 \quad \text{so that} \quad \log x + \log y - \log (a - z) = \log c$$

or $xy = c(a - z)$, c being an arbitrary constant.

Ex. 8(a). *Solve* $yz^2(x^2 - yz)dx + zx^2(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$. **[Meerut 1997]**

(b) $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$.

[Lucknow 2006; Sagar 2003, 05; Rajasthan 2002, 07; Agra 1998; Gorakhpur 2003,05; [Delhi Maths (G) 2001; 2004; Meerut 1996; Rohilkhand 1997]

Sol. (*a*) As usual, verify that the given equation is integrable.

Dividing each term by $x^2y^2z^2$, the given equation becomes

$$(1/y)dx - (z/x^2)dx + (1/z)dy - (x/y^2)dy + (1/x)dz - (y/z^2)dz = 0$$

or $$\left(\frac{dx}{y}-\frac{x}{y^2}\,dy\right)+\left(\frac{dy}{z}-\frac{y}{z^2}\,dz\right)+\left(\frac{dz}{x}-\frac{z}{x^2}\,dx\right)=0,\ \text{on rearranging}$$

or $$\frac{ydx-xdy}{y^2}+\frac{zdy-ydz}{z^2}+\frac{xdz-zdx}{x^2}=0 \qquad \text{or} \qquad d\left(\frac{x}{y}\right)+d\left(\frac{y}{z}\right)+d\left(\frac{z}{x}\right)=0.$$

Integrating, $(x/y)+(y/z)+(z/x)=c$ or $x^2z+y^2x+z^2y=cxyz.$

(*b*) As usual, verify that the given equation is integrable.

Dividing each term by x^2y^2, the given equation becomes

$$\frac{dx}{y}-\frac{ydx}{x^2}-\frac{zdx}{x^2}+\frac{dy}{x}-\frac{zdy}{y^2}-\frac{xdy}{y^2}+\frac{dz}{x}+\frac{dz}{y}=0$$

or $$\left(\frac{dx}{y}-\frac{xdy}{y^2}\right)+\left(\frac{dy}{x}-\frac{ydx}{x^2}\right)+\left(\frac{dz}{x}-\frac{zdx}{x^2}\right)+\left(\frac{dz}{y}-\frac{zdy}{y^2}\right)=0,\ \text{on re-arranging}$$

or $$\frac{ydx-xdy}{y^2}+\frac{xdy-ydx}{x^2}+\frac{xdz-zdx}{x^2}+\frac{ydz-zdy}{y^2}=0$$

or $$d(x/y)+d(y/x)+d(z/x)+d(z/y)=0.$$

Integrating, $(x/y)+(y/x)+(z/x)+(z/y)=c$ or $x^2+y^2+z(x+y)=cxy.$

Ex. 9. *Solve* $(2x^2+2xy+2xz^2+1)dx+dy+2zdz=0.$ **[Delhi Maths (Hons) 1993, 2001]**

Sol. As usual verify that the given equation is integrable. On rearranging,

$$2x(x+y+z^2)dx+(dx+dy+2zdz)=0 \qquad \text{or} \qquad \frac{dx+dy+2zdz}{x+y+z^2}=-2xdx.$$

Integrating, $\log(x+y+z^2)-\log c=-x^2$ or $x+y+z^2=ce^{-x^2}.$

Ex. 10. *Solve* (*a*) $xz^3dx-zdy+2ydz=0.$ **[Delhi Maths (G) 1996]**

(*b*) $(x^2z-y^3)dx+3xy^2dy+x^3dz=0.$ **[Himanchal 2004; Delhi Maths (G) 1995, 96]**

Sol. (*a*) As usual verify that the given equation is integrable.

Dividing each term of the given equation by z^3, we get

$$xdx-\left(\frac{dy}{z^2}-\frac{2ydz}{z^3}\right)=0 \qquad \text{or} \qquad 2xdx-2\,d\left(\frac{1}{z^2}.\,y\right)=0.$$

Integrating, $x^2-2\,(y/z^2)=c$ or $x^2z^2-2y=cz^2.$

(*b*) Dividing each term by x^2, the given equation becomes

$$zdx-\frac{y^3}{x^2}dx+\frac{3y^2}{x}dy+xdz=0 \qquad \text{or} \qquad (zdx+xdz)+\left(\frac{3y^2\,dy}{x}-\frac{y^3\,dx}{x^2}\right)=0.$$

Integrating, $xz+(y^3/x)=c$ or $x^2z+y^3=cx.$

Ex. 11. *Solve* $2xz\,dx+zdy-dz=0.$ **[Delhi Maths (Hons) 1998]**

Sol. Dividing by z, $2xdx+dy-(1/z)dz=0.$

Integrating, $x^2+y-\log z=c$, c being an arbitrary constant.

Ex. 12. (*a*) *Solve* $2yzdx-3zxdy-4xydz=0.$ **[Delhi Maths 2003]**

(*b*) *Solve* $yzdx+2xzdy-3xydz=0.$ **[Delhi Maths (Hons) 1993]**

Sol. (*a*) Dividing each term by xyz, the given equation becomes

$(2/x)dx-(3/y)dy-(4/z)dz=0$ so that $2\log x-3\log y-4\log z=\log c$

or $\log x^2=\log y^3+\log z^4+\log c$ or $x^2=cy^3z^4.$

(*b*) Proceed as in part (*a*). **Ans.** $x^2y=cz.$

Ex. 13. *Solve* $(x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0$. **[Delhi BA (Prog) II 2011]**

Sol. As usual verify that the given equation is integrable.

One rearranging, the given equation can be written as

$$x\,dx + 2y\,dy + z\,dz - 3(y\,dx + x\,dy) - (z\,dx + x\,dz) = 0.$$

Integrating, $x^2/2 + y^2 + z^2/2 - 3xy - xz = c/2$

or $x^2 + 2y^2 + z^2 - 6xy - 2xz = c$, c being an arbitrary constant.

Ex. 14. *Very that* $x(y^2 - a^2)dx + y(x^2 - z^2)sdy - z(y^2 - a^2)dz = 0$ *is integrable and solve it.* **[G.N.D.U. (Amritar) 2004, 07; Himanchal 2003, 04, 05]**

Sol. As usual verify that the given equation is integrable.

Dividing each term by $(y^2 - a^2)\,(x^2 - z^2)$, the given equation becomes

$$\frac{2(xdx - zdz)}{x^2 - z^2} + \frac{2ydy}{y^2 - a^2} = 0 \quad \text{or} \quad d\{\log (x^2 - z^2)\} + d\{\log (y^2 - a^2)\} = 0.$$

Integrating, $\log (x^2 - z^2) + \log (y^2 - a^2) = \log c$ or $(x^2 - z^2)\,(y^2 - a^2) = c$.

Ex. 15. *Solve* $\dfrac{yz\,dx}{x^2 + y^2} - \dfrac{xz\,dy}{x^2 + y^2} - \tan^{-1}\dfrac{y}{x}\,dz = 0.$

Sol. As usual, verify that the given equation is integrable. On rearranging it, we get

$$\frac{z(ydx - xdy)}{x^2 + y^2} - \tan^{-1}\frac{y}{x}\,dz = 0 \quad \text{or} \quad \frac{1}{\tan^{-1}(y/x)} \cdot \frac{1}{1 + (y^2/x^2)} \cdot \frac{xdy - ydx}{x^2} + \frac{dz}{z} = 0.$$

Integrating, $\log \tan^{-1}(y/x) + \log z = \log c$ or $\tan^{-1}(y/x) = c/z$

or $y/x = \tan(c/z)$ or $y = x \tan(c/z)$, c being an arbitrary constant.

Ex. 16. *Solve* (a) $xdx + ydy - \sqrt{(a^2 - x^2 - y^2)}\,dz = 0$. **[Gujrat 1997]**

(b) $zdz + (x - a)dx = \sqrt{[h^2 - z^2 - (x - a)^2]}\,dy$. **[Agra 1994; Rajasthan 2000, 07]**

Sol. (a) On rearranging, $[a^2 - (x^2 + y^2)]^{-1/2}\,(2xdx + 2ydy) - 2dz = 0$ ...(1)

Putting $x^2 + y^2 = t$ so that $2x\,dx + 2y\,dy = dt$. Then (1) gives $(a^2 - t)^{-1/2}\,dt - 2dz = 0$.

Integrating, $-2(a^2 - t)^{1/2} - 2z = 2c$ or $-(a^2 - t^2)^{1/2} = z + c$.

Squaring, $a^2 - t^2 = (z + c)^2$ or $t^2 + (z + c)^2 = a^2$ or $x^2 + y^2 + (z + c)^2 = a^2$.

(b) On rearranging, $[h^2 - z^2 - (x - a)^2]^{-1/2}\,[2zdz + 2(x - a)dx] = 2dy$.

Putting $z^2 + (x - a)^2 = t$ so that $2z\,dz + 2(x - a)dx = dt$

$\therefore$ $(h^2 - t)^{-1/2}dt = 2dy$ so that $-2(h^2 - t)^{1/2} = 2y - 2c$

or $(h^2 - t)^{1/2} = c - y$ or $h^2 - t = (c - y)^2$ or $t + (c - y)^2 = h^2$

or $z^2 + (x - a)^2 + (c - y)^2 = h^2$, when c is an arbitrary constant.

Ex. 17. *Solve* $\dfrac{xdx + ydy + zdz}{(x^2 + y^2 + z^2)^{1/2}} + \dfrac{z\,dx - x\,dz}{x^2 + z^2} + 3a\,x^2dx + 2b\,y\,dy + c\,dz = 0.$

Sol. We notice by inspection that all the parts are exact differentials. Hence the given equation must be exact and so integrable. Integrating, we obtain

$(x^2 + y^2 + z^2)^{1/2} + \tan^{-1}(x/z) + ax^3 + by^2 + cz = k$, where k is an arbitary constant.

Ex. 18. *Solve* $(x^2 - y^2 - z^2 + 2xy + 2xz)dx + (y^2 - z^2 - x^2 + 2yz + 2yx)dy + (z^2 - x^2 - y^2 + 2zx + 2zy)dz = 0.$

Sol. Adding and subtracting x^2dx, y^2dy, z^2dz in first, second and third term respectively and simplifying, we get

$[-(x^2 + y^2 + z^2) + 2x(x + y + z)]dx + [-(x^2 + y^2 + z^2) + 2y(x + y + z)]dy + [-(x^2 + y^2 + z^2) + 2z(x + y + z)]dz = 0$

or $-(x^2 + y^2 + z^2)\,(dx + dy + dz) + 2(x + y + z)\,(xdx + ydy + zdz) = 0$

or $$\frac{dx+dy+dz}{x+y+z}=\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}.$$

Integrating, $\log(x+y+z)=\log(x^2+y^2+z^2)+\log c$

or $x+y+z=c(x^2+y^2+z^2)$, c being arbitrary constant.

Ex. 19. *Solve* $\frac{y+z-2x}{(y-x)(z-x)}dx+\frac{z+x-2y}{(z-y)(x-y)}dy+\frac{x+y-2z}{(x-z)(y-z)}=0$

[Delhi Maths (G) 2004]

Sol. Re-writing, the given euqation yields

$$\frac{(y-x)+(z-x)}{(y-x)(z-x)}dx+\frac{(z-y)+(x-y)}{(z-y)(x-y)}dy+\frac{(x-z)+(y-z)}{(x-z)(y-z)}dz=0$$

or $$\left(\frac{1}{z-x}+\frac{1}{y-x}\right)dx+\left(\frac{1}{x-y}+\frac{1}{z-y}\right)dy+\left(\frac{1}{y-z}+\frac{1}{x-z}\right)dz=0$$

or $$-\frac{dx-dy}{x-y}-\frac{dy-dz}{y-z}-\frac{dz-dx}{z-x}=0$$

Integrating, $-[\log(x-y)+\log(y-z)+\log(z-x)]=-\log c$

or $(x-y)(y-z)(z-x)=c$, c being an arbitrary constant.

Ex. 20. *Find $f(y)$ such that the total differential equation $\{(yz+z)/x\}dx-zdy+f(y)dz=0$ is integrable. Hence solve it.* **[Indore 2000; Meerut 1994; Pune 2000]**

Sol. Multiplying throughout by x, the given equation reduces to

$$(yz+z)dx-xzdy+xf(y)dz=0. \quad ...(1)$$

Comparing (1) with $Pdx+Qdy+Rdz=0$, we have

$$P=yz+z, \quad Q=-xz \quad \text{and} \quad R=xf(y). \quad ...(2)$$

Suppose that (1) is integrable so that the following condition of integrability is satisfied by it.

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \quad ...(3)$$

Using (2) and denoting df/dy by f', (3) gives

$$(yz+z)(-x-xf')-xz[f-(y+1)]+xf[z-(-z)]=0.$$

or $xz(1+y)f'=xzf$ or $(df)/f=(dy)/(y+1)$.

Integrating, $\log f=\log(y+1)+\log k$ or $f=k(y+1)$, where k is arbitrary constant.

Thus the required value of $f(y)$ is $k(y+1)$ Putting this value $f(y)$ in (1), we get

$$z(y+1)dx-xzdy+xk(y+1)dz=0. \quad ...(4)$$

Dividing by $xz(1+y)$, (4) becomes $(1/x)dx-(dy)/(y+1)+(k/z)dz=0$.

Integrating, $\log x-\log(y+1)+k\log z=\log c$. Hence $xz^k=c(y+1)$ is required solution.

Ex. 21. *Find $f(z)$ such that $[(y^2+z^2-x^2)/2x]dx-y\,dy+f(z)dz=0$ is integrable. Hence solve it.*

[Himanchal 2001; Indore 2002; Karnataka 2002; P.C.S. (U.P) 2002; Rajasthan 2003]

Sol. Multiplying throughout by $2x$, the given equation reduces to

$$(y^2+z^2-x^2)dx-2xydy+2xf(z)dz=0. \quad ...(1)$$

Comparing (1) with $Pdx+Qdy+Rdz=0$, we have

$$P=y^2+z^2-x^2, \quad Q=-2xy, \quad R=2xf(z). \quad ...(2)$$

Suppose that (1) is integrable so that the following condition of integrability is satisfied by it.

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \qquad ...(3)$$

Using (2) and denoting $f(z)$ by f, (3) gives

$$(y^2+z^2-x^2)\,(0-0)-2xy\,(2f-2z)+2xf\,[2y-(-2y)]=0$$

or $\quad -4xy(f-z)+8xfy=0 \quad$ or $\quad f=-z.$

Putting $f(y)=-z$ in (1) and re–writing, $\quad (y^2+z^2-x^2)dx-2xydy-2xzdz=0. \qquad ...(4)$

Adding and subtracting x^2dx, (4) reduces to

$$(x^2+y^2+z^2)dx-x\,(2xdx+2ydy+2zdz)=0 \quad \text{or} \quad \frac{dx}{x}=\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}$$

Integrating, $\quad \log x+\log c=\log(x^2+y^2+z^2) \quad$ or $\quad xc=x^2+y^2+z^2.$

Ex. 22. (*a*) *Solve* $x^2dx^2+y^2dy^2-z^2dz^2+2xydxdy=0.$ **[Meerut 2002, 04]**

(*b*) $dx\,dy\,dz=0$ **[Meerut 2004, 05, 06, 07]**

Sol. Re–writing, $(xdx+ydy)^2-(zdz)^2=0 \quad$ or $\quad (xdx+ydy+zdz)\,(xdx+ydy-zdz)=0. \quad ...(1)$

Equation (1) can be resolved into two component equations

$$xdx+ydy+zdz=0 \qquad ...(2)$$

and $\quad xdx+ydy-zdz=0. \qquad ...(3)$

Integrating (2) and (3), $\quad x^2+y^2+z^2=c_1 \quad$ and $\quad x^2+y^2-z^2=c_2.$

Hence the complete solution of the given differential equation is

$$(x^2+y^2+z^2-c_1)\,(x^2+y^2-z^2-c_2)=0.$$

(*b*) Given equation can be resolved in the following three component equations :

$$dx=0, \quad dy=0 \quad \text{and} \quad dz=0$$

Integrating these, $\quad x-c_1=0, \quad y-c_2=0 \quad$ and $\quad z-c_3=0,$

where c_1, c_2 and c_3 are arbitrary constants. Hence the complete solution of the given equation is given by $\quad (x-c_1)\,(y-c_2)\,(z-c_3)=0.$

Ex. 23. (*i*) *Solve* $(z+z^2)\cos x\,(dx/dt)-(z+z^2)\,(dy/dt)+(1-z^2)\,(y-\sin x)\,(dz/dt)=0.$

(*ii*) *Solve* $(z+z^2)\cos x\,dx-(z+z^2)dy+(1-z^2)\,(y-\sin x)dz=0.$

Sol. (*i*) The given differential equation can be re–written as

$$(z+z^2)\cos x\,dx-(z+z^2)dy+(1-z^2)\,(y-\sin x)dz=0$$

or $\quad (z+z^2)\,(\cos x\,dx-dy)+(1-z^2)\,(y-\sin x)dz=0$

or $\quad \frac{\cos x\,dx-dy}{y-\sin x}+\frac{1-z^2}{z+z^2}\,dz=0 \quad$ or $\quad \frac{dy-\cos x\,dx}{y-\sin x}-\frac{1-z}{z}\,dz=0$

Integrating, $\quad \log(y-\sin x)-\int\left(\frac{1}{z}-1\right)dz=\log c$

or $\quad \log(y-\sin x)-\log z+z=\log c \quad$ or $\quad (y-\sin x)/(cz)=-z$

or $\quad y-\sin x=cz\,e^{-z}$, c being an arbitrary constant.

(*ii*) This equation is same as given in part (*i*).

Ex. 24. *Solve* $(e^xy+\cos x)dx+(e^x+e^yz)dy+e^zdz=0.$

Sol. On rearranging, $\quad (ye^xdx+e^xdy)+\cos x\,dx+(ze^ydy+e^ydz)=0.$

Integrating, $\quad ye^x+\sin x+ze^y=c$, c being an arbitrary constant.

Ex. 25. *Solve* $(2x^2y+2xy^2+2xyz+1)dx+(x^3+x^2y+x^2z+2xyz+2y^2z+2yz^2+1)dy+(xy^2+y^3+y^2z+1)dz=0.$

Sol. On rearranging the given reduces to

$$[2xy(x+y+z)+1]dx+[x^2(x+y+z)+2yz(x+y+z)+1]dy+[y^2(x+y+z)+1]dz=0$$

or $(x + y + z)\,(2xydx + x^2dy + 2yzdy + y^2dz) + (dx + dy + dz) = 0$

or $(2xydx + x^2dx) + (2yzdy + y^2dz) + \dfrac{dx + dy + dz}{x + y + z} = 0.$

Integrating, $x^2y + y^2z + \log(x + y + z) = c$, c being an arbitrary constant.

Ex. 26. (*a*) *Solve* $(2x + y^2 + 2xz)(dx/dt) + 2xy(dy/dt) + x^2(dz/dt) = 1$.

[Kolkata 2002, 05, 06; Meerut 2003, 04, Vikram 2002]

(*b*) $y(y + z)dx + z(u - x)dy + y(x - u)dz + y(y + z)du = 0$.

Sol. (*a*) Re–writing, $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = dt$

or $2xdx + (y^2dx + 2xydy) + (2xzdx + x^2dz) - dt = 0.$

Integrating, $x^2 + xy^2 + zx^2 - t = c$, c being an arbitrary constant

(*b*) Re–writing, $(y + z)\,(zdx + ydu) + z(u - x)dy + y(x - u)dz = 0.$

Adding and subtracting $(y + z)\,(xdz + udy)$ and simplifying, we get

$$(y + z)\,(zdx + ydu + xdz + udy) - (xz + uy)\,(dy + dz) = 0$$

or $$\frac{z\,dx + y\,du + x\,dz + u\,dy}{x\,z + u\,y} = \frac{dy + dz}{y + z}.$$

Integrating, $\log(xz + uy) = \log(y + z) + \log c$ or $xz + uy = c(y + z)$.

Ex. 27. *Solve* $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$.

[Meerut 2005, 07, 11; Garhwal 1996; Karnataka 2001, 06: Rajasthan 2005; Vikram 2001]

Sol. We have, $y^2 + yz + z^2 = (y + z)^2 - yz = (y + z)^2 + x(y + z) - (xy + yz + zx)$

$$= (y + z)\,(x + y + z) - (xy + yz + zx).$$

Similarly, $z^2 + zx + x^2 = (z + x)\,(x + y + z) - (xy + yz + zx)$

and $x^2 + xy + y^2 = (x + y)\,(x + y + z) - (xy + yz + zx).$

Using these new forms of $y^2 + yz + z^2$ etc, the given equation becomes

$(x + y + z)\,\{(y + z)dx + (z + x)dy + (x + y)dz\} - (xy + yz + zx)\,(dx + dy + dz) = 0$

or $(x + y + z)\,\{(ydx + xdy) + (ydz + zdy) + (zdx + xdz)\} - (xy + yz + zx)\,(dx + dy + dz) = 0$

or $(x + y + z)\,d(xy + yz + zx) = (xy + yz + zx)\,d(x + y + z)$

or $$\frac{d(xy + yz + zx)}{xy + yz + zx} = \frac{d(x + y + z)}{x + y + z}.$$

Integrating, $\log(xy + yz + zx) = \log(x + y + z) + \log c$

or $xy + yz + xz = c(x + y + z)$, c being an arbitrary constant.

Ex. 28. *Solve* $y(x + 4)\,(y + z)\,dx - x(y + 3z)dy + 2xy\,dz = 0$. **[Delhi Maths (Hons) 2007]**

Sol. Dividing by $xy(y + z)$, the given equation reduces to

$$\frac{x+4}{x}dx + \frac{2ydz - y(y+3z)dy}{y(y+z)} = 0 \quad \text{or} \quad \frac{x+4}{x}dx + \frac{(2y\,dz + 2y\,dy) - 3(y+z)dy}{y(y+z)} = 0$$

or $$\left(1 + \frac{4}{x}\right)dx + 2\frac{dz + dy}{y + z} - 3\frac{dy}{y} = 0$$

Integrating, $x + 4\log x + 2\log(y + z) - 3\log y = \log c$

or $$\log\frac{x^4(y+z)^2}{y^3c} = -x \quad \text{or} \quad \frac{x^4(y+z)^2}{y^3c} = e^{-x}$$

or $e^x x^4(y + z)^2 = cy^3$, c being an arbitrary constant.

EXERCISE 3 (A)

Solve the following differential equations :

1. **(*a*)** $(y + z)dx + (z + x)dy + (x + y)dz = 0$. **Ans.** $xy + yz + zx = c$.

(*b*) $y\,dx + x\,dy + 2z\,dz = 0$. **[Pune 2010]** **Ans.** $xy + z^2 = c$.

2. $(y + z)dx + dy + dz = 0$. **[Agra 2001, 02]** **Ans.** $y + z = c\,e^{-x}$.

3. $(yz + 2x)dx + (zx + 2y)dy + (xy + 2z)dz = 0$. **Ans.** $x^2 + y^2 + z^2 + xyz = c$.

4. $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = du$. **Ans.** $x^2 + y^2x + x^2z = u + c$.

5. $(x - y)dx - x\,dy + z\,dz = 0$. **[Delhi Maths (G) 2006]** **Ans.** $x^2 - 2xy + z^2 = c$.

6. $(2x^3y + 1)dx + x^4dy + x^2 \tan z\,dz = 0$. **Ans.** $x^2y - (1/x) + \log \sec z = c$.

7. $(y + b)(z + c)\,dx + (x + a)(z + c)\,dy + (x + a)(y + b)\,dz = 0$.

Ans. $(x + a)(y + b)(z + c) = c$, c being an arbitrary constant

8. $(x^2+z^2)(xdx+ydy+zdz)+\sqrt{(x^2 + y^2 + z^2)}\,(zdx-xdz)=0$. **Ans.** $\sqrt{(x^2 + y^2 + z^2)} + \tan^{-1}(x/z) = c$.

9. $(2xy + z^2)dx + (x^2 + 2yz)dy + (y^2 + 2xz)dz = 0$. **Ans.** $x^2y + y^2z + z^2x = c$

10. $dx + dy + (x + y + z + 1)dz = 0$. **[Meerut 2005]** **Ans.** $(x + y + z)e^z = c$.

11. $(y - z)(y + z - 2x)dx + (z - x)(z + x - 2y)dy + (x - y)(x + y - 2z)dz = 0$.

Ans. $y^2x - z^2x + z^2y - x^2y + x^2z - y^2z = c$.

12. $(2x^3 - z)zdx + 2x^2yzdy + x(z + x)dz = 0$. **Ans.** $x^2 + y^2 + \log z + (z/x) = c$.

13. $(y + a)^2dx + zdy - (y + a)dz = 0$. **Ans.** $(x + c)(y + a) = z$.

14. $(a + z)ydx + (a + z)xdy - xydz = 0$. **Ans.** $xy = c(a + z)$

15. $(2xyz + y^2z + z^2y)\,dx + (x^2z + 2xyz + xz^2)dy + (x^2y + xy^2 + 2xyz)dz = 0$. **Ans.** $xyz\,(x + y + z) = c$

16. $a^2y^2z^2dx + b^2z^2x^2dy + c^2x^2y^2dz = 0$ **[Himanchal 2001]** **Ans.** $a^2/x + b^2/y + c^2/z = k$

17. $y(1 + z^2)dx - x(1 + z^2)\,dy + (x^2 + y^2)\,dz = 0$ **[Kurukshetra 1998]**

Ans. $x^2 + 4y^2 - 12\,xy - 4\,xy + 2z^2 = c$

18. $2y\,dx = 2\,xdy + y^2dz$ **[Meerut 2006]** **Ans.** $x^2 + 2yz + 2z^2 = c$

19. $yz\,dx + zx\,dy + xy\,dz = 0$ **[U.P. (P.C.S.) 2005]** **Ans.** $xyz = c$

20. $(2yz + y^2z + yz^2)dx + (x^2z + 2xy\,z + zx^2)dy + (x^2y + xy^2 + 2xyz)dz = 0$ **[Indore 2001, 02]**

Ans. $x^2yz + xy^2z + xyz^2 = c$

21. If $\boldsymbol{X}$ is vector such that $\boldsymbol{X} \cdot \text{curl}\,\boldsymbol{X} = 0$ and μ is an arbitrary function of x, y, z then prove that $(\mu\boldsymbol{X}) \cdot \text{curl}\,(\mu\boldsymbol{X}) = 0$. **[Amravati 2005; Kerala 2002, 05, 06]**

3.8. Special Method II. Solution of homogeneous equation.

The equation $Pdx + Qdy + Rdz = 0$ is called a homogeneous equation if P, Q, R are homogeneous functions of x, y, z of the same degree.

There are two following working rules to solve such equations.

Working rule I. Solution by use of an integrating factor (I.F.)

Step 1. As usual, verify that the given equation is integrable.

Step 2. Calculate $Px + Qy + Rz$. If it is not equal to zero, then $1/(Px + Qy + Rz)$ is taken as I.F. (Integrating factor) of the given equation. Thus, I.F. $= 1/D$, where $D = Px + Qy + Rz$.

Step 3. Multiply the given equation by I.F. (say $1/D$) where D denotes the denominator of I.F. Find $d(D)$ *i.e.* total differential of D. Now add and subtract $d(D)$ from the numerator. Write the given equation in the form $\dfrac{d(D) \pm \ldots}{D} = 0$ or $\dfrac{d(D)}{D} \pm \ldots = 0$

and then integrate. Remember that the several terms in the resulting equations will be exact differential and hence rules of Art. 3.6 have to be used while regrouping the remaining terms.

Working rule II. The first method fails when $Px + Qy + Rz = 0$. In such cases we apply the following method which is applicable to all homogeneous equations.

Step 1. Do same as done in step 1 of working rule I.

Step 2. Put $x = zu$, $y = zv$ so that $dx = udz + zdu$ and $dy = zdv + vdz$. Substituting these in the given equation two cases may arise.

Case I. If the coefficient of dz is zero, we shall have an equation in only two variables u and v. By regrouping properly, it can be easily integrated.

Case II. If the coefficient of dz is not zero, then we shall be able to separate z from u and v. Thus the resulting equation will be of the form

$$\frac{f_1(u, v)\, du + f_2(u, v)\, dv}{f(u, v)} + \frac{dz}{z} = 0. \qquad ...(A)$$

We now denote $f(u, v)$ by D and find $d(D)$. Add and subtract $d(D)$ as done in step 3 of method I. Remember all this is done only in first term. Finally we integrate. After integration u and v are replaced by x/z and y/z respectively so as to get the desired solution in x, y and z.

Note. Sometimes integration of (A) is possible without assuming D etc. Hence we use D only when it helps to integrate equation (A).

3.9. Solved examples based on homogeneous equations (see Art. 3.8)

Ex. 1. *Solve* $(yz + z^2)dx - xzdy + xydz = 0$. **[Delhi Maths (Hons) 2007; Meerut 1997 Guwahati 2007; Gorakhpur 2003, 07; Gwalior 2003; Rohilkhand 2000; Mysore 2004]**

Sol. Given $(yz + z^2)dx - xzdy + xydz = 0$. ...(1)

Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we have

here $P = yz + z^2$, $Q = -xz$, $R = xy$ and let $D = Px + Qy + Rz$.

The condition of integrability is satisfied. (Do yourself)

Now, $D = x(yz + z^2) - xyz + xyz = xz(y + z) \neq 0$. ...(2)

Multiplying (1) by integrating factor $1/D$, $\dfrac{(yz + z^2)dx - xzdy + xydz}{D} = 0$. ...(3)

Now, $d(D) = d[xz(y + z)] = (zdx + xdz)\,(y + z) + xz(dy + dz)$

or $d(D) = z(y + z)dx + x(y + 2z)dz + xz\,dy$. ...(4)

Re–writting, the numerator of (3)

$= d(D) - d(D) + (yz + z^2)dx - xzdy + xydz = d(D) - 2xz(dy + dz)$, by (4).

$\therefore$ (3) becomes $\dfrac{d(D)}{D} - \dfrac{2xz(dy + dz)}{D} = 0$

or $\dfrac{d(D)}{D} - \dfrac{2xz(dy + dz)}{xy(y + z)} = 0$ so that $\dfrac{d(D)}{D} - \dfrac{2(dy + dz)}{y + z} = 0.$

Integrating, $\log D - 2 \log (y + z) = \log c$ or $D = c(y + z)^2$

or $xz(y + z) = c(y + z)^2$ or $xz = c(y + z)$,

which is the required solution, c being an arbitrary constant.

Alternative Method. Verify the usual condition of integrability.

Let $x = uz$ and $y = z$ so that $dx = u\,dz + z\,du$ and $dy = dz + z\,d$.

Putting these values of x, y, dx, dy in (1), we get

$$(z^2 + z^2)\,(u\,dz + z\,du) - uz^2(\ dz + z\,d\ ) + uz^2\,dz = 0$$

$$(\ + 1)z^3\,du - uz^3 d\ + (\ + 1)uz^2 dz = 0.$$

Dividing by $u(\ +1)z^3$, we get $\dfrac{du}{u}-\dfrac{d}{v+1}+\dfrac{dz}{z}=0.$

Integrating, $\log u-\log(\ +1)+\log z=\log c$ or $uz=c(\ +1)$

or $(x/z)\times z=c(1+y/z)$ or $xz=c(y+z)$, as $u=x/z$ and $=y/z$.

Ex. 2. *Solve* $z^2dx+(z^2-2yz)dy+(2y^2-yz-xz)dz=0$. **[Meerut 1993]**

Sol. Verify yourself the condition of integrability. Since the given equation is homogeneous, we put

$x=uz$ and $y=\ z$, so that $dx=z\,du+u\,dz$ and $dy=z\,d\ +\ dz$. ...(1)

Putting these in the given equation, we get

$$z^2(z\,du+u\,dz)+z^2(1-2\ )(z\,d\ +\ dz)+(2\ ^2-\ -u)z^2\,dz=0$$

or $z^3\,du+z^3(1-2\ )d\ +[u+\ (1-2\ )+2\ ^2-\ -u]z^2\,dz=0$

or $z^3\,du+z^3(1-2\ )d\ +(0\times z^2)dz=0$ or $du+(1-2\ )d\ =0.$

Integrating, $u+\ -\ ^2=c$ or $(x/z)+(y/z)-(y^2/z^2)=c$, by (1)

or $(x+y)z-y^2=cz^2$, c being an arbitrary constant.

Ex. 3. $2(y+z)dx-(x+y)dy+(2y-x+z)dz=0$. **[Bangalore 2003, 07]**

Sol. The condition of integrability is satisfied. (Do yourself).

Since the given equation is homogeneous, we put

$x=uz$ and $y=\ z$ so that $dx=zdu+udz$ and $dy=zd\ +\ dz$. ...(1)

Putting these in the given equation, we get

$$2(\ +1)z(zdu+udz)-(u+1)\,z\,(zd\ +\ dz)+(2\ -u+1)\,zdz=0$$

or $2z^2(\ +1)du-z^2(u+1)d\ +[2u(\ +1)-\ (u+1)+(2\ -u+1)]zdz=0$

or $z^2[2(\ +1)du-(u+1)d\ ]+(u\ +u+\ +1)zdz=0$

or $z^2[2(\ +1)du-(u+1)d\ ]+(u+1)(\ +1)\,dz=0.$

Dividing by $z^2(u+1)(v+1)$, $2\dfrac{du}{u+1}-\dfrac{d}{\ +1}+\dfrac{dz}{z}=0.$

Integrating, $2\log(u+1)-\log(\ +1)+\log z=\log c$

or $(u+1)^2z=c(\ +1)$ or $(x/z+1)^2\,z=c(y/z+1)$, by (1)

i.e. $(x+z)^2=c(x+z)$, which is the required solution, c being an arbitrary constant.

Ex. 4. $yz(y+z)dx+zx(x+z)dy+xy(x+y)dz=0$. **[Delhi Maths Hons. 1990; Himanchal 2002; Kurukshetra 2000, 01; Delhi B.Sc. (Prog) II 2011]**

Sol. As usual verify the condition of integrability. Since the given equation is homogeneous,

Put $x=uz$ and $y=\ z$ so that $dx=zdu+udz$ and $dy=zd\ +\ dz$. ...(1)

Putting these in the given equation, we get

$$(\ +1)z^3\,(zdu+udz)+u(u+1)z^3\,(zd\ +\ dz)+u\ (u+\ )z^3dz=0$$

or $[\ (\ +1)du+u(u+1)d\ ]z^4+[u\ (\ +1)+u\ (u+1)+u\ (u+\ )]z^3dz=0$

or $[\ (\ +1)du+u(u+1)d\ ]z^4+2u\ (u+\ +1)z^3\,dz=0$

Dividing by $u\ (u+\ +1)z^4$, we get $\dfrac{(\ +1)du}{u(u+\ +1)}+\dfrac{(u+1)d}{(u+\ +1)}+2\dfrac{dz}{z}=0.$

or $\left(\dfrac{1}{u}-\dfrac{1}{u+\ +1}\right)du+\left(\dfrac{1}{-}\dfrac{1}{u+\ +1}\right)d\ +2\dfrac{dz}{z}=0$ or $\dfrac{du}{u}+\dfrac{d}{---}\dfrac{du+d}{u+\ +1}+2\dfrac{dz}{z}=0.$

Integrating, $\log u+\log\ -\log(u+\ +1)+2\log z=\log c$

or $u\ z^2=c(u+\ +1)$ or $(x/z)\times(y/z)\times z^2=c(x/z+y/z+1)$, by (1)

i.e. $xyz=c(x+y+z)$, which is the required solution, c being an arbitrary constant.

Ex. 5. *Solve* $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$. **[Delhi B.Sc. (Prog) II 2009] [I.A.S. 1999; Delhi Maths (G) 1993; Garhwal 1994; Meerut 2002; Rohilkhand 1995; Calicut 2000; G.N.D.U. Amritsar 2002; Gwalior 2004; Lucknow 2001, 04; Mumbai 2004; Nagarjuna 2000, 04, 07; Sagar 2001, 04; Vikram 2000]**

Sol. As usual, verify that the given equation is integrable.

Since the given equation is homogeneous, we put

$x = uz$ and $y = z$ so that $dx = zdu + udz$ and $dy = zd + dz$. ...(1)

Putting these in the given equation, we get

$$(^2 +)z^2 (zdu + udz) + (u + 1)z^2 (zd + dz) + (^2 - u)z^2 dz = 0$$

or $$(^2 +)z^3 du + (u + 1)z^3 d + [u(^2 +) + (u + 1) + ^2 - u]z^2 dz = 0$$

or $$z^3[(^2 +)du + (u + 1)d] + (u ^2 + u + + ^2)z^2 dz = 0$$

or $$z^3[(^2 +)du + (u + 1)d] + (u + 1) (^2 +)z^2 dz = 0.$$

Dividing by $z^3(u + 1)(^2 +)$, we get

$$\frac{(^2 +)du + (u+1)d}{(u+1)(^2 +)} + \frac{dz}{z} = 0 \quad \text{or} \qquad \frac{du}{u+1} + \frac{d}{^2 +} + \frac{dz}{z} = 0$$

or $$\frac{du}{u+1} + \left[\frac{1}{} - \frac{1}{+1}\right] d + \frac{dz}{z} = 0, \text{ by resolving into partial fractions}$$

Integrating, $\log (u + 1) + \log - \log (+ 1) + \log z = \log c$

or $(u + 1) z = c(+ 1)$ or $\left(x/z + 1\right) \times (y/z) \times z = c\left(y/z + 1\right)$, by (1)

or $(x + z)y = c(y + z)$, which is the required solution, c being an arbitrary constant.

Ex. 6. *Solve* $(2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$. **[Agra 2002, 03; Garhwal 2005; Delhi Maths (H) 2005; Delhi Maths (G) 1998; Kolkata 2002]**

Sol. As usual, verify that the given equation is integrable.

Since the given equation is homogeneous, we put

$x = uz$ and $y = z$ so that $dx = zdu + udz$ and $dy = zd + dz$ ...(1)

Putting these in the given equation, we get

$$(2uz^2 - z^2) (udz + zdu) + (2 z^2 - uz^2) (dz + zd) - (u^2z^2 - u z^2 + ^2z^2)dz = 0$$

or $$(2u -) (udz + zdu) + (2 - u) (dz + zd) - (u^2 - u + ^2)dz = 0$$

or $$z[(2u -)du + (2 - u)d] + [u(2u -) + (2 - u) - (u^2 - u + ^2)]dz = 0$$

or $z[2udu - (ud + du) + 2 d] + (u^2 - u + ^2)dz = 0$ or $z[du^2 - d(u) + d ^2] + (u^2 - u + ^2)dz = 0$

or $$\frac{d(u^2 - u + ^2)}{u^2 - u + ^2} + \frac{dz}{z} = 0.$$

Integrating, $\log (u^2 - u + ^2) + \log z = \log c$ or $z(u^2 - u + ^2) = c$

or $$z\left(\frac{x^2}{z^2} - \frac{x}{z} \times \frac{y}{z} + \frac{y^2}{z^2}\right) = c \qquad \text{or} \qquad x^2 - xy + y^2 = cz.$$

Ex. 7. *Solve* (*a*) $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$. **[Meerut 1996]**

(*b*) $y(y + z)dx + x(x - z)dy + x(x + y)dz = 0$.

Sol. (*a*) As usual, verify that the given equation is integrable.

Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$P = z(z - y)$, $Q = z(z + x)$ and $R = x(x + y)$. ...(1)

The given equation is homogeneous and we have

$Px + Qy + Rz = zx(z - y) + zy(z + x) + zx(x + y)$, using (1)

$= z(xz - xy + yz + xy + x^2 + xy) = z(xz + yz + x^2 + xy) = z(z + x)\,(y + x) \neq 0,$

showing that I.F. of the given equation $= \dfrac{1}{Px + Qy + Rz} = \dfrac{1}{z(z + x)(x + y)}$.

Multiplying the given equation by the above I.F., we have

$$\frac{(z - y)dx}{(x + y)\,(z + x)} + \frac{dy}{x + y} + \frac{xdz}{z(z + x)} = 0$$

or $$\left(\frac{1}{x + y} - \frac{1}{x + z}\right)dx + \frac{dy}{x + y} + \left(\frac{1}{z} - \frac{1}{z + x}\right)dz = 0 \quad \text{or} \quad \frac{dx + dy}{x + y} - \frac{dx + dz}{x + z} + \frac{dz}{z} = 0.$$

Integrating, $\log (x + y) - \log (x + z) + \log z = \log c$

or $z(x + y) = c(x + z)$, c being an arbitrary constant.

(*b*) Proceed as in part (*a*). **Ans.** $x(y + z) = c(x + y)$

Ex. 8. *Solve* $(y^2z - y^3 + x^2y)dx - (x^2z + x^3 - xy^2)dy + (x^2y - xy^2)dz = 0$.

Sol. As usual, verify that the given equation is integrable.

Since the given equation is homogeneous, we put

$x = uz$ and $y = z$ so that $dx = zdu + udz$ and $dy = zd + dz$. ...(1)

Putting these in the given equation, we get

$({}^2z^3 - {}^3z^3 + u^2 z^3)\,(udz + zdu) - (u^2z^3 + u^3z^3 - u\,{}^2z^3)(dz + zd) + (u^2vz^3 - u\,{}^2z^3)dz = 0$

or $({}^2 - {}^3 + u^2)\,(udz + zdu) - (u^2 + u^3 - u\,{}^2)\,(dz + zd) + (u^2 - u\,{}^2)dz = 0$

or $z[({}^2 - {}^3 + u^2)du - (u^2 + u^3 - u\,{}^2)d] + [u({}^2 - {}^3 + u^2) - (u^2 + u^3 - u\,{}^2) + (u^2 - u\,{}^2)]dz = 0$

or $z[({}^2 - {}^3 + u^2)du - (u^2 + u^3 - u\,{}^2)d] + 0 \times dz = 0$ or $(- {}^2 + u^2)\,du - (u + u^2 - {}^2)ud = 0$

or ${}^2du + (u^2 - {}^2)\,du - u^2d - (u^2 - {}^2)ud = 0$ or ${}^2du - u^2d + (u^2 - {}^2)\,(du - ud) = 0$

or $$\frac{du}{u^2} - \frac{d}{{}^2} + \left(\frac{u^2 - {}^2}{u^2}\right)\left(\frac{du - ud}{{}^2}\right) = 0, \text{ on dividing by } u^2\,{}^2$$

or $u^{-2}du - {}^{-2}d + \{1 - (u/\,)^{-2}\}\,d\{u/\,\} = 0$ or $u^{-2}du - {}^{-2}d + \{1 - t^{-2}\}dt = 0$, taking $t = u/\,$.

Integrating, $-u^{-1} + {}^{-1} + t + t^{-1} = c$ or $-(1/u) + (1/\,) + (u/\,) + (u/\,)^{-1} = c$

or $-(z/x) + (z/y) + (x/y) + (y/x) = c$, using (1)

or $x^2 + y^2 - zy + xz = cxy$, c being an arbitrary constant.

EXERCISE 3 (B)

Solve the following differential equations :

1. $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$. **[Rohilkhand 1997]**

[**Hint :** Do as in Ex. 8 of Art. 3.9. For solution by inspection see Ex. 8(*b*) of Art. 3.7]

Ans. $x^2 + y^2 + z(x + y) = cxy$

2. $(x - y)dx - xdy + zdz = 0$. **[Meerut 2011]** **Ans.** $x^2 - 2xy + z^2 = c$

[**Hint :** Do as in Ex. 3 of Art 3.9. We now solve it by inspection also.

Re–writing, $xdx - (ydx + xdx) + zdz = 0$ or $2xdx - 2d(xy) + 2zdz = 0$.

Integrating, $x^2 - 2xy + z^2 = c$, c being an arbitrary constant.]

3. $yz^2(x^2 - yz)dx + x^2z(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$. **[Meerut 1997]**

[**Hint :** Do as in Ex. 8 of Art. 3.9. For solution be inspection see Ex. 8(a) of Art 3.7.]

Ans. $x^2z + y^2x + z^2y = cxyz$

4. $(x^2-y^2-z^2+2xy+2xz)dx+(y^2-z^2-x^2+2yz+2yx)dy+(z^2-x^2-y^2+2xz+2zy)dz=0.$

[**Hint :** Put $x=uz$ and $y=vz$. Alternatively, solution can be obtained by inspection. For solution refer Ex. 18 of Art. 3.7] **Ans.** $x^2+y^2+z^2=c(x+y+z)$

5. $(2z^2-xy+y^2)zdx+(2z^2+x^2-xy)zdy-(x+y)(xy+z^2)dz=0.$ **Ans.** $(x+y)^2z=c(z^2-xy)$

6. $2(2y^2+yz-z^2)dx+x(4y+z)dy+x(y-2z)dz=0.$ **Ans.** $x^2(2y^2+yz-z^2)=c$

3.10. Special Method III. Use of auxiliary equations.

Let $$Pdx+Qdy+Rdz=0 \qquad ...(1)$$

be the given equation. Its condition of integrability is

$$P(\partial Q/\partial z-\partial R/\partial y)+Q(\partial R/\partial x-\partial P/\partial z)+R(\partial P/\partial y-\partial Q/\partial x)=0. \qquad ...(2)$$

Comparing (1) and (2), we obtain simultaneous equations, known as auxiliary equations

$$\frac{dx}{\partial Q/\partial z-\partial R/\partial y}=\frac{dy}{\partial R/\partial x-\partial P/\partial z}=\frac{dz}{\partial P/\partial y-\partial Q/\partial x}. \qquad ...(3)$$

Equations (3) are solved by methods of chapter 2. Let $u=c_1$ and $\quad=c_2$ be two integrals so obtained. With these formulate the following equation

$$A\,du+B\,d\ =0. \qquad ...(4)$$

Compare (1) and (4) and thus get values of A and B. Put these values of A and B in (4) and then integrate the resulting equation. Now substitute the values of u and $\quad$ in the relation after integration. We thus obtain the required general solution.

Note 1. Method III discussed in Art 3.10 will fail in case the given equation (1) in exact, *i.e.*, when $\partial Q/\partial z=\partial R/\partial y$, $\partial R/\partial x=\partial P/\partial z$ and $\partial P/\partial y=\partial Q/\partial x$.

Note 2. This method is generally applied when solution by method I of Art. 3.6 and method II of Art. 3.8 are not convenient.

3.11. Solved examples based on special method III of Art. 3.10.

Ex. 1. *Solve* $xz^3dx-zdy+2ydz=0.$

Sol. Given $$xz^3dx-zdy+2ydz=0. \qquad ...(1)$$

Comparing (1) with $Pdx+Qdy+Rdz=0$, here $P=xz^3,\quad Q=-z,\quad R=2y.$ ...(2)

The auxiliary equations of the given equation are

$$\frac{dx}{\partial Q/\partial z-\partial R/\partial y}=\frac{dy}{\partial R/\partial x-\partial P/\partial z}=\frac{dz}{\partial P/\partial y-\partial Q/\partial x}.$$

or $\dfrac{dx}{-1-2}=\dfrac{dy}{0-3xz^2}=\dfrac{dz}{0}$, using (2) or $\dfrac{dx}{1}=\dfrac{dy}{xz^2}=\dfrac{dz}{0}.$ ...(3)

Taking the third member of (3), we have

$dz=0$ so that $z=c_1=u$, say ...(4)

Taking the first and 2nd members of (3), we have

$xz^2dx-dy=0$ or $2xu^2dx-2dy=0$, using (4)

Integrating, $x^2u^2-2y=c_2=\ $, say or $x^2z^2-2y=\ $, using (4) ...(5)

Substituting the value of u and $\quad$ from (4) and (5) in $Adu+Bd\ =0$, we get

$Adz+Bd(x^2z^2-2y)=0$ or $Adz+B(2xz^2dx+2x^2zdz-2dy)=0$

or $$2Bxz^2dx-2Bdy+(A+2Bx^2z)dz=0. \qquad ...(6)$$

Comparing (1) and (6), $xz^3=2Bxz^2$, $-z=-2B$, $2y=A+2Bx^2z.$

$\therefore$ $B=z/2$ and $A=2y-2Bx^2z=2y-x^2z^2$

or $B=u/2$ and $A=-\ $, using (4) and (5).

Substituting these values of A and B in $Adu+Bd\ =0$, we get

$- \quad du + (1/2) \times ud \quad = 0$ or $(1/\ )d \quad = (2/u)du.$

Integrating, $\log \quad = 2 \log u + \log c$ or $= cu^2.$ ...(7)

Putting the values of u and from (4) and (5) in (7), we get

$x^2z^2 - 2y = cz^2$, which is the required general solution.

Ex. 2. *Solve* $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0.$ **[Delhi Maths (G) 1998]**

Sol. Given $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0.$ ...(1)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = 2xz - yz, \quad Q = 2yz - zx, \quad R = -(x^2 - xy + y^2)$ ...(2)

The auxiliary equations of the given equation are

$$\frac{dx}{\partial Q/\partial z - \partial R/\partial y} = \frac{dy}{\partial R/\partial x - \partial P/\partial z} = \frac{dz}{\partial P/\partial y - \partial Q/\partial x}$$

or $$\frac{dx}{(2y-x)-(x-2y)} = \frac{dy}{(-2x+y)-(2x-y)} = \frac{dz}{-z-(-z)}, \text{ using (2)}$$

or $$\frac{dx}{2(2y-x)} = \frac{dy}{2(y-2x)} = \frac{dz}{0}. \quad ...(3)$$

Taking the third member of (3), we have

$dz = 0$ so that $z = c_1 = u,$ say ...(4)

Taking the first and second members of (3), we have

$(y - 2x)dx = (2y - x)dy$ or $ydx + xdy - 2xdx - 2ydy = 0.$

Integrating, $xy - x^2 - y^2 = c_2 = \ ,$ say ...(5)

We now proceed to determine two functions A and B in such a manner so that the given equation (1) becomes idential with $A\ du + B\ d \quad = 0.$ ...(6)

Using (4) and (5), (6) reduces to

$Adz + Bd(xy - x^2 - y^2) = 0$ or $Adz + B(xdy + ydx - 2xdx - 2ydy) = 0$

or $(By - 2xB)dx + (xB - 2yB)dy + Adz = 0.$ ...(7)

Comparing (7) with (1), we have

$B(y - 2x) = z(2x - y), \quad B(x - 2y) = z(2y - x)$ and $A = -(x^2 - xy + y^2),$

giving $B = -z = -u$ and $A = \ ,$ using (4) and (5)

Putting these values of A and B in (6), we have

$du - ud \quad = 0$ or $(1/\ )d \quad - (1/u)du = 0.$

Integrating, $\log \quad - \log u = \log c$ or $= cu$

or $xy - x^2 - y^2 = cz$, by (4) and (5), c being an arbitrary constant.

Ex. 3. *Solve* $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0.$ **[Garhwal 1996]**

Sol. Given $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0.$ ...(1)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = y^2 + yz + z^2, \quad Q = z^2 + zx + x^2$ and $R = x^2 + xy + y^2.$ ...(2)

The auxiliary equations of the given equation are

$$\frac{dx}{\partial Q/\partial z - \partial R/\partial y} = \frac{dy}{\partial R/\partial x - \partial P/\partial z} = \frac{dz}{\partial P/\partial y - \partial Q/\partial x}$$

$$\frac{dx}{(x+2z)-(x+2y)} = \frac{dy}{(2x+y)-(y+2z)} = \frac{dz}{(2y+z)-(2x+z)}.$$

or $$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}. \quad ...(3)$$

Each member of (3) $= \dfrac{dx + dy + dz}{z - y + x - z + y - x} = \dfrac{dx + dy + dz}{0}$

so that $dx + dy + dz = 0.$

Integrating, $x + y + z = c_1 = u$, say ...(4)

Again, using multipliers $z + y$, $x + z$, $y + x$, each member of (3)

$$= \frac{(z + y)dx + (x + z)dy + (y + x)dz}{(z + y)(z - y) + (x + z)(x - z) + (y + x)(y - x)}$$

$$= \frac{(xdy + ydx) + (ydz + zdy) + (zdx + xdz)}{0} = \frac{d(xy) + d(yz) + d(zx)}{0}$$

so that $d(xy) + d(yz) + d(zx) = 0.$

Integrating, $xy + yz + zx = c_2 =$, say ...(5)

We now proceed to determine two functions A and B in such a manner so that the given equation (1) becomes identical with $A\,du + B\,d$ $= 0.$ ...(6)

Using (4) and (5), (6) reduces to

$$Ad(x + y + z) + Bd(xy + yz + zx) = 0$$

or $A(dx + dy + dz) + B(ydx + xdy + zdy + ydz + xdz + zdx) = 0$

or $\{A + B(y + z)\}dx + \{A + B(z + x)\}dy + \{A + B(x + y)\}dz = 0.$ ...(7)

Comparing (7) with (1), we have

$$A + B(y + z) = y^2 + yz + z^2 \quad ...(8)$$

$$A + B(z + x) = z^2 + xz + x^2 \quad ...(9)$$

$$A + B(x + y) = x^2 + xy + y^2 \quad ...(10)$$

Subtracting (9) from (8), we get $B(y - x) = y^2 - x^2 + z(y - x) = (y - x)(x + y + z)$

$\therefore$ $B = x + y + z = u$, by (4) ...(11)

From (8), $A = y^2 + yz + z^2 - B(y + z) = y^2 + yz + z^2 - (x + y + z)(y + z)$, by (1)

$= y^2 + yz + z^2 - (y^2 + z^2 + 2yz + xy + zx)$

Thus, $A = -(xy + yz + zx)$ or $A = -$, by (5) ...(12)

Using (11) and (12), (6) becomes

$-\,du + ud$ $= 0$ or $(1/\ )d$ $= (1/u)du.$

Integrating, $\log$ $= \log u + \log c$ or $= cu$

or $xy + yz + zx = c(x + y + z)$, by (4) and (5).

Ex. 4. (*a*) *Solve* $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0.$

[Delhi Maths (G) 1993; Garhwal 1994; Meerut 1997, 1998; Rohilkhand 1995]

(*b*) $z(z - y)dx + (z + x)zdy + x(x + y)dz = 0.$ **[Meerut 1996]**

Sol. Try yourself. **Ans.** (*a*) $(x + z)y = c(y + z)$; (*b*) $z(x + y) = c(x + y)$

Ex. 5. *Solve* $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0.$ **Ans.** $x^3 + y^3 = e^{2z} + ce^z$

3.12. General method of solving Pdx + Qdy + Rdz = 0 by taking one variable as constant.

Step 1. First verify the condition of integrability.

Step 2. We now treat one of the variables, say z, as a constant *i.e.* $dz = 0$, then the resulting equations is reduced to $Pdx + Qdy = 0.$ ...(1)

We should select a proper variable to be constant so that the resulting equation in the remaining variables is easily integrable. Thus this selection will vary from problem to problem. The present discussion is for the choice z = constant. For other cases the necessary changes have to be made in the entire procedure.

Step 3. Let the solution of (1) by $u(x, y) = f(z)$, where $f(z)$ is an arbitrary function of z. Note that in place of taking merely an absolute constant, we have taken $f(z)$. This is possible because the arbitrary function $f(z)$ is constant with respect to x and y. This is in keeping with out starting assumption, namely z = constant. Thus the solution of (1) is of the from

$$u(x, y) = f(z). \qquad ...(2)$$

Step 4. We now differentiate (2) totally with respect to x, y, z and then compare the result with the given equation $Pdx + Qdy + Rdz = 0$.

After comparing we shall get an equation in two variables f and z. If the coefficient of df or dz involve functions of x and y, it will always be possible to remove them by using (2).

Step 5. Solve the equation got in step 4 and obtain f. Putting this value of f in (2), we shall get the requried solution of the given equation.

Remarks. Many questions can be solved by this method, but the method may become tedius in some problems. Number of exercises based on methods I, II and III can be done by this method. We shall apply this method whenever there is no difficulty in solving the equation obtained by treating one variable as constant.

3.13. Solved examples based on Art. 3.12.

Ex. 1 (*a*). *Solve* $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$. **[Delhi Maths (G) 1993; Meerut 2007]**

Sol. Given $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0. \qquad ...(1)$

As usual, verify that (1) satisfies the condition of integrability.

Let z be treated as constant, so that $dz = 0$. Then (1) becomes

$$3x^2dx + 3y^2dy = 0. \qquad ...(2)$$

Integrating (2) $x^3 + y^3 = f(z)$, say, ...(3)

where the constant of integration has been taken as a function $f(z)$ of z because we have treated z as constant.

Differentiating (3), $3x^2dx + 3y^2dy - f'(z)dz = 0. \qquad ...(4)$

Comparing (4) with (1), we have

$f'(z) = x^3 + y^3 + e^{2z}$ or $f'(z) = f(z) + e^{2z}$, by (2)

or $(df/dz) - f = e^{2z}$, which is a linear equation. Its I.F. $= e^{\int(-1)dz} = e^{-z}$ and hence its solution is

$$fe^{-z} = \int [e^{2z} . e^{-z}]\, dz + c = \int e^z + c = e^z + c$$

or $f(z) = e^{2z} + ce^z$ or $x^3 + y^3 = e^{2z} + ce^z$, using (3),

which is the required solution of (1), c being an arbitrary constant.

Ex. 1 (*b*). *Verify that the condition of integrability for the differential equation* $(2x + y + 2xz)dx + 2xy\,dy + x^2dz = 0$. *Also solve it.* **[Delhi Maths (Prog.) 2007]**

Sol. Given $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = 0 \qquad ... (1)$

For the first part, refer Ex. 1 (*b*) of Art. 3.7

Second Part: Let x be treated as constant so that $dx = 0$.

Then, (1) reduces to $2xy\,dy + x^2dz = 0$ or $2y\,dy + x\,dz = 0$

Integrating it, $y^2 + xz = f(x) \qquad ... (2)$

where $f(x)$ is taken as constant of integration as x is treated as constant. Differentiating (2), we have

$2y\,dy + x\,dz + zdx = f'dx$ or $x(z - f')\,dx + 2xy\,dy + x^2dz = 0 \qquad ... (3)$

Comparing (3) with (1), we get $2x + y^2 + 2xz = xz - xf'$

or $2x + y^2 + xz = -xf'$ or $2x + f = -xf'$, using (2)

or $x(df/dx) + f = -2x$ or $(df/dx) + (1/x) \times f = -2 \qquad ... (4)$

which is a linear equation whose *I.F.* $= e^{\int(1/x)dx} = e^{\log x} = x$.

$\therefore$ Solution of (4) is $xf = \int (-2x)\, dx + c$ or $xf = -x^2 + c$

or $x(y^2 + xz) = -x^2 + c$ or $x^2 + xy^2 + x^2z = c$, using (2)

which is the equired solution, c being an orbitrary constant.

Ex. 2. *Solve* $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$.

[Gwalior 2005; Indorer 1999; Jiwaji 2004; Punjab 2001, 05; Rajastan 2006; Sager 2002; Bangalore 1995; Delhi Maths (Hons) 2008; Meerut 1997]

Sol. Given $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdx = 0.$ ...(1)

As usual, verify that (1) is integrable. Let x be treated as constant, so that $dx = 0$. Then (1) becomes $dy + 2zdz = 0.$ ...(2)

Integrating (1), $y + z^2 = f(x)$, (say) ...(3)

where the constant of integration has been taken as a function $f(x)$ of x because we have treated x as constant

Differentiating (3), $dy + 2zdz - f'(x)dx = 0.$ ...(4)

Comparing (4) with (1), $f'(x) = -(2x^2 + 2xy + 2xz^2 + 1)$

or $f'(x) = -(2x^2 + 1) - 2x(y + z^2) = -(2x^2 + 1) - 2xf(x)$, using (3).

or $(df/dx) + 2xf = -(2x^2 + 1)$, which is a linear equation. Its I.F. $= e^{\int (2x)dx} = e^{x^2}$ and hence its solution is $f\, e^{x^2} = \int e^{x^2} \{-(2x^2 + 1)\}dx + c = -\int x(2xe^{x^2})\, dx - \int e^{x^2}\, dx + c.$ ...(5)

Now, $\int 2x\, e^{x^2}\, dx = \int e^t\, dt = e^t = e^{x^2}$, putting $x^2 = t$ and $2xdx = dt$ ...(6)

Applying the formula of integration by parts in first term on R.H.S. of (5) and noting that $\int 2x\, e^{x^2}\, dx = e^{x^2}$ as proved in (6), we have

$$f\, e^{x^2} = -\left[xe^{x^2} - \int (1.e^{x^2})dx \right] - \int e^{x^2}\, dx + c = -xe^{x^2} + c$$

or $f(x) = -x + c\, e^{-x^2}$ or $y + z^2 = -x + c\, e^{-x^2}$, by (3).

Ex. 3. *Solve* $(x^2 + y^2 + z^2)dx - 2xy\, dy - 2xz\, dz = 0$.

[Delhi Maths (H) 1999; Delhi Maths (G) 2005]

Sol. Given $(x^2 + y^2 + z^2)dx - 2xy\, dy - 2xz\, dz = 0$ ...(1)

As usual, verify that (1) is integrable. Let x be treated as constant, so that $dx = 0$. Then (1) becomes $-2xydy - 2xzdz = 0$ or $2ydy + 2zdz = 0.$ ...(2)

Integrating (2), $y^2 + z^2 = f(x)$, (say) ...(3)

where $f(x)$ is taken as constant of integration as x is treated as constant.

Differentiating (3), $2ydy + 2zdz = f'(x)dx$ or $xf'(x)dx - 2xydy - 2xzdz = 0,$...(4)

Comparing (4) with (1), $xf'(x) = x^2 + y^2 + z^2$ or $xf'(x) = x^2 + f(x)$, by (3)

or $(df/dx) - (1/x)f = x$, which is a differential linear equation

Its I.F. $= e^{\int (-1/x)dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$ and its solution is

$$(1/x) \times f = \int \{x \times (1/x)\}dx + c = x + c \quad \text{or} \quad f(x) = x^2 + cx$$

or $y^2 + z^2 = x^2 + cx$, using (3); c being an arbitrary constant.

Ex. 4. *Solve* $xz^3dx - z\, dy + 2y\, dz = 0$. **[Delhi Maths (Hons) 1996]**

Sol. Given $xz^3dx - z\, dy + 2y\, dz = 0.$ ...(1)

Let x be treated as constant, so that $dx = 0$. Then (1) becomes

$-zdy + 2ydz = 0$ or $-(1/y)dy + (2/z)dz = 0.$ ...(2)

Integrating (2), $-\log y + 2\log z = \log f(x)$ or $z^2/y = f(x)$, ...(3)

when $f(x)$ is taken as constant of integration as x is treated as constant.

Differentiating (3), $\dfrac{y\times 2zdz - z^2dy}{y^2} = f'(x)$ or $-\left(\dfrac{y^2}{z}\right)\times f'(x) - zdy + 2ydz = 0$. ...(4)

Comparing (4) with (1), $-(y^2/z)f'(x) = xz^3$

or $f'(x) = -x(z^2/y)^2$ or $f'(x) = -x[f(x)]^2$, using (3)

or $df/dx = -xf^2$ or $-f^{-2}\,df = x\,dx$.

Integrating, $(1/f) = (x^2/2) + c$ or $y/z^2 = (x^2/2) + c$, using (3)

or $2y = x^2z^2 + 2cz^2$, c being an arbitrary constant

Ex. 5. *Solve* $y^2z(x\cos x - \sin x)dx + x^2z(y\cos y - \sin y)dy + xy(y\sin x + x\sin y + xy\cos z)dz = 0$. **[Meerut 1998]**

Sol. As usual verify that the given equation is integrable. Let z be treated as constant, so that $dz = 0$. Then the given equation becomes

$$y^2z(x\cos x - \sin x)dx + x^2z(y\cos y - \sin y)dy = 0$$

or $\dfrac{x\cos x - \sin x}{x^2}dx + \dfrac{y\cos y - \sin y}{y^2}dy = 0$ or $d\left(\dfrac{\sin x}{x}\right) + d\left(\dfrac{\sin y}{y}\right) = 0$.

Integrating it, $\dfrac{\sin x}{x} + \dfrac{\sin y}{y} = f(z)$, say ...(1)

where $f(z)$ is taken as constant of integration as z is treated as constant.

Differentiating (1), $\dfrac{x\cos x - \sin x}{x^2}dx + \dfrac{y\cos y - \sin y}{y^2}dy = f'(z)\,dz$

or $zy^2(x\cos x - \sin x)dx + zx^2(y\cos y - \sin y)dy - x^2y^2zf'(z)dz = 0$. ...(2)

Comparing (2) with the given equation, we have

$$-x^2y^2zf'(z) = xy(y\sin x + x\sin y + xy\cos z)$$

or $-zf'(z) = \dfrac{\sin x}{x} + \dfrac{\sin y}{y} + \cos z = f(z) + \cos z$, using (1)

or $\dfrac{df}{dz} + \dfrac{1}{z}f = -\dfrac{\cos z}{z}$, which is a linear differential equation.

Its I.F. $= e^{\int (1/z)dz} = e^{\log z} = z$ and its solution is

$$zf(z) = \int z\left(-\frac{\cos z}{z}\right)dz + c = -\sin z + c \quad \text{or} \quad z\left(\frac{\sin x}{x} + \frac{\sin y}{y}\right) = c - \sin z, \text{ using (1).}$$

Ex. 6. *Solve* $(e^xy + e^z)dx + (e^yz + e^x)dy + (e^y - e^xy - e^yz)dz = 0$.

Sol. As usual, verify that the given equation is integrable.

Let z be treated as constant, so that $dz = 0$. Then the given equation becomes

$(e^xy + e^z)dx + (e^yz + e^x)dy = 0$ or $(e^xydx + e^xdy) + e^zdx + ze^ydy = 0$

or $d(e^xy) + e^zdx + ze^ydy = 0$. ...(1)

Integrating (1), $e^xy + e^zx + e^yz = f(z)$, say ...(2)

where $f(z)$ is taken as constant of integration as z is treated as constant.

Differentiating (2), $ye^xdx + e^xdy + xe^zdz + e^zdx + ze^ydy + e^ydz = f'(z)dz$

or $(e^xy + e^z)dx + (e^yz + e^x)dy + [xe^z + e^y - f'(z)]dz = 0$. ...(3)

Comparing (3) with the given equation, we have

$e^y - e^x y - e^y z = xe^z + e^y - f'(z)$ or $f'(z) = [e^x y + e^z x + e^y z] = f(z)$, by (2)

or $df/dz = f(z)$ or $(1/f)df = dz$.

Integrating, $\log f - \log c = z$ or $f(z) = ce^z$ or $e^x y + e^z x + e^y z = ce^z$, using (2).

Ex. 7. *Find $f(y)$ if $f(y)dx - zx\,dy - xy \log y\,dz = 0$ is integrable. Find the corresponding integral.* **[Meerut 2011]**

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, have $P = f(y)$, $Q = -zx$, $R = -xy \log y$. If the given equation is integrable, the following condition of integrability must be satisfied by P, Q, R

$$P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x) = 0.$$

i.e. $f(y)[-x - (-x \log y - x)] - xz(-y \log y - 0) - xy \log y[f'(y) - (-z)] = 0$

or $x \log y\, f(y) - xy \log y\, f'(y) = 0$ or $f(y) - yf'(y) = 0$

or $f'(y)/f(y) = 1/y$ whose integration gives, $\log f(y) = \log y + \log c$

i.e. $f(y) = cy$, which is the required value of $f(y)$.

Then the given equation becomes $cy\,dx - zx\,dy - xy \log y\,dz = 0$. ...(1)

Treating z as constant *i.e.* $dz = 0$, (1) becomes

$cy\,dx - zx\,dy = 0$ or $(c/x)dx - (z/y)dy = 0$.

Integrating and remembering that z is to be regarded as constant, we get

$c \log x - z \log y = F(z)$, where $F(z)$ is an arbitrary function of z. ...(2)

Differentiating (2), $(c/x)\,dx - (z/y)\,dy - \log y\,dz = F'(z)\,dz$

or $cy\,dx - zx\,dy - [xy \log y + xy\,F'(z)]\,dz = 0$. ...(3)

Comparing (1) and (3), we get $xy \log y + xy\,F'(z) = xy \log y$ or $F'(z) = 0$.

Integrating, $F(z) = c'$, where c' is an arbitrary constant.

Putting $F(z) = c'$ in (2) the required solution is

$c \log x - z \log y = c'$ or $x^c = ky^z$, taking $c' = \log k$.

Ex. 8. *Solve* $(z + z^3) \cos x\,dx - (z + z^3)dy + (1 - z^2)(y - \sin x)dz = 0$

or $(z + z^3) \cos x\,(dx/dt) - (z + z^3)\,(dy/dt) + (1 - z^2)(y - \sin x)\,(dz/dt) = 0$.

[Allahabad 2005; Kolkata 2007; Nagpur 2004, 06]

Sol. Treating z as constant so that $dz = 0$, the given equation becomes

$(z + z^3) \cos x\,dx - (z + z^3)dy = 0$ or $\cos x\,dx - dy = 0$.

Integrating, $\sin x - y = f(z)$, where f is an arbitrary function. ...(1)

Differentiating (1), $\cos x\,dx - dy - (df/dz)\,dz = 0$

or $(z + z^3) \cos x\,dx - (z + z^3)\,dy - (z + z^3)\,f'(z)\,dz = 0$ ...(2)

Comparing (2) and the given equation, we have

$(1 - z^2)(y - \sin x) = -(z + z^3)\,f'(z)$ or $(1 - z^2)(-f) = -(z + z^3)\,(df/dz)$, using (1)

or $\dfrac{df}{f} = \dfrac{1 - z^2}{z(1 + z^2)}dz$ or $\dfrac{df}{f} = \left(\dfrac{1}{z} - \dfrac{2z}{z^2 + 1}\right)dz.$

Integrating, $\log f = \log z - \log (z^2 + 1) + \log c$ or $(z^2 + 1)f = cz$

or $(\sin x - y)(z^2 + 1) = cz$, by (1), c being on arbitrary constant.

Ex. 9. *Verify that the following equation is integrable and find its primitive :*

$$zy\,dx + (x^2y - zx)\,dy + (x^2z - xy)\,dz = 0.$$

Sol. Given $zy\,dx + (x^2y - zx)\,dy + (x^2z - xy)\,dz = 0$. ...(1)

Treating x as constant so that $dx = 0$, (1) reduces to

$(x^2y - zx)dy + (x^2z - xy)\,dz = 0$ or $x^2(ydy + zdz) - x(zdy + ydz) = 0$. ...(2)

Integrating (2) and remembering that x is being regarded as constant, we get

$$(x^2/2)\times(y^2+z^2)-xyz=f(x),\ f \text{ being an arbitrary function.} \quad ...(3)$$

Differentiating (3), we have

$$x\,dx(y^2+z^2)+(x^2/2)\times(2ydy+2zdz)-xy\,dz-yzdx-zx\,dy=f'(x)dx$$

or $$[x(y^2+z^2)-yz-f'(x)]dx+(x^2y-xy)dy+(x^2z-xy)dz=0.$$

Comparing the above equation with (1), we have

$x(y^2+z^2)-yz-f'(x)=yz$ or $x(y^2+z^2)-2yz=f'(x)$

or $(x^2/2)\times(y^2+z^2)-xyz=(x/2)\times f'(x)$ or $f=(x/2)\times f'(x)$ by (3)

or $f=(x/2)\times(df/dx)$ or $(1/f)df=(2/x)dx$ so that $f(x)=cx^2$.

Putting this value of $f(x)$ in (3), the required primitive is

$(x^2/2)\times(y^2+z^2)-xyz=cx^2$ or $x^2(y^2+z^2-2c)=2xyz.$

EXERCISE 3 (C)

Solve the following differential equations :

1. $z^2dx+(z^2-2yz)dy+(2y^2-yz-zx)dz=0.$ **Ans.** $xz+yz-y^2=cz^2$

2. $(y^2+yz)dx+(z^2+zx)dy+(y^2-xy)dz=0.$ **Ans.** $y(x+z)=c(y+z)$

3. $2yzdx+zxdy-xy(1+z)dz=0.$ **Ans.** $x^2y=cze^z$

4. $(mz-ny)dx+(nx-lz)dy+(ly-mx)dz=0.$ **[Rajasthan 2001, 07] Ans.** $nx-lz=c(mz-ny)$

5. $z(z+y^2)dx+z(z+x^2)dy-xy(x+y)dz=0.$ **[Pune 2010] Ans.** $x(y^2+z)=z(x+y)(1-cy)$

6. $z(x^2-yz-z^2)dx+(x+z)xzdy+x(z^2-x^2-xy)dz=0.$ **Ans.** $z^{-1}(x+y)+x^{-1}(y+z)=c$

7. $xdy-ydx-2x^2zdz=0.$ **Ans.** $y=x(c-z^2)$

8. $3ydx=3xdy+y^2dz.$ **Ans.** $x=y\log(cz)$

9. $zy(1+4xz)dx-xz(1+2xz)dy-xydz=0.$ **[Delhi 2008, 09, 10] Ans.** $x(1+2xz)=cyz$

10. $yz\log x\,dx-zx\log z\,dy+xy\,dz=0.$ **Ans.** $x\log z=cy$

11. $y^2dx-2x^2dy+(xy-zy^3)dz=0.$ **[Bangalore 1994] Ans.** $2y^{-1}-x^{-1}=ce^z$

12. $yz\,dx+(xz-yz^3)dy-2xy\,dz.$ **[Bangalore 2005]**

3.14. Solution of Pdx + Qdy + Rdz = 0, if it is exact and homogeneous of degree n ≠ 1.

Theorem. *$xP+yQ+zR=c$ is the solution of $Pdx+Qdy+Rdz=0$, when it is exact and homogeneous of degree $n\neq 1$.* **[Bilaspur 2001; Meerut 1998; Guwahati 1996]**

Proof : Give solution is $xP+yQ+zR=c.$ ...(1)

Differentiating (1), we obtain

$$\left(P+x\frac{\partial P}{\partial x}+y\frac{\partial Q}{\partial x}+z\frac{\partial R}{\partial x}\right)dx+\left(x\frac{\partial P}{\partial y}+Q+y\frac{\partial Q}{\partial y}+z\frac{\partial R}{\partial y}\right)dy+\left(x\frac{\partial P}{\partial z}+y\frac{\partial Q}{\partial z}+R+z\frac{\partial R}{\partial z}\right)dz=0 \quad ....(2)$$

Since $Pdx+Qdy+Rdz=0$ is exact, we have

$\partial P/\partial y=\partial Q/\partial x,$ $\partial Q/\partial z=\partial R/\partial y$ and $\partial R/\partial x=\partial P/\partial z.$ ...(3)

Using relation (3), (2) may be re–written as

$$\left(P+x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}+z\frac{\partial P}{\partial z}\right)dx+\left(Q+x\frac{\partial Q}{\partial x}+y\frac{\partial Q}{\partial y}+z\frac{\partial Q}{\partial z}\right)dy+\left(R+x\frac{\partial R}{\partial x}+y\frac{\partial R}{\partial y}+z\frac{\partial R}{\partial z}\right)dz=0 \quad ...(4)$$

Since $Pdx+Qdy+Rdz=0$ is homogeneous of degree n, it follows that P, Q and R are all homogeneous functions of degree n. Using Euler's theorem on homogeneous functions $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ of degree n, we get

$$\left.\begin{aligned} x(\partial P/\partial x) + y(\partial P/\partial y) + z(\partial P/\partial z) &= nP, \\ x(\partial Q/\partial x) + y(\partial Q/\partial y) + z(\partial Q/\partial z) &= nQ \\ \text{and} \quad x(\partial R/\partial x) + y(\partial R/\partial y) + z(\partial R/\partial z) &= nR. \end{aligned}\right\} \quad ...(5)$$

Using (5), (4) reduces to

$(P + nP)dx + (Q + nQ)dy + (R + nR)dz = 0$ or $(n + 1)(Pdx + Qdy + Rdz) = 0$

or $Pdx + Qdy + Rdz = 0$, as $n \neq -1$ so that $(n + 1) \neq 0$ ... (6)

which is given differential equation and hence (1) is solution of (6), as required.

ILLUSTRATIVE SOLVED EXAMPLE

Solve $(x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0$. **[Guwahati 1996]**

Sol. Given $(x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0$. ...(1)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = x - 3y - z$, $Q = 2y - 3x$ and $R = z - x$. ...(2)

(1) is homogeneous equation of degree $n = 1 \neq -1$. Also from (2), we get

$(\partial P/\partial y) = -3 = (\partial Q/\partial x)$, $(\partial Q/\partial z) = 0 = (\partial R/\partial y)$, $(\partial R/\partial x) = -1 = (\partial P/\partial z)$. ...(3)

(3) shows that (1) is exact. Thus, (1) is exact and homogeneous of degree $n = 1 \neq -1$. Hence, solution of (1) is given by

$xP + yQ + zR = c$ or $x(x - 3y - z) + y(2y - 3x) + z(z - x) = c$

or $x^2 + 2y^2 + z^2 - 6xy - 2xz = c$, c being an arbitrary constant.

3.15. The non–integrable single equation

Suppose that the condition of integrability is not satisfied by the equation

$$Pdx + Qdy + Rdz = 0. \quad ...(1)$$

Then (1) represents a family of curves orthogonal to the family represented by the equations

$$dx/P = dy/Q = dz/R. \quad ...(2)$$

However, in the present case there exists no family of surfaces orthogonal to the second family of curves. In such cases we can find an infinite number of curves that lie on any given surface and satisfy (1). However, note that the method of finding the above mentioned infinite number of curves is equally applicable to integrable equation (1).

3.16. Working Rule for finding the curves represented by the solution of non–integrable total differential equation

$$Pdx + Qdy + Rdz = 0 \quad ...(1)$$

which lie on given surface $f(x, y, z) = c$. ....(2)

Step 1. Verify that (1) is not integrable.

Step 2. Differentiate (2) and then eliminate z and dz from the equation so obtained with help of (1) and (2).

Step 3. Integrate the equation obtained in step 2. This equation will involve x and y only. The resulting solution and (2) together represent the desired curves.

3.17. Solved Examples based on working rule 3.16.

Ex. 1. *Find the curves represented by the solution of* $y\,dx + (z - y)\,dy + x\,dz = 0$, *which lie in the plane* $2x - y - z = 1$. **[Allhabad 2006; Bangalore 2003, 07]**

Sol. Given $y\,dx + (z - y)\,dy + x\,dz = 0$ ...(1)

and $2x - y - z = 1$. ...(2)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, here $P = y$, $Q = z - y$ and $R = x$ ...(3)

Using (3), $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= y(1 - 0) + (z - y)(1 - 0) + x(1 - 0) = z + x \neq 0$.

This shows that the condition of integrability is not satisfied by (1).

Differentiating (2), $2dx - dy - dz = 0$ or $dz = 2dx - dy$. ...(4)

Using (4) to eliminate dz, (1) gives $y\,dx + (z - y)\,dy + x(2dx - dy) = 0$

or $ydx + (2x - y - 1 - y)dy + x(2dx - dy) = 0$, using (2) to eliminate z

or $(y + 2x)dx + (x - 2y - 1)dy = 0$ or $(ydx + xdy) + 2xdx - 2ydy - dy = 0$.

Integrating, $xy + x^2 - y^2 - y = c$, c being an arbitrary constant. ...(5)

The required curves are given by the intersection of plane (2) and rectangular hyperbolic cylinders (5).

Ex. 2. *Show that there is no single integral of $dz = 2ydx + xdy$. Prove that the curves of this equation that lie in the plane $z = x + y$ lie also on surfaces of the family $(x - 1)^2(2y - 1) = c$.*

[Agra 2001, 03, 07; Meerut 2006]

Sol. Given $2ydx + xdy - dz = 0$ ...(1)

and $z = x + y$. ...(2)

As in Ex. 1, show that the condition of integrability is not satisfied

Differentiating (2), $dz = dx + dy$. ...(3)

Using (3), (1) gives $2ydx + xdy - (dx + dy) = 0$

or $(2y - 1)dx + (x - 1)dy = 0$ ...(3)

or $\{2/(x - 1)\}dx + \{2/(2y - 1)\}\,dy = 0$

Integrating, $2 \log (x - 1) + \log (2y - 1) = \log c$ or $(x - 1)^2(2y - 1) = c$.

Ex. 3. *Show that the curves of* $xdx + ydy + c\sqrt{(1 - x^2/a^2 - y^2/b^2)}\,dz = 0$ ...(1)

that lie on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ ...(2)

lie also on the family of concentric spheres $x^2 + y^2 + z^2 = a^2$.

Sol. Verify that (1) does not satisfy the condition of integrability. Form (2), $1 - x^2/b^2 - y^2/b^2 = z^2/c^2$. Using this, (1) may be re–written as

$$2xdx + 2ydy + 2zdz = 0$$

Integrating, $x^2 + y^2 + z^2 = a^2$, a being an arbitrary constant. ...(3)

Hence the curves of (1) that lie on (2) also lie on the family of concentric spheres (3).

Ex. 4. *Find the orthogonal projection on the plane of xz of curve which lie on the paraboloid $3z = x^2 + y^2$ and satisfy the equation $2dz = (x + z)dx + y\,dy$.* **[Kurukshetra 1997]**

Sol. Given $(x + z)dx + y\,dy - 2dz = 0$ ...(1)

and $x^2 + y^2 - 3z = 0$. ...(2)

Verify yourself that (1) does not satisfy the usual condition of integrability. Differentiating (2), we get $2x\,dx + 2y\,dy - 3dz = 0$. ...(3)

Multiplying (1) by 3 and (3) by 2 and then subtracting the resulting equations, we get

$$3(x + z)dx + 3y\,dy - 3(2x\,dx + 2y\,dy) = 0$$

or $-x\,dx - y\,dy + 3z\,dx = 0$ or $x\,dx + y\,dy = (x^2 + y^2)dx$, using (2)

or $\dfrac{2xdx + 2ydy}{x^2 + y^2} = 2dx$ so that $\log (x^2 + y^2) - \log c = 2x$

or $x^2 + y^2 = ce^{2x}$ or $3z = ce^{2x}$, by (2) ...(4)

which gives the required projection on xy plane.

EXERCISE 3 (D)

1. Obtain the system of the curves lying on the system of surfaces $zx = c$ and satisfying the differential equation $yz\,dx + z^2dy + y(z + x)dz = 0$. **Ans.** $x = c/y$, $zx = c$

2. Show that the differential equation $3ydx + (z - 3y)dy + xdz = 0$ is not integrable.

Prove that the projection on the plane of xy of the curves that satisfy the equation and lie on the plane $2x + y - z = a$ are the rectangular hyperbolas $x^2 + 3xy - y^2 - ay = b$.

[Meerut 2003, 05, 06; Allahabad 2001, 07; Lucknow 2003, 05, 06]

3. Find the equation of the cylinder, with generators parallel to the axis of y, passing through the point (2, 1, –1), and also through a curve that lies on the sphere $x^2 + y^2 + z^2 = 4$ and satisfies the equation $(xy + 2xz)dz + y^2dy + (x^2 + yz)dz = 0$.

3.18. Geometrical Interpretation of Pdx + Qdy + Rdz = 0

[G.N.D.U. (Amritsar) 1997; Lucknow 2003]

The given differential equation expresses that the tangent to a curve is perpendicular to a certain line, the direction cosines of this tangent line and another line being proportional to dx, dy, dz and P, Q, R respectively. Suppose that the equation

$$Pdx + Qdy + Rdz = 0. \qquad ...(1)$$

satisfies the condition of integrability and that its solution is

$$F(x, y, z) = c. \qquad ...(2)$$

Since (2) has one arbitrary constant, it represents a single infinity of surfaces. Choosing this constant in an appropriate manner, (2) can be made to pass through any given point of space. If a point is moving upon this surface in any direction, its co–ordinates and direction cosines of its path at any moment (which are proportional to dx, dy, dz since the point moves along tangent) must satisfy (1), since (2) is the integral of (1). Again for each point (x, y, z) there will be an infinite number of values of dx, dy, dz which will satisfy (1). Thus it follows that a point which is moving in such a manner that its co–ordinates and the direction cosines of its path always satisfy (1) can pass through any point in an infinity of directions. However, while passing through any point, it must remain on the particular surface given by (2) which passes through the point. Thus, infinite number of such possible curves which it can describe through that point must lie on that surface.

3.19. To show that the locus of Pdx + Qdy + Rdz = 0 is orthogonal to the locus of (dx)/P = (dy)/Q = (dz)/R.

The equation $$Pdx + Qdy + Rdz = 0 \qquad ...(1)$$

means, geometrically, that a straight line whose direction cosines are proportional to dx, dy, dz, is perpendicular to a line whose direction cosines are proportional to P, Q, R. As a consequence a point which satisfies (1) must move in a direction at right angles to a line whose direction cosines are proportional to P, Q, R. On the other hand, the equations

$$(dx)/P = (dy)/Q = (dz)/R. \qquad ...(2)$$

mean, geometrically, that a straight line whose direction cosines are proportional to dx, dy, dz is parallel to a line whose direction cosines are proportional to P, Q, R. As a consequence, a point which satisfies (2) must move in a direction parallel to line whose direction cosines are proportional to P, Q, R.

From the above discussion it follows that the curves traced out by the points that are moving according to the condition (1) are orthogonal to the curves traced out by the points that are moving according to the conditions (2). The former curves are any of the curves upon the surfaces given by (1). Thus, geometrically, the curves represented by (2) are normal to the surfaces represented by (1). In case (1) is not integrable, there cannot exist a family of surfaces which is orthogonal to all lines that form the locus of (2).

3.20. Total differential equations containing more that three variables.

The total differential equation in four variables is

$$Pdx + Qdy + Rdz + Tdt = 0, \qquad ...(1)$$

where P, Q, R and T are the functions of x, y, z and t.

Now (1) is integrable if it is integrable in any three variables also. Accordingly, the condition of integrability of (1) can be obtained by treating t, x, y, z as constants by turn.

Let t be regarded as constant so that $dt = 0$. Then (1) gives

$$Pdx + Qdy + Rdz = 0. \qquad ...(2)$$

The condition of integrability of (2) is

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \quad ...(3)$$

Similarly, taking x, y and z as constants successively, the conditions of integrability are

$$Q\left(\frac{\partial R}{\partial t}-\frac{\partial T}{\partial z}\right)+R\left(\frac{\partial T}{\partial y}-\frac{\partial Q}{\partial t}\right)+T\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=0, \quad ...(4)$$

$$R\left(\frac{\partial P}{\partial t}-\frac{\partial T}{\partial x}\right)+P\left(\frac{\partial T}{\partial z}-\frac{\partial R}{\partial t}\right)+T\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=0 \quad ...(5)$$

and

$$P\left(\frac{\partial Q}{\partial t}-\frac{\partial T}{\partial y}\right)+Q\left(\frac{\partial T}{\partial x}-\frac{\partial P}{\partial t}\right)+T\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \quad ...(6)$$

Hence (1) is integrable if conditions (3), (4), (5) and (6) are satisfied.

Remark 1. Proceeding as above the conditions of integrability can be derived for the total differential equations containing more than four variables.

Remark 2. Multiplying equation (3) by T, (4) by $(-P)$, (5) by Q and (6) by $(-R)$ and adding, we have $T \times (3) - P \times (4) + Q \times (5) - R \times (6) = 0$,
showing that the conditions (3) to (6) are not independent. Thus any three of the conditions (3) to (6) give the conditions of integrability of (1).

Remark 3. If given total differential equation contains n indepdent variables, there will be $(1/2) \times (n-1)(n-2)$ independent conditions for its integrability.

Methods of solving (1) : First of all satisfy any of three conditions (3) to (6) and hence declare that (1) is integrable.

Now proceed by any of the following three methods to solve (1) :

Method I. Solution by inspection. This method is the same as discussed in Art. 3.6 for three variables. Read that article carefully.

Method II. Homogeneous equation. Let P, Q, R and T be functions of x, y, z and t of the same degree.

Put $x = ut$, $y = vt$, $z = wt$ in (1) and then integrate the transformed equation as in the case of three variables (Refer Art. 3.8.)

Alternative method for solving homogeneous equation

If $(Px + Qy + Rz + Tt) \neq 0$, then we may make use of the integrating factor (I.F) given by I.F. $= 1/(Px + Qy + Rz + Tt)$. Procedure is same as explained in Art. 3.8.

Method III. Treating two variables as constants.

This method is merely an extension of the method adopted in Art. 3.12 for three variables. Take two suitable variables, z and t (say) as constants. Then $dz = 0$ and $dt = 0$. Now integrate the reduced equation and take the constant of integration as the function of the variables which were kept constant. Thus, for the above assumption we take $f(z, t)$ as constant of integration. Now proceed as discussed in Art. 3.12.

To understand the above methods, read carefully the following examples.

3.21. Solved examples based on Art. 3.20.

Ex. 1. *Solve* : $z(y + z)dx + z(u - x)dy + y(x - u)dy + y(y + z)du = 0$.

Sol. Re–writing the given equation, $(y + z)(zdx + ydu) + z(u - x)dy + y(x - u)dz = 0$

or $(y + z)(zdx + ydu) + (y + z)(xdz + udy) - (y + z)(xdz + udy) + z(u - x)dy + y(x - u)dz = 0$

or $(y + z)(zdx + ydu + xdz + udy) - (xz + uy)(dy + dz) = 0$, by simplification

or $$\frac{zdx + ydu + xdz + udy}{xz + uy} = \frac{dy + dz}{y + z}. \quad ...(1)$$

Integrating (1), $\log(xz + uy) = \log(y + z) + \log c$ or $xz + uy = c(y + z)$.

Ex. 2. *Solve :* $(2x + y^2 + 2xz)(dx/dt) + 2xy(dy/dt) + x^2(dz/dt) = 1$.

Sol. Re–writing the given equation, $(2x + y^2 + 2xz)dx + 2xydy + x^2dz - dt = 0$

or $2xdx + (y^2dx + 2xydy) + (2xzdx + x^2dz) - dt = 0$ or $d(x^2) + d(xy^2) + d(x^2z) - dt = 0$.

Integrating, $x^2 + xy^2 + x^2z - t = c$, c being an arbitrary constant.

Ex. 3. Solve $t(y + z)dx + t(y + z + 1)dy + tdz - (y + z)dt = 0$.

Sol. Given $t(y + z)dx + t(y + z + 1)dy + tdz - (y + z)dt = 0$. ...(1)

Comparing (1) with $Pdx + Qdy + Rdz + Tdt = 0$, we have

$P = t(y + z)$, $Q = t(y + z + 1)$, $R = t$, $T = -(y + z)$. ...(2)

Now, $$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= t(y + z)(t - 0) + t(y + z + 1)(0 - t) + t(t - 0) = 0. \quad ...(3)$$

Similarly, we can prove that

$$P\left(\frac{\partial Q}{\partial t} - \frac{\partial T}{\partial y}\right) + Q\left(\frac{\partial T}{\partial x} - \frac{\partial P}{\partial t}\right) + T\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad ...(4)$$

$$R\left(\frac{\partial P}{\partial t} - \frac{\partial T}{\partial x}\right) + P\left(\frac{\partial T}{\partial z} - \frac{\partial R}{\partial t}\right) + T\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = 0 \quad ...(5)$$

and $$Q\left(\frac{\partial R}{\partial t} - \frac{\partial T}{\partial z}\right) + R\left(\frac{\partial T}{\partial y} - \frac{\partial Q}{\partial t}\right) + T\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = 0. \quad ...(6)$$

Hence (1) satisfies the conditions of integrability (3), (4), (5) and (6). Therefore, (1) must be integrable. We now solve (1) by the following two methods :

Method I. (By inspection). Re–writing (1), we have

$(y + z)dx + (y + z + 1)dy + dz - (y + z)(1/t)dt = 0$ or $(y + z)(dx + dy) + (dy + dz) - (y + z)(1/t)dt = 0$

or $$dx + dy + \frac{dy + dz}{y + z} - \frac{dt}{t} = 0.$$

Integrating, $x + y + \log(y + z) - \log t = \log c$, c being arbitrary constant.

or $\log\left(\frac{y + z}{ct}\right) = -(x + y)$ or $y + z = cte^{-(x + y)}$ or $(y + z)e^{x + y} = ct$.

Method II. (Regarding two variables as constants)

Treat z and t as constants so that $dz = 0$ and $dt = 0$. Then (1) reduces to

$$t(y + z)dx + t(y + z + 1)dy = 0 \quad \text{or} \quad dx + \left(1 + \frac{1}{y + z}\right)dy = 0. \quad ...(7)$$

Integrating (7), $x + y + \log(y + z) = f(z, t)$, say ...(8)

Differentiating (8), we have $$dx + dy + \frac{dy + dz}{y + z} = df$$

or $(y + z)dx + (y + z)dy + dy + dz - (y + z)df = 0$

or $t(y + z)dx + t(y + z + 1)dy + tdz - (y + z)tdf = 0$. ...(9)

Comparing (9) with (1), $-(y + z)t\,df = -(y + z)dt$ or $df = (1/t)dt$. ...(10)

Integraing (10), $f = \log t + \log c$ or $f = tc$.

or $x + y + \log(y + z) = \log(tc)$, using (8)

or $\log\left(\frac{y + z}{tc}\right) = -(x + y)$ or $\frac{y + z}{tc} = e^{-(x + y)}$ or $(y + z)e^{x + y} = tc$.

Ex. 4. *Solve :* $z(y+z)dx + z(t-x)dy + y(x-t)dz + y(y+z)dt = 0$.

Sol. Given $z(y+z)dx + z(t-x)dy + y(x-t)dz + y(y+z)dt = 0$. ...(1)

Comparing (1) with $Pdx + Qdy + Rdz + Tt = 0$, we have

$P = z(y+z)$, $Q = z(t-x)$, $R = y(x-t)$, $T = y(y+z)$. ...(2)

As in Ex. 3, we can show that the conditions of integrability are satisfied.

We now solve (1) by the following two methods.

Method I. (Regardings two variables as constants)

Treat y and z as constants so that $dy = 0$ and $dz = 0$. Then (1) reduces to

$z(y+z)dx + y(y+z)dt = 0$ or $zdx + ydt = 0$. ...(3)

Integrating (3), $zx + yt = f(y, z)$, (say). ...(4)

Differentiating (1), $zdx + tdy + xdz + ydt = df$ or $(y+z)(zdx + tdy + xdz + ydt) = (y+z)df$

or $z(y+z)dx + t(y+z)dy + x(y+z)dz + y(y+z)dt = (y+z)df$. ...(5)

Comparing (5) with (1), we have

$t(y+z)dy + x(y+z)dz - (y+z)df = z(t-x)dy + y(x-t)dz$.

or $(ty + xz)dy + (ty + xz)dz = (y+z)df$

or $(ty + xz)(dy + dz) = (y+z)df$ or $f \times (dy + dz) = (y+z)df$, using (4)

or $(1/f)df = (dy + dz)/(y+z)$. ...(6)

Integrating (6), $\log f = \log (y+z) + \log c$ or $f = c(y+z)$

or $zx + yt = c(y+z)$, using (4), c being an arbitrary constant.

Method II. (Homogeneous equation. Use of integrating factor)

Here P, Q, R and T are functions of x, y, z and t of the same degree 2.

Also, $D = Px + Qy + Rz + Tt = xz(y+z) + yz(t-x) + yz(x-t) + ty(y+z)$

$= xz(y+z) + ty(y+z) + yz(t - x + x - t) = (y+z)(xz + ty) \neq 0$.

$\therefore$ Integrating factor = I.F. $= \dfrac{1}{D} = \dfrac{1}{(y+z)(xz+ty)}$. ...(3)′

Multiplying the given equation (1) by I.F., we have

$$\frac{y(y+z)dx + x(t-x)dy + y(x-t)dz + y(y+z)dt}{(y+z)(xz+ty)} = 0. \quad \text{...(4)}'$$

Now, the numerator of (4)′

$= [(y+z)(zdx + xdz + ydt + tdy) + (xz + ty)(dy + dz)] + z(t-x)dy + y(x-t)dz$

$- (y+z)(xdz + tdy) - (xz + ty)(dy + dz)$

$= d\{(y+z)(xz+ty)\} - 2(xz+ty)(dy+dz)$.

Using the above value of numerator of (4)′, (4)′ becomes

$$\frac{d\{(y+z)(xz+ty)\} - 2(xz+ty)(dy+dz)}{(y+z)(xz+ty)} = 0 \quad \text{or} \quad \frac{d[(y+z)(xz+ty)]}{(y+z)(xz+ty)} - 2\frac{dy+dz}{y+z} = 0 \quad \text{....(5)}'$$

Integrating (5)′, $\log \{(y+z)(xz+ty)\} - 2 \log (y+z) = \log c$

or $\log \dfrac{(y+z)(xz+ty)}{(y+z)^2} = \log c$ or $xz + ty = c(y+z)$.

Ex. 5. *Show that the equation* $y \sin w\, dx + x \sin w\, dy - xy \sin w\, dz - xy \cos w\, dw = 0$ *satisfies the conditions of integrability and obtain its integral.* **[Meerut 1999]**

Sol. Given $y \sin w\, dx + x \sin w\, dy - xy \sin w\, dz - xy \cos w\, dw = 0$. ...(1)

Compare (1) with $Pdx + Qdy + Rdz + Wdw = 0$, we have

$P = y \sin w$, $Q = x \sin w$, $R = -xy \sin w$ and $W = -xy \cos w$. ...(2)

Here, $$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$$= y\sin w\,(0 + x\sin w) + x\sin w\,(-y\sin w - 0) + (-xy\sin w)(\sin w - \sin w) = 0. \quad ...(3)$$

Similarly, we can prove that

$$P\left(\frac{\partial Q}{\partial w}-\frac{\partial W}{\partial y}\right)+Q\left(\frac{\partial W}{\partial x}-\frac{\partial P}{\partial w}\right)+W\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0, \quad ...(4)$$

$$R\left(\frac{\partial P}{\partial w}-\frac{\partial W}{\partial x}\right)+P\left(\frac{\partial W}{\partial z}-\frac{\partial R}{\partial w}\right)+W\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=0 \quad ...(5)$$

and $$Q\left(\frac{\partial R}{\partial w}-\frac{\partial W}{\partial z}\right)+R\left(\frac{\partial W}{\partial y}-\frac{\partial Q}{\partial w}\right)+W\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=0 \quad ...(6)$$

Hence (1) satisfies the conditions of integrability (3), (4), (5) and (6). Hence (1) is integrable.

Treat z and w as constants so that $dz = 0$ and $dw = 0$. Then (1) reduces to

$y \sin w\, dx + x \sin w\, dy = 0$ or $ydx + xdy = 0.$ ...(7)

Integrating (7), $xy = f(z, w)$, (say). ...(8)

Differentiating (8), we have $y\,dx + x\,dy = df.$

or $y \sin w\, dx + x \sin w\, dy - \sin w\, df = 0.$ ...(9)

Comparing (9) with (1), $\sin w\, df = xy \sin w\, dz + xy \cos w\, dw$

or $df = xy(dz + \cot w\, dw)$ or $df = f \times (dz + \cot w\, dw)$, using (8)

or $(1/f)df = dz + \cot w\, dw.$ ...(10)

Integrating, $\log f = z + \log \sin w + \log c$, c being an arbitrary constant.

or $f = c \sin w\, e^z$ or $xy = c \sin w\, e^z$, using (8).

3.22. Working rule (based on Art. 3.3) for solving $Pdx + Qdy + Rdz = 0$ ...(i)

Step 1: Verify that (*i*) satisfies the condition of integrability (8) of Art. 3.3

Step 2: Let z = constant so that $dz = 0$. Then, (*i*) yields $Pdx + Qdy = 0$ ...(*ii*)

Step 3: Let solution of (*ii*) be $f(x, y, z) = V$ = constant ...(*iii*)

Step 4: From (*ii*) and (*iii*), $\lambda P = \partial V / \partial x$ and $\lambda Q = \partial V / \partial y$. Determine λ.

Step 5: $$\lambda P\,dx + \lambda Q\,dy + \lambda R\,dz = (\partial V/\partial x)\,dx + (\partial V/\partial y)\,dy + (\partial V/\partial z)\,dz + (\lambda R - \partial V/\partial z)\,dz$$

$$\Rightarrow \quad \lambda P\,dx + \lambda Q\,dy + \lambda R\,dz = dV + S\,dz, \quad \text{where} \quad S = \lambda R - (\partial V/\partial z) \quad ...(iv)$$

Compute S in terms of z and V.

Step 6: Given equation (*i*) reduces to $dV + S\,dz = 0$ ...(*v*)

Substitute the value of S (obtained in step 5) in (*v*) and then integrate it. Finally, using (*iii*), we obtain the required solution.

Remark. Select a proper variable to be constant so that the resulting equation is easily integrable. Accordingly, necessary corresponding changes must be done in the above working rule. The following example will make this point clear.

Solved example : *Solve the following partial differential equation: zy(1 + 4xz)dx – xz(1 + 2xz) dy – xy dz = 0* **[Delhi B.A. (Prog) II 2010; Delhi Maths (H) 2008, 09]**

Solution. Comparing the given equation with $P\,dx + Q\,dy + R\,dz = 0$, we have

$P = zy\,(1 + 4xz)$, $Q = -xz\,(1 + 2xz)$ and $R = -xy$ ...(1)

Hence, $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= zy\,(1 + 4xz)\,(-x - 4x^2z + x) - xz(1+ 2xz)\,(-y - y - 8xyz) - xy\,(z + 4xz^2 + z + 4xz^2) = 0$

Thus, the given equation is integrable. Let x = constant. Then, the given equation yields

$$-xz\,(1 + 2xz)\,dy - xydz = 0 \quad \text{or} \quad \frac{dy}{y} + \frac{dz}{z(1+2xz)} = 0 \quad \text{or} \quad \frac{dy}{y} + \left(\frac{1}{z} - \frac{2x}{1+2xz}\right)dz = 0$$

Integrating, $\log y + \log z - \log(1 + 2xz) = \log c$ or $(yz)/(1 + 2xz) = c$ = constant

Let us take $V = (yz)/(1 + 2xz)$ ... (2)

Now, $\lambda Q = \partial V/\partial y \quad \Rightarrow \quad -\lambda xz\,(1+2xz) = z/(1 + 2xz)$, using (1) and (2)

Thus, $\lambda = -\{1/x(1+2xz)^2\}$ ... (3)

Hence, $S = \lambda P - \dfrac{\partial V}{\partial x} = -\dfrac{zy(1+4xz)}{x(1+2xz)^2} + \dfrac{2z^2y}{(1+2xz)^2}$, using (1), (2) and (3)

or $S = -\dfrac{yz(1+4xz-2xz)}{x(1+2xz)^2} = -\dfrac{yz(1+2xz)}{x(1+2xz)^2} = -\dfrac{yz}{x(1+2xz)} = -\dfrac{V}{x}$, using (2) …(4)

Now, the given equation reduces to $dV + S\,dx = 0$ or $dV - (V/x)dx = 0$, by (4)

or $(1/V)\,dV = (1/x)\,dx$ so that $\log V = \log x + \log A$ or $V = xA$ or $(yz)/(1+2xz) = xA$, which is the required solution, A being an arbitrary constant

Note: The reader can solve problems of Art 3.13 and Exercise 3 (*C*) by using the above working rule of Art. 3.22.

Miscellaneous Problems on Chapter 3

1. Solve $yz(1+x)dx + z\,xy\,(1+y)dy + xy\,(1+z)dz = 0$ **[Delhi B.A. (Prog). II 2010]**

Ans. $x + y + z + y^2/2 + \log(xz) = c.$

2. State true or false with justification : If curl $(P, Q, R) = 0$, then $Pdx + Qdy + Rdz = 0$ is exact. **[Pune 2010]**

Hints. Refer theorem on page 3.3.

4

Riccati's Equation

4.1. Introduction. A differential equation of the form

$$dy/dx = P + Qy + Ry^2 \qquad i.e. \qquad y_1 = P + Qy + Ry^2, \quad ...(1)$$

where P, Q and R are functions of x, is known as the generalized Riccati's equation. Throughout this chapter suffixes will denote differentiation with respect to x. In 1841, the French mathematician Liouville proved that (1) is one of the simplest differential equations of the first order and first degree that cannot, in general, be integrated by quadratures. Due to historical and theoretical importance and occurrence of (1) in appalications, the study of certain aspect of (1) becomes quite useful.

4.2. General solution of Riccati's equation namely,

$$y_1 = P + Qy + Ry^2. \quad ...(1)$$

We introduce another dependent variable u such that

$$y = -u_1/(Ru) = -u_1(Ru)^{-1}. \quad ...(2)$$

Differentiating (2) w.r.t 'x', $y_1 = -u_2(Ru)^{-1} + u_1(Ru)^{-2}[R_1u + Ru_1]$. ...(3)

Using (2) and (3), (1) becomes

$$-\frac{u_2}{Ru} + \frac{R_1u_1}{R^2u} + \frac{u_1^2}{Ru^2} = P + Q\left(-\frac{u_1}{Ru}\right) + R\left(\frac{u_1^2}{R^2u^2}\right)$$

or $$-Ru_2 + R_1u_1 = PR^2u - Qu_1R$$

or $$Ru_2 - (QR + R_1)u_1 + PR^2u = 0, \quad ...(4)$$

which is linear differential equation of the second order. We know that the general solution of (4) is of the form

$$u = Af(x) + BF(x), \text{ where } A \text{ and } B \text{ are arbitrary constants.} \quad ...(5)$$

$\therefore$ (2) gives $$y = -\frac{Af_1 + BF_1}{R(Af + BF)} = \frac{(A/B)(-f_1/R) + (-F_1/R)}{(A/B)f + F}$$

which is of the form $$y = \frac{cg(x) + G(x)}{cf(x) + F(x)} \quad ...(6)$$

where $c = (A/B)$ is an arbitrary constant. Hence the general solution of (1) is of the form (6).

SOLVED EXAMPLES BASED ON ART. 4.2

Ex. 1. *Solve* (*i*) $y_1 = -2 - 5y - 2y^2$ (*ii*) $x^2y_1 + 2 - 2xy + x^2y^2 = 0$.

Sol. (*i*) Given $y_1 = -2 - 5y - 2y^2$. ... (1)

Comparing (1) with $y_1 = P + Qy + Ry^2$, here $R = -2$ and so we take

$$y = -u_1/(Ru) = -u_1/(-2u) = u_1(2u)^{-1}. \quad ...(2)$$

Differentiating (2) w.r.t. 'x', $y_1 = u_2(2u)^{-1} - u_1(2u)^{-2}.(2u_1)$. ...(3)

Using (2) and (3) in (1), $$\frac{u_2}{2u} - \frac{u_1^2}{2u^2} = -2 - 5\left(\frac{u_1}{2u}\right) - 2\left(\frac{u_1}{2u}\right)^2$$

or $u_2 + 5u_1 + 4u = 0$ or $(D^2 + 5D + 4)u = 0$, ...(4)

where $D \equiv d/dx$. The usual auxiliary equation is $D^2 + 5D + 4 = 0$, giving $D = -4, -1$ and hence solution of (4) is

$$u = Ae^{-x} + Be^{-4x} \quad \text{so that} \quad u_1 = -Ae^{-x} - 4Be^{-4x}. \quad ...(5)$$

Putting these values of u and u_1 in (2), the required general solution of (1) is

$$y = \frac{u_1}{2u} = -\frac{-Ae^{-x} - 4Be^{-4x}}{2(Ae^{-x} + Be^{-4x})} = -\frac{(A/B)e^{3x} + 4}{2[(A/B)e^{3x} + 1]}$$

or $2y(1 + ce^{3x}) = -(4 + ce^{3x})$, where $c(= A/B)$ is an arbitrary constant.

Part (*ii*) Put the equation in standard form by dividing by x^2 and get $y_1 = -(2/x^2) + (2y/x) - y^2$. Now proceed as above, **Ans.** $y(cx + x^2) = c + 2x$.

4.3. Theorem. *The cross–ratio of any four particular integrals of a Riccati's equation is independent of x.*

Proof. We know that the general solution of Riccati's equation

$$y_1 = P + Qy + Ry^2 \quad ...(1)$$

is of the form $y = (cg + G)/(cf + F)$. ...(2)

where g, G, f, F are appropriate functions of x and c is an arbitrary constant. Let $p(x)$, $q(x)$, $r(x)$, $s(x)$ be any four integrals of (1). Then these can be obtained from (2) by giving suitable values to c, say α, β, γ and δ. Thus, we have

$$p = \frac{\alpha g + G}{\alpha f + F}, \quad q = \frac{\beta g + G}{\beta f + F}, \quad r = \frac{\gamma g + G}{\gamma f + F}, \quad s = \frac{\delta g + G}{\delta f + F}$$

Then, $p - q = \dfrac{(\alpha - \beta)(gF - fG)}{(\alpha f + F)(\beta f + F)}$, on simplifying.

Similarly $(p - s)$, $(r - s)$, $(r - q)$ can be obtained. Hence the cross–ratio

$$\frac{(p - q)(r - s)}{(p - s)(r - q)} = \frac{(\alpha - \beta)(\gamma - \delta)}{(\alpha - \delta)(\gamma - \beta)} = k, \text{ say,}$$

where k is indepedent of x. While forming the above cross–ratio, all the factors involving x cancel out and we are left with a mere constant. This completes the proof.

4.4. Method of solving Riccati's equation when three particular integrals are known.

Let the three particular integrals be $q(x), r(x), s(x)$ and let the corresponding values of c be β, γ, δ. Then as explained in Art. 4.3, we have

$$y = \frac{cg + G}{cf + F}, \quad q = \frac{\beta g + G}{\beta f + F}, \quad r = \frac{\gamma g + G}{\gamma f + F}, \quad s = \frac{\delta g + G}{\delta f + F}$$

and then $$\frac{(y - q)(r - s)}{(y - s)(r - q)} = \frac{(c - \beta)(\gamma - \delta)}{(c - \delta)(\gamma - \beta)} = k, \text{ say}$$

where k is independent of x. Thus, the general solution of Riccati's eqn. $y_1 = P + Qy + Ry^2$ is given by

$$\frac{(y - q)(r - s)}{(y - s)(r - q)} = k, \quad ...(1)$$

where k is an arbitrary constant. It should be noted that in the present situation solution has been obtained without quadratures (*i.e.* integrations)

An illustration : *Show that 1, x, x^2 are three integrals of $x(x^2 - 1)y_1 + x^2 - (x^2 - 1)y - y^2 = 0$, and hence obtain the general solution $y(x + k) = x + kx^2$.*

Solution. Re–writing the given eqn. in standard form, we have

$$y_1 = -\frac{x}{x^2-1} + \frac{1}{x}y + \frac{1}{x(x^2-1)}y^2. \qquad ...(1)'$$

Putting $y = 1$ so that $y_1 = 0$ in (1)′, we have

$$0 = -\frac{x}{x^2-1} + \frac{1}{x}\cdot 1 + \frac{1}{x(x^2-1)}\cdot 1 = 0,$$

showing that 1 is an integral of (1)′. Similarly we see that x and x^2 are also integrals of (1). Take $q(x) = 1$, $r(x) = x$ and $s(x) = x^2$. Then using formula (1) of above method of Art. 4.4, the general solution is

$$\frac{(y-1)(x-x^2)}{(y-x^2)(x-1)} = k \qquad \text{or} \qquad \frac{(y-1)(-x)}{(y-x^2)} = k$$

or $k(y - x^2) + x(y - 1) = 0$ or $y(k + x) = x + kx^2$,

where k is an arbitrary constant.

4.5. Method of solving Riccati's equation when two particular integrals are known.

Let $q(x)$ and $r(x)$ be two known integrals of the Riccati's equation

$$y_1 = P + Qy + Ry^2 \qquad ...(1)$$

so that $$q_1 = P + Qq + Rq^2 \qquad ...(2)$$

and $$r_1 = P + Qr + Rr^2. \qquad ...(3)$$

From (1) and (2), we get

$y_1 - q_1 = (y - q)Q + (y^2 - q^2)R$ or $y_1 - q_1 = (y - q)\{Q + (y + q)R\}$

or $$(y_1 - q_1)/(y - q) = Q + (y + q)R. \qquad ...(4)$$

Similarly, $$(y_1 - r_1)/(y - r) = Q + (y + r)R. \qquad ...(5)$$

From (4) and (5), we get $$\frac{y_1 - q_1}{y - q} - \frac{y_1 - r_1}{y - r} = (q - r)R$$

Integrating this, we get $$\log (y - q) - \log (y - r) = c + \int (q - r)R \; dx$$

or $$\log [(y - q)/(y - r)] = c + \int (q - r)R \; dx, \qquad ...(6)$$

which is the required general solution and it requires only one integration.

An illustration. *Show that there are two values of the constant k for which k/x is an integral of* $x^2(y_1 + y^2) = 2$, *and hence obtain the general solution.*

Solution. Re–writing the given equation in standard form

$$y_1 = P + Qy + Ry^2 \qquad ...(1)$$

we get $$y_1 = (2/x^2) - y^2, \qquad ...(2)$$

Let $q(x)$ and $r(x)$ be two–particular integrals of (1), then as above we have (to be proved in examination for complete solution)

$$\log[(y - q)/(y - r)] = c + \int (q - r)R \; dx. \qquad ...(3)$$

Let $y(x) = k/x$ so that $y_1 = -k/x^2$. Putting these in (2), we get

$-k/x^2 = 2/x^2 - k^2/x^2$ or $k^2 - k - 2 = 0$ so that $k = 2, -1$.

Hence $2/x$ and $-1/x$ are two particular integrals of (2). We take

$q(x) = 2/x$ and $r(x) = -1/x.$ ...(4)

Comparing (1) and (2), here $R = -1$ ...(5)

Using (4) and (5), (3) gives the desired solution as

$$\log\frac{y-(2/x)}{y+(1/x)} = \log k + \int\left(\frac{2}{x}+\frac{1}{x}\right)(-1)\,dx, \text{ taking } c = \log k$$

or $$\log\frac{xy-2}{xy+1} = \log k - 3\log x \qquad \text{or} \qquad \left(\frac{xy-2}{xy+1}\right)x^3 = k$$

or $x^3(xy-2) = k(xy+1)$, k being arbitrary constant.

4.6. Method of solving Riccati's equation when one particular integral is known

Let $q(x)$ be the known integral of the given Riccati's equation

$$y_1 = P + Qy + Ry^2 \qquad ...(1)$$

so that $$q_1 = P + Qq + Rq^2. \qquad ...(2)$$

Let v be another dependent variable such that

$$y = q(x) + 1/v. \qquad ...(3)$$

$\therefore$ $$y_1 = q_1 - v_1/v^2 = P + Qq + Rq^2 - v_1/v^2, \text{ by (2)} \qquad ...(4)$$

Using (3) and (4), (1) becomes

$$P + Qq + Rq^2 - \frac{v_1}{v^2} = P + Q\left(q+\frac{1}{v}\right) + R\left(q^2 + \frac{2q}{v} + \frac{1}{v^2}\right)$$

or $$-\frac{v_1}{v} = \frac{Q}{v} + R\left(\frac{2q}{v}+\frac{1}{v^2}\right) \qquad \text{or} \qquad \frac{dv}{dx} + (Q+2Rq)v = -R, \qquad ...(5)$$

which is linear differential equation of first order and first degree in v and x. Its integrating factor I.F. is given by I.F. $= e^{\int (Q+2Rq)\,dx}$ and hence the required general solution is

$$v\,e^{\int (Q+2Rq)\,dx} = -\int \mathrm{Re}^{\int (Q+2Rq)\,dx} + c,$$

where c is an arbitrary constant and v is to be replaced by $1/(y-q)$ [by using (3)].

Remark. The above mentioned method is very useful in practice.

4.7 Solved examples based on Art. 4.6.

Ex. 1. *Solve* $$y_1 = \cos x - y\sin x + y^2. \qquad ...(1)$$

Sol. Take $y = \sin x$ so that $y_1 = \cos x$. Putting these in (1), we get

$\cos x = \cos x - \sin^2 x + \sin^2 x$, showing that $\sin x$ is a particular solution of (1).

Let $y = \sin x + 1/v$ so that $y_1 = \cos x - v_1/v^2$.

Using (2), (1) becomes

$$\cos x - v_1/v^2 = \cos x - \sin x\,(\sin x + 1/v) + (\sin x + 1/v)^2$$

or $$-\frac{v_1}{v^2} = \frac{\sin x}{v} + \frac{1}{v^2} \qquad \text{or} \qquad \frac{dv}{dx} = (\sin x)\,v = -1 \qquad ...(3)$$

whose I.F. $= e^{\int \sin x\,dx} = e^{-\cos x}$ and so solution of (3) is

$$ve^{-\cos x} = c - \int e^{-\cos x}\,dx. \qquad ...(4)$$

From (2), $v = 1/(y - \sin x)$ and hence (4) gives

$$\frac{e^{-\cos x}}{y - \sin x} = c - \int e^{-\cos x}\,dx$$

Ex. 2. *Solve* $y_1 = 2 + (1/2) \times (x - 1/x)y - y^2/2$ ...(1)

Sol. Take $y = x + 1/x$ so that $y_1 = 1 - 1/x^2$. Then (1) gives

$$1 - 1/x^2 = 2 + \tfrac{1}{2}(x - 1/x)(x + 1/x) - \tfrac{1}{2}(x + 1/x)^2$$

or $1 - 1/x^2 = 2 + \tfrac{1}{2}(x - 1/x^2) - \tfrac{1}{2}(x^2 + 2 + 1/x^2)$ or $1 - 1/x^2 = 1 - 1/x^2$.

Hence $x + 1/x$ is a particular integral of (1). So we take

$y = x + 1/x + 1/v$ so that $y_1 = 1 - 1/x^2 - v_1/v^2$. ...(2)

Using (2) in (1), we get

$$1 - \frac{1}{x^2} - \frac{v_1}{v^2} = 2 + \frac{1}{2}\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x} + \frac{1}{v}\right) - \frac{1}{2}\left(x + \frac{1}{x} + \frac{1}{v}\right)^2$$

or $$1 - \frac{1}{x^2} - \frac{1}{v^2} = 2 + \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) + \frac{1}{2v}\left(x - \frac{1}{x}\right) - \frac{1}{2}\left[\left(x + \frac{1}{x}\right)^2 + \frac{2}{v}\left(x + \frac{1}{x}\right) + \frac{1}{v^2}\right]$$

or $$\frac{dv}{dx} - \left(\frac{x}{2} + \frac{3}{2x}\right)v = \frac{1}{2}, \text{ on simplification.} \quad ...(3)$$

Its I.F. $= e^{-(x/2 + 3/2x)dx} = e^{-x^2/4 - (3/2)\log x} = x^{-3/2}\, e^{-x^2/4}$. Hence solution of (3) is

$$vx^{-3/2}\, e^{-x^2/4} = c + \int \frac{1}{2} x^{-3/2}\, e^{-x^2/4}\, dx. \quad ...(4)$$

From (2), $1/v = y - x - 1/x$ so that $v = 1/(xy - x^2 - 1)$. ...(5)

Hence from (4) and (5), the required solution is

$$(x^{-3/2}\, e^{-x^2/4})/(xy - x^2 - 1) = c + \frac{1}{2}\int x^{-3/2}\, e^{-x^2/4}\, dx,$$

EXERCISE

1. $y_1 = P(1 - xy) - y^2$, P is a function of x.

Hint. $1/x$ is a particular integral. The solution is $\{1/(x^2y - x)\}\, e^{-\int xP dx} = c - \int x^{-2}\, e^{-\int xP dx}\, dx$

2. $x(1 - x^3)y_1 = x^2 + y - 2xy^2$, x^2 is an integral. **Ans.** $(x^4 - x)/(y - x^2) = c - 2x^3/3$

3. $y_1 = 1 + y^2$, $\tan x$ is an integral. **Ans.** $y(c - \tan x) = c \tan x + 1$.

4. $y_1 = 2x - (x^2 + 1)y + y^2$, $x^2 + 1$ is an integral.

5. $x^3y_1 = x^2y + y^2 - x^2$, x is an integral. **Ans.** $y(ce^{2/x} - 1) = x + cx\, e^{2/x}$

6. $y_1 = x + y(1 - 2x) - y^2(1 + x)$, 1 is an integral. **Ans.** $y = 1 + 1/(x + ce^x)$

7. $x(x - 1)y_1 - (2x + 1)y + y^2 + 2x = 0$, x is a solution. **Ans.** $y = (x^2 + c)/(x + c)$

8. $y_1 = x + [(1/x) - x^3]y + xy^2$, x^2 is a solution. **Ans.** $y = x^2 - (xe^{x^2/4})/(c + \int x^2\, e^{x^2/4}\, dx$

9. $y_1 = e^{2x} - ye^x + e^{-x}y^2$, e^x is a solution. **Ans.** $y = e^x - (e^{2x - e^x}) \Big/ \left[c + \int e^{x - e^x}\, dx\right]$

5

Chebyshev Polynomials

5.1. Chebyshev Polynomials

The Chebyshev polynomials of first kind, $T_n(x)$, and second kind, $U_n(x)$ are defined by

$$T_n(x) = \cos(n \cos^{-1} x) \quad \ldots (1)$$

and

$$U_n(x) = \sin(n \cos^{-1} x) \quad \ldots (2)$$

where n is a non-negative integer.

Remark. Chebyshev polynomials are also known as Tchebicheff, Tchebieheff or Tschebysheff.

5.2. Theroem. *$T_n(x)$ and $U_n(x)$ are independent solutions of Chebyshev's equation*

$$(1-x^2)(d^2y/dx^2) - x(dy/dx) + n^2 y = 0. \quad \textbf{[Kanpur, 2005; Garhwal 2005]}$$

Proof. The Chebyshev's equation is $(1-x^2)(d^2y/dx^2) - x(dy/dx) + n^2 y = 0 \quad \ldots (1)$

To show that $T_n(x)$ is a solution of (1) : By definition, we have

$$T_n(x) = \cos(n \cos^{-1} x) \quad \ldots (2)$$

$$\therefore \quad \frac{d}{dx}T_n(x) = \frac{d}{dx}\cos(n\cos^{-1}x) = -\sin(n\cos^{-1}x)\cdot n \cdot \frac{-1}{(1-x^2)^{1/2}}$$

or

$$\frac{d}{dx}T_n(x) = \frac{n}{(1-x^2)^{1/2}}\sin(n\cos^{-1}x) \quad \ldots (3)$$

and

$$\frac{d^2}{dx^2}T_n(x) = \frac{d}{dx}\left(\frac{d}{dx}T_n(x)\right) = n\frac{d}{dx}[(1-x^2)^{-1/2}\sin(n\cos^{-1}x)]$$

$$= n\left[-\frac{1}{2}(1-x^2)^{-3/2}(-2x)\cdot\sin(n\cos^{-1}x) + (1-x^2)^{-1/2}\cos(n\cos^{-1}\cdot x)\cdot n\cdot\frac{1}{(1-x^2)^{1/2}}\right]$$

Thus,

$$\frac{d^2}{dx^2}T_n(x) = \frac{nx}{(1-x^2)^{3/2}}\sin(n\cos^{-1}x) - \frac{n^2}{1-x^2}\cos(n\cos^{-1}x). \quad \ldots (4)$$

Using (2), (3) and (4), we have

$$(1-x^2)\frac{d^2}{dx^2}T_n(x) - x\frac{d}{dx}T_n(x) + n^2T_n(x)$$

$$= \frac{nx}{(1-x^2)^{1/2}}\sin(n\cos^{-1}x) - n^2\cos(n\cos^{-1}x) - \frac{nx}{(1-x^2)^{1/2}}\sin(n\cos^{-1}x) + n\cos(n\cos^{-1}x) = 0,$$

showing that $T_n(x)$ is a solution of (1).

To show that $U_n(x)$ is a solution of (1) : Proceed as above. Left as an exercise for students.

To show that $T_n(x)$ and $U_n(x)$ are independent solutions of (1) : We have, by definition

$T_n(x) = \cos(n \cos^{-1} x)$ and $U_n(x) = \sin(n \cos^{-1} x)$.

$\therefore\ T_n(1) = \cos(n \cos^{-1} 1) = \cos(n \times 0) = 1$ and $U_n(1) = \sin(n \cos^{-1} 1) = \sin(n \times 0) = 0.$

Hence $U_n(x)$ cannot be expressed as a constant multiple of $T_n(x)$. This shows that $T_n(x)$ and $U_n(x)$ are independent solutions of (1).

Remark. Two solutions $u(x)$ and $v(x)$ are said to be linearly independent if $u(x)$ cannot be expressed as a constant multiple of $v(x)$ *i.e.*, $u(x) = k\,v(x)$, where k is a constant.

5.3. Orthogonal properties of Chebyshev polynomials.

To show that

$$(i)\ \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{(1-x^2)}}dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases} \qquad (ii)\ \int_{-1}^{1} \frac{U_m(x)U_{(x)}}{\sqrt{(1-x^2)}}dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

[*Kanpur 2011*]

Proof. (*i*) We have, by definition

$T_m(x) = \cos(m \cos^{-1} x)$ and $T_n(x) = \cos(n \cos^{-1} x)$. ... (1)

$\therefore\ T_m(\cos\theta) = \cos(m\cos^{-1}\cos\theta) = \cos m\theta,\qquad T_n(\cos\theta) = \cos(n\cos^{-1}\cos\theta) = \cos n\theta$ ... (2)

Let $$I = \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{(1-x^2)}}dx.$$ ... (3)

Putting $x = \cos\theta$ so that $dx = -\sin\theta\, d\theta$ and using (2), (3), reduces to

$$I = \int_{\pi}^{0} \frac{\cos m\theta \cos n\theta}{\sin\theta}(-\sin\theta)\, d\theta \quad \text{or} \quad I = \int_{\pi}^{0} \cos m\theta \cos n\theta\, d\theta \qquad \ldots (4)$$

Case 1. Let $m \neq n$ so that $(m-n) \neq 0$. then, (4) gives

$$I = \frac{1}{2}\int_0^{\pi} 2\cos m\theta \cos n\theta d\theta = \frac{1}{2}\int_0^{\pi} [\cos(m+n)\,\theta + \cos(m-n)\,\theta]\, d\theta = \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n}\right]_0^{\pi} = 0$$

Case 2. Let $m = n \neq 0$. Then (4) gives

$$I = \int_0^{\pi} \cos^2 m\theta\, d\theta = \int_0^{\pi} \frac{1+\cos 2m\theta}{2} d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2m\theta}{2m}\right]_0^{\pi} = \frac{\pi}{2}.$$

Case 3. Let $m = n = 0$. Then $\cos m\theta = \cos n\theta = 1$. Then (4) gives

$$I = \int_0^{\pi} (1)\,(1) dx = [\theta]_0^{\pi} = \pi.$$

From cases 1, 2 and 3, the required result (*i*) follows.

Part (*ii*). We have, by definition

$U_m(x) = \sin(m \cos^{-1} x)$ and $U_n(x) = \sin(n \cos^{-1} x)$ ... (1)

$\therefore\ U_m(\cos\theta) = \sin(m\cos^{-1}\cos\theta) = \sin m\theta,\qquad U_n(\cos\theta) = \sin(n\cos^{-1}\cos\theta) = \sin n\theta.$ ... (2)

Let $$J = \int_{-1}^{1} \frac{U_m(x)\,U_n(x)}{\sqrt{1-x^2}}dx.$$ ... (3)

Putting $x = \cos\theta$ so that $dx = -\sin\theta d\theta$ and using (2), (3) gives

$$J = \int_{\pi}^{0} \frac{\sin m\theta \sin n\theta}{\sin\theta}(-\sin\theta d\theta) \quad \text{or} \quad J = \int_{\pi}^{0} \sin m\theta \sin n\theta\, d\theta \qquad \ldots (4)$$

Case 1. Let $m \neq n$ so that $(m-n) \neq 0$. Then, (4) gives

$$J = \frac{1}{2}\int_0^{\pi} 2\sin m\theta \sin n\theta d\theta = \frac{1}{2}\int_0^{\pi} [\cos(m-n)\theta - \cos(m+n)\theta]\, d\theta = \frac{1}{2}\left[\frac{\sin(m-n)\theta}{m-n} - \frac{\cos(m+n)\,\theta}{m+n}\right]_0^{\pi} = 0.$$

Case 2. Let $m = n \neq 0$. Then (4) gives

$$J = \int_0^{\pi} \sin^2 m\theta \, d\theta = \int_0^{\pi} \frac{1-\cos 2m\theta}{2} d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2m\theta}{2m}\right]_0^{\pi} = \frac{\pi}{2}.$$

Case 3. Let $m = n = 0$. Then $\sin m\theta = \sin n\theta = 0$. Hence (4) gives

$$J = \int_0^{\pi} (0)\,(0)\, d\theta = 0.$$

From cases 1, 2 and (3), the required result (ii) follows

5.4. Recurrence relations (formulae) [Kanpur 2011]

I. $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$

II. $(1 - x^2)\, T_n'(x) = -nx\, T_n(x) + nT_{n-1}(x).$

III. $U_{n+1}(x) - 2x\, U_n(x) + U_{n-1}(x) = 0.$

IV. $(1 - x^2)\, U_n'(x) = -nx\, U_n(x) + nU_{n-1}(x).$

Proof 1. We have, by definition $T_n(x) = \cos\,(n \cos^{-1} x)$ ... (1)

$\therefore$ $T_n(\cos\theta) = \cos\,(n\cos^{-1}\cos\theta) = \cos n\theta$ ... (2)

so that $T_{n+1}(\cos\theta) = \cos(n+1)\theta$ and $T_{n-1}(\cos\theta) = \cos(n-1)\theta$ ... (3)

We are to show that $T_{n+1}(x) - 2x\, T_n(x) + T_{n-1}(x) = 0.$ ... (4)

Replacing x by $\cos\theta$ in (4), we must now prove that

$$T_{n+1}(\cos\theta) - 2\cos\theta\, T_n(\cos\theta) + T_{n-1}(\cos\theta) = 0 \quad \text{... (5)}$$

i.e. $\cos(n+1)\theta - 2\cos\theta\cos n\theta + \cos(n-1)\theta = 0,$ by (2) and (3)

i.e., $\cos(n+1)\theta + \cos(n-1)\theta - 2\cos\theta\cos n\theta = 0.$ ... (6)

Now, L.H.S. (6) $= 2\cos n\theta\cos\theta - 2\cos\theta\cos n\theta = 0,$

which proves (6) and hence (4) is true.

II. From (1), $T_n'(x) = -\sin(n\cos^{-1}x)\cdot\dfrac{-n}{\sqrt{(1-x^2)}}$ or $T_n'(\cos\theta) = \sin(n\cos^{-1}\cos\theta)\cdot\dfrac{n}{\sqrt{(1-\cos^2\theta)}}$

Thus, $T_n'(\cos\theta) = (n\sin n\,\theta)/\sin\theta$ ... (7)

We are to show that $(1-x^2)T_n'(x) = -nxT_n(x) + n\,T_{n-1}(x).$ ... (8)

Putting $x = \cos\theta$ and using (3) and (7), (8) may be re-written as

$$\sin^2\theta\frac{n\sin n\theta}{\sin\theta} = -n\cos\theta\cos n\theta + n\cos(n-1)\theta \quad \text{or} \quad \sin\theta\sin n\theta = \cos(n-1)\theta - \cos\theta\cos n\theta. \quad \text{...}$$

(9)

R.H.S. of (9) $= \cos(n\theta-\theta) - \cos\theta\cos n\theta = \cos n\theta\cos\theta + \sin n\theta\sin\theta - \cos\theta\cos n\theta$

$= \sin n\theta\sin\theta$ = L.H.S. of (9),

which proves (9) and hence (8) is true.

III and IV. Proceed as above. Left as exercises for students.

5.5. Some theorems on Chebyshev polynomials

Theorem I. *To show that* (i) $T_n(x) = (1/2)\times[\{x + i(1-x^2)^{1/2}\}^n + \{x - i(1-x^2)^{1/2}\}^n].$

(ii) $U_n(x) = -(i/2)\times[\{x + i(1-x^2)^{1/2}\}^n - \{x - i(1-x^2)^{1/2}\}^n].$

Proof. (*i*) Putting $x = \cos\theta,$ and using definition, we have

$$T_n(x) = \cos(n\cos^{-1}x), \ = \cos(n\cos^{-1}\cos\theta) = \cos n\theta = (e^{in\theta} + e^{-in\theta})/2$$
$$= (1/2)\times\{(e^{i\theta})^n + (e^{-i\theta})^n\} = (1/2)\times\{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\}$$
$$= (1/2)\times[\{\cos\theta + i(1-\cos^2\theta)^{1/2}\}^n + \{\cos\theta - i(1-\cos^2\theta)^{1/2}\}^n]$$
$$= (1/2)\times[\{x + i(1-x^2)^{1/2}\}^n + \{x - i(1-x^2)^{1/2}\}^n], \text{ as } x = \cos\theta$$

Part (*ii*) Putting $x = \cos\theta,$ and using definition, we have

$$U_n(x) = \sin(n\cos^{-1}x)$$
$$= \sin n\theta = \frac{1}{2i}(e^{in\theta} - e^{in\theta}) = \frac{1}{2i}\{(e^{i\theta})^n - (e^{-i\theta})^n\} = \frac{1}{2i}\{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n\}$$
$$= [\{\cos\theta + i(1-\cos^2\theta)^{1/2}\}^n - \{\cos\theta - i(1-\cos^2\theta)^{1/2}\}^n]/2i$$
$$= \frac{1}{2i}[\{x + i(1-x^2)^{1/2}\}^n - \{x - i(1-x^2)^{1/2}\}^n] = \frac{-i}{2}[\{x + i(1-x^2)^{1/2}\}^n - \{x - i(1-x^2)^{1/2}\}^n], \text{ as } x = \cos\theta$$

Theorem II. *To show that* (*i*) $T_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{(2r)!\,(n-2r)!}(1-x^2)^r x^{n-2r}$

(*ii*) $U_n(x) = \sum_{r=0}^{[(n-1)/2]} (-1)^r \frac{n!}{(2r+1)!\,(n-2r-1)!}(1-x^2)^{r+1/2} x^{x-2r-1}.$

Proof. (*i*) As in part (*i*) of Art. 5.5, we have

$$T_n(x) = \frac{1}{2}[\{x + i(1-x^2)^{1/2}\}^n + \{x - i(1-x^2)^{1/2}\}^n] = \frac{1}{2}\left[\sum_{s=0}^{n} {}^nC_s x^{n-s}\{i(1-x^2)^{1/2}\}^s + \sum_{s=0}^{n} {}^nC_s x^{n-s}\{-i(1-x^2)^{1/2}\}^s\right]$$

[Since by binomial theorem, we have

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_s a^{n-s}b^s + \ldots + {}^nC_n b^n = \sum_{s=0}^{n} {}^nC_s a^{n-s}b^s\ ]$$

$$\therefore \qquad T_n(x) = (1/2)\times\sum_{s=0}^{n} {}^nC_s x^{n-s}(1-x^2)^{s/2} i^s\{1 + (-1)^s\}. \qquad \ldots (1)$$

But
$$1 + (-1)^s = \begin{cases} 0, \text{ if } s \text{ is odd} \\ 2, \text{ if } s \text{ is even} \end{cases} \qquad \ldots (2)$$

Using (2), (1) reduces to
$$T_n(x) = \sum_{s \text{ even}, \le n} {}^nC_s x^{n-s}(1-x^2)^{s/2} i^s \qquad \ldots (3)$$

Since s is even in (3), we take s = 2r where r is an integer. Hence $s \le n \ \Rightarrow \ 2r \le n \ \Rightarrow \ r \le n/2.$ Now, if n is even, r goes from 0 to $n/2$, while if n is odd, r goes from 0 to $(n-1)/2$; that is, in all cases, r goes from 0 to $[n/2]$, where

$$[n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd.} \end{cases}$$

Hence with $s = 2r$ and the above arguments, (3) reduces to

$$T_n(x) = \sum_{r=0}^{[n/2]} {}^nC_{2r} x^{n-2r}(1-x^2)^r i^{2r} = \sum_{r=0}^{[n/2]} \frac{n!}{(n-2r)!\,(2r)!} x^{n-2r}(1-x^2)^r(-1)^r, \text{ as } i^{2r} = (i^2)^r = (-1)^r$$

Part (*ii*). As in part (*ii*) of Art. 5.5, we have

$$U_n(x) = (-i/2)\times[\{x + i(1-x^2)^{1/2}\}^n - \{x - i(1-x^2)^{1/2}\}^n]$$

$$= (-i/2)\times\left[\sum_{s=0}^{n} {}^nC_s x^{n-s}\{i(1-x^2)^{1/2}\}^s - \sum_{s=0}^{n} {}^nC_s x^{n-s}\{-i(1-x^2)^{1/2}\}^s\right], \text{ using binomial theorem}$$

$$= (-i/2)\times \sum_{s=0}^{n} {}^nC_s x^{n-s}(1-x^2)^{s/2} i^s \{1-(-1)^s\}. \quad \ldots (4)$$

But $$1-(-1)^s = \begin{cases} 0, \text{ if } s \text{ is even} \\ 2, \text{ if } s \text{ is odd} \end{cases} \quad \ldots (5)$$

Using (5), (4) reduces to $$U_n(x) = -i \sum_{s \text{ odd}, s\le n} {}^nC_s x^{n-s}(1-x^2)^{s/2} i^s. \quad \ldots (6)$$

Since s is odd in (6), we take $s = 2r + 1$, where r is an integer. Hence $s \le n \Rightarrow 2r+1 \le n \Rightarrow r \le (n-1)/2$. Now, if n is odd, r goes from 0 to $(n - 1)/2$, while if n is even r goes from 0 to $(n - 2)/2$; that is in all cases, r goes from 0 to $[(n - 1)/2]$, where

$$[(n-1)/2] = \begin{cases} (n-1)/2, \text{ if } n \text{ is odd} \\ (n-2)/2, \text{ if } n \text{ is even.} \end{cases}$$

Hence with $s = 2r + 1$ and the above arguments, (6) reduces to

$$U_n(x) = -i \sum_{r=0}^{[(n-1)/2]} {}^nC_{2r+1} x^{n-2r-1}(1-x^2)^{r+1/2} i^{2r+1} = \sum_{r=0}^{[(n-1)/2]} (-1)^r \frac{n!}{(2r+1)!(n-2r-1)!}(1-x^2)^{r+1/2} x^{n-2r-1}$$

$$[\because\ i^{2r+1} = (i^2)^r \times i = (-1)^r i \text{ and } i^2 = -1]$$

5.6. First few Chebyshev polynomials.

We have $$T_n(x) = \cos(n \cos^{-1} x) \quad \ldots (1)$$

and $$U_n(x) = \sin(n \cos^{-1} x) \quad \ldots (2)$$

Putting $n = 0$ and 1 successively in (1), we get

$T_0(x) = \cos(0 \times \cos^{-1} x) = \cos 0 = 1$ and $T_1(x) = \cos(\cos^{-1} x) = x$.

Putting $n = 2$ in part (*i*) of theorem II of Art. 5.5, we have

$$T_2(x) = \sum_{r=0}^{1} (-1)^r \frac{2!}{(2r)!(2-2r)!}(1-x^2)^r x^{2-2r}, \text{ as } [n/2] = n/2 \text{ if } n \text{ is even}$$

$$= x^2 - (1 - x^2) = 2x^2 - 1.$$

Next, putting $n = 3$ in part (*i*) of theorem II of Art. 5.5, we have

$$T_3(x) = \sum_{r=0}^{1} (-1)^r \frac{3!}{(2r)!(3-2r)!}(1-x^2)^r x^{3-3r}, \quad \text{as} \quad [n/2] = (n-1)/2, \text{ if } n \text{ is odd}$$

$$= x^3 - 3(1 - x^2) \,.\, x = 4x^3 - 3x.$$

and so on. Thus, we see that $T_n(x)$ is a polynomial of degree n.

Now, putting $n = 0$ and 1 successively in (2), we get

$$U_0(x) = \sin(0 \times \cos^{-1} x) = \sin 0 = 0 \quad \text{and} \quad U_1(x) = \sin\cos^{-1} x = \sin\sin^{-1}(1-x^2)^{1/2} = (1-x^2)^{1/2}.$$

Putting $n = 2$ in part (*ii*) of theorem II of Art 5.5, we have

$$U_2(x) = \sum_{r=0}^{0} (-1)^r \frac{2!}{(2r+1)!(2-2r-1)!}(1-x^2)^{r+1/2} x^{2-2r-1} = 2(1-x^2)^{1/2} \cdot x = 2x\sqrt{(1-x^2)}.$$

Putting $n = 3$ in part (*ii*) of theorem II of Art. 5.5, we have

$$T_3(x) = \sum_{r=1}^{1} (-1)^r \frac{3!}{(2r+1)!(3-2r-1)!}(1-x^2)^{r+1/2} x^{3+2r-1}$$

$$= 3\,(1-x^2)^{1/2} \cdot x^2 + (-1)^1\,(1-x^2)^{3/2} = (4x^2-1)(1-x^2)^{1/2}. \quad (\because 0! = 1)$$

Remark 1. We observe that $U_n(x)$ is not a polynomial. However, if we define the Chebyshev polynomial of second kind by

$$U_u^*(x) = \sin\{(n+1)\cos^{-1} x\}(1-x^2)^{1/2} = \{1/(1-x^2)^{1/2}\}/U_{n+1}, \qquad \ldots (3)$$

then $U_n^*(x)$ will be polynomial of degree n. This can be easily verified by using (3).

Remark 2. Putting $x = \cos\theta$ in (1) and (2), we have

$$T_n(\cos\theta) = \cos n\theta \qquad \text{and} \qquad U_n(\cos\theta) = \sin n\theta. \quad \ldots (4)$$

Thus, we conclude that Chebashev polynomials can be used to obtain explansions of $\cos n\theta$ and $\sin n\theta / \sin\theta$ in terms of $\cos\theta$.

5.7. Generating functions for Chebyshev polynomials.

$$(i)\ \frac{1-t^2}{1-2tx-t^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)\,t^n \qquad (ii)\ \frac{(1-x^2)^{1/2}}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_{n+1}(x)\,t^n.$$

Proof. (*ii*) To prove the desired result we must show that in the expansion of L.H.S. in ascending powers of t the coefficient of t^n for $n \geq 1$ is $2T_n(x)$ and the coefficient of t^n is $T_0(x)$ for $n = 0$. Putting $x = \cos\theta = (e^{i\theta} + e^{-i\theta})/2$, we find that

L.H.S. of the required result

$$= \frac{1-t^2}{1-(e^{i\theta}+e^{-i\theta})t+t^2} = \frac{1-t^2}{1-e^{i\theta}t-e^{-i\theta}t+e^{i\theta}t\cdot e^{-i\theta}t} = \frac{1-t^2}{(1-e^{i\theta}t)-e^{-i\theta}t\,(1-e^{i\theta}t)} = \frac{1-t^2}{(1-e^{i\theta}t)\,(1-e^{-i\theta}t)}$$

$$= (1-t^2)\,(1-e^{i\theta}t)^{-1}(1-e^{-i\theta}t)^{-1} = (1-t^2)\sum_{r=0}^{\infty}(e^{i\theta}t)^r \sum_{s=0}^{\infty}(e^{-i\theta}t)^s$$

$$= (1-t^2)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{i(r-s)\theta}t^{r+s} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{i(r-s)\theta}t^{r+s} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{i(r-s)\theta}t^{r+s+2} \qquad \ldots (1)$$

Note that we need not consider the summation for the coefficient of t^0. Taking $r = 0$, $s = 0$ in the first summation in (1), we find that

Coefficient in t^n in L.H.S. of the required result $= e^{i(0-0)\theta} = e^0 = 1 = T_0(x)$. $[\because\ T_0(x) = 1]$

For $n \geq 1$, we get the coefficient of t^n by taking $r + s = n$, (*i.e.*, $s = n - r$, so that $s \geq 0$ $\Rightarrow$ $n - r \geq 0$ $\Rightarrow$ $r \leq n$ for the total coefficient of t^n) in the first summation in (1) and $r + s + 2 = n$, (*i.e.*, $s = n - r - 2$ so that $s \geq 0$ $\Rightarrow$ $n - r - 2 \geq 0$ $\Rightarrow$ $r \leq n - 2$ for the total coefficient of t^n), in the second summation in (1).

$\therefore$ the total coefficient of t^n in (1)

$$= \sum_{r=0}^{n} e^{i\{r-(n-r)\}\theta} - \sum_{r=0}^{n-2} e^{i\{r-(n-r-2)\}\theta} = e^{-in\theta}\sum_{r=0}^{n} e^{2ir\theta} - e^{-i(n-2)\theta}\sum_{r=0}^{n-2} e^{2ir\theta}$$

$$= e^{-in\theta}[1 + e^{2\theta i} + e^{4\theta i} + \ldots \text{to } (n+1) \text{ terms}]\ - e^{-i(n-2)\theta}[1 + e^{2\theta i} + e^{4\theta i} + \ldots \text{to } (n-1) \text{ terms}]$$

$$= e^{-in\theta}\frac{1-(e^{2i\theta})^{n+1}}{1-e^{2i\theta}} - e^{-i(n-2)\theta}\frac{1-(e^{2i\theta})^{n-1}}{1-e^{2i\theta}}$$

$[\because$ For geometric series, $a + ar + ar^2 + \ldots$ to n term $= \{a(1-r^n)\}/(1-r)]$

$$= \frac{e^{-in\theta} - e^{i(n+2)\theta}}{1-e^{2i\theta}} - \frac{e^{-i(n-2)\theta} - e^{in\theta}}{1-e^{2i\theta}} = \frac{e^{-in\theta} - e^{-i(n-2)\theta} + e^{in\theta} - e^{i(n+2)\theta}}{1-e^{2i\theta}}$$

$$= \frac{e^{-in\theta}(1-e^{2i\theta}) + e^{in\theta}(1-e^{2i\theta})}{(1-e^{2i\theta})} = \frac{(1-e^{2i\theta})\,(e^{-in\theta}+e^{in\theta})}{(1-e^{2i\theta})} = e^{in\theta} + e^{-in\theta}$$

$$= 2\cos n\,\theta, \text{ as } \cos n\,\theta = (e^{in\theta} + e^{-in\theta})/2$$

$$= 2T_n(x). \qquad [\because\ T_n(x) = \cos(n\cos^{-1}x) = \cos(n\cos^{-1}\cos\theta) = \cos n\theta\,]$$

From the above arguments, the required result follows.

Part (ii). To Prove the desired result we must show that in the expansion of L.H.S. in ascending powers to t, the coefficient of t^n for $n \geq 1$ is $U_{n+1}(x)$.

Now, L.H.S. $= \dfrac{\sqrt{(1-x^2)}}{1-2tx+t^2} = \dfrac{\sqrt{(1-\cos^2\theta)}}{1-2t\cos\theta+t^2}$, putting $x = \cos\theta$

$$= \frac{\sin\theta}{1-t(e^{i\theta}+e^{-i\theta})+t^2} = \frac{\sin\theta}{1-e^{i\theta}t-e^{-i\theta}t+e^{i\theta}t\times e^{-i\theta}t} = \frac{\sin\theta}{(1-e^{i\theta}t)-e^{-i\theta}t(1-e^{i\theta}t)} - \frac{\sin\theta}{(1-e^{i\theta}t)\,(1-e^{-i\theta}t)}$$

$$= \sin\theta\,(1-e^{i\theta}t)^{-1}(1-e^{-i\theta}t)^{-1} = \sin\theta \sum_{r=0}^{\infty}(e^{i\theta}t)^r \sum_{s=0}^{\infty}(e^{-i\theta}t)^s = \sin\theta \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{i(r-s)\theta}t^{r+s} \qquad \ldots (2)$$

We get the coefficient of t^n by taking $r + s = n$ (*i.e.*, $s = n - r$, so that $s \geq 0 \Rightarrow n - r \geq 0$ $\Rightarrow$ $r \leq n$ for the total coefficient of t^n)

$\therefore$ The total coefficient of t^n in (2) $= \sin\theta \sum_{r=0}^{n} e^{i[r-(n-r)]\theta} = \sin\theta \sum_{r=0}^{n} e^{-in\theta}\cdot e^{2r\theta i}$

$$= \sin\theta\, e^{-in\theta} \sum_{r=0}^{n} e^{2r\theta i} = \sin\theta\, e^{-in\theta}[1 + e^{2i\theta} + e^{4i\theta} + e^{6i\theta} + \ldots \text{ to } (n+1) \text{ terms}]$$

$$= \sin\theta\, e^{-in\theta}\,\frac{1-(e^{2i\theta})^{n+1}}{1-e^{2i\theta}} = \frac{e^{i\theta}-e^{-i\theta}}{2i} \times \frac{e^{-in\theta}(1-e^{2i\theta(n+1)})}{-e^{i\theta}(e^{i\theta}-e^{-i\theta})}$$

$$= \frac{e^{-i(n+1)\theta}[e^{2i\theta(n+1)}-1]}{2i} = \frac{e^{i(n+1)\theta}-e^{-i(n+1)\theta}}{2i} = \sin(n+1)\,\theta$$

$$= \sin\{(n+1)\cos^{-1}x\} = U_{n+1}(x) \text{ by definition, as } x = \cos\theta \Rightarrow \theta = \cos^{-1}x$$

Form the above argument, we get the required result.

5.8. Special values of the Chebyshev polynomials.

To show that (*i*) $T_n(1) = 1$, $T_n(-1) = (-1)^n$, $T_{2n}(0) = (-1)^n$, $T_{2n+1}(0) = 0$.
(*ii*) $U_n(1) = 0$, $U_n(-1) = 0$, $U_{2n}(0) = 0$, $U_{2n+1}(0) = (-1)^n$.

Proof (*i*) We have, by definition $T_n(x) = \cos(n\cos^{-1}x)$. ... (1)

Putting $x = 1$ in (1), we have $T_n(1) = \cos(n\cos^{-1}1) = \cos(n \times 0) = 1$.

Putting $x = -1$ in (1), we have $T_n(-1) = \cos[n\cos^{-1}(-1)] = \cos(n\pi) = (-1)^n$.

Replacing x by 0 and n by $2n$ in (1), we have

$$T_{2n}(0) = \cos(2n\cos^{-1}0) = \cos\{(2n)\times(\pi/2)\} = \cos n\pi = (-1)^n.$$

Replacing x by 0 and n by $2n + 1$ in (1), we have

$$T_{2n+1}(0) = \cos[(2n+1)\cos^{-1}0] = \cos\{(2n+1)\times(\pi/2)\} = 0.$$

Part (ii) Proceed as above yourself.

5.9. Illustrative Solved Examples.

Ex. 1. *Show that* $T_m\{T_n(x)\} = T_n\{T_m(x)\} = T_{nm}(x)$.

Sol. We have, by definition

$$T_m\{T_n(x)\} = T_m[\cos(n\cos^{-1}x)]$$
$$= \cos[m\cos^{-1}\{\cos(n\cos^{-1}x)\}], \text{ by definition again}$$
$$= \cos(nm\cos^{-1}x). \quad \ldots(1)$$

Again, $T_n\{T_m(x)\} = T_n[\cos(m\cos^{-1}x)]$, by definition

$$= \cos[n\cos^{-1}\{\cos(m\cos^{-1}x)\}], \text{ by definition again}$$
$$= \cos(nm\cos^{-1}x), \quad \ldots(2)$$

Finally, $T_{mn}(x) = \cos(mn\cos^{-1}x)$, by definition ... (3)

From (1), (2) and (3), we get the required result.

Ex. 2. *Show that* $(1-x^2)^{1/2}\,T_n(x) = U_{n+1}(x) - xU_n(x)$.

Sol. By definition, $T_n(x) = \cos(n\cos^{-1}x)$ and $U_n(x) = \sin(n\cos^{-1}x)$ ... (1)

Putting $x = \cos\theta$, (1) reduces to $T_n(\cos\theta) = \cos n\theta$ and $U_n(\cos\theta) = \sin n\theta$. ... (2)

Replacing x by $\cos\theta$, the required result takes the following form

$$\sin\theta T_n(\cos\theta) = U_{n+1}(\cos\theta) - \cos\theta\, U_n(\cos\theta), \text{ } i.e., \text{ } \sin\theta\cos n\theta = \sin(n+1)\theta - \cos\theta\sin n\theta, \text{ by (2)} \quad \ldots(3)$$

Now, R.H.S. of (3) $= \sin(n\theta+\theta) - \cos\theta\sin n\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta - \cos\theta\sin n\theta$

$= \sin\theta\cos n\theta$ = L.H.S. of (3)

This proves the required result.

Ex. 3. *Show that* $\sum_{r=0}^{n} T_{2r}(x) = \frac{1}{2}\left(1 + \frac{1}{(1-x^2)^{1/2}}U_{2n+1}(n)\right)$. **[Kanpur 2007, 11]**

Sol. Let $x = \cos\theta$. Then by definition, we have

$$\sum_{r=0}^{n} T_{2r}(x) = \sum_{r=0}^{n}\cos(2r\cos^{-1}x) = \sum_{r=0}^{n}\cos(2r\cos^{-1}\cos\theta) = \sum_{r=0}^{n}\cos 2r\theta$$

Now, $\sum_{r=0}^{n} T_{2r}(x)$ = Real part of $\sum_{r=0}^{n} e^{2ri\theta}$. ... (1)

But $\sum_{r=0}^{n} e^{2ri\theta} = 1 + e^{2i\theta} + e^{4i\theta} + \ldots$ to $(n+1)$ terms

$$= \frac{1-(e^{2i\theta})^{n+1}}{1-e^{i\theta}} \qquad [\because\ a + ar + ar^2 + \ldots \text{ to } n \text{ terms} = \frac{a(1-r^n)}{1-r}]$$

$$= \frac{(1-e^{i(2n+2)\theta})\,(1-e^{-2i\theta})}{(1-e^{2i\theta})\,(1-e^{-2i\theta})} = \frac{(1-e^{i(2n+2)\theta})\,(1-e^{-2i\theta})}{1+1-(e^{i\theta}+e^{-i\theta})}$$

$$= \frac{\{1-\cos(2n+2)\theta - i\sin(2n+2)\theta\}\,\{1-\cos 2\theta + i\sin 2\theta\}}{2-2\cos 2\theta}, \text{ as } e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$\therefore$ (1) reduces to

$$\sum_{r=0}^{n} T_{2r}(x) = \frac{1-\cos(2n+2)\theta - \cos 2\theta + \cos(2n+2)\theta\cos 2\theta + \sin(2n+2)\theta\sin 2\theta}{2(1-\cos 2\theta)}$$

$$= \frac{(1-\cos 2\theta) - \cos(2n+2)\theta\,(1-\cos 2\theta) + \sin(2n+2)\theta\,\sin 2\theta}{2(1-\cos 2\theta)}$$

$$= \frac{1}{2}\left[1 - \cos(2n+2)\theta + \frac{\sin(2n+2)\theta \sin 2\theta}{1-\cos 2\theta}\right] = \frac{1}{2}\left[1 - \cos(2n+2)\theta + \frac{\sin(2n+2)\theta \cdot 2\sin\theta\cos\theta}{2\sin^2\theta}\right]$$

$$= \frac{1}{2}\left[1 + \frac{\sin(2n+2)\theta\cos\theta - \cos(2n+2)\theta\sin\theta}{\sin\theta}\right] = \frac{1}{2}\left[1 + \frac{\sin\{(2n+2)\theta - \theta\}}{\sin\theta}\right]$$

$$= \frac{1}{2}\left[1 + \frac{\sin(2n+1)\theta}{(1-\cos^2\theta)^{1/2}}\right] = \frac{1}{2}\left[1 + \frac{U_{2n+1}(x)}{(1-x^2)^{1/2}}\right]$$

$$[\because \; x = \cos\theta \text{ so that } U_{2n+1}(x) = \sin[(2n+1)\cos^{-1}\cos\theta] = \sin(2n+1)\theta]$$

Ex. 4. *Show that* $\{T_n(x)\}^2 - T_{n+1}(x)\;\; T_{n-1}(x) = 1 - x^2$.

Sol. We have, by definition $T_n(x) = \cos(n \cos^{-1} x)$.

$$\therefore \qquad T_n(\cos\theta) = \cos(n\cos^{-1}\cos\theta) = \cos n\theta. \qquad \ldots (1)$$

With $x = \cos\theta$, the required result takes the form

$$\{T_n(\cos\theta)\}^2 - T_{n+1}(\cos\theta)\, T_{n-1}(\cos\theta) = 1 - \cos^2\theta, \; \textit{i.e.}, \; \cos^2 n\theta - \cos(n+1)\theta\cos(n-1)\theta = \sin^2\theta, \; \ldots(2)$$

L.H.S. of (2) $= \cos^2 n\theta - \cos(n\theta+\theta)\cos(n\theta-\theta)$

$= \cos^2 n\theta - (\cos^2 n\theta - \sin^2\theta)$, as $\cos(A+B)\cos(A+B) = \cos^2 A - \sin^2 A$

$= \sin^2\theta$ = R.H.S. of (2).

This proves the required result.

EXERCISE

1. Show that (*i*) $(1-x^2)^{1/2} U_n(x) = xT_n(x) - T_{n+1}(x)$.

(*ii*) $T_{m+n}(x) + T_{m-n}(x) = 2T_m(x)\, T_n(x)$. **[Kanpur 2006]**

2. Show that (*i*) $T_m'(x) = \dfrac{n}{(1-x^2)^{1/2}} U_n(x)$. **[Kanpur 2009]**

(*ii*) $2\{T_n(x)\}^2 = 1 + T_{2n}(x)$.

3. Show that $\{1/(1-x^2)^{1/2}\}U_n(x)$ satisfies $(1-x^2)y'' + 3xy' + (n^2-1)y = 0$.

[Kanpur 2006, 07]

4. Use Chebyshev's differential equation and the equation of exercise 3 above to show

that $$T_n(x) = \frac{n}{2} \sum_{r=0}^{[n/2]} (-1)^r \frac{(n-r-1)!}{r!(n-2r)!}(2x)^{n-2r}$$

and $$U_n(x) = \sqrt{(1-x^2)} \sum_{r=0}^{[(n-1)/2]} (-1)^r \frac{(n-r-1)!}{r!(n-2r-1)!}(2x)^{n-2r-1}.$$

5. Show that the set of Chebyshev polynomials $T_n(x) = \cos(n \cos^{-1} x)$, (n = 0, 1, 2 .....) is orthogonal on the interval (–1, 1) with respect to the weight function $p(x) = 1/(1-x^2)^{1/2}$.

6. Prove that (*i*) $\int_{-1}^{1} x^6 (1-x^2)^{-1/2} T_8(x)dx = 0$. (*ii*) $T_n(x) - 2x\; T_{n-1}(x) + T_{n-2}(x) = 0$.

7. Show that Chebyshev's polynomials $T_n(x) = \cos(n \cos^{-1} x)$ are solutions of

$$(1 - x^2)\,(d^2y/dx^2) - x(dy/dx) + n^2y = 0.$$

8. Prove that $T_n(x) = \sum_{m=0}^{[n/2]} \dfrac{(-1)^m n!\,(1-x)^m x^{n-2m}}{(2m)!\;\;(n-2m)!}$ **[Kanpur 2007]**

9. Prove that $T_n(x) - 2x\; T_{n-1}(x) + T_{n-2}(x) = 0$ **[Kanpur 2010]**

6

Beta and Gamma Functions

6.1. INTRODUCTION

In this chapter we propose to discuss the Gamma and Beta functions. These functions arise in the solution of physical problems and are also of great importance in various branches of mathematical analysis. The reader in strongly advised to master important results of these functions in order to understand the topics coverved by this book.

6.2. Euler's integrals. Beta and Gamma functions

Beta function. Definition

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad \text{for } m>0, \quad n>0$

is known as the *Beta function* and is denoted by $B(m, n)$ [read as "Beta m, n"]. Beta function is also called the *Eulerian integral of the first kind*.

Thus, $$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m>0,\ n>0 \qquad \ldots (1)$$

Gamma function. Definition **(Agra 2000)**

The definite integral $\int_0^\infty e^{-x}x^{n-i}dx, \text{ for } n>0$

is known as the *Gamma function* and is denoted by $\Gamma(n)$ [read as "Gamma n"]. Gamma function is also called the *Eulerain integral of the second kind*.

Thus, $$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx, \text{ for } n>0 \qquad \ldots (2)$$

Remark. The integral (1) is valid only for $m > 0$ and $n > 0$ and the integral (2) is valid only for $n > 0$, because it is for just these values of m and n that the above integrals are convergent.

6.3. Properties of Gamma function. **(Agra 1999)**

I. To show that $\Gamma(1) = 1$.

Proof. By the definition of Gamma function, $$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx,\ n>0 \ \ldots (1)$$

From (1), $$\Gamma(1)=\int_0^\infty e^{-x}x^{1-1}dx = \int_0^\infty e^{-x}dx = \left[-e^{-x}\right]_0^\infty = 1.$$

II. To show that $\Gamma(n+1) = n\Gamma(n)$, $n > 0$. **(Agra 1998, Rohilkhand 1997, Delhi Physics (H) 2000)**

Proof. We have from the definition of Gamma function,

$$\Gamma(n+1)=\int_0^\infty e^{-x}x^{n+1-1}dx = \int_0^\infty x^n e^{-x}dx$$

$$= \left[x^n (-e^{-x}) \right]_0^\infty - \int_0^\infty (nx^{n-1})\,(-e^{-x})\,dx, \text{ on integrating by parts}$$

$$\therefore \qquad \Gamma(n+1) = -\lim_{x\to\infty} \frac{x^n}{e^x} + 0 + n\int_0^\infty e^{-x} x^{n-1}\,dx. \qquad \ldots(1)$$

Now, we have $\left(\because \lim_{x\to 0} x^n e^{-x} = 0 \text{ as } n > 0 \right)$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^n}{1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^n} + \frac{1}{1!\,x^{n-1}} + \ldots + \frac{1}{n!} + \frac{x}{(n+1)!} + \ldots} =$$

0

Also, by definition $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx.$

Using the above facts (1) reduces to $\Gamma(n+1) = n\,\Gamma(n).$

III. If n is a non-negative integer, then $\Gamma(n+1) = n!$.

Proof. We known that for $n > 0$, we have (from property II)

$\Gamma(n+1) = n\Gamma(n) = n\,\Gamma(n-1+1) = n\,(n-1)\Gamma(n-1)$, by property II again

$= n\,(n-1)\,(n-2)\,\Gamma\,(n-2)$, by property II again

$= n\,(n-1)\,(n-2)\ldots 3\cdot 2\cdot 1\,\Gamma\,(1)$

(by repeated use of property II and the fact that n is positive integer)

$= n!$, as $\Gamma(1) = 1$

Remark. *Gauss's Pi-function* is denoted by $\Pi\,(n)$ and is defined by $\Pi\,(n) = \Gamma(n+1)$. When n is +ve integer, $\Pi\,(n) = n!$.

6.4. Extension of definition of Gamma function $\Gamma(n)$ for n < 0.

When $n > 0$, we known that $\Gamma(n+1) = n\,\Gamma(n)$

so that $\Gamma(n) = \Gamma(n+1)/n.$ ... (1)

Let $-1 < n < 0$. Then $-1 < n \Rightarrow n + 1 > 0$ so that $\Gamma(n+1)$ is well defined by definition 6.2 and so R.H.S. of (1) is well defined. Thus $\Gamma(n)$ is defined for $-1 < n < 0$ by (1). Similarly, $\Gamma(n)$ is given by (1) for $-2 < n < -1$, $-3 < n < -2$ and so on. Thus (1) defines $\Gamma(n)$ for all values of n except $n = 0, -1, -2, -3, \ldots$

Property : To show that $\Gamma(n) = \infty$, if n is zero or a negative integer.

Proof. Putting $n = 0$ in (1), we get $\Gamma(0) = \Gamma(1)/0 \quad \Rightarrow \quad \Gamma(0) = \infty$ ... (2)

Again, putting $n = -1$ in (1), we get $\Gamma(-1) = \dfrac{\Gamma(0)}{-1} = \infty,$ by (2) ... (3)

Next putting $n = -2$ in (1) and using (3), we get $\Gamma(-2) = \dfrac{\Gamma(-1)}{-2} = \infty,$

and so on. Thus, we find that $\Gamma(n)$ is ∞ if n is zero or negative integer.

6.5. Theorem. To show that $\Gamma(1/2) = \sqrt{\pi}$. **[Agra 2000, 05; Meerut 2007]**

Proof. From definition of gamma function, $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, n > 0$ ... (1)

Replacing n by 1/2 in (1), we have $\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2} dt = 2\int_0^\infty e^{-u^2} du$ ... (2)

[Putting $t = u^2$ so that $dt = 2u\, du$]

$\therefore\ \Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx$ and $\Gamma(1/2) = 2\int_0^\infty e^{-y^2} dy.$ ... (3)

[Limits remaining the same, we can write x or y as the variable in the integrand of (2)].

Multiplying the corresponding sides of two equations of (3), we get

$$[\Gamma(1/2)]^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, d\theta\, dr$$

(on changing the variables to polar co-ordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $dx\, dy = r\, d\theta\, dr$. The area of integration is the positive quadrant of xy– plane).

$$\therefore\ [\Gamma(1/2)]^2 = 2\int_0^{\pi/2} \left\{\int_0^\infty 2e^{-r^2} r\, dr\right\} d\theta = 2\int_0^{\pi/2} \left\{\int_0^\infty e^{-v}\, dv\right\} d\theta, \text{ putting } r^2 = v \text{ so that } 2r\, dr = dv$$

Hence, $[\Gamma(1/2)]^2 = 2\int_0^{\pi/2} \left[-e^{-v}\right]_0^\infty d\theta = 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = \pi$

Thus, $[\Gamma(1/2)]^2 = \pi$ so that $\Gamma(1/2) = \sqrt{\pi}.$ ... (4)

Remark. From (3) and (4), $2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$ ... (5)

6.6. Transformation of Gamma function.

Form I. To show that $\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}} dx$, $n > 0$. **(Meerut 1996)**

Proof. By definition, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0.$ ... (*i*)

Put $x^n = t$ so that $nx^{n-1}\, dx = dt$. Then (*i*) gives

$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-t^{1/n}} dt$ or $\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}} dx.$ ... (*ii*)

Particular Case. Put $n = 1/2$ in (*ii*). Then, $\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx.$ ... (*iii*)

Form II. Show that $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$, $n > 0$, $k > 0$. **(Meerut 1996)**

Proof. By definition, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0.$ ... (*i*)

Put $x = kt$ so that $dx = k\, dt$, Then (*i*) gives

$\Gamma(n) = \int_0^\infty e^{-kt} k^{n-1} t^{n-1} k\, dt$ or $\Gamma(n) = k^n \int_0^\infty e^{-kx} x^{n-1} dx$

or $$\int_0^\infty e^{-kx}x^{n-1}dx = \frac{\Gamma(n)}{k^n}, \quad n>0, \quad k>0.$$

Form III. To show that $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$, $n>0$. [Meerut 1997, Kumaun 2000, Agra 2008; Garhwal 2003 Purvanchal 2005, Delhi Physics (H) 2001]

Proof. By definition, $$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx. \qquad \ldots (i)$$

Put $e^{-x} = t$ so that $-e^{-x}\,dx = dt$. Then (i) gives

$$\Gamma(n) = -\int_1^0 \left(\log\frac{1}{t}\right)^{n-1} dx = \int_0^1 \left(\log\frac{1}{t}\right)^{n-1} dt, \quad \text{as} \quad e^{-x} = t \quad \Rightarrow \quad e^x = \frac{1}{t} \quad \Rightarrow \quad x = \log\frac{1}{t}$$

$$\therefore \qquad \Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx, \quad n>0.$$

Form IV. To show that $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}dx$, $n>0$.

Proof. By definition, $$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx. \qquad \ldots (i)$$

Put $x = t^2$ so that $dx = 2t\,dt$. Then (i) gives

$$\Gamma(n) = \int_0^x e^{-t^2}(t^2)^{n-1}2t\,dt \qquad \text{or} \qquad \Gamma(n) = 2\int_0^\infty e^{-t^2}t^{2n-1}dt \qquad \text{or} \qquad \Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}dx.$$

6.7. Solved examples based on Gamma function

Ex. 1. *Evaluate* (i) $\int_0^\infty x^4e^{-x}dx$ $\qquad$ (ii) $\int_0^\infty x^6e^{-2x}dx.$

Sol. (i) $\int_0^\infty x^4e^{-x}dx = \int_0^\infty e^{-x}x^{5-1}dx = \Gamma(5) = 4! = 24$, by definition of Gamma function

(ii) Let $I = \int_0^\infty x^6e^{-2x}dx.$ Put $2x = t$ so that $dx = \frac{1}{2}dt.$ Then, we have

$$I = \int_0^\infty \left(\frac{t}{2}\right)^6 e^{-t}\cdot\frac{1}{2}dt = \frac{1}{2^7}\int_0^\infty e^{-t}t^{7-1}dt = \frac{1}{2^7}\Gamma(7), \text{ by definition of Gamma function.}$$
$$= (1/2^7) \times 6! = 45/8$$

Ex. 2. *Compute* (i) $\Gamma\left(-\frac{1}{2}\right)$ **[Delhi, Physics (H) 2001; Agra 2006]** (ii) $\Gamma\left(-\frac{3}{2}\right)$ (iii) $\Gamma\left(-\frac{5}{2}\right)$.

(iv) Prove that $\Gamma(-9/2) = -(32\sqrt{\pi})/945$ **[Kanpur 2004]**

Sol. We know that $$\Gamma(n) = \Gamma(1+n)/n \qquad \ldots (i)$$

Part (*i*). Putting $n = -\frac{1}{2}$ in (1), $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(1/2)}{(-1/2)} = \frac{\sqrt{\pi}}{(-1/2)} = -2\sqrt{\pi}$, as $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Part (*ii*). Putting $n = -3/2$ in (1), we have

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{(-3/2)} = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = -\frac{2}{3}\left(-2\sqrt{\pi}\right) = \frac{4\sqrt{\pi}}{3}, \text{ using part } (i)$$

Part (*iii*). Putting $n = -\frac{5}{2}$ in (*i*), $\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}\right)}{(-5/2)} = -\frac{2}{5}\left(\frac{4\sqrt{\pi}}{3}\right)$, using part (*ii*)

Part (*iv*) Left as an exercise

Ex. 3. *If n is a positive integer, prove that* $2^n \Gamma(1+1/2) = 1\cdot 3\cdot 5 \ldots (2n+1)\sqrt{\pi}$.

[Delhi Physics (H) 2002]

Sol. Using the formula $\Gamma(n+1) = n\Gamma(n), \quad n > 0.$ ... (1)

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(n-\frac{1}{2}+1\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{3}{2}+1\right), \text{ using (1)}$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right) = \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\Gamma\left(\frac{2n-3}{2}\right) = \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)$$

[By repeated application of (1) and noting that $(2n-1)$, $(2n-3)$, ... are all odd].

$$\therefore \qquad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)\,(2n-3)\ldots 5\cdot 3\cdot 1}{2^n}\sqrt{\pi}, \quad \text{as} \quad \Gamma(1/2) = \sqrt{\pi}$$

$$\therefore \qquad 2^n\Gamma(n+1/2) = 1\cdot 3\cdot 5\ldots\ (2n-1)\sqrt{\pi}$$

Ex. 4. *If n is a positive integer and* $m > -1$, *prove that* $\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$.

Sol. Let $I = \int_0^1 x^m (\log x)^n\, dx$. Put $\log x = -t$ so that $x = e^{-t}$ and $dx = -e^{-t}\, dt$.

$$\therefore \qquad I = \int_\infty^0 (e^{-t})^m (-t)^n (-e^{-t}\, dt) \qquad [\because\ \log 0 = -\infty \text{ and } \log 1 = 0]$$

$$= (-1)^n \int_0^\infty e^{-(m+1)t} t^{(n+1)-1} dt = (-1)^n \cdot \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \text{ provided } m+1 > 0 \textit{ i.e., } m > -1$$

[using form II of Art. 6.6]

$$= \frac{(-1)^n\, n!}{(m+1)^{n+1}}. \qquad [\because\ \Gamma(n+1) = n!,\ n \text{ being the integer}]$$

Ex. 5 (*a*) *With certain limitations on the values of a, b, m and n, prove that*

$$\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1} dx\, dy = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}.$$

Sol. Let $I = \int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1} dx\, dy$ ... (1)

or $I = \int_0^\infty e^{-ax^2} x^{2m-1} dx \times \int_0^\infty e^{-by^2} y^{2n-1} dy = I_1 \times I_2$ ... (2)

where $I_1 = \int_0^\infty e^{-ax^2} x^{2m-1} dx$ ... (3)

and $I_2 = \int_0^\infty e^{-by^2} y^{2n-1} dy.$ ... (4)

Put $ax^2 = t$, *i.e.*, $x = (t/a)^{1/2}$ so that $dx = dt/2\sqrt{at}$. Then (3) becomes

$$I_1 = \int_0^\infty e^{-t}\left[\frac{t}{a}\right]^{(2m-1)/2}\frac{dt}{2\sqrt{at}} = \frac{1}{2a^m}\int_0^\infty e^{-t}t^{m-1}dt$$

$= \Gamma(m)/2a^m$, by definition of Gamma function, taking $m > 0$, $a > 0$

Similarly, $I_2 = \Gamma(n)/2b^n$, if $n > 0$, $b > 0$

$\therefore$ From (1) and (2), we obtain $I = I_1 \times I_2 = \dfrac{\Gamma(m)\,\Gamma(n)}{4a^m b^n}$.

Ex. 5 (*b*) *Show that* $\int_0^\infty e^{-ax^2}x^{2n-1}dx = \dfrac{\Gamma(n)}{2a^n}$ **(Purvanchal 2007)**

[**Hint.** Same as I_1 of Ex. 5(*a*)]

Ex. 6. Evaluate $\int_0^1 \dfrac{dx}{\sqrt{(-\log x)}}$ **[Meerut 1996, Agra 1998]**

Sol. Put $-\log x = t$ so that $x = e^{-t}$ and $dx = -e^{-t}\,dt$.

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \int_\infty^0 \frac{-e^{-t}dt}{\sqrt{t}} = \int_0^\infty e^{-t}t^{-1/2}dt = \int_0^\infty e^{-t}t^{(1/2)-1}dt = \Gamma(1/2) = \sqrt{\pi}$$

Ex. 7. Evaluate $\int_0^\infty t^{-3/2}(1-e^{-t})\,dt$

Sol. $$\int_0^\infty t^{-3/2}(1-e^{-t})\,dt = \left[(1-e^{-t})\left(\frac{t^{-1/2}}{-1/2}\right)\right]_0^\infty - \int_0^\infty (e^{-t})\left(\frac{t^{-1/2}}{-1/2}\right)dt$$

$$= 0 + 2\int_0^\infty e^{-t}t^{(1/2)-1}dt \;=\; 2\Gamma(1/2) = 2\sqrt{\pi}$$

Ex. 8. *Evaluate* (*i*) $\int_0^\infty x^m e^{-a\,x^n}dx$ (*ii*) $\int_0^\infty x^m e^{-x^n}dx$.

Sol. Part (*i*). Put $ax^n = t$ so that $x = t^{1/n}/a^{1/n} = (t/a)^{1/n}$, $dx = (1/a^{1/n}) \times (1/\mathrm{n}) \times t^{(1/n)-1}\,dt$

$$\therefore \quad \int_0^\infty x^m e^{-a\,x^n}dx = \int_0^\infty \left(\frac{t}{a}\right)^{m/n} e^{-t}\,\frac{1}{a^{1/n}}\cdot\frac{1}{n}t^{(1/n)-1}\,dt = \int_0^\infty \frac{e^{-t}}{a^{(m/n)+(1/n)}}t^{(m/n)+(1/n)-1}\cdot\frac{1}{n}dt$$

$$= \frac{1}{n\,a^{(m+1)/n}}\int_0^\infty e^{-t}t^{[(m+1)/n]-1}dt \;= \frac{1}{n\,a^{(m+1)/n}}\Gamma\left(\frac{m+1}{n}\right),\ \text{by definition of Gamma function}$$

Part (*ii*). Put $a = 1$ in part (*i*) **Ans.** $(1/2)\times\Gamma((m+1)/n)$

Ex. 9. *Show that* $\int_0^\infty \dfrac{x^c}{c^x}dx = \dfrac{\Gamma(c+1)}{(\log c)^{c+1}}, c > 0$ **(Purvanchal 2006)**

Sol. $$\int_0^\infty \frac{x^c}{c^x}dx = \int_0^\infty x^c c^{-x}dx = \int_0^\infty x^c[e^{\log_e c}]^{-1}dx,\ \text{as}\ c = e^{\log_e c},\ \text{if}\ c > 0$$

$$= \int_0^\infty x^{(c+1)-1}e^{-\log_e c}dx = \frac{\Gamma(c+1)}{(\log_e c)^{c+1}} \quad \left[\because \int_0^\infty x^{n-1}e^{-kx}dx = \frac{\Gamma(n)}{k^n},\ n > 0, k > 0\right]$$

Ex. 10. *Show that* $[\Gamma(1/2)]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta.$

Sol. Let $I = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy.$ ...(1)

$\therefore \quad I = 4\int_0^\infty \int_0^\infty e^{-x^2}e^{-y}dx\,dy = \left(2\int_0^\infty e^{-x^2}dx\right)\left(2\int_0^x e^{-y^2}dy\right) = \Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right),$ refer Art. 6.5

$\therefore \quad I = [\Gamma(1/2)]^2$ ... (2)

Again, put $x = r\cos\theta,\ y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $dxdy = r\,d\theta\,dr.$

Furthermore, the region of integration in integral I is the first quadrant of *xy* plane and so in polar coordinates the corresponding limits will be $r = 0$ to $r = \infty$ and $\theta = 0$ to $\theta = \pi/2$ for the same first quadrant. Hence $I = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,d\theta\,dr.$ ... (3)

From (1), (2) and (3), the required result follows

Ex. 11. Show that $\int_0^\infty \exp(2ax - x^2)dx = \frac{1}{2}\sqrt{\pi}\exp a^2,$ where $\exp k = e^k$.

Sol. $\int_0^\infty \exp(2ax - x^2)dx = \int_0^\infty e^{2ax-x^2}dx = \int_0^\infty e^{a^2-(x^2-2ax+a^2)}dx = \int_0^\infty e^{a^2-(x-a)^2}dx$

$= e^{a^2}\int_0^\infty e^{-(x-a)^2}dx = e^{a^2}\int_0^\infty e^{-t^2}dt,$ on putting $x - a = t$ and $dx = dt$

$\therefore \quad \int_0^\infty \exp(2ax - x^2)dx = \exp a^2 \int_0^\infty e^{-t^2}dt.$ ... (1)

Now, $\Gamma(n) = \int_0^\infty e^{-u}u^{n-1}du. \quad \Rightarrow \quad \Gamma(1/2) = \int_0^\infty e^{-u}u^{-1/2}du.$ ... (2)

Putting $u = t^2$ so that $du = 2t$ in (2), we get $\Gamma(1/2) = \int_0^\infty (e^{-t^2}\cdot t^{-1}\cdot 2t)\,dt$

or $\sqrt{\pi} = 2\int_0^\infty e^{-t^2}dt$ or $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$ ... (3)

Using (3), (1) reduces to $\int_0^\infty \exp(2ax - x^2)dx = \frac{1}{2}\sqrt{\pi}\exp a^2.$

EXERCISE 6(A)

1. Prove that (*i*) $\int_0^\infty e^{-4x}x^{3/2}dx = \frac{3\sqrt{\pi}}{128}$ (*ii*) $\int_0^\infty e^{-x^2}x^2dx = \frac{\sqrt{\pi}}{4}$

2. Show that $\Gamma\left(-\frac{15}{2}\right) = \frac{2^8\sqrt{\pi}}{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15}$

3. Show that if *n* is a positive integer, then $\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n 2^n\sqrt{\pi}}{1\cdot3\cdot5\ldots(2n-1)}$

4. Prove that $\int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1}dx = \frac{\Gamma(m)}{n^m},\ m > 0,\ n > 0.$ **[Garhwal 2003]**

5. Prove that $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,dt = \left(\frac{\pi}{s}\right)^{1/2}$, $s > 0$

6. Prove that (*i*) $\int_0^\infty \sqrt{x}\, e^{-x^3}\,dx = \frac{\sqrt{\pi}}{3}$ (*ii*) $\int_0^\infty 3^{-4x^2}\,dx = \frac{\sqrt{\pi}}{4\sqrt{(\log 3)}}$

7. Prove that $\int_0^\infty e^{-ax} x^n\,dx = \frac{\Gamma(n+1)}{a^{n+1}}$, $n > -1$, $a > 0$.

6.8. Symmetrical property of Beta function, i.e., B(m, n) = B(n, m).

Proof. By the definition of Beta function, we have

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx, \text{ as } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = B(n, m), \text{ by the def. of Beta function.}$$

$\therefore$ $B(m, n) = B(n, m)$.

6.9. Evaluation of *B*(*m*, *n*) in an explicit form when *m* or *n* is a positive integer

By the definition of Beta function, $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$. ... (1)

The following three cases arise:

Case 1. When only *n* is positive integer. If $n = 1$, (1) gives

$$B(m, 1) = \int_0^1 x^{m-1}(1-x)^{1-1}\,dx = \int_0^1 x^{m-1}\,dx = \left[\frac{x^m}{m}\right]_0^1 = \frac{1}{m}, \quad \text{... (2)}$$

showing that $B(m, n)$ can be evaluated when $n = 1$.

Now, let $n > 1$. Then from (1), we have

$$B(m, n) = \int_0^1 (1-x)^{n-1} x^{m-1}\,dx = \left[(1-x)^{n-1}\cdot\frac{x^m}{m}\right]_0^1 - \int_0^1 (n-1)(1-x)^{n-2}.(-1)\frac{x^m}{m}\,dx$$

$$= 0 + \frac{n-1}{m}\int_0^1 x^m (1-x)^{n-2}\,dx, \quad \text{as} \quad n > 1, \quad \text{so} \quad \lim_{x\to 0}(1-x)^{n-1}\frac{x^m}{m} = 0$$

$$= \frac{n-1}{m}\int_0^1 x^{(m+1)-1}(1-x)^{(n-1)-1}\,dx = \frac{n-1}{m}B(m+1, n-1), \text{ by def. of Beta function}$$

Thus, $B(m, n) = \frac{n-1}{m}B(m+1, n-1)$. ... (3)

Replacing m by $m + 1$ and n by $n - 1$ in (3), we get

$$B(m+1, n-1) = \frac{n-1-1}{m+1}B(m+2, n-2). \quad \text{... (4)}$$

Using (4), (3) becomes $B(m, n) = \frac{n-1}{m}\cdot\frac{n-2}{m+1}B(m+2, n-2)$ ... (5)

Since n is a positive integer and $n > 1$, after applying the above process repeatedly, we get

$$B(m, n) = \frac{n-1}{m}\cdot\frac{n-2}{m+1}\cdot\frac{n-3}{m+1}\ldots\frac{1}{m+n-2}B(m+n-1, 1). \quad \text{... (6)}$$

Replacing m by $m + n - 1$ is (2), we get $$B(m+n-1, 1) = \frac{1}{m+n-1} \quad \text{... (7)}$$

Using (7), (6) becomes $$B(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2} \cdot \frac{1}{m+n-1}$$

or $$B(m, n) = \frac{(n-1)!}{m(m+1)\,(m+2)...(m+n-2)\,(m+n-1)} \quad \text{... (8)}$$

Case II. When only *m* is a positive integer.

Since the Beta function is symmetrical in m and n, *i.e.*, $B(m, n) = B(n, m)$, hence from case *I*, interchanging m and n in (8), we get

$$B(m, n) = \frac{(m-1)!}{n(n+1)\,(n+2)...(n+m-2)\,(n+m-1)} \quad \text{... (9)}$$

Case III. When both *m* and *n* are positive integers.

Since n is a positive integer, so by case *I*, we have

$$B(m, n) = \frac{(n-1)!}{m(m+1)\,(m+2)...(m+n-2)\,(m+n-1)}$$

$$= \frac{[1\,.\,2\,.\,3\,...\,(m-1)]\,(n-1)!}{1\,.\,2\,.\,3\,....\,(m-1)m(m+1)(m+2)...(m+n-2)\,(m+n-1)} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

6.10. Transformation of Beta function

From I. To show that $B(m, n) = \int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}$, $m > 0$, $n > 0$

Proof. By definition, $$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx. \quad \text{... (1)}$$

Put $x = 1/(1 + t)$, so that $dx = -dt/(1 + t)^2$. Then, from (1), we have

$$B(m, n) = -\int_\infty^0 \frac{1}{(t+1)^{m+1}}\left(1-\frac{1}{1+t}\right)^{n-1}\frac{dt}{(1+t)^2} = \int_0^\infty \frac{1}{(t+1)^{m+1}}\left(\frac{1}{1+t}\right)^{n-1} dt = \int_0^\infty \frac{t^{n-1}dt}{(1+t)^{m+n}}$$

or $$B(m, n) = \int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} \quad \text{... (2)}$$

Since m and n are interchangeable in Beta function, (2) gives

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx. \quad \text{... (3)}$$

Thus, (2) and (3) $\Rightarrow$ $$B(m, n) = \int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} \quad \text{... (4)}$$

Deduction. Show that $B(m,n) = \frac{1}{2}\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ **[Delhi 2008; Kanpur 2004]**

Proof. Adding (3) and (4), we have

$$2B(m,n) = \int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx \quad \Rightarrow \quad B(m,n) = \frac{1}{2}\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$$

From II. To show that $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx.$ **[Delhi Phy (H) 2002, Kumaun 2002]**

Proof. From form I, we have

$$B(m, n) = \int_0^\infty \frac{x^{m-1} dx}{(1+x)^{m+n}} = \int_0^1 \frac{x^{m-1} dx}{(1+x)^{m+n}} + \int_1^\infty \frac{x^{m-1} dx}{(1+x)^{m+n}}. \quad \text{... (1)}$$

Put $x = 1/t$, so that $dx = -1/t^2\, dt$. Then, we have

$$\int_1^\infty \frac{x^{m-1} dx}{(1+x)^{m+n}} = \int_1^0 \frac{(1/t)^{m-1}(-1/t^2)\, dt}{(1+1/t)^{m+n}} = \int_0^1 \frac{t^{n-1} dt}{(1+t)^{m+n}} = \int_0^1 \frac{x^{n-1} dx}{(1+x)^{m+n}}. \quad \text{... (2)}$$

Using (2), (1) reduces $B(m, n) = \int_0^1 \frac{x^{m-1} dx}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n-1} dx}{(1+x)^{m+n}} = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx.$

From III. To show that $\int_0^\infty \frac{x^{m-1} dx}{(ax+b)^{m+n}} = \frac{B(m, n)}{a^m b^n}$ **[Delhi Maths (H) 2006]**

Proof. From form I, we have $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ ... (1)

Put $x = (at)/b$ so that $dx = (a\, dt)/b.$ Then, (1) reduces to

$$B(m, n) = \int_0^\infty \frac{(at/b)^{m-1} \times (a/b)}{(1+at/b)^{m+n}} dt = a^m b^n \int_0^\infty \frac{t^{m-1} dt}{(at+b)^{m+n}} = a^m b^n \int_0^\infty \frac{x^{m-1} dx}{(ax+b)^{m+n}}.$$

$$\therefore \quad \int_0^\infty \frac{x^{m-1} dx}{(ax+b)^{m+n}} = \frac{B(m, n)}{a^m b^n}.$$

From IV. To show that $\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{B(m, n)}{2a^m b^n}$

Proof. Put $x = \tan^2\theta$ so that $dx = 2\tan\theta\sec^2\theta\, d\theta.$ Then from form III above, we have

$$\frac{B(m, n)}{a^m b^n} = \int_0^{\pi/2} \frac{\tan^{2m-2}\theta \cdot 2\tan\theta \cdot \sec^2\theta\, d\theta}{(a\tan^2\theta + b)^{m+n}}$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2m-2}\theta (\sin\theta/\cos\theta)(\cos\theta)^{2m+2n} d\theta}{\cos^{2m-2}\theta \cos^2\theta (a\sin^2\theta + b\cos^2\theta)^{m+n}} = 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}$$

$$\therefore \quad \int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{B(m,n)}{2a^m b^n}$$

Form V. To show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1} dx}{(x+a)^{m+n}} = \frac{B(m, n)}{a^n (1+a)^m}$

Proof. Put $\frac{x}{1+a} = \frac{t}{t+a}$ so that $dx = a(1+a)\frac{dt}{(t+a)^2}$. Then (1) reduces to

$$B(m, n) = \int_0^1 (1+a)^{m-1} \left(\frac{t}{t+a}\right)^{m-1} a^{n-1} \left(\frac{1-t}{1+t}\right)^{n-1} \frac{a(a+1)\,dt}{(t+a)^2}$$

$$= a^n (1+a)^m \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}} dt = a^n (1+a)^m \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx$$

$$\therefore \quad \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx = \frac{B(m, n)}{a^n (1+a)^m}.$$

Form VI. To show that $\int_b^a (x-b)^{m-1}(a-x)^{n-1} dx = (a-b)^{m+n-1} B(m, n)$, $m > 0$, $n > 0$

[Delhi Maths (H) 2009]

Proof. Put $x = \frac{t-b}{a-b}$ so that $dx = \frac{dt}{a-b}$. Then (1) gives

$$B(m, n) = \int_b^a \left(\frac{t-b}{a-b}\right)^{m-1} \left(\frac{a-t}{a-b}\right)^{n-1} \frac{dt}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_b^a (t-b)^{m-1}(a-t)^{n-1} dt = \frac{1}{(a-b)^{m+n-1}} \int_b^a (x-b)^{m-1}(a-x)^{n-1} dx$$

$$\therefore \quad \int_b^a (x-b)^{m-1}(a-x)^{n-1} dx = (a-b)^{m+n-1} B(m, n) \qquad \ldots (1)$$

Remark 1. By putting $a = 1$, $b = -1$ in (1), we get

$$\int_{-1}^1 (x+1)^{m+1}(1-x)^{n-1} dx = 2^{m+n-1} B(m, n) = 2^{m+n-1} \frac{(m)\,\Gamma(n)}{\Gamma(m+n)}$$ **[Delhi Maths(H) 2002]**

Remark 2. Putting $b = 5$, $a = 7$, $m = 7$, $n = 4$ in (1), we get

$$\int_5^7 (x-5)^6 (7-x)^3 dx = 2^{10} B(7, 4).$$

Form VII. To show that (i) $\int_0^1 \frac{x^{m-1}(1-x)^{n-1} dx}{\{a+(b-a)x\}^{m+n}} = \frac{1}{a^n b^m} B(m, n)$

(ii) $\int_0^1 \frac{x^{m-1}(1-x)^{n-1} dx}{(b+cx)^{m+n}} = \frac{1}{(b+c)^m b^n} B(m, n), m > 0,\ n > 0.$ **[Delhi Maths (H) 2006, 09]**

Proof. By definition, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ ... (1)

Let $\frac{a}{y} - \frac{b}{x} = a - b$ so that $x = \frac{by}{a+(b-a)y}$ ... (2)

From (2), $dx = \frac{b[a+(b-a)y] - by(b-a)}{\{a+(b-a)y\}^2} dy$ or $dx = \frac{ab\,dy}{[a+(b-a)y]^2}$ ... (3)

Again from (2), we see that when $x = 1, y = 1$ and when $x = 0, y = 0$. Again, using (2), we have

$$1 - x = 1 - \frac{by}{a+by-ay} = \frac{a(1-y)}{a+(b-a)} \qquad \ldots (4)$$

Using (2), (3) and (4), (1) gives

$$B(m,n)=\int_0^1\left\{\frac{by}{a+(b-a)y}\right\}^{m-1}\left\{\frac{a(1-y)}{a+(b-a)y}\right\}^{n-1}\frac{ab\,dy}{\{a+(b-a)y\}^2}$$

$$=a^nb^m\int_0^1\frac{y^{m-1}(1-y)^{n-1}dy}{\{a+(b-a)y\}^{m+n}}=a^nb^m\int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{\{a+(b-a)x\}^{m+n}}$$

$$\therefore\quad\int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{\{a+(b-a)x\}^{m+n}}=\frac{1}{a^nb^m}B(m,n). \qquad \ldots(5)$$

Part (ii). Interchanging a and b in (5), $\displaystyle\int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{\{b+(a-b)x\}^{m+n}}=\frac{1}{a^mb^n}B(m,n)$ ... (6)

Putting $a-b=c$, *i.e.*, $a=b+c$ in (6), we get

$$\int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{(b+cx)^{m+n}}=\frac{1}{(b+c)^mb^n}B(m,n). \qquad \ldots(7)$$

6.11. Relation between Beta and Gamma Functions.

$B(m,n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, $m>0$, $n>0$. **[Agra 2000, Meerut 2004, Delhi Maths (H) 2000, 04]**

Proof. From form II of Art. 6.6, $\displaystyle\frac{\Gamma(m)}{z^m}=\int_0^\infty e^{-zx}x^{m-1}dx$ ... (1)

or $\displaystyle\Gamma(m)=\int_0^\infty z^me^{-zx}x^{m-1}dx.$ ... (2)

Multiplying both sides of (2) by $e^{-z}\,z^{n-1}$, $\displaystyle\Gamma(m)e^{-z}z^{n-1}=\int_0^\infty e^{-z(1+x)}z^{m+n-1}x^{m-1}dx.$ ... (3)

Integrating both sides of (3) w.r.t. z from 0 to ∞, we have

$$\Gamma(m)\int_0^\infty e^{-z}z^{n-1}dz=\int_0^\infty\left\{\int_0^\infty e^{-z(1+x)}z^{m+n-1}x^{m-1}dx\right\}dz \quad\text{or}\quad \Gamma(m)\Gamma(n)=\int_0^\infty\left\{\int_0^\infty e^{-z(1+x)}z^{m+n-1}dz\right\}x^{m-1}\,dx$$

[using definition of Gamma function on L.H.S. and interchanging the order of integration on R.H.S.]

or $\displaystyle\Gamma(m)\Gamma(n)=\int_0^\infty\frac{\Gamma(m+n)}{(1+x)^{m+n}}x^{m-1}dx,$ by (1)

$$=\Gamma(m+n)\int_0^\infty\frac{x^{m-1}dx}{(1+x)^{m+n}}=\Gamma(m+n)B(m,n),\ \text{by form I of Art. 6.10.}$$

$$\therefore\quad B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Deduction IA. To show that $\Gamma(n)\,\Gamma(1-n)=\pi/\sin n\pi,\ 0<n<1$

[Agra 1999, Delhi Maths (H) 2002, 08; Delhi Physics (H) 2000; Kanpur 2006]

Proof. We know that $\displaystyle B(m,n)=\int_0^\infty\frac{x^{n-1}dx}{(1+x)^{m+n}},\ m>0,\ n>0.$ ... (1)

The relation between Beta and Gamma is $B(m, n) = [\Gamma(m)\,\Gamma(n)]/\Gamma(m+n)$ ... (2)

From (1) and (2), $$\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$ ... (3)

Taking $m + n = 1$ so that $m = 1 - n$, (3) reduces to

$$\int_0^\infty \frac{x^{n-1}dx}{1+x} = \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)}, 0 < n < 1.$$ as $m > 0 \Rightarrow 1 - n > 0 \Rightarrow n < 1$; Also $n > 0$

But we know that $$\int_0^\infty \frac{x^{n-1}dx}{1+x} = \frac{\pi}{\sin n\pi}$$ and $\Gamma(1) = 1.$

$\therefore$ $$\pi/\sin n\pi = \Gamma(1-n)\Gamma(n), \quad 0 < n < 1$$

Deduction IB. To show that $\Gamma(1+n)\,\Gamma(1-n) = (n\pi)/\sin n\pi$.

Proof. L.H.S. $= n\,\Gamma(n)\,\Gamma(1-n)$, as $\Gamma(n+1) = n\,\Gamma(n)$

$= (n\pi)/\sin n\pi$, by deduction IA.

Deduction II. To show that $\Gamma(1/2) = \sqrt{\pi}$.

Proof. We have just proved that $\Gamma(n)\Gamma(1-n) = \pi/\sin n\pi$ ... (1)

Putting $n = 1/2$ in (1), we obtain

$\Gamma(1/2)\Gamma(1-1/2) = \pi/\sin(\pi/2)$ or $[\Gamma(1/2)]^2 = \pi$ or $\Gamma(1/2) = \sqrt{\pi}$

Deduction III. To show that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$ **[Meerut 1996; Rohilkhand 1999, Delhi Maths (H) 2005]**

Proof. From the definition of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$ and so $$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-1/2}dx.$$ ... (1)

Let $x = t^2$ so that $dx = 2t\,dt$. Then (1) becomes

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t^2}(t^2)^{-1/2}2t\,dt$$ or $$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-t^2}dt.$$ or $$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}dx.$$ ...(2)

Also $$\Gamma(1/2) = \sqrt{\pi}.$$ ... (3)

From (2) and (3), we obtain $$2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$$ or $$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

Deduction IV. (*i*) $B(x+1, y) = \frac{x}{x+y}B(x, y)$ (*ii*) $B(x, y+1) = \frac{y}{x+y}B(x, y)$

[Delhi Maths (H) 2009]

Proof. (*i*) $$B(x+1, y) = \frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+1+y)} = \frac{x\Gamma(x)\,\Gamma(y)}{(x+y)\,\Gamma(x+y)},$$ as $\Gamma(n+1) = n\,\Gamma(n)$

$$= \frac{x}{x+y}B(x, y).$$

(*ii*) Proceed as in part (*i*)

Deduction V. To show that for $m > 0, n > 0$, $B(m, n) = B(m + 1, n) + B(m, n + 1)$.

Proof. Using results (*i*) and (*ii*) of deduction IV, we have **[Agra 2006, 07]**

$$B(m+1, n) + B(m, n+1) = \frac{m}{m+n}B(m, n) + \frac{n}{m+n}B(m, n) = \frac{m+n}{m+n}B(m, n) = B(m, n)$$

Deduction VI. To show that

(*i*) $\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)} = \frac{B(m,n)}{2}$, $m>0$, $n>0$

[Delhi Maths (H) 2008; Purvanchal 2005; Agra 2008]

(*ii*) $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$, $p>-1, q>-1$ **[Agra 1999]**

(*iii*) $\int_0^{\pi/2} \sin^p\theta\, d\theta = \int_0^{\pi/2} \cos^p\theta\, d\theta = \frac{1\cdot 3\cdot 5\ldots(p-1)}{2\cdot 4\cdot 6\ldots p}\frac{\pi}{2}$, if p is even +ve integer

$= \frac{2\cdot 4\cdot 6\ldots(p-1)}{1\cdot 3\cdot 5\ldots p}$, is p is odd +ve integer

(*iv*) $\int_0^{\pi/2} \sin^{p-1}\theta\cos^{q-1}\theta\, d\theta = \frac{\Gamma(p/2)\Gamma(q/2)}{2\Gamma\left(\frac{p+q}{2}\right)}$

(*v*) $\int_0^{\pi/2} \sin^{p-1}\theta\, d\theta = \int_0^{\pi/2} \cos^{p-1}\theta\, d\theta = \frac{\Gamma(p/2)\Gamma(1/2)}{2\Gamma\left(\frac{p+1}{2}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(p/2)}{\Gamma\left(\frac{p+1}{2}\right)}$, as $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Proof. (*i*) By definition of Beta function, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Let $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$. Then, we have

$B(m,n) = \int_0^{\pi/2} \sin^{2m-2}(1-\sin^2\theta)^{n-1}(2\sin\theta\cos\theta d\theta)$ or $\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{B(m,n)}{2}$.

$\therefore\ \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$, as $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ ... (1)

Part (*ii*). Let $p = 2m-1$ and $q = 2n-1$, so that $m = (p+1)/2$ and $n = (q+1)/2$.

Then (1) becomes $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$ ... (2)

Part (*iii*) Replacing q by 0 in (2), $\int_0^{\pi/2} \sin^p\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$ ... (3)

Next, putting $p = 0$ and $q = p$ in (2), $\int_0^{\pi/2} \cos^p\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$ ... (4)

Let p be even, say $p = 2r$. Then R.H.S. of (3) or (4)

$$= \frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2r+2}{2}\right)} = \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(r+1)} = \frac{\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,.\,r\,(r-1)\,(2r-3)\ldots 3\,.\,2\,.\,1}$$

$$= \frac{(2r-1)\,(2r-3)\ldots 3\,.\,1}{(2r)\,(2r-2)\,(2r-4)\ldots 6\,.\,4\,.\,2}\,\frac{\pi}{2} = \frac{1\,.\,3\,.\,5\ldots(2r-3)\,(2r-1)}{2\,.\,4\,.\,6\ldots(2r-2)\,(2r)}\,\frac{\pi}{2}, \text{ as } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{1\,.\,3\,.\,5\ldots(p-3)\,(p-1)}{2\,.\,4\,.\,6\ldots(p-2)\,p}\,\frac{\pi}{2}, \text{as } \; p = 2r \qquad \ldots (5)$$

Next, let $p = 2r + 1$ *i.e.*, odd +ve integer. Then R.H.S. of (3) and (4)

$$= \frac{\Gamma(r+1)\Gamma(1/2)}{2\Gamma\left(r+\frac{3}{2}\right)} = \frac{r(r-1)\ldots 3\,.\,2\,.\,1\sqrt{\pi}}{2.\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{2\,.\,4\,.\,6\ldots(2r-2)\,(2r)}{1\,.\,3\,.\,5\ldots(2r-1)\,(2r+1)} = \frac{2\,.\,4\,.\,6\ldots(p-1)}{1\,.\,3\,.\,5\ldots p}, \qquad \ldots (6)$$

since $2r + 1 = p$. Thus from (3), (4), (5) and (6) the required results follow.

Part (*iv*) Let $2m = p$ and $2n = q$ so that $m = p/2$ and $n = q/2$. Then (1) becomes

$$\int_0^{\pi/2} \sin^{p-1}\theta\,\cos^{q-1}\theta\,d\theta = \frac{\Gamma(p/2)\,\Gamma\,(q/2)}{2\Gamma\left(\frac{p+q}{2}\right)} \qquad \ldots (7)$$

Part (*v*) Replacing q by 1 in (7),

$$\int_0^{\pi/2} \sin^{p-1}\theta\,d\theta = \frac{\Gamma(p/2)\Gamma(1/2)}{2\Gamma\left(\frac{p+1}{2}\right)} \qquad \ldots (8)$$

Next, replacing p by 1 and q by p, in (7),

$$\int_0^{\pi/2} \cos^{p-1}\theta\,d\theta = \frac{\Gamma(1/2)\Gamma(p/2)}{2\Gamma\left(\frac{1+p}{2}\right)} \qquad \ldots (9)$$

From (8) and (9), the required results follow.

6.12. Solved Examples

Ex. 1. *Evaluate the following Integrals :*

(*i*) $\int_0^1 x^4(1-x)^2\,dx$ (*ii*) $\int_0^2 \frac{x^2 dx}{\sqrt{(2-x)}}$,

(*iii*) $\int_0^a y^4\sqrt{(a^2-y^2)}\,dy$ (*iv*) $\int_0^2 x(8-x^3)^{1/3}\,dx.$ **[Agra 2000, 03]**

Sol. We know that $$\int_0^1 x^{m-1}(1-x)^{n-1}dx = B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \ldots (1)$$

Part. (*i*). $\int_0^1 x^4(1-x)^2\,dx = \int_0^1 x^{5-1}(1-x)^{3-1}\,dx = \frac{\Gamma\,(5)\,\Gamma\,(3)}{\Gamma\,(5+3)} = \frac{4\,!\,2\,!}{7\,!} = \frac{4!\times 2}{7\times 5\times 4!\times 6} = \frac{1}{105}$

Part. (*ii*) Let $I = \int_0^2 x^2(2-x)^{-1/2}\,dx.$ Let $x = 2t$, so that $dx = 2dt$, Then

$$I = \int_0^1 (2t)^2(2-2t)^{-1/2}(2dt) = 4\sqrt{2}\int_0^1 t^2(1-t)^{-1/2}\,dt = 4\sqrt{2}\int_0^1 t^{3-1}(1-t)^{1/2-1}\,dt$$

$$= 4\sqrt{2}\,\frac{\Gamma(3)\Gamma(1/2)}{\Gamma(3+1/2)} = 4\sqrt{2}\,\frac{2!\,\Gamma(1/2)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)} = \frac{64\sqrt{2}}{15}$$

Part (*iii*) Let $I = \int_0^a y^4\sqrt{(a^2-y^2)}\,dy$. Let $y^2 = a^2t$, so that $dy = \frac{a^2dt}{2y} = \frac{a\,dt}{2\sqrt{t}}$. Then

$$I = \int_0^1 (a^2t)^2\sqrt{(a^2-a^2t)}\,\frac{(a\,dt)}{2\sqrt{t}} = \frac{a^6}{2}\int_0^1 t^{3/2}(1-t)^{1/2}\,dt = \frac{a^6}{2}\int_0^1 t^{(5/2)-1}(1-t)^{(3/2)-1}\,dt$$

$$= \frac{a^6}{2}\times\frac{\Gamma(5/2)\Gamma(3/2)}{\Gamma(5/2+3/2)} = \frac{a^6}{2}\times\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{3!} = \frac{\pi a^6}{32}$$

Part (*iv*) Let $I = \int_0^2 x(8-x^3)^{1/3}\,dx$. Put $x^3 = 8t$ or $x = 2t^{1/3}$ so that $dx = (2/3) \times t^{-2/3}$ dt

$$\therefore\quad I = \int_0^1 (2t^{1/3})(8-8t)^{1/3}(2/3)t^{-2/3}dt = \frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3}\,dt = \frac{8}{3}\int_0^1 t^{(2/3)-1}(1-t)^{(4/3)-1}\,dt$$

$$= \frac{8}{3}\times\frac{\Gamma(2/3)\Gamma(4/3)}{\Gamma(2/3+4/3)} = \frac{8}{3}\times\frac{\Gamma(1-1/3)\Gamma(1+1/3)}{\Gamma(2)} = \frac{8}{3}\Gamma\left(1-\frac{1}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right),\ \text{as}\quad \Gamma(n+1) = n\,\Gamma(n)$$

$$= \frac{8}{9}\times\frac{\pi}{\sin(\pi/3)} = \frac{16\pi}{2\sqrt{3}},\ \text{as}\ \ \Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin n\pi}$$

Ex. 2. *Show that* (*a*) $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{\Gamma(1/n)}{\Gamma(1/2+1/n)}\cdot\frac{\sqrt{\pi}}{n}$. **[Meerut 2004, Purvanchal 2006]**

(*b*) $\int_0^1 \frac{dx}{(1-x^4)^{1/2}} = \frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma(1/4)}{\Gamma(1/2+1/4)}$ **[Meerut 2007]**

Sol. (*a*) Let $I = \int_0^1 \frac{dx}{(1-x^n)^{1/2}}$. Putting $x^n =$ so that $x = t^{1/n}$ and $dx = (1/n)t^{(1/n)-1}\,dt$, we get

$$I = \int_0^1 \frac{1}{(1-t)^{1/2}}\cdot\frac{1}{n}t^{(1/n)-1}dt = \frac{1}{n}\int_0^1 t^{(1/n)-1}(1-t)^{(1/2)-1}\,dt = \frac{1}{n}B\left(\frac{1}{n},\frac{1}{2}\right)$$

$$= \frac{1}{n}\,\frac{\Gamma(1/n)\Gamma(1/2)}{\Gamma(1/n+1/2)} = \frac{\sqrt{\pi}\,\Gamma(1/n)}{n\,\Gamma(1/n+1/2)}.$$

(*b*) Taking $n = 4$ in part (*a*), we get the required result.

Ex. 3. *Show that* $\int_0^a \frac{dx}{(a^n-x^n)^{1/n}} = \frac{\pi}{n\sin(\pi/n)}$. **[Meerut 1998]**

Sol. Let $I = \int_0^a \frac{dx}{(a^n-x^n)^{1/n}}$. Putting $x^n = a^n\,t$ so that $x = at^{1/n}$ and $dx = a(1/n)\,t^{(1/n)-1}\,dt$, gives

$$I = \int_0^1 \frac{1}{(a^n-a^nt)^{1/n}}\frac{a}{n}t^{(1/n)-1}dt = \frac{1}{n}\int_0^1 t^{(1/n)-1}(1-t)^{-1/n}\,dt = \frac{1}{n}\int_0^1 t^{(1/n)-1}(1-t)^{(1-1/n)-1}\,dt$$

$$= \frac{1}{n} B\left(\frac{1}{n}, 1-\frac{1}{n}\right), \quad = \frac{1}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+1-\frac{1}{n}\right)} = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{n} \quad = \frac{1}{n} \cdot \frac{\pi}{\sin(\pi/n)}.$$

$[\because \ \Gamma(p)\Gamma(1-p) = \pi/\sin p\pi\,]$

Ex. 4. *Show that* $\int_0^\infty \frac{x\,dx}{1+x^6} = \frac{1}{6} B\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{\pi}{3\sqrt{3}}$ **[Delhi Maths (H) 2007, 08]**

Sol. Let $I = \int_0^\infty \frac{x\,dx}{1+x^6}$ Putting $x^6 = t$ so that $x = t^{1/6}$ and $dx = (1/6)t^{-5/6}\,dt$, we get

$$I = \int_0^\infty \frac{t^{1/16} \cdot (1/6)t^{-5/6}}{1+t} = \frac{1}{6}\int_0^\infty \frac{t^{-2/3}dt}{1+t} \quad = \frac{1}{6}\int_0^\infty \frac{t^{(1/3)-1}}{(1+t)^{1/3+2/3}}\,dt$$

$$= \frac{1}{6} B\left(\frac{1}{3}, \frac{2}{3}\right) \qquad \left| \ \because \ \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} = B(m,n) \right.$$

$$= \frac{1}{6} \times \frac{\Gamma(1/3)\Gamma(2/3)}{\Gamma(1/3+2/3)} = \frac{1}{6}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{1}{6} \times \frac{\pi}{\sin(\pi/3)} = \frac{\pi\sqrt{3}}{9} \qquad \left| \ \because \ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \right.$$

$$= (\pi\sqrt{3})/9.$$

Ex. 5. *Evaluate* (*i*) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx$ (*ii*) $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+1}}\,dx$

(*iii*) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx.$ **[Garhwal 2000]**

Sol. Part (*i*) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^4\,dx}{(1+x)^{15}} + \int_0^\infty \frac{x^9}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^{5-1}dx}{(1+x)^{5+10}} + \int_0^\infty \frac{x^{10-1}dx}{(1+x)^{10+5}}$

$= B\ (5, 10) + B\ (10, 5)$, as $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n)$

$= 2B\ (5, 10)$, as $B\ (5, 10) = B\ (10, 5)$

$= 2\frac{\Gamma(5)\,\Gamma\,(10)}{\Gamma\,(5+10)} = \frac{2 \times 4\,! \times 9\,!}{14\,!} = \frac{1}{5005}.$

Part (*ii*) $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} - \int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}}$

$= B\ (m, n) - \mathrm{B}\ (n, m) = 0.$ $[\because \ B(m, n) = B(n, m)]$

Part (*iii*) $\int_0^\infty \frac{x^8(1-x^6)dx}{(1+x)^{24}} = \int_0^\infty \frac{x^8dx}{(1+x)^{24}} - \int_0^\infty \frac{x^{14}dx}{(1+x)^{24}} = \int_0^\infty \frac{x^{9-1}dx}{(1+x)^{9+15}} - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}$

$= B\ (9, 15) - B\ (15, 9) = 0$ $[\because \ B(9, 15) = B\ (15, 9)]$

Ex. 6. *Show by means of Beta function, that* $\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^{-\alpha}} = \frac{\pi}{\sin \pi\alpha}, 0 < \alpha < 1$

Sol. Let $I = \int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^{-\alpha}}$ ... (1)

Putting $x - t = (z - t)y$ so that $x = t + (z - t)\, y$ and $dx = (z - t)dy$, (1) becomes

$$I = \int_0^1 \frac{(z-t)dy}{[z-t-(z-t)y]^{1-\alpha}[(z-t)y]^{-\alpha}} = \int_0^1 \frac{(z-t)dy}{(z-t)^{1-\alpha}(1-y)^{1-\alpha}(z-t)^{-\alpha}\, y^{-\alpha}}$$

$$= \int_0^1 (1-y)^{\alpha-1} y^{\alpha} dy = \int_0^1 y^{(1-\alpha)-1}(1-y)^{\alpha-1} dy = B(1-\alpha, \alpha), \text{ by definition of Beta function.}$$

$$= \frac{\Gamma(1-\alpha)\,\Gamma(\alpha)}{\Gamma(1-\alpha+\alpha)} = \frac{\pi}{\sin \pi\alpha}, \qquad \text{as} \qquad \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

Ex. 7. *Prove that* (*a*) $\int_0^{\pi/2} \sqrt{\tan\theta}\; d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi\sqrt{2}}{2}$ **[Kumaun 2000 Meerut 2004]**

(*b*) $\int_0^{\pi/2} \tan^n x\, dx = \frac{\pi}{2} \sec\frac{n\pi}{2}, -1 < n < 1.$

Sol. Part (*a*) $\int_0^{\pi/2} \sqrt{\tan\theta}\; d\theta = \int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{1/2} d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$

$$= \frac{\Gamma\left(\frac{1+1/2}{2}\right)\Gamma\left(\frac{1-1/2}{2}\right)}{2\Gamma\left(\frac{1/2-1/2+2}{2}\right)}$$

Refer deduction VI (ii) of Art. 6.11.

$$= \frac{\Gamma(3/4)\Gamma(1/4)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2}\frac{\pi}{\sin(\pi/4)} = \frac{\pi\sqrt{2}}{2}$$

Part (*b*) $\int_0^{\pi/2} \tan^n x\, dx = \int_0^{\pi/2} \sin^n x \cos^{-n} x\, dx$

$$= \frac{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma\left(\frac{n-n+2}{2}\right)}$$

Refer deduction VI (ii) of Art. 6.11
Here $(1 + n)/2 > 0$ and $(1 - n)/2 > 0$
$\Rightarrow n > -1$ and $n < 1 \Rightarrow -1 < n < 1$.

$$= \frac{1}{2}\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(1-\frac{1+n}{2}\right) = \frac{1}{2}\frac{\pi}{\sin\left(\frac{1+n}{2}\right)\pi} = \frac{\pi}{2\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)} = \frac{\pi}{2\cos\frac{n\pi}{2}}$$

$= (\pi/2) \times \sec(n\pi/2)$, where $-1 < n < 1$. $\quad [\because\ \Gamma(p)\Gamma(1-p) = \pi/\sin\, p\pi\,]$

Ex. 8. *If* $p > 0, q > 0, m + 1 > 0, n + 1 > 0$, *prove* $\int_0^p x^m (p^q - x^q)^n dx = \frac{p^{nq+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$

Sol. Let $$I = \int_0^p x^m (p^q - x^q)^n dx \quad \ldots (1)$$

Putting $x^q = p^q t$ so that $x = pt^{1/q}$ and $dx = (p/q)\, t^{(1/q)-1}\, dt$, (1) reduces to

$$I = \int_0^1 (pt^{t/q})^m (p^q - p^q t)^n (p/q) t^{(1/q)-1} dt = \frac{p^m \cdot p^{nq} \cdot p}{q} \int_0^1 t^{(m/q)+(1/q)-1} (1-t)^{(n+1)-1} dt$$

$$= \frac{p^{nq+m+1}}{q} B\left(\frac{m+1}{q}, n+1\right) = \frac{p^{nq+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$$

Ex. 9. *Compute* $I = \int_0^\infty x^2 e^{-x^4} dx \cdot \int_0^\infty e^{-x^4} dx.$

Sol. Putting $x^4 = t$ so that $x = t^{1/4}$ and $dx = (1/4)\, t^{-3/4}\, dt$, we get

$$I = \int_0^\infty (t^{1/4})^2 e^{-t} \left(\frac{1}{4} t^{-3/4}\right) dt \cdot \int_0^\infty e^{-t} \frac{1}{4} t^{-3/4} dt = \frac{1}{16} \int_0^\infty e^{-t} t^{-1/4} dt \cdot \int_0^\infty e^{-t} e^{-3/4} dt$$

$$= \frac{1}{16} \int_0^\infty e^{-t} t^{(3/4)-1} \cdot \int_0^\infty e^{-t} t^{(1/4)-1} dt = \frac{1}{16} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right) = \frac{1}{16} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1 - \frac{1}{4}\right)$$

$$= \frac{1}{16} \cdot \frac{\pi}{\sin(\pi/4)} = \frac{\pi\sqrt{2}}{16}. \qquad \left| \because\ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \right.$$

Ex. 10. *Show that* $I = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$ **[Delhi B.Sc. (Prog.) 2009]**

[Garhwal 2002, Meerut 1998, Delhi Maths (H) 2005, 08]

Sol. We know that $$\int_0^{\pi/2} \sin^p \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{p+2}{2}\right)} \quad \ldots (1)$$

$$I = \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta \cdot \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta = \frac{\Gamma\left(\frac{1/2+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{1/2+2}{2}\right)} \cdot \frac{\Gamma\left(\frac{-1/2+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{-1/2+2}{2}\right)}, \text{ using (1)}$$

$$= \frac{\Gamma(3/4)\sqrt{\pi}}{2\Gamma(5/4)} \cdot \frac{\Gamma(1/4)\sqrt{\pi}}{2\Gamma(3/4)} = \frac{\pi\Gamma(1/4)}{4\Gamma(1+1/4)} = \frac{\pi\Gamma(1/4)}{4 \times (1/4) \times \Gamma(1/4)} = \pi.$$

EXERCISE 6 B

1. Prove that (*a*) $\int_0^1 \frac{dx}{(1-x^3)^{1/3}} = \frac{2\pi}{3\sqrt{3}}$ (*b*) $\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{\pi}{\sin(\pi/n)}$

(*c*) $\int_0^1 (1-x^n)^{1/n} dx = \frac{\{\Gamma(1/n)\}^2}{2\Gamma(2/n)}$ (*d*) $\int_0^1 x^{m-1}(1-x^k)^n dx = \frac{1}{a} \frac{n!\Gamma(m/k)}{\Gamma(n+1+m/k)}$

(*e*) $\int_0^1 x^{m-1}(1-x^2)^{n-1}dx = \frac{1}{2}B\left(\frac{1}{2}m, n\right)$ **[Garhwal 2003]** (*f*) $\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\Gamma(1/4)\Gamma(1/2)}{4\sqrt{2}\,\Gamma(3/4)}$

(*g*) $\int_0^1 \frac{x^{n-1}dx}{\sqrt{(1-x^2)}} = \frac{\sqrt{\pi}\,\Gamma\{(n-1)/2\}}{2\Gamma(n/2)}$ (*h*) $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{1}{6\sqrt{2\pi}}[\Gamma(1/4)]^2$

[Delhi Math (H) 2003, Meerut 2007]

2. Prove that (*a*) $\int_0^1 \left(\frac{1}{x}-1\right)^{1/4} dx = B\left(\frac{5}{4}, \frac{3}{4}\right) = \frac{\pi}{2\sqrt{2}}$ (*b*) $\int_0^\infty \frac{dt}{\sqrt{t}(1+t)} = B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$

3. Show that $\int_0^1 \frac{x^n dx}{\sqrt{(1-x^2)}} = \frac{1.3.5...(n-1)}{2.4.6...n} \times \frac{\pi}{2}$ or $\frac{2.4.6...(n-1)}{1.3.5...n}$

according as n is even or odd positive integer.

4. Show that if p and q are positive, then

$B(p, q) = 2\int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta\, d\theta = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and deduce that $\int_0^\pi e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$

5. Prove that (*i*) $B(l, m)\, B(l + m, n) = B(m, n)\, B(m + n, l) = B(n, l)\, B(n + l, m)$.

(*ii*) $B(l, m)\, B(l + m, n)\, B(l + m + n, p) = \frac{\Gamma(l)\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(l+m+n+p)}$

(*iii*) $l\, B(l, m + l) = mB\,(l + l, m)$. **[Agra 2005]**

6. Prove that $\int_{-a}^{a} (a+x)^{m-1}(a-x)^{n-1}dx = (2a)^{m+n-1}B(m, n)$

7. Prove $\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}dx = 2^{p+q-1}B(p, q)$

8. Using the integral $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}, 0 < n < 1,$ prove that $\Gamma(n)\,\Gamma(1-n) = \pi/\sin n\pi,$ $0 < n < 1$ Hence obtain value of $\Gamma(1/2)$.

9. Prove that $\int_{-1}^{1} \frac{(x+1)^{a-1}(1-x)^{b-1}}{(x+2)^{a+b}}dx = \frac{2^{a+b-1}}{3a}B(a, b), a > 0, b > 0$

10. Show that the perimeter of a loop of the curve $r^n = a^n \cos n\theta$ can be expressed as

$$(a/n) \times 2^{(1/n)-1}\{\Gamma(1/2n)/\Gamma(1/n)\}$$

11. Show that the area enclosed by the curve $x^4 + y^4 = 1$ is $[\Gamma(1/4)]^2/2\sqrt{\pi}$.

12. With the help of double integral, prove that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2.$

[Delhi Maths (H) 2007]

6.13. Legendre-Duplication Formula. $\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n),\ n>0.$

[Delhi Maths (H) 2005, 07, Delhi Phy (H) 2002 Agra 2000, 01, 02, 03, 06, 08; Meerut 1999]

Proof. We know that $B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$ where $m>0,\ n>0.$ ... (1)

Putting $m=n$ in (1), we get $B(n,n)=[\Gamma(n)]^2/\Gamma(2n)$ ... (2)

By the definition of the Beta function, $B(n,n)=\int_0^1 x^{n-1}(1-x)^{n-1}dx.$ ... (3)

Putting $x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta\,d\theta,$ (1) gives

$$B(n,n)=\int_0^{\pi/2}(\sin^2\theta)^{n-1}(1-\sin^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\,d\theta=2\int_0^{\pi/2}(\sin\theta\cos\theta)^{2n-1}d\theta$$

$$=2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^{2n-1}d\theta=\frac{1}{2^{2n-2}}\int_0^{\pi/2}\sin^{2n-1}2\theta\,d\theta=\frac{1}{2^{2n-2}}\int_0^{\pi}\sin^{2n-1}\phi\frac{d\phi}{2}=\frac{1}{2^{2n-1}}\int_0^{\pi}\sin^{2n-1}\phi\,d\phi$$

[On putting $2\theta=\phi$ and $d\theta=(d\phi)/2$]

$$=\frac{1}{2^{2n-1}}\times 2\int_0^{\pi/2}\sin^{2n-1}\phi\,d\phi,\qquad \text{as}\quad \int_0^{2a}f(x)dx=2\int_0^a f(x)\,dx\quad\text{when}\quad f(2a-x)=f(x)$$

$$=\frac{1}{2^{2n-2}}\int_0^{\pi/2}\sin^{2n-1}\phi\,(\cos\phi)^0\,d\phi=\frac{1}{2^{2n-2}}\frac{\Gamma\left(\frac{2n-1+1}{2}\right)\cdot\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2n-1+0+2}{2}\right)}$$

$$\therefore\qquad B(n,n)=\frac{1}{2^{2n-1}}\frac{\Gamma(n)\sqrt{\pi}}{\Gamma(n+1/2)},\ \text{as}\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\qquad \text{... (3)}$$

Equating two values of $B\ (n,\ n)$ given by (2) and (3), we obtain

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)}=\frac{1}{2^{2n-1}}\frac{\Gamma(n)\sqrt{\pi}}{\Gamma(n+1/2)}\qquad\text{or}\qquad \Gamma(n)\Gamma(n+1/2)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n).\qquad\text{... (4)}$$

Deduction 1. To show that $B(n,n)=\frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma(n+1/2)},\ n>0$

Proof. From (3), we get the required result.

Deduction II For all positive real values of $p,\ 2^p\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)=\sqrt{\pi}\,\Gamma(p+1).$

Proof. Putting $2n-1=p$ so that $n=(p+1)/2$ in (4), we get

$$\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^p}\Gamma\,(p+1)\qquad\text{or}\qquad 2^p\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)=\sqrt{\pi}\,\Gamma(p+1).$$

Deduction III. When n is positive integer, to show that

[Delhi Maths (H) 2003]

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}.$$

Proof. Let n be positive integer, then $\dfrac{\Gamma(2n)}{\Gamma(n)} = \dfrac{(2n-1)!}{(n-1)} = \dfrac{(2n)\,(2n-1)!}{2\cdot n\,(n-1)!} = \dfrac{(2n)!}{2\cdot n!}$

Now, from the duplication formula (4) and the above result, we have

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\cdot\frac{\Gamma(2n)}{\Gamma(n)} = \frac{\sqrt{\pi}}{2^{2n-1}}\cdot\frac{(2n)!}{2\cdot n!} = \frac{(2n)!}{2^{2n}\,n!}\sqrt{\pi}.$$

6.14. SOLVED EXAMPLES.

Ex. 1. Express $\Gamma(1/6)$ in terms of $\Gamma(1/3)$.

Sol. From the duplication formula, $\Gamma(n)\Gamma\left(n+\dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n).$ ... (1)

Putting n = 1/6 in (1), we get

$$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}} \quad \text{or} \quad \Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}\Gamma(2/3)} \quad \text{... (2)}$$

Now, we know that $\Gamma(n)\Gamma(1-n) = \pi/\sin n\pi.$ ... (3)

Putting n = 1/3 in (3) we get

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin(\pi/3)} = \frac{2\pi}{\sqrt{3}} \quad \text{or} \quad \Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}\,\Gamma(1/3)} \quad \text{... (4)}$$

Substituting the value of $\Gamma(2/3)$ given by (4) in (2), we get

$$\Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}}\cdot\frac{\sqrt{3}\,\Gamma(1/3)}{2\pi} \quad \text{or} \quad \Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{3}}{2^{1/3}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{3}\right)\right]^2.$$

Ex. 2. Prove that $\Gamma(n)\Gamma\left(\dfrac{1-n}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma(n/2)}{2^{1-n}\cos(n\pi/2)}, 0 < n < 1.$

Sol. We know that $\Gamma(m)\,\Gamma(1-m) = \pi/\sin m\pi,\ 0 < m < 1$ ... (1)

and $\Gamma(m)\Gamma\left(m+\dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}},\ m > 0,$ ... (2)

Putting $m = (n+1)/2$ in (1), we get

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right) = \frac{\pi}{\sin\{(n+1)\pi/2\}} = \frac{\pi}{\sin(\pi/2+n\pi/2)} = \frac{\pi}{\cos(n\pi/2)} \quad \text{... (3)}$$

Putting $m = n/2$ in (2), we get $\Gamma\left(\dfrac{n}{2}\right)\Gamma\left(\dfrac{n+1}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{n-1}}$ ... (4)

Dividing the corresponding sides of (3) and (4), we get

$$\frac{\Gamma[(1-n)/2]}{\Gamma(n/2)} = \frac{\pi}{\cos(n\pi/2)}\times\frac{2^{n-1}}{\sqrt{\pi}\,\Gamma(n)} \quad \text{or} \quad \Gamma(n)\Gamma\left(\frac{1-n}{2}\right) - \frac{\sqrt{\pi}\,\Gamma(n/2)}{2^{1-n}\cos(n\pi/2)}$$

Ex. 3. Prove that $B(m,m)B(m+1/2, m+1/2) = (\pi m^{-1})/2^{4m-1}$

Ex. 4. By evaluating $I = \int_0^{\pi/2}\sin^{2p}x\,dx$ and $J = \int_0^{\pi/2}\sin^{2p}x\,dx$ derive the Legendre's duplication formula for gamma function. **[Kanpur 2006]**

7

Power Series

7.1. INTRODUCTION

In this chapter we propose to study the theory of power series which is very useful tool in the study of analyis. We shall present a summary of the pertinent results of infinite series specially power series. These will be used in solving second order differential equations in the next chapter 8, namely, 'Integration in series'.

7.2. *SUMMARY OF USEFUL RESULTS

In what follows, we shall deal with infinit series and hence we shall write simply Σu_n to denote $\sum_{n=1}^{\infty} u_n$ etc.

List A : Results related to convergence of infinite series of positive terms

A-1. Let $\sum_{n=1}^{\infty} u_n$ be an infinite series and let $S_n = \sum_{i=1}^{n} u_i$. Then $\sum_{n=1}^{\infty} u_n$ is said to be convergent or divergent according as the sequence $< S_n >$ is convergent or divergent.

A-2. Geometric series. The positive term infinite geometric series $1 + r + r^2 + .. + r^n + ...$, $(r \geq 0)$ is convergent if and only if $r < 1$.

A-3. Harmonic series. The positive term series $\Sigma 1/n^{\lambda}$ is converted iff $\lambda > 1$.

A-4. Comparisian test. If Σu_n and Σv_n are two positive term series such that $\lim_{n\to\infty}(u_n / v_n) = l \neq 0,$ then the two series Σu_n and Σv_n have identical behaviours in relation to convergence.

A-5. D'Alembert's ratio test. Let Σu_n be a positive term series such that $\lim_{n\to\infty}(u_{n+1} / u_n) = l.$ Then the series is (*i*) convergent if $l < 1$ (*ii*) divergent if $l > 1$. (*iii*) No firm decision if $l = 1$. Also, if $l = \infty,$ then the series is divergent.

A-6 Cauchy's *n*th root test. Let Σu_n be a positive term series and let $\lim_{n\to\infty}(u_n)^{1/n} = l.$ Then the series is (*i*) convergent if $l < 1$ (*ii*) divergent if $l > 1$ (*iii*) No firm decision if $l = 1$.

A-7. Raabe's test. Let Σu_n be a positive term series and let $\lim_{n\to\infty} n\{(u_n / u_{n+1}) - 1\} = l.$ Then the series is (*i*) convergent if $l > 1$ (*ii*) divergent if $l < 1$ (*iii*) No firm decision if $l = 1$.

A-8. Logarithmic test. Let Σu_n be a positive term series and let $\lim_{n\to\infty} n \log\{u_n / u_{n+1}) = l.$ Then the series is (*i*) convergent if $l > 1$ (*ii*) divergent if $l < 1$ (*iii*) No firm decision if $l = 1$.

---

*For all results of this article, refer chapters 6, 7 and 15 of Real Analysis by Shanti Narayan and M.D. Raisinghania, published by S.Chand & Co. New Delhi.

List B : Results related to convergence of infinite series with positive and negative terms

B-1. Absolutely convergent series. A series Σu_n is said to be absolutely convergent if the positive term series $\Sigma |u_n|$ formed by the moduli of the terms of the series is convergent.

B-2. Alternating series. A series whose terms are alternatively positive and negative is referred to an alternating series.

B-3. Leibnitz's test. Let $< u_n >$ be a sequence such that for each natural number n (i) $u_n \geq 0$ (ii) $u_{n+1} \leq u_n$ (iii) $\lim_{n\to\infty} u_n = 0$. Then the alternating series $\Sigma(-1)^{n-1} u_n$ is convergent.

B-4. Every absolutely convergent series is convergent. The converse need not be true.

List C. Results related uniform convergence of infinite series of functions.

C-1. A series $\Sigma f_n(x)$ will converge uniformly in $[a, b]$, if there exists a convergent series ΣM_n of numbers such that $\forall\, x \in [a, b], |f_n(x)| \leq M_n$

C-2. The sum of a uniformly convergent series of continuous functions is continuous.

C-3. If $\Sigma f_n(x)$ be a uniformly convergent series of intergrable functions in $[a, b]$, then the series is term by integrable, that is,

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx$$

C-4. If $\Sigma f_n(x)$ be a uniformly convergent series of differentiable functions, then the series is term by term differentiable, *i.e.*,

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x)$$

7.3. POWER SERIES

A series of the form

$$\sum_{n=0}^{\infty} a_n x^n \qquad \ldots (1)$$

is known as real infinite power series where $a_0, a_1, \ldots, a_n, \ldots$ are real coefficients free from x, and x is the real variable. More generally $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is taken to represent a general power series. Since with a shift of origin to x_0 *i.e.*, with change of variable $x - x_0$ to x this precisely reduces to the form (1), hence without any loss of generality our studies shall be confined to the form (1).

For the sake of brevity we shall write $\Sigma a_n x^n$ instead of $\sum_{n=0}^{\infty} a_n x^n$

7.4. SOME IMPORTANT FACTS ABOUT THE POWER SERIES $\Sigma a_n x^n$.

(i) For all values of the coefficients, every power series converges for $x = 0$. Hence if a power series converges for no value other than $x = 0$, we say that the given power series is *nowhere convergent.* For example, the power series $\Sigma n^n x^n$ is nowhere convergent.

(ii) If a given series converges for all values of x, we say that the given power series is *everywhere convergent*. For example, the power series $\Sigma (x^n / n!)$ is everywhere convergent.

(iii) If the given power series converges for some value of x and diverges for other values of x, then the set of all values of x for which it is convergent is known as its *region of convergence*.

7.5. RADIUS OF CONVERGENCE AND INTERVAL OF CONVERGENCE

If a given power series does not converge everywhere or nowhere, then a definite positive number R exists such that the given power series converges (indeed absolutely) for every $|x| < R$ and diverges for every $|x| > R$. Such a number R is known as the *radius of convergence* and the interval $]-R, R[$, the *interval of convergence*, of the given power series.

7.6. FORMULAS FOR DETERMINING THE RADIUS OF CONVERGENCE

Theorem I. *If the power series* $\Sigma a_n x^n$ *is such that* $a_n \neq 0$ *for all n and* $\lim_{n\to\infty} |a_{n+1}/a_n| = 1/R$, *then* $\Sigma a_n x^n$ *is convergent (indeed absolutely) for* $|x| < R$ *and divergent for* $|x| > R$..

[Delhi Maths (H) 2006]

Proof. Let $u_n = a_n x^n$ so that $u_{n+1} = a_{n+1} x^{n+1}$. Then, we have

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}\, x}{a_n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{R} \qquad \ldots (1)$$

$\therefore$ By D' Alembert's ratio test, $\Sigma a_n x^n$ converges absolutely if $|x|/R < 1$, *i.e.*, $|x| < R$. Also, $\Sigma a_n x^n$ diverges if $|x| > R$.

Theorem II. *If the power series* $\Sigma a_n x^n$ *is such that* $a_n \neq 0$ *for all n and* $\lim_{n\to\infty} |a_n|^{1/n} = 1/R$, *then* $\Sigma a_n x^n$ *is convergent (indeed absolutely) for* $|x| < R$ *and divergent for* $|x| > R$.

Proof. According to Cauchy's second theorem on limits, if $< |a_n| >$ is a sequence of positive constants, then

$$\lim_{n\to\infty} |u_n|^{1/n} = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|, \qquad \ldots (1)$$

provided the limit on the right side of (1) exists, whether finite or infinite. Also given that

$$\lim_{n\to\infty} |u_n|^{1/n} = 1/R \qquad \ldots (2)$$

$$\therefore \quad (1) \text{ and } (2) \Rightarrow \quad \lim_{n\to\infty} |u_{n+1}/u_n| = 1/R \qquad \ldots (3)$$

Using (3), the result of the theorem follows from theorem I.

In view of the above discussion, the radius of convergence R of the power series $\Sigma a_n x^n$ can be determined as follows :

$$R = 1 \div \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \quad \text{or} \quad R = 1 \div \lim_{n\to\infty} |a_n|^{1/n} \qquad \ldots (*)$$

$R = \infty$ if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$ or $\lim_{n\to\infty} |a_n|^{1/n} = 0$

$R = 0,$ if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$ or $\lim_{n\to\infty} |a_n|^{1/n} = \infty$

Note 1. Every power series converges absolutely within its interval of convergence.

Note 2. Observe that formula (*) is derived with the supposition of existence of the finite limit $\lim |a_n / a_{n+1}|$, that is, with the supposition that the power series $\Sigma a_n x^n$ contains all powers of x. Indeed for the power series $\Sigma\{(2x)^{2n+1}/(2n+1)\}$, the coefficients of even powers of x are equal to zero, $a_{2m} = 0$ and hence $\lim_{n\to\infty}(a_{2n+1}/a_{2n}) = \infty$ and $\lim_{n\to\infty}(a_{2n+2}/a_{2n+1}) = 0$. This shows that we cannot apply the formula (*) to the given power series. However, a direct application of D' Alembert's ratio test leads to the desired result:

Here, let $u_n = (2x)^{2n+1}/(2n+1)$ so that $u_{n+1} = (2x)^{2n+3}/(2n+3)$

$$\therefore \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{2n+3}}{(2x)^{2n+1}}\times\frac{2n+1}{2n+3}\right| = 4|x|^2 \lim_{n\to\infty}\left|\frac{2+1/n}{2+3/n}\right| = 4|x|^2$$

Therefore, by D' Alembert's ratio test, the given power series converges absolutely if

$4|x|^2<1$ or $|x|^2<1/4$ or $|x|<1/2.$

Note. If the given power series is given in general form $\Sigma a_n(x-x_0)^n$, then formula (*) is used to find the radius of convergence R. In such a case, we say that the given power series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$. The interval of convergence is given by $]x_0-R, x_0+R[$.

7.7. SOLVED EXAMPLES BASED ON ART. 7.6.

Ex. 1. *Find the radius of convergence of the following series*

(*i*) $\frac{x}{2}+\frac{1\cdot3}{2\cdot5}x^2+\frac{1\cdot3\cdot5}{2\cdot5\cdot8}x^3+....$ **[Delhi Maths (H) 2006]**

(*ii*) $1+\frac{a\cdot b}{1\cdot c}+\frac{a(a+1)\,b\,(b+1)}{1\cdot2\,c\,(c+1)}+...$ **[Delhi Maths (H) 2003]**

Sol. (*i*) Let the given series be denoted by $\sum_{n=1}^{\infty} a_n x^n$.

Then, here $a_n = \frac{1\cdot3\cdot5\,...\,(2n-1)}{2\cdot5\cdot8...(3n-1)}$ and $a_{n+1} = \frac{1\cdot3\cdot5\,...\,(2n-1)\,(2n+1)}{2\cdot5\cdot8\,...(3n-1)\,(3n+2)}$

$$\therefore \text{ Radius of convergence } = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{3n+2}{2n+1}\right| = \lim_{n\to\infty}\left|\frac{3+2/n}{2+1/n}\right| = \frac{3}{2}$$

(*ii*) Omitting the first term, let the given series be denoted by $\Sigma a_n x^n$. Then, here we have

$$a_n = \frac{a\,(a+1)...(a+n-1)\,b\,(b+1)\,...\,(b+n-1)}{1\cdot2\,...\,n\,c\,(c+1)\,...\,(c+n-1)}$$

and $$a_{n+1} = \frac{a\,(a+1)...(a+n-1)\,(a+n)\,b\,(b+1)\,...\,(b+n-1)\,(b+n)}{1\cdot2\,...\,n\,(n+1)\,c\,(c+1)\,...\,(c+n-1)\,(c+n)}$$

Radius of convergence $= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,(c+n)}{(a+n)\,(b+n)}\right| = \lim_{n\to\infty}\left|\frac{(1+1/n)\,(1+c/n)}{(1+a/n)\,(1+b/n)}\right| = 1$

Ex. 2. *Find the radius of convergence the exact interval of convergence of the power series* $\Sigma\frac{(n+1)}{(n+2)\,(n+3)}x^n.$

Sol. Let the given series be denoted by $\Sigma a_n x^n$ or Σu_n. Then, we have

$a_n = (n+1)\,/\,\{(n+2)\,(n+3)\}$ and $a_{n+1} = (n+2)\,/\,\{(n+3)\,(n+4)\}$

$$\therefore \quad \text{R = radius of convergence } = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)\,(n+4)}{(n+2)^2} = 1$$

Hence the given series converges for $|x| <$ and diverges for $|x| > 1$. We now investigate the nature of the given power series when $|x| = 1$, *i.e.*, when $x = 1$ and $x = -1$.

For $x = 1$, $$u_n = \frac{(n+1)}{(n+2)(n+3)} = \frac{1}{n} \times \frac{(1+1/n)}{(1+2/n)(1+3/n)} \quad \ldots (1)$$

Let the companion series Σv_n be such that $v_n = 1/n.$

Then, $\lim_{n\to\infty}(u_n / v_n) = 1$, which is finite and non zero.

Again, $\Sigma v_n = \Sigma(1/n)$ is a divergent series. So by comparison test Σu_n diverges for $x = 1$.

Next, for $x = -1$, the given series is an alternating series for which $u_{n+1} < u_n$ for each natural number n and $\lim_{n\to\infty} u_n = 0,$ by (1). Hence, by Leibnitz's test the given series converges for $x = -1$. Hence the exact interval of convergence is $[-1, 1[$.

Ex. 3. *Determine the interval of convergence of the power series* $\Sigma\{(1/n) \times (-1)^{n+1}(x-1)^n\}$.

Sol. Let the given series be denoted by $\Sigma a_n (x - x_0)^n$. Then, we have

$$a_n = (-1)^{n+1}/n \qquad \text{and} \qquad a_{n+1} = (-1)^{n+2}/(n+1).$$

$$\therefore \quad R = \text{radius of convergence} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|-\frac{n+1}{n}\right| = 1$$

Since the given power series is about the point $x = x_0 = 1$, the interval of convergence is

$$x_0 - R < x < x_0 + R, \qquad i.e., \qquad -1 + 1 < x < 1 + 1, \qquad i.e., \qquad 0 < x < 2.$$

For $x = 2$, the given series reduces to the alternating series $\Sigma(-1)^{n-1}/n$ $(= \Sigma(-1)^{n-1}u_n,$ say) for which $u_{n+1} < u_n$ for each natural number n and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(1/n) = 0.$ Hence by Leibnitz's test the given series is convergent when $x = 2$.

Next, for $x = 0$, clearly the given series diverges. Hence the exact interval of convergence is $]0, 2]$.

EXERCISE 7 (A)

Determine the radius of convergence and the exact interval of convergence of each of the following power series.

1. (*i*) $\Sigma\dfrac{nx^n}{(n+1)^2}$ (*ii*) $\Sigma\dfrac{2^n x^n}{n!}$ (*iii*) $\Sigma\dfrac{x^n}{n^3}$ (*iv*) $\Sigma\dfrac{x^n}{n^n}$

2. (*i*) $\Sigma\dfrac{(2n)!x^{2n}}{(n!)^2}$ (*ii*) $\Sigma\dfrac{(-1)^n x^{2n}}{(n!)^2 2^{2n}}$ (*iii*) $\Sigma(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$

(*iv*) $\Sigma\dfrac{(n!)^2 x^{2n}}{(2n)!}$ (*v*) $\Sigma(-1)^n\dfrac{x^{2n+1}}{(2n+1)}$

3. (*i*) $\dfrac{(x-1)^n}{2^n}$ (*ii*) $\Sigma\dfrac{(-1)^n(x-1)^n}{2^n(3n-1)}$ (*iii*) $\Sigma\dfrac{n!(x+2)^n}{n^n}$

4. If the power series $\Sigma a_n x^n$ has radius of convergence R, then prove that, for any positive integer m, $\Sigma a_n x^{mn}$ has radius of convergence $R^{1/m}$.

ANSWERS

1. (*i*) $R = 1$, $[-1, 1[$; (*ii*) $R = \infty$, R; (*iii*) $R = 1$, $[-1, 1]$ (*iv*) $R = \infty$, R

2. (*i*) R = 1/4; (*ii*) $R = \infty$, R; (*iii*) $R = 1$, $[-1, 1]$ (*iv*) $R = 4$, $]-4, 4[$ (*v*) $R = 1$, $[-1, 1]$

3. (*i*) $R = 2$, $]-1, 3[$, (*ii*) $R = 2$, $]-1, 3]$ (*iii*) $R = e$, $]-2-e, -2+e[$

7.8. Some theorems about power series $\Sigma a_n x^n$

Theorem I. *If a power series* $\Sigma a_n x^n$ *converges for* $x = x_0$, *then*

(*i*) *it is absolutely convergent in the interval* $|x| < |x_0|$

(*ii*) *it is uniformly convergent in the interval* $|x| \le |x_1|$, *where* $|x_1| < |x_0|$.

[Delhi Maths (H) 2004]

Solution (*i*) $\Sigma a_n x_0^n$ is convergent $\Rightarrow \lim_{n\to\infty} a_n x_0^n = 0$

$\Rightarrow$ there exists a positive integer m such that

$|a_n x_0^n - 0| < 1 \quad \forall\, n \ge m$ so that $|a_n| < 1/|x_0|^n, \quad \forall\, n \ge m$ ... (1)

Now, $|a_n x^n| = |a_n||x|^n < |x|^n / |x_0|^n, \quad \forall\, n \ge m,$ by (1)

Then, $|a_n x^n| < (|x|/|x_0|)^n, \quad \forall\, n \ge m$ ... (2)

The series on the R.H.S. of (2) converges for $|x| < |x_0|$ (being a geometric series with common ratio < 1). Hence, by the comparison test $\Sigma a_n x^n$ is convergent for $|x| < |x_0|$. Therefore, $\Sigma a_n x^n$ is absolutely convergent for $|x| < |x_0|$

(*ii*) Let $M_n = |x_1|^n / |x_0|^n$. Then ΣM_n converges since $|x_1| < |x_0|$ (being a geometric series with common ratio < 1).

Now, $|a_n x_n| = |a_n||x|^n < |x|^n / |x_0|^n, \quad \forall\, n \ge m$ using (1)

$< |x_1|^n / |x_0|^n, \forall\, n \ge m$ since $|x| \le |x_1|$

Thus, $|a_n x^n| < M_n, \forall\, n \in N,$ where ΣM_n converges.

Hence, from the Weierstrass *M*-test, it follows that $\Sigma a_n x^n$ is uniformly convergent in the interval $|x| \le |x_1|$ where $|x_1| < |x_0|$.

Theorem II. *If a power series* $\Sigma a_n x^n$ *converges for* $|x| < R$ *and if a function f(x) is defined as* $f(x) = \Sigma a_n x^n, |x| < R,$ *then* $f(x) = \Sigma a_n x^n$ *converges uniformly on* $[-R+\varepsilon, R-\varepsilon]$ *for every* $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ be any given number. Then, we have

$|x| \le R - \varepsilon \quad \Rightarrow \quad |a_n x^n| \le |a_n| (R-\varepsilon)^n$ ... (1)

Since every power series converges absolutely within its interval of convergence, it follows that $\Sigma a_n (R-\varepsilon)^n$ converges absolutely. Hence by Weierstrass's M-test it follows that the series $\Sigma a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$.

Theorem III. *The series obtained by integrating and differentiating power series term by term has the same radius of convergence as the original series.*

Proof. Let R be the radius of convergence of the given power series $\sum_{n=0}^{\infty} a_n x^n$. ... (1)

On integrating (1) term by term, we get $\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ ... (2)

Let R' be the radius of convergence of (2). Then, we have

$$R = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}} \quad \text{and} \quad R' = \lim_{n\to\infty} \frac{(n+1)^{1/n}}{|a_n|^{1/n}} \quad \text{... (3)}$$

Let $l = \lim_{n\to\infty} (n+1)^{1/n}$ so that $\log l = \lim_{n\to\infty} \frac{\log(1+n)}{n} = \lim_{n\to\infty} \frac{1/(n+1)}{1}$, by L' Hopital's rule

$$\Rightarrow \quad \log l = 0 \quad \Rightarrow \quad l = e^0 = 1 \quad \Rightarrow \quad \lim_{n\to\infty} (n+1)^{1/n} = 1 \quad \text{... (4)}$$

Using (4), (3) $\Rightarrow$ $R' = R$.

Next, differentiating (1) term by term, we get $\sum_{n=1}^{\infty} n a_n x^{n-1}$ ... (5)

Let R'' be the radius of convergence of (5). Then

$$R'' = \lim_{n\to\infty} \frac{1}{n^{1/n} |a_n|^{1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} \times \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}} \quad \text{... (6)}$$

Let $m = \lim_{n\to\infty} n^{1/n}$ so that $\log m = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{(1/n)}{1}$, by L' Hopital's rule

$$\Rightarrow \quad \log m = 0 \quad \Rightarrow \quad m = e^0 = 1 \quad \Rightarrow \quad \lim_{n\to\infty} n^{1/n} = 1 \quad \text{... (7)}$$

From (3), (6) and (7), we have $R'' = R$.

Exercise. Show that both the power series $\sum_{n=0}^{\infty} a_n x^n$ and corresponding series of derivatives $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence. **[Delhi Maths (H) 2001]**

[**Hint.** Refer first part of the above theorem III.]

Theorem IV. *Let the given power series* $\Sigma a_n x^n$ *converges for* $|x| < R$ *and let* $f(x) = \Sigma a_n x^n$. *Then* (*i*) *f*(*x*) *is continuous in* $]-R, R[$. **[Delhi Maths (H) 1995]**

(*ii*) $\Sigma a_n x^n$ *can be integrated term by term in* $]-R, R[$.

(*iii*) $\Sigma a_n x^n$ *can be differentiated term by term in* $]-R, R[$.

[**Note :** If $f(x) = \Sigma a_n x^n$, then *f*(*x*) is known as the *sum function* of the series.]

Proof (*i*) Since each term of the series $\Sigma a_n x^n$ is continuous on $]-R, R[$ and $\Sigma a_n x^n$ is uniformly convergent on $[-R+\varepsilon, R-\varepsilon]$, hence the sum function *f*(*x*) of $\Sigma a_n x^n$ is also continuous.

(*ii*) Since each term of $\Sigma a_n x^n$ is continuous on $[-R, R[$ and $\Sigma a_n x^n$ is uniformly covergent in $[-R+\varepsilon, R-\varepsilon]$, hence $\Sigma a_n x^n$ is term by term integrable.

(*iii*) Since each term of $\Sigma a_n x^n$ in continuous, possess continuous derivatives in $]-R, R[$ and $\Sigma a_n x^n$ is uniformly convergent in $[-R+\varepsilon, R-\varepsilon]$, hence $\Sigma a_n x^n$ is term by term differentiable.

Exercise. Let the power series $\Sigma a_n x^n$ converge for $|x| < R$, and $f(x) = \Sigma a_n x^n, |x| < R$, prove that $\Sigma a_n x^n$ converge uniformly on $[-R+\varepsilon, R-\varepsilon]$, no matter which $\varepsilon > 0$ is chosen, and that the function *f* is continuous and differentiable on [–*R*, *R*]. **[Delhi Maths (H) 2007]**

8

Integration In Series

8.1. INTRODUCTION

In may happen that a given linear differential equation comes under none of the standard classes which are all of some particular form, and thus we may fail to express its solution in terms of elementary functions, namely, polynomials, rational functions, exponentials, trigonometric fucntions, hyperbolic functions, logarithms etc. In such situations we have to find a convergent series arranged according to powers of the independent variable, which will approximately express the value of the dependent variable. The solution in the form of an infinite series is called '*integration in series*'. In this chapter we propose to discuss some methods of getting solution in the form of infinite series for second order linear equation.

8.2. Some basic definitions. **(Jabalpur 2004)**

Power series. An infinite series of the form

$$\sum_{n=0}^{\infty} C_n (x - x_0)^n = C_0 + C_1(x - x_0) + C_2(x - x_0)^2 + \ldots \quad \ldots(1)$$

is called a power series in $(x - x_0)$. In particular, a power series in x is an infinite series

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \ldots \quad \ldots(2)$$

For example, the exponential function e^x has the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

The power series (1) converges (absolutely) for $| x | < R$, where

$$R = \lim_{n \to \infty} \left| \frac{C_n}{C_{n+1}} \right|, \text{ provided the limit exists.} \quad \ldots(3)$$

R is said to be the *radius of convergence* of power series (1). The interval $(-R, R)$ is said to be the *interval of convergence*.

Since $R = \infty$ for the power series (2), hence the interval of convergence of the power series (2) is $(-\infty, \infty)$ *i.e.* the real line.

In what follows we shall use the following results :

(*i*) *A power series represents a continuous function within its interval of convergence.*

(*ii*) *A power series can be differentiated termwise within its interval of convergence.*

For more results about power series, refer chapter 7.

Analytic function. A function $f(x)$ defined on an interval containing the point $x = x_0$ is called *analytic at* x_0 if its Taylor series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0) \quad \ldots(4)$$

exists and converges to $f(x)$ for all x in the interval of convergence of (4).

Hence, we find that all polynomial functions, e^x, sin x, cos x, sinh x and cosh are analytic everywhere. A rational function is analytic except at those values of x at which its denominator is zero. For example, the rational function defined by $x/(x^2 - 3x + 2)$ is analytic everywhere except at $x = 1$ and $x = 2$.

8.3. Ordinary and singular points. [Nagpur 1996, Ravishankar 1993]

Definitions. A point $x = x_0$ is called an *ordinary point* of the equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad ...(1)$$

if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

If the point $x = x_0$ is not an ordinary point of the differential equation (1), then it is called a *singular point* of the differential equation of (1). There are two types of singular points :

(*i*) regular singular points and (*ii*) irregular singular points.

A singular point $x = x_0$ of the differential equation (1) is called a *regular singular point* of the differential equation (1) if both $(x - x_0)\,P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$.

A singular point, which is not regular is called an *irregular singular point*.

8.4. Solved examples based on Art. 8.3.

Ex. 1. *Determine whether $x = 0$ is an ordinary point or a regular singular point of the differential equation $2x^2(d^2y/dx^2) + 7x(x + 1)\,(dy/dx) - 3y = 0$.* **[Delhi Maths (Hons) 1993, 2000]**

Sol. Dividing by $2x^2$, the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{7(x+1)}{2x}\frac{dy}{dx} - \frac{3}{2x^2}y = 0. \qquad ...(1)$$

Comparing (1) with standard equation $y'' + P(x)\,y' + Q(x)\,y = 0$, we have

$$P(x) = [7(x + 1)]/2x \quad \text{and} \quad Q(x) = -3/(2x^2). \qquad ...(2)$$

Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$, so both $P(x)$ and $Q(x)$ are not analytic at $x = 0$. Thus $x = 0$ is not an ordinary point and so $x = 0$ is a singular point.

Also, $(x - 0)P(x) = 7(x + 1)/2$ and $(x - 0)^2\,Q(x) = -3/2$, showing that both $(x - 0)P(x)$ and $(x - 0)^2Q(x)$ are analytic at $x = 0$. Therefore $x = 0$ is a regular singular point.

Ex. 2. *Show that $x = 0$ is an ordinary point of $(x^2 - 1)y'' + xy' - y = 0$, but $x = 1$ is a regular singular point.* **[Ranchi 2010]**

Sol. Dividing by $(x^2 - 1)$, the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{x}{(x-1)(x+1)}\frac{dy}{dx} - \frac{1}{(x-1)(x+1)}y = 0. \qquad ...(1)$$

Comparing (1) with with standard equation $y'' + P(x)y' + Q(x)y = 0$, we have

$$P(x) = x/[(x - 1)(x + 1)] \quad \text{and} \quad Q = -1/\{(x - 1)\,(x + 1)\}.$$

Since both $P(x)$ and $Q(x)$ are analytic at $x = 0$, so $x = 0$ is an ordinary point of the given equation (1).

Since both $P(x)$ and $Q(x)$ are undefined at $x = 1$, so they are not analytic at $x = 0$. Thus $x = 1$ is not an ordinary point and so $x = 1$ is a singular point.

Also $(x - 1)P(x) = x/(x + 1)$ and $(x - 1)^2Q(x) = -(x - 1)/(x + 1)$, showing that both $(x - 1)P(x)$ and $(x - 1)^2Q(x)$ are analytic at $x = 1$. Therefore $x = 1$ is a regular singular point.

Ex. 3. *Show that $x = 0$ and $x = -1$ are singular points of $x^2(x + 1)^2y'' + (x^2 - 1)y' + 2y = 0$, where the first is irregular and the other is regular.*

Sol. Dividing by $x^2(x + 1)^2$, the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{x-1}{x^2(x+1)}\frac{dy}{dx} + \frac{2}{x^2(x+1)^2}y = 0. \qquad ...(1)$$

Comparing (1) with standard equation $y'' + P(x)y' + Qy = 0$, we get

$P(x) = (x-1)/[x^2(x+1)]$ and $Q(x) = 2/[x^2(x+1)^2]$.

Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$ and $x = -1$, so they are not analytic at $x = 0$ and $x = -1$. Hence $x = 0$ and $x = -1$ are both singular points.

Also $(x-0)P(x) = (x-1)/[x(x+1)]$ and $(x-0)^2Q(x) = 2/(x+1)^2$, showing that $P(x)$ is not analytic at $x = 0$ and so $x = 0$ is an irregular singular point.

Again, $(x+1)P(x) = (x-1)/x^2$ and $(x+1)^2Q(x) = 2/x^2$, showing that both $(x+1)P(x)$ and $(x+1)^2Q(x)$ are analytic at $x = -1$ and hence $x = -1$ is a regular singular point.

Ex. 4. *Discuss the singularities of the equation* $x^2y'' + xy' + (x^2 - n^2)y = 0$ *at* $x = 0$ *and* $x = \infty$. **(Delhi Physics (Hons.) 2000, 02; Bhopal 2010)**

Sol. Discussion about singularity at $x = 0$. Re-writing the given equation

$$y'' + (1/x)y' + \{(x^2 - n^2)/x^2\}y = 0 \qquad ... (1)$$

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, $P(x) = 1/x$ and $Q(x) = (x^2 - n^2)/x^2$.

Here $(x-0)\,P(x) = 1$ and $(x-0)^2\,Q(x) = x^2 - n^2$, showing that both $(x-0)\,P(x)$ and $(x-0)^2\,Q(x)$ are analytic at $x = 0$. Therefore, $x = 0$ is a regular singular point.

Discussion about singularity at $x = \infty$.

Let $x = 1/t$ or $t = 1/x$. Then, $dt/dx = -1/x^2$ ... (2)

Now, $$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}, \text{ by (2)} \qquad ... (3)$$

and $$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\left(-\frac{1}{x^2}\right), \text{ by (2) and (3)}$$

or $$y'' = \left(-t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dx}\right)\times(-t^2) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} \qquad ... (4)$$

Using (2), (3) and (4), the given equation reduces to

$$\frac{1}{t^2}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \frac{1}{t}\left(-t^2\frac{dy}{dt}\right) + \left(\frac{1}{t^2} - n^2\right)y = 0 \quad \text{or} \quad t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \frac{1-n^2t^2}{t^2}y = 0$$

or $$(d^2y/dt^2) + (1/t)\times(dy/dt) + \{(1-n^2t^2)/t^4\} = 0 \qquad ... (5)$$

Comparing (5) with $(d^2y/dt^2) + P(t)\,(dy/dt) + Q(t)y = 0$, here

$P(t) = 1/t$ and $Q(t) = (1-x^2t^2)/t^4$. Then, we have

$(t-0)\,P(t) = 1$ and $(t-0)^2\,Q(t) = (1 - n^2\,t^2)/t^2$.

Since $(t-0)^2\,Q(t)$ is not analytic at $t = 0$, so $t = 0$ is irregular singular point of (5). In view of (2), $x = \infty$ is an irregular singular point of the given equation.

EXERCISE 8 (A)

1. Show that $x = 0$ is an ordinary point of $y'' - xy' + 2y = 0$.

2. Determine whether $x = 0$ is an ordinary point or regular singular point for the differential equation $2x^2y'' - xy' + (x - 5)y = 0$. **Ans.** $x = 0$ is regular singular point

3. Show that $x = 0$ is an ordinary point of $(x^2 + 1)y'' + xy' - xy = 0$.

4. Show that $x = 0$ is a regular singular point of $x^2y'' + xy' + (x^2 - 1/4)y = 0$.

5. Show that $x = 0$ is a regular singular point and $x = 1$ is an irregular singular point of $x(x-1)^3y'' + 2(x-1)^3y' + 3y = 0$.

6. Verify that origin is regular singular point of the equation $2x^2y'' + xy' - (x + 1)y = 0$.

7. Determine the nature of the point $x = 0$ for the equations

(*i*) $xy'' + y \sin x = 0$ **[Nagpur 1996]** (*ii*) $x^3y'' + y \sin x = 0$ **[Nagpur 2005]**

Ans. (*i*) Regular singular point (*ii*) Irregular singular point

8. Determine the singular points and their nature for the following differential equations :

(*i*) $3xy'' + 2x(x - 1)y' + 5y = 0$ (*ii*) $y'' + (1 - x)y' + (1 - x)^2 y = 0$ **[Utkal 2003]**

Ans. (*i*) $x = 0$ is regular singular point (*ii*) There is no singular point.

8.5. Power series solution in power of $(x - x_0)$ or the power series solution near the ordinary point $x = x_0$ or power series solution about the ordinary point $x = x_0$.

Let the given equation be $y'' + P(x)y' + Q(x)y = 0.$ ...(1)

If $x = x_0$ is an ordinary point of (1), then (1) has two non–trivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} C_n (x - x_0)^n \qquad ...(2)$$

and these power series converge in some interval of convergence $| x - x_0 | < R$, (where R is the radius of convergence of (2)) about x_0. In order to get the coefficients C_n's is (2), we take

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n . \qquad ...(3)$$

Differentiating twice in succession, (3) gives

$$y' = \sum_{n=1}^{\infty} n\, C_n (x - x_0)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)\, C_n (x - x_0)^{n-2} . \qquad ...(4)$$

Putting the above values of y, y' and y'' in (1), we get an equation of form

$$A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + ... + A_n(x - x_0)^n + ... = 0, \qquad ...(5)$$

where the coefficients A_0, A_1, A_2 .... etc. are now some functions of the coefficients C_0, C_1, C_2, ... etc.

Since (5) is an identity, all the coefficients A_0, A_1, A_2, ... of (5) must be zero, *i.e.*,

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = 0, \quad ...., \quad A_n = 0. \qquad ...(6)$$

Solving equation (6), we obtain the coefficients of (3) in terms of C_0 and C_1. Substituting these coefficients in (3), we obtain the required series solution of (1) in powers of $(x - x_0)$.

8.6. Solved examples based on Art. 8.5.

Ex. 1. *Find the power series solution of the equation* $(x^2 + 1)y'' + xy' - xy = 0$ in powers of x (*i.e.* about $x = 0$). **[Delhi B.Sc. (Hons.) 1993, 2000, 06, 07]**

Sol. Given that $(x^2 + 1)y'' + xy' - xy = 0.$ ...(1)

Dividing by $(x^2 + 1)$, (1) can be written in standard form as

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{x}{x^2+1}y = 0. \qquad ...(2)$$

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, here we have

$P(x) = x/(x^2 + 1)$ and $Q(x) = -x/(x^2 + 1)$,

showing that $P(x)$ and $Q(x)$ are analytic at $x = 0$. So $x = 0$ is an ordinary point. Therefore, to solve (1), we take power series $y = C_0 + C_1x + C_2x^2 + C_3x^3 + ... = \sum_{n=0}^{\infty} C_n\, x^n$ ...(3)

Differentiating (3) twice in succession w.r.t. 'x', we get

$y' = \sum_{n=1}^{\infty} nC_n\, x^{n-1}$ and $y'' = \sum_{n=1}^{\infty} n(n-1)\, C_n\, x^{n-2}$ ....(4)

Substituting the above values of y, y' and y'' in (1), we get

$$(x^2 + 1)\sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2} + x\sum_{n=1}^{\infty} nC_n\, x^{n-1} - x\sum_{n=0}^{\infty} C_n\, x^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n\, x^n + \sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2} - \sum_{n=1}^{\infty} nC_n\, x^n - \sum_{n=0}^{\infty} C_n\, x^{n+1} = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n\, x^n + \sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}\, x^n + \sum_{n=1}^{\infty} nC_n\, x^n - \sum_{n=1}^{\infty} C_{n-1}\, x^n = 0$$

or $$2C_2 + (6C_3 + C_1 - C_0)x + \sum_{n=2}^{\infty}[n(n-1)C_n + (n+2)(n+1)C_{n+2} + nC_n - C_{n-1}]x^n = 0. \quad ...(5)$$

Since (5) is an identity, equating the constant term and the coefficients of various powers of x to zero, we get $2C_2 = 0$ so that $C_2 = 0$ ...(6)

$6C_3 + C_1 - C_0 = 0$ so that $C_3 = (C_0 - C_1)/6$ ...(7)

$$n(n-1)C_n + (n+2)(n+1)C_{n+2} + nC_n - C_{n-1} = 0 \quad \text{for all} \quad n \geq 2$$

or $$C_{n+2} = \frac{C_{n-1} - n^2C_n}{(n+1)(n+2)}, \quad \text{for all} \quad n \geq 2. \quad ...(8)$$

The above relation (8) in known as *recurrence relation.*

Putting $n = 2$ is (8), $C_4 = (C_1 - 4C_2)/12 = C_1/12$, as $C_2 = 0$. ...(9)

Putting $n = 3$ in (8), $C_5 = -\frac{9C_3}{20} = -\frac{9}{20}\left(\frac{C_0 - C_1}{6}\right) = -\frac{3}{40}(C_0 - C_1)$. ...(10)

Putting the above values of C_2, C_3, C_4, C_5, ... etc. in (3), we have

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + ... \text{ ad. inf.}$$

or $$y = C_0 + C_1x + (1/6) \times (C_0 - C_1)x^3 + (1/12) \times C_1x^4 - (3/40) \times (C_0 - C_1)x^5 + ...$$

or $$y = C_0\left(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + ...\right) + C_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - ....\right),$$

which is the required solution near $x = 0$, where C_0 and C_1 are arbitrary constants.

Ex. 2(*a*). *Find the solution in series of* $(d^2y/dx^2) + x(dy/dx) + x^2y = 0$ about $x = 0$.

[Delhi Maths (Hons.) 2005, 08; Ranchi 2010]

(*b*) *Solve* $y'' - xy' + x^2y = 0$ *in powers of* x **[Guwahati 2007]**

Sol. (*a*) Given that $y'' + xy' + x^2y = 0.$ ...(1)

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = x$ and $Q(x) = x^2$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, it follows that $x = 0$ is an ordinary point. To solve (1), we take

$$y = C_0 + C_1x + C_2x^2 + C_2x^3 + ... = \sum_{n=0}^{\infty} C_nx^n. \quad ...(2)$$

Differentiating (2) twice in succession w.r.t. 'x',

$y' = \sum_{n=1}^{\infty} C_n nx^{n-1}$ and $y'' = \sum_{n=2}^{\infty} C_n n(n-1)x^{n-2}$. ...(3)

Putting the above values of y, y' and y'' is (1), $\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} C_n n x^{n-1} + x^2 \sum_{n=0}^{\infty} C_n x^n = 0$

or $$\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} C_n n x^n + \sum_{n=0}^{\infty} C_n x^{n+2} = 0$$

or $$\sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} C_n n x^n + \sum_{n=2}^{\infty} C_{n-2} x^n = 0$$

or $$2C_2 + (6C_3 + C_1)x + \sum_{n=2}^{\infty} [(n+1)(n+2)C_{n+2} + nC_n + C_{n-2}]x^n = 0. \quad ...(4)$$

Since (4) is an identity, equating the constant term and the coefficients of various powers of x to zero, we get $2C_2 = 0$ so that $C_2 = 0$ ...(5)

$6C_3 + C_1 = 0$ so that $C_3 = -(C_1/6)$ ...(6)

$$(n+1)(n+2)C_{n+2} + nC_n + C_{n-2} = 0, \quad \text{for all} \quad n \geq 2$$

or $$C_{n+2} = -\frac{nC_n + C_{n-2}}{(n+1)(n+2)} \quad \text{for all} \quad n \geq 2. \quad ...(7)$$

Putting $n = 2$ in (7), $C_4 = -\frac{2C_2 + C_0}{12} = -\frac{C_0}{12}$, by (5). ...(8)

Putting $n = 3$ in (7), $C_5 = -\frac{3C_3 + C_1}{20} = -\frac{-3\times(C_1/6) + C_1}{20} = -\frac{C_1}{40}$, by (6)

Putting $n = 4$ in (7), $C_6 = -\frac{4C_4 + C_2}{30} = -\frac{-(C_0/3)}{30} = \frac{C_0}{90}$, by (8)

and so as. Putting these values in (1), we get

$$y = C_0 + C_1x - (1/6)\times C_1x^3 - (1/12)\times C_0x^4 - (1/40)\times C_1x^5 + (1/90)\times C_0x^6 + ...$$

or $$y = C_0\left(1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 - ....\right) + C_1\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - ....\right),$$

which is the required general solution about $x = 0$, where C_0 and C_1 are arbitrary constants.

***(b)* Ans.** $y = C_0(1 - x^4/12 - x^6/90 + ...) + C_1(x + x^3/16 - x^5/40 + ...)$

Ex. 3. *Find the general power series solution near $x = 0$ of the Legendre's equation $(1 - x^2)(d^2y/dx^2) - 2x(dy/dx) + p(p+1)y = 0$, where p is an arbitrary constant.*

[Delhi Maths (Hons.) 2006]

Sol. Given $(1 - x^2)y'' - 2xy' + p(p+1)y = 0.$ ...(1)

or $$y'' - [(2x)/(1 - x^2)]y' + [p(p+1)/(1 - x^2)]y = 0. \quad ...(2)$$

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, we have

$P(x) = -(2x)/(1 - x^2)$ and $Q(x) = p(p+1)/(1 - x^2)$,

showing that both $P(x)$ and $Q(x)$ are analytic at $x = 0$ and hence $x = 0$ is an ordinary point of (1).

To solve (1), let $y = C_0 + C_1x + C_2x^2 + C_3x^3 + ... = \sum_{n=0}^{\infty} C_n x^n$. ...(3)

Differentiating (3) twice in succession w.r.t. 'x', we get

$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} C_n n(n-1)x^{n-2}$. ...(4)

Putting the above values of y, y' and y'' in (1), we get

$$(1-x^2)\sum_{n=2}^{\infty}C_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty}C_n nx^{n-1} + p(p+1)\sum_{n=0}^{\infty}C_n x^n = 0$$

or $$\sum_{n=2}^{\infty}C_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty}C_n n(x-1)x^n - 2\sum_{n=1}^{\infty}C_n nx^n + p(p+1)\sum_{n=0}^{\infty}C_n x^n = 0$$

or $$\sum_{n=0}^{\infty}C_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty}C_n n(n-1)x^n - \sum_{n=1}^{\infty}2C_n nx^n + \sum_{n=0}^{\infty}p(p+1)C_n x^n = 0$$

or $$[2C_2+p(p+1)C_0]+[6C_3-2C_1+p(p+1)C_1]x+\sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2}-n(n-1)C_n-2nC_n-p(p+1)C_n]x^n = 0$$

or $$[2C_2+p(p+1)C_0]+[6C_3+(p^2+p-2)C_1]x+\sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2}-\{n(n-1)+2n-p(p+1)\}C_n]x^n = 0$$

or $$[2C_2+p(p+1)C_0]+[6C_3+(p-1)(p+2)C_1]x+\sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2}+\{(p^2-n^2)+(p-n)\}C_n]x^n = 0$$

or $$[2C_2+p(p+1)C_0]+[6C_3+(p-1)(p+2)C_1]x+\sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2}+(p-n)(p+n+1)C_n]x^n = 0. \quad ...(5)$$

Since (5) is an identity, we equate the coefficients of various powers of x to zero and obtain

$2C_2 + p(p+1)C_0 = 0$ so that $C_2 = -\dfrac{p(p+1)}{2!}C_0$ ...(6)

$6C_3 + (p-1)(p+2)C_1 = 0$ so that $C_3 = -\dfrac{(p-1)(p+2)}{3!}C_1$ ...(7)

$$(n+1)(n+2)C_{n+2} + (p-n)(p+n+1)C_n = 0$$

or $$C_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)}C_n, \quad \text{for} \quad n \geq 2. \qquad ...(8)$$

Putting $n = 2, 3, ...$ in (8) and using (6) and (7), we have

$$C_4 = -\frac{(p-2)(p+3)}{4\cdot 3}C_2 = \frac{p(p-2)(p+1)(p+3)}{4!}C_0$$

$$C_5 = -\frac{(p-3)(p+4)}{5\cdot 4}C_3 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!}C_1$$

and so on. Putting the above values of C_2, C_3, C_4, C_5, ... in (3), we get

$$y = C_0 + C_1x - \frac{p(p+1)}{2!}C_0x^2 - \frac{(p-1)(p+2)}{3!}C_1x^3 + \frac{p(p-2)(p+1)(p+3)}{4!}C_0\,x^4 + ...$$

or $$y = C_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 - .... \text{ad. inf}\right]$$

$$+ C_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - .... \text{ad. inf}\right],$$

which is the required general solution, C_0 and C_1 being arbitrary constants.

Ex. 4. *Solve $y'' - xy' - py = 0$, where p is any constant.*

Sol. Given $y'' - xy' - py = 0.$ ...(1)

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = -x$ and $Q(x) = -p$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, so $x = 0$ is an ordinary point. To solve (1), we take

$$y = C_0 + C_1x + C_2x^2 + C_2x^3 + C_4x^4 + ... = \sum_{n=0}^{\infty}C_n x^n. \qquad ...(2)$$

Differentiating (2) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} C_n n(n-1)x^{n-2}. \qquad ...(3)$$

Putting the above values of y, y' and y'' in (1), we have

$$\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} C_n n x^{n-1} - p\sum_{n=0}^{\infty} C_n x^n = 0 \quad \text{or} \quad \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=0}^{\infty} pC_n x^n = 0$$

or $$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}x^n - \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=0}^{\infty} pC_n x^n = 0$$

or $$(2C_2 - pC_0) + (6C_3 - C_1 - pC_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)C_{n+2} - nC_n - pC_n] = 0$$

or $$(2C_2 - pC_0) + \{6C_3 - (p+1)C_1\}x + \sum_{n=2}^{\infty}[(n+2)(n+1)C_{n+2} - (p+n)C_n] = 0. \quad ...(4)$$

Since (4) is an identity, we equate the coefficients of various powers of x to zero and obtain

$2C_2 - pC_0 = 0$ so that $C_2 = (p/2)C_0$ ...(5)

$6C_3 - (p+1)C_1 = 0$ so that $C_3 = [(p+1)/6]C_1$ ...(6)

$(n+2)(n+1)C_{n+2} - (p+n)C_n = 0$ so that $C_{n+2} = \dfrac{p+n}{(n+1)(n+2)}C_n$, for all $n \geq 2$

...(7)

Putting $n = 2, 3, 4, ...$ in (7) and using (5) and (6), we get

$$C_4 = \frac{p+2}{3\cdot 4}C_2 = \frac{p+2}{3\cdot 4}\times\frac{p}{2}C_0 = \frac{p(p+2)}{4!}C_0, \; C_5 = \frac{p+3}{4\cdot 5}C_3 = \frac{p+3}{4\cdot 5}\times\frac{p+1}{2\cdot 3}C_1 = \frac{(p+1)(p+3)}{5!}C_1$$

and so on. Putting these values in (3), we obtain

$$y = C_0 + C_1 x + \frac{p}{2}C_0 x^2 + \frac{p+1}{6}C_1 x^3 + \frac{p(p+2)}{4!}C_0 x^4 + \frac{(p+1)(p+3)}{5!}C_1 x^5 + ..$$

or $$y = C_0\left[1 + \frac{p}{2!}x^2 + \frac{p(p+2)}{4!}x^4 + \ldots\right] + C_1\left[x + \frac{p+1}{3!}x^3 + \frac{(p+1)(p+3)}{5!}x^5 + \ldots\right]$$

which is the required solution, C_0 and C_1 being arbitrary constants.

Ex. 5(a). *Find the general solution of* $y'' + (x-3)y' + y = 0$ near $x = 2$.

(b) *Obtain power series solution of* $y'' + (x-1)y' + y = 0$ in powers of $(x-2)$.

Sol. Given $$y'' + (x-3)y' + y = 0. \qquad ...(1)$$

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = x - 3$ and $Q(x) = 1$. Since both $P(x)$ and $Q(x)$ are analytic at $x = 2$, so $x = 2$ is an ordinary point of (1). To find solution near $x = 2$, we shall find series solution in powers of $(x-2)$. We assume that

$$y = C_0 + C_1(x-2) + C_2(x-2)^2 + C_3(x-2)^3 + ... = \sum_{n=0}^{\infty} C_n (x-2)^n. \qquad ...(2)$$

Differentiating (2) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} nC_n (x-2)^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n-1)C_n (x-2)^{n-2}. \quad ...(3)$$

Putting the above values of y, y' and y'' in (1), we get

$$\sum_{n=2}^{\infty} n(n-1)C_n(x-2)^{n-2} + (x-3)\sum_{n=1}^{\infty} nC_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n(x-2)^{n-2} + [(x-2)-1]\sum_{n=1}^{\infty} nC_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n(x-2)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-2)^n - \sum_{n=1}^{\infty} nC_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or $$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}(x-2)^n + \sum_{n=1}^{\infty} nC_n(x-2)^n - \sum_{n=0}^{\infty} (n+1)C_{n+1}(x-2)^n + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or $$(2C_2 - C_1 + C_0) + \sum_{n=2}^{\infty} [(n+2)(n+1)C_{n+2} + nC_n - (n+1)C_{n+1} + C_n](x-2)^n = 0. \quad ...(4)$$

which is an identity. Equating to zero the coefficients of various powers of $(x-2)$, we get

$2C_2 - C_1 + C_0 = 0$ so that $C_2 = (C_1 - C_0)/2$. ...(5)

$(n+2)(n+1)C_{n+2} + (n+1)C_n - (n+1)C_{n+1} = 0$ for all $n \geq 1$

or $C_{n+2} = (C_{n+1} - C_n)/(n+2)$, for all $n \geq 1$. ...(6)

Putting $n = 1, 2, 3, ...$ in (6) and using (5) etc., we get

$$C_3 = \frac{C_2 - C_1}{3} = \frac{1}{3}\left[\frac{C_1 - C_0}{2} - C_1\right] = -\frac{C_0 + C_1}{6} \quad ...(7)$$

$$C_4 = \frac{C_3 - C_2}{4} = \frac{1}{4}\left[-\frac{C_0 + C_1}{6} - \frac{C_1 - C_0}{2}\right] = \frac{1}{12}C_0 - \frac{1}{6}C_1. \quad ...(8)$$

and so on. Putting these values in (2), the required solution near $x = 2$ is

$$y = C_0 + C_1(x-2) + \left(\frac{C_1 - C_0}{2}\right)(x-2)^2 - \left(\frac{C_0 + C_1}{6}\right)(x-2)^3 + \left(\frac{1}{12}C_0 - \frac{1}{6}C_1\right)(x-2)^4 + ...$$

or $y = C_0[1 - (1/2) \times (x-2)^2 - (1/6) \times (x-2)^3 - (1/12) \times (x-2)^4 + ...$ ad. inf.]

$+ C_1[(x-2) + (1/2) \times (x-2)^2 - (1/6) \times (x-2)^3 - (1/6) \times (x-2)^4 + ...$ ad. inf.]

(*b*) Do as in part (*a*) yourself.

Ex. 6. *Find the power series solution in powers of* $(x-1)$ *of the initial value problem* $xy'' + y' + 2y = 0$, $y(1) = 1$, $y'(1) = 2$. **[Purvanchal 2007; CDLU 2004]**

Sol. Given equation is $y'' + (1/x)y' + (2/x)y = 0$. ...(1)

Comparing (1) with $y'' + P(x)y' + Q(y) = 0$, here $P(x) = 1/x$ and $Q(x) = 2/x$, which are analytic at $x = 1$. Hence $x = 1$ is an ordinary point of (1). To find solution near $x = 1$, we shall find series solution in powers of $(x-1)$.

Let $$y = C_0 + C_1(x-1) + C_2(x-1)^2 + C_3(x-1)^3 + ... = \sum_{n=0}^{\infty} C_n(x-1)^n. \quad ...(2)$$

Differentiating (2) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} nC_n(x-1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-2}. \quad ...(3)$$

Putting these values of y, y' and y'' in given equation, we get

$$x\sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} C_n(x-1)^n = 0$$

or $$[(x-1)+1]\sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} C_n(x-1)^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} C_n(x-1)^n = 0$$

or $$\sum_{n=1}^{\infty} (n+1)nC_{n+1}(x-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}(x-1)^n$$

$$+ \sum_{n=0}^{\infty} (n+1)C_{n+1}(x-1)^n + 2\sum_{n=0}^{\infty} C_n(x-1)^n = 0,$$

which is an identity. Equating to zero the coefficients of various powers of $(x-1)$, we get

$2C_2 + C_1 + 2C_0 = 0$ so that $C_2 = -(C_1 + 2C_0)/2$. ...(4)

and $(n+1)nC_{n+1} + (n+2)(n+1)C_{n+2} + (n+1)C_{n+1} + 2C_n = 0$, for all $n \geq 1$

or $(n+1)(n+2)C_{n+2} + (n+1)^2C_{n+1} + 2C_n = 0$, for all $n \geq 1$

or $$C_{n+2} = -\frac{(n+1)^2C_{n+1} + 2C_n}{(n+1)(n+2)}, \text{ for all } n \geq 1. \quad ...(5)$$

Given that $y = 1$ and $y' = 2$ when $x = 1$. Hence putting $x = 1$ in (2) and (3), we have

$C_0 = 1$ and $C_1 = 2$. ...(6)

Using (6), (4) gives $C_2 = -(2+2)/2 = -2$. ...(7)

Putting $n = 1, 2, 3, \ldots$ (5) and using (6) and (7) etc., we get

$$C_3 = -\frac{2^2C_2 + 2C_1}{2\cdot 3} = -\frac{4\times(-2) + (2\times 2)}{2\cdot 3} = \frac{2}{3},\ C_4 = -\frac{3^2C_3 + 2C_2}{3\cdot 4} = -\frac{9\times(2/3) + 2\times(-2)}{3\cdot 4} = -\frac{1}{6}$$

$$C_5 = -\frac{4^2C_4 + 2C_3}{4\cdot 5} = -\frac{16\times(-1/6) + 2\times(2/3)}{4\cdot 5} = \frac{1}{15}.$$

and so on. Putting these values in (2), we have

$$y = 1 + 2(x-1) - 2(x-1)^2 + (2/3)\times(x-1)^3 - (1/6)\times(x-1)^4 + (1/15)\times(x-1)^5 + \ldots.$$

Note. Ex. 5 and 6 can also be solved by shifting the origin. The following example is given to provide an alternative method of solving 5 and 6.

Ex. 7. *Find the power series solution of the initial value problem* $(x^2-1)y'' + 3xy' + xy = 0$, $y(2) = 4$, $y'(2) = 6$.

Sol. Given equation is $(x^2-1)y'' + 3xy' + xy = 0$. ...(1)

Dividing by (x^2-1), (1) gives $y'' + [(3x)/(x^2-1)]y' + [x/(x^2-1)]y = 0$. ...(2)

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = (3x)/(x^2-1)$ and $Q(x) = x/(x^2-1)$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 2$, so $x = 2$ is an ordinary point of (1).

Since the initial values of (1) are prescribed at $x = 2$ and $x = 2$ is an ordinary point, hence we shall find the required solution near $x = 2$, *i.e.*, in powers of $(x-2)$.

Let $$y = C_0 + C_1(x-2) + C_2(x-2)^2 + \ldots. = \sum_{n=0}^{\infty} C_n(x-2)^n. \quad ...(3)$$

We now shift the origin to $x = 2$ by writing $t = x - 2$ so that $x = t + 2$ ...(4)

Then $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{dy}{dt}$ and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d^2y}{dt^2}$. ...(5)

Using (4) and (5), (1) reduces to $[(t + 2)^2 - 1](d^2y/dt^2) + 3(t + 2)(dy/dt) + (t + 2)y = 0$

or $(t^2 + 4t + 3)(d^2y/dt^2) + (3t + 6)(dy/dt) + (t + 2)y = 0.$ ...(6)

Also, (3) reduces to $y = C_0 + C_1t + C_2t^2 + \ldots. = \sum_{n=0}^{\infty} C_n t^n$. ...(7)

Differentiating (7) twice in succession w.r.t. 't', we get

$dy/dt = \sum_{n=1}^{\infty} nC_n t^{n-1}$ and $d^2y/dt^2 = \sum_{n=2}^{\infty} n(n-1)C_n t^{n-2}$. ...(8)

Using (7) and (8), (6) reduces to

$$(t^2 + 4t + 3)\sum_{n=2}^{\infty} n(n-1)C_n t^{n-2} + (3t + 6)\sum_{n=1}^{\infty} nC_n t^{n-1} + (t + 2)\sum_{n=0}^{\infty} C_n t^n = 0$$

or $\sum_{n=2}^{\infty} n(n-1)C_n t^n + \sum_{n=2}^{\infty} 4n(n-1)C_n t^{n-1} + \sum_{n=2}^{\infty} 3n(n-1)C_n t^{n-2} + \sum_{n=1}^{\infty} 3nC_n t^n + \sum_{n=1}^{\infty} 6nC_n t^{n-1} + \sum_{n=0}^{\infty} C_n t^{n+1} + \sum_{n=0}^{\infty} 2C_n t^n = 0$

or $\sum_{n=2}^{\infty} n(n-1)C_n t^n + \sum_{n=1}^{\infty} 4(n+1)nC_{n+1} t^n + \sum_{n=0}^{\infty} 3(n+2)(n+1)C_{n+2} t^n$

$$+ \sum_{n=1}^{\infty} 3nC_n t^n + \sum_{n=0}^{\infty} 6(n+1)C_{n+1} t^n + \sum_{n=1}^{\infty} C_{n-1} t^n + \sum_{n=0}^{\infty} 2C_n t^n = 0$$

or $(6C_2 + 6C_1 + 2C_0) + (8C_2 + 18C_3 + 3C_1 + 12C_2 + C_0 + 2C_1)t$

$$+ \sum_{n=2}^{\infty} [n(n-1)C_n + 4n(n+1)C_{n+1} + 3(n+1)(n+2)C_{n+2} + 3nC_n + 6(n+1)C_{n+1} + C_{n-1} + 2C_n]t^n = 0$$

or $2(3C_2 + 3C_1 + C_0) + (18C_3 + 20C_2 + 5C_1 + C_0)t$

$$+ \sum_{n=2}^{\infty} [3(n+1)(n+2)C_{n+2} + 2(2n+3)(n+1)C_{n+1} + (n^2 + 2n + 2)C_n + C_{n-1}]t^n = 0 \quad ...(9)$$

From (3), $y' = C_1 + 2C_2(x - 2) + 3C_3(x - 2)^2 + \ldots.$ ...(10)

Putting $x = 2$ in (3) and (10) and using the given initial conditions, namely, $y = 4$ and $y' = 6$ when $x = 2$, we get $C_0 = 4$ and $C_1 = 6$. Hence (9) reduces to

$$2(3C_2 + 22) + (18C_3 + 20C_2 + 34)t + \sum_{n=2}^{\infty} \{3(n+1)(n+2)C_{n+2} + 2(2n+3)(n+1)C_{n+1} + (n^2 + 2n + 2)C_n + C_{n-1}\}t^n = 0,$$

which is an identity in t. Equating to zero the coefficients of various powers of t, we have

$2(3C_2 + 22) = 0$ so that $C_2 = -(22/3)$. ...(11)

$18C_3 + 20C_2 + 34 = 0$ or $18C_3 - 20 \times (22/3) + 34 = 0$ or $C_3 = 169/27$ ...(12)

and $3(n+1)(n+2)C_{n+2} + 2(2n+3)(n+1)C_{n+1} + (n^2 + 2n + 2)C_n + C_{n-1} = 0$, for all $n \geq 2$. ...(13)

Putting $n = 2$ in (13), we get $36C_4 + 42C_3 + 10C_2 + C_1 = 0$

or $36C_4 + 42 \times (169/27) + 18 \times (-22/3) + 6 = 0 \Rightarrow C_4 = 344/81$

Putting the above values in (3), the required solution is

$y = 4 + 6(x - 2) - (22/3) \times (x - 2)^2 + (169/27) \times (x - 2)^3 + (344/81) \times (x - 2)^4 + \ldots.$

Ex. 8. *Solve* $y'' - 2x^2y' + 4xy = x^2 + 2x + 4$ *in powers of* x.

Sol. Given equation is $y'' - 2x^2y' + 4xy = x^2 + 2x + 4.$ ...(1)

Clearly $x = 0$ is an ordinary point of (1). To solve (1) it, let

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 + ... = \sum_{n=0}^{\infty} C_n x^n . \qquad ...(2)$$

Differentiating (2) twice in succession w.r.t. 'x', we have

$$y' = \sum_{n=1}^{\infty} nC_n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} . \qquad ...(3)$$

Substituting these values of y, y' and y'' in (1), we have

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - 2x^2 \sum_{n=1}^{\infty} nC_n x^{n-1} + 4x \sum_{n=0}^{\infty} C_n x^n = x^2 + 2x + 4$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=1}^{\infty} 2nC_n x^{n+1} + \sum_{n=0}^{\infty} 4C_n x^{n+1} - x^2 - 2x - 4 = 0$$

or $$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2} x^n - \sum_{n=2}^{\infty} 2(n-1)C_{n-1} x^n + \sum_{n=1}^{\infty} 4C_{n-1} x^n - x^2 - 2x - 4 = 0$$

or $$(2C_2 - 4) + (6C_3 + 4C_0 - 2)x + (12C_4 + 2C_1 - 1)x^2 + \sum_{n=3}^{\infty} [(n+2)(n+1)C_{n+2} - 2(n-1)C_{n-1} + 4C_{n-1}]x^n = 0. \quad ...(4)$$

Equating to zero the coefficients of various powers of x in (4), we get

$2C_2 - 4 = 0$ so that $C_2 = 2$, ...(5)

$6C_3 + 4C_0 - 2 = 0$ so that $C_3 = (1/3) - (2C_0/3)$ ...(6)

$12C_4 + 2C_1 - 1 = 0$ so that $C_4 = (1/12) - (C_1/6)$ ...(7)

and $$(n+2)(n+1)C_{n+2} - 2(n-1)(C_{n-1} + 4C_{n-1} = 0, \text{ for all } n \geq 3. \qquad ...(8)$$

Putting $n = 3, 4, 5, ...$ in (8) and using (5), (6), (7), etc, we get

$20C_5 - 4C_2 + 4C_2 = 0$ so that $C_5 = 0$, ...(9)

$30C_6 = 2C_3$ so that $C_6 = \frac{1}{15}\left(\frac{1}{3} - \frac{2}{3}C_0\right) = \frac{1}{45} - \frac{2}{45}C_0$

$42C_7 = 4C_4$ so that $C_7 = \frac{2}{21}\left(\frac{1}{12} - \frac{1}{6}C_1\right) = \frac{1}{125} - \frac{1}{63}C_1$

and so on. Putting these values in (2), the required solution is

$$y = C_0 + C_1x + 2x^2 + \left(\frac{1}{3} - \frac{2C_0}{3}\right)x^3 + \left(\frac{1}{12} - \frac{C_1}{6}\right)x^4 + \left(\frac{1}{45} - \frac{2C_0}{45}\right)x^6 + \left(\frac{1}{126} - \frac{C_1}{63}\right)x^7 + ...$$

or $$y = C_0\left(1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 ....\right) + C_1\left(x - \frac{1}{6}x^4 - \frac{1}{63}x^7 ....\right) + 2x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{1}{126}x^7 + ...$$

Ex. 9. (*i*) *Explain the method of integrating in series for solving a first order differential equation.*

(*ii*) *Find a power series solution of the form* $\sum_{n=0}^{\infty} a_n x^n$ *for the differential equation* $y' = 2xy$.

[Nagpur 2005]

(*iii*) *Solve* $y' = x^2 - 4x + y + 1$ *satisfying* $y = 3$ *when* $x = 2$.

Sol. (*i*) The Picard's theorem of Art. 1.6 (refer chapter 1) for a differential equation of the form $$dy/dx = f(x, y) \qquad ... (1)$$ gives a sufficient condition for a solution. In the proof using power series, y is found in the form of a Taylor series

$$y = a_0 + a_1 (x - x_0) + a_2(x - x_0)^2 + ... + a_n (x - x_0)^n + ... \qquad ... (2)$$

where for convenience y_0 has been replaced by a_0. This series has the following properties:

(*a*) It satisfies the differential equation (1) (*b*) It has the value y_0 when $x = x_0$.

(*c*) It is convergent for all values of x sufficiently near $x = x_0$.

To find the solution of (1) satisfying the condition $y = y_0$ when $x = 0$, we assume the solution to be of the form

$$y = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ... = \sum_{n=0}^{\infty} a_nx^n \quad ... (3)$$

in which $a_0 = y_0$ and the remaining a's are constant to be determined.

Substitute the assumed series (3) in (1) and proceed to find a_1, a_2, a_3, ... a_n ... as usual.

Remark. If we are to find the solution of (1) satisfying $y = y_0$ when $x = x_0$, we modify the above procedure as follows

Make the substitution $x - x_0 = v$, that is, $x = v + x_0$, $dy/dx = dy/dv$ resulting in

$$dy/dv = F(y, v) \quad ... (4)$$

Use the above procedure to obtain the solution of (4) satisfying $y = y_0$ when $v = 0$. Finally make the substitution $v = x - x_0$ in the solution.

(*ii*) Given $y' = 2xy$ ... (1)

Let (1) possess series solution $y = \sum_{n=0}^{\infty} a_nx^n$ ... (2)

From (2), $y' = \sum_{n=0}^{\infty} na_nx^{n-1}$ ... (3)

Inserting (2) and (3) into (1), we have

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ... + na_nx^{n-1} + ... = 2x\left(a_0 + a_1x + a_2x^2 + ... + a_{n-2}x^{n-2} + ...\right)$$

Collecting like powers of x yields.

$$a_1 + (2a_2 - 2a_0)x + (3a_3 - 2a_1)x^2 + (4a_4 - 2a_2)x^3 + (5a_5 - 3a_3)x^4 + (6a_6 - 5a_4)x^5 + ... + (na_n - 2a_{n-2})x^{n-1} + ... = 0$$

In order that this series vanish for all values of x in some region surrounding $x = 0$, it is necessary and sufficient that coefficients of each power of x vanish. Thus, we obtain

$a_1 = 0$, $2a_2 - 2a_0 = 0$, $3a_3 - 2a_1 = 0$, $4a_4 - 2a_2 = 0$, $5a_5 - 2a_3 = 0$, $6a_6 - 2a_4 = 0$

$\Rightarrow a_1 = 0$, $a_2 = a_0$, $a_3 = (2/3)a_1 = 0$, $a_4 = (1/2)a_2 = (1/2)a_0$, $a_6 = (1/3)a_4 = (1/3) \times (a_0/2) = a_0/3!$ and so on

In general, $a_{2n+1} = 0$ and $a_{2n} = a_0/n!$, for all $n = 1, 2, 3$

Substituting these values into (2), we obtain the required power series solution.

$y = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + ... + a_{2n}x^{2x} + ...$ or $y = a_0 + a_0x^2 + (a_0/2!)x^4 + (a_0/3!)x^6 + ... + (a_0/n!)x^{2n} + ...$

or $y = a_0\,(1 + x^2/1! + x^4/2! + x^6/3! + x^{2n}/n! + ...)$, a_0 being an arbitrary constant

(*iii*) Given $dy/dx = x^2 - 4x + y + 1$ ... (1)

where $y = 3$ when $x = 2$ ... (2)

Let $x = v + 2$. Then (1) and (2) reduce to

$$dy/dv = v^2 + y - 3 \quad ... (3)$$

and $y = 3$ when $v = 0$ ... (4)

We now proceed with (3) and (4) as in part (*ii*). We assume the series solution

$$y = 3 + a_1v + a_2v^2 + a_3v^3 + a_4v^4 + ... + a_nv^n + ... \quad ... (5)$$

From (5), $dy/dv = a_1 + 2a_2v + 3a_3v^2 + 4a_4v^3 + ... + na_nv^{n-1} + ...$ ... (6)

Inserting (5) and (6) into (3), we have, as before

$$a_1 + (2a_2 - a_1)v + (3a_3 - a_2 - 1)v^2 + (4a_4 - a_3)v^3 + ... + (na_n - a_{n-1})v^{n-1} + ... = 0$$

Equating each of the coefficients to zero, we obtain

$a_1 = 0, 2a_2 - a_1 = 0$ so that $a_2 = 0; 3a_3 - a_2 - 1 = 0$ so that $a_3 = 1/3; 4a_4 - a_3 = 0$ so that $a_4 = 1/12$, and so on.

Again, $na_n - a_{n-1} = 0 \quad \Rightarrow \quad a_n = (1/n)a_{n-1}$ for all $n \geq 1$ ... (7)

Using the recursion formula (7), we have

$$a_n = \frac{1}{n}a_{n-1} = \frac{1}{n(n-1)}a_{n-2} = \cdots = \frac{1}{n(n-1)(n-2)\cdots 4}a_3 = \frac{2}{n!}, \quad \text{for all } n \geq 2$$

Substituting the above values of the coefficients into (5), we have

$$y = 3 + v^3/3 + v^4/12 + \ldots + (2/n!)v^n + \ldots \quad \ldots (8)$$

Replacing v by $x - 2$, (8) gives the required solution

$$y = 3 + (2/3!) \times (x-2)^3 + (2/4!) \times (x-2)^4 + \ldots + (2/n!) \times (x-2)^n + \ldots$$

Ex. 10. *Solve by power series method :* $y' - y = 0$. **[Sagar 2004]**

Sol. Given $y' - y = 0$ ... (1)

Assume that a solution of (1) is given by power series

$$y = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n + \ldots = \sum_{n=0}^{\infty} C_nx^n \quad \ldots (2)$$

Differentiating (2) w.r.t. 'x', $y' = \sum_{n=1}^{\infty} n\, C_nx^{n-1}$

Substituting the above values of y and y' in (1), we get $\sum_{n=1}^{\infty} n\, C_nx^{n-1} - \sum_{n=0}^{\infty} C_nx^n = 0$

or $(C_1 + 2C_2x + 3C_3x^2 + \ldots.) - (C_0 + C_1x + C_2x^2 + \ldots) = 0$

or $(C_1 - C_0) + (2C_2 - C_1)x + (3C_3 - C_2)x^2 + \ldots = 0 \quad \ldots (3)$

Since (3) is an identity, we must have

$C_1 - C_0 = 0, \quad 2C_2 - C_1 = 0, \quad 3C_3 - C_2 = 0, \ldots. \quad \ldots (4)$

Solving (4), $C_1 = C_0, \quad C_2 = C_1/2 = C_0/2, \quad C_3 = C_2/3 = C_0/3!, \ldots$

Substituting these values in (2), we obtain

$y = C_0(1 + x + x^2/2! + x^3/3! + \ldots)$ or $y = C_0e^x$,

which is the required solution, C_0 being an arbitrary constant.

EXERCISE 8 (B)

Find the series solution of the following equations :

1. $(1 - x^2)y'' + 2xy' - y = 0$ about $x = 0$. **[Purvanchal 2007; Meerut 2000]**

Ans. $y = C_0(1 + x^2/2 - x^4/24 + \cdots) + C_1(x - x^3/6 - x^5/120 + \cdots)$

2. $(2 + x^2)y'' + xy' - (1 + x)y = 0$ near $x = 0$. **(Delhi Maths (H) 2002)**

Ans. $y = C_0(1 + x^2/4 + x^3/12 - 3x^4/96 + \cdots) + C_1(x + x^4/24 + \cdots)$

3. $(1 + x^2)y'' + xy' - y = 0$ near $x = 0$. **[Delhi Maths (Hons) 1999]**

Ans. $y = C_0(1 + x^2/2 - x^4/8 + x^6/15 + \cdots) + C_1x.$

4. $(x^2 - 1)y'' + xy' - y = 0$ near $x = 0$. **Ans.** $y = C_0\{1 + (x^2/2) + (x^4/4) + \ldots.\} + C_1x$

5. $(x^2 - 1)y'' + 4xy' + 2y = 0$ near $x = 0$. **Ans.** $y = C_0(1 + x^2 + x^4 + \ldots.) + C_1(x + x^3 + x^5 + \ldots)$

6. $(1 - x^2)y'' + 2xy' - y = 0$ about $x = 0$.

Ans. $y = C_0(1 + x^2/2! - x^4/4! + \ldots) + C_1(1 - x^3/3! - x^5/5! + \ldots)$

7. $y'' - xy' + 2y = 0$ near $x = 1$.

Ans. $y = C_0[1 - (x-1)^2 - (1/3) \times (x-1)^3 - ....] + C_1[(x-1) + (1/2) \times (x-1)^2 + ...]$

8. $(x^2 - 1)y'' + 3xy' + xy = 0, y(0) = 2, y'(0) = 3$. **Ans.** $y = 2 + 3x + (11/6) \times x^3 + (1/4) \times x^4 - ....$

9. (*i*) $(1 - x^2)y'' + 2y = 0$ near $x = 0$. **Ans.** $y = C_0(1 - x^2) + C_1(x - x^3/3 - x^5/5 - x^7/35 - ....$

(*ii*) $(1-x^2)y'' + 2y = 0, y(0) = 4, y'(0) = 5$. **Ans.** $y = 4 + 5x - 4x^2 - (5/3) \times x^3 - (1/3) \times x^5 + ...$

10. $(x^2 + 2x)y'' + (x+1)y' - y = 0$ near $x = -1$.

Ans. $y = C_0[1 - (1/2) \times (x+1)^2 - (1/8) \times (x+1)^4 - (1/16) \times (x+1)^6 + ...] + C_1(x+1)$

11. $y'' - xy' = e^{-x}, y(0) = 2, y'(0) = -3$.

[Hint : Use the expansion $e^{-x} = 1 - (x/1!) + (x^2/2!) - (x^3/3!) + ...$

Ans. $y = 2 - 3x + (1/2) \times x^2 - (2/3) \times x^3 + (1/8) \times x^4 - ....$

Ex. 12(*a*). *Solve* $y'' + x^2y = 2 + x + x^2$ *about* $x = 0$.

Sol. Proceed as in Ex. 8. of Art. 8.6. Its general solution is given by

$$y = C_0\left(1 - \frac{x^4}{12} + \frac{x^8}{672} - ....\right) + C_1\left(x - \frac{x^5}{20} + \frac{x^9}{1440} - ....\right) + x^2 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{30} - \frac{x^7}{252} - ....$$

Ex. 12(*b*). *Apply power method to solve* $y'' - y = x$.

Sol. Proceed as in Ex. 8. of Art 8.6. Its general solution is given by

$$y = C_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + C_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) + \left(\frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)$$

13. $y'' - 4y = 0$ near $x = 0$ **[Agra 2007]**

8.7. Series solution about regular singular point x = 0. Frobenius Method.

If $x = 0$ is regular point, we shall use the Frobenius method for finding series solution about $x = 0$. However if we wish to find the series solution about regular singular point $x = a$, then we first shift the origin to point $x = a$ and later on proceed as before.

If $x = 0$ is an irregular point of the given equation, then discussion of solution of the equation is beyond the scope of this book.

We now discuss Frobenius method. Consider the differential equation of the form

$$y_2 + \frac{F(x)}{x} y_1 + \frac{G(x)}{x^2} y = 0, \qquad \text{where} \qquad y_1 = \frac{dy}{dx}, \qquad y_2 = \frac{d^2y}{dx^2} \qquad ...(1)$$

and the functions $F(x)$ and $G(x)$ are analytic at $x = 0$. Then the following method for solving (1) is called Frobenius method. We assume a trial solution

$$y = x^r \sum_{m=0}^{\infty} c_m x^m = x^r(c_0 + c_1x + c_2x^2 + \ldots), \text{ where } c_0 \neq 0. \qquad ...(2)$$

Differentiating (2) term by term, we have

$$y_1 = rc_0x^{r-1} + (r+1)c_1x^r + ... = x^{r-1}[rc_0 + (r+1)c_1x + ...]$$

$$y_2 = r(r-1)c_0x^{r-2} + r(r+1)c_1x^{r-1} + ... = x^{r-2}[r(r-1)c_0 + (r+1)rc_1x + ...]$$

Since $F(x)$ and $G(x)$ are analytic at $x = 0$, we can write

$F(x) = a_0 + a_1x + a_2x^2 + ...$ and $G(x) = b_0 + b_1x + b_2x^2 + ...$

Putting the values of $y, y_1, y_2, F(x)$ and $G(x)$ in (1) and then multiplying both sides by x^2, gives

$$x^r[r(r-1)c_0 + ...] + (a_0 + a_1x + ...)x^r(rc_0 + ...) + (b_0 + b_1x + ...)x^r(c_0 + c_1x + c_2x^2 + ...) = 0. \qquad ...(3)$$

Since (3) is an identity, we can equate to zero the coefficients of various powers of x. This will give us a system of equations involving the unknown coefficients c_m. The smallest power is x^r, and the corresponding equation is $[r(r-1) + a_0r + b_0]c_0 = 0$.

Since by assumption $c_0 \neq 0$, we obtain $r^2 + (a_0 - 1)r + b_0 = 0$. ...(4)

This important quadratic equation is known as the *indicial equation* of (1). We shall see that

this method will give rise to a fundamental system of solutions ; one of these solutions will always be of the form (2), but for the form of the other solutions there will be three different possibilities corresponding to the following cases :

Case I. The roots of the indicial equation are distinct and do not differ an integer.

Let r_1 and r_2 be the roots of (4). We substitute $r = r_1$ in the above mentioned system of equations and obtain a solution $u(x) = x^{r_1}(c_0 + c_1x + \ldots.)$.

Proceeding similarly by using $r = r_2$, we obtain a second solution $v(x) = x^{r_2}(c_0' + c_1'x + \ldots.)$ where $c'_0, c_1', \ldots.$ are the new values of the coefficients corresponding to $r = r_2$. Since $r_1 - r_2$ is not an integer, u/v is never constant. Thus u and v are two independent solutions of (1). Hence the general solution will be $y = au + bv$, where a and b are arbitrary constants.

Case II. The indicial equation has equal roots. The indicial equation (4) has double root r if

$$(a_0 - 1)^2 - 4b_0 = 0 \quad \text{giving} \quad r = (1 - a_0)/2. \qquad ...(5)$$

We first obtain the coefficients $c_1, c_2\ldots.$ successively from the system of equations connecting them. Thus we obtain a first solution $u(x) = x^r(c_0 + c_1x + \ldots.),$ ...(6)

wherein we write $r = (1 - a_0)/2$ afterwards.

To determine another solution we use the method of variation of parameters, that is, we replace the constant c in the solution cu by a function $w(x)$ to be determined such that

$$v(x) = u(x)\, w(x) \qquad ...(7)$$

is solution of (1). Differentiating (7), we get

$$v_1 = u_1w + uw_1 \quad \text{and} \quad v_2 = u_2w + 2u_1w_1 + uw_2. \qquad ...(8)$$

Since $v(x)$ is a solution of (1) by assumption, $x^2 v_2 + F(x)\ x\ v_1 + G(x)\ v = 0.$ ...(9)

Putting the values of v, v_1 and v_2 from (7) and (8) in (9), we get

$$x^2(u_2w + 2u_1w_1 + uw_2) + xF(x)[u_1w + uw_1] + G(x)uw = 0$$

or $$[x^2u_2 + F(x)xu_1 + G(x)u]w + x^2(2u_1w_1 + uw_2) + xF(x)uw_1 = 0$$

Since $u(x)$ is a solution of (1), we have $x^2u_2 + F(x)xu_1 + G(x)u = 0.$

$\therefore$ The above equation reduces to $x^2(2u_1w_1 + uw_2) + xF(x)uw_1 = 0.$

Dividing by x^2u and putting the value of $F(x)$ gives, $w_2 + \left(2\frac{u_1}{u} + \frac{a_0}{x} + \ldots.\right)w_1 = 0.$ ... (10)

In what follows we write dots to represent terms which are constant or involve positive powers of x. Now from (6), we obtain

$$\frac{u_1}{u} = \frac{x^{r-1}[rc_0 + (r+1)c_1x + \ldots.]}{x^r[c_0 + c_1x + \ldots.]} = \frac{r}{x} + \ldots.$$

$\therefore$ (10) becomes $w_2 + \left(\frac{2r + a_0}{x} + \ldots.\right)w_1 = 0.$ ...(11)

But from (5), $2r + a_0 = 1$. Hence the above equation reduces to

$w_2 + \left(\frac{1}{x} + \ldots.\right)w_1 = 0$ or $\frac{w_2}{w_1} = -\frac{1}{x} + \ldots.$

Integrating, $\log w_1 = -\log x + \ldots.$ or $w_1 = \frac{1}{x}e^{(\ldots)}$

Expanding the exponential function in powers of x and integrating once more, we obtain the expression of w always in the following form $w = \log x + k_1x + k_2x^2 + \ldots.$

Putting this value of w in (7), we obtain the desired another independent solution $v(x)$. Then the general solution is $y = au + bv$, a and b being arbitrary constants.

Case III. The roots of the indicial equation differ by an integer.

Let the roots r_1 and r_2 of the indicial equation differ by an integer, say $r_1 = r$ and $r_2 = r - p$, where p is a positive integer, then as before one solution will be given by

$$u(x) = x^{r_1}\,(c_0 + c_1x + c_2x^2 + \ldots.)$$

corresponding to the root r_1. However, while, dealing with the root r_2 it may not be possible to determine another independent solution $v(x)$ as in case I. In such cases we determine $v(x)$ by using method outlined in case II. Thus as before we first obtain (11).

From (5), $r_1 + r_2 = -(a_0 - 1)$, using theory of equations

Since $r_1 = r$, $r_2 = r - p$, this gives $2r + a_0 = p + 1$. Thus (11) becomes

$$\frac{w_2}{w_1} = -\left(\frac{p+1}{x} + \ldots.\right).$$

Integrating, $\log w_1 = -(p + 1) \log x + \ldots.$ or $w_1 = x^{-(p+1)}e^{(\ldots)}$

Expanding the exponential function and simplifying, we get

$$w_1 = \frac{1}{x^{p+1}} + \frac{k_1}{x^p} + \ldots. + \frac{k_p}{x} + k_{p+1} + k_{p+2}x + \ldots.$$

Integrating again, $$w = -\frac{1}{px^p} - \ldots. + k_p \log x + k_{p+1}x + \ldots.$$

Putting this value of w in (7), we obtain another solution $v(x)$ and as usual the general solution $y = au + bv$, where a and b are arbitrary constants.

8.8. Working rule for solution by Frobenius method

Consider linear differential equation of order two $f(x)y'' + g(x)y' + r(x)y = 0.$ ...(1)

Step 1. Suppose that a trial solution of (1) be of the form

$$y = x^k(c_0 + c_1x + c_2x^2 + \ldots. + c_mx^m + \ldots.) \quad \textit{i.e.} \quad y = x^k \sum_{m=0}^{\infty} c_mx^m. \qquad ...(2)$$

Thus we take $$y = \sum_{m=0}^{\infty} c_m x^{m+k}, \quad \text{where} \quad c_0 \neq 0. \qquad ...(3)$$

Step 2. Differentiate (3) and obtain

$$y' = \sum_{m=0}^{\infty} c_m(m+k)x^{m+k-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m(m+k)(m+k-1)x^{m+k-2}. \qquad ...(4)$$

Using (3) and (4), (1) reduces to an identity.

Step 3. Equating to zero the coefficient of the smallest power of x in the identity obtained in step 2 above, we obtain a quadratic equation is k. The quadratic equation so obtained is called the *indicial equation.*

Step 4. Solve the indicial equation. The following cases arise :

(*i*) The roots of indicial equation unequal and not differing by an integer.

(*ii*) The roots of indicial equation unequal, differing by an integer and making a coefficient of y indeterminate.

(*iii*) The roots of indicial equation unequal, differing by an integer and making a coefficient of y infinite.

(*iv*) The roots of indicial equation equal.

Step 5. We equate to zero the coefficient of general power (*e.g.* x^{k+m}, x^{k+m-1} etc. whichever may be the lowest) in the identity obtained in step 2. The equation so obtained will be called the *recurrence relation*, because it connects together the coefficients c_m, c_{m-2} or c_m, c_{m-1} etc.

Step 6. If the recurrence relation connects c_m and c_{m-2}, then we, in general, determine c_1 by equating to zero the coefficient of the next higher power (than already used for getting the indicial equation). On the other hand, if the recurrence relation connects c_m and c_{m-1}, this step may be omitted.

Step 7. After getting various coefficinets with help of steps 5 and 6 above, solution is obtained by substituting these in (2) or (3) above. Other necessary working will be shown in details and necessary modifications in method will be discussed as we proceed with different four cases outlined in step 4. The readers are advised to study carefully the first example of each case. In each problem the two series of solution should be linearly independent.

8.9. Examples of Type–1 on Frobenius method. Roots of indicial equation unequal and not differing by an integer :

In this connection the following rule should be noted carefully.

Rule. *Let k_1 and k_2 be the roots of the indicial equation. It k_1 and k_2 do not differ by an integer. Then, in general, two independent solutions u and v are obtained by putting $k = k_1$ and k_2 in the series for y. Then the general solution is $y = au + bv$, where a and b are arbitrary constants.*

Ex. 1. *Solve in series :* $9x(1-x)y'' - 12y' + 4y = 0$. **[Delhi Maths (H) 2007, 08; Meerut 1997]**

Sol. Given equation is $9x(1-x)y'' - 12y' + 4y = 0.$ ...(1)

Dividing by $9x(1-x)$, (1) can be put in standard form

$$\frac{d^2y}{dx^2} - \frac{4}{3x(1-x)}\frac{dy}{dx} + \frac{4}{9x(1-x)}y = 0.$$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, we have $P(x) = -4/[3x(1-x)]$ and $Q(x) = 4/[9x(1-x)]$. Since $P(x)$ and $Q(x)$ are not both analytic at $x = 0$, so $x = 0$ is not ordinary point of (1). Again $xP(x) = -4/[3(1-x)]$ and $x^2Q(x) = 4x/(1-x)$, showing that both $P(x)$ and $Q(x)$ are analytic at $x = 0$. So $x = 0$ is a regular singular point of (1). To find solution of (1), we take

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \text{ where } c_0 \neq 0. \quad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2}. \quad ...(3)$$

Substituting the series (2) and (3) in (1), we have

$$9x(1-x)\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} - 12\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} + 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\text{or} \quad 9x\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} - 9x^2\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} - 12\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} + 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\text{or} \quad \sum_{m=0}^{\infty} c_m \{9(k+m)(k+m-1) - 12(k+m)\}x^{k+m-1} + \sum_{m=0}^{\infty} c_m \{4 - 9(k+m)(k+m-1)\}x^{k+m} = 0. \quad ...(4)$$

But $9(k+m)(k+m-1) - 12(k+m) = 3(k+m)(3k+3m-7)$ ...(5)

and $4 - 9(k+m)(k+m-1) = 4 - 9(k+m)^2 + 9(k+m)$

$= -[9(k+m)^2 - 9(k+m) - 4] = -[9(k+m)^2 - 12(k+m) + 3(k+m) - 4]$

$= -[3(k+m)\{3(k+m) - 4\} + 3(k+m) - 4] = -\{3(k+m) - 4\}\{3(k+m) + 1\}$

Thus, $4 - 9(k+m)(k+m-1) = -(3k+3m-4)(3k+3m+1).$ ...(6)

Using (5) and (6), (4) can be re–written as

$$3\sum_{m=0}^{\infty} c_m (k+m)(3k+3m-7)x^{k+m-1} - \sum_{m=0}^{\infty} c_m (3k+m-4)(3k+3m+1)x^{k+m} = 0, \quad ...(7)$$

which is an identity in x. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (7) gives the indicial equation

$$3c_0k(3k-7)=0 \quad \text{or} \quad k(3k-7)=0 \quad [\because\ c_0 \neq 0]$$

Thus $k=0$ and $7/3$, ...(8)

which are unequal and not differing by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m-1} and obtain

$$3c_m(k+m)(3k+3m-7)-c_{m-1}[3k+3(m-1)-4]\times[3k+3(m-1)+1]=0$$

or
$$c_m=\frac{3k+3m-2}{3(k+m)}c_{m-1}. \qquad ...(9)$$

Taking $m=1$ in (9) gives
$$c_1=\frac{c_0}{3}\times\frac{3k+1}{k+1}. \qquad ...(10)$$

Next, taking $m=2$ in (9) gives $c_2=\dfrac{3k+4}{3(k+2)}\ c_1=\dfrac{c_0}{3^2}\times\dfrac{(3k+1)(3k+4)}{(k+1)(k+2)}$, by (10)

...(11)

and so on. Putting these values in (2) *i.e.*, $y=x^k(c_0+c_1x+c_2x^2+....)$, gives

$$y=c_0x^k\left[1+\frac{1}{3}\frac{3k+1}{k+1}x+\frac{1}{3^2}\frac{(3k+1)(3k+4)}{(k+1)(k+2)}x^2+....\right] \qquad ...(12)$$

Putting $k=0$ and replacing c_0 by a in (12), $y=a\left(1+\dfrac{1}{3}x+\dfrac{1\cdot4}{3\cdot6}x^2+....\right)=au$, say

Next, putting $k=\dfrac{7}{2}$ and replacing C_0 by b in (12), $y=bx^{7/3}\left(1+\dfrac{8}{10}x+\dfrac{8\cdot11}{10\cdot13}x^2+....\right)=bv$, say.

The required solution is given by $y=au+bv$, *i.e.*

$$y=a\left(1+\frac{1}{3}x+\frac{1\cdot4}{3\cdot6}x^2+....\right)+bx^{7/3}\left(1+\frac{8}{10}x+\frac{8\cdot11}{10\cdot13}x^2+....\right).$$

Ex. 2. *Solve the Bessel equation $x^2y''+xy'+(x^2-n^2)y=0$ in series, taking $2n$ as non–integral.* **[Delhi 1997; G.N.D.U. Amritsar 2010; Ranchi 2010]**

Sol. Given
$$x^2y''+xy'+(x^2-n^2)y=0. \qquad ...(1)$$

Dividing by x^2, $y''+(1/x)y'+\{(x^2-n^2)/x^2\}y=0.$

Comparing it with $y''+P(x)y'+Q(x)y=0$, here $P(x)=1/x$ and $Q(x)=(x^2-n^2)/x^2$ so that $xP(x)=1$ and $x^2Q(x)=x^2-n^2$. Thus both $P(x)$ and $Q(x)$ are analytic at $x=0$ and so $x=0$ is a regular singular point of (1). Let the series solution of (1) be of the form

$$y=\sum_{m=0}^{\infty}C_mx^{k+m}, \quad \text{where } C_0\neq0. \qquad ...(2)$$

$$\therefore\quad y'=\sum_{m=0}^{\infty}C_m(k+m)x^{k+m-1} \quad \text{and} \quad y''=\sum_{m=0}^{\infty}C_m(k+m)(k+m-1)x^{k+m-2}. \qquad ...(3)$$

Substitution for y, y', y'' in (1), we have

$$x^2\sum_{m=0}^{\infty}C_m(k+m)(k+m-1)x^{k+m-2}+x\sum_{m=0}^{\infty}C_m(k+m)x^{k+m-1}+(x^2-n^2)\sum_{m=0}^{\infty}C_mx^{k+m}=0$$

or
$$\sum_{m=0}^{\infty}C_m(k+m)(k+m-1)x^{k+m}+\sum_{m=0}^{\infty}C_m(k+m)x^{k+m}+\sum_{m=0}^{\infty}C_mx^{k+m+2}-n^2\sum_{m=0}^{\infty}C_mx^{k+m}=0$$

or $$\sum_{m=0}^{\infty} C_m\{(k+m)(k+m-1)+(k+m)-n^2\}x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m\{(k+m)^2-n^2\}x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m(k+m+n)(k+m-n)x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0, \quad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^k, (4) gives the indicial equation $C_0(k+n)(k-n) = 0$ or $(k+n)(k+n) = 0$ $[\because C_0 \neq 0]$ so that $k = n$ and $-n$. ...(5)

Since $2n$ is non–integral (given), the roots given by (5) are unequal and not differing by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m} and obtain

$$C_m(k+m+n)(k+m-n) + C_{m-2} = 0$$

giving $$C_m = -\frac{1}{(k+m+n)(k+m-n)} C_{m-2} \quad ...(6)$$

[Since (6) gives relationship between C_m and C_{m-2}, we proceed to find C_1 as explained in step 6 in Art. 8.8].

Equating to zero the coefficient of x^{k+1} in (4) gives $C_1(k+1+n)(k+1-n) = 0$ giving $C_1 = 0$ for both $k = n$ and $k = -n$. Then from (6) and $C_1 = 0$, we have

$$C_1 = C_3 = C_5 = ... = 0. \quad ...(7)$$

Further, taking $n = 2$ in (6) gives $$C_2 = -\frac{1}{(k+2+n)(k+2-n)} C_0. \quad ...(8)$$

Next, taking $n = 4$ in (6) and using (8) gives

$$C_4 = -\frac{1}{(k+4+n)(k+4-n)} C_2 = \frac{1}{(k+2+n)(k+2-n)(k+4+n)(k+4-n)} C_0 \quad ...(9)$$

and so on. Putting these values in (2), *i.e.*, $y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ...)$, gives

$$y = c_0 x^k\left[1 - \frac{x^2}{(k+2+n)(k+2-n)} + \frac{x^4}{(k+2+n)(k+2-n)(k+4+n)(k+4-n)} - ....\right]. \quad ...(10)$$

Putting $k = n$ and replacing C_0 by a in (10), gives

$$y = ax^n\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4\cdot 8(n+1)(n+2)} - ....\right] = au \text{ , say.}$$

Next, putting $k = -n$ and replacing C_0 by b in (10) gives

$$y = bx^{-n}\left[1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4\cdot 8(1-n)(2-n)} - ....\right] = bv \text{ , say.}$$

Required general series solution is given by $y = au + bv$, where a and b are arbitrary constants.

Ex. 3. *Solve the following differential equations in series :*

(*a*) $2x^2y'' - xy' + (1-x^2)y = 0$ **[Garhwal 2010]**

(*b*) $2x^2y'' - xy' + (1-x^2)y = x^2$. **[Meerut 1996, 98]**

Sol. (*a*) Given $$2x^2y'' - xy' + (1-x^2)y = 0 \quad ...(1)$$

Dividing by x^2, (1) takes standard form $y'' - (1/2x)y' + \{(1-x^2)/2x^2\}y = 0$.

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = -(1/2x)$ and $Q(x) = \{(1-x^2)/2x^2\}$ so that $xP(x) = -(1/2)$ and $x^2Q(x) = (1-x^2)/2$. Thus both $P(x)$ and $Q(x)$ are analytic at $x = 0$ and so $x = 0$ is a

regular singular point. Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}. \qquad ...(3)$$

Substitution for y, y', y'' in (1), we have

$$2x^2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + (1-x^2) \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m) x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{2(k+m)(k+m-1) - (k+m) + 1\} x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{2(k+m)^2 - 3(k+m) + 1\} x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m (2k+2m-1)(k+m-1) x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0, \qquad ...(4)$$

$[\because 2(k+m)^2 - 3(k+m) + 1 = 2(k+m)^2 - 2(k+m) - (k+m) + 1 = 2(k+m)[(k+m)-1] - [(k+m)-1]$
$= (k+m-1)[(2(k+m)-1] = (k+m-1)[(2k+2m-1)]$

(4) is an identity. Equating to zero the coefficient of the smallest power of x namely x^k, (4) gives the indicial equation $C_0(2k-1)(k-1) = 0$ or $(2k-1)(k-1) = 0$ $[\because C_0 \neq 0]$ so that $k = 1$, and $k = 1/2$, ...(5) which are unequal and not differing by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m} and obtain $C_m (2k+2m-1)(k+m-1) - C_{m-2} = 0$

giving $$C_m = \frac{1}{(2k+2m-1)(k+m-1)} C_{m-2}. \qquad ...(6)$$

To obtain C_1, we now equate to zero the coefficient of x^{k+1} and get $C_1(2k+1)k = 0$ so that $C_1 = 0$ for both roots $k = 1$, and $k = 1/2$ of the indicial equation. Then from (6) and $C_1 = 0$, we have

$$C_1 = C_3 = C_5 = ... = 0. \qquad ...(7)$$

Further, taking $n = 2$ in (6) gives $$C_2 = \frac{1}{(2k+3)(k+1)} C_0. \qquad ...(8)$$

Next, taking $n = 4$ in (6) and using (8) gives

$$C_4 = \frac{1}{(2k+5)(k+3)} C_2 = \frac{1}{(k+1)(k+3)(2k+3)(2k+5)} C_0$$

and so on. Putting these values in (2), *i.e.*, $y = x^k(C_0 + C_1 x + C_1 x^2 + C_3 x^3 + ...)$, gives

$$x = C_0 x^k \left[1 + \frac{x^2}{(k+1)(2k+3)} + \frac{x^4}{(k+1)(k+3)(2k+3)(2k+5)} + ... \right] \qquad ...(10)$$

Putting $k = 1$ and replacing C_0 by a in (10) gives

$$y = ax \left[1 + \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 9} + ... \right] = au \text{, say}$$

Next, putting $k = 1/2$ and replacing C_0 by b in (10) gives

$$y = bx^{1/2}\left[1 + \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} + ...\right] = bv \text{ , say}$$

The required series solution is given by $y = au + bv$ where a and b are arbitrary constants.

Part (b) Given $2x^2y'' - xy' + (1 - x^2)y = x^2$. ...(11)

Since R.H.S. is involved in (11), the general solution of (11) is made up of complementary function (C.F.) and particular integral (P.I.) as usual. To find C.F. of (11), we solve

$$2x^2y'' - xy' + (1 - x^2)y = 0,$$

which is the same as equation (1) of part (a). So as above, C.F. is $au + bv$. Next assume that the particular solution of (11) is of the form [similar to (2)].

$$y = x^k \sum_{m=0}^{\infty} A_m x^m, \qquad \text{where} \qquad A_0 \neq 0, \qquad ...(12)$$

Find y' and y'' and then substitute values of y, y' and y'' in (11). Since L.H.S. of (11) and (1) is the same, proceeding as in part (a) we shall get [compare with (4) of part (a)]

$$\sum_{m=0}^{\infty} A_m(2k + 2m - 1)(k + m - 1)x^{k+m} - \sum_{m=0}^{\infty} A_m x^{k+m+2} = x^2, \qquad ...(13)$$

which is an identity. Hence the leading term (the term containing the smallest power of x) of L.H.S. of (13) must be x^2 and the coefficients of each of the remaining terms of L.H.S. of (13) must vanish. These conditions are satisfied by taking

$$k = 2, \qquad A_0(k - 1)(2k - 1) = 1 \qquad \text{and} \qquad A_1 = 0 \qquad ...(14)$$

and

$$A_m = \frac{A_{m-2}}{(2k + 2m - 1)(k + m - 1)}. \qquad ...(15)$$

Note that (15) is similar to the recurrence relation (6) of part (a). Since $k = 2$, (14) gives $A_0 = 1/3$. Since $A_1 = 0$, so (15) gives

$$A_1 = A_3 = A_5 = ... = 0. \qquad ...(16)$$

Putting $m = 2$ and $k = 2$ in (15), we obtain $A_2 = \frac{A_0}{7\cdot 3} = \frac{1}{7\cdot 3\cdot 3\cdot 1}$, as $A_0 = \frac{1}{3\cdot 1}$. ...(17)

Putting $m = 4$ and $k = 2$ in (15) and using (17), we obtain

$$A_4 = \frac{A_2}{11\cdot 5} = \frac{1}{11\cdot 5\cdot 7\cdot 3\cdot 3\cdot 1}, \qquad ...(18)$$

Putting these values in (12), *i.e.*, $y = x^k(A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + ...)$, we obtain

$$\text{P.I.} = \frac{x^2}{3\cdot 1} + \frac{x^4}{7\cdot 3\cdot 3\cdot 1} + \frac{x^6}{11\cdot 5\cdot 7\cdot 3\cdot 3\cdot 1} + ... = f(x) \text{ , say.}$$

Hence the required solution is given by $y = au + bv + f(x)$, a and b being arbitrary constants.

Ex. 4. *Solve* $xy'' + (x + n)y' + (n + 1)y = 0$, *where n is not an integer.* **[Meerut 1993]**

Sol. Given $xy'' + (x + n)y' + (n + 1)y = 0.$ ...(1)

As usual verify that $x = 0$ is a regular singular point of (1). To solve (1), take

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad \text{where} \quad C_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} (k + m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} (k + m)(k + m - 1)C_m x^{k+m-2}. \quad ...(3)$$

Putting the above values of y, y' and y'' in (1), we have

$$x\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2}+(x+n)\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1}+(n+1)\sum_{m=0}^{\infty}C_m x^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1}+\sum_{m=0}^{\infty}(k+m)C_m x^{k+m}+\sum_{m=0}^{\infty}n(k+m)C_m x^{k+m-1}+\sum_{m=0}^{\infty}(n+1)C_m x^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1+n)C_m x^{k+m-1}+\sum_{m=0}^{\infty}(k+m+n+1)C_m x^{k+m}=0. \quad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, the above identity (4) gives the indicial equation

$C_0 k(k-1+n)=0$ so that $k=0$ and $1-n$, as $C_0\neq 0$.

Given that n is not an integer. So the roots 0 and $1-n$ of the indicial are unequal and do not differ by an integer. Next, we equate to zero the coefficient of x^{k+m-1} in the above identity (4) and obtain the recurrence relation

$$(k+m)(k+m+n-1)C_m+(k+m+n)C_{m-1}=0$$

so that $$C_m=-\frac{k+m+n}{(k+m)(k+m+n-1)}c_{m-1}. \quad ...(5)$$

Putting $m=1, 2, 3, ...$ in (5), we have $$C_1=-\frac{k+n+1}{(k+1)(k+n)}C_0, \quad ...(6)$$

$$C_2=-\frac{k+n+2}{(k+2)(k+n+1)}C_1=\frac{k+n+2}{(k+2)(k+n+1)}\times\frac{k+n+1}{(k+1)(k+n)}C_0, \text{ by (6)}$$

or $$C_2=\frac{k+n+2}{(k+1)(k+2)(k+n)}C_0, \quad ...(7)$$

$$C_3=-\frac{k+n+3}{(k+3)(k+n+2)}C_2=-\frac{(k+n+3)}{(k+3)(k+n+2)}\times\frac{(k+n+2)}{(k+1)(k+2)(k+n)}C_0, \text{ using (7)}$$

or $$C_3=-\frac{k+n+3}{(k+1)(k+2)(k+3)(k+n)}C_0 \quad ...(8)$$

and so on. Putting these values of C_1, C_2, C_3, ... in (2), we have

$$y=C_0x^k\left[1-\frac{k+n+1}{(k+1)(k+n)}x+\frac{k+n+2}{(k+1)(k+2)(k+n)}x^2-\frac{k+n+3}{(k+1)(k+2)(k+3)(k+n)}x^3+....\right] \quad ...(9)$$

Putting $k=0$ and replacing C_0 by a in (9), we get

$$y=a\left[1-\frac{n+1}{n}x+\frac{n+2}{n}\frac{x^2}{2!}-\frac{n+3}{n}\frac{x^3}{3!}+....\right]=au, \text{ say}$$

Putting $k=1-n$ and replacing C_0 by b in (9), we get

$$y=bx^{1-n}\left[1-\frac{2}{2-n}x+\frac{3}{(2-n)(3-n)}x^2-\frac{4}{(2-n)(3-n)(4-n)}x^3-....\right]=bv, \text{ say}$$

The required solution is $y=au+bv$, where a and b are arbitrary constants.

Ex. 5. *Verify that the origin is a regular singular point of* $2x^2y'' + xy' - (x+1)y = 0$ *and find two independent Frobenius series solutions of it.* **[Lucknow 1995]**

Sol. Given $2x^2y'' + xy' - (x+1)y = 0.$ ...(1)

Dividing by $2x^2$, (1) takes the standard form $y'' + (1/2x)y' - \{(x+1)/2x^2\}y = 0.$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 1/2x$ and $Q(x) = -(x+1)/2x^2$ so that $xP(x) = 1/2$ and $Q(x) = -(x+1)/2$. Since $xP(x)$ and $x^2Q(x)$ are both analytic, so $x = 0$ is a regular singular point of x. The solve (1), we take

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \quad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2}. \quad ...(3)$$

Putting the above values of y, y' and y'' in (1), we get

$$2x^2\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} + x\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} - (x+1)\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}2(k+m)(k+m-1)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)C_m x^{k+m} - \sum_{m=0}^{\infty}C_m x^{k+m+1} - \sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)(k+m-1) + (k+m) - 1\}C_m x^{k+m} - \sum_{m=0}^{\infty}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)^2 - (k+m) - 1\}C_m x^{k+m} - \sum_{m=0}^{\infty}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m) + 1\}(k+m-1)C_m x^{k+m} - \sum_{m=0}^{\infty}C_m x^{k+m+1} = 0. \quad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely, x^k, the above identity (4) in x gives the indicial equation, namely,

$C_0(2k+1)(k-1) = 0$ so that $k = 1$ and $-1/2$, as $C_0 \neq 0$.

Here the difference of these roots $= 1 - (-1/2) = 3/2 \neq$ not an integer.

Next, we equate to zero the coefficient of x^{k+m} in (4) and obtain the recurrence relation

$$\{2(k+m) + 1\}(k+m-1)C_m - C_{m-1} = 0$$

so that $$C_m = \frac{1}{(2k+2m+1)(k+m-1)}C_{m-1}. \quad ...(5)$$

Putting $m = 1, 2, 3, ...$ in (5), we have $C_1 = \{1/k\,(2k+3)\}C_0, \quad ...(6)$

$$C_2 = \frac{1}{(2k+5)(k+1)}C_1 = \frac{1}{(2k+3)(2k+5)k(k+1)}C_0, \text{ by (6).} \quad ...(7)$$

and so on. Putting these values in (2), we get

$$y = C_0x^k\left[1 + \frac{x}{(2k+3)k} + \frac{x^2}{(2k+3)(2k+5)k(k+1)} + ...\right]. \quad ...(8)$$

Putting $k = 1$ and replacing C_0 by a in (8), we get

$$y = ax[1 + x/5 + x^2/70 + ...] = au, \text{ say}$$

Next, putting $x = -1/2$ and replacing C_0 by b in (8), we get

$$y = bx^{-1/2}[1 - x - (x^2/2) + ...] = bv, \text{ say.}$$

The required solution is $y = au + bv$, a and b being arbitrary constants.

Ex. 6. *Show that* $x = 0$ *is a regular singular point of* $(2x + x^3)y'' - y' - 6xy = 0$ *and find its solution about* $x = 0$. **[Delhi Maths (H) 1995, 96; Meerut 1996]**

Sol. Given $(2x + x^3)y'' - y' - 6xy = 0.$ ...(1)

Dividing $(2x + x^3)$, (1) can be put in standard form $y'' - [1/(2x + x^3)\}y' - \{6/(2 + x^2)\}y = 0.$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = -1/(2x + x^3)$ and $Q(x) = -6/(2 + x^2)$ so that $xP(x) = -1/(2 + x^2)$ and $x^2Q(x) = -(6x^2)/(2 + x^2)$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point of (1). Let series solution of (1) be

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \quad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} (k+m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} (k+m)(k+m-1)C_m x^{k+m-2}.$$

...(3)

Putting the above values of y, y' and y'' in (1), we get

$$(2x + x^3)\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} - \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} - 6x\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}2(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m+1} - \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} - \sum_{m=0}^{\infty}6C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)(k+m-1) - (k+m)\}C_m x^{k+m-1} + \sum_{m=0}^{\infty}\{(k+m)(k+m-1) - 6\}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)^2 - 3(k+m)\}C_m x^{k+m-1} + \sum_{m=0}^{\infty}\{(k+m)^2 - (k+m) - 6\}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(2k+2m-3)C_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m-3)(k+m+2)C_m x^{k+m+1} = 0. \quad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, the above identity (4) gives the indicial equation, namely

$C_0k(2k - 3) = 0$ so that $k = 0$ and $3/2$, as $C_0 \neq 0$.

Here the difference of these roots $= (3/2) - 0 = 3/2 \neq$ not an integer.

Here the difference of the powers of x in (4) $= (k + m + 1) - (k + m - 1) = 2$. Hence we equate to zero the coefficient of x^k in the identity (4) and obtain

$C_1(k + 1)(2k - 1) = 0$ so that $C_1 = 0$ for both $k = 0$ and $k = 3/2$.

Next, equating to zero the coefficient of x^{k+m-1} in (4), we get

$$(k + m)(2k + 2m - 3)C_m + (k + m - 5)(k + m)C_{m-2} = 0$$

or $$C_m = -\frac{k+m-5}{2k+2m-3}C_{m-2}. \quad ...(5)$$

Putting $m = 3, 5, 7, ...$ in (5) and noting that $C_1 = 0$, we get

$$C_1 = C_3 = C_5 = C_7 = ... = 0. \quad ...(6)$$

Next, putting $m = 2, 4, 6, ...$ in (5), we have

$$C_2 = -\frac{k-3}{2k+1}C_0, \qquad C_4 = -\frac{k-1}{2k+5}C_2 = \frac{(k-1)(k-3)}{(2k+1)(2k+5)}C_0, ... \quad ...(7)$$

Putting these values in (2), we have

$$y = C_0 x^k \left[1 - \frac{k-3}{2k+1}x^2 + \frac{(k-1)(k-3)}{(2k+1)(2k+5)}x^4 - \ldots.\right]. \qquad \ldots(8)$$

Putting $k = 0$ and replacing C_0 by a in (8), $y = a[1 + 3x^2 + (3/5) \times x^4 - \ldots.] = au$, say

Putting $k = 3/2$ and replacing C_0 by b in (8), $y = x^{3/2}[1 + (3/8) \times x^2 - (3/128) \times x^4 + \ldots.] = bv$, say.

Hence the required solution is $y = au + bv$, a, b being arbitrary constants.

Ex. 7. *Solve the hypergeometric equation* $x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0$ *near* $x = 0$, if γ is not an integer. **[Meerut 1999, Kanpur 2006]**

Sol. Given $x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0. \qquad \ldots(1)$

Dividing by $x(1-x)$, (1) gives $\dfrac{d^2y}{dx^2} + \dfrac{\gamma - (\alpha + \beta + 1)x}{x(1-x)}\dfrac{dy}{dx} - \dfrac{\alpha\beta}{x(1-x)}y = 0.$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, we have

$$P(x) = \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \qquad \text{and} \qquad Q(x) = -\frac{\alpha\beta}{x(1-x)}.$$

Since $xP(x)$ and $x^2Q(x)$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point of (1).

Let the series solution of (1) be $y = \sum_{m=0}^{\infty} c_m x^{k+m}$, where $c_0 \neq 0 \qquad \ldots (2)$

$$\therefore \quad y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} \qquad \ldots(3)$$

Putting the above values of y, y', y'' in (1) gives

$$(x - x^2)\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} + [\gamma - (\alpha + \beta + 1)x]\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} - \alpha\beta \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\text{or} \quad \sum_{m=0}^{\infty} c_m \{(k+m)(k+m-1) + \gamma(k+m)\}x^{k+m-1} - \sum_{m=0}^{\infty} c_m \{(k+m)(k+m-1) + (\alpha+\beta+1)(k+m) + \alpha\beta\}x^{k+m} = 0$$

$$\text{or} \quad \sum_{m=0}^{\infty} c_m (k+m)(k+m-1+\gamma)x^{k+m-1} - \sum_{m=0}^{\infty} c_m \{(k+m)^2 + (\alpha+\beta)(k+m) + \alpha\beta\}x^{k+m} = 0$$

$$\text{or} \quad \sum_{m=0}^{\infty} c_m (k+m)(k+m-1+\gamma)x^{k+m-1} - \sum_{m=0}^{\infty} c_m (k+m+\alpha)(k+m+\beta)x^{k+m} = 0. \qquad \ldots(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (4) gives the indicial equation

$$c_0 k(k-1+\gamma) = 0 \qquad \text{or} \qquad k(k-1+\gamma) = 0 \qquad [\because\ c_0 \neq 0]$$

so the roots of the indicial equation are $k = 0$ $1 - \gamma$, which are unequal and not differing by an integer because by assumption γ is not an integer.

To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m-1}. Then we have

$$c_m (k+m)(k+m-1+\gamma) - c_{m-1}(k+m-1+\alpha)(k+m-1+\beta) = 0$$

$$\text{or} \quad c_m = \frac{(k+m-1+\alpha)(k+m-1+\beta)}{(k+m)(k+m-1+\gamma)} c_{m-1}. \qquad \ldots(5)$$

Case 1. When k = 0. Putting $m = 1, 2, 3, \ldots.$ succesively in (5), we get

$$c_1 = \frac{\alpha\beta}{1\cdot\gamma}c_0, \qquad c_2 = \frac{(\alpha+1)(\beta+1)}{2\cdot(\gamma+1)}c_1 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}c_0,$$

$$c_3 = \frac{(\alpha+2)(\beta+2)}{3(\gamma+2)}\,c_2 = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}c_0 \text{ and so on.}$$

Putting these values and $k = 0$ and replacing c_0 by a in (2) gives

$$y = a\left[1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots.\right] \quad \ldots(6)$$

If we take $a = 1$ in (6), the series on the right hand side of (6) is called *hypergeometric series* and is represented by ${}_2F_1(\alpha, \beta\,;\,\gamma\,;\,x)$. Thus we see that ${}_2F_1(\alpha, \beta\,;\,\gamma\,;\,x)$ is a solution of (1).

Case 2. When k = 1 – γ. Then (5) reduces to

$$c_m = \frac{(1-\gamma+m-1+\alpha)(1-\gamma+m-1+\beta)}{(1-\gamma+m)(1-\gamma+m-1+\gamma)}c_{m-1}$$

or
$$c_m = \frac{(\alpha'+m-1)(\beta'+m-1)}{m(\gamma'+m-1)}c_{m-1}. \quad \ldots(7)$$

where $\alpha' = 1-\gamma+\alpha$, $\beta' = 1-\gamma+\beta$ and $\gamma' = 2-\gamma$. …(8)

Replacing m by 1, 2, 3, ... successively in (7) gives as before

$$c_1 = \frac{\alpha'\beta'}{1\cdot\gamma'}c_0, \qquad c_2 = \frac{(\alpha'+1)(\beta'+1)}{2\cdot(\gamma'+1)}c_1 = \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{1\cdot 2\cdot\gamma'(\gamma'+1)}c_0 \text{ etc.}$$

Hence putting $k = 1-\gamma$, using the above values of $c_1, c_2, \ldots.$ in (2) and replacing c_0 by b gives

$$y = bx^{1-\gamma}\left[1 + \frac{\alpha'\cdot\beta'}{1\cdot\gamma'}x + \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{1\cdot 2\cdot\gamma'(1+\gamma')}x^2 + \ldots.\right] \quad \ldots(9)$$

If we take $b = 1$ in (9), the series on the R.H.S. of (9) would $x^{\gamma-1}\,{}_2F_1(\alpha', \beta'; \gamma'\,;\,x)$ *i.e.* $x^{\gamma-1}\,{}_2F_1(1-\gamma+\alpha,\ 1-\gamma+\beta\,;\,2-\gamma\,;\,x)$ which is another independent solution of (1). Hence the general series solution of (1) is

$$y = a\ {}_2F_1(\alpha, \beta\,;\,\gamma\,;\,x) + b\ x^{\gamma-1}\,{}_2F_1(1-\gamma+\alpha,\ 1-\gamma+\beta : 2-\gamma : x),$$

where a and b are arbitrary constants. ${}_2F_1(\alpha, \beta; \gamma; x)$ is called *hypergeometric function*.

Ex. 8. *Find the series solution of* $4xy'' + 2y' + y = 0$.

[Bilaspur 2004, Purvanchal 2005, Ravishankar 1998, 2004, Vikram 2004]

Sol. Given $4xy'' + 2y' + y = 0$ ...(1)

Re-writing (1), $y'' + (1/2x)y' + (1/4x)y = 0$ ... (2)

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, we have

$P = 1/2x$ and $Q(x) = 1/4x$ ... (3)

Since $P(x)$ and $Q(x)$ are not with analytic at $x = 0$, so $x = 0$ is not ordinary point of (1).

Also, $xP(x) = 1/2$ and $x^2Q(x) = x/4$,

showing that both $P(x)$ and $Q(x)$ are analytic at $x = 0$. Hence $x = 0$ is a regular singular point of (1). We, therefore, use Frobenium method to solve (1). Let a solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0 \quad \ldots (4)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1)x^{k+m-2} \quad \ldots (5)$$

Substituting the series (4) and (5) in (1), we have

$$4x\sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1)x^{k+m-2} + 2\sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \left\{4(k+m)\,(k+m-1) + 2(k+m)\right\} x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \left\{2(k+m)\,(2k+2m-1)\right\} x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0, \qquad \ldots (6)$$

which is an identity in x. Equating to zero the coefficients of the smallest power of x, namely, x^{k-1}, (6) gives the indicial equation

$$2C_0 k(2k-1) = 0 \qquad \text{giving} \qquad k = 0,\ 1/2 \qquad [\because\ C_0 \neq 0]$$

Here roots of indicial equation are unequal and do not differ by integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m-1} and obtain

$$2C_m(k+m)\,(2k+2m-1) + C_{m-1} = 0$$

or $$C_m = -\frac{C_{m-1}}{2(k+m)\,(2k+2m-1)},\ m = 1, 2, 3, \ldots \qquad \ldots (7)$$

Taking $m = 1$ in (7), $$C_1 = -\frac{C_0}{2(k+1)\,(2k+1)} \qquad \ldots (8)$$

Next, Taking $m = 2$ in (7), we have

$$C_2 = -\frac{C_1}{2(k+2)\,(2k+3)} = \frac{C_0}{4(k+1)\,(k+2)\,(2k+1)(2k+3)},\ \text{by (8)} \qquad \ldots (9)$$

Taking $m = 3$, in (7) and using (9), $C_3 = -\dfrac{C_0}{8(k+1)\,(k+2)\,(k+3)\,(2k+1)\,(2k+3)(2k+5)}$

Substituting the above values of C_1, C_2, C_3, ... in (4), we get

$$y = C_0 x^m \left\{1 - \frac{x}{2(k+1)\,(2k+1)} + \frac{x^2}{4(k+1)\,(k+2)\,(2k+1)\,(2k+3)} - \frac{x^3}{8(k+1)\,(k+2)\,(k+3)\,(2k+1)\,(2k+3)\,(2k+5)} + \ldots.\right\} \qquad \ldots (10)$$

Taking $k = 0$ and replacing C_0 by a in (10), we have

$$y = a\left(1 - \frac{x}{1\cdot 2} + \frac{x^2}{1\cdot 2\cdot 3\cdot 4} - \frac{x^3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots\right) = a\left(1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \ldots\right) = a\cos x^{1/2}$$

Taking $k = 1/2$ and replacing C_0 by b in (10), we have

$$y = bx^{1/2}\left(1 - \frac{x}{2\cdot 3} + \frac{x^2}{2\cdot 3\cdot 4\cdot 5} - \frac{x^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \ldots\right) = b\left(\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} - \frac{(\sqrt{x})^7}{7!} + \ldots\right) = b\sin x^{1/2}$$

Hence the required solution is $y = a\cos x^{1/2} + b\sin x^{1/2}$, a and b being arbitrary constants.

Ex. 9. *Using Frobenius method solve the differential equation* $d^2y/dx^2 + (1/4x) \times (dy/dx) + (1/8x^2)y = 0$ **[Delhi Maths (Hons.) 1993 , 2006]**

Sol. Re-writing the given differential equation, $8x^2y'' + 2xy' + y = 0$ ... (1)

Let a series solution of (1) be of the form $y = \sum_{m=0}^{\infty} C_m x^{k+m}, C_0 \neq 0$ ... (2)

$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1}$ and $y'' = \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m-2}$

Substituting the above values of y, y' and y'' in (1), we get

$$8x^2 \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m-2} + 2x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $\sum_{m=0}^{\infty} C_m \{8(k+m)\,(k+m-1) + 2(k+m) + 1\} x^{k+m} = 0$ or $\sum_{m=0}^{\infty} C_m \{8(k+m)^2 - 6\,(k+m) + 1\} x^{k+m} = 0$

or $$\sum_{m=0}^{\infty} C_m \{4\,(k+m) - 1\} \{2(k+m) - 1\} x^{k+m} = 0, \quad \text{... (3)}$$

which is an identity in x. Equating to zero the coefficient of the smallest power in x, namely, x^k, (6) gives the indicial equation

$C_0(4k - 1)\,(2k - 1) = 0$ giving $k = 1/2,\ 1/4,$ as $C_0 \neq 0$

which are unequal and do not differ by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m} in (3) and obtain

$$C_m \{4(k + m) - 1\} \{2(k + m) - 1\} = 0, \quad \text{for all} \quad m \geq 1 \quad \text{... (4)}$$

Relation (4) is satisfied by both values of $k = 1/2$ and $k = 1/4$ by choosing $C_m = 0$ for $m \geq 1$. Hence for $k = 1/2$ and $k = 1/4$, (2) reduces to

$y = x^k (C_0 + C_1x + C_2x^2 + ...)$ or $y = C_0x^k$ ... (5)

Putting $k = 1/2$ and replacing C_0 by a in (5), we have $y = ax^{1/2}$ ... (6)

Next, putting $k = 1/4$ and replacing C_0 by b in (5), we have $y = bx^{1/4}$ ... (7)

The required solution is $y = ax^{1/2} + bx^{1/4}$, where a and b are arbitrary constants.

EXERCISE 8 (C)

Find the series solution following equations near $x = 0$.

1. $2x(1 - x)y'' + (1 - x)y' + 3y = 0.$ **Ans.** $y = a\{1 - 3x + 3x^2/(1 \cdot 3) - ....\} + bx^{1/2}(1 - x)$

2. $2xy'' + (x + 1)y' + 3y = 0.$ **[Delhi Maths (H) 2005]**

Ans. $y = a\,(1 - 3x + 2x^2 - 2x^3/3...) + bx^{1/2}\,(1 - 7x/6 + 21x^2/40 - 11x^3/80\ ....)$

3. $2x^2y'' - xy' + (x^2 + 1)y = 0.$

4. $3xy'' + 2y' + x^2y = 0.$

8.10. Examples of Type 2 on Frobenius method. Roots of indicial equation unequal, differing by an Integer and making a coefficient of y indeterminate :

In this connection the following rule should be noted.

Rule : *If the indicial equation has two roots* k_1 *and* k_2 *(say* $k_1 < k_2$*) and if the one of the coefficients of y becomes indeterminate when* $k = k_2$*, the complete solution is given by putting* $k = k_2$ *in y, which then contains two arbitrary constants. The result of putting* $k = k_1$ *in y merely gives a numerial multiple of one of the series contained in the first solution. Hence we reject the solution obtained by putting* $k = k_1$.

Ex. 1. *Find solution near* $x = 0$ *of* $x^2y'' + (x + x^2)y' + (x - 9)y = 0$.

Sol. Given $x^2y'' + (x + x^2)y' + (x - 9)y = 0.$ ...(1)

Dividing by x^2, (1) can be put in standard form as $y'' + \{(1 + x)/x\}y' + \{(x - 9)/x^2\}y = 0$.

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = (1 + x)/x$ and $Q(x) = (x - 9)/x^2$ so that $xP(x) = 1 + x$ and $x^2Q(x) = x - 9$. Since both $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a regular singular point of (1). Let the series solution of (1) be

$$y = x^k(C_0 + C_1x + C_2x^2 + \text{.... ad. inf.}) = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad \text{where } C_0 \neq 0. \qquad ...(2)$$

$$\therefore y' = \sum_{m=0}^{\infty}(k + m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty}(k + m)(k + m - 1)C_m x^{k+m-2}.$$

...(3)

Putting the above values of y, y' and y'' in (1), we have

$$x^2 \sum_{m=0}^{\infty}(k + m)(k + m - 1)C_m x^{k+m-2} + (x + x^2) \sum_{m=0}^{\infty}(k + m)C_m x^{k+m-1} + (x - 9) \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

$$\text{or } \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)C_m x^{k+m+1} + \sum_{m=0}^{\infty} C_m x^{k+m+1} - 9\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

$$\text{or } \sum_{m=0}^{\infty}\{(k + m)(k + m - 1) + (k + m) - 9\}C_m x^{k+m} + \sum_{m=0}^{\infty}\{(k + m) + 1\}C_m x^{k+m+1} = 0$$

$$\text{or } \sum_{m=0}^{\infty}\{(k + m)^2 - 3^2\}C_m x^{k+m} + \sum_{m=0}^{\infty}(k + m + 1)C_m x^{k+m+1} = 0$$

$$\text{or } \sum_{m=0}^{\infty}(k + m + 3)(k + m - 3)C_m x^{k+m} + \sum_{m=0}^{\infty}(k + m + 1)C_m x^{k+m+1} = 0. \qquad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely x^k, the above identity (4) gives the indicial equation

$(k + 3)(k - 3)C_0 = 0$ so that $k = 3, -3$ as $C_0 \neq 0$.

Next, equating to zero the coefficient of x^{k+m} in (4), we get

$$(k + m + 3)(k + m - 3)C_m + (k + m)C_{m-1} = 0$$

$$\text{or } C_m = -\frac{(k + m)}{(k + m + 3)(k + m - 3)}C_{m-1}. \qquad ...(5)$$

Putting $m = 1, 2, 3, \ldots$ in (5), we get $C_1 = -\dfrac{k + 1}{(k + 4)(k - 2)}C_0$,

$$C_2 = -\frac{(k + 2)}{(k + 5)(k - 1)}C_1 = \frac{(k + 1)(k + 2)}{(k - 1)(k - 2)(k + 4)(k + 5)}C_0,$$

$$C_3 = -\frac{(k + 3)}{(k + 6)k}C_2 = -\frac{(k + 1)(k + 2)(k + 3)}{k(k - 1)(k - 2)(k + 4)(k + 5)(k + 6)}C_0$$

and so on. Putting these values in (2), we have

$$y = C_0x^k\left[1 - \frac{(k+1)}{(k-2)(k+4)}x + \frac{(k+1)(k+2)}{(k-1)(k-2)(k+4)(k+5)}x^2 - \frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k+4)(k+5)(k+6)}x^3 + \text{.... ad. inf.}\right] \qquad ...(6)$$

Putting $k = 3$ and replacing C_0 by a in (6), we have

$$y = ax^3\left[1 - \frac{4}{2\cdot 7}x + \frac{4\cdot 5}{2\cdot 1\cdot 7\cdot 8}x^2 + \frac{4\cdot 5\cdot 6}{3\cdot 2\cdot 1\cdot 7\cdot 8\cdot 9}x^3 - \text{....}\right] = au, \text{ say}$$

Putting $k = -3$ and replacing C_0 by b in (6), we have

$$y = bx^{-3}\{1 - (2/5)x + (1/20)x^2\} = bv, \text{ say.}$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Important note. When $x = 0$ is an ordinary point of $y'' + P(x)y' + Q(x)y = 0$, we have already explained the method of solution in Art. 8.5. We have solved many problems based on Art. 8.5. All those problems can also be solved by Frobenius method as given in Art. 8.10 and explained in above solved Ex. 1. We now solve the same problems by method of solved Ex. 1 as follows.

Ex. 2. *Solve in series* $(1 - x^2)y'' - xy' + 4y = 0$.

Sol. Given $$(1 - x^2)y'' - xy' + 4y = 0. \quad ...(1)$$

Clearly $x = 0$ is an ordinary point of (1). Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \quad C_0 \neq 0. \quad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} \quad ...(3)$$

Putting the above values of y, y', y'' in (1) gives

$$\text{or} \quad (1 - x^2) \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + 4 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

$$\text{or} \quad \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m) x^{k+m} + 4 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

$$\text{or} \quad \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m [(k+m)(k+m-1) + (k+m) - 4] x^{k+m} = 0$$

$$\text{or} \quad \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m [(k+m)^2 - 4] x^{k+m} = 0$$

$$\text{or} \quad \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m (k+m+2)(k+m-2) x^{k+m} = 0, \quad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-2}, (4) gives the indicial equation

$$C_0 k(k-1) = 0 \quad \text{or} \quad k(k-1) = 0 \quad [\because C_0 \neq 0]$$

giving $k = 1$ and $k = 0$. These are unequal and differ by an integer. To get the recurrence relation, we equate to zero the coefficient of x^{k+m-2}. Thus,

$$C_m (k+m)(k+m-1) - C_{m-2}(k+m)(k+m-4) = 0$$

giving $$C_m = \frac{k+m-4}{k+m-1} C_{m-2}. \quad ...(5)$$

Next, we equate to zero the coefficient of x^{k-1} and get $$C_1(k+1)k = 0. \quad ...(6)$$

If we take $k = 0$, (6) shows that C_1 is indeterminate. With $k = 0$ and using (5), we can express C_2, C_4, C_6 .... in terms of C_0 and C_3, C_5, C_7 .... in terms of C_1 if we assume C_1 to be finite. Thus,

$$C_m = \{(m-4)/(m-1)\}\, C_{m-2}$$

so that $C_2 = -2C_0$, $C_4 = 0$ and hence $C_6 = C_8 = ... = 0$

and $C_3 = (-C_1)/(2) = -C_1/2$, $C_5 = C_3/(4) = -C_1/8$, $C_7 = (3C_5)/(6) = -C_1/16$,

and so on. Putting $k = 0$ and these values in (2), *i.e.*

$$y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + ...)$$

i.e. $$y = x^0(C_0 + C_2x^2 + C_4x^4 + ...) + x^0(C_1x + C_3x^3 + C_5x^5 + ...), \text{ we get}$$

$$y = C_0(1 - 2x^2) + C_1(x - x^3/2 - x^5/8 + x^7/16 - ....) \quad ...(7)$$

which is the required series solution, C_0 and C_1 being two arbitrary constants.

Remarks. The reader can easily verify that the root $k = 1$ of the indicial equation gives another solution as usual (Refer examples of Type 1). But this solution, will be a constant multiple of one of the series occurring in (7). So we reject it.

Ex. 3. *Solve in series the Legendre's equation* $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$

[Delhi Maths (Hons.) 1996, Nagpur 1996, Utkal 2003]

Sol. Given $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ ...(1)

Here $x = 0$ is an ordinary point of (1). Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \qquad ...(2)$$

$$\therefore \; y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}. \qquad ...(3)$$

Putting the above values of y, y', y'' into (1) gives

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - x^2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

$$- 2x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + n(n+1) \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} \{(k+m)(k+m-1) + 2(k+m) - n(n+1)\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m \{(k+m)^2 + (k+m) - n^2 - n)\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m \{(k+m+n)(k+m-n) + (k+m-n)\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m (k+m-n)(k+m+n+1) x^{k+m} = 0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-2}, (4) gives the identical equation

$$c_0 k(k-1) = 0 \quad \text{or} \quad k(k-1) = 0, \quad [\because \; c_0 \neq 0]$$

giving $k = 1$ and $k = 0$. These are unequal and differ by an integer. To get the recurrence relation, we equate to zero the coefficient of x^{k+m-2}. Thus

$$C_m (k+m)(k+m-1) - C_{m-2}(k+m-2-n)(k+m-2+n+1) = 0$$

so that $$C_m = \frac{(k+m-2-n)(k+m-1+n)}{(k+m)(k+m-1)} C_{m-2}. \qquad ...(5)$$

Next, equating to zero the coefficient of x^{k-1} gives $C_1(k+1)k = 0$. ...(6)

If we take $k = 0$, (6) shows that C_1 is indeterminate. For $k = 0$, (5) gives

$$C_m = \frac{(m-2-n)(m-1+n)}{m(m-1)} C_{m-2}. \qquad ...(7)$$

We now express C_2, C_4, C_6, ... in terms of C_0 and C_3, C_5, C_7, ... in terms of C_1 by assuming that C_1 is finite. Putting $m = 2$ in (7) gives

$$C_2 = \frac{(-2)(n+1)}{2\cdot 1} C_0 = -\frac{n(n+1)}{2!} C_0 . \qquad ...(8)$$

Putting $m = 4$ in (7) and using (8) gives

$$C_4 = \frac{(2-n)(3+n)}{3\cdot 2} C_2 = \frac{(n-2)n(n+1)(n+3)}{4!} C_0 . \qquad ...(9)$$

Next, putting $m = 3$ in (7) gives

$$C_3 = \frac{(1-n)(2+n)}{3\cdot 2} C_1 = -\frac{(n-1)(n+2)}{3!} C_1 . \qquad ...(10)$$

Again, putting $m = 5$ in (7) and using (10) gives

$$C_5 = \frac{(3-n)(4+n)}{5\cdot 4} C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1$$

and so on. Now (2) can be re–written as

$$y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ...)$$

or $$y = x^0(C_0 + C_2x^2 + C_4x^4 + ...) + x^0(C_1x + C_3x^3 + C_5x^5 + ...) \qquad [\because\ k = 0]$$

Using the values of C_2, C_3, C_4, C_5, ... as given by (8), (9), (10), (11) etc. in the above equation gives

$$y = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - ....\right] + C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + ...\right]$$

which is the required general series solution, C_0 and C_1 being arbitrary constants.

Ex. 4. *Find the series solution of* $x^2(d^2y/dx^2) - x\,(dy/dx) - (x^2 + 5/4)y = 0$ about $x = 0$. **[Kurukshetra 2004]**

Sol. Given $$x^2y'' - xy' - (x^2 + 5/4)y = 0 \qquad ...(1)$$

Re-writing (1), $$y'' - (1/x)y' - (1 + 5/4x^2)y = 0 \qquad ... (2)$$

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, we have

$P(x) = -(1/x)$ and $Q(x) - (1 + 5/4x^2)$ ... (3)

$x = 0$ is not an ordinary point of (1) because neither $P(x)$ nor $Q(x)$ is analytic at x. Again, from (3),

$x\,P(x) = -1$ and $x^2Q(x) = -(x^2 + 5/4)$,

which are both analytic at $x = 0$. So $x = 0$ is a regular singular point of (1). To find solution of (1), we choose a series solution of (1) about $x = 0$ in the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0 \qquad ... (4)$$

$$\therefore\quad y' = \sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} \qquad ... (5)$$

Substituting the above values of y, y' and y'' in (1), we get

$$x^2\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} - x\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} - (x^2 + 5/4)\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m} - \sum_{m=0}^{\infty} C_m(k+m)x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} - \frac{5}{4}\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1)-(k+m)-5/4\} x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-2)-5/4\} x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0, \quad \ldots (6)$$

which is an identity in x. Equating to zero the coefficient of the smallest power of x, namely, x^k, (6) gives the indicial equation

$C_0\{k(k-2)-5/4\} = 0$ or $4k^2 - 8k - 5 = 0$ or $(2k-5)(2k+1) = 0$, as $C_0 \neq 0$

Hence $k = 5/2, -1/2$ are roots of the indicial equation. These are distinct roots and differ by an integer.

We begin with determination of the solution corresponding to the smallest root $k = -1/2$.

Here the exponents in x^{k+m} and x^{k+m+2} differ by 2 in identity (6). So we first equote to zero the coefficient of x^{k+1}, and obtain $9\{(k+1)(k-1)-(5/4)\} = 0 \Rightarrow C_1 = 0$, since the second factor does not vanish for both $k = -1/2$ and $k = 5/2$.

For recurrence relation, equating to zero the coefficient of x^{k+m}, (6) yields

$$C_m \{(k+m)(k+m-2)-5/4\} C_m - C_{m-2} = 0, \text{ for all } m \geq 2 \quad \ldots (7)$$

Putting $m = 3$ and $k = -1/2$ in (7), $O \times C_3 - C_1 = 0$ or $O \times C_3 = 0$, as $C_1 = 0$

Hence, C_3 can be chosen as any arbitrary constant. Putting $k = -1/2$ in (7), we have

$$\{(m-1/2)(m-5/2)-5/4\} C_m - C_{m-2} = 0, \text{ for all } m \geq 2$$

Thus, $$C_m = \frac{1}{m(m-3)} C_{m-2}, \quad \text{for all } m \geq 2 \text{ and } m \neq 3 \quad \ldots (8)$$

Putting $m = 2$ in (8), $C_2 = -(1/2)\, C_0$

Putting $m = 4$ in (8), $C_4 = (1/4) \times C_2 = -(1/8) \times C_0$, by (9)

Similarly, $C_5 = (1/10) \times C_3$, $C_6 = -(1/144) \times C_0$, $C_7 = (1/280) \times C_3, \ldots$.

Substituting the above values in (4) and taking $k = -1/2$, we get

$$y = x^{-1/2}\left\{C_0 - (1/2)C_0x^2 + C_3x^3 - (1/8)C_0x^4 + (1/10)C_3x^5 - (1/144)C_0x^6 + (1/280)C_3x^7 - (1/5760)C_0x^8 + \ldots\right)$$

or $$y = C_0x^{-1/2}(1 - x^2/2 - x^4/8 - x^6/144 - x^8/5760 + \ldots) + C_3x^{5/2}(1 + x^2/10 + x^4/280 + \ldots), \ldots (10)$$

which is the required solution containing a_0 and a_3 as arbitrary constants.

Remark. The reader can easily verify that the roots $k = 5/2$ of the indicial equation gives another solution $C_0x^{5/2}(1 + x^2/10 + x^4/280 + \ldots)$, which already occurs in (10). So we need not consider $k = 5/2$.

EXERCISE 8 (D)

Find the series solution of the following equations near $x = 0$.

1. $(x^2 + 1)y'' + xy' - xy = 0$ about $x = 0$. **[Delhi Maths (Hons.) 1993]**

Ans. $y = C_0\{1 + (1/6)x^3 - (3/40)x^5 + \ldots.\} + C_1\{x - (1/6)x^3 + (1/12)x^4 + \ldots.\}$

2. $y'' + xy' + x^2y = 0$ near $x = 0$. **[Delhi Maths (Hons.) 1996]**

Ans. $y = C_0\{1 - (1/12)x^4 + (1/90)x^6 - \ldots.\} + C_1\{x - (1/6)x^3 - (1/40)x^5 - \ldots\}$

3. $y'' - xy' - py = 0$, where p is any constant, near $x = 0$.

Ans. $$y = C_0\left\{1 + \frac{p}{2!}x^2 + \frac{p(p+2)}{4!}x^4 + \ldots\right\} + C_1\left\{x + \frac{p+1}{3!}x^3 + \frac{(p+1)(p+3)}{5!}x^5 + \ldots\right\}$$

4. $(1 - x^2)y'' + 2xy' - y = 0$ about $x = 0$.

Ans. $y = C_0\{(1 + (1/2)x^2 - (1/24)x^4 - ....\} + C_1\{x - (1/6)x^3 - (1/120)x^5 - ....\}$

5. $(2 + x^2)y'' + xy' - (1 + x)y = 0$ near $x = 0$.

Ans. $y = C_0\{1 + (1/4)x^2 + (1/12)x^3 - ....\} + C_1\{x + (1/24)x^4 + ...\}$

6. $(1 + x^2)y'' + xy' - y = 0$ near $x = 0$. **Ans.** $y = C_0\{1 + (1/2)x^2 - (1/8)x^4 + ...\} + C_1x$

7. $(x^2 - 1)y'' + 4xy' + 2y = 0$ near $x = 0$. **Ans.** $y = C_0(1 + x^2 + x^4 + ...) + C_1(x + x^3 + x^5 + ...)$

8. $y'' + x^2y = 0$.

Ans. $y = a\left(1 - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 7\cdot 8}x^8 - ....\right) + b\left(x - \frac{1}{4\cdot 5}x^5 + \frac{1}{4\cdot 5\cdot 8\cdot 9}x^9 - ....\right)$.

9. $(1 - x^2)y'' + 2xy' + y = 0$. **[Guwahati 2007; Meerut 1997]**

Ans. $y = a[1 - (1/2)x^2 + (1/8)x^4 + (1/80)x^6 + ...] + b[x - (1/2)x^3 + (1/40)x^5 + ...$

10. Find two independent Frobenius series solutions of the equation $xy'' + 2y' + xy = 0$.
[Delhi B.Sc. (Hons) II 2011] **Ans.** $(\cos x)/x$ and $(\sin x)/x$

8.11. Examples of Type 3 on Frobenius method. Roots of indicial equation, unequal, differing by an integer and making a coefficient of y infinite.

In this connection the following rule should be noted carefully.

Rule. *If the indicial equation has two unequal roots k_1 and k_2 (say $k_1 > k_2$) differing by an integer, and if some of the coefficients of y become infinite when $k = k_2$, we modify the form of y by replacing c_0 by do $(k - k_2)$. We then obtain two independent solutions by putting $k = k_2$ in the modified form of y and $\partial y/\partial k$. The result of putting $k = k_1$ in y gives a numerical multiple of that obtained by putting $k = k_2$ and hence we reject the solution obtained by putting $k = k_1$ in y.*

Ex. 1. *Solve in series $x(1 - x)y'' - 3xy' - y = 0$ near $x = 0$.* **[Delhi Maths (H) 2009]**

Sol. Given $$x(1 - x)y'' - 3xy' - y = 0. \quad ...(1)$$

Dividing by $x(1 - x)$, (1) yields $$\frac{d^2y}{dx^2} - \frac{3}{1-x}\frac{dy}{dx} - \frac{1}{x(1-x)}y = 0.$$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, hence $P(x) = -3/(1 - x)$ and $Q(x) = -1/\{x(1 - x)\}$ so that $xP(x) = -3x/(1 - x)$ and $x^2Q(x) = -x/(1 - x)$. Since $xP(x)$ and $x^2Q(x)$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point of (1). Let the series solution of (1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \text{ where } c_0 \neq 0. \quad ...(2)$$

$$\therefore y' = \sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2}. \quad ...(3)$$

Putting the above values of y, y', y'' into (1) gives

$$(x - x^2)\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} - 3x\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} - \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-1} - \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m} - 3\sum_{m=0}^{\infty} c_m (k+m)x^{k+m} - \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-1} - \sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) + 3(k+m) + 1\}x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-1} - \sum_{m=0}^{\infty} c_m \{(k+m)^2 + 2(k+m) + 1\}x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-1} - \sum_{m=0}^{\infty} c_m (k+m+1)^2 x^{k+m} = 0. \quad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (4) gives the indicial equation

$$c_0 k(k-1) = 0 \quad \text{or} \quad k(k-1) = 0. \quad [\because\ c_0 \neq 0]$$

which gives $k = 0$ and $k = 1$. These are unequal and differ by an integer. Next, to find the recurrence relation we equate to zero the coefficient of x^{k+m-1} and obtain

$$c_m(k+m)(k+m-1) - c_{m-1}(k+m)^2 = 0 \quad \text{or} \quad c_m = \frac{k+m}{k+m-1} c_{m-1}. \quad ...(5)$$

Putting $m = 1$ in (5) gives $c_1 = \{(k+1)/k\} C_0$. ...(6)

Putting $m = 2$ in (5) and using (6) gives $c_2 = \frac{k+2}{k+1} c_1 = \frac{k+2}{k} c_0$. ...(7)

Putting $m = 3$ in (5) and using (7) gives $c_3 = \frac{k+3}{k+2} c_2 = \frac{k+3}{k} c_0$. ...(8)

Putting these values in (2), *i.e.*, $y = x^k(c_0 + c_1x + c_2x^2 + ...)$, gives

$$y = c_0 x^k \left[1 + \frac{k+1}{k}x + \frac{k+2}{k}x^2 + \frac{k+3}{k}x^3 + ...\right]. \quad ...(9)$$

If we put $k = 0$ in (9), we find that due to presence of the factor k in their denominators, the coefficients becomes infinite. To remove this difficulty, we write $c_0 = k$ do in (9). Then (9) becomes

$$y = d_o\, x^k[k + (k+1)x + (k+2)x^2 + (k+3)k^3 + ...] \quad ...(10)$$

Putting $k = 0$ and replacing do by a in (10) gives

$$y = a(x + 2x^2 + 3x^3 + ...) = au, \text{ say} \quad ...(11)$$

To obtain a second solution, if we put $k = 1$ in (9) we obtain

$$y = c_0(x + 2x + 3x^2 + ...) \quad ...(12)$$

which is not distinct (*i.e.* not linearly independent because ratio of the two series in (11) and (12) is a constant) from (11). Hence (12) will not serve the purpose of a second solution. In such a case the second independent solution is given by $(\partial y/\partial k)_{k=0}$. Differentiating (10) partially w.r.t. 'k'

$$\partial y/\partial k = do\, x^k \log x\, [k + (k+1)x + (k+2)x^2 + ...] + do\, x^k\, [1 + x + x^2 + ...]. \quad ...(13)$$

Putting $k = 0$ and replacing do by b (13), gives

$$(\partial y/\partial k)_{k=0} = b \log x\, (x + 2x^2 + 3x^3 + ...) + b(1 + x + x^2 + ...)$$

or $$(\partial y/\partial k)_{k=0} = b[u \log x + (1 + x + x^2 + ...)] = bv, \text{ by (11)} \quad ...(14)$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Ex. 2. *Solve in series the Bessel's equation of order* 2, *near* $x = 0$, $x^2y'' + xy' + (x^2 - 4)y = 0$.

[Delhi Maths (Hons.) 1995, Meerut 1995]

Sol. Given $x^2y'' + xy' + (x^2 - 4)y = 0.$ ...(1)

Dividing by x^2, $y'' + (1/x)y' + [(x^2 - 4)/x^2]y = 0.$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 1/x$ and $Q(x) = (x^2 - 4)/x^2$ so that $xP(x) = 1$ and $x^2Q(x) = x^2 - 4$. Since $xP(x)$ and $x^2Q(x)$ are both analytic at $x = 0$, hence $x = 0$ is a regular singular point of (1). Let the series solution of (1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \text{ where } c_0 \neq 0. \quad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} \quad ...(3)$$

Putting the above values of y, y', y'' into (1) gives

$$x^2\sum_{m=0}^{\infty}c_m(k+m)(k+m-1)x^{k+m-2}+x\sum_{m=0}^{\infty}c_m(k+m)x^{k+m-1}+x^2\sum_{m=0}^{\infty}c_m x^{k+m}-4\sum_{m=0}^{\infty}c_m x^{k+m}=0$$

or $\sum_{m=0}^{\infty}c_m\{(k+m)(k+m-1)+(k+m)-4\}x^{k+m}+\sum_{m=0}^{\infty}c_m x^{k+m+2}=0$ or $\sum_{m=0}^{\infty}c_m\{(k+m)^2-4\}x^{k+m}+\sum_{m=0}^{\infty}c_m x^{k+m+2}=0$

or $$\sum_{m=0}^{\infty}c_m(k+m+2)(k+m-2)x^{k+m}+\sum_{m=0}^{\infty}c_m x^{k+m+2}=0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^k, gives the indicial equation $c_0(k+2)(k-2)=0$ or $(k+2)(k-2)=0$ $[\because\ c_0\neq 0]$

This gives $k = 2$ and $k = -2$. These are unequal and differ by an integer. For the recurrence relation, we equate to zero the coefficient of x^{k+m} and get

$$c_m(k+m+2)(k+m-2)+c_{m-2}=0 \quad \text{so} \quad c_m=-\frac{1}{(k+m+2)(k+m-2)}c_{m-2}. \qquad ...(5)$$

To determine c_1, we equate to zero the coefficient of x^{k+1} and get $c_1(k+3)(k-1)=0$ giving $c_1 = 0$ for both the roots $k = 2$ and $k = -2$ of the indicial equation. Now using $c_1 = 0$ and (5), we get

$$c_1 = c_3 = c_5 = c_7 = ... = 0. \qquad ...(6)$$

Next, putting $m = 2, 4, 6, ...$ in (5) and simplifying, we have

$c_2 = -c_0/k(k+4)$, $\quad c_4 = -c_2/(k+2)(k+6) = c_0/k(k+2)(k+4)(k+6)$,

$c_6 = -c_4/(k+4)(k+8) = -c_0/k(k+2)(k+4)^2(k+6)(k+8)$

and so on. Putting these values in (2), *i.e.*,

$y = x^k(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + ...)$, we get

$$y = c_0x^k\left\{1-\frac{x^2}{k(k+4)}+\frac{x^4}{k(k+2)(k+4)(k+6)}-\frac{x^6}{k(k+2)(k+4)^2(k+6)(k+8)}+....\right\} \qquad ...(7)$$

Putting $k = 2$ in (7) yields $y = c_0x^2\left\{1-\frac{x^2}{2\cdot 6}+\frac{x^4}{2\cdot 4\cdot 8\cdot 6}-\frac{x^6}{2\cdot 4\cdot 6^2\cdot 8\cdot 10}+....\right\} = c_0\,\omega$, say ...(8)

Next, if we put $k = -2$ in (7), the coefficients of x^4, x^6, ... become infinite. To get rid of this difficulty, we put $c_0 = do(k+2)$ in (7) and obtain modified solution as

$$y = do\,x^k\left\{(k+2)-\frac{(k+2)x^2}{k(k+4)}-\frac{x^4}{k(k+4)(k+6)}-\frac{x^6}{k(k+4)^2(k+6)(k+8)}+....\right\} \qquad ...(9)$$

Putting $k = -2$ and replacing do by a in (9) gives

$$y = ax^{-2}\left\{0-0\cdot x^2+\frac{x^4}{(-2)(2)(4)}-\frac{x^6}{(-2)(2)^2(4)(6)}+....\right\} \text{ or } y=\frac{-ax^2}{16}\left\{1-\frac{x^2}{2\cdot 6}+\frac{x^4}{2\cdot 4\cdot 8\cdot 6}-....\right\}=au, \text{ say} \qquad ...(10)$$

Now, (8) and (10) $\Rightarrow$ $u=-\frac{x^2}{16}\left\{1-\frac{x^2}{2\cdot 6}+\frac{x^4}{2\cdot 4\cdot 8\cdot 6}-....\right\}=-\frac{\omega}{16}$,

showing that w and u are dependent solutions. Hence we must find one more independent solution in order to obtain the required general solution.

To get another independent solution, substituting (9) into the L.H.S. of (1) and simplifying, we find $x^2y'' + xy' + (x^2 - 4)y = d_0(k-2)(k+2)^2x^k$. ... (11)

Differentiating both sides of (11) partially w.r.t. k, we get

$$\frac{\partial}{\partial k}\left[x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-4)y\right]=d_0\frac{\partial}{\partial k}[(k+2)^2\cdot x^k(k-2)]$$

$$\text{or}\left[x^2\frac{d^2}{dx^2}+x\frac{d}{dx}+(x^2-4)\right]\frac{\partial y}{\partial k}=2d_0(k+2)\cdot x^k(k-2)+d_0(k+2)^2[x^k\log x\cdot(k-2)+x^k\cdot 1]\quad ...(12)$$

The presence of the factor $(k+2)$ in each term on R.H.S. of (12) shows that a second solution is $(\partial y/\partial k)_{k=-2}$, where y is given by (9). Differentiating (9) partially w.r.t. 'k' we get

$$\frac{\partial y}{\partial k}=d_0x^k\log x\left\{(k+2)-\frac{(k+2)x^2}{k(k+4)}+\frac{x^4}{k(k+4)(k+6)}-\frac{x^6}{k(k+4)^2(k+6)(k+8)}+....\right\}$$

$$+d_0x^k\left\{1-\frac{(k+2)x^2}{k(k+4)}\left(\frac{1}{k+2}-\frac{1}{k}-\frac{1}{k+4}\right)+\frac{x^4}{k(k+4)(k+6)}\left(-\frac{1}{k}-\frac{1}{k+4}-\frac{1}{k+6}\right)\right.$$

$$\left.-\frac{x^6}{k(k+4)^2(k+6)(k+8)}\times\left(-\frac{1}{k}-\frac{2}{k+4}-\frac{1}{k+6}-\frac{1}{k+8}\right)+....\right\}\quad ...(13)$$

$$\left[\text{To find }\frac{d}{dk}\left\{\frac{k+2}{k(k+4)}\right\},\text{ we proceed as follows : Take } z=\frac{k+2}{k(k+4)}\text{ so that } \log z=\log\left[\frac{k+4}{k(k+4)}\right]\right.$$

or $$\log z=\log(k+4)-\log k-\log(k+4).$$

Differentiating it w.r.t. 'k', we get

$$\frac{1}{z}\frac{dz}{dk}=\frac{1}{k+4}-\frac{1}{k}-\frac{1}{k+4}\quad\Rightarrow\quad\frac{d}{dk}\left\{\frac{k+2}{k(k+4)}\right\}=\frac{k+2}{k(k+4)}\left(\frac{1}{k+4}-\frac{1}{k}-\frac{1}{k+4}\right)\text{ etc.}$$

Other terms can be similarly differentiated easily]

Putting $k=-2$ and replacing do by b in (13) gives

$$\left(\frac{\partial y}{\partial k}\right)_{k=-2}=bx^{-2}\log x\left\{0-0.x^2+\frac{x^4}{(-2)(2)(4)}-\frac{x^6}{(-2)(2)^2(4)(6)}+....\right\}+bx^{-2}\left\{1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot4}\left(\frac{1}{4}\right)\right.$$

$$\left.+\frac{x^6}{2^3\cdot4\cdot6}\left(\frac{1}{2}-1+\frac{1}{4}-\frac{1}{6}\right)+....\right\}=b\left[u\log x+x^{-2}\left\{1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot4^2}-....\right\}\right]=bv,\text{ by (10)}$$

The required solution is $y=au+bv$, where a and b are arbitrary constants.

Ex. 3. *Solve Bessel's equation of first order,* $x^2y''+xy'+(x^2-1)y=0$ in series near $x=0$.

[Delhi Maths (H) 2007; Ravishankar 2005; Meerut 1994; Agra 2006]

Sol. Given $$x^2y''+xy'+(x^2-1)y=0.\quad ...(1)$$

As in Ex. 2, show that $x=0$ is a regular singular point of (1). Let the series solution of (1) be

of the form $$y=x^k(C_0+C_1x+C_2x^2+....)=\sum_{m=0}^{\infty}C_mx^{k+m},\ C_0\neq0.\quad ...(2)$$

$$\therefore\quad y'=\sum_{m=0}^{\infty}(k+m)C_mx^{k+m-1}\qquad\text{and}\qquad y''=\sum_{m=0}^{\infty}(k+m)(k+m-1)C_mx^{k+m-2}.\quad ...(3)$$

Putting the above values of y, y' and y'' in (1), we have

$$x^2 \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} + x\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} + (x^2-1)\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)C_m x^{k+m} + \sum_{m=0}^{\infty}C_m x^{k+m+2} - \sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}[(k+m)(k+m-1)+(k+m)-1]C_m x^{k+m} + \sum_{m=0}^{\infty}C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty}[(k+m)^2-1]C_m x^{k+m} + \sum_{m=0}^{\infty}C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty}(k+m-1)(k+m+1)C_m x^{k+m} + \sum_{m=0}^{\infty}C_m x^{k+m+2} = 0. \quad ...(4)$$

(4) is an identity. Equating to zero the coefficient of the smallest power of x, namely x^k, we obtain the indicial equation. $C_0(k-1)(k+1) = 0$ so that $k = 1$ and $k = -1$ as $C_0 \neq 0$. These are unequal and differ by an integer. Here the difference in powers of x in (4) is 2. Hence we equate to zero the coefficient of x^{k+1} in identity (4) and obtain.

$k(k+2)C_1 = 0$ giving $C_1 = 0$ for both the roots $k = 1$ and $k = -1$.

Next, equating to zero the coefficient of x^{k+m} in (4), $(k+m-1)(k+m+1)C_m + C_{m-2} = 0$

or $$C_m = -\frac{1}{(k+m+1)(k+m+1)}C_{m-2}. \quad ...(5)$$

Putting $m = 3, 5, 7, ...$ in (5) and noting that $C_1 = 0$, $C_1 = C_3 = C_5 = C_7 = ... = 0.$ ...(6)

Next, putting $m = 2, 4, 6, ...$ in (5), we get $$C_2 = -\frac{1}{(k+1)(k+3)}C_0, \quad ...(7)$$

$$C_4 = -\frac{1}{(k+3)(k+5)}C_2 = \frac{1}{(k+1)(k+3)^2(k+5)}C_0, \text{ by (7)}$$

and so on. Putting these values in (2), we have

$$y = C_0 x^k\left\{1 - \frac{x^2}{(k+1)(x+3)} + \frac{x^4}{(k+1)(k+3)^2(k+5)} + ...\right\} \quad ...(8)$$

Now if we take $k = -1$ in the above series, the coefficients become infinite because of the factor $(k+1)$ in the denominator. To get rid of this difficulty, we put $C_0 = d_0(k+1)$ in (8) and get the modified solution as

$$y = d_0 x^k\left\{(k+1) - \frac{x^2}{k+3} + \frac{x^4}{(k+3)^2(x+5)} + ...\right\}. \quad ...(9)$$

Differentiating (9) partially w.r.t. 'k', we have

$$\frac{\partial y}{\partial k} = d_0 x^k \log x\left\{(k+1) - \frac{x^2}{k+3} + \frac{x^4}{(k+3)^2(x+5)} + ...\right\} + d_0 x^k\left[1 + \frac{x^2}{(k+3)^2} - \left\{\frac{2}{(k+3)^3(k+5)} + \frac{1}{(k+3)(k+5)^2}\right\}x^4 + ...\right] ...(10)$$

Putting $k = -1$ and replacing do by a, (9) gives

$$y = ax^{-1}\left\{0 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + ...\right\} = au \text{ , say.} \quad ...(11)$$

Putting $k = -1$ and replacing *do* by b, (10) gives

$$\left(\frac{\partial y}{\partial k}\right)_{k=-1} = bx^{-1}\log x\left\{0-\frac{x^2}{2}+\frac{x^4}{2^2\cdot 4}+\frac{x^6}{2^2\cdot 4^2\cdot 6}+....\right\} + bx^{-1}\left\{1+\frac{x^2}{2^2}-\frac{1}{2^2\cdot 4}\left(\frac{2}{2}+\frac{1}{4}\right)x^4+...\right\}$$

or $$\left(\frac{\partial y}{\partial k}\right)_{k=-1} = bu\log x + bx^{-1}\left\{1+\frac{x^2}{2^2}-\frac{1}{2^2\cdot 4}\left(\frac{2}{2}+\frac{1}{4}\right)x^4+....\right\} = bv\text{, say. [using (11)]} \quad ...(12)$$

Putting $k = 1$ in (9), we get

$$y = do\,x\left[2-\frac{x^2}{4}+\frac{x^4}{4^2\cdot 6}+...\right] \quad \text{or} \quad y = -2^2\,do\,x^{-1}\left[-\frac{x^2}{2}+\frac{x^4}{2^2\cdot 4}-\frac{x^6}{2^2\cdot 4^2\cdot 6}+...\right]. \quad ...(13)$$

From (11) and (13), we find that these are linearly dependent solutions. Thus out of three solutions (11), (12) and (13) only two are linearly independent. From (11) and (12) the required general solution is $y = au + bv$, a and b being arbitrary constants.

Ex. 4. *Find the series solution near* $x = 0$ *of* $(x + x^2 + x^3)y'' + 3x^2y' - 2y = 0$.

Sol. Given $$(x + x^2 + x^3)y'' + 3x^2y' - 2y = 0. \quad ...(1)$$

Dividing by $(x + x^2 + x^3)$, (1) can be put in standard form

$$y'' + \{3x/(1 + x + x^2)\}y' - \{2/(x + x^2 + x^3)\}y = 0.$$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 3x/(1 + x + x^2)$ and $Q(x) = -2/(x + x^2 + x^3)$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, hence $x = 0$ is a regular singular point of (1). Let series solution of (1) be of the form

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad \text{where} \quad C_0 \neq 0. \quad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2}. \quad ...(3)$$

Putting the above values of y, y' and y'' in (1), we have

$$(x + x^2 + x^3)\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} + 3x^2\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} - 2\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m}$$

$$+ \sum_{m=0}^{\infty}(k+m)(k+m+1)C_m x^{k+m+1} + \sum_{m=0}^{\infty}3(k+m)C_m x^{k+m+1} - \sum_{m=0}^{\infty}2C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}[(k+m)(k+m-1)-2]C_m x^{k+m} + \sum_{m=0}^{\infty}[(k+m)(k+m+1)+3(k+m)]C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}[(k+m)^2-(k+m)-2]C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)(k+m+2)C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m-2)(k+m+1)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)(k+m+2)C_m x^{k+m+1} = 0. \quad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, in identity (4), the indicial equation is given by $C_0k(k - 1) = 0$ giving $k = 0$ and $k = 1$ as $C_0 \neq 0$. These roots are unequal and differ by an integer. Next, equating to zero the coefficient of x^k in (4), we get

$(k+1)kC_1 + (k-2)(k+1)C_0 = 0$ so that $C_1 = -[(k-2)/k]C_0$. ...(5)

Finally, equating to zero the coefficient of x^{k+m-1} in (4), we get

$(k+m)(k+m-1)C_m + (k+m-3)(k+m)C_{m-1} + (k+m-2)(k+m)C_{m-2} = 0$, for all $m \geq 2$

or $(k+m-1)C_m + (k+m-3)C_{m-1} + (k+m-2)C_{m-2} = 0$ for all $m \geq 2$.

[Note that $(k+m) \neq 0$ for $k = 0$ and 1 and for all $m \geq 2$]

or $$C_m = -\frac{1}{k+m-1}[(k+m-3)C_{m-1} + (k+m-2)C_{m-2}], \quad \text{for all} \quad m \geq 2 \qquad ...(6)$$

Putting $m = 2$ in (6), we have

$$C_2 = -\frac{1}{k+1}[(k-1)C_1 + kC_0] = -\frac{1}{k+1}\left[-\frac{(k-1)(k-2)}{k}C_0 + kC_0\right], \text{ by (5)}$$

or $$C_2 = -\frac{C_0}{k(k+1)}[k^2 - (k-1)(k-2)] = -\frac{3k-2}{k(k+1)}C_0. \qquad ...(7)$$

Putting $m = 3$ in (6), we have

$$C_3 = -\frac{1}{k+2}[kC_2 + (k+1)C_1] = -\frac{1}{k+2}\left[-\frac{k(3k-2)C_0}{k(k+1)} - \frac{(k+1)(k-1)}{k}C_0\right]$$

or $$C_3 = \frac{C_0}{k+2}\left[\frac{3k-2}{k+1} + \frac{k^2-1}{k}\right] = \frac{k^3+3k^2-5k-2}{k(k+1)(k+2)}$$

and so on putting the above values in (2), we have

$$y = C_0 x^k\left[1 - \frac{k-2}{k}x - \frac{3k-2}{k(k+1)}x^2 + \frac{k^3+3k^2-5k-2}{k(k+1)(k+2)}x^3 + ..\right] \qquad ...(8)$$

Now if we put $k = 0$ in the above series, the coefficients become infinite because of the factor k in the denominator. To get rid of this difficulty, we put $C_0 = d_0 k$ in (8) and get the modified solution as

$$y = d_0 x^k\left[k - (k-2)x - \frac{3k-2}{k+1}x^2 + \frac{k^3+3k^2-5k-2}{(k+1)(k+2)}x^3 + ...\right] \qquad ...(9)$$

Differentiating (9) partially w.r.t. 'k', we get

$$\frac{\partial y}{\partial k} = d_0 x^k \log x\left[k - (k-2)x - \frac{3k-2}{k+1}x^2 + \frac{k^3+3k^2-5k-2}{(k+1)(k+2)}x^3 + ...\right] + d_0 x^k[1 - x$$

$$-\left\{\frac{3}{k+1} - \frac{(3k-2)}{(k+1)^2}\right\}x^2 + \left\{\frac{3k^2+6k-5}{(k+1)(k+2)} - \frac{k^3+3k^2-5k-2}{(k+1)^2(k+2)} - \frac{k^3+3k^2-5k-2}{(k+1)(k+2)^2}\right\}x^3 + ...\Bigg]$$

...(10)

Putting $k = 0$ and replacing *do* by *a* in (9), we get

$$y = a(2x + 2x^2 - x^3 + ...) = au, \text{ say.} \qquad ...(11)$$

Putting $k = 0$ and replacing *do* by *b* in (10), we get

$(\partial y/\partial k)_{k=0} = b \log x\,(2x + 2x^2 - x^3 + ...) + b(1 - x - 5x^2 - x^3 + ...)$

or $(\partial y/\partial k)_{k=0} = b[\log x\,(2x + 2x^2 - x^3 + ...) + (1 - x - 5x^2 - x^3 + ...)] = bv$, say. ...(12)

Putting $k = 1$ in (9), we get

$$y = d_0 x[1 + x - x^2/2 - x^3/3 + ...] = (d_0/2) \times [2x + 2x^2 - x^3 + ...] \qquad ...(13)$$

From (11) and (13), we find that these are linearly dependent solutions. Thus out of three solutions (11), (12) and (13) only two are linearly independent. From (11) and (12), the required general solution is $y = au + bv$, a and b being arbitrary constants.

Ex. 5. *Solve* (*a*) $(x^2 - x)y'' + 3y' - 2y = 0$ near $x = 0$.

(*b*) $(x^2 - x)y'' + 3y' - 2y = x + (3/x^2)$ near $x = 0$.

Sol. (*a*) Given $$(x^2 - x)y'' + 3y' - 2y = 0 \quad \ldots (1)$$

Re-writing (1), $$y'' + \{3/(x^2 - x)\}y' - \{2/(x^2 - x)\}y = 0$$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, $P(x) = 3/(x^2 - x)$ and $Q(x) = -\{2/(x^2 - x)\}$, showing that $P(x)$ and $Q(x)$ are undefined at $x = 0$ and so $P(x)$ and $Q(x)$ are not analytic at $x = 0$. Thus, $x = 0$ is not an ordinary point and therefore $x = 0$ is a singular point.

Also, $(x - 0)\,P(x) = 3/(x - 1)$ and $(x - 0)^2 Q(x) = -\{2x / (x - 1)\}$, showing that both $(x - 0)\,P(x)$ and $(x - 0)^2\,Q(x)$ are both analytic at $x = 0$. so $x = 0$ is a regular singular point.

To solve (1), we assume a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0 \quad \ldots (2)$$

$$(2) \Rightarrow \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \Sigma C_m (k+m)\,(k+m-1) x^{k+m-2}$$

Substituting the above values of y, y' and y'' in (1), we get

$$(x^2 - x) \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m-1} + 3 \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} - 2 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m-1} + 3 \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} - 2 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)\,(k+m-1) - 2\} x^{k+m} - \sum_{m=0}^{\infty} C_m \{(k+m)\,(k+m-1) - 3(k+m)\} x^{k+m-1} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m-2)\,(k+m+1) x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-4) x^{k+m-1} = 0$$

or $$-k(k-4) C_0 x^{k-1} + \{(k-2)\,(k+1) C_0 - (k-3)\,(k+1) C_1\} x^k + \{(k-1)\,(k+2)\,C_1 - (k-2)\,(k+2) C_2\}\, x^{k+1} + \ldots.$$

$$\ldots\ldots + \{(k+m-3)\,(k+m)\,C_{m-1} - (k+m-4)\,(k+m)\,C_m x^{k+m-1} + \ldots = 0 \quad \ldots (3)$$

(3) is identity in x. Equating to zero the coefficient of the lowest power of x, namely x^{k+1}, the indicial equation is $-k\,(k-4)\,C_0 = 0$ giving $k = 0, 4$ as $C_0 \neq 0$

Equating to zero the coefficient of x^{k+m-1}, the recurrence relation is given by

$$C_m = \frac{k+m-3}{k+m-4} C_{m+1}, \quad \text{for all } m \geq 1 \quad \ldots (4)$$

Putting m = 1, 2, 3, ... in (4), we have

$$C_1 = \frac{k-2}{k-3} C_0, \qquad C_2 = \frac{k-1}{k-2} \quad C_1 = \frac{k-1}{k-2} \times \frac{k-2}{k-3} \quad C_0 = \frac{k-1}{k-3} C_0,$$

$$C_3 = \frac{k}{k-1}C_2 = \frac{k}{k-1}\times\frac{k-1}{k-3}\ C_0 = \frac{k}{k-3}C_0, \qquad C_4 = \frac{k+1}{k}C_3 = \frac{k+1}{k}\times\frac{k}{k-3}\ C_0 = \frac{k+1}{k-3}C_0$$

and so on. Substituting these values of C_1, C_2, C_3, ... in (2), we get

$$y = C_0x^k\left\{1+\frac{k-2}{k-3}x+\frac{k-1}{k-3}x^2+\frac{k}{k-3}x^3+\frac{k+1}{k-3}x^4+...\right\}, \qquad ... (5)$$

The roots $k = 0, 4$ of the indicial equation differ by an integer. However, when $k = 0$ the expected vanishing of the denominator in the coefficent of x^4 does not occur since the factor k appears in both numerator and denominator (refer calculation of C_4) and thus cancels out. Observe that the coefficient C_3 of x^3 is zero when $k = 0$.

Putting $k = 0$ and replacing C_0 by a, (5) gives

$$y = a\ (1 + 2x/3 + x^2/3 - x^4/3 - 2x^5/3 - x^6\ ...) = au, \text{ say,}$$

Putting $k = 4$ and replacing C_0 by b, (5) gives

$$y = bx^4\ (1 + 2x + 3x^2 + 4x^3 + 5x^4 + ...) = bv, \text{ say}$$

Clearly, $$u = (1 + 2x/3 + x^2/3) - v/3 \qquad ... (6)$$

The required general solution is $y = au + bv$, that is,

$$y = a(1+2x/3+x^2/3)+(b-a/3)v, \text{ using (6)}$$

or $y = A\ (x^2 + 2x + 3) + Bx^4\ (1 + 2x + 3x^2 + 4x^3 + ...)$, taking $A = a/3$ and $B = b - a/3$

or $y = A(x^2 + 2x + 3) + Bx^4\ (1 - x)^{-2}$, A and B being arbitrary constants. ... (7)

(*b*) Given $$(x^2-x)y''+3y'-2y = x+3/x^2 \qquad ... (i)$$

Here $x = 0$ is rgular singular point of (i). To solve (*i*), we assume a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0 \qquad ... (ii)$$

$$\therefore \qquad y' = \sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} \qquad y'' = \sum_{m=0}^{\infty} C_m(k+m)\ (k+m-1)x^{k+m-2}$$

Substituting the above values of y, y' and y'' in (1) and proceeding as in part (*a*) gives

$$-k(k-4)C_0x^{k-1}+\{(k-2)\ (k+1)C_0-(k-3)(k+1)C_1\}x^k+\{(k-1)\ (k+2)C_1-(k-2)\ (k+2)C_2\}x^{k+1}$$

$$+...+\{(k+m-3)\ (k+m)\ C_{m-1}-(k+m-4)\ (k+m)\ C_m\}x^{k+m-1}+...=x+3/x^2 \ \ ... (iii)$$

***To determine complementary function of* (*i*) :** Setting the left member of (*iii*) equal to zero,

$$-k(k-4)\ C_0x^{k-1}+\{(k-2)\ (k+1)\ C_0-(k-3)\ (k+1)\ C_1\}\ x^k+...$$

$$+\{(k+m-3)\ (k+m)\ C_{m-1}-(k+m-4)\ (k+m)\ C_m\}\ x^{k+m-1}+...=0 \qquad ... (iv)$$

Starting with (*iv*), we now proceed as in part (*a*) upto equation (6) of part (*a*). Thus, as in part (*a*) we obtain $$C_m = \{(k+m-3)/(k+m-4)\}\ C_{m-1} \text{ for all } m \geq 1 \qquad ... (v)$$

and $$y = C_0x^k\left(1-\frac{k-2}{k-3}x+\frac{k-1}{k-3}x^2+\frac{k}{k-3}x^3+\frac{k+1}{k-3}x^4+...\right) \qquad ... (vi)$$

which satisfies $$(x^2-x)y''+3y'-2y = -k(k-4)C_0x^{k-1} \qquad ... (vii)$$

The required complementary function (C.F.) of (*i*)

$= au + bv = a\ (1 + 2x/3 + x^3/3) + (b - a/3)v$

$= A(x^2 + 2x + 3) + Bx^4\ (1 + 2x + 3x^2 + 4x^3 + ...)$, setting $A = a/3$ and $B = b - a/3$

$= A\ (x^2 + 2x + 3) + Bx^4\ (1 - x)^{-2}$

To determine particular integral (P.I.). of (*i*) : To this end we shall consider each of the terms of the right member of the given equation (*i*).

Setting the right member of (*vii*) equal to x, that is, $-k(k-4)\,C_0x^{k-1} = x$, identically,

we have $k = 2$ and $C_0 = 1/4$. Then recurrence relation (*v*) takes the form

$$C_m = \{(m-1)/(m-2)\}\,C_{m-1}, \quad \text{for all} \quad m \geq 1 \qquad \ldots (viii)$$

Putting $m = 1, 2, 3, ..$ in (*viii*) yields $C_1 = C_2 = C_3 = ... = 0$.

Putting $k = 2$, $C_0 = 1/4$ and using the fact that $C_m = 0$ for all $m \geq 1$, from (*ii*), we have P.I. corresponding to the term x is $x^2/4$.

Again, setting the right member of (*vii*) equal to $3/x^2$, *i.e.*, $-k(k-4)C_0x^{k-1} = 3/x^2$, identically, we have $k = -1$ and $C_0 = -3/5$. Then recurrence relation (*v*) takes the form

$$C_m = \{(m-4)/(m-5)\}\,C_{m-1}, \quad \text{for all} \quad m \geq 1 \qquad \ldots (ix)$$

Putting $m = 1, 2, 3, \ldots$ in (*ix*), we have $C_1 = 3C_0/4, C_2 = C_0/2, C_3 = C_0/4, C_4 = C_5 = C_8 = \ldots = 0$.

Putting these values in (*ii*) and taking $k = -1$, $C_0 = -3/5$, we have

$\therefore$ Particular integral corresponding to the term $3/x^2$ is $-(3/5) \times x^{-1}\,(1 + 3x/4 + x^2/2 + x^3/4)$

Hence the complete integral of (1) is y = C.F. + total P.I.

or $y = A(x^2 + 2x + 3) + Bx^4\,(1-x)^{-2} + x^2/4 - 3/5x - 9/20 - 3x/10 - 3x^2/20$

or $y = A(x^2 + 2x + 3) + Bx^4\,(1-x)^{-2} + x^2/4 - 3/5x - (3/20) \times (x^2 + 2x + 3)$

or $y = C(x^2 + 2x + 3) + Bx^4\,(1-x)^{-2} - (3/5x) + (x^2/4)$,

where $C = (A - 3/20)$. Here C and B are arbitrary constants.

EXERCISE 8 (E)

Find the series solution of the following equations near $x = 0$.

1. $x(1-x)y'' - (1 + 3x)y' - y = 0$. **Ans.** $y = a(1\cdot 2x^2 + 2\cdot 3x^3 + 3\cdot 4x^4 + \ldots)$

$$+b\{\log x(1\cdot 2x^2 + 2\cdot 3x^3 + 3\cdot 4x^4 + ...) + (-1 + x + 3x^2 + 5x^3 + \ldots)\}$$

2. $x^2y'' - 3xy' + (3 + 4x)y = 0$. **[Meerut 1997]**

3. $x^2y'' + xy' - (1 + x^2)y = 0$.

8.12. Examples of Type 4 on Frobenius method. Roots of indicial equation equal.

In this connection the following rule should be noted carefully.

Rule. *If the indicial equation has two equal roots $k = k'$, we obtain two independent solutions by substituting this value of k in y and $\partial y/\partial k$.*

Ex. 1. *Solve the Bessel's equation of order zero $xy'' + y' + xy = 0$ in series.*

[Delhi Maths (H) 2009; Meerut 1996, G.N.D.U. Amritsar 1997, Garhwal 2005]

Sol. Given $xy'' + y' + xy = 0$. ...(1)

Dividing by x, $y'' + (1/x)y' + y = 0$. Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 1/x$ and $Q(x) = 1$, so that $xP(x) = 1$and $x^2Q(x) = x^2$, which are analytic at $x = 0$. So $x = 0$ is a regular singular point. Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad \text{where} \quad C_0 \neq 0. \qquad \ldots(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}. \qquad \ldots(3)$$

Putting the above values of y, y', y'' into (1) gives

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-1} + \sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) + (k+m)\}x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)^2 x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m+1} = 0, \quad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (4) gives the indicial equation $C_0 k^2 = 0$ or $k^2 = 0$ $(\because\ C_0 \neq 0)$. So $k = 0, 0$ (equal roots.)

For recurrence relation, equating to zero the coefficient of x^{k+m-1} in (4) gives

$$C_m(k+m)^2 + C_{m-2} = 0 \quad \text{or} \quad C_m = -\frac{C_{m-2}}{(k+m)^2}. \quad ...(5)$$

To compute C_1, equating to zero the coefficient of x^k gives $C_1(k+1)^2 = 0$ so that $C_1 = 0$ (for $k = 0$ is root of indicial equation). Using $C_1 = 0$ and (5), we get

$$C_1 = C_3 = C_5 = C_7 = ... = 0. \quad ...(6)$$

Putting $m = 2, 4, 6, ...$ in turn in (5) and simplifying, we get

$C_2 = -C_0/(k+2)^2$, $C_4 = -C_2/(k+4)^2 = C_0/(k+2)^2(k+4)^2$,

$C_6 = -C_4/(k+6)^2 = -C_0/(k+2)^2(k+4)^2(k+6)^2$,

and so on. Putting these values in (2), *i.e.* $y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + ...)$,

$$y = C_0 x^k \left\{1 - \frac{x^2}{(x+2)^2} + \frac{x^4}{(k+2)^2(k+4)^2} - \frac{x^6}{(k+2)^2(k+4)^2(k+6)^2} + \ldots\right\} \quad ...(7)$$

Putting $k = 0$ and replacing C_0 by a in (7) gives

$$y = a\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^4 \cdot 6^2} + ...\right) = au \text{ say} \quad ...(8)$$

To get another independent solution, substituting, (7) into the L.H.S. of (1) and simplifying, we find $x^2y'' + y' + xy = C_0 k^2 x^{k-1}$. ...(9)

Differentiating both sides of (9) partially w.r.t. 'k' gives

$$\frac{\partial}{\partial k}\left[x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy\right] = \frac{C_0}{x}\frac{\partial}{\partial k}(k^2x^k) \quad \text{or} \quad \left[x^2\frac{d^2}{dx^2} + \frac{d}{dx} + x\right]\frac{\partial y}{\partial k} = \frac{C_0}{x}[2kx^k + k^2 \cdot x^k \log x]. \quad ...(10)$$

The presence of the factor k in each term on R.H.S. of (10) shows that a second solution is $(\partial y/\partial k)_{k=0}$, where y is given by (7). Differentiating partially w.r.t. 'k' (7) gives

$$\partial y / \partial k = C_0 x^k \log x \{1$$

$$-\frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2(k+4)^2} - \frac{x^6}{(k+2)^2(k+4)^2(k+6)^2} + ...\Big\} + c_0 x^k \left\{-\frac{x^2}{(k+2)^2} \times \frac{(-2)}{k+2}\right.$$

$$\left. + \frac{x^4}{(k+2)^2(k+4)^2}\left[-\frac{2}{k+2} - \frac{2}{k+4}\right] - \frac{x^6}{(k+2)^2(k+4)^2(k+6)^2}\left[-\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6}\right] + ...\right\} ...(11)$$

[To find $\frac{d}{dx}\left[\frac{1}{(k+2)^2(k+4)^2(k+6)^2}\right]$, we have proceeded as follows. Suppose that

$z = 1/[(k+2)^2 (k+4)^2 (k+6)^2]$ so that $\log z = -2\log(k+2) - 2\log(k+4) - 2\log(k+6)$.

Differentiating w.r.t. 'k', we have

$$\frac{1}{z}\frac{dz}{dk} = -\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6} \quad \text{or} \quad \frac{dz}{dk} = z\left(-\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6}\right),$$

where $z = 1/\{(k+2)^2 (k+4)^2 (k+6)^2\}$; etc. Other terms can be similarly differentiated easily]

Putting $k = 0$ and replacing c_0 by b in (11) gives

$$(\partial y/\partial k)_{k=0} = b\log x\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \ldots.\right) + b\left\{\frac{x^2}{2^2} - \frac{x^2}{2^2\cdot 4^2}\left(1+\frac{1}{2}\right) + \frac{x^2}{2^2\cdot 4^2\cdot 6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \ldots.\right\}$$

$$= b\left[u\log x + \left\{\frac{x^2}{2^2} - \frac{x^2}{2^2\cdot 4^2}\left(1+\frac{1}{2}\right) + \ldots.\right\}\right] = bv \text{ , (say)} \qquad \text{[using (8)]}$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Ex. 2. *Solve in series* $(x - x^2)y'' + (1 - 5x)y' - 4y = 0$.

[Agra 2009; Delhi Maths (H) 2005; Nagpur 1996, Meerut 1995, 96]

Sol. Given $(x - x^2)y'' + (1 - 5x)y' - 4y = 0.$ ...(1)

Dividing by $(x - x^2)$, (1) gives $y'' + \{(1-5x)/(x-x^2)\}y' - \{4/(x-x^2)\}y = 0.$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = (1 - 5x)/(x - x^2)$ and $Q(x) = -4/(x - x^2)$ so that $xP(x) = (1 - 5x)/(1 - x)$ and $x^2Q(x) = -4x/(1 - x)$. Since $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a regular singular point of (1).

Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0. \qquad \ldots(2)$$

$$\therefore\ y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}. \quad \ldots(3)$$

Putting the above values of y, y' y'' into (1) gives

$$(x - x^2)\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2} + (1-5x)\sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} - 4\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) + (k+m)\}x^{k+m-1} - \sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) + 5(k+m) + 4\}x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} C_m \{(k+m)^2 + 4(k+m) + 4\}x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} C_m (k+m+2)^2 x^{k+m} = 0, \qquad \ldots(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (4) gives the indicial equation

$C_0 k^2 = 0$ or $k^2 = 0$, as $C_0 \neq 0$. Thus $k = 0, 0$ (equal roots).

Finally, equating to zero the coefficient of x^{k+m-1} in (4), we get

$$C_m (k+m)^2 - C_{m-1}(k+m+1)^2 = 0 \quad \text{or} \quad C_m = \frac{(k+m+1)^2}{(k+m)^2} C_{m-1} \qquad \ldots(5)$$

Putting $m = 1, 2, 3, \ldots.$ in turn in (5) and simplifying $C_1 = [(k+2)^2/(k+1)^2]C_0$,

$$C_2 = \frac{(k+3)^2}{(k+2)^2} C_1 = \frac{(k+3)^2}{(k+2)^2} \times \frac{(k+2)^2}{(k+1)^2} C_0 = \frac{(k+3)^2}{(k+1)^2} C_0, \quad C_3 = \frac{(k+4)^2}{(k+3)^2} C_2 = \frac{(k+4)^2}{(k+3)^2} \times \frac{(k+3)^2}{(k+1)^2} C_0 = \frac{(k+4)^2}{(k+1)^2} C_0$$

and so on. Putting these in (2), *i.e.*, $y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + ...)$, gives

$$y = x^k C_0 \left\{1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \frac{(k+4)^2}{(k+1)^2} x^3 + ... \right\}. \quad ...(6)$$

Putting $k = 0$ and replacing C_0 by a in (6) gives

$$y = a(1 + 2^2x + 3^2x^2 + 4^2x^3 + ...) = au, \text{ say} \quad ...(7)$$

Differentiating partially w.r.t. 'k', (6) gives

$$\frac{\partial y}{\partial k} = C_0 x^k \log x \left\{1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \frac{(k+4)^2}{(k+1)^2} x^3 + ... \right\}$$

$$+ C_0 x^k \left\{ \frac{(k+2)^2}{(k+1)^2} x \left[\frac{2}{k+2} - \frac{2}{k+1} \right] + \frac{(k+3)^2}{(k+1)^2} x^2 \left[\frac{2}{k+3} - \frac{2}{k+1} \right] + ... \right\}$$

Putting $k = 0$ and replacing C_0 by b gives

$$(\partial y/\partial k)_{k=0} = b \log x(1 + 2^2x + 3^2x^2 + 4^2x^3 + ...) + b\{2^2 x(1-2) + 3^2 \cdot x^2 \cdot 2(2/3 - 2) + ...\}$$

$$= b[u \log x - 2(1 \cdot 2x + 2 \cdot 3x^2 + ...)] = bv, \text{ say} \quad \text{[using (7)]}$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Ex. 3. *Find the series solution of* $(x - x^2)y'' + (1 - x)y' - y = 0$ near $x = 0$.

[Delhi Maths (Hons.) 1998, 2008; Ravishenker 2010]

Sol. Given $$(x - x^2)y'' + (1 - x)y' - y = 0. \quad ...(1)$$

As usual, $x = 0$ is a regular singular point of (1). Let its series solution be

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \quad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} (k+m) C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m-2}. \quad ...(3)$$

Putting the above values of y, y' and y'' in (1), we get

$$(x - x^2) \sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m-2} + (1 - x) \sum_{m=0}^{\infty} (k+m) C_m x^{k+m-1} - \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m-1} - \sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m}$$

$$+ \sum_{m=0}^{\infty} (k+m) C_m x^{k+m-1} - \sum_{m=0}^{\infty} (k+m) C_m x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} [(k+m)(k+m-1) + (k+m)] C_m x^{k+m-1} - \sum_{m=0}^{\infty} [(k+m)(k+m-1) + (k+m) + 1] C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} (k+m)^2 C_m x^{k+m-1} - \sum_{m=0}^{\infty} [(k+m)^2 + 1] C_m x^{k+m} = 0, \quad ...(4)$$

which is an identity is x. Equating to zero the coefficient of the smallest power of x, namely x^{k-1} in (4), the indicial equation is

$C_0k^2 = 0$ so that $k = 0, 0$ as $C_0 \neq 0$.

Equating to zero the coefficient of x^{k+m-1} in (4), we get

$$(k+m)^2C_m - \{(k+m-1)^2+1\}C_{m-1} = 0, \quad \text{for all} \quad m \geq 1$$

or $$C_m = \frac{(k+m-1)^2+1}{(k+m)^2}C_{m-1}, \qquad \text{for all} \qquad m \geq 1 \qquad ...(5)$$

Putting $m = 1, 2, 3, ...$ in (5), we have $C_1 = \{(k^2+1)/(k+1)^2\}C_0$, ...(6)

$$C_2 = \frac{(k+1)^2+1}{(k+2)^2}C_1 = \frac{(k^2+1)\{(k+1)^2+1\}}{(k+1)^2(k+2)^2}C_0, \text{ by (6)} \qquad ...(7)$$

$$C_3 = \frac{(k+2)^2+1}{(k+3)^2}C_2 = \frac{(k^2+1)\{(k+1)^2+1\}\{(k+2)^2+1\}}{(k+1)^2(k+2)^2(k+3)^2}C_0$$

and so on. Putting these values in (2), we have

$$y = C_0x^k\left[1 + \frac{k^2+1}{(k+1)^2}x + \frac{(k^2+1)\{(k+1)^2+1\}}{(k+1)^2(k+2)^2}x^2 + \frac{(k^2+1)\{(k+1)^2+1\}\{(k+2)^2+1\}}{(k+1)^2(k+2)^2(k+3)^2}x^3 + ...\right] \qquad ...(8)$$

Differentiating (8) partially w.r.t. 'k', we have

$$\frac{\partial y}{\partial k} = C_0x^k \log x\left\{1 + \frac{k^2+1}{(k+1)^2}x + \frac{(k^2+1)[(k+1)^2+1]}{(k+1)^2(k+2)^2}x^2 + \frac{(k^2+1)\{(k+1)^2+1\}\{(k+2)^2+1\}}{(k+1)^2(k+2)^2(k+3)^2}x^3 + ...\right\}$$

$$+ C_0x^k\left\{\left[\frac{2k}{(k+1)^2} - \frac{2(k^2+1)}{(k+1)^3}\right]x + \left[\frac{2k\{(k+1)^2+1\} + 2(k+1)(k^2+1)}{(k+1)^2(k+2)^2}\right.\right.$$

$$\left.\left. - \frac{2(k^2+1)\{(k+1)^2+1\}}{(k+1)^3(k+2)^2} - \frac{2(k^2+1)\{(k+1)^2+1\}}{(k+1)^2(k+2)^3}\right]x^2 + ...\right\} \qquad ...(9)$$

Putting $k = 0$ and replacing C_0 by a in (8), $y = a\left(1 + x + \frac{2}{4}x^2 + \frac{2\cdot5}{4\cdot9}x^3 + ...\right) = au$, say.

Putting $k = 0$ and replacing C_0 by b in (9), we get

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b \log x\left(1 + x + \frac{2}{4}x^2 + \frac{2\cdot5}{4\cdot9}x^3 + ...\right) + b(-2x - x^2 - ....)$$

or $$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b\left\{\log x\left(1 + x + \frac{2}{4}x^2 + \frac{2\cdot5}{4\cdot9}x^3 + ...\right) + (-2x - x^2 - ....)\right\} = bv, \text{ say}$$

The required solution is $y = au + bv$, a, b being arbitrary constants.

Ex. 4. *Solve in series the following differential equations* :

(*a*) $x(x-1)y'' + (3x-1)y' + y = 0$. **[Delhi Physics (H) 2001]**

(*b*) $x(x-1)y'' + (3x-1)y' + y = 4x$ **[Jabalpur 2004]**

Sol. (*a*) Given $$(x^2 - x)y'' + (3x-1)y' + y = 0 \qquad ... (1)$$

As usual show that $x = 0$ is regular singular point. To solve (1), we assume a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0 \qquad ... (2)$$

$$\therefore\ y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m-2}$$

Substituting the above values of y, y' and y'' in (1), we get

$$(x^2 - x) \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m-2} + (3x-1) \sum_{m=0}^{\infty} C_m (k-m) x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \left\{(k+m)\,(k+m-1) + 3(k+m) + 1\right\} x^{k+m} - \sum_{m=0}^{\infty} C_m \left\{(k+m)\,(k+m-1) + (k+m)\right\} x^{k+m-1} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m+1)^2 x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m)^2 x^{k+m-1} = 0$$

or $$-C_0 k^2 x^{k-1} + (C_0 - C_1)(k+1)^2 x^k + \ldots\ldots + (C_{m-1} - C_m)(k+m)^2 x^{k+m-1} + \ldots = 0 \qquad \ldots (3)$$

(3) is identity in x. Equating to zero the coefficient of the lowest power of x, namely, x^{k-1}, the identical equation is $-C_0 k^2 = 0$ giving $k = 0, 0$ (equal roots), as $C_0 \neq 0$

Equating to zero the coefficient of x^{k+m-1}, the recurrence relation is given by

$$C_{m-1}(k+m)^2 - C_m (k+m)^2 = 0, \qquad \text{giving} \qquad C_m = C_{m-1} \text{ for all } m \geq 1 \qquad \ldots (4)$$

Putting $m = 1, 2, 3, \ldots$ in (4), we have

$C_1 = C_0$, $C_2 = C_1 = C_0$, $C_3 = C_2 = C_0$ and so on. Substituting these values in (2), we have

$y = C_0 x^k + C_1 x^{k+1} + C_2 x^{k+2} + \ldots$ or $y = C_0 x^k (1 + x + x^2 + \ldots)$

or $$y = C_0 x^k (1-x)^{-1}. \qquad \ldots (5)$$

Putting $k = 0$ and replacing C_0 by a in (5), we get $y = a(1-x)^{-1}$ ... (6)

Since the indicial equation has equal roots, so the second independent solution of (1) is given by $(\partial y / \partial k)_{k=0}$ (refer Art. 8.12).

Differentiating (5) partially w.r.t. 'k', yields $\partial y / \partial k = x^k C_0 (1-x)^{-1} \log x$ ... (7)

Putting $k = 0$ and replacing C_0 by b in (7), we get $(\partial y / \partial k)_{k=0} = b(1-x)^{-1} \log x$ ... (8)

From (6) and (8), the required solution of (1) is

$$y = a(1-x)^{-1} + b(1-x)^{-1} \log x = (a + b \log x)(1-x)^{-1}, \qquad \ldots (9)$$

where a and b are arbitrary constants.

(*b*) Given $(x^2 - x)y'' + (3x-1)y' + y = 4x$ ... (*i*)

Here $x = 0$ is regular singular point of (*i*). To solve (1), assume a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0 \qquad \ldots (ii)$$

$$\therefore\ y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m-1}$$

Substituting the above values of y, y' and y'' in (*i*) and proceeding exactly as in part (*a*), we get

$$-C_0 k^2 x^{k-1} + (C_0 - C_1)(k+1)^2 x^k + \ldots + (C_{m-1} - C_m)(k+m)^2 x^{k+m-1} + \ldots = 4x \qquad \ldots (iii)$$

To determine complementary function of (*i*) Setting the left number of (*iii*) equal to zero,

$$-C_0 k^2 x^{k-1} + (C_0 - C_1)(k+1)^2 x^k + \ldots + (C_{m-1} - C_m)(k+m)^2 x^{k+m-1} + \ldots = 0 \qquad \ldots (iv)$$

Starting with (*iv*), we now proceed as in part (*a*) upto equation (8) of part (*a*). Thus, we get

$$C_m = C_{m-1}, \qquad \text{for all} \qquad m \geq 1 \qquad \ldots (v)$$

and $$y = C_0 x^k (1-x)^{-1} \qquad \ldots (vi)$$

which satisfies $$(x^2 - x)y'' + (3x - 1)\, y' + y = -C_0 k^2 x^{k-1} \qquad \ldots (vii)$$

The required complementary function (C.F.) is given by

$$\text{C.F. } (i) = a(1-x)^{-1} + b(1-x)^{-1} \log x = (a + b \log x)\,(1-x)^{-1}$$

To determine particular integral of (*i*) To this end we shall consider the term of the right member of the given equation (*i*). Thus, setting the right member of (*vii*) equal to $4x$, that is,

$-C_0 k^2 x^{k-1} = 4x$, identically, we have $k = 2$ and $C_0 = -1$.

The recurrence relation (*v*) yields $C_1 = C_2 = C_3 = ... = C_0$, as before

Putting $k = 2$, $C_0 = -1$ and $C_1 = C_2 = C_3 = ... = C_0$ in (*ii*), we have

$$\text{P. I. of } (i) = \sum_{m=0}^{\infty} C_m x^{2+m} = x^2 (C_0 + C_1 x + C_2 x^2 + \ldots) = x^2\, C_0\, (1 + x + x^2 + \ldots) = -x^2\,(1-x)^{-1}$$

Hence the required complete integral of (*i*) is $y = \text{C.F.} + \text{P.I.}$, *i.e.*,

$y = (a + b \log x)\,(1-x)^{-1} - x^2\,(1-x)^{-1}$, a, b being arbitrary constants.

EXERCISE 8 (F)

Find the series solutions of the following equations near $x = 0$.

1. $xy'' + (1 + x)y' + 2y = 0$. **[Garhwal 2005]**

Ans. $y = a(1 - 2x + (3/2!)x^2 - (4/3!)x^3 + ...) + b[\log x\{1 - 2x + (3/2!) \times x^2 - (4/3!) \times x^3 + ...\}$
$+ 2(2 - 1/2)x - (3/2!) \times (-1/3 + 2 + 1/2)x^2 + ...\}$

2. $x^2y'' - x(1 + x)y' + y = 0$. **[Delhi Maths (Hons.) 1998]**

Ans. $y = ax[1 + x + (1/2) \times x^2 + (1/2 \cdot 3) \times x^3 + ...] + b[\log x\ \{1 + x + (1/2) \times x^2$
$+ (1/2 \cdot 3) \times x^3 + ...\} + x^2\ \{-1 - (3/4) \times x + ...\}]$

3. $4(x^4 - x^2)y'' + x^3y' - y = 0$.

Ans. $$y = ax^{1/2}\left(1 + \frac{1 \cdot 3}{4^2}x^2 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4^2 \cdot 8^2}x^4 + \ldots\right) + b\left[x^{1/2} \log x\left(1 + \frac{1 \cdot 3}{4^2}x^2 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4^2 \cdot 8^2}x^4 + \ldots\right)\right.$$
$$\left. + 2x^{1/2}\left\{\frac{1 \cdot 3}{4^2}\left(1 + \frac{1}{3} - \frac{1}{2}\right)x^2 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4^2 \cdot 8^2}\left(1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right)x^4 + ...\right\}\right]$$

4. $xy'' + (p - x)y' - y = 0$ when (*i*) $p = 1$ (*ii*) p is not an integer.

Ans. (*i*) $$y = a\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right) + b\left[\log x\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\right.$$
$$\left. - \left\{x + \frac{x^2}{1 \cdot 2}\left(1 + \frac{1}{2}\right) + \frac{x^3}{1 \cdot 2 \cdot 3}\left(1 + \frac{1}{2} + \frac{2}{3}\right) + ...\right\}\right]$$

(*ii*) $$y = a\left(1 + \frac{x}{p} + \frac{x^2}{p(p+1)} + ...\right) + b\left\{\log x\left(1 + \frac{x}{p} + \frac{x^2}{p(p+1)} + ...\right) + x^{1-p}\left(1 + x/1! + x^2/2! + ...\right)\right.$$

5. $xy'' + y' + x^2y = 0$. **[Delhi Maths (Hons.) 2001]** **Ans.** $y = au + bv$, where

$$u = 1 - \frac{x^3}{3^2} + \frac{x^6}{3^4(2!)^2} - \frac{x^9}{3^6(3!)^2} + ... \text{ and } v = u \log x + 2\left[\frac{x^3}{3^3} - \frac{1}{3^5(2!)^2}\left(1 + \frac{1}{2}\right)x^6 + ...\right]$$

6. $xy'' + y' - y = 0$. **[G.N.D.U. Amritsar 2010]**

7. $xy'' + y' + y = 0$ **[Meerut 1992]**

8. $x(1 - x^2)y'' + (1 - 3x^2)y' - xy = 0$. **[Meerut 1994]**

8.13. Series solution about regular singular point at infinity.

Suppose the given equation is $y'' + P(x)\, y' + Q(x)\, y = 0$. ...(1)

We propose to find the solution of (1) for large values of the independent variable, *i.e.,* about $x = \infty$. For this purpose we change the independent variable from x to t with help of the following transformation : $x = 1/t$ *i.e.* $t = 1/x$. ...(2)

Clearly large values of x correspond to small values of t. Using (2), we re–write (1) and obtain the transformed equation near $t = 0$, say

$$d^2y/dt^2 + P_1(t)\,(dy/dt) + Q_1(t)\, y = 0. \quad ...(3)$$

Then the given equation (1) is said to have regular singular point at $x = \infty$ if the transformed equation (3) has regular singular at $t = 0$.

Working rule for solving (1) near $x = \infty$. Assume transformation (2). With help of (2), we obtain the transformed equation (3). We verify that $t = 0$ is regular singular point of (3). Solve (3) by Frobenius method (refer Art. 8.8). Finally putting $t = 1/x$ in the solution, we obtain the required general solution of (1) near $x = \infty$.

8.14. Solved examples based on Art. 8.13

Ex. 1. *Show that $x = \infty$ is a regular singular point of* $x^2y'' + 4xy' + 2y = 0$.

Sol. Given $x^2y'' + 4xy' + 2y = 0$. ...(1)

Let $x = 1/t$ or $t = 1/x$ so that $dt/dx = -1/x^2$. ...(2)

Now, $$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}, \text{ by (2)} \quad ...(3)$$

and $$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\times\left(-\frac{1}{x^2}\right), \text{ by (2) and (3)}$$

or $$y'' = \left(-t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt}\right)(-t^2) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}. \quad ...(4)$$

Using (2), (3) and (4), (1) transforms to $$\frac{1}{t^2}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \frac{4}{t}\left(-t^2\frac{dy}{dt}\right) + 2y = 0$$

or $$(d^2y/dt^2) - (2/t)(dy/dt) + (2/t^2)y = 0. \quad ...(5)$$

Comparing (5) with $(d^2y/dt^2) + P(t)(dy/dt) + Q(t)y = 0$, here $P(t) = -2/t$ and $Q(t) = 2/t^2$ so that $tP(t) = -2$ and $t^2Q(t) = 2$. Since $tP(t)$ and $t^2Q(t)$ are the both analytic at $t = 0$, so $t = 0$ is a regular singular point of (5). In view of (2), $x = \infty$ is a regular singular point of (1).

Ex. 2. *Find the power series solution of* $(1 - x^2)(d^2y/dt^2) - 2x(dy/dx) + 6y = 0$ about $x = \infty$. **[Delhi Maths (Hons.) 1997]**

Sol. Given $(1 - x^2)(d^2y/dx^2) - 2x(dy/dx) + 6y = 0$. ...(1)

Let $x = 1/t$ or $t = 1/x$ so that $dt/dx = -1/x^2$. ...(2)

Now, $$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}, \text{ by (2).} \quad ...(3)$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\times\left(-\frac{1}{x^2}\right), \text{ by (2) and (3).}$$

or $$\frac{d^2y}{dx^2} = t^2\frac{d}{dt}\left(t^2\frac{dy}{dt}\right) = t^2\left(2t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right). \qquad ...(4)$$

Using (2), (3) and (4), (1) is transformed to

$$\left(1 - \frac{1}{t^2}\right)\left(2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}\right) - \frac{2}{t}\left(-t^2\frac{dy}{dt}\right) + 6y = 0$$

or $$t^2(t^2 - 1)(d^2y/dt^2) + 2t^3(dy/dt) + 6y = 0. \qquad ...(5)$$

Dividing by $t^2(t^2 - 1)$, $d^2y/dt^2 + \{2t/(t^2 - 1)\}(dy/dt) + [6/\{t^2(t^2 - 1)\}]y = 0$.

Comparing it with $(d^2y/dt^2) + P(t)\,(dy/dt) + Q(t)y = 0$, here $P(t) = 2t/(t^2 - 1)$ and $Q(t) = 6/[t^2(t^2 - 1)]$ so that $tP(t) = 2t^2/(t^2 - 1)$ and $t^2Q(t) = 6/(t^2 - 1)$. Since $tP(t)$ and $t^2Q(t)$ are both analytic at $t = 0$, so $t = 0$ is a regular singular point of (5). To solve (5), let its series solution be

$$y = t^k(C_0 + C_1t + C_2t^2 + ...) = \sum_{m=0}^{\infty} C_m t^{k+m}, \quad \text{where} \quad C_0 \neq 0. \qquad ...(6)$$

$$\therefore \quad dy/dt = \sum_{m=0}^{\infty}(k+m)C_m t^{k+m-1} \quad \text{and} \quad d^2y/dt^2 = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2}.$$

Putting these values of y, dy/dt and d^2y/dt^2 in (5), we get

$$(t^4 - t^2)\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2} + 2t^3\sum_{m=0}^{\infty}(k+m)C_m t^{k+m-1} + 6\sum_{m=0}^{\infty}C_m t^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2} - \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m} + \sum_{m=0}^{\infty}2(k+m)C_m t^{k+m+2} + 6\sum_{m=0}^{\infty}C_m t^{k+m} = 0$$

or $$-\sum_{m=0}^{\infty}\{(k+m)(k+m-1) - 6\}C_m t^{k+m} + \sum_{m=0}^{\infty}\{(k+m)(k+m-1) + 2(k+m)\}C_m t^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty}\{(k+m)^2 - (k+m) - 6\}C_m t^{k+m} - \sum_{m=0}^{\infty}\{(k+m)^2 + (k+m)\}C_m t^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty}(k+m-3)(k+m+2)C_m t^{k+m} - \sum_{m=0}^{\infty}(k+m)(k+m+1)C_m t^{k+m+2} = 0, \qquad ...(7)$$

which is an identity. Equating to zero the coefficient of the smallest power of t, namely t^k, (7) gives the indicial equation $C_0(k - 3)(k + 2) = 0$ so that $k = 3$ and $k = -2$ as $C_0 \neq 0$. The roots of indicial equation are unequal and differ by an integer.

Next equating to zero the coefficient of t^{k+1} in (7), we get $(k - 2)(k + 3)C_1 = 0$ giving $C_1 = 0$ for both $k = 3$ and $k = -2$. Finally, equating to zero the coefficient of t^{k+m} in (4), we get

$$(k + m - 3)(k + m + 2)C_m - (k + m - 2)(k + m - 1)C_{m-2} = 0$$

or $$C_m = \frac{(k+m-2)(k+m-1)}{(k+m-3)(k+m+2)}C_{m-2} \quad \text{for all } m \geq 2. \qquad ...(8)$$

Putting $m = 3, 5, 6, ...$ in (8) and noting that $C_1 = 0$, we get

$$C_1 = C_3 = C_5 = C_7 = ... = 0. \qquad ...(9)$$

Next, putting $m = 2, 4, 6, ...$ in (8), we have $$C_2 = \frac{k(k+1)}{(k-1)(k+4)}C_0,$$

$$C_4 = \frac{(k+2)(k+3)}{(k+1)(k+6)}C_2 = \frac{k(k+1)(k+2)(k+3)}{(k-1)(k+1)(k+4)(k+6)}C_0, \text{ using (10)}$$

or $$C_4 = \frac{k(k+2)(k+3)}{(k-1)(k+4)(k+6)}C_0, \qquad \text{since } (k+1) \neq 0.$$

and so on. Putting these values in (6), we have

$$y = t^k C_0\left[1 + \frac{k(k+1)}{(k-1)(k+4)}x^2 + \frac{k(k+2)(k+3)}{(k-1)(k+4)(k+6)}x^4 + \ldots.\right]. \qquad ...(11)$$

Putting $k = 3$ in (11) and replacing C_0 by a, we get

$$y = at^3\left[1 + \frac{3\cdot 4}{2\cdot 7}t^2 + \frac{3\cdot 5\cdot 6}{2\cdot 7\cdot 9}t^4 + \ldots.\right] = au \text{ , say} \qquad ...(12)$$

Putting $k = -2$ in (11) and replacing C_0 by b, we get $y = bt^{-2}[1 - t^2/3] = bv$, say. ...(13)

The required solution is $y = au + bv$, *i.e.*

$$y = at^3\left(1 + \frac{3\cdot 4}{2\cdot 7}t^2 + \frac{3\cdot 5\cdot 6}{2\cdot 7\cdot 9}t^4 + \ldots.\right) + \frac{b}{t^2}\left(1 - \frac{t^2}{3}\right)$$

i.e., $$y = \frac{a}{x^3}\left(1 + \frac{3\cdot 4}{2\cdot 7}\frac{1}{x^2} + \frac{3\cdot 5\cdot 6}{2\cdot 7\cdot 9}\frac{1}{x^4} + \ldots.\right) + bx^2\left(1 - \frac{1}{3x^2}\right), \text{ as, } t = \frac{1}{x}.$$

Ex. 3. *Find the power series solution of* $(1 - x^2)(d^2y/dx^2) - 2x(dy/dx) + 12y = 0$ at $x = \infty$. **[Delhi Maths (Hons.) 1998]**

Hint. Proceed just like Ex. 2. Its general solution is

$$y = ax^3(1 - 3/5x^2) + (b/x^4)\times(1 + 10/9x^2 + 35/33x^4 + \ldots).$$

Ex. 4. *Solve in series the equation* $x^3y'' + x(1 - x)y' + y = 0$ near $x = \infty$.

Sol. Given $$x^3(d^2y/dx^2) + x(1 - x)(dy/dx) + y = 0. \qquad ...(1)$$

Let $x = 1/t$ or $t = 1/x$ so that $dt/dx = -1/x^2$. ...(2)

Now, $$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}, \text{ by (2).} \qquad ...(3)$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\times\left(-\frac{1}{x^2}\right), \text{ by (2) and (3)}$$

or $$\frac{d^2y}{dx^2} = t^2\frac{d}{dt}\left(t^2\frac{dy}{dt}\right) = t^2\left(2t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right). \qquad ...(4)$$

Using (2), (3) and (4), (1) is transformed to

$$\frac{1}{t^3}\left(2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}\right) + \frac{1}{t}\left(1 - \frac{1}{t}\right)\left(-t^2\frac{dy}{dt}\right) + y = 0$$

$t(d^2y/dt^2) + 2(dy/dt) - (t - 1)(dy/dt) + y = 0$ or $t(d^2y/dt^2) + (3 - t)(dy/dt) + y = 0$. ...(5)

We now solve (5) in series about $t = 0$ which is a regular singular point of (5). (Prove yourself as in solved Ex. 2). Let a series solution of (5) be

$$y = t^k(C_0 + C_1t + C_2t^2 + \ldots.) = \sum_{m=0}^{\infty} C_m t^{k+m}, \text{ where } C_0 \neq 0. \qquad ...(6)$$

$$\therefore \quad dy/dt = \sum_{m=0}^{\infty}(k+m)C_m t^{k+m-1} \quad \text{and} \quad d^2y/dt^2 = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2}.$$

Putting these values of y, dy/dt and d^2y/dt^2 in (5), we get

$$t\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2}+(3-t)\sum_{m=0}^{\infty}(k+m)C_m t^{k+m-1}+\sum_{m=0}^{\infty}C_m t^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-1}+\sum_{m=0}^{\infty}3(k+m)C_m t^{k+m-1}-\sum_{m=0}^{\infty}(k+m)C_m t^{k+m}+\sum_{m=0}^{\infty}C_m t^{k+m}=0$$

or $$\sum_{m=0}^{\infty}\{(k+m)(k+m-1)+3(k+m)\}C_m t^{k+m-1}-\sum_{m=0}^{\infty}\{(k+m)-1\}C_m t^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m+2)C_m t^{k+m-1}-\sum_{m=0}^{\infty}(k+m-1)C_m t^{k+m}=0, \qquad ...(7)$$

which is an identity. Equating to zero the coefficients of the smallest power of t, namely t^{k-1}, (7) gives the indicial equation $C_0k(k+2)=0$ so that $k=0,-2$ as $C_0\neq 2$.

Equating to zero the coefficients of t^{k+m-1} in (7), we get

$$(k+m)(k+m+2)C_m-(k+m-2)C_{m-1}=0$$

so that $$C_m=\frac{k+m-2}{(k+m)(k+m+2)}C_{m-1}, \quad \text{for all} \quad m\geq 1. \qquad ...(8)$$

Putting $m = 1, 2, 3, 4, ...$ in (8) we get $$C_1=\frac{k-1}{(k+1)(k+3)}C_0, \qquad ...(9)$$

$$C_2=\frac{k}{(k+2)(k+4)}C_1=\frac{k(k-1)}{(k+1)(k+2)(k+3)(k+4)}C_0, \text{ using (9)} \qquad ...(10)$$

$$C_3=\frac{k+1}{(k+3)(k+5)}C_2=\frac{k(k-1)}{(k+2)(k+3)^2(k+4)(k+5)}C_0, \text{ using (10)} \qquad ...(11)$$

$$C_4=\frac{k+2}{(k+4)(k+6)}C_3=\frac{k(k-1)}{(k+3)^2(k+4)^2(k+5)(k+6)}C_0, \text{ using (11)}$$

and so on. Putting these values in (6), we have

$$y=C_0t^k\left[1+\frac{k-1}{(k+1)(k+3)}t+\frac{k(k-1)}{(k+1)(k+2)(k+3)(k+4)}t^2\right.$$
$$\left.+\frac{k(k-1)}{(k+2)(k+3)^2(k+4)(k+5)}t^3+\frac{k(k-1)}{(k+3)^2(k+4)^2(k+5)(k+6)}t^4+\ldots\right] \qquad ...(12)$$

If we put $k = -2$ in (12), then the coefficients of t^2 and t^3 become infinite. To get rid of this difficulty, we put $C_0 = d_0(k + 2)$ in (12) and obtain the modified solution

$$y=d_0t^k\left[(k+2)+\frac{(k-1)(k+2)}{(k+1)(k+3)}t+\frac{k(k-1)}{(k+1)(k+3)(k+4)}t^2\right.$$
$$\left.+\frac{k(k-1)}{(k+3)^2(k+4)(k+5)}t^3+\frac{k(k-1)(k+2)}{(k+3)^2(k+4)^2(k+5)(k+6)}t^4+\ldots.\right]. \qquad ...(13)$$

Differentiating (13) partially w.r.t. 'k', we get

$$\frac{\partial y}{\partial k}=d_0t^k\log t\left[(k+2)+\frac{(k-1)(k+2)}{(k+1)(k+3)}t+\frac{k(k-1)}{(k+1)(k+3)(k+4)}t^2+\frac{k(k-1)}{(k+3)^2(k+4)(k+5)}t^3\right.$$

$$+\frac{k(k-1)(k+2)}{(k+3)^2(k+4)^2(k+5)(k+6)}t^4+....\Big]+d_0t^k\Big[1+\Big\{\frac{2k+1}{(k+1)(k+3)}-\frac{(k-1)(k+2)}{(k+1)^2(k+3)}-\frac{(k-1)(k+2)}{(k+1)(k+3)^2}\Big\}t$$

$$+\Big\{\frac{2k-1}{(k+1)(k+3)(k+4)}-\frac{k(k-1)}{(k+1)^2(k+3)(k+4)}-\frac{k(k-1)}{(k+1)(k+3)^2(k+4)}-\frac{k(k-1)}{(k+1)(k+3)(k+4)^2}\Big\}t^2$$

$$+\Big\{\frac{2k-1}{(k+3)^2(k+4)(k+5)}-\frac{2k(k-1)}{(k+3)^3(k+4)(k+5)}-\frac{k(k-1)}{(k+3)^2(k+4)^2(k+5)}-\frac{k(k-1)}{(k+3)^2(k+4)(k+5)^2}\Big\}t^3+....\Big] \quad ...(14)$$

Putting $k = -2$ in (13) and replacing do by a, $y = at^{-2}(-3t^2 + t^3) = au$, say ...(15)

Putting $k = -2$ in (14) and replacing *do* by b, we get

$$(\partial y/\partial t)_{t=-2} = bt^{-2}(-3t^2 + t^3)\log t + bt^{-2}[1 + 3t + 4t^2 - (11/3)t^3 + ...]$$

or $$(\partial y/\partial t)_{t=-2} = bt^{-2}[(-3t^2 + t^3)\log t + 1 + 3t + 4t^2 - (11/3)t^3 + ...) = bv, \text{ say.} \quad ...(16)$$

Finally putting $k = 0$ in (13), $y = d_0[2 - (2/3)t] = (-2/3)d_0\, t^{-2}(-3t^2 + t^3)$. ...(17)

From (15) and (17), we see that these solutions are not independent. So out of three solutions (15), (16) and (17) only two are linearly independent, namely (15) and (16). So the required solution is

$y = au + bv$, *i.e.* $y = at^{-2}(-3t^2 + t^3) + bt^{-2}[(-3t^2 + t^3)\log t + 1 + 3t + 4t^2 - (11/3)t^3 + ...]$

or $$y = ax^2\left(-\frac{3}{x}+\frac{1}{x^3}\right)+bx^2\left[\left(-\frac{3}{x}+\frac{1}{x^3}\right)\log\frac{1}{x}+1+\frac{3}{x}+\frac{4}{x^2}-\frac{11}{3x^3}+...\right], \quad \text{as} \quad t=\frac{1}{x}$$

Ex. 5. *Find the solution in series of the equation* $(1 - x^2)(d^2y/dx^2) + 2x(dy/dx) - y = 0$ about $x = \infty$. **[Delhi Maths (Hons.) 1995]**

8.15. Series solution in descending powers of independent variable.

So far we obtained series solutions in ascending powers of the independent variable. However, the following cases may arise :

(*i*) There exists no solution of the form $\sum_{m=0}^{\infty} C_m x^{k+m}$

(*ii*) The usual Frobenius method may break down.

(*iii*) The series solution obtained by earlier methods does not converge within a particular range of values of independent variable.

In such cases we obtain series solution in descending powers of independent variable. Sometimes the series solutions in descending powers are desirable and are more useful in practice.

Working Rule. For details see Art. 8.8. However, the following changes should be noted carefully.

(*i*) We assume a trial solution of the form $y = x^k(c_0 + c_1x^{-1} + c_2x^{-2} + ...) = \sum_{m=0}^{\infty} c_m x^{k-m}$, $c_0 \neq 0$

(*ii*) For indicial equation the coefficient of the highest power of x in the identity is equated to zero.

(*iii*) For recurrence relation the coefficient of the higher power, in the identity is equated to zero.

Another Working Rule. In order to get a series solution in descending powers of x of the differential equation $f(x)(d^2y/dx^2) + g(x)(dy/dx) + h(x)y = 0$, ...(1)

we change the variable from x to t by putting $x = 1/t$ so that

$$dy/dx = (dy/dt)(dt/dx) = (dy/dt)(-1/x^2) = -t^2\,(dy/dt) \quad ...(2)$$

From (2), $$d/dx \equiv -t^2(d/dt). \quad ...(3)$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -t^2\frac{d}{dt}\left(-t^2\frac{dy}{dt}\right), \text{ using (2) and (3)}$$

or $$\frac{d^2y}{dx^2} = -t^2\left[-2t\frac{dy}{dt} - t^2\frac{d^2y}{dt^2}\right] = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}. \quad ...(4)$$

Using $x = 1/t$, (2) and (4), (1) transforms in the form

$$F(t)(d^2y/dt^2) + G(t)(dy/dt) + H(t)y = 0. \quad ...(5)$$

We now solve (5) by assuming series solution in the form $y = \sum_{m=0}^{\infty} c_m t^{k+m}$

Proceed as explained in Art. 8.8. After getting solution replace t by $1/x$ to get the desired series in descending powers of x.

8.16. Solved examples based on Art. 8.15

Ex. 1. *Integrate in descending series the Legendre's equation.*

Sol. The differential equation of the form $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ ...(1) is called Legendre's equation, where n is a positive integer.

Let the series solution of (1) be of the form $y = \sum_{m=0}^{\infty} c_m x^{k-m}$, where $c_0 \neq 0$. ...(2)

$$\therefore \quad y' = \sum_{m=0}^{\infty} c_m(k-m)x^{k-m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2}. \quad ...(3)$$

Putting the above values of y, y', y'' in (1) gives

$$(1-x^2)\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - 2x\sum_{m=0}^{\infty} c_m(k-m)x^{k-m-1} + n(n+1)\sum_{m=0}^{\infty} c_m x^{k-m} = 0$$

or $$\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty}\{(k-m)(k-m-1) + 2(k-m) - n(n+1)\}x^{k-m} = 0$$

or $$\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m\{(k-m)^2 - n^2 + (k-m) - n\}x^{k-m} = 0$$

or $$\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m} - \sum_{m=0}^{\infty} c_m(k-m-n)(k-m+n+1)x^{k-m} = 0 \quad ...(4)$$

which is an identity. Equating to zero the coefficient of the highest power of x, namely x^k, (4) gives the indicial equation $c_0(k - n)(k + n + 1) = 0$ or $(k - n)(k + n + 1) = 0$ $[\because c_0 \neq 0]$ giving $k = n, -(n + 1)$. For recurrence relation, we equate to zero the coefficient of x^{k-m} and obtain

$$c_{m-2}[k - (m - 1)][k - (m - 2) - 1] - c_m(k - m - n)(k - m + n + 1) = 0$$

or $$c_{m-2}(k - m + 2)(k - m + 1) + c_m(n - k + m)(n + k - m + 1) = 0$$

or $$c_m = -\frac{(k-m+2)(k-m+1)}{(n-k+m)(n+k-m+1)}c_{m-2}. \quad ...(5)$$

Equating to zero the coefficient of x^{k-1} in (4) gives $c_1(k - 1 - n)(k + n) = 0$ so that $c_1 = 0$ for both $k = n$ and $k = -(n + 1)$. Using $c_1 = 0$ and (5), we note that

$$c_1 = c_3 = c_5 = c_7 = ... = 0 \quad ...(6)$$

Putting $m = 2$ in (5) gives $$c_2 = -\frac{k(k-1)}{(n-k+2)(n+k-1)}c_0 \quad ...(7)$$

Putting $m = 4$ in (5) and using (7) gives

$$c_4 = -\frac{(k-2)(k-3)}{(n-k+4)(n+k-3)}c_2 = \frac{k(k-1)(k-2)(k-3)}{(n-k+2)(n+k-1)(n-k+4)(n+k-3)}c_0$$

and so on. Putting these values in (2), *i.e.* $y = c_0x^k + c_1x^{k-1} + c_2x^{k-2} + c_3x^{k-3} + c_4x^{k-4} + ...$, gives

$$y = c_0\left[x^k - \frac{k(k-1)x^{k-2}}{(n-k+2)(n+k-1)} + \frac{k(k-1)(k-2)(k-3)x^{k-4}}{(n-k+2)(n+k-1)(n-k+4)(n+k-3)} + \ldots\right]$$

Putting $k = n$ and $-(n+1)$ in turn and replacing c_0 by a and b respectively, the above equation gives

$$y = a\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right] = au, \text{ say} \quad ...(8)$$

$$y = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4(2n+3)(2n+5)}x^{-n-5} + \ldots\right] = bv, \text{ say} \quad ...(9)$$

The required solution is $y = au + bv$, a, b being arbitrary constants.

Ex. 2. *Find a series solution in descending powers of x of* $4x^3y'' + 6x^2y' + y = 0$.

Sol. Let t be another independent variable such that $x = 1/t$. Then we have (See Art. 8.15)

$$\frac{4}{t^3}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \frac{6}{t^2}\left(-t^2\frac{dy}{dt}\right) + y = 0 \quad \text{or} \quad 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

Let solution of (2) be the form $y = \sum_{m=0}^{\infty} c_m t^{k+m}$, where $c_0 \neq 0$.

Now, proceed as explained in Art. 8.8 and obtain the required solution

$y = a(1 - t/2! + t^2/4! + ...) + bt^{1/2}(1 - t/3! + t^2/5! - ....)$, where $t = 1/x$.

8.17. Method of differentiation. In this connection, remember the following two results.

(*i*) **Maclaurin's theorem.** $y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots$

(*ii*) **Leibnitz theorem.** If u and v be two functions, then

$D^n(uv) = u_n\ v + {}^nc_1\ u_{n-1}v_1 + {}^nc_2\ u_{n-2}\ v_2 + ... + u\ v_n$, where $u_n = d^nu/dx^n$ and $v_n = d^nv/dx^n$.

Ex. 1. *Solve* $(1 - x^2)y_2 - xy_1 + m^2y = 0$ *where* $x = 0$, $y = 0$, $dy/dx = m$. **[Meerut 1995]**

Sol. Given that $(1 - x^2)y_2 - xy_1 + m^2y = 0$. ...(1)

Differentiating (1) n times by Leibnitz's theorem w.r.t. 'x', we have

$$y_{n+2}(1 - x^2) + {}^nC_1\ y_{n+1}(-2x) + {}^nC_2\ y_n(-2) - [y_{n+1}x + {}^nC_1\ y_n.1] + m^2y_n = 0$$

or $$(1 - x^2)y_{n+2} - 2nx\,y_{n+1} - \frac{n(n-1)}{2}y_n \times 2 - xy_{n+1} - ny_n + m^2y_n = 0$$

or $$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0. \quad ...(2)$$

Given that $(y)_0 = 0$ and $(y_1)_0 = m$. ...(3)

Putting $x = 0$ in (1), $(y_2)_0 = -m^2(y)_0 = 0$, by (3). ...(4)

Putting $x = 0$ in (2), $(y_{n+2})_0 = (n^2 - m^2)(y_n)_0$. ...(5)

Putting $n = 2, 4, 6, ...$ in (5) and noting that $(y_2)_0 = 0$, we get

$$(y_2)_0 = (y_4)_0 = (y_6)_0 = ... = 0. \quad ...(6)$$

Putting $n = 1, 3, 5, ...$ in (5), we get

$$(y_3)_0 = (1^2 - m^2)(y_1)_0 = (1^2 - m^2)m, \text{ by (3)} \quad ...(7)$$

$$(y_5)_0 = (3^2 - m^2)(y_3)_0 = (3^2 - m^2)(1^2 - m^2)m, \text{ by (7)}$$

and so on. Now, from Maclaurin's theorem, $y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots.$

or $y = mx + (1^2 - m^2)m(x^3/3!) + (3^2 - m^2)(1^2 - m^2)m(x^5/5!) + \ldots.$

Ex. 2. *Solve* $(1 - x^2)y_2 - xy_1 - a^2y = 0$, where $x = 0$, $y = 1$ and $dy/dx = 0$.

Hint. Do as in Ex. 1. **Ans.** $y = 1 + ax + (a^2/2!)x^2 + a(a^2 + 1^2)(x^3/3!) + \ldots.$

Ex. 3. *Solve* $(1 + x^2)y_2 + 2xy_1 = 0$ where $x = 0$, $y_1 = 1$, $y = 0$.

Hint. Do as in Ex. 1. **Ans.** $y = x - (x^3/3) + (x^5/5) - (x^7/7) + \ldots.$

Ex. 4. *Solve* $xy_2 + y_1 + my = 0$ given that $y = 1$ when $x = 0$. **[Meerut 1994, 95]**

Sol. Put $x = 0$, $y = 1$ in given equation, we get $y_1 = 0$. Now do as it Ex. 1. **Ans.** $y = e^{-mx}$

MISCELLANEOUS PROBLEMS IN CHAPTER 8.

Ex. 1. *Find the series solutions of the following equation about* $x = 0$.

(*i*) $2x^2y'' + xy' + (x^2 - 1)y = 0$ **[MDU Rohtak 2004]**

Ans. $y = ax\,(1 - x^2/14 + x^4/616 + \ldots) + bx^{-1/2}\,(1 - x^2/2 + x^4/40 + \ldots)$

(*ii*) $x^2y'' - xy' + (x^2 + 8/9)y = 0$

Ans. $y = ax^{4/3}\,(1 - 3x^2/16 + 9x^4/896 + \ldots) + bx^{2/3}(1 - 3x^2/8 + 9x^4/320 + \ldots)$

(*iii*) $xy'' + y' - y = 0$ **[MDU Rohtak 2005]**

Ans. $y = (a + b\log x)\,(1 + x + x^2/(2!)^2 + \ldots) - 2b\{x + (x^2/(2!)^2) \times (1 + 1/2) + \ldots.\}$

(*iv*) $y'' + x^2y = 0$ **Ans.** $y = a\left(1 - \frac{x^4}{4\cdot 3} + \frac{x^8}{8\cdot 7\cdot 4\cdot 3} + \ldots\right) + b\left(x - \frac{x^5}{5\cdot 4} + \frac{x^9}{9\cdot 8\cdot 5\cdot 4}\ldots\right)$

Ex. 2. Find the power series solution of $xy'' + y' + 2y = 0$ about $x = 1$, *i.e.*, in power of $(x - 1)$. **[KU Kurukshetra 2005]**

Ans. $y = a\left\{1 - (x-1)^2 + (2/3)\times(x-1)^3 - (1/3)\times(x-1)^4 + \ldots\right\} + b\left\{(x-1) - (1/2)\times(x-1)^2 + (1/12)\times(x-1)^4 + \ldots\right\}$

Ex. 3. *Find the general solution of the differential equation* $(x-1)^2y'' + (x-1)y' - 4y = 0$ *in powers of x using the Frobenius method.* **[GATE 2002]**

Ex. 4. *Find the general solution of* $\left(1 + x^2\right)y'' + 2xy' - 2y = 0$ in terms of power series in x.

[Nagpur 2005]

Ex. 5. *Show that the differential equation* $x^2y'' - 3xy' + (4x + 4)y = 0$ *has only one Frobenius series solution and find it.* **[Nagpur 2005]**

OBJECTIVE PROBLEMS ON CHAPTER 8.

Write (a), (b), (c) or (d), whichever is correct.

1. *Singular points are* (*a*) *regular* (*b*) *irregular* (*c*) regular or irregular (*d*) none of these

Sol. Ans. (*c*). Refer Art. 8.3. **[Agra 2005, 06, 08]**

2. *For the differential equatin* $(x-1)\,(d^2y/dx^2) + (\cot \pi x)\,(dy/dx) + (\text{cosec}^2 \pi x)y = 0,$ *which of the following statement is true ?* (*a*) 0 *is regular and* 1 *is irregular* (*b*) 0 *is irregular and* 1 *is regular* (*c*) *Both* 0 *and* 1 *are regular* (*d*) *Both* 0 *and* 1 *are irregular.* **[Gate 2006]**

Sol. Ans. (*a*) Re-writing the given equation, $\frac{d^2y}{dx^2}+\frac{\cot \pi x}{x-1}\frac{dy}{dx}+\frac{\text{cosec}^2 \pi x}{x-1}y=0,$ ... (1)

Comparing (1) with standard equation $y''+P(x)y'+Q(x)y=0,$ here

$P(x)=(\cot \pi x)/(x-1)$ and $Q(x)=(\text{cosec}^2 \pi x)/(x-1)$ ... (2)

Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$, so $x = 0$ is not an ordinary point. Also,

$$(x-0)P(x)=\frac{x(\cot \pi x)}{x-1}=\frac{\cos \pi x}{\pi(x-1)}\times\frac{\pi x}{\sin \pi x} \quad \Rightarrow \quad \lim_{x\to 0}(x-0)P(x)=-\frac{1}{\pi}$$

and $$(x-0)^2Q(x)=\frac{x^2\text{cosec}^2 \pi x}{x-1}=\left(\frac{\pi x}{\sin \pi x}\right)^2\times\frac{1}{\pi^2(x-1)} \Rightarrow \lim_{x\to 0}(x-0)^2Q(x)=-\frac{1}{\pi^2}$$

Thus, $(x – 0)\ P(x)$ and $(x – 0)^2\ Q(x)$ are both analytic at $x = 0$ and hence $x = 0$ is a regular point. Again, from (2), we see that both $P(x)$ and $Q(x)$ are undefined at $x = 1$, so $x = 1$ is not an ordinary point. Also, we have

$(x-1)P(x)=\cot \pi x$ and $(x-1)^2Q(x)=(x-1)\ \text{cosec}^2 \pi x$

Clearly, $(x – 1)\ P(x)$ is not analytic at $x = 1$. Hence $x = 1$ is irregular point.

3. *Let* $y_1(x)$ and $y_2(x)$ *be two linearly independent solutions of* $xy''+y'+x^2y=0,$ *in the neighbourhood of* $x = 0$. *If* $y_1(x)$ *is a power series around* $x = 0$, then (*a*) $y_2(x)$ *is bounded around* $x = 0$ (*b*) $y_2(x)$ *is unbounded around* $x = 0$. (*c*) $y_2(x)$ has a power series solution (*d*) $y_2(x)$ has solution of the form $\sum_{n=0}^{\infty} b_n x^{n+r}$ *where* $r \neq 0$ and $b_0 \neq 0$. **[GATE 2003]**

4. *For the differential equation* $x^2(1-x)y''+xy'+y=0$

(*a*) $x = 1$ is an ordinary point (*b*) $x = 1$ is a regular singular point

(*c*) $x = 0$ is an irregular singular point. (*d*) $x = 0$ is an ordinary point.

[GATE 2005]

Sol. Ans. (*b*) Proceed as in Ex. 1 of Art. 8.4.

5. *It is required to find the solution of* $2x(2+x)y''-2(3+x)y'+xy=0$ *around* $x = 0$. *The roots of the indicial equation are*

(a) 0, 1/2 *(b)* 0, 2 *(c)* 1/2, 1/2 *(d)* 0, –1/2 **[GATE 2005]**

Sol. Ans. (*b*) See working rule of Art. 8.8.

6. *If* $y=\sum_{m=0}^{\infty} a_m x^m$ *is a solution of* $y''+xy'+3y=0,$ *then* a_m/a_{m+2} *equals :*

(*a*) $\frac{(m+1)(m+2)}{m+3}$ (*b*) $-\frac{(m+1)(m+2)}{m+3}$ (*c*) $-\frac{m(m-1)}{m+3}$ (*d*) $\frac{m(m-1)}{m+3}$ **[GATE 2004]**

Sol. Ans. (*b*). $y = \sum_{m=0}^{\infty} a_m x^m \quad \Rightarrow \quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

Substituting these values in the given equation, we obtain

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{m=1}^{\infty} m a_m x^{m-1} + 3\sum_{m=0}^{\infty} a_m x^m = 0 \quad \text{or} \quad \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^m + 3\sum_{m=0}^{\infty} a_m x^m = 0 \quad \ldots (1)$$

Equating coefficient of x^m to zero in the above identity, we get

$$(m+2)\,(m+1)a_{m+2} + m a_m + 3a_m = 0 \Rightarrow \qquad \frac{a_m}{a_{m+2}} = -\frac{(m+2)\,(m+1)}{m+3}$$

7. *The indicial equation for* $x(1+x^2)y'' + (\cos x)y' + (1-3x+x^2)y = 0$ *is*

(*a*) $r^2 - r = 0$ (*b*) $r^2 + r = 0$ (*c*) $r^2 = 0$ (*d*) $r^2 - 1 = 0$ **[GATE 2004]**

Sol. Ans. (*c*). Use working rule of Art. 8.8.

8. For the equation $x(x-1)y'' + (\sin x)y' + 2x(x-1)y = 0$, consider the following statements. A: $x = 0$ is a regular singular point; B: $x = 1$ is a regular singular points. Then (a) both A and B are true (b) A is false but B is true (c) A is true but B is false (d) both A and B are false.**[Gate 2008]**

Sol. Ans. (*d*). Proceed as in solved Ex.1, page 8.4

9. In the equation $x(x-1)^3 y'' + 2(x-1)^3 y' + 3y = 0$ the singular point is

(*a*) $x = 1$ (*b*) $x = -1$ (*c*) $x = 0$ (*d*) None of these **[Agra 2009]**

Sol. (*c*) Proved as in Ex-1, page 8.2

10. Solve the differential equation $y'' + x^2 y' + 2xy = 0$ by the power series method.

[Lucknow 2010]

11. Write the different forms of the series solutions of the differential equation $P_0(x)\,y'' + P_1(x)\,y' + P_2(x)\,y = 0$ when

(*i*) $x = a$ is an ordinary point and (*ii*) when $x = a$ in a regular sigularity. **[Ranchi 2010]**

9

Legendre Polynomials

PART I : LEGENDRE FUNCTION OF THE FIRST KIND

9.1. Legendre's equation and its solution. [Bilaspur 1996; Meerut 1994]

The differential equation of the form $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ ...(1)

is called *Legendre's equation*, where n is a positive interger. We now solve (1) in series of descending powers of x. Let the series solution of (1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k-m}, \qquad \text{where} \qquad c_0 \neq 0 . \qquad ...(2)$$

Differnetiating (2) and then putting the values of y, dy/dx and d^2y/dx^2 into (1), we have

$$(1 - x^2) \sum_{m=0}^{\infty} c_m (k - m)\ (k - m - 1)\ x^{k-m-2} - 2x \sum_{m=0}^{\infty} c_m (k - m)\ x^{k-m-1} + n(n + 1) \sum_{m=0}^{\infty} c_m x^{k-m} = 0$$

or $$\sum_{m=0}^{\infty} c_m (k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m \{(k-m)(k-m-1) + 2(k-m) - n(n+1)\}x^{k-m} = 0 \quad ....(3)$$

Now, $(k - m)\ (k - m - 1) + 2\ (k - m) - n(n + 1)$

$$= (k - m)^2 - (k - m) + 2(k - m) - n(n + 1) = (k - m)^2 - n^2 + (k - m) - n$$

$$= (k - m + n)\ (k - m - n) + (k - m - n) = (k - m - n)\ (k - m + n + 1).$$

Hence (3) may be re–written as

$$\sum_{m=0}^{\infty} c_m (k - m)\ (k - m - 1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m (k - m - n)\ (k - m + n + 1)x^{k-m} = 0. \qquad ...(4)$$

(4) is an identity. To get the indicial equation, we equate to zero the coefficient of the highest power of x, namely x^k in (4) and obtain

$$c_0(k - n)\ (k + n + 1) = 0 \qquad \text{or} \qquad (k - n)\ (k + n + 1) = 0, \quad \text{as} \quad c_0 \neq 0. \qquad ...(5)$$

So the roots of (5) are $k = n$, $-(n + 1)$. They are unequal and differ by an integer. The next lower power of x is $k - 1$. So we equate to zero the coefficient of x^{k-1} in (4) and obtain

$$c_1(k - 1 - n)\ (k + n) = 0. \qquad ...(6)$$

For $k = n$ and $-(n + 1)$, neither $(k - 1 - n)$ nor $(k + n)$ is zero. So from (6), $c_1 = 0$. Finally, equating to zero the coefficient of x^{k-m} in (4), we have

$$c_{m-2}(k - m + 2)\ (k - m + 1) - c_m(k - m - n)\ (k - m + n + 1) = 0$$

$$c_m = \frac{(k - m + 2)\ (k - m + 1)}{(k - m - n)\ (k - m + n + 1)} c_{m-2} . \qquad ...(7)$$

Putting $m = 3, 5, 7, ...$ in (7) and noting that $c_1 = 0$, we have

$$c_1 = c_3 = c_5 = c_7 = ... = 0, \qquad ...(8)$$

which hold good for both $k = n$ and $k = -(n + 1)$.

To obtain c_2, c_4, c_6 .... etc., we consider two cases.

Case I. When k = n. Then, (7) becomes $c_m = -\frac{(n-m+2)(n-m+1)}{m(2n-m+1)} c_{m-2}$. ...(9)

Putting $m = 2, 4, 6$ .... in (9), we have

$$c_2 = -\frac{n(n-1)}{2(2n-1)} c_0, \qquad c_4 = -\frac{(n-2)(n-3)}{4(2n-3)}\ c_2 = \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)} c_0$$

and so on. Re–writting (2), we have for $k = n$

$$y = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + c_3 x^{n-3} + c_4 x^{n-4} + \ldots\ldots \quad ...(10)$$

Using (8) and the above values of c_2, c_4, c_6. etc., (10) becomes (after replacing c_0 by a)

$$y = a\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)} x^{n-4} - \ldots. \right]. \quad ...(11)$$

Case II. When k = –(n + 1). Then, (7) becomes $c_m = \frac{(n+m-1)(n+m)}{m(2n+m+1)} c_{m-2}$. ... (12)

Putting $m = 2, 4, 6, \ldots$ in (12), we have

$$c_2 = \frac{(n+1)(n+2)}{2(2n+3)} c_0, \qquad c_4 = \frac{(n+3)(n+4)}{4(2n+5)}\ c_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot (2n+3)(2n+5)} c_0$$

and so on. For $k = -(n + 1)$, (2) gives

$$y = c_0 x^{-n-1} + c_1 x^{-n-2} + c_2 x^{-n-3} + c_3 x^{-n-4} + c_4 x^{-n-5} + \ldots \quad ...(13)$$

Using (8) and the above values of c_2, c_4, c_6....etc., (13) becomes (after replacing c_0 by b)

$$y = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot (2n+3)(2n+5)} x^{-n-5} + \ldots \right]. \quad ...(14)$$

Thus, two independent solutions of (1) are given by (11) and (14). If we take $a = [1\cdot 3\cdot 5\ldots(2n-1)]/n!$, the solution (11) is denoted by $P_n(x)$ and is called *Legendre's function of the first kind* or *Legendre's polynomial of degree n*. Notice that (11) is a terminating series and so it gives rise to a polynomial of degree n. Thus $P_n(x)$ is a solution of (1), Again, if we take $b = n/[1\cdot 3\cdot 5\ldots(2n+1)]$ the solution (14) is denoted by $Q_n(x)$ and is called *Legendre's function of the second kind*. Since n is a positive integer, (14) is an infinite or non–terminating series and hence $Q_n(x)$ is not a polynomial. Thus $P_n(x)$ and $Q_n(x)$ are two linearly independent solutions of (1). Hence the general solution of (1) is

$$y = A\,P_n(x) + B\,Q_n(x), \text{ where } A \text{ and } B \text{ are arbitrary constants.} \quad ...(15)$$

Remark 1. When there is no confusion regarding the variable x, we shall use a shorter notation P_n for $P_n(x)$ and P_n' for $dP_n(x)/dx$, Q_n for $Q_n(x)$ and Q_n' for $dQ_n(x)/dx$ etc.

Another form of Legendre's polynomial $P_n(x)$

Legendre's polynomial of degree n is denoted and defined by

$$P_n(x) = \frac{1\cdot 3\cdot 5\ldots.(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2\cdot(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} x^{n-4} - \ldots. \right] \quad ...(1)$$

We now re–write (1) in a compact form. The general term of polynomial (1) is given by

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n}\cdot(-1)^r\ \frac{n(n-1)\ldots(n-2r+1)}{2\cdot 4\cdot 2r(2n-1)(2n-3)\ldots(2n-2r+1)} x^{n-2r}. \quad ...(2)$$

Now, $1\cdot3\cdot5...(2n-1)$,

$$= \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6...(2n-1)\,.\,2n}{2\cdot4\cdot6\,...\,2n} = \frac{(2n)!}{(2\cdot1)\,(2\cdot2)\,(2\cdot3)\,...(2\cdot n)} = \frac{(2n)!}{2^n\cdot1\cdot2\cdot3\,...n} = \frac{(2n)!}{2^n\cdot n!}. \quad ...(3)$$

Also, $n(n-1)\,...(n-2r+1)$

$$= \frac{n(n-1)\,(n-2r+1)\,(n-2r)\,(n-2r-1)\,...\,3\cdot2\cdot1}{(n-2r)\,(n-2r-1)\,...3\cdot2\cdot1} = \frac{n!}{(n-2r)\,!} \quad ...(4)$$

and $$2\cdot4\cdot6\,...\,2r = (2\cdot1)\,(2\cdot2)\,(2\cdot3)\,...\,(2\cdot r) = 2^r\cdot r!. \quad ...(5)$$

Finally, $(2n-1)\,(2n-3)\,...(2n-r+1)$

$$= \frac{(2n)\,(2n-1)\,(2n-2)\,(2n-3)\,...\,(2n-2r+2)\,(2n-2r+1)}{(2n)\,(2n-2)\,(2n-4)\,...(2n-2r+2)} \times \frac{(2n-2r)\,!}{(2n-2r)\,!}$$

$$= \frac{(2n)\,(2n-1)\,(2n-2)\,...\,(2n-2r+1)\,(2n-2r)\,(2n-2r-1)\,...\,3\cdot2\cdot1}{2\cdot n\cdot2(n-1)\cdot2(n-2)\,...2\,(n-r+1)\,.\,(2n-2r)\,!}$$

$$= \frac{(2n)\,!}{2^n\cdot n(n-1)\,(n-2)\,...(n-r+1)\,(2n-2r)\,!}$$

$$= \frac{(2n)\,!}{2^n(2n-2r)\,!} \times \frac{(n-r)\,(n-r-1)\,...\,3\cdot2\cdot1}{n(n-1)\,(n-2)\,...\,(n-r+1)\,(n-r)\,(n-r-1)\,...\,3\cdot2\cdot1}$$

$$= \frac{(2n)!}{2^n\,(2n-2r)!} \times \frac{(n-r)!}{n!}. \quad ...\,(6)$$

Using (3), (4), (5) and (6), the general term (2) becomes

$$\frac{(2n)\,!}{2^n\,n\,!}(-1)^r\,\frac{n\,!}{(n-2r)\,!} \times \frac{1}{2^r\,r\,!} \times \frac{2^n(2n-2r)\,!\,n\,!}{(2n)\,!\,(n-r)\,!}\,x^{n-2r}$$

i.e. $$(-1)^r\,\frac{(2n-2r)\,!}{2^n\,r\,!\,(n-r)\,!\,(n-2r)\,!}\,x^{n-2r}. \quad ...(7)$$

Since (1) is polynomial of degree n, r must be chosen so that $n-2r \geq 0$, *i.e.*, $r \leq n/2$.

Thus, if n is even, r goes from 0 to $\frac{1}{2}n$, while if n is odd, r goes from 0 to $\frac{1}{2}(n-1)$; that is, for the complete polynomial (1), r goes from 0 to $\left[\frac{1}{2}n\right]$, where

$$\left[\tfrac{1}{2}n\right] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases}$$

Hence the Legendre polynomial of degree n is given by

$$P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r\,\frac{(2n-2r)\,!}{2^r\,r\,!\,(n-r)\,!\,(n-2r)\,!}\,x^{n-2r}. \quad ...(8)$$

9.2. Legendre's function of the first kind or Legendre's polynomial of degree n. [Kanpur 2011, Ranchi 2010]

Definition. Legendre's polynomial of degree n is denoted and defined by

$$P_n(x) = \frac{1\cdot3\cdot5\,...(2n-1)}{n\,!}\left[x^n - \frac{n(n-1)}{2(2n-1)}\,x^{n-2} + \frac{n(n-1)\,(n-2)\,(n-3)}{2\cdot4\cdot(2n-1)\,(2n-3)}\,x^{n-4} - ...\right] \quad ...(1)$$

or $$P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}, \qquad ...(2)$$

where $$[n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases} \qquad ...(3)$$

Legendre's function of the second kind. Definition. This is denoted and defined by $Q_n(x) =$

$$\frac{n!}{1\cdot 3\cdot 5 \ldots (2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot (2n+3)(2n+5)} x^{-(n+5)} + \ldots \right]. \qquad ...(4)$$

Determination of first few Legendre's polynomials.

Putting $n = 0, 1, 2, 3, 4, 5, ...$ in result (1), we have

$$P_0(x) = \frac{1}{0!}x^0 = 1, \qquad P_1(x) = \frac{1}{1!}x^1 = x, \qquad P_2(x) = \frac{1\cdot 3}{2!}\left[x^2 - \frac{2\cdot 1}{2\cdot 3}x^0 \right] = \frac{1}{2}(3x^2 - 1),$$

[Bhopal 2010]

$$P_3(x) = \frac{1\cdot 3\cdot 5}{3!}\left[x^3 - \frac{3\cdot 2}{2\cdot 5}x^1 \right] = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1\cdot 3\cdot 5\cdot 7}{4!}\left[x^4 - \frac{4\cdot 3}{2\cdot 7}x^2 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 7\cdot 5}x^0 \right] = \frac{1}{8}(35x^4 - 30x^2 + 3),$$

$$P_5(x) = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{5!}\left[x^5 - \frac{5\cdot 4}{2\cdot 9}x^3 + \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 9\cdot 7}x^1 \right] = \frac{1}{8}(63x^5 - 70x^3 + 15x) \qquad \textbf{[Kanpur 2004]}$$

Ex. 1. *Express* $2 - 3x + 4x^2$ *in terms of Legendre polynomials.* **[Bangalore 1995]**

Sol. We have $1 = P_0(x)$, $x = P_1(x)$, $(3x^2 - 1)/2 = P_2(x) \Rightarrow x^2 = [2P_2(x) + 1]/3$ ...(1)

Now, $2 - 3x + 4x^2 = 2P_0(x) - 3P_1(x) + (4/3) \times [2P_2(x) + 1]$, by (1)

$= 2P_0(x) - 3P_1(x) + (8/3) \times P_2(x) + (4/3) \times P_0(x) = (10/3) \times P_0(x) - 3P_1(x) + (8/3) \times P_2(x)$.

Ex. 2. *Show that* $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, $P_3(x) = (5x^3 - 3)/2$ *and also expand* $x^4 + 2x^3 + 2x^2 - x - 3$ *in terms of Legendre's polynomials.* **[Bhopal 2004, Kakitiya 1997]**

Ex. 3. *Show that* $x^3 = (2/5) \times P_3(x) + (3/5) \times P_2(x)$. **[Nagpur 1996]**

9.3. Generating function for Legendre polynomials. [Nagpur 2005]

Theorem. *To show* $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n\, P_n(x), \; | x | \le 1, \; | z | < 1$ **[M.D.U. Rohtak 2006, Delhi Physics (H) 2000; Kanpur 2005; Banaglore 1994; Nagpur 2003; Meerut 1994, 96]**

Or To show that $P_n(x)$ *is the coefficient of* z^n *in the exapansion of* $(1 - 2xz + z^2)^{-1/2}$ *in assending powers of* z. **[Garhwal 2004; Meerut 1998, Ravishankar 2010; Ranchi 2010]**

[**Note :** $(1 - 2xz + z^2)^{-1/2}$ is called the *generating function* for Legendre polynomial $P_n(x)$].

Proof. (*a*) since $| z | < 1$ and $| x | \le 1$, we have

$$(1 - 2xz + z^2)^{-1/2} = [1 - z(2x - z)]^{-1/2}$$

$$= 1 + \tfrac{1}{2}\, z(2x - z) + \frac{1\cdot 3}{2\cdot 4}\, z^2(2x - z)^2 + ... + \frac{1\cdot 3 \ldots (2n-3)}{2\cdot 4 \ldots (2n-2)} z^{n-1}\; (2x - z)^{n-1}$$

$$+ \frac{1\cdot 3 \ldots (2n-1)}{2\cdot 4 \ldots (2n)}\; z^n (2x - z)^n + \ldots \qquad ...(1)$$

Now, the coefficient of z^n in $\dfrac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}z^n(2x-z)^n$

$$= \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}(2x)^n = \frac{1\cdot 3\cdot 5\ldots(2n-1)\cdot 2^n\cdot x^n}{(2\cdot 1)(2\cdot 2)(2\cdot 3)\ldots(2\cdot n)} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n\cdot n!}2^n\cdot x^n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}x^n. \quad \ldots(2)$$

Again, the coefficient of z^n in $\dfrac{1\cdot 3\ldots(2n-3)}{2\cdot 4\ldots(2n-2)}z^{n-1}(2x-z)^{n-1}$

$$= \frac{1\cdot 3\ldots(2n-3)}{(2\cdot 1)(2\cdot 2)\ldots 2(n-1)}\{-(n-1)(2x)^{n-2}\}$$

$$= -\frac{1\cdot 3\ldots(2n-3)}{2^{n-1}\,1\cdot 2\cdot 3\ldots(n-1)}\cdot\frac{2n-1}{n}\cdot\frac{n}{2n-1}[(n-1)\times 2^{n-2}\times x^{n-2}], \text{ on multiplying and dividing by } (2n-1)/n$$

$$= -\frac{1\cdot 3\ldots(2n-1)}{n!}\cdot\frac{n(n-1)}{2(2n-1)}x^{n-2} \quad \ldots(3)$$

and so on. Using (2), (3), ... we see that the coefficient of z^n in the expansion of $(1-2xz+z^2)^{-1/2}$, namely (1) is given by $\dfrac{1\cdot 3\ldots(2n-1)}{n!}\left[x^n-\dfrac{n(n-1)}{2(2n-1)}x^{n-2}+\dfrac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4}-\ldots\right]$

i.e., $P_n(x)$, by definition of Legendre polynomial.

We find that $P_1(x)$, $P_2(x)$, ... will be the coefficients of z, z^2,.... in the expansion of $(1-2xz+z^2)^{-1/2}$. Thus, we may write

$$(1-2xz+z^2)^{-1/2} = 1 + zP_1(x) + z^2P_2(x) + \ldots + z^nP_n(x) + \ldots \quad \text{or} \quad (1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^nP_n(x).$$

9.4. Solved examples based on Art. 9.2 and 9.3.

Ex. 1. *Prove that* (*a*) : $1 + \frac{1}{2}P_1(\cos\theta) + \frac{1}{3}P_2(\cos\theta) + \ldots = \log[(1+\sin\frac{1}{2}\theta)/\sin\frac{1}{2}\theta]$.

(*b*) $\sum_{n=0}^{\infty}\dfrac{P_n(\cos\theta)}{n+1} = \log\{1+\operatorname{cosec}(\theta/2)\}$ **[Ravishankar 2004]**

Sol. (*a*) From the generating function, $\sum_{n=0}^{\infty} z^nP_n(x) = (1-2xz+z^2)^{-1/2}$. ...(1)

Integrating (1) w.r.t. z from 0 to 1, $\sum_{n=0}^{\infty}\int_0^1 z^nP_n(x)\,dz = \int_0^1\dfrac{dz}{\sqrt{(1-2xz+z^2)}}$. ...(2)

Replacing x by $\cos\theta$ on both sides, (2) gives

$$\sum_{n=0}^{\infty}P_n(\cos\theta)\int_0^1 z^n\,dz = \int_0^1\frac{dz}{\sqrt{(1-2z\cos\theta+z^2)}} \quad \text{or} \quad \sum_{n=0}^{\infty}P_n(\cos\theta)\left[\frac{z^{n+1}}{n+1}\right]_0^1 = \int_0^1\frac{dz}{\sqrt{[(z-\cos\theta)^2+\sin^2\theta]}}$$

$$\text{or} \quad \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \left[\log\left\{(z-\cos\theta)+\sqrt{[(z-\cos\theta)^2+\sin^2\theta]}\right\}\right]_0^1$$

$$= \log\left\{(1-\cos\theta)+\sqrt{[(1-\cos\theta)^2+\sin^2\theta]}\right\} - \log(1-\cos\theta) = \log\left\{(1-\cos\theta)+\sqrt{[2(1-\cos\theta)]}\right\} - \log(1-\cos\theta)$$

$$= \log\frac{(1-\cos\theta)+\sqrt{2}\sqrt{(1-\cos\theta)}}{1-\cos\theta} = \log\frac{\sqrt{(1-\cos\theta)}\,\sqrt{(1-\cos\theta)}+\sqrt{2}\,\sqrt{(1-\cos\theta)}}{\sqrt{(1-\cos\theta)}\,\sqrt{(1-\cos\theta)}}$$

$$= \log\frac{\sqrt{(1-\cos\theta)}+\sqrt{2}}{\sqrt{(1-\cos\theta)}} = \log\frac{\sqrt{(2\sin^2\frac{1}{2}\theta)}+\sqrt{2}}{\sqrt{(2\sin^2\frac{1}{2}\theta)}} = \log\frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$$

$$\therefore \qquad \frac{P_0(\cos\theta)}{1} + \tfrac{1}{2}P_1(\cos\theta) + \tfrac{1}{3}P_2(\cos\theta) + \ldots = \log\frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$$

or $$1 + \tfrac{1}{2}P_1(\cos\theta) + \tfrac{1}{3}P_2(\cos\theta) + \ldots = \log\frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}, \quad \text{as} \quad P_0(\cos\theta) = 1 \qquad \ldots(3)$$

(*b*) Proceed as in part (*a*) upto equation (3). Re-writing (3), we have

$$\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \log\{1+\operatorname{cosec}(\theta/2)\}.$$

Ex. 2. *Prove that* (*i*) $P_n(1) = 1$ **[Agra 2010; Purvanchal 2005, Meerut 2007, 11; Bhopal 2010; Lucknow 2010; Nagpur 2010]**

(*ii*) $P_n(-1) = (-1)^n$ **[Agra 2009; Kanpur 2007]**

(*iii*) $P_n'(1) = \frac{1}{2}n(n+1)$. **[Delhi Physics (H) 2001]**

(*iv*) $P'_n(-1) = (-1)^{n-1} \times \frac{1}{2}n(n+1)$. **[Agra 2009; Gulbarge 2005]**

(*v*) $P_n(-x) = (-1)^n P_n(x)$. Deduce $P_n(-1) = (-1)^n$. **[Lucknow 2010, Nagpur 2010 Agra 2010; Kanpur 2005, 06; Kakitiya 1997; Meerut 2005; Purvanchal 2005]**

Sol. The generating function formula is $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n\, P_n(x), \mid z \mid < 1, \mid x \mid \le 1$. ...(1)

Part (i). Putting $x = 1$ in (1), we have

$$(1-2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1) \qquad \text{or} \qquad (1-z)^{-1} = \sum_{n=0}^{\infty} z^n P_n(1).$$

Since $\mid z \mid < 1$, the binomial theorem can be used for expansion of $(1-z)^{-1}$.

$$\therefore \qquad 1 + z + z^2 + \ldots. + z^n + \ldots = \sum_{n=0}^{\infty} z^n P_n(1). \qquad \ldots(2)$$

Equating the coefficient of z^n from both sides, (2) gives $P_n(1) = 1$

Part (ii). Putting $x = -1$ in (1), we have as before

$$(1+2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-1) \qquad \text{or} \qquad (1+z)^{-1} = \sum_{n=0}^{\infty} z^n P_n(-1)$$

or $$1 - z + z^2 \ldots + (-1)^n z^n + \ldots. = \sum_{n=0}^{\infty} z^n P_n(-1). \qquad \ldots(3)$$

Equation the coefficients of z^n from both sides, (3) gives $P_n(-1) = (-1)^n$.

Part (iii). Since $P_n(x)$ satisfies Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, we get $(1-x^2)\,P''_n(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$. ...(4)

Putting $x = 1$ in (4) and using $P_n(1) = 1$, we get

$$0 - 2P'_n(1) + n(n+1) = 0 \qquad \text{or} \qquad P'_n(1) = \tfrac{1}{2}n(n+1).$$

Part (iv). Putting $x = -1$ in (4) and using $P_n(-1) = (-1)^n$, we get

$0 + 2P'_n(-1) + n(n+1)(-1)^n = 0$ or $P'_n(-1) = -(-1)^n \times \frac{1}{2} n(n+1)$.

or $P_n'(-1) = (-1)^{n-1} \times \frac{1}{2} n(n+1)$ $[\because -(-1)^n = -(-1)^{n-1}(-1) = (-1)^{n-1}]$

Part (v). Replacing x by $-x$ in (1), $(1 + 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-x)$. ...(5)

Next, replacing z by $-z$ in (1), $(1 + 2xz + z)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n P_n(x)$. ...(6)

From (5) and (6), $\sum_{n=0}^{\infty} z^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n z^n P_n(x)$. ...(7)

Equating the coefficients of z^n from both sides of (8), we get

$P_n(-x) = (-1)^n P_n(x)$. ...(8)

Deduction. Replacing x by 1 and noting that $P_n(1) = 1$, (8) gives $P_n(-1) = (-1)^n$.

Note. When n is odd, $(-1)^n = -1$ and so (8) becomes $P_n(-x) = -P_n(x)$. Thus, $P_n(x)$ is an odd function of x when n is odd. Similarly, $P_n(x)$ is an even function of x when n is even.

Ex. 3. *Prove that* (*i*) $P_{2m+1}(0) = 0$ (*ii*) $P_{2m}(0) = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}$ **[Kanpur 2008]**

i.e. $P_{2m}(0) = \{(-1)^m 1 \cdot 3 \cdot 5 \cdot \cdots (2m-1)\}/2^m m!$ **[Nagpur 1995, 2002]**

(*iii*) $P_n(0) = 0$, if n is odd. **[Kanpur 2007; Agra 2006; Meerut 1996]**

and (*iv*) $P_n(0) = \frac{(-1)^{n/2} n!}{2^n \{(n/2)!\}^2}$, *if n is even.* **[Kanpur 2007; Meeerut, 1994, 95, 96]**

Sol. We have $\sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2)^{-1/2}, \ |z| < 1, \ |x| \le 1$. ...(1)

Putting $x = 0$ in (1), $\sum_{n=0}^{\infty} z^n P_n(0) = (1 + z^2)^{-1/2}$, *i.e.*,

$$\sum_{n=0}^{\infty} z^n P_n(0) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}(z^2)^n \quad \text{or} \quad \sum_{n=0}^{\infty} z^n P_n(0) = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n n!} z^{2n}.$$

...(2)

Part (i). Note that the R.H.S. of (2) consists of even powers of z alone. So equating the coefficients of z^{2m+1} from both sides of (2), we have $P_{2m+1}(0) = 0$. ...(3)

Part (ii). Equating the coefficients of z^{2m} from both sides of (2), we get

$$P_{2m}(0) = (-1)^m \frac{1 \cdot 3 \cdot 5 \ldots (2m-1)}{2^m m!} = (-1)^m \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots (2m-1)(2m)}{2^m m! \, 2 \cdot 4 \cdot 6 \ldots (2m)}$$

$$= (-1)^m \frac{(2m)!}{2^m m! (2 \cdot 1)(2 \cdot 2)(2 \cdot 3) \ldots (2 \cdot m)} = (-1)^m \frac{(2m)!}{2^m m!} \cdot \frac{1}{2^m m!} = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2} \quad \ldots(4)$$

Part (iii). Proceed as in part (*i*). Here $2m + 1 = n$ = odd. So from (3) $P_n(0) = 0$, if n is odd.

Part (iv). Proceed as in part (*ii*) upto (4). Here $2m = n$ = even so that $m = n/2$. Putting $m = n/2$ in (4), we have $$P_n(0) = (-1)^{n/2}\frac{n!}{2^n\{(n/2)!\}^2}.$$

Ex. 4. *Express* $x^4 + 2x^3 + 2x^2 - x - 3$ *in terms of Legendre's polynomials.*

Sol. We have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, $P_3(x) = (5x^3 - 3x)/2$, and $P_4(x) = (35x^4 - 30x^2 + 3)/8$.

These $\Rightarrow$ $x^4 = (8/35)P_4(x) + (6/7)x^2 - (3/35)$, ...(1)

$x^3 = (2/5)P_3(x) + (3/5)x$, $x^2 = (2/3)P_2(x) + (1/3)$, ...(2)

$x = P_1(x)$ and $1 = P_0(x)$ ...(3)

Now, $x^4 + 2x^3 + 2x^2 - x - 3 = (8/35)P_4(x) + (6/7)x^2 - (3/35) + 2[(2/5)P_3(x) + (3/5)x] + 2x^2 - x - 3$

[Putting values of x^4 and x^3 with help of (1) and (2)]

$= (8/35)P_4(x) + (4/5)P_3(x) + (20/7)x^2 + (1/5)x - (108/35)$

$= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{20}{7}\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + \frac{1}{5}P_1(x) - \frac{108}{35}$, using (2) and (3)

$= (8/35)P_4(x) + (4/5)P_3(x) + (40/21)P_2(x) + (1/5)P_1(x) - (224/105)P_0(x)$, using (3)

Ex. 5. *Prove that* $P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right) + P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right) + \ldots. P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right)$.

Sol. We have $$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x). \quad \ldots (1)$$

Replacing x by $1/2$ and $-1/2$ successively, (1) gives

$$(1 - z + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n\left(\tfrac{1}{2}\right) \quad \ldots(2)$$

and $$(1 + z + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n\left(-\tfrac{1}{2}\right). \quad \ldots(3)$$

Next, replacing z by z^2 in (3), $$(1 + z^2 + z^4)^{-1/2} = \sum_{n=0}^{\infty} z^{2n} P_n\left(-\tfrac{1}{2}\right). \quad \ldots(4)$$

But $1 + z^2 + z^4 = (1 + z^2)^2 - z^2 = (1 + z^2 + z)(1 + z^2 - z)$

$\therefore$ $(1 + z^2 + z^4)^{-1/2} = (1 + z + z^2)^{-1/2} \cdot (1 - z + z^2)^{-1/2}$

or $$\sum_{n=0}^{\infty} z^{2n} P_n\left(-\tfrac{1}{2}\right) = \sum_{n=0}^{\infty} z^n P_n\left(-\tfrac{1}{2}\right) \cdot \sum_{n=0}^{\infty} z^n P_n\left(\tfrac{1}{2}\right), \text{ by (2), (3) and (4)}$$

or $$\sum_{n=0}^{\infty} z^{2n} P_n\left(-\tfrac{1}{2}\right) = \left[P_0\left(-\tfrac{1}{2}\right) + z P_1\left(-\tfrac{1}{2}\right) + \ldots.. + z^{2n-1} P_{2n-1}\left(-\tfrac{1}{2}\right) + z^{2n} P_{2n}\left(-\tfrac{1}{2}\right) + \ldots\right] \times \left[P_0\left(\tfrac{1}{2}\right) + z P_1\left(\tfrac{1}{2}\right) + \ldots + z^{2n-1} P_{2n-1}\left(\tfrac{1}{2}\right) + z^{2n} P_{2n}\left(\tfrac{1}{2}\right) + \ldots\right]$$

Equating the coefficients of z^{2n} from both sides of the above equation, we get the desired result.

Ex. 6. *Prove that* $\dfrac{1 - z^2}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) z^n P_n$.

[Kanpur 2005; Purvanchal 2007; Meerut 2010]

Sol. We have $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n$. ...(1)

Differentiating both sides of (1) w.r.t. 'z', $-\frac{1}{2}(1 - 2xz + z^2)^{-3/2}(-2x + 2z) = \sum_{n=0}^{\infty} n\, z^{n-1} P_n$

or $(x - z)(1 - 2xz + z^2)^{-3/2} = \sum_{n=0}^{\infty} n\, z^{n-1} P_n$. ...(2)

Multiplying both sides of (2) by $2z$, $2z(x - z)(1 - 2xz + z^2)^{-3/2} = 2\sum_{n=0}^{\infty} n\, z^n P_n$. ...(3)

Adding (1) and (3), $\frac{1}{(1 - 2xz + z^2)^{1/2}} + \frac{2z(x - z)}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} z^n P_n + \sum_{n=0}^{\infty} 2nz^n P_n$

or $\frac{1 - 2xz + z^2 + 2z(x - z)}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) z^n P_n$ or $\frac{1 - z^2}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) z^n P_n$.

Ex. 7. *Prove that* $\frac{1 + z}{z\sqrt{(1 - 2xz + z^2)}} - \frac{1}{z} = \sum_{n=0}^{\infty} (P_n + P_{n+1}) z^n$.

Sol. We have, $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n$. ...(1)

$\therefore$ L.H.S. of the required result $= (1/z) \times (1 - 2xz + z^2)^{-1/2} + (1 - 2xz + z^2)^{-1/2} - (1/z)$

$= \frac{1}{z}\sum_{n=0}^{\infty} z^n P_n + \sum_{n=0}^{\infty} z^n P_n - \frac{1}{z}$, by (1) ...(2)

But $\sum_{n=0}^{\infty} z^n P_n = P_1 + zP_1 + z^2 P_2 + ... + z^n P_n + z^{n+1} P_{n+1} + ... = 1 + z(P_1 + zP_2 + ... + z^n P_{n+1} + ...)$, as $P_0 = 1$

Thus, $\sum_{n=0}^{\infty} z^n P_n = 1 + z \sum_{n=0}^{\infty} z^n P_{n+1}$. ...(3)

Using (3) in (2), the L.H.S. of the required result

$= \frac{1}{z}\left[1 + z\sum_{n=0}^{\infty} z^n P_{n+1}\right] + \sum_{n=0}^{\infty} z^n P_n - \frac{1}{z} = \sum_{n=0}^{\infty} z^n P_{n+1} + \sum_{n=0}^{\infty} z^n P_n = \sum_{n=0}^{\infty} (P_n + P_{n+1}) z^n$

= R.H.S. of the required result.

Ex. 8. *Prove that* $(1 - 2xz + z^2)^{-1/2}$ *is a solution of the equation* $z\frac{\partial^2 (zv)}{\partial z^2} + \frac{\partial}{\partial x}\left[(1 - x^2)\frac{dv}{dx}\right] = 0$.

Sol. Let $v = (1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n$. ...(1)

$\therefore$ $zv = z\sum_{n=0}^{\infty} z^n P_n = \sum_{n=0}^{\infty} z^{n+1} P_n$.

$$\therefore \qquad \frac{\partial^2 (zv)}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left[\sum_{n=0}^{\infty} z^{n+1} P_n\right] = \sum_{n=0}^{\infty} n(n+1)\, z^{n-1} P_n .$$

$$\therefore \qquad z\frac{\partial^2 (zv)}{\partial z^2} = z\sum_{n=1}^{\infty} n(n+1)\, z^{n-1} P_n = \sum_{n=1}^{\infty} n(n+1)\, z^n P_n . \qquad ...(2)$$

From (1), $$\frac{\partial v}{\partial x} = \sum_{n=0}^{\infty} \frac{\partial}{\partial x}(z^n\, P_n) = \sum_{n=0}^{\infty} z^n\, P_n' . \qquad ...(3)$$

$$\therefore \qquad \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\sum_{n=0}^{\infty} z^n\, P_n' = \sum_{n=0}^{\infty} z^n\, P_n'' . \qquad ...(4)$$

But $$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] = (1-x^2)\frac{\partial^2 v}{\partial x^2} - 2x\frac{\partial v}{\partial x}$$

$$\therefore \qquad \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] = (1-x^2)\sum_{n=0}^{\infty} z^n\, P_n'' - 2x\sum_{n=0}^{\infty} z^n\, P_n', \text{ using (3) and (4)} \qquad ...(5)$$

Adding the corresponding sides of (2) and (5), we have

$$z\frac{\partial^2 (zv)}{\partial v^2} + \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\} = \sum_{n=0}^{\infty} n(n+1)\, z^n\, P_n + \sum_{n=0}^{\infty} (1-x^2) z^n\, .\, P_n'' - \sum_{n=0}^{\infty} z^n (2x\, P_n')$$

$$= \sum_{n=0}^{\infty} z^n [(1-x^2)\, P_n'' - 2x\, P_n' + n(n+1)\, P_n] = 0,$$

as P_n is a solution of Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

9.5. Trigonometrical series for $P_n(x)$.

To show that $P_n(\cos\theta)$

$$= \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^{n-1}\cdot n!}\left[\cos n\theta + \frac{1\cdot n}{1\cdot(2n-1)}\cos(n-2)\,\theta + \frac{1\cdot 3\cdot n(n-1)}{1\cdot 2\cdot(2n-1)\,(2n-3)}\cos(n-4)\,\theta + \ldots\right]$$

Sol. Since $\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)^{-1/2}$, we have

$$\sum_{n=0}^{\infty} z^n P_n(\cos\theta) = (1 - 2z\cos\theta + z^2)^{-1/2} = [1 - z(e^{i\theta} + e^{-i\theta}) + z^2]^{-1/2}$$

$$= [1 - ze^{i\theta} - ze^{-i\theta} + ze^{i\theta}\, .\, ze^{-i\theta}]^{-1/2} = (1 - ze^{i\theta})^{-1/2}\,(1 - ze^{-i\theta})^{-1/2}$$

$$= \left[1 + \tfrac{1}{2}ze^{i\theta} + \frac{1\cdot 3}{2\cdot 4}z^2e^{2i\theta} + \ldots + \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}z^n e^{ni\theta} + \cdots\right]$$

$$\times\left[1 + \tfrac{1}{2}ze^{-i\theta} + \frac{1\cdot 3}{2\cdot 4}z^2e^{-2i\theta} + \ldots + \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}z^n e^{-ni\theta} + \cdots\right].$$

Equating the coefficients of z^n from both sides, we get

$$P_n(\cos\theta) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}(e^{ni\theta} + e^{-ni\theta}) + \frac{1\cdot 3\cdot 5\ldots(2n-3)\,.\,1}{2\cdot 4\cdot 6\ldots(2n-2)\,.\,2}[e^{(n-2)i\theta} + e^{-(n-2)i\theta}]$$

$$+\frac{1\cdot 3\cdot 5\ldots(2n-5)\cdot 1\cdot 3}{2\cdot 4\cdot 6\ldots(2n-2)\cdot 2\cdot 4}[e^{(n-4)i\theta}+e^{-(n-4)i\theta}]+\cdots$$

$$=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\left[2\cos n\theta+\frac{2n}{2n-1}\cdot\frac{1}{2}\cdot 2\cos(n-2)\theta\right.$$

$$\left.+\frac{2n\,.\,(2n-2)}{(2n-1)\,(2n-3)}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot 2\cos(n-4)\theta+\cdots.\right] \quad \ldots(A)$$

$$=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^{n-1}\cdot n\,!}\left[\cos n\theta+\frac{1\,.\,n}{1\cdot(2n-1)}\cos(n-4)\theta+\frac{1\cdot 3\cdot n(n-1)}{1\cdot 2\cdot(2n-1)\,(2n-3)}\cos(n-4)\theta+\cdots\right]$$

SOLVED EXAMPLE BASED ON ART 9.5

Example. *Prove that* $\displaystyle\int_0^{\pi} P_n(\cos\theta)\cos n\theta\,d\theta=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\cdot\pi=B\left(n+\tfrac{1}{2},\tfrac{1}{2}\right).$

Sol. From equation (A) of Art 9.5, we have

$$P_n(\cos\theta)=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\left[2\cos n\theta+\frac{2n}{2n-1}\cdot\frac{1}{2}\cdot 2\cos(n-2)\theta+\frac{2n\cdot(2n-2)}{(2n-1)\,(2n-3)}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot 2\cos(n-4)\theta+\cdots\right]$$

$$\therefore\ \int_0^{\pi}P_n(\cos\theta)\cos n\theta\,d\theta=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\left[2\int_0^{\pi}\cos^2 n\theta\,d\theta+\frac{2n}{2n-1}\cdot\frac{1}{2}\cdot 2\int_0^{\pi}\cos(n-2)\theta\cos n\theta\,d\theta+\cdots\right]$$

$$=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\int_0^{\pi}(1+\cos 2n\theta)\,d\theta\text{, since for } m\neq n\text{, we have}$$

$$\int_0^{\pi}\cos mx\cos nx\,dx=\tfrac{1}{2}\int_0^{\pi}[\cos(m+n)x+\cos(m-n)x]dx=\frac{1}{2}\left[\frac{\sin(m+n)\,x}{m+n}+\frac{\sin(m-n)\,x}{m-n}\right]_0^{\pi}=0$$

$$\therefore\quad\int_0^{\pi}P_n(\cos\theta)\cos n\theta\,d\theta=\frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots(2n)}\pi=\frac{\frac{2n-1}{2}\cdot\frac{2n-3}{2}\ldots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\,.\,\sqrt{\pi}}{1\,.\,2\,.\,3\ldots n}$$

$$=\frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)}=\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}+\frac{1}{2}\right)}=B\left(n+\tfrac{1}{2},\tfrac{1}{2}\right)\qquad\left[\because B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

EXERCISE 9 (A)

1. Show that Legendre's polynomial $P_n(x)$ satisfies the Legendre's equation
$$(1-x^2)\,(d^2y/dx^2)-2x(dy/dx)+n(n+1)y=0.$$

2. Show that the general solution of Legendre's equation $(1-x^2)y''-2xy'+n(n+1)y=0$, where n is a positive integer is $y=AP_n(x)+BQ_n(x)$, where $P_n(x)$ and $Q_n(x)$ have their usual meanings.

3. Express the following expressions in terms of Legendre's polynomial to show that

(*i*) $x^2=(1/3)\times P_0(x)+(2/3)\times P_2(x)$. (*ii*) $x^3=(2/5)\times P_3(x)+(3/5)\times P_1(x)$.

(*iii*) $x^4=(8/35)\times P_4(x)+(4/7)\times P_2(x)+(1/5)\times P_0(x)$. **(Nagpur 2003, Bilaspur 2004)**

(*iv*) $x^4-3x^2+x=-(4/5)\times P_0(x)+P_1(x)-(10/7)\times P_2(x)+(8/35)\times P_4(x)$.

9.6. Laplace's definite integrels for $P_n(x)$

(I) Laplace's first integral $P_n(x)$. When is n is a positive integer, then

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x \pm (x^2-1)^{1/2}\cos\phi]^n \, d\phi \,. \qquad \textbf{(Agra 2010; Meerut 2005, 06)}$$

Proof. From integral calculus, $\int_0^{\pi} \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{(a^2-b^2)^{1/2}}$, when $a^2 > b^2$. ...(1)

Let $a = 1 - zx$ and $b = z(x^2-1)^{1/2}$. Then $a^2 - b^2 = (1-zx)^2 - z^2(x^2-1) = 1 - 2xz + z^2$.

Using these values of a, b and $a^2 - b^2$, (1) becomes

$$\pi(1-2zx+z^2)^{-1/2} = \int_0^{\pi}\left[1 - zx \pm z(x^2-1)^{1/2}\cos\phi\right]^{-1} d\phi$$

or
$$\pi\sum_{n=0}^{\infty} z^n P_n(x) = \int_0^{\pi}(1-zt)^{-1}\, d\phi, \qquad \text{if} \qquad t = x \pm (x^2-1)^{1/2}\cos\phi$$

$$= \int_0^{\pi}(1 + zt + z^2t^2 + \ldots)\, d\phi = \int_0^{\pi}\sum_{n=0}^{\infty}(zt)^n d\phi = \sum_{n=0}^{\infty} z^n \int_0^{\pi} t^n \, d\phi \,.$$

$$\therefore \qquad \pi\sum_{n=0}^{\infty} z^n P_n(x) = \sum_{n=0}^{\infty} z^n \int_0^{\pi}[x \pm \sqrt{(x^2-1)}\cos\phi]^n \, d\phi. \qquad ...(2)$$

Equating coefficients of z^n from both sides, (2) gives

$$\pi P_n(x) = \int_0^{\pi}[x \pm (x^2-1)^{1/2}\cos\phi]^n \, d\phi \qquad \text{or} \qquad P_n(x) = \frac{1}{\pi}\int_0^{\pi}[x \pm (x^2-1)^{1/2}\cos\phi]^n \, d\phi \quad ...(3)$$

Deductions : Show that (*i*) $P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n \, d\phi$

(*ii*) $P_1(x) = \frac{1}{\pi}\int_0^{\pi}\{x + (x^2-1)^{1/2}\cos\theta\}\, d\theta$

Proof. (i) Let $x = \cos\theta$. Then, we have

$$(x^2-1)^{1/2} = \sqrt{(\cos^2\theta - 1)} = \sqrt{\{(-1)(1-\cos^2\theta)\}} = \sqrt{\{(i^2\sin^2\theta)\}} = i\sin\theta$$

With these value and +ve sign, (3) gives $P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n d\phi$.

Part (ii). Take $n = 1$ and +ve sign in (3). Thus, we obtain the required result.

(II) Laplace's second integral for $P_n(x)$. When n is a positive integer,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{[x \pm (x^2-1)^{1/2}\cos\phi]^{n+1}} \,. \qquad \textbf{[Kanpur 2007; Meerut 2007]}$$

Proof. From integral calculus, $\int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{(a^2-b^2)^{1/2}}$, where $a^2 > b^2$. ...(1)

Let $a = zx - 1$ and $b = (x^2-1)^{1/2}$. Then, $a^2 - b^2 = (zx-1)^2 - z^2(x^2-1) = 1 - 2zx + z^2$

Using these values of a, b and $a^2 - b^2$, (1) gives

$$\pi(1-2zx+z^2)^{-1/2} = \int_0^{\pi}\left[-1 + zx \pm z(x^2-1)^{1/2}\cos\phi\right]^{-1} d\phi$$

or $$\frac{\pi}{z}\left(1-2x\,\frac{1}{z}+\frac{1}{z^2}\right)^{-1/2} = \int_0^{\pi}[-1+z\,\{x\pm(x^2-1)^{1/2}\cos\phi\}]^{-1}\,d\phi.$$

...(2)

Let $$t = x\pm(x^2-1)^{1/2}\cos\phi.$$ ...(3)

We know that $$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x).$$ ...(4)

Replacing z by $\frac{1}{z}$ in (4), we have $$\left(1-2x\times\frac{1}{z}+\frac{1}{z^2}\right)^{-1/2} = \sum_{n=0}^{\infty}\frac{1}{z^n}P_n(x).$$ ...(5)

Using (3) and (5), we have

$$\frac{\pi}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}P_n(x) = \int_0^{\pi}(-1+zt)^{-1}d\phi = \int_0^{\pi}(zt)^{-1}\left(1-\frac{1}{zt}\right)^{-1}d\phi = \int_0^{\pi}\frac{1}{zt}\sum_{n=0}^{\infty}\left(\frac{1}{zt}\right)^n d\phi = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_0^{\pi}\frac{d\phi}{t^{n+1}}.$$

$$\therefore \quad \sum_{n=0}^{\infty}\frac{\pi}{z^{n+1}}P_n(x) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_0^{\pi}\frac{d\phi}{\{x\pm(x^2-1)^{1/2}\cos\phi\}^{n+1}}.$$ ...(6)

Equating the coefficients of $1/z^{n+1}$ from sides, (6) gives

$$\pi P_n(x) = \int_0^{\pi}\frac{d\phi}{\{x\pm(x^2-1)^{1/2}\cos\phi\}^{n+1}} \quad \text{or} \quad P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{x\pm(x^2-1)^{1/2}\cos\phi\}^{n+1}}$$ ...(7)

Deduction. Replacing n by $-(n+1)$ in (7), we have

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{x\pm(x^2-1)^{1/2}\cos\phi\}^{-n}} = \int_0^{\pi}\frac{1}{\pi}\{x\pm(x^2-1)^{1/2}\cos\phi\}^n\,d\phi$$

$= P_n(x)$, by form I of Laplace's integral

Thus, $$P_n(x) = P_{-(n+1)}(x).$$

9.7. Some bounds on $P_n(x)$. If $-1 < x < 1$ and n is any positive integer, then

(*i*) $|P_n(x)| < 1$ (*ii*) $|P_n(x)| < \{\pi/2n(1-x^2)\}^{1/2}$.

Proof. If $-1 < x < 1$, we have

$$|x+(x^2-1)^{1/2}\cos\phi| = |x+i(1-x^2)^{1/2}\cos\phi| = [x^2+(1-x^2)\cos^2\phi]^{1/2}$$
$$= [\cos^2\phi + x^2(1-\cos^2\phi]^{1/2} = [1-\sin^2\phi+x^2\sin^2\phi]^{1/2}.$$

Thus, $$|x+(x^2-1)^{1/2}\cos\phi| = [1-(1-x^2)\sin^2\phi]^{1/2}.$$ ...(1)

(*i*) If $\phi \neq 0$ or $\phi \neq \pi$, $(1) \Rightarrow |x+(x^2-1)^{1/2}\cos\phi| < 1.$ ...(2)

Now, $$|P_n(x)| = \left|\frac{1}{\pi}\int_0^{\pi}[x+(x^2-1)^{1/2}\cos\phi]^n\,d\phi\right|,$$ using Laplace's first integral for $P_n(x)$

$$\Rightarrow \quad |P_n(x)| \le \frac{1}{\pi}\int_0^{\pi}|x+(x^2-1)^{1/2}\cos\phi|^n\,d\phi < \frac{1}{\pi}\int_0^{\pi}1.\,d\phi,$$ using (2)

Thus, $$|P_n(x)| < 1, \quad \text{if} \quad -1 < x < 1.$$

(*ii*) We have, $$|P_n(x)| = \left|\frac{1}{\pi}\int_0^{\pi}[x+(x^2-1)^{1/2}\cos\phi]^n d\phi\right| \le \frac{1}{\pi}\int_0^{\pi}|x+(x^2-1)\cos\phi|^n\,d\phi$$

or $$|P_n(x)| \le \frac{1}{\pi}\int_0^{\pi} [1-(1-x^2)\sin^2\phi]^{n/2} d\phi, \text{ using (1)}$$

or $$|P_n(x)| \le \frac{2}{\pi}\int_0^{\pi/2} [1-(1-x^2)\sin^2\phi]^{n/2} d\phi. \qquad ...(3)$$

But, we know that, if $0 < \phi < \pi$, then $\sin\phi > 2\phi/\pi$.

This $\Rightarrow 1-(1-x^2)\sin^2\phi < 1-(1-x^2)\times(2\phi/\pi)^2 \Rightarrow 1-(1-x^2)\sin^2\phi < \exp[-4\phi^2(1-x^2)/\pi^2]$

[Here $4\phi^2(1-x^2)/\pi^2 < 1$ and $\exp a = e^a$]

Then, $(3) \Rightarrow |P_n(x)| < \frac{2}{\pi}\int_0^{\pi/2} \exp[-2n\phi^2(1-x^2)/\pi^2]\, d\phi$

Thus, $$|P_n(x)| < \frac{2}{\pi}\int_0^{\infty} \exp[-2n\phi^2(1-x^2)/\pi^2]\, d\phi. \qquad ...(4)$$

Putting $[2n\phi^2 \quad (1-x^2)/\pi^2]^{1/2} = t$ so that $d\phi = [\pi/\{2n(1-x^2)\}^{1/2}]dt$, (4) becomes

$$|P_n(x)| < \frac{2}{\pi}\cdot\frac{\pi}{\{2n(1-x^2)\}^{1/2}}\int_0^{\infty} e^{-t^2}dt = \frac{2}{\{2n(1-x^2)\}^{1/2}}\times\frac{\sqrt{\pi}}{2} \Rightarrow |P_n(x)| < \left[\frac{\pi}{2n(1-x^2)}\right]^{1/2}.$$

9.8. Orthogonal properties of Legendre's polynomials

Prove that (*i*) $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$ if $m \ne n$.

[Agra 2006, 08, 09, 10; Kanpur 2006, 08; Meerut 2010, 11]

(*ii*) $\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$. **[Agra 2009, 10; Meerut 1992, 93]**

(*iii*) $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 \text{ if, } m \ne n \\ 2/(2n+1), \text{ if } m = n. \end{cases}$

[Agra 2005, Bilaspur 1998; Ravishankar 1996; Nagpur 2005]

or $$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2n+1}\delta_{mn}, \qquad \text{where} \qquad \delta_{mn} = \begin{cases} 0 \text{ if } m \ne n \\ 1 \text{ if } m = n. \end{cases}$$

Here δ_{mn} is called Kronecker delta. It is also denoted by δ_n^m or δ^{mn} or δ_{mn}.

[Kanpur 2006; Delhi Physics (H) 2000]

Proof (i). When m ≠ n. Since $P_m(x)$ and $P_n(x)$ satisfy Legendre's equation we have

$$(1-x^2)P_m'' - 2xP_m' - m(m+1)P_m = 0 \qquad ...(1)$$

and $$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0. \qquad ...(2)$$

Multiplying (1) by P_n and (2) by P_m and then subtracting the resulting equations, we have

$$(1-x^2)(P_nP_m' - P_mP_n') - 2x(P_nP_m' - P_n'P_m) + [m(m+1) - n(n+1)]P_mP_n = 0$$

or $$(1-x^2)\frac{d}{dx}(P_nP_m' - P_n'P_m) - 2x(P_nP_m' - P_mP_n') = (n^2 - m^2 + n - m)\, P_mP_n$$

or $$\frac{d}{dx}\{(1-x^2)(P_nP_m' - P_n'P_m)\} = (n-m)(n+m+1)P_mP_n.$$

Integrating both sides w.r.t. 'x' from -1 to 1, we get

$$(n-m)(n+m+1)\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \left[(1-x^2)(P_nP_m' - P_n'P_m)\right]_{x=-1}^{x=1}$$

$$\therefore \qquad \int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0, \quad \text{as} \quad m \ne n. \qquad ...(3)$$

(ii). When m = n. Then the required result takes the form $\int_{-1}^{1} [P_n(x)]^2\, dx = 2/(2n+1)$.

To prove this, start with generating function $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$. ...(4)

Also, (4) may be re-written as $(1 - 2xz + z^2)^{-1/2} = \sum_{m=0}^{\infty} z^m P_m(x)$ ...(5)

Multiplying the corresponding sides of (4) and (5), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)\, P_n(x)\, z^{m+n} = (1 - 2xz + z^2)^{-1}$$

Integrating both sides of the above equation w.r.t. 'x' we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_m(x)\, P_n(x)\, dx\right\} z^{m+n} = \int_{-1}^{1} (1 - 2xz + z^2)^{-1}\, dx \qquad ...(6)$$

Making use of (3), (6) reduces to

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1} \{P_n(x)\}^2\, dx\right] z^{2n} = \int_{-1}^{1} \frac{dx}{1+z^2-2xz} = \left[\frac{\log(1+z^2-2zx)}{-2z}\right]_{-1}^{1} = -\frac{1}{2z}[\log(1-z)^2 - \log(1+z)^2]$$

$$= -(1/2z) \times [2\log(1-z) - 2\log(1+z)] = (1/z) \times [\log(1+z) - \log(1-z)]$$

$$= \frac{1}{z}\left[\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\right) - \left(-z - \frac{z^2}{2} - \frac{z^2}{3} - \ldots\right)\right] = \frac{2}{z}\left(z - \frac{z^3}{3} + \frac{z^5}{5} + \ldots\right) = \frac{2}{z}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}.$$

$$\therefore \quad \sum_{n=0}^{\infty}\left[\int_{-1}^{1} \{P_n(x)\}^2\, dx\right] = \sum_{n=0}^{\infty}\frac{2}{2n+1} z^{2n}. \qquad ...(7)$$

Equating coefficients of z^{2n} from both sides, (7) gives $\int_{-1}^{1} [P_n(x)]^2\, dx = 2/(2n+1)$. ... (8)

(*iii*) Combining results (3) and (8), we have

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, \text{ if } m = n \\ 2/(2n+1), \text{ if } m = n \end{cases} \qquad \text{or} \qquad \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} = Kronecker delta $= \begin{cases} 0, \text{ if } m \neq n \\ 1, \text{ if } m = n \end{cases}$

9.9. Recurrence relations (formulae). To show that

I. $nP_n = (2n-1)\, xP_{n-1} - (n-1)P_{n-2}$, $n \geq 2$ **[Delhi Physics (H) 2000; Agra 2005; Bilaspur 1998, Purvanchal 2005, Kanpur 2006, 09]**

or $(n+1)P_{n+1} = (2n+1)\, xP_n - nP_{n-1}$, $n \geq 1$ **[Nagpur 2005; Kakitiya 1997; Agra 2005, 06; KU Kurukshetra 2006; Meerut 2006; Ravishankar 2004; Vikram 2004]**

or $xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)$. **[Utkar 2003; Agra 1998]**

II. $nP_n = xP'_n - P'_{n-1}$. **[Purvanchal 2004; Kanpur 2011; Vikram 2000; Bangalore 1995; Meerut 1996]**

III. $(2n+1)P_n = P'_{n+1} - P'_{n-1}$. **[Bilaspur 1993; Bangalore 1992, 93]**

IV. $(n+1)P_n = P'_{n+1} - xP'_n$. or $P_n' - xP'_{n-1} = nP_{n-1}$. **[Kanpur 2007]**

V. $(1-x^2)P_n' = n(P_{n-1} - xP_n)$ or $(x^2-1)P'_n = n\,x\,P_n - nP_{n-1}$.

[Delhi Physics (H) 2001]

VI. $(1-x^2)\,P'_n = (n+1)\,(xP_n - P_{n+1})$. **[Meerut 1997]**

Proof I. From generating function, we have $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$. ...(1)

Differentiating both sides of (1) w.r.t. 'z', we get

$$-\tfrac{1}{2}(1 - 2xz + z^2)^{-3/2}\,(-2x + 2z) = \sum_{n=0}^{\infty} n\,z^{n-1}P_n(x). \quad ...(2)$$

Multiplying both sides by $1 - 2xz + z^2$, (2) gives

$$(x - z)\,(1 - 2xz + z^2)^{-1/2} = (1 - 2xz + z^2)\sum_{n=0}^{\infty} n\,z^{n-1}P_n(x)$$

or $$(x - z)\sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2)\sum_{n=0}^{\infty} n\,z^{n-1}P_n(x), \text{ by (1)}$$

or $$x\sum_{n=0}^{\infty} z^n P_n - \sum_{n=0}^{\infty} z^{n+1}P_n(x) = \sum_{n=0}^{\infty} n\,z^{n-1}P_n - 2x\sum_{n=0}^{\infty} n\,z^n P_n + \sum_{n=0}^{\infty} n\,z^{n+1}P_n.$$

Equating coefficients of z^n from both sides, we get

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1}$$

or $$(n+1)P_{n+1} = (2n+1)\,xP_n - nP_{n-1}. \quad ...(3)$$

Replacing n by $n-1$ in (3), we get $nP_n = (2n-1)\,x\,P_{n-1} - (n-1)P_{n-2}$. ...(4)

Again (3) can be re–arranged to give another form $xP_n = \dfrac{n+1}{2n+1}P_{n+1} + \dfrac{n}{2n+1}P_{n-1}$.

II. We have $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$. ...(1)

Differentiating (1) w.r.t. 'z', we get $-\tfrac{1}{2}(1 - 2xz + z^2)^{-3/2}\,(-2x + 2z) = \sum_{n=0}^{\infty} n\,z^{n-1}P_n$

or $$(x - z)\,(1 - 2xz + z^2)^{-3/2} = \sum_{n=0}^{\infty} n\,z^{n-1}P_n. \quad ...(2)$$

Again, differentating (1) w.r.t. 'x' and simplifying, we get

$z(1 - 2xz + z^2)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n'$. or $z(x - z)\,(1 - 2xz + z^2)^{-3/2} = (x - z)\sum_{n=0}^{\infty} z^n P_n'$

or $z\sum_{n=0}^{\infty} n\,z^{n-1}P_n = (x - z)\sum_{n=0}^{\infty} z^n P_n'$, by (2) or $\sum_{n=0}^{\infty} n\,z^n P_n = x\sum_{n=0}^{\infty} z^n P_n' - \sum_{n=0}^{\infty} z^{n+1}P_n'$.

Equating coefficient of z^n on both sides we get $nP_n = xP'_n - P'_{n-1}$.

III. From recurrence relation I, $(2n+1)\,xP_n = (n+1)P_{n+1} + nP_{n-1}$.

Differentiating it w.r.t. 'x', $(2n+1)\,xP'_n + (2n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1}$

or $$(2n+1)\,(nP_n + P'_{n-1}) + (2n+1)P_n = (n+1)P'_{n+1} - nP'_{n-1}$$

$[\because$ from recurrence II, $xP'_n = nP_n + P'_{n-1}]$

or $$(2n+1)\,(n+1)P_n = (n+1)P'_{n+1} - (n+1)P'_{n-1}$$

or $$(2n+1)P_n = P'_{n+1} - P'_{n-1}. \qquad ...(1)$$

Replacing n by $n-1$ in (1), we have $(2n-1)P_{n-1} = P'_n - P'_{n-2}$

or $$\frac{dP_n(x)}{dx} = \frac{dP_{n-2}(x)}{dx} + (2n-1)P_{n-1}(x) \qquad ...(2)$$

(1) and (2) are the required forms of the results.

IV. From recurrence relations II and III, we get

$$nP_n = xP_n' - P'_{n-1} \qquad ...(1)$$

and $$(2n+1)P_n = P'_{n+1} - P'_{n-1} \qquad ...(2)$$

Subtracting (1) from (2), $(n+1)P_n = P'_{n+1} - xP'_n$

V. From recurrence relations II and IV, we get

$$nP_n = xP'_n - P'_{n-1} \qquad ...(1)$$

and $$(n+1)P_n = P'_{n+1} - xP'_n. \qquad ...(2)$$

Replacing n by $n-1$ in (2), $nP_{n-1} = P'_n - xP'_{n-1}. \qquad ...(3)$

Multiplying both sides of (1) by x, $xnP_n = x^2P'_n - xP'_{n-1}. \qquad ...(4)$

Subtracting (4) from (3), we have

$n(P_{n-1} - xP_n) = (1-x^2)P'_n$ or $(x^2-1)P'_n = nxP_n - nP_{n-1}.$

VI. From recurrence relations I and V, we have

$$(2x+1)xP_n = (n+1)P_{n+1} + nP_{n-1} \qquad ...(1)$$

and $$(1-x^2)P'_n = n(P_{n-1} - xP_n). \qquad ...(2)$$

Re–writting (1), $[(n+1)+n]xP_n = (n+1)P_{n+1} + nP_{n-1}$

or $$(n+1)\,(xP_n - P_{n+1}) = n(P_{n-1} - xP_n). \qquad ...(3)$$

Now, from (2) and (3), $(1-x^2)P'_n = (n+1)\,(xP_n - P_{n+1}).$

9.10. Beltrami's result.

Prove that $(2n+1)\,(x^2-1)\,P_n' = n(n+1)(P_{n+1} - P_{n-1}).$ **[Ravishankar 1998]**

Sol. Recurrence relations V and VI are

$$(1-x^2)P'_n = n(P_{n-1} - xP_n) \qquad ...(1)$$

and $$(1-x^2)P'_n = (n+1)\,(xP_n - P_{n+1}) \qquad ...(2)$$

Multiplying (1) by $n+1$ and (2) by n and adding, we get

$$(n+1)\,(1-x^2)P'_n + n(1-x^2)P_n' = n(n+1)P_{n-1} - n(n+1)P_{n+1}$$

or $$(2n+1)\,(1-x^2)P'_n = n(n+1)\,(P_{n-1} - P_{n+1})$$

or $$(2n+1)\,(x^2-1)P'_n = n(n+1)\,(P_{n+1} - P_{n-1}),$$

which is known as *Beltrami's result.*

9.11. Christoffel's summation formula.

Prove that $$\sum_{k=0}^{l}(2k+1)\,P_k(x)\,P_k(y) = \frac{l+1}{x-y}\{P_{l+1}(x)\,P_l(y) - P_l(x)\,P_{l+1}(y)\}.$$

[Bilaspur 1997, Jodhpur 2004; Ravishankar 2000; Garhwal 2005]

Deduce that $$\sum_{k=0}^{l}(2k+1)P_k(x) = \frac{l+1}{x-1}\{P_{l+1}(x) - P_l(x)\}.$$

Sol. From recurrence relation I, we have

$$(2k+1)\,xP_k(x) = (k+1)P_{k+1}(x) + kP_{k-1}(x) \qquad ...(1)$$

and $$(2k+1)\,yP_k(y) = (k+1)P_{k+1}(y) + kP_{k-1}(y). \qquad ...(2)$$

Multiplying (1) by $P_k(y)$ and (2) by $P_k(x)$ and then substracting, we get

$$(2k+1)\,(x-y)P_k(x)P_k(y)$$
$$= (k+1)\,[P_{k+1}(x)P_k(y) - P_{k+1}(y)\,P_k(x)] - k[P_{k-1}(x)\,P_k(y) - P_{k-1}(y)\,P_k(x)]. \quad ...(3)$$

Replacing k by 0, 1, 2, 3, ..., $l-1$, l successively in (3) and adding the resulting equations, we have

$$(x-y)\sum_{k=0}^{l}(2k+1)\,P_k(x)\,P_k(y) = (l+1)\,[P_{l+1}(x)\,P_l(y) - P_{l+1}(y)\,P_l(x)]$$

or
$$\sum_{k=0}^{l}(2k+1)\,P_k(x)\,P_k(y) = \frac{l+1}{x-y}[P_{l+1}(x)\,P_l(y) - P_{l+1}(y)\,P_l(x)]. \quad ...(4)$$

Deduction. Replace y by 1 in (4) and use $P_l(1) = P_{l+1}(1) = P_k(1) = 1$ to get the required result.

9.12. Christoffel's expansion. To show that $P'_n = (2n-1)P_{n-1} - (2n-5)P_{n-3} + (2n-9)P_{n-5} + ...$, the last terms of the series being $3P_1$ or P_0 according as n is even or odd.

or
$$P_n'(x) = \sum_{r=0}^{[\frac{1}{2}(n-1)]}(2n-4r-1)\,P_{n-2r-1}(x), \quad \text{where} \quad [\tfrac{1}{2}(n-1)] = \begin{cases}(n-1)/2, \text{ if } n \text{ is even}\\ (n-2)/2, \text{ if } n \text{ is odd}\end{cases}$$

Sol. Replacing n by $n-1$ in recurrence relation $(2n+1)P_n = P'_{n+1} - P'_{n-1}$, we have

$$P'_n = (2n-1)P_{n-1} + P'_{n-2}. \quad ...(1)$$

Case I. Let n be even, Replacing n by n, $n-2$, $n-4$, ..., 4, 2, successively in (1) and using the fact that $P_0 = 1$ and $P'_0 = 0$, we get

$$P'_n = (2n-1)P_{n-1} + P'_{n-2},$$
$$P'_{n-2} = (2n-5)P_{n-3} + P'_{n-4},$$
$$P'_{n-1} = (2n-9)\,P_{n-5} + P'_{n-6},$$
$$.... \qquad .... \quad .... \quad ....$$
$$P'_4 = 7P_3 + P'_2$$

and
$$P'_2 = 3P_1 + P'_0$$

Adding these and simplifying, $P'_n = (2n-1)P_{n-1} + (2n-5)P_{n-3} + ... + 3P_1$. ...(2)

Case II. Let *n* is odd. Replacing n by n, $n-2$, $n-4$, ..., 5, 3 successively in (1) and using the fact that $P_1(x) = x$, $P_0 = 1$ so that $P'_1 = 1 = P_0$, we get

$$P'_n = (2n-1)P_{n-1} + P'_{n-2}$$
$$P'_{n-2} = (2n-5)P_{n-3} + P'_{n-4}$$
$$P'_{n-4} = (2n-9)P_{n-5} + P'_{n-6}$$
$$... \qquad .... \quad .... \quad ....$$
$$... \qquad .... \quad .... \quad ....$$
$$P_5' = 9P_4 + P'_3$$

and
$$P_3' = 5P_2 + P'_1 = 5P_2 + P_0.$$

Adding these and simplifying, we get $P'_n = (2n-1)P_{n-1} + (2n-5)P_{n-3} + ... + 5P_2 + P_0$. ...(3)

Combining (2) and (3), we have $P'_n(x) = \sum_{r=0}^{[\frac{1}{2}(n-1)]}(2n-4r-1)\,P_{n-2r-1}(x)$.

9.13. Solved examples based on Art. 9.8 and 9.9

Ex. 1. *If $P_n(x)$ is a Legendre polynomial of degree n and α is such that $P_n(\alpha) = 0$. Show that $P_{n-1}(\alpha)$ and $P_{n+1}(\alpha)$ are of opposite signs.* **[Purvanchal 2006]**

Sol. From recurrence relation I, $(2n+1)\,xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$. ...(1)

Given that $P_n(\alpha) = 0.$ ... (2)

Putting $x = \alpha$ in (1) and using (2), $(2n + 1)\ \alpha\ .\ 0 = (n + 1)P_{n+1}(\alpha) + nP_{n-1}(\alpha)$

or $$\frac{P_{n+1}(\alpha)}{P_{n-1}(\alpha)} = -\frac{n}{n+1}.$$ ...(3)

Since n is a positive integer, so R.H.S. of (3) is negative. Hence (3) shows that $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ must be of opposite signs.

Ex. 2. *Prove that* $P'_{n+1} + P'_n = P_0 + 3P_1 + ... + (2n + 1)P_n$.

or $$\sum_{r=0}^{n} (2r+1)\, P_r(x) = P'_{n+1}(x) + P'_n(x).$$ **[Meerut 1993]**

Sol. From recurrence relation III, we have $(2n + 1)P_n = P'_{n+1} - P'_{n-1}.$ ...(1)

Replacing n by 1, 2, ...$n - 1$, n successively in (!), we get

$$3P_1 = P'_2 - P'_0,$$
$$5P_2 = P'_3 - P'_1,$$
$$7P_3 = P'_4 - P'_2,$$
$$\text{... ....}$$
$$\text{... ... ...}$$
$$(2n-1)P_{n-1} = P'_n - P'_{n-2}$$

and $$(2n+1)P_n = P'_{n+1} - P'_{n-1}.$$

Adding these and noting that in the sum of right hand sides all the terms cancel except the first two of the second column and the last two of the first column, we get

$$3P_1 + 5P_2 + 7P_3 + ... + (2n + 1)P_n = -P_0' - P_1' + P_n' + P'_{n+1}.$$ ...(2)

Since $P_0 = 1$ and $P_1 = x$, we have $P_0' = 0$ and $P_1' = 1 = P_0$. Using these results in (2), we have

$$3P_1 + 5P_2 + 7P_3 + ... + (2n + 1)P_n = 0 - P_0 + P_n' + P'_{n+1}$$

or $$P_0 + 3P_1 + 5P_2 + ...+ (2n + 1)P_n = P'_n + P'_{n+1}$$

or $$\sum_{r=0}^{n} (2r+1)\, P_r(x) = P'_n(x) + P'_{n+1}(x).$$

Ex. 3. *Prove that* (*i*) $c + \int P_n\, dx = (P_{n+1} - P_{n-1})\ /(2n + 1).$

(*ii*) $\int_x^1 P_n\, dx = (P_{n-1} - P_{n+1})\ /(2n + 1).$

Sol. From recurrence relations III, we have

$(2n + 1)P_n = P'_{n+1} - P'_{n-1}$ or $$P_n = \frac{1}{2n+1}\frac{d}{dx}(P_{n+1} - P_{n-1}).$$ ...(1)

(*i*) Integrating (1), $c + \int P_n\, dx = (P_{n+1} - P_{n-1})/(2n + 1)$, c being an arbitrary constant

(*ii*) Integrating both sides of (1) w.r.t. 'x' between limits of x to 1, we get

$$\int_x^1 P_n(x)\, dx = \frac{1}{2n+1}\Big[P_{n+1}(x) - P_{n-1}(x)\Big]_x^1 = \frac{1}{2n+1}[P_{n+1}(1) - P_{n-1}(1) - \{P_{n+1}(x) - P_{n-1}(x)\}]$$
$$= \{P_{n-1}(x) - P_{n+1}(x)\}/(2n + 1), \text{ as } P_{n+1}(1) = P_{n-1}(1) = 1.$$

Ex. 4. *Prove that* $P_0^2 + 3P^2_1 + 5P^2_2 + ... + (2n + 1)P_n^2$
$= (n + 1)[P_n(x)\ P'_{n+1}(x) - P_{n+1}\ (x)\ P'_n(x)] = (n + 1)^2\ [\{P_n(x)\}^2 + (1 - x^2)\ \{P'_n(x)\}^2.$

Sol. From Christoffel's summation formula (Refer Art 9.11), we have

$$(x-y)\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(y) = (n+1)[P_{n+1}(x)\,P_n(y) - P_{n+1}(y)\,P_n(x)]. \quad ...(1)$$

Putting $y = x + h$, where h is a small quantity in (1), we obtain

$$-h\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(x+h) = (n+1)[P_{n+1}(x)\,P_n(x+h) - P_{n+1}(x+h)\,P_n(x)].$$

Expanding by Taylor's theorem on both sides, we have

$$-h\sum_{r=0}^{n}(2r+1)\,P_r(x)\,\{P_r(x) + h\,P_r'(x) + \ldots.\}$$

$$= (n+1)\left[P_{n+1}(x)\left\{P_n(x) + hP'_n(x) + \frac{h^2}{2!}P''_n(x) + \ldots.\right\} - P_n(x)\left\{P_{n+1}(x) + hP'_{n+1}(x) + \frac{h^2}{2!}P''_{n+1}(x) + \ldots.\right\}\right]$$

$$= -h(n+1)\left[\{P_n(x)\,P'_{n+1}(x) - P'_n(x)\,P_{n+1}(x)\} + \text{terms containing } h\right]$$

or $$\sum_{r=0}^{n}(2r+1)\,P_r(x)\,\{P_r(x) + h\,P_r'(x) + \ldots.\} = (n+1)\,[P_n(x)\,P'_{n+1}(x) - P'_n(x)\,P_{n+1}(x)\} + \text{terms containing } h].$$

Taking limit on both sides as $h \to 0$, we obtain

$$\sum_{r=0}^{\infty}(2r+1)\,[P_r(x)]^2 = (n+1)\,[P_n(x)P'_{n+1}(x) - P'_n(x)\,P_{n+1}(x)]$$

or $$P_0^2(x) + 3P_1^2(x) + 5P^2{}_2(x) + \ldots.. + (2n+1)\,P^2{}_n(x) = (n+1)\,[P_n(x)\,P'_{n+1}(x) - P'_n(x)\,P_{n+1}(x)] \quad ...(2)$$

Now, $(n+1)^2\,\{P_n(x)\}^2 + (1-x^2)\,\{P'_n(x)\}^2$

$= (n+1)P_n(x)\,[(n+1)P_n(x)] + P'_n(x)\,\{(1-x^2)\,P'_n(x)\}$

$= (n+1)P_n(x)\,[P'_{n+1}(x) - xP'_n(x)] + P'_n(x)\,[(n+1)\,\{xP_n(x) - P_{n+1}(x)\}]$

(Using recurrence formulas IV and VI)

Thus, $(n+1)^2\,\{P_n(x)\}^2 + (1-x^2)\,\{P'_n(x)\}^2 = (n+1)\,[P_n(x)P'_{n+1}(x) - P_{n+1}(x)P'_n(x)]$ ...(3)

From (2) and (3), we have

$$P_0^2(x) + 3P_1^2(x) + \ldots + (2n+1)\,P_n^2(x) = (n+1)^2\,\{P_n(x)\}^2 + (1-x^2)\,\{P'_n(x)\}^2.$$

Ex. 5. *Evaluate :* (*i*) $\int_0^1 P_n(x)\,dx$, when n is odd. **[Jiwaji 2004]**

(*ii*) $\int_0^1 P_n(x)\,dx$, when n is e*ven.* **[Gulbarga 2005]**

Sol. (*i*) From recurrence relation III, we have $(2n+1)P_n = P'_{n+1} - P'_{n-1}$

or $$P_n(x) = \frac{1}{2n+1}\frac{d}{dx}[P_{n+1}(x) - P_{n-1}(x)].$$

Integrating both sides w.r.t. 'x' from 0 to 1, we get

$$\int_0^1 P_n(x)\,dx = \frac{1}{2n+1}\Big[P_{n+1}(x) - P_{n-1}(x)\Big]_0^1 = \frac{1}{2n+1}\Big[P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(0) + P_{n-1}(0)\Big] \quad ....(1)$$

But $P_{n+1}(1) = 1$, $P_{n-1}(1) = 1$ and $P_{2l}(0) = \dfrac{(-1)^l}{2^{2l}}\,\dfrac{(2l)!}{(l!)^2}$. ...(2)

Since n is odd, $n-1$ and $n+1$ are both even. Taking $2l = n-1$, *i.e.*, $l = (n-1)/2$ in (2),

we have $$P_{n-1}(0) = \frac{(-1)^{(n-1)/2}}{2^{n-1}} \frac{(n-1)!}{[\{\frac{1}{2}(n-1)\}!]^2}. \quad ...(3)$$

Next, taking $2l = n + 1$ *i.e.* $l = (n+1)/2$ in (2), we have

$$P_{n+1}(0) = \frac{(-1)^{(n+1)/2}}{2^{n+1}} \frac{(n+1)!}{[\{\frac{1}{2}(n+1)\}!]^2} = -\frac{(-1)^{(n-1)/2}(n+1)\,n\,(n-1)!}{2^2 \cdot 2^{n-1}\left[\frac{n+1}{2} \cdot \frac{n}{2} \cdot \left(\frac{n-1}{2}\right)!\right]^2}. \quad ...(4)$$

Using (2), (3) and (4) in (1), we have

$$\int_0^1 P_n(x)\,dx = \frac{1}{2n+1} \times \frac{(-1)^{(n-1)/2}}{2^{n-1}} \times \frac{(n-1)!}{\left[\left(\frac{n-1}{2}\right)!\right]^2}\left[\frac{n}{n-1}+1\right]$$

$$= \frac{(-1)^{(n-1)/2}\,(n-1)!}{2^n\left(\frac{n}{2}+\frac{1}{2}\right)\left(\frac{n}{2}-\frac{1}{2}\right)!\left(\frac{n}{2}-\frac{1}{2}\right)!} = \frac{(-1)^{(n-1)/2}\,(n-1)!}{2^n\left(\frac{n}{2}+\frac{1}{2}\right)!\left(\frac{n}{2}-\frac{1}{2}\right)!} \qquad [\because\ (p+1)\,p\,! = (p+1)!]$$

Part (ii). Proceed upto equation (1) as in part (*i*). Since n is even, $n-1$ and $n+1$ are both odd and so $P_{n-1}(0) = 0$, $P_{n+1}(0) = 0$, $P_{n+1}(1) = 0$ and $P_{n-1}(1) = 0$ ...(5)

Making use of (5), (1) reduces to $\int_0^1 P_n(x)\,dx = 0$, when n is even.

Ex. 6. *Prove that* $\int_{-1}^{1} x\,P_n(x)\,P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$.

[Purvanchal 2007; Gulbarga 2005; Sagar 2004; Kanpur 2007]

Sol. From recurrence relation *I*, $$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x). \quad ...(1)$$

Multiplying both sides of (1) by $P_{n-1}(x)$ and then integrating w.r.t. x from –1 to 1, we get

$$\int_{-1}^{1} x\,P_n(x)\,P_{n-1}(x)\,dx = \frac{n+1}{2n+1}\int_{-1}^{1} P_{n+1}(x)\,P_{n-1}(x)\,dx + \frac{n}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2 dx. \quad ...(2)$$

But, $$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n+1), \text{ if } m = n \end{cases} \quad ...(3)$$

Making use of (3), (2) becomes

$$\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = 0 + \frac{n}{2n+1} \times \frac{2}{2(n-1)+1} = \frac{2n}{(2n+1)(2n-1)} \quad \text{or} \quad \frac{2n}{4n^2-1}.$$

Ex. 7. *Prove that* $\int_{-1}^{1} (1-x^2)\,P_l'\,.\,P_m'\,dx = \begin{cases} 0, \text{ if } l \neq m \\ \frac{2l(l+1)}{2l+1}, \text{ if } l = m \end{cases}$

Or *Prove that* $\int_{-1}^{1} (1-x^2)\,P_l'\,P_m'\,dx = \frac{2l(l+1)}{2l+1}\delta_{lm}$, where $\delta_{lm} = \begin{cases} 0, \text{ if } l \neq m \\ 1, \text{ if } l = m \end{cases}$

[Indore 2004; Purvanchal 2004; Ravishakar 2005]

Sol. Case I. Let $l \neq m$. Then integrating by parts, we have

$$\int_{-1}^{1} [(1-x^2)\,P_l']\,P_m'\,dx = \left[(1-x^2)P_l'.P_m\right]_{-1}^{1} - \int_{-1}^{1} [(1-x^2)\,P_l'' - 2x\,P_l']\,P_m\,dx = -\int_{-1}^{1} [(1-x^2)\,P_l'' - 2x\,P_l']\,P_m\,dx. \quad ...(1)$$

Since P_l satisfies Legendre's equation $(1-x^2)y'' - 2xy' + l(l+1)y = 0$, hence

$$(1-x^2)\,P_l'' - 2xP_l' + l(l+1)P_l = 0 \quad \text{or} \quad (1-x^2)P_l'' - 2xP_l' = -l(l+1)P_l. \quad ...(2)$$

But $$\int_{-1}^{1} P_l\,P_m\,dx = 0, \text{ if } l \neq m \quad ...(3)$$

Using (2), (1) reduces to

$$\int_{-1}^{1} (1-x^2)\,P_l'\,P_m'\,dx = l(l+1)\int_{-1}^{1} P_l\,P_m\,dx = 0, \text{ using (3).} \quad ...(4)$$

Case II. Let $l = m$. Then the required result takes the form

$$\int_{-1}^{1} (1-x^2)\,(P_l')^2\,dx = \frac{2l(l+1)}{2l+1}. \quad \textbf{[Agra 2010]} \quad ...(5)$$

We have, by using integration by parts,

$$\int_{-1}^{1} (1-x^2)\,(P_l')^2\,dx = \int_{-1}^{1} [(1-x^2)\,P_l']\,.\,P_l'\,dx$$

$$= \left[(1-x^2)\,P_l'P_l\right]_{-1}^{1} - \int_{-1}^{1}\left[(1-x^2)P_l'' - 2xP_l'\right]P_l\,dx = 0 + l(l+1)\int_{-1}^{1}(P_l)^2\,dx, \text{ using (2)}$$

$$= l(l+1)\,.\,\frac{2}{2l+1} = \frac{2l(l+1)}{2l+1}.$$

Combining (4) and (5) and using symbol δ_{lm}, we get $$\int_{-1}^{1} (1-x^2)\,P_l'\,P_m'\,dx = \frac{2l(l+1)}{2l+1}\,\delta_{lm}.$$

Ex. 8. *Prove that* $$(2n+1)\,(x^2-1)P'_n = n(n+1)\,(P_{n+1} - P_{n-1})$$

and hence prove that $\int_{-1}^{1} (x^2-1)\,P_{n+1}\,P_n'\,dx = \dfrac{2n\,(n+1)}{(2n+1)\,(2n+3)}$ **[Agra 2009; Nagpur 2010]**

Sol. Refer Art. 9.10 to show that $$(2n+1)\,(x^2-1) = n(n+1)\,(P_{n+1} - P_{n-1}). \quad ...(1)$$

From (1), $$(x^2-1)P'_n = \frac{n\,(n+1)}{2n+1}\,(P_{n+1} - P_{n-1}). \quad ...(2)$$

Multiplying both sides of (2) by P_{n+1} and then integrating w.r.t. x from -1 to 1, we have

$$\int_{-1}^{1} (x^2-1)\,P_{n+1}\,P_n'\,dx = \frac{n(n+1)}{2n+1}\int_{-1}^{1} P_{n+1}(P_{n+1} - P_{n-1})dx = \frac{n\,(n+1)}{2n+1}\left[\int_{-1}^{1} P_{n+1}^2\,dx - \int_{-1}^{1} P_{n+1}\,P_{n-1}\,dx\right] \quad ...(3)$$

But $$\int_{-1}^{1} P_m\,P_n\,dx = \begin{cases} 0, & \text{if } m \neq n \\ n/(2n+1), & \text{if } m = n \end{cases} \quad ...(4)$$

Using (4), (3) reduces to

$$\int_{-1}^{1} (x^2-1)\,P_{n+1}\,P_n'\,dx = \frac{n\,(n+1)}{2n+1}\left[\frac{2}{2(n+1)+1} - 0\right] = \frac{2n\,(n+1)}{(2n+1)\,(2n+3)}$$

Ex. 9. (a). *Show that* $\int_{-1}^{1} x^2\,P_{n+1}(x)\,P_{n+1}(x)\,dx = \dfrac{2n\,(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}$

[Kanpur 2007, 08]

(b) *Deduce the value of* $\int_0^1 x^2 P_{n+1}(x) P_{n+1}(x)\,dx$. *i.e., prove that* $\int_0^1 x^2 P_{n+1}(x) P_{n+1}(x)\,dx = \frac{n(n+1)}{(4n^2-1)(2n+3)}$.

Sol. (*a*) From recurrence relation *I*, $(2n-1)xP_{n-1} = nP_n + (n-1)P_{n-2}$. ...(1)

Replacing *n* by *n* + 2 in (1), $(2n+3)xP_{n+1} = (n+2)P_{n+2} + (n+1)P_n$. ...(2)

Multiplying the corresponding sides of (1) and (2), we get

$$(2n-1)(2n+3)x^2 P_{n-1}P_{n+1} = n(n+1)P_n^2 + n(n+2)P_{n+2}P_n + (n-1)(n+2)P_{n-2}P_{n+2} + (n-1)(n+1)P_{n-2}P_n. \quad ...(3)$$

Also, $$\int_{-1}^1 P_m P_n\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n+1), \text{ if } m = n \end{cases} \quad ...(4)$$

Integrating (3) w.r.t. '*x*' from –1 to 1 and using (4), we get

$$(2n-1)(2n+3)\int_{-1}^1 x^2 P_{n-1}(x) P_{n+1}dx = n(n+1)\cdot\frac{2}{2n+1} + 0 + 0$$

or $$\int_{-1}^1 x^2 P_{n-1}(x) P_{n+1}dx = \frac{2n(n+1).}{(2n+3)(2n-1)(2n+1)} \quad ...(5)$$

Part (b). Deduction. Since $P_n(x)$ is a polynomial of degree *n*, so $x^2P_{n-1}(x)\,P_{n+1}(x)$ is a polynomial of degree 2 + (*n* – 1) + (*n* + 1) *i.e.* 2(*n* + 1). Since 2(*n* + 1) is even, we see that $x^2P_{n-1}\,P_{n+1}$ is an even function of *x* and hence

$$\int_{-1}^1 x^2 P_{n-1}(x) P_{n+1}dx = 2\int_0^1 x^2 P_{n-1} P_{n+1}\,dx$$

or $$\int_0^1 x^2 P_{n-1}(x) P_{n+1}dx = \tfrac{1}{2}\int_{-1}^1 x^2 P_{n-1} P_{n+1}\,dx = \frac{n(n+1)}{(2n+3)(4n^2-1)},$$ by (5).

Ex. 10. *Prove that* $\int_{-1}^1 x^2 P_n^2\,dx = \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$.

Sol. From recurence relation I, $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$.

Squaring both sides, we have

$$(2n+1)^2 x^2 P_n^2 = (n+1)^2 P^2_{n+1} + n^2 P^2_{n-1} + 2n(n+1)P_{n+1}P_{n-1}. \quad ...(1)$$

Also, $$\int_{-1}^1 P_m P_n\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n+1), \text{ if } m = n \end{cases} \quad ...(2)$$

Integrating both sides of (1) w.r.t. '*x*' between the limits –1 to 1 and using (2), we have

$$(2n+1)^2\int_{-1}^1 x^2 P_n^2\,dx = (n+1)^2\frac{2}{2(n+1)+1} + n^2\frac{2}{2(n-1)+1} + 0$$

$$\therefore \int_{-1}^1 x^2 P_n^2\,dx = \frac{2}{(2n+1)^2}\left[\frac{(n+1)^2}{2n+3} + \frac{n^2}{2n-1}\right] = \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$$

(on resolving into partial fractions)

Ex. 11. *Prove that* $xP_n' = nP_n + (2n-3)P_{n-2} + (2n-7)P_{n-4} + \ldots$ and hence or otherwise show that (*a*) $\int_{-1}^1 x\,P_n\,P_n'\,dx = (2n)/(2n+1)$.

(*b*) $\int_{-1}^1 x\,P_n\,P_m'\,dx$ = either 0 or 2 or $(2n)/(2n+1)$.

Sol. From recurrence relation II, we have

$xP'_n = nP_n + P'_{n-1}$ or $xP'_n - P'_{n-1} = nP_n$. ...(1)

Again from recurrence relation III, we have

$P'_{n+1} = (2n + 1)P_n + P'_{n-1}$ or $P'_{n+1} - P'_{n-1} = (2n + 1)P_n$ ...(2)

Replacing n by $n - 2$, $n - 4$, $n - 6$, ... successively in (2), we get

$$\begin{aligned} P'_{n-1} - P'_{n-3} &= (2n - 3)P_{n-2} \\ P'_{n-3} - P'_{n-5} &= (2n - 7)P_{n-4} \\ \text{....} \qquad & \qquad \text{.... ....} \\ \text{...} \qquad & \qquad \text{.... ....} \end{aligned} \qquad \text{...(3)}$$

Adding (1) and (3) and simplifying, we get

$$xP'_n = nP_n + (2n - 3)P_{n-2} + (2n - 7)P_{n-4} + \text{....} \qquad \text{...(4)}$$

Part (a). Multiplying both sides of (4) by P_n, we get

$$xP_nP'_n = nP_n^2 + (2n - 3)P_{n-2}P_n + (2n - 5)P_{n-4}P_n + \text{...} \qquad \text{...(5)}$$

Also $$\int_{-1}^{1} x\, P_m\, P_n\, dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n + 1), \text{ if } m = n \end{cases}. \qquad \text{...(6)}$$

Integrating both sides of (5) w.r.t. x from –1 to 1 and using (6), we have

$$\int_{-1}^{1} x\, P_n\, P_n'\, dx = n \cdot \frac{2}{2n+1} + 0 + 0 + \ldots = \frac{2n}{2n+1}. \qquad \text{...(7)}$$

Part (b). Replaing n by m in (4), we get

$$xP'_m = mP_m + (2m - 3)P_{m-2} + (2m - 7)P_{m-4} + \text{...} \qquad \text{...(8)}$$

Multiplying both sides of (8) by P_n, we get

$$xP_nP'_m = mP_mP_n + (2m - 3)P_{m-2}P_n + (2m - 7)P_{m-4}P_n + \text{...} \qquad \text{...(9)}$$

Integrating both sides of (9) w.r.t. 'x' from –1 to 1 and using (6), three cases arise:

Case I. When n is different from m, $m - 2$, $m - 4$, ... and so on. Then

$$\int_{-1}^{1} x\, P_n\, P_m'\, dx = 0 + 0 + 0 + \ldots = 0.$$

Case II. When $n = m$. Then, $$\int_{-1}^{1} x\, P_n\, P_n'\, dx = n \cdot \frac{2}{2n+1} + 0 + 0 + \ldots = \frac{2n}{2n+1}.$$

Case III. When $n = m - 2$. Then $n \neq m$, $n \neq (m - 4)$, $n \neq (m - 6)$, ... and so on. So we obtain

$$\int_{-1}^{1} x\, P_n\, P_m'\, dx = 0 + (2m-3) \times \frac{2}{2n+1} = [2(n+2)-3] \times \frac{2}{2n+1} \qquad \left[\because\ n = m - 2 \Rightarrow m = n + 2\right]$$

$$= (2n + 1) \times \frac{2}{2n+1} = 2.$$

Similarly we can prove that $\int_{-1}^{1} x\, P_n\, P_m'\, dx = 2$, when $n = m - 4$ or $n = m - 6$ etc.

Thus $\int_{-1}^{1} x\, P_n\, P_m'\, dx = 0$ or 2 or $\frac{2n}{2n+1}$.

Ex. 12. *Prove that* $\int_{-1}^{1} (P_n')^2\, dx = n(n + 1)$.

Sol. From Christoffel's expansion, we get

$$P'_n = (2n - 1)P_{n-1} + (2n - 5)P_{n-3} + (2n - 9)P_{n-5} + \text{...} \qquad \text{...(1)}$$

The last term on R.H.S. of (1) is $3P_1$ or P_0 according as n is even or odd.

But $(a + b + c + d + ...)^2 = (a^2 + b^2 + c^2 + ...) + 2\Sigma ab.$ ...(2)

Squaring both sides of (1) and using (2), we have

$$P'_n{}^2 = (2n-1)^2 P^2_{n-1} + (2n-5)^2 P^2_{n-3} + (2n-9)^2 P^2_{n-5} + ... + 3^2 P_1^2 + 2\Sigma(2n-1)(2n-5)P_{n-1}P_{n-3}, \text{ if } n \text{ is even} \quad ...(3A)$$

or $$P'_n{}^2 = (2n-1)^2 P^2_{n-1} + (2n-5)^2 P^2_{n-3} + ... + P_0^2 + 2\Sigma(2n-1)(2n-5)P_{n-1}P_{n-3}, \text{ if } n \text{ is odd} \quad ...(3B)$$

Also, $$\int_{-1}^{1} P_m P_n \, dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n+1), \text{ if } m = n \end{cases}. \quad ...(4)$$

We consider two cases :

Case I. When n is even. Integrating both sides of (3A) w.r.t. 'x' from -1 to 1 and using (4),

$$\int_{-1}^{1} (P_n')^2 \, dx = (2n-1)^2 \cdot \frac{2}{2(n-1)+1} + (2n-5)^2 \cdot \frac{2}{2(n-3)+1} + ... + 3^2 \frac{2}{2\cdot 1+1} + 0 + 0 + ...$$

$$= 2[(2n-1) + (2n-5) + ... + 3]. \quad ...(5)$$

Let m be the number of terms in A.P. on R.H.S. of (5). Then, we have

$3 = (2n-1) + (m-1) \times (-4)$ so that $m = n/2$.

But sum of A.P. $= \frac{number\ of\ terms}{2}$(first term + last term)

$\therefore$ $(2n-1) + (2n-5) + ... + 3 = \frac{(n/2)}{2}(2n - 1 + 3) = \frac{1}{2}n(n+1).$

Hence (5) reduces to $\int_{-1}^{1} P_n'^2 \, dx = 2 \times \frac{1}{2} n(n+1) = n(n+1).$

Case II. When n is odd. Integrating both sides of (3B) w.r.t. 'x' from -1 to 1 ans using (4),

$$\int_{-1}^{1} P_n'^2 \, dx = (2n-1)^2 \cdot \frac{2}{2(n-1)+1} + (2n-5)^2 \cdot \frac{2}{2(n-3)+1} + ... + \frac{2}{2\cdot 0+1}$$

$$= 2[(2n-1) + (2n-5) + ... + 1]. \quad ...(6)$$

Let p be the number of terms in A.P. on R.H.S. of (6). Then, we have

$1 = (2n-1) + (p-1) \times (-4)$ so that $p = (n+1)/2$.

As before, $(2n-1) + (2n-5) + ... + 1 = \frac{\{(n+1)/2\}}{2}(2n - 1 + 1) = \frac{1}{2}n(n+1).$

Hence (6) reduces to $\int_{-1}^{1} P_n'^2 \, dx = 2 \times \frac{1}{2} n(n+1) = n(n+1).$

Thus for all values of n, we have $\int_{-1}^{1} P_n'^2 \, dx = n(n+1).$

Ex. 13. Show that, when $|z| < 1$ and $|x| \leq 1$,

$$\int_{-1}^{1} P_n(x) \, (1 - 2zx + z^2)^{-1/2} \, dx = (2z^n)/(2n+1). \quad \textbf{[Meerut 2005]}$$

Sol. We know that $(1 - 2zx + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n \, P_n(x).$

Multiplying both sides by $P_n(x)$, we have

$$P_n(x).\ (1 - 2zx + z^2)^{-1/2} = P_n(x)\ [P_0(x) + zP_1(x) + ... + z^nP_n(x) + ...].$$

Integrating both sides w.r.t. 'x' between –1 to 1, we get

$$\int_{-1}^{1} P_n(x)\ (1 - 2zx + z^2)^{-1/2}\ dx = \int_{-1}^{1} P_0(x)\ P_n(x)\ dx + z\int_{-1}^{1} P_1(x)\ P_n(x)dx\ + ...$$

$$+ z^n \int_{-1}^{1} [P_n(x)]^2\ dx + z^{n+1}\int_{-1}^{1} P_{n+1}(x)\ P_n(x)\ dx + ... \quad ...(1)$$

But $\int_{-1}^{1} P_m(x)\ P_n(x)\ dx = 0,$ if $m \neq n$ and $\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}.$ ...(2)

Using (2), (1) reduces to $\int_{-1}^{1} P_n(x)\ (1 - 2zx + z^2)^{-1/2}\ dx = \frac{2z^n}{2n+1}.$

Ex. 14. Evaluate $\int_{-1}^{1} P_3^2(x)\ dx$. **[Nagpur 1995, 96]**

Sol. Since $\int_{-1}^{1} P_n^2(x)\ dx = 2/(2n + 1),$ so $\int_{-1}^{1} P_3^2(x)\ dx = 2/(6 + 1) = 2/7.$

EXERCISE 9 (B)

1. Evaluate $(i)\int_{-1}^{1} x\ P_n^2(x)\ dx$ $(ii)\ \int_{-1}^{1} x\ P_n(x)\ P_{n+1}(x)\ dx$ $(iii)\ \int_{-1}^{1} x^3\ P_4(x)\ dx$.

Ans. (i) 0 (ii) $2(n + 1)/[(2n + 1)\ (2n + 3)]$ (iii) 0.

2. Prove that $\int_{-1}^{1} (1 - x^2)\ P_m'\ P_n'\ dx = 0,$ if $m \neq n.$

3. If $u_n = \int_{-1}^{1} x^{-1}\ P_n(x)\ P_{n-1}(x)\ dx,$ show that $nu_n + (n - 1)u_{n-1} = 2$ and hence evaluate u_n.

Ans. $u_n = 2/n,$ if n is even; $u_n = 0$ if n is odd

4. Prove that $\int_{-1}^{1} x\ P_n(x)\ P_n'(x)\ dx = \frac{2n}{2n+1}.$

5. Obtain the relation : $xP_n'(x) = P_{n+1}'(x) - (n+1)P_n(x)$ **[Guwahati 2007]**

9.14. Rodrigue's formula. To show that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$

[Punjab 2005; Purvanchal 2006; Gulbarga 2005; Ultak 2003; Nagpur 1996; Meerut 2007, 11; Garhwal 2004; Bilaspur 1998; Bhopal 2004, 10; KU Kurukshetra 2005; Agra 2010; MDU Rohtak 2005; Kanpur 2008, 11; Ranchi 2010; Lucknow 2010]

Proof. By the definition of Legendre polynomial, we get

$$P_n(x) = \sum_{n=0}^{[n/2]} (-1)^r\ \frac{(2n - 2r)!\ x^{n-2r}}{2^n\ r!\ (n - r)!\ (n - 2r)!}. \quad ...(1)$$

where $\left[\tfrac{1}{2}n\right] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases}$ ...(2)

Now, by binomial theorem, $(x^2 - 1)^n = \sum_{r=0}^{n} {}^nC_r\ (x^2)^{n-r}(-1)^r = \sum_{r=0}^{n} {}^nC_r\ (-1)^r x^{2n-2r}.$

$$\therefore \qquad \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n \times n!}\sum_{r=0}^{n} {}^nC_r(-1)^r \frac{d^n}{dx^n}x^{2n-2r}. \qquad ...(3)$$

But $$\frac{d^n}{dx^n}x^m = 0 \text{ if } m < n \quad \text{and} \quad \frac{d^n}{dx^n}x^m = \frac{m!}{(m-n)!}x^{m-n}, \text{ if } m \geq n \qquad ...(4)$$

$$\therefore \qquad \frac{d^n}{dx^n}x^{2n-2r} = 0, \text{ if } 2n-2r < n, \quad i.e., \quad r > \frac{n}{2}. \qquad ...(5)$$

Making use of (5) in (3), we see that we must replace $\sum_{r=0}^{n}$ by $\sum_{r=0}^{n/2}$ if n is even and by $\sum_{r=0}^{(n-1)/2}$ if n is odd. *i.e.* we must replace $\sum_{r=0}^{n}$ by $\sum_{r=0}^{[n/2]}$. Hence (3) reduces to

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1) = \frac{1}{2^n\, n!}\sum_{r=0}^{[n/2]} {}^nC_r(-1)^r \frac{d^n}{dx^n}x^{2n-2r} = \frac{1}{2^n\, n!}\sum_{r=0}^{[n/2]} {}^nC_r(-1)^r \frac{(2n-2r)!}{(2n-2r-n)!}x^{2n-2r-n}, \text{ by (4)}$$

$$= \sum_{r=0}^{[n/2]} \frac{1}{2^n\, n!}\frac{n!(-1)^r}{r!(n-r)!}\cdot\frac{(2n-2r)!}{(n-2r)!}x^{n-2r} = P_n(x), \text{ using (1).}$$

9.15. Solved examples based on Art. 9.14.

Ex. 1. *Using Rodrigue's furmula, find values of* $P_0(x)$, $P_1(x)$, $P_2(x)$ *and* $P_3(x)$. **[MDU Rohtak 2004; Bangalore 1995]**

Sol. Rodrigue's formula is given by $$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n. \qquad ... (1)$$

Putting $n = 0$ in (1), $$P_0(x) = \frac{1}{2^0\, 0!}(x^2-1)^0 = 1.$$

Putting $n = 1$ in (1), $$P_1(x) = \frac{1}{2^1 . 1!}\frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = x.$$

Putting $n = 2$ in (1), we have

$$P_2(x) = \frac{1}{2^2 \times 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left[\frac{d}{dx}(x^2-1)^2\right] = \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)\times 2x\right] = \frac{1}{2}\frac{d}{dx}(x^3-x) = \frac{1}{2}(3x^2-1).$$

Putting $n = 3$ in (1), we have

$$P_3(x) = \frac{1}{2^3 \times 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^2}{dx^2}\left[\frac{d}{dx}(x^2-1)^3\right] = \frac{1}{48}\frac{d^2}{dx^2}\left[3(x^2-1)^2 \times 2x\right]$$

$$= \frac{1}{8}\frac{d}{dx}\left[\frac{d}{dx}x(x^2-1)^2\right] = \frac{1}{8}\frac{d}{dx}\left[(x^2-1)^2 + x\times 2(x^2-1)\times 2x\right] = \frac{1}{8}\times\frac{d}{dx}(5x^4-6x^2+1)$$

$$= (1/8)\times(20x^3 - 12x) = (1/2)\times(5x^3 - 3x).$$

Ex. 2. *If* $x > 1$, *show that* $P_n(x) < P_{n+1}(x)$. **[Garhwal 2005; Ravishankar 2000]**

Sol. We are to prove $$P_n(x) < P_{n+1}(x). \qquad ...(1)$$

Since $x > 1$, from Rodrigues's formula, we find that $P_n(x) > 0$ for each value of n. We now use the mathematical induction to prove (1).

Now $x > 1 \Rightarrow 1 < x \Rightarrow P_0(x) < P_1(x)$ $[\because P_0(x) = 1$ and $P_1(x) = x]$

This implies that (1) is true for $n = 0$. Let (1) be true for $n - 1$. Then, we have

$$P_{n-1} < P_n \quad \text{so that} \quad P_{n-1}/P_n < 1. \qquad ...(2)$$

From recurrence relation I, $(2n + 1)xP_n = (n + 1)P_{n+1} + nP_{n-1}$

or $$\frac{(2n+1)x}{n+1} = \frac{P_{n+1}}{P_n} + \frac{n}{n+1}\frac{P_{n-1}}{P_n} \quad \text{or} \quad \frac{P_{n+1}}{P_n} = \frac{(2n+1)x}{n+1} - \frac{n}{n+1}\frac{P_{n-1}}{P_n}$$

or $$\frac{P_{n+1}}{P_n} > \frac{2n+1}{n+1} - \frac{n}{n+1}, \text{ using (2) and noting that } x > 1$$

$\therefore P_{n+1}/P_n > 1$, so that $P_n < P_{n+1}$ $[\because P_n > 0$ for each $n.]$

This shows that (1) is true for n whenever (1) is true for $n - 1$. Hence (1) is true for each n by induction.

Ex. 3. *Prove that* (*i*) $\int_{-1}^{1} P_n(x)\,dx = 2$, if $n = 0$ **[Guwahati 2007]**

(*ii*) $\int_{-1}^{1} P_n(x)\,dx = 0$, if $n \geq 1$.

Sol. (*i*) When $n = 0$, $P_n(x) = P_0(x) = 1$. Hence $\int_{-1}^{1} P_n(x)\,dx = \int_{-1}^{1} dx = 2.$

Part (ii) Using Rodrigues' formula, we have

$$\int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n \,.\, n!}\int_{-1}^{1} D^n(x^2-1)^n\,dx \text{, where } D^n \equiv d^n/dx^n$$

$$= \frac{1}{2^n\, n!}\left[D^{n-1}(x^2-1)^n\right]_{-1}^{1} = \frac{1}{2^n\, n!}\left[D^{n-1}\{(x-1)^n\,(x+1)^n\}\right]_{-1}^{1}$$

$$= \frac{1}{2^n\, n!}\left[D^{n-1}(x-1)^n \cdot (x+1)^n + {}^{n-1}C_1\, D^{n-2}(x-1)^n\, D(x+1)^n + \ldots + (x-1)^n \cdot D^{n-1}(x+1)^n\right]_{-1}^{1}$$

$[\because$ By Leibnitz Theorem, $D^n(uv) = D^n u \,.\, v + {}^nC_1 D^{n-1}\, u.\, Dv + \ldots + u \,.\, D^n v]$

$$= \frac{1}{2^n \,.\, n!}\left[n!\,(x-1)\,(x+1)^n + \ldots + n!\,(x+1)\,(x-1)^n\right]_{-1}^{1}$$

$$= 0 \qquad \left[\because D^n(ax+b)^m = a^n \frac{m!}{(m-n)!}(ax+b)^{m-n}\right]$$

Ex. 4. *If* $m > n - 1$ and n is a positive integer, prove that

$$\int_0^1 x^m P_n(x)\,dx = \frac{m(m-1)\,(m-2)\ldots(m-n+2)}{(m+n+1)\,(m+n-1)\ldots(m-n+3)}$$

Sol. Using Rodrigues' formula, we have

$$\int_0^1 x^m P_n(x)\,dx = \frac{1}{2^n\, n!}\int_0^1 x^m\, D^n(x^2-1)^n\,dx \text{, where } D^n \equiv d^n/dx^n$$

$$= \frac{1}{2^n\, n!}\left[\left\{x^m D^{n-1}(x^2-1)^n\right\}_0^1 - m\int_0^1 x^{m-1}\, D^{n-1}(x^2-1)^n\,dx\right] \qquad ...(1)$$

$$= \frac{(-1)^1\, m}{2^n\, n!}\int_0^1 x^{m-1}\, D^{n-1}(x^2-1)^n\,dx$$

$[\because$ the first term in (1) vanishes on using Leibnitz theorem as shown in Ex. 3.]

$$= \frac{(-1)^1 m}{2^n\, n\,!}\left[\left\{x^{m-1}\, D^{n-2}(x^2-1)^n\right\}_0^1 - (m-1)\int_0^1 x^{m-2}\, D^{n-2}(x^2-1)^n\, dx\right]$$

$$= \frac{(-1)^2 m(m-1)}{2^n\, n\,!}\int_0^1 x^{m-2}\, D^{n-2}(x^2-1)^n\, dx \qquad ...(2)$$

[$\because$ the first term is again zero as shown in Ex. 3.]

$$= \frac{(-1)^n\, m(m-1)\ldots(m-n+1)}{2^n\, n!}\int_0^1 x^{m-n}\,(x^2-1)^n\, dx\,, \text{ on continuing the similar steps } n-2 \text{ times more}$$

$$= \frac{(-1)^n\, m(m-1)\ldots(m-n+1)}{2^n\, n\,!}\int_0^1 x^{m-n}\,(-1)^n(1-x^2)^n\, dx$$

$$= \frac{m(m-1)\ldots(m-n+1)}{2^n\, n\,!}\int_0^1 (t^{1/2})^{m-n}\,(1-t)^n\,\frac{dt}{2t^{1/2}}\,, \text{ taking } x^2 = t \text{ so that } dx = \frac{dt}{2x} = \frac{dt}{2t^{1/2}}$$

$$= \frac{m(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}\int_0^1 t^{\frac{1}{2}(m-n-1)}\,(1-t)^n\, dt = \frac{m(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}\int_0^1 t^{\frac{1}{2}(m-n+1)-1}\,(1-t)^{(n+1)-1}\, dt$$

$$= \frac{m(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}\,\frac{\Gamma\left(\frac{m-n+1}{2}\right)\Gamma(n+1)}{\Gamma\left(\frac{m-n+1}{2}+n+1\right)} \qquad \left(\because \int_0^1 t^{p-1}(1-t)^{q-1}\, dt = B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\right)$$

$$= \frac{m(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}\,\frac{\Gamma\left(\frac{m-n+1}{2}\right).\, n\,!}{\Gamma\left(\frac{m+n+3}{2}\right)}$$

$$= \frac{m(m-1)\ldots(m-n+1)\,\Gamma\left(\frac{m-n+1}{2}\right)}{2^{n+1}\times\frac{m+n+1}{2}\cdot\frac{m+n-1}{2}\ldots\frac{m-n+1}{2}\,\Gamma\left(\frac{m-n+1}{2}\right)} \qquad [\because\ \Gamma(p+1) = p\Gamma(p)]$$

$$= \frac{m(m-1)\ldots(m-n+1)}{(m+n+1)\,(m+n-1)\ldots(m-n+1)}$$

Ex. 5. (*i*) If $m < n$, show that $\int_{-1}^1 x^m\, P_n(x)\, dx = 0$.

[Ranchi 2010, Ravishankar 2010, Kanpur 2011]

Deduce that $\int_{-1}^1 x^4 P_6(x)dx = 0$ **[Meerut 2006]**

(*ii*) Prove that $\int_{-1}^1 x^n\, P_n(x)\, dx = \frac{\Gamma(1/2)\,\Gamma(n+1)}{2^n\,\Gamma(n+3/2)} = \frac{2^{n+1}(n\,!)^2}{(2n+1)\,!}$. **[Kanpur 2006, 10]**

Sol. (*i*) Using Rodrigues' formula, we have

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{1}{2^n \cdot n!}\int_{-1}^{1} x^m\, D^n(x^2-1)^n\,dx \text{, where } D^n \equiv d^n/dx^n$$

$$= \frac{(-1)^2\, m\,(m-1)}{2^n \cdot n!}\int_{-1}^{1} x^{m-2}\, D^{n-2}(x^2-1)^n\,dx, \text{ doing upto equation (2) as in Ex. 4}$$

$$= \frac{(-1)^m\, m\,(m-1)....3\cdot 2\cdot 1}{2^n \cdot n!}\times\int_{-1}^{1} x^{m-m}\, D^{n-m}(x^2-1)^n\,dx \qquad ...(A)$$

(On continuing the similar steps $m-2$ times more and noting that $m < n$)

$$= \frac{(-1)^m\, m!}{2^n \cdot n!}\int_{-1}^{1}\frac{d}{dx}\left\{\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right\}dx$$

$$= \frac{(-1)^m\, m!}{2^n \cdot n!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right]_{-1}^{1} = \frac{(-1)^m\, m!}{2^n \cdot n!}\left[D^{n-m-1}\{(x-1)^n\cdot(x+1)^n\}\right]_{-1}^{1} = 0$$

[By using Leibnitiz theorem and simplifying as before]

Deduction : Taking $m = 4$ and $n = 6$ in the above result, we get the required result because $4 < 6$ satisfies condition $m < n$.

Part (ii). Here $m = n$. So proceeding as above upto (A), we obtain

$$\int_{-1}^{1} x^n P_n(x)\,dx = \frac{(-1)^n\, n!}{2^n\, n!}\int_{-1}^{1} x^{n-n}D^{n-n}(x^2-1)^n\,dx = \frac{(-1)^n}{2^n}\int_{-1}^{1}(x^2-1)^n\,dx = \frac{(-1)^n}{2^n}\int_{-1}^{1}(-1)^n(1-x^2)^n\,dx$$

$$= \frac{1}{2^n}\times 2\int_{0}^{1}(1-x^2)^n\,dx = \frac{1}{2^{n-1}}\int_{0}^{\pi/2}\cos^{2n+1}\theta\,d\theta, \text{ putting } \quad x=\sin\theta \quad \text{and} \quad dx=\cos\theta\,d\theta$$

$$= \frac{1}{2^{n-1}}\times\frac{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2n+3}{2}\right)}, \qquad \text{as} \qquad \int_{0}^{\pi/2}\cos^p\theta\sin^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$= \frac{1}{2^n}\times\frac{n!\sqrt{\pi}}{\frac{2n+1}{2}\cdot\frac{2n-1}{2}.....\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{n!}{2^n}\times\frac{2^{n+1}}{(2n+1)(2n-1)...3\cdot 1} \qquad (\because\ \Gamma(n+1) = n!)$$

$$= 2(n!)\times\frac{(2n)(2n-2)...4\cdot 2}{(2n+1)(2n)(2n-1)(2n-2)...4\cdot 3\cdot 2\cdot 1}$$

$$= 2(n!)\times\frac{(2\cdot n)\cdot[2\cdot(n-1)]...(2\cdot 2)\cdot(2\cdot 1)}{(2n+1)!} = 2(n!)\times\frac{2^n\, n!}{(2n+1)!} = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

Ex. 6. *Deduce from Rodrigue's formula* $\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{(-1)^n}{2^n\, n!}\int_{-1}^{1}(x^2-1)^n\, f^{(n)}(x)\,dx$.

Sol. Using Rodrigue's formula, we have

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{1}{2^n\,n!}\int_{-1}^{1} f(x)\,D^n(x^2-1)^n, \text{ where } D^n \equiv d^n/dx^n$$

$$= \frac{1}{2^n\,n!}\left[\left\{f(x)\,D^{n-1}(x^2-1)^n\right\}_{-1}^{1} - \int_{-1}^{1} f'(x)\,D^{n-1}(x^2-1)^n\,dx\right] \quad ...(1)$$

(On integration by parts)

$$= \frac{(-1)^1}{2^n\,n!}\int_{-1}^{1} f'(x)\,D^{n-1}(x^2-1)^n\,dx$$

(the first term in (1) vanishes on using Leibnitz theorem as explained in Ex. 3)

$$= \frac{(-1)^1}{2^n\,n!}\left[\left\{f'(x)\,D^{n-2}(x^2-1)^n\right\}_{-1}^{1} - \int_{-1}^{1} f''(x)\,D^{n-2}(x^2-1)^n\,dx\right], \text{ on integration by parts again}$$

$$= \frac{(-1)^2}{2^n\,n!}\int_{-1}^{1} f''(x)\,D^{n-2}(x^2-1)^n\,dx \qquad (\because \text{ the first term vanishes as shown in Ex. 3})$$

$$= \frac{(-1)^n}{2^n\,n!}\int_{-1}^{1} f^{(n)}(x)\,D^{n-n}(x^2-1)^n\,dx, \qquad \text{on continuing the similar steps } n-2 \text{ times more}$$

$$= \frac{(-1)^n}{2^n\,n!}\int_{-1}^{1} (x^2-1)^n\,f^{(n)}(x)\,dx.$$

Ex.7. *Using Rodrigue's formula, show that $P_n(x)$ satisfies*

$$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n(x)\right\} + n(n+1)P_n(x) = 0 \qquad \textbf{[CDLU 2004]}$$

Sol. Rodrigue's formula is $$P_n(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n \quad ... (1)$$

Let $$y = (x^2-1)^n \quad ... (2)$$

Differentiating (2) w.r.t. 'x', $y_1 = 2nx(x^2-1)^{n-1}$ so that $(x^2-1)y_1 = 2nx\,(x^2-1)^n$

or $$(x^2-1)\,y_1 = 2nxy, \text{ using (2)} \quad ... (3)$$

Differentiating (3) w.r.t. 'x', $$(x^2-1)y_2 + 2xy_1 = 2n\,(xy_1 + y)$$

or $$(x^2-1)\,y_2 + 2(1-n)xy_1 - 2ny = 0 \quad ... (4)$$

Differentiating both sides of (4) w.r.t. 'x' n times, we have

$$D^n\{(x^2-1)y_2\} + 2(1-n)\,D^n(xy_1) - 2nD^n(y) = 0, \text{ where } D^n \equiv d^n/dx^n \quad ... (5)$$

Using Leibnitz' theorem, (5) yields

$$y_{n+2}\,(x^2-1) + {}^nC_1\,y_{n+1}(2x) + {}^nC_2\,y_n \cdot 2 + 2(1-n)\,(y_{n+1}x + {}^nC_1\,y_n \cdot 1) - 2ny_n = 0$$

or $$(x^2-1)\,y_{n+2} + 2x\,y_{n+1} + \{n(n-1) + 2n(1-n) - 2n\}\,y_n = 0$$

or $$(1-x^2)\,y_{n+2} - 2x\,y_{n+1} + n(n+1)\,y_n = 0$$

or $$\frac{d}{dx}\left\{(1-x^2)y_{n+1}\right\} + n(n+1)y_n = 0 \qquad \text{or} \qquad \frac{d}{dx}\left\{(1-x^2)\times\left(\frac{dy_n}{dx}\right)\right\} + n(n+1)y_n = 0$$

or $$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}\left(\frac{d^n}{dx^n}(x^2-1)^n\right)\right\} + n(n+1)\frac{d^n}{dx^n}(x^2-1)^n = 0, \text{ using (2)}$$

Dividing by $2^n n!$ and using (1), we get $$\frac{d}{dx}\{(1-x^2)\frac{d}{dx}P_n(x)\} + n(n+1)P_n(x) = 0$$

Ex. 8. Using Rodrigue's formula, prove that

(*i*) $(1-t)^n P_n\left(\frac{1+t}{1-t}\right) = \sum_{k=0}^{\infty} \left({}^n c_k\right)^2 t^k$ **(Ravishankar 1993)**

(*ii*) $P_n(\cosh u) \geq 1$ (*ii*) $|P_n(x)| \geq 1$, if $|x| \geq 1$. **(Bilaspur 1994)**

Sol. (*i*) Rodrigues formula is given by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ ... (1)

Let $D \equiv d/dx$. Then (1) can be re-written as

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n = \frac{1}{2^n n!} D^n \left\{(x+1)^n (x-1)^n\right\}$$

or $P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} {}^n C_k D^{n-k}(x+1)^n D^k (x-1)^n$, by Leibnitz's rule ... (2)

From Differential calcules, we know that $D^n (ax+b)^m = \frac{m!}{(m-n)!}(ax+b)^{m-1} a^n$ ... (3)

Using (3), $D^{n-k}(x+1)^n = \frac{n!}{[n-(n-k)]!}(x+1)^{n-(n-k)} = \frac{n!}{k!}(x+1)^k$ ... (4)

and $D^k (x-1)^n = \frac{n!}{(n-k)!}(x-1)^{n-k}$ ... (5)

Using (4) and (5), (3) reduces to

$$P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} {}^n C_k \frac{n!}{k!}(x+1)^k \frac{n!}{(n-k)!}(x-1)^{n-k} \quad \text{or} \quad P_n(x) = \sum_{k=0}^{\infty} ({}^n C_k)^2 \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k \quad \text{... (6)}$$

Put $x = \frac{1+t}{1-t}$ so that $x-1 = \frac{2t}{1-t}$ and $x+1 = \frac{2}{1-t}$ ... (7)

Using (7), (6) reduces to

$$P_n\left(\frac{1+t}{1-t}\right) = \sum_{k=0}^{\infty} ({}^n C_k)^2 \left(\frac{t}{1-t}\right)^{n-k} \frac{1}{(1-t)^k} \quad \text{or} \quad (1-t)^n P_n\left(\frac{1+t}{1-t}\right) = \sum_{k=0}^{\infty} ({}^n C_k)^2 t^k \quad \text{... (8)}$$

(*ii*) Let $t = \tanh^2 \frac{u}{2}$, then $t \geq 0$ and $\frac{1+t}{1-t} = \cosh u \geq 1$.

Hence (8) reduces to $P_n(\cosh u) = (1+t)^{-n} \sum_{k=0}^{n} ({}^n C_k)^2 t^k$ ... (9)

Since the first term on R.H.S. of (9) is 1 and all the subsequent terms are positive, hence (9) yields $P_n(\cosh u) \geq 1$, as required

(*iii*) Left as an exercise.

Ex. 9. *Show that all the roots of* $P_n(x) = 0$ *are real and lie between* -1 *and* 1. **(Bilaspur 1998)**

Sol. Let $f(x) = (x^2-1)^n = (x-1)^n (x+1)^n$. ...(1)

From (1), we notice that $f(x)$ vanishes for $x = 1$ and $x = -1$, hence by *Rolle's theorem, $f'(x)$ must vanish at least once for some value α of x lying between –1 and 1. Now, from (1), we have

$$f'(x) = n(x-1)^{n-1}(x+1)^n + n(x-1)^n (x+1)^{n-1}. \qquad ...(2)$$

(2) shows that $f'(x)$ vanishes at $x = 1$ and $x = -1$. But we have just proved that $f'(x)$ vanishes at $x = \alpha$, where $-1 < \alpha < 1$. Hence applying Rolle's theorem to function f' two times, we conclude that $f''(x)$ must vanish at $x = \beta$ such that $-1 < \beta < \alpha$ and also at $x = \gamma$ such that $\alpha < \gamma < 1$. Proceeding likewise we conclude that $f^{(n)}(x) = 0$ must have n real roots lying between –1 and 1.

Using (1), Rodrigue's formula gives

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n \cdot n!}\cdot f^{(n)}(n). \qquad ...(3)$$

$$\therefore \qquad f^{(n)}(x) = 0 \qquad \Rightarrow \qquad P_n(x) = 0, \quad \text{by (3).} \qquad ...(4)$$

Since $f^n(x) = 0$ has n real roots lying between –1 and 1, so (4) shows that $P_n(x) = 0$ has n real roots between –1 and 1.

Remarks. The roots of $P_n(x) = 0$ are also known as *zeros of $P_n(x)$.*

Ex. 10. *Prove that all the roots of $P_n(x)$ are distinct.*

Sol. If possible, let the roots of $P_n(x) = 0$ be not all different. Then at least two roots must be equal. Let α be the repeated root, then from the theory of equations, we have

$$P_n(\alpha) = 0 \qquad \text{and} \qquad P'_n(\alpha) = 0. \qquad ...(1)$$

Since $P_n(x)$ satisfies Legendre's equation, $(1-x^2)P''_n - 2xP'_n + n(n+1)P_n = 0.$ ...(2)

Differentiating r times and using Leibnitz theorem, (2) gives

$$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) + {}^rC_1 \times (-2x) \times \frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_2 \times (-2) \times \frac{d^r}{dx^r}P_n(x)$$
$$-2\left[x\frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_1 \times 1 \times \frac{d^r}{dx^r}P_n(x)\right] + n(n+1)\frac{d^r}{dx^r}P_n(x) = 0$$

or $$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x({}^rC_1 + 1)\frac{d^{r+1}}{dx^{r+1}}P_n(x) - \{2 \times {}^rC_2 + 2 \times {}^rC_1 - n(n+1)\}\frac{d^r}{dx^r}P_n(x) = 0$$

....(3)

Putting $r = 0$ and $x = \alpha$ in (3) and using (1), we get

$$(1-\alpha^2)P''_n(\alpha) - 0 - 0 = 0 \qquad \text{or} \qquad P''_n(\alpha) = 0. \qquad ...(4)$$

Next, putting $r = 1$ and $x = \alpha$ in (3) and using (1) and (4), we get

$$(1-\alpha^2)P'''_r(\alpha) - 0 - 0 = 0 \qquad \text{or} \qquad P'''_n(\alpha) = 0. \qquad ...(5)$$

Putting $r = 2, 3, ..., n-3, n-2$ in (3) and doing as above stepwise, we finally arrive at

$$P_n^{(n)}(\alpha) = 0 \qquad \textit{i.e.} \qquad \left[\frac{d^n}{dx^n}P_n(x)\right]_{x=\alpha} = 0. \qquad ...(6)$$

But $$P_n(x) = \frac{1 \cdot 3 \cdot 5 ...(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + ...\right]$$

$$\therefore \qquad \frac{d^n}{dx^n}P_n(x) = \frac{1 \cdot 3 \cdot 5 ...(2n-1)}{n!} \times n! \qquad \Rightarrow \qquad \left[\frac{d^n}{dx^n}P_n(x)\right]_{x=\alpha} \neq 0. \qquad ...(7)$$

***Rolle's theorem:** If $f(x)$ vanishes for $x = a$ and $x = b$, then $f'(x)$ vanishes at least once for some value of x between a and b.

Since (6) and (7) are contradictory results, it follows that our assumption about not distinct roots of $P_n(x)$ is absurd. Hence all the roots of $P_n(x) = 0$ must be distinct.

EXERCISE 9 (C)

1. Define Legendre's differential equation and show that $y = \frac{d^n}{dx^n}(x^2 - 1)^n$ satisfies it.

2. Using Rodrigue's formula, prove that $P'_{n+1} - P'_{n-1} = (2n + 1)P_n$.

3. Show that $P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2 - 1)^n$ is a solution of the Legendre's equation $(1 - x^2)y_2 - 2xy_1 + n(n + 1)y = 0$, where n is +ve integer. Hence or otherwise show that

(*i*) $xP'_n(x) - P'_{n-1}(x) = nP_n(x)$. (*ii*) $(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x)$.

4. Use Rodrigue's formula to derive the orthogonal property for $P_n(x)$ and show that

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

5. Prove that the function $y = \frac{d^n}{dx^n}(x^2 - 1)^n$ satisfies the Legendre's differential equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$. Hence obtain Rodrigue's formula for Legendre–polynomial $P_n(x)$.

Using this formula prove that $\int_{-1}^{1} x^m P_n(x)\, dx = 0$ for $m < n$.

9.16. Legendre series for f(x) when f(x) is a polynomial.

Theorem (a). If $f(x)$ is a polynomial of degree n, then $f(x) = \sum_{r=0}^{n} C_r P_r(x)$, ...(*i*)

where $c_r = (r + 1/2)\int_{-1}^{1} f(x) P_r(x)\, dx.$ ...(*ii*)

(b) If $f(x)$ is even (or odd), only those C_r with even (or odd) suffixes are non–zero.

Proof (a). Since $f(x)$ is a polynomial of degree n, we write

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0. \quad ...(i)$$

Again, we know that $P_n(x)$ is a polynomial of degree n of the form

$$P_n(x) = k_n x^n + k_{n-1}x^{n-1} + ... + k_1 x + k_0. \quad ...(1)$$

Consider $f(x) - (a_n/k_n) P_n(x)$. Two cases may arise :

Case (i). $f(x) - (a_n/k_n)P_n(x) = 0$ so that $f(x) = (a_n/k_n) P_n(x)$, which proves the required result (*i*).

Case (ii). $f(x) - (a_n/k_n) P(x) = g_{n-1}(x)$, $g_{n-1}(x)$ being a polynomial of degree $n - 1$. Taking $c_n = a_n/k_n$, we may write $f(x) = c_n P_n(x) + g_{n-1}(x)$. ...(3)

Taking $g_{n-1}(x)$ in place of $f(x)$ and proceeding as above, we have

$$g_{n-1}(x) = c_{n-1}P_{n-1}(x) + g_{n-2}(x). \quad ...(4)$$

Making use of (4), (3) may be re–written as $f(x) = c_n P(x) + c_{n-1}P_{n-1}(x) + g_{n-2}(x)$. ...(5)

Making use of similar method for $g_{n-2}(x)$ etc., we finally obtain (noting that $P_0(x) = 1$).

$$f(x) = c_n P_n(x) + c_{n-1}P_{n-1}(x) + ... + c_1 P_1(x) + c_0 P_0(x)$$

or $$f(x) = \sum_{r=0}^{\infty} c_r P_r(x). \quad ...(6)$$

Since $\sum_{r=0}^{\infty} c_r P_r(x) = \sum_{s=0}^{\infty} c_s P_s(x)$, (6) gives $f(x) = \sum_{s=0}^{\infty} c_s P_s(x).$ ...(7)

Multiplying both sides of (7) by $P_r(x)$ and then integrating w.r.t. 'x' from –1 to 1, we have

$$\int_{-1}^{1} f(x)\, P_r(x)\, dx = \sum_{s=0}^{n} \left\{ c_s \int_{-1}^{1} P_s(x)\, P_r(x)\, dx \right\}. \qquad ...(8)$$

But $$\int_{-1}^{1} P_s(x)\, P_r(x)\, dx = \begin{cases} 0, \text{ if } r \neq s \\ 2/(2r+1), \text{ if } r = s \end{cases} \qquad ...(9)$$

Using (9), (8) reduces to $$\int_{-1}^{1} f(x)\, P_r(x)\, dx = c_r \times \frac{2}{2r+1}$$

so that $$c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} f(x)\, P_r(x)\, dx. \qquad ...(10)$$

Part (b). We now prove that if $f(x)$ is even, only those c_r with even suffixes are non–zero. We know that $P_r(x)$ is even when r is even, and odd when r is odd. Thus $f(x)\, P_r(x)$ is even when r is even and odd when r is odd. But, it is known that $\int_{-1}^{1} F(x)\, dx = 0$, if $F(x)$ is odd. Hence (10) show that $c_r = 0$ if r is odd. Thus if r is even, only those c_r with even suffixes are non–zero.

Similarly, we can prove that if $f(x)$ is odd, only those c_r with odd suffixes are non–zero.

9.17. Solved examples based on Art 9.16

Ex. 1. *If $f(x)$ is a polynomial of degree less than l, prove that* $\int_{-1}^{1} f(x)\, P_l(x)\, dx = 0.$

Sol. Let $f(x)$ be a polynomial of degree n such that $n < l$. Then we have (Do upto equation (6) as explained in Art 9.16) $$f(x) = \sum_{r=0}^{n} c_r\, P_r(x)$$

Multiplying both sides by $P_l(x)$ and then integrating from –1 to 1, we have

$$\int_{-1}^{1} f(x)\, P_l(x)\, dx = \sum_{r=0}^{n} \left\{ c_r \int_{-1}^{1} P_r(x)\, P_l(x)\, dx \right\}. \qquad ...(A)$$

Since $n < l$, for each $r = 0, 1, 2, ..., n$ we find that $r \neq l$. But we know that

$$\int_{-1}^{1} P_r(x)\, P_l(x) dx = 0 \quad \text{if} \quad r \neq l.$$

Hence (A) reduces to $$\int_{-1}^{1} f(x)\, P_l(x) dx = 0.$$

Ex. 2. *Expand $f(x) = x^2$ in a series of the form $\Sigma c_r\, P_r(x)$*

Sol. Since x^2 is a polynomial of degree two, from Legendre series, we have

$$x^2 = \sum_{r=0}^{2} c_r\, P_r(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x), \qquad ...(1)$$

where $$c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} x^2\, P_r(x)\, dx. \qquad ...(2)$$

But $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. ...(3)

Putting $r = 0, 1, 2$ successively in (2) and using (3), we have

$$c_0 = \tfrac{1}{2} \int_{-1}^{1} x^2\, dx = \tfrac{1}{2} \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{3},\ c_1 = \frac{3}{2} \int_{-1}^{1} x^3\, dx = 0,\ c_2 = \frac{1}{2} \times \frac{5}{2} \int_{-1}^{1} x^2 (3x^2 - 1)\, dx = \tfrac{5}{4} \left[3 \times \frac{x^5}{5} - \frac{x^3}{3} \right]_{-1}^{1} = \frac{2}{3}.$$

With the above values of c_0, c_1 and c_2, (1) gives $x^3 = (1/3) \times P_0(x) + (2/3) \times P_2(x)$.

Ex. 3. *Expand* $x^4 - 3x^2 + x$ *in a series of form* $\Sigma c_r P_r(x)$.

Ans. $x^4 - 3x^2 + x = -(4/5) \times P_0(x) + P_1(x) - (10/7) \times P_2(x) + (8/35) \times P_4(x)$.

9.18. Expansion of function f(x) in a series of Legendre Polynomials.

Supposing the expansion of $f(x)$ in a series of Legendre polynomials to be possible, we write

$$f(x) = \sum_{r=0}^{\infty} c_r P_r(x). \qquad ...(1)$$

But (1) may also be expressed as $$f(x) = \sum_{s=0}^{\infty} c_s P_s(x) \qquad ...(2)$$

where $c_0, c_1, c_2, ... c_r, ...$ are constants. Multiplying both sides of (2) by $P_r(x)$ and then integrating it w.r.t. 'x' from –1 to 1, we have

$$\int_{-1}^{1} f(x) P_r(x)\, dx = \sum_{s=0}^{\infty} \left\{ c_s \int_{-1}^{1} P_r(x) P_s(x)\, dx \right\}. \qquad ...(3)$$

But $$\int_{-1}^{1} P_r(x) P_s(x)\, dx = \begin{cases} 0, \text{ if } r \neq s \\ 2/(2r+1), \text{ if } r = s \end{cases} \qquad ...(4)$$

Using (4), (3) reduces to $\int_{-1}^{1} f(x) P_r(x)\, dx = c_r \times \dfrac{2}{2r+1} \Rightarrow c_r = \left(r + \dfrac{1}{2}\right) \int_{-1}^{1} f(x) P_r(x)\, dx$.

Remark 1. Fourier–Legendre expansion of f(x). If $f(x)$ be defined from $x = -1$ to $x = 1$, then

$$f(x) = \sum_{r=0}^{\infty} c_r P_r(x), \qquad \text{where} \qquad c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} f(x) P_r(x)\, dx.$$

Remark 2. Suppose that function f is continuous and has continuous derivatives in [–1, 1], then we prove that series (1) converges uniformly in [–1, 1] and the series expansion (1) is unique.

Ex. 1. *Expand* $f(x)$ *in the form* $\sum_{r=0}^{\infty} c_r P_r(x)$, *where* $f(x) = \begin{cases} 0, \text{ where } -1 < x < 0 \\ 1, \text{ where } 0 < x < 1. \end{cases}$

Sol. Given that $$f(x) = \begin{cases} 0, \text{ if } -1 < x < 0 \\ 1, \text{ if } 0 < x < 1 \end{cases} \qquad ... (1)$$

We know that $$f(x) = \sum_{r=0}^{\infty} c_r P_r(x), \text{ where} \qquad ...(2)$$

$$c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} f(x) P_r(x)\, dx = \frac{2r+1}{2} \left[\int_{-1}^{0} f(x) P_r(x)\, dx + \int_{0}^{1} f(x) P_r(x)\, dx \right]$$

$$\therefore \qquad c_r = \frac{2r+1}{2} \int_{0}^{1} P_r(x)\, dx, \text{ by (1)} \qquad ...(3)$$

Putting $r = 0, 1, 2, ...$ successively in (3), we get

$$c_0 = \frac{1}{2} \int_0^1 P_0(x)\, dx = \frac{1}{2} \int_0^1 (1)\, dx = \frac{1}{2}, \qquad c_1 = \frac{3}{2} \int_0^1 P_1(x)\, dx = \frac{3}{2} \int_0^1 x\, dx = \frac{3}{4},$$

$$c_2 = \frac{5}{2}\int_0^1 P_2(x)\,dx = \int_0^1 \frac{3x^2-1}{2}\,dx = 0, \qquad c_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \frac{7}{2}\int_0^1 \frac{5x^2-2x}{2}\,dx = -\frac{7}{16}$$

and so on. Using these values in (2), we get

$f(x) = (1/2) \times P_0(x) + (3/4) \times P_1(x) - (7/16) \times P_3(x) + ... + c_r P_r(x) + ...$, where c_r is given by (3)

EXERCISE 9 (D)

1. Expand $f(x)$ in a series of Legendre polynomials, if

(*i*) $f(x) = x, 0 < x < 1; f(x) = 0, -1 < x < 0.$ (*ii*) $f(x) = 0, -1 \le x < \alpha; f(x) = 1, \alpha < x \le 1.$

(*iii*) $f(x) = \frac{1}{2}, 0 < x < 1; f(x) = -\frac{1}{2}, -1 < x < 0.$ (*iv*) $f(x) = 2x + 1, 0 < x, \le 1; f(x) = 0, -1 \le x < 0.$

(*v*) Given $f(x) = |x|$ for $-1 \le x \le 1$.

2. If $f(x) = \sum_{r=0}^{\infty} C_r\, P_r(x)$, obtain Parseval's identity $\int_{-1}^{1} [f(x)]^2\,dx = \sum_{r=0}^{\infty} \frac{C_r^2}{2r+1}$

and illustrate it by making use of $f(x) = x^4 - 3x^2 + x$.

3. Obtain the first three terms in the expansion of the following function f in terms of Legendre's polynomial: $f(x) = \begin{cases} 0, \text{ if } -1 < x < 0 \\ x, \text{ if } 0 < x < 1. \end{cases}$

4 (a). Express $P(x) = x^4 + 2x^3 + 5x^2 - x - 2$ in terms of Legendre's polynomials.

(b). Prove that $x^4 + 3x^3 - x^2 + 5x - 2 = -(31/15) \times P_0(x) + (34/5) \times P_1(x) - (2/21) \times P_2(x) - (6/5) \times P_3(x) + (8/35) \times P_4(x)$.

5. Prove that, for all x, (*a*) $x^2 = (1/3) \times P_0(x) + (2/3) \times P_2(x)$.

(*b*) $x^3 = (3/5) \times P_1(x) + (2/5) \times P_3(x)$.

(*c*) $x^4 = (1/5) \times P_0(x) + (4/7) \times P_2(x) + (8/35) \times P_4(x)$.

(*d*) $x^5 = (3/7) \times P_1(x) + (4/9) \times P_3(x) + (8/63) \times P_5(x)$.

6. Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ in terms of Legendre's polynomials. **[Kanpur 2005]**

7. Express $f(x) = x^4 + 2x^3 - 2x^2 - x - 3$ in terms of Legendre's polynomials. **[Kanpur 2011]**

ANSWERS

1. (*i*) $f(x) = (1/4) + (1/2) \times P_1(x) + (5/16) \times P_2(x) + ... + C_r P_r(x) + ...$ where $C_r = \left(r + \frac{1}{2}\right)\int_0^1 x\, P_r(x)\,dx$.

(*ii*) $f(x) = \frac{1}{2}(1 - \alpha) - \sum_{r=0}^{\infty} [P_{r+1}(\alpha) - P_{r-1}(\alpha)]\, P_r(x)$.

(*iii*) $C_r = 0$ if r is even, and $C_r = (-1)^{r-1/2} \frac{\left(r + \frac{1}{2}\right)(r-1)!}{2^r \{(r+1)/2\}!\,\{(r-1)/2\}!}$, if r is odd

(*iv*) $f(x) = P_0(x) + (7/4) \times P_1(x) + (5/8) \times P_2(x) - (7/16) P_2(x) + ...$

(*v*) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(4n+1)(2n-2)!}{2^{2n}(n+1)!\,(n-1)!} P_{2n}(x)$.

3. $f(x) = -(1/4) \times P_0(x) + (1/6) \times P_1(x) + (1/16) \times P_2(x)$.

4 (a). $P(x) = (8/35) \times P_4(x) + (4/5) \times P_3(x) + (82/21) \times P_2(x) + (1/5) \times P_1(x) - (2/15) \times P_0(x)$.

6. $f(x) = (8/5) \times P_3(x) + (32/5) \times P_2(x) + 7\, P_1(x) + 4P_0(x)$.

9.19. Even and odd fuctions. An important observation.

(*i*) **Even function.** f is called an even function of x if $f(-x) = f(x)$. Suppose an even function f can be sum of a series of functions $f_1, f_2, ..., f_n, ...$ such that

$$f(x) = f_1(x) + f_2(x) + ... + f_n(x) + ... \qquad ....(1)$$

Then all functions $f_1, f_2, ..., f_n$ must be even functions of x because if any of the above function is not even, then $f(-x) \neq f(x)$ and so f ceases to be an even function.

(*ii*) **Odd function.** f is called an odd functions of x if $f(-x) = -f(x)$. Suppose an odd function f can be sum of a series as (1).

Then all the functions $f_1, f_2, ..., f_n$ must be odd functions of x because if any of the above function is not odd, then $f(-x) \neq -f(x)$ and so f ceases to be an odd function.

9.20. Expansion of x^n in Legendre's polynomials. (Bilaspur 1994, 96)

Let $$x^n = C_n P_n(x) + C_{n-2} P_{n-2}(x) + C_{n-4} P_{n-4}(x) + ... + C_r P_r(x) + ..., \quad ...(1)$$

where $P_n(x)$, $P_{n-2}(x)$, ... are all even or odd functions of x according as x^n is an even or odd function (Refer Art 9.19 for more details). Again $P_{n-1}(x)$, $P_{n-3}(x)$, ... cannot occur in the proposed series (1) because they are odd or even functions of x according as x^n is even or odd function x respectively. Since $P_n(x)$ contains terms of degree n and lower, hence the expansion (1) cannot contain any P with suffix higher than n.

Multiplying both sides of (1) by $P_m(x)$ and integrating between the limits –1 and 1, we have

$$\int_{-1}^{1} x^n P_m(x)\,dx = C_m \int_{-1}^{1} [P_m(x)]^2\,dx, \quad \text{as} \quad \int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2m+1), \text{ if } m = n \end{cases}$$

or $$\int_{-1}^{1} x^n P_m(x)\,dx = C_m \cdot \frac{2}{2m+1} \quad \text{or} \quad C_m = \frac{(2m+1)}{2} \int_{-1}^{1} x^n P_m(x)\,dx \quad ...(2)$$

$$(2) \Rightarrow C_m = \frac{(2m+1)}{2} \int_{-1}^{1} x^n \left[\frac{1}{2^m\, m!} \frac{d^m}{dx^m} (x^2-1)^m \right] dx, \text{ by Rodrigue's formula}$$

$$= \frac{(2m+1)}{2^{m+1}\, m!} \int_{-1}^{1} x^n \left[\frac{d^m}{dx^m} (x^2-1)^m \right] dx = \frac{(2m+1)}{2^{m+1}\, m!} \left[\left\{ x^n \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m \right\}_{-1}^{1} - \int_{-1}^{1} n\, x^{n-1} \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m\,dx \right]$$

(Integrating by parts taking x^n as first function)

$$= (-1) \frac{(2m+1)n}{2^{m+1}\, m!} \int_{-1}^{1} x^{n-1} \left\{ \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m \right\} dx = (-1)^2 \frac{(2m+1)}{2^{m+1}\, m!} n(n-1) \int_{-1}^{1} x^{n-1} \left\{ \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m \right\} dx$$

[Again Integrating by parts taking x^{n-1} as first function and simplifying as before]

$$= (-1)^m \frac{(2m+1)}{2^{m+1}\, m!} n(n-1) \ldots (n-m+1) \int_{-1}^{1} x^{n-m} (x^2-1)^m\,dx$$

[Repeating the above process of integration by parts $m-2$ times more]

$$= (-1)^m \frac{(2m+1)}{2^{m+1}\, m!} n(n-1) \ldots (n-m+1) (-1)^m \int_{-1}^{1} x^{n-m} (1-x^2)^m\,dx$$

$$\therefore \quad C_m = \frac{(2m+1)}{2^{m+1}\, m!} n(n-1)(n-2) \ldots (n-m+1) \int_{-1}^{1} x^{n-m} (1-x^2)^m\,dx. \quad ...(3)$$

In our discussion m can be one of integers $n, n-2, n-4, ...$ only and hence $(n-m)$ can be one of the integers 0, 2, 4, 6, ... etc only. Hence $x^{n-m}(1-x^2)^m$ is an even function of x.

$$\therefore \quad \int_{-1}^{1} x^{n-m} (1-x^2)^m\,dx = 2 \int_{0}^{1} x^{n-m} (1-x^2)^m\,dx. \quad ...(4)$$

Using (4), (3) reduces to

$$C_m = \frac{(2m+1)}{2^{m+1}\,m!}\cdot\frac{n!}{(n-m)!}\cdot 2\int_0^1 x^{n-m}\,(1-x^2)^m\,dx = \frac{(2m+1)}{2^{m+1}\,m!}\cdot\frac{n!}{(n-m)!}\int_0^1 x^{n-m-1}\,(1-x^2)^m\cdot 2x\,dx$$

$$= \frac{(2m+1)}{2^{m+1}\,m!}\cdot\frac{n!}{(n-m)!}\cdot\int_0^1 t^{(n-m-1)/2}\,(1-t)^m\,dt \qquad [\text{Putting } x^2 = t \text{ and } 2x\,dx = dt]$$

$$= \frac{(2m+1)}{2^{m+1}\,m!}\cdot\frac{n!}{(n-m)!}\cdot\int_0^1 t^{(n-m+1)/2-1}\,(1-t)^{(m+1)-1}\cdot dt$$

$$= \frac{(2m+1)}{2^{m+1}\,m!}\cdot\frac{n!}{(n-m)!}\cdot\frac{\Gamma[(n-m+1)/2]\,\Gamma(m)}{\Gamma[(n-m+1)/2+m]} \quad \left[\because B(p,q) = \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\right]$$

$$= \frac{(2m+1)\,n!}{2^{m+1}\,(n-m)!}\times\frac{\Gamma\{(n-m+1)/2\}}{\frac{n+m+1}{2}\cdot\frac{n+m-1}{2}\cdot\frac{n+m-3}{2}\ldots\frac{n-m+1}{2}\,\Gamma\left(\frac{n-m+1}{2}\right)}$$

$$= \frac{(2m+1)}{(n-m)!}\cdot\frac{n(n-1)\,(n-2)\ldots(n-m+2)\,(n-m+1)\,(n-m)!}{(n+m+1)\,(n+m-1)\ldots(n-m+3)\,(n-m+1)}.$$

Thus, $$C_m = \frac{n(n-1)\,(n-2)\ldots(n-m+2)}{(n+m+1)\,(n+m-1)\ldots(n-m+3)}(2m+1). \qquad \ldots(5)$$

Putting $m = n, (n-2), (n-4), \ldots$ in (5), we obtain

$$C_n = \frac{n(n-1)\,(n-2)\ldots 3\cdot 2}{(2n+1)\,(2n-1)\ldots 5\cdot 3}(2n+1) = \frac{n!}{3\cdot 5\cdot(2n-1)\,(2n+1)}(2n+1),$$

$$C_{n-2} = \frac{n(n-1)\ldots 5\cdot 4}{(2n-1)\,(2n-3)\ldots 7\cdot 5}(2n-3) = \frac{n!}{3\cdot 5\cdot 7\cdot(2n-1)\,(2n+1)}\,\frac{(2n+1)}{2}(2n-3),$$

$$C_{n-4} = \frac{n(n-1)\ldots 7\cdot 6}{(2n-3)\,(2n-5)\ldots 9\cdot 7}(2n-7) = \frac{6\cdot 7\ldots(n-1)n}{7\cdot 9\ldots(2n-5)\,(2n-3)}(2n-7)$$

$$= \frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\ldots(n+1)n}{3\cdot 5\cdot 7\ldots(2n-1)\,(2n+1)}\cdot\frac{(2n+1)\,(2n-1)}{2\,.\,4}(2n-7) = \frac{n!}{3\cdot 5\cdot 7\ldots(2n+1)}\cdot\frac{(2n+1)\,(2n-1)}{2\,.\,4}(2n-7)$$

and so on. Putting these values in (1), we have

$$x^n = \frac{n!}{3\cdot 5\ldots(2n+1)}\left[(2n+1)\,P_n(x) + (2n-3)\cdot\frac{(2n+1)}{2}\,P_{n-2}(x) + (2n-7)\frac{(2n+1)(2n-1)}{2\cdot 4}P_{n-4}(x)\right.$$

$$\left. + \ldots + \frac{1}{(n+1)}P_0(x)\right], \text{ if } n \text{ is even} \qquad \ldots 6(A)$$

and $$x^n = \frac{n!}{3\cdot 5\ldots(2n+1)}\left[(2n+1)\,P_n(x) + (2n-3)\cdot\frac{(2n+1)}{2}P_{n-2}(x)\right.$$

$$\left. + (2n-7)\frac{(2n+1)\,(2n-1)}{2\cdot 4}P_{n-4}(x) + \ldots + \frac{3P_1(x)}{(n+2)}\right], \text{ if } n \text{ is odd} \qquad \ldots 6\,(B)$$

To compute C_0 and C_1 it is convenient to use (2) as shown below. From (2), we have

$$C_0 = \frac{1}{2}\int_{-1}^{1} x^n\, P_0(x)\, dx = \frac{1}{2}\int_{-1}^{1} x^n\, dx = \frac{1}{2}\cdot 2\int_0^1 x^n\, dx = \left[\frac{n^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}.$$

[using the facts that $P_0(x) = 1$ and n is even for C_0]

and $$C_1 = \frac{3}{2}\int_{-1}^{1} x^n\, P_1(x)\, dx = \frac{2}{3}\int_{-1}^{1} x^n \cdot x\, dx, \text{ as } P_1(x) = x$$

$$= \frac{3}{2}\cdot 2\int_0^1 x^{n+1}\, dx, \text{ as } n \text{ is odd so } (n+1) \text{ is even for } C_1$$

$$= 3\left[\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{3}{n+2}.$$

Remark. The result of this article can also put in compact form

$$x^n = \frac{n!}{2^n}\sum_{k=0}^{[n/2]} \frac{(2n-4k+1)\, P_{n-2k}(x)}{k!\,(3/2)_{n-k}}, \qquad \text{where} \qquad [n/2] = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

and the symbol $(\alpha)_n$ is defined as below $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)$.

Corollary. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$, where $a_0, a_1, \ldots, a_n, \ldots$ are constants. Then $f(x)$ can be expanded in Legendre's polynomials in the form

$$f(x) = \sum_{n=0}^{\infty} C_nP_n(x), \quad \text{where}$$

$$C_n = \frac{n!}{3\cdot 5\ldots(2n+1)}\left[a_n + \frac{(n+1)(n+2)}{2(2n+3)}a_{n+2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}a_{n+4} + \ldots\right]$$

Proof. Given $$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + a_{n+1}x^{n+1} + \ldots \qquad \ldots(7)$$

Assume that $$f(x) = \sum_{n=0}^{\infty} C_nP_n(x). \qquad \ldots(8)$$

With help of formula (4), we replace every power of x in (7) by its expansion in terms of Legendre's polynomials and then we collect the terms involving $P_n(x)$.

Clearly when $x^n, x^{n+2}, x^{n+4}, \ldots$ are expanded in terms of Legendre's polynomials, each one of them involves a term containing $P_n(x)$. Again $P_n(x)$ will not be involved in any expansion containing power of x less than n. Thus, we see that only $a_nx^n, a_{n+2}x^{n+2}, a_{n+4}x^{n+4}, \ldots$ etc in (7) will contain $P_n(x)$. Now, using (6), we have

$$a_nx^n = a_n\frac{n!}{3\cdot 5\ldots(2n+1)}\left[(2n+1)\, P_n(x) + \ldots\ldots\right]$$

$$a_{n+2}x^{n+2} = a_{n+2}\frac{(n+2)!}{3\cdot 5\ldots(2n+5)}\left[(2n+5)P_{n+2}(x) + (2n+1)\frac{(2n+5)}{2}P_n(x) + \ldots\right]$$

$$a_{n+4}x^{n+4} = a_{n+4}\frac{(n+4)!}{3\cdot 5\ldots(2n+9)}\left[(2n+9)\, P_{n+4}(x)\right.$$

$$\left. + (2n+5)\frac{(2n+9)}{2}P_{n+2}(x) + (2n+1)\,.\,\frac{(2n+9)(2n+7)}{2\cdot 4}P_n(x) + \ldots.\right]$$

Putting the above values in (7) and equating the coefficients of $P_n(x)$ from (7) and (8),

$$C_n = \frac{n!}{3\cdot 5\ldots(2n+1)}(2n+1)a_n + \frac{(n+2)!}{3\cdot 5\ldots(2n+5)}\left[\frac{(2n+1)(2n+5)}{2}\right]a_{n+2}$$

$$+\frac{(n+4)!}{3\cdot 5\ldots(2n+9)}\left[(2n+1)+\frac{(2n+9)(2n+7)}{2\,.\,4}\right]a_{n+4}+\ldots$$

or $$C_n = \frac{n!}{3\cdot 5\ldots(2n-1)}\left[a_n + \frac{(n+1)(n+2)}{2(2n+3)}a_{n+2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}a_{n+4}+\ldots\right]$$

9.21. Solved examples based on Art 9.20

Ex. 1. *Prove that for all x* (*a*) $x^4 = (8/35) \times P_4(x) + (4/7) \times P_2(x) + (1/5) \times P_0(x)$.
(*b*) $x^5 = (8/63) \times P_5(x) + (4/9) \times P_3(x) + (3/7) \times P_1(x)$.

Sol. We have, $$x^n = \frac{n!}{3\cdot 5\ldots(2n+1)}\left[(2n+1)P_n(x) + (2n-3)\frac{(2n+1)}{2}P_{n-2}(x)\right.$$

$$\left.+(2n-7)\frac{(2n+1)(2n-1)}{2\cdot 4}P_{n-4}(x)+\ldots\right] \ldots(1)$$

(*a*) Putting $n = 4$ in (1), we have

$$x^4 = \frac{4!}{3\cdot 5\cdot 7\cdot 9}\left[9P_4(x) + 5\times\frac{9}{2}P_2(x) + 1\times\frac{9\cdot 7}{2\cdot 4}P_0(x)\right] = \frac{8}{35}P_4(x)+\frac{4}{7}P_2(x)+\frac{1}{5}P_0(x)$$

(*b*) Putting $n = 5$ in (1), we have

$$x^5 = \frac{5!}{3\cdot 5\cdot 7\cdot 9\cdot 11}\left[11P_5 + 7\times\frac{11}{2}P_3(x) + 3\times\frac{11\cdot 9}{2\cdot 4}P_1(x)\right] = \frac{8}{63}P_5(x)+\frac{4}{9}P_3(x)+\frac{3}{7}P_1(x)$$

Ex. 2. *Prove* (*a*) $x^2 = (1/3) \times P_0(x) + (2/3) \times P_1(x)$. (*b*) $x^3 = (3/5) \times P_1(x) + (2/5) \times P_3(x)$.

Ex. 3. *Express* $f(x) = x^4 + 3x^2 - x^2 + 5x - 2$ *in terms of Legendre's polynomials.*

Sol. As in Ex. 1 and 2, prove yourself that

$x^4 = (8/35)P_4(x) + (4/7)P_2(x) + (1/5)P_0(x)$, $x^3 = (3/5)P_1(x) + (2/5)P_3(x)$,

$x^2 = (1/3)P_0(x) + (2/3)P_1(x)$, $x = P_1(x)$ and $1 = P_0(x)$.

$\therefore$ $f(x) = (8/35)P_4(x) + (4/7)P_2(x) + (1/5)P_0(x) + 3[(3/5)P_1(x) + (2/5)P_3(x)] - [(1/3)P_0(x) + (2/3)P_1(x)] + 5P_1(x) - 2P_0(x)$, on putting values of x^4, x^3, x^2, x and 1

$= (8/35)P_4(x) + (6/5)P_3(x) + [(4/7) - (2/3)]P_2(x) + (9/5 + 5)P_1(x) + [(1/5) - (1/3) - 2]P_0(x)$

$= (8/35)P_4(x) + (6/5)P_3(x) - (2/21)P_2(x) + (34/5)P_1(x) - (32/15)P_0(x)$.

EXERCISE

Express the following polynomials in terms of Legendre polynomials :

1. $x^3 + 2x^2 - x - 3$ **(KU Kurukshetra 2004)**

2. $4x^3 - 2x^2 - 3x + 8$ **(KU Kurukshtra 2005)**

3. $x^4 + 3x^3 - x^2 + 5x - 2$

4. $x^4 + 2x^3 - 5x^2 - x - 2$

5. $1 + x - x^2$ **(Gulbarga 2005, Purvanchal 2004)**

6. $5x^3 + x$ **[Agra 2007]**

ANSWERS

1. $(2/5) \times P_3(x) + (4/3) \times P_2(x) - (2/5) \times P_1(x) - (7/3) \times P_0(x)$

2. $(8/5) \times P_3(x) - (4/3) \times P_2(x) - (3/5) \times P_1(x) + (22/3) \times P_0(x)$

3. $(8/35) \times P_4(x) + (6/5) \times P_3(x) - (2/21) \times P_2(x) + (34/5) \times P_1(x) - (32/25) \times P_0(x)$

4. $(8/35) \times P_4(x) + (4/5) \times P_3(x) + (82/21) \times P_2(x) + (1/5) \times P_1(x) - (2/15) \times P_0(x)$

5. $(2/3) \times P_0(x) + (1/3) \times P_0(x)$ **6.** $2P_3(x) + 4\, P_1(x)$

MISCELLANEOUS EXAMPLES ON CHAPTER 9

1. *Verify that the Legendre polynomise* $P_4(x) = (35x^3 - 30x^2 + 3)/8$ *satisfies the Legendre equation when the parameter n is equal to* 4.

[Sol. The Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ for $n = 4$ becomes

$$(1-x^2)y'' - 2xy' + 20y = 0 \qquad \ldots (1)$$

Let $$y = P_4(x) = (35x^4 - 30x^2 + 3)/8 \qquad \ldots (2)$$

From (2), $y' = (35x^3 - 15x)/2$ and $y'' = (105x^2 - 15)/2$

Substituting the above value of y, y' and y'' in (1), we get

$(1 - x^2) \times (1/2) \times (105x^2 - 15) - (2x) \times (1/2) \times (35x^3 - 15x) + 20 \times (1/8) \times (35x^4 - 15x^2 + 3) = 0$

or $(-105/2 - 35 + 175/2)x^4 + (105/2 + 15/2 + 15 - 75)\, x^2 + (-15/2 + 15/2) = 0$

or $0 = 0$, which is true. Hence $P_4(x)$ is a solution of (1).]

2. Prove that $P_n(\cos\theta) = \sum_{r=0}^{p} {}^nC_r \cos(n-2r)\theta$, where $p = n/2$ or $(n-1)/2$ according as n is even or odd. Deduc that $|P_n(\cos\theta)| \le 1$

3. Prove that $\displaystyle\int_{-1}^{1} x^m P_n(x)dx = \begin{cases} 0, \text{ if } m < n \\ \dfrac{m!\,\Gamma(m/2 - n/2 + 1/2)}{2^n (m-n)!\,\Gamma(m/2 + n/2 + 3/2)}, \text{ if } m - n\ (\ge 0) \text{ is even} \\ 0, \text{ if } m - n\ (> 0) \text{ is odd} \end{cases}$

4. If $\displaystyle U_n = \int_{-1}^{1} P_n(x)P_{n-1}(x)\,\frac{dx}{x}$, show that $(n+1)\, U_{n+1} + n\, U_n = 2$. Hence evaluate U_n.

5. If R denotes the operator

$$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}\right\},$$ then show that $\displaystyle\int_{-1}^{1} P_n(x)R\{f(x)\}\,dx = -n(n+1)\int_{-1}^{1} P_n(x)f(x)\,dx$

provided that $f(x)$ and $f'(x)$ are finite at $x = \pm 1$. Deduce that $\displaystyle\int_{-1}^{1} \log(1-x)P_n(x)\,dx = -\frac{2}{n(n+1)}$.

6. If n is a positive integer, show that $\displaystyle\int_{-1}^{1} P_n(x)(1-2xz+z^2)^{-1/2}\,dx = \frac{2z^n}{2n+1}$

and hence making use of Rodrigue's formula, deduce that

$$\int_{-1}^{1} (1-x^2)^n (1-2xz+z^2)^{-n-1/2}\,dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

From this result, deduce $\int_{-1}^{1} \frac{P_n(x)dx}{(1-x)^{1/2}} = \frac{2\sqrt{2}}{2n+1}$

7. Using Rodrigue's formula show that

(*i*) $P_n(-x) = (-1)^n P_n(x)$ (*ii*) $P_n(1) = 1$ **(Bangalore 2005)**

OBJECTIVE PROBLEMS IN CHAPTER 9

Ex 1. *The value of* $(1/2^n n!) \times \{d^n(x^2 - 1)^n/dx^n\}$ is (*a*) 0 (*b*) 1 (*c*) $P_n(x)$ (*d*) *None of these*.

Sol. Ans. (*c*) Refer Art. 9.14. **[Agra 2005, 06]**

Ex. 2. *Let* $P_n(x)$ *be the Legendre polynomial of degree* $n \geq 0$. If $1 + x^{10} = \sum_{n=0}^{10} C_n P_n(x)$, *then* C_5 *equals* : (*a*) 0 (*b*) 2/11 (*c*) 1 (*d*) 11/2 **[GATE 2004]**

Sol. Ans. (*a*) Refer theorem 9.16. Here $f(x) = 1 + x^{10}$ is a polynomial of degree 10, which is even. Hence, only those C_n with even suffuses are non-zero.

Ex. 3. *Let* $y = \psi(x)$ *be a bounded solution for the equation* $(1-x^2)y'' - 2xy' + 30y = 0.$

Then (*a*) $\int_{-1}^{1} x^3 \psi(x)dx \neq 0$ (*b*) $\int_{-1}^{1} (1 + x^3 + x^4)\, \psi(x) \neq 0$

(*c*) $\int_{-1}^{1} x^3 \psi(x)\, dx = 0$ (*d*) $\int_{-1}^{1} x^{2n} \psi(x)dx = 0$ for all $n \in N$ **[GATE 2003]**

Sol. Ans. (d) Re-writing given equation, $(1-x^2)y'' - 2xy' + 5(5+1)y = 0$ ... (1)

Comparing (1) with $(1-x^2)y'' - 2xy' + n(n+1)y = 0,$ we find that (1) is Legendre equation with $n = 5$. Since $\psi(x)$ is a bounded solution of (1), we have $\psi(x) = P_5(x) = (63x^2 - 70x^3 + 15x)/8,$ by Art. 9.2. Since $x^{2n}\psi(x)$ is an odd function, conclusion (*d*) is true.

Ex. 4. Let $P_n(x)$ denote the Legendre polynmial of degree *n*. If

$f(x) = \begin{cases} x, -1 \leq x \leq 0 \\ 0, 0 \leq x \leq 1 \end{cases}$, and $f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + ...$, then

(*a*) $a_0 = -1/4$, $a_1 = -1/2$ (*b*) $a_0 = -1/4$, $a_1 = 1/2$

(*c*) $a_0 = 1/2$, $a_1 = -1/4$ (*d*) $a_0 = -1/2$, $a_1 = -1/4$ **(GATE 2005)**

Sol. Ans. (*b*). Here $f(x) = \sum_{r=0}^{\infty} a_r P_r$. Proceed exactly as in Ex. 1 of Art. 9.18 to get a_0, a_1.

PART II : ASSOCIATED LEGENDRE FUNCTIONS.

9.22. Associated Legendre Functions

Theorem. If *z* is a solution of Legendre equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ ...(1)

then to show that $(1 - x^2)^{m/2} \frac{d^m z}{dx^m}$ is a solution of the equation

$$(1 - x^2)y'' - 2xy' + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\} y = 0\,. \quad ...(2)$$

Proof. Since z is a solution of (1), we get $(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0.$ ...(3)

Differentiation (3) m times with help of Leibnitz' theorem, we have

$$\frac{d^m}{dx^m}\left[\frac{d^2z}{dx^2}.(1-x^2)\right] + \frac{d^m}{dx^m}\left[\frac{dz}{dx}(-2x)\right] + n(n+1)\frac{d^mz}{dx^m} = 0$$

or $\frac{d^{m+2}z}{dx^{m+2}}(1-x^2) + {}^mC_1\frac{d^{m+1}z}{dx^{m+1}}(-2x) + {}^mC_2\frac{d^mz}{dx^m}(-2) + \frac{d^{m+1}z}{dx^{m+1}}(-2x) + {}^mC_1\frac{d^mz}{dx^m}(-2) + n(n+1)\frac{d^mz}{dx^m} = 0$

or $(1-x^2)\frac{d^{m+2}z}{dx^{m+2}} - 2x(m+1)\frac{d^{m+1}z}{dx^{m+1}} + \{n(n+1) - m(m-1) - 2m\}\frac{d^mz}{dx^m} = 0$

or $(1-x^2)\frac{d^{m+2}z}{dx^{m+2}} - 2x(m+1)\frac{d^{m+1}z}{dx^{m+1}} + \{n(n+1) - m(m+1)\}\frac{d^mz}{dx^m} = 0$ ...(4)

Let $u = d^mz/dx^m.$ ...(5)

Then (4) reduces to

$$(1-x^2)u'' - 2x(m+1)u'' + [n(n+1) - m(m+1)\}u = 0.$$ ...(6)

Let $v = (1-x^2)^{m/2}u$ ...(7)

so that $u = (1-x^2)^{-m/2}\,v.$ ...(8)

From (8), $u' = (1-x^2)^{-m/2}\,v' + (-m/2) \times (1-x^2)^{-(m/2)-1}(-2x)\,v.$

or $u' = (1-x^2)^{-m/2}\,v' + mvx(1-x^2)^{-(m/2)-1}.$

From (9), $u'' = (1-x^2)^{-m/2}v'' + (-m/2) \times (1-x^2)^{(m/2)-1} \times (-2x) \times v'$

$+ mvx\{-(m/2) - 1\}\,(1-x^2)^{-(m/2)-1}(-2x) + m(1-x^2)^{-(m/2)-1}(xv' + v).$ ...(10)

Substituting the values of u, u' and u'' given by (8), (9) and (10) into (6), we have

$(1-x^2)^{-(m/2)+1}v'' + mx(1-x^2)^{-m/2}v' + mx^2v(m+2)(1-x^2)^{-(m/2)-1} + mx(1-x^2)^{-m/2}v' + mv(1-x^2)^{-m/2}$

$- 2(m+1)x\,(1-x^2)^{-m/2}v' - 2m(m+1)\,x^2v(1-x^2)^{-m/2-1} + \{n(n+1) - m^2 + m\}\,(1-x^2)^{-m/2}v = 0.$

Dividing throughout by $(1-x^2)^{-(m/2)}$, the above equation becomes

$$(1-x^2)v'' - 2xv''\left[n(n+1) - m^2 - m - \frac{2m(m+1)}{1-x^2}x^2 + m + \frac{m(m+2)}{1-x^2}x^2\right]v = 0.$$

or $(1-x^2)v'' - 2xv' + \{n(n+1) - m^2/(1-x^2)\}\,v = 0$ ... (11)

Thus, using (11), (7) and (5), we get $v = (1-x^2)^{m/2}\,u = (1-x^2)^{m/2}\frac{d^mz}{dx^m}$

is a solution of (11) and hence it is solution of (2).

Remark 1. Equation (2) is called the *associated Legendre equation*. Since $P_n(x)$ and $Q_n(x)$ are solutions of (3), we conclude that

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \quad \text{and} \quad Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x)$$

are solutions of (2). $P_n^m(x)$ and $Q_n^m(x)$ are called the *associated Legendre's functions of degree n and order m of the first and second kind* respectively. Since these are independent solutions of (2), the general solutions of (2) is $y = AP_n^m(x) + BQ_n^m(x)$, where A and B are arbitrary constants.

Note that if $m > n$, $P_n^m(x) = 0$. The functions $Q_n^m(x)$ are unbounded for $x = \pm 1$.

Remark 2. For $m \geq 0,$ we define $P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x).$

Using Rodeigue's formula this gives $P_n^m(x) = (1-x^2)^{m/2}\frac{1}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n,.$

where R.H.S.is well defined for negative values of m such that $m+n \geq 0$ *i.e.*, $m \geq -n$.

Thus, we define $$P_n^m(x) = \begin{cases} (1-x^2)^{m/2}\dfrac{d^m}{dx^m}P_n(x), \text{ for } m \geq 0 \\ (1-x^2)^{m/2}\dfrac{1}{2^n n!}\dfrac{d^{m+n}}{dx^{m+n}}(x^2-1)^n, \text{for } m \geq n. \end{cases}$$

Corollary. *To find the general solutions of the following equations:*

(*a*) $$\Theta'' + \cot\theta\,.\,\Theta' + \{n(n+1) - m^2/\sin^2\theta\}\Theta. \qquad ...(12)$$

(*b*) $$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\sin\theta\,\Theta = 0 \qquad ...(13A)$$

or $$\Theta'' + \cot\theta.\,\Theta' + n(n+1)\Theta = 0. \qquad ...(13B)$$

Proof. (*a*) Put $\cos\theta = \mu$. Then, $$\Theta' = \frac{d\Theta}{d\theta} = \frac{d\Theta}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta\,.\,\frac{d\Theta}{d\mu} \qquad ...(14)$$

$$\therefore \qquad \frac{d}{d\theta} \equiv -\sin\theta\,.\,\frac{d}{d\mu} \qquad ...(15)$$

Now using (14), $\Theta'' = \dfrac{d}{d\theta}\left(\dfrac{d\Theta}{d\theta}\right) = -\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\mu}\right) = -\cos\theta\dfrac{d\Theta}{d\mu} - \sin\theta\dfrac{d}{d\theta}\left(\dfrac{d\Theta}{d\mu}\right)$

$= -\cos\theta\dfrac{d\Theta}{d\mu} - \sin\theta \times (-\sin\theta)\dfrac{d}{d\mu}\left(\dfrac{d\Theta}{d\mu}\right)$, by (15)

$= -\cos\theta\dfrac{d\Theta}{d\mu} + \sin^2\theta\dfrac{d^2\Theta}{d\mu^2} = -\mu\dfrac{d\Theta}{d\theta} + (1-\mu^2)\dfrac{d^2\Theta}{d\mu^2}$, as $\sin^2\theta = 1 - \cos^2\theta = 1 - \mu^2$

Using these values of Θ' and Θ'' in (12), we have

$$-\mu\frac{d\Theta}{d\mu} + (1-\mu^2)\frac{d^2\Theta}{d\mu^2} + \frac{\cos\theta}{\sin\theta}\left(-\sin\theta\frac{d\Theta}{d\mu}\right) + \left\{n(n+1) - \frac{m^2}{1-\cos^2\theta}\right\}\Theta = 0$$

or $$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0, \text{ as } \mu = \cos\theta \qquad ...(16)$$

which is same as (2). Hence the general solution of (16) is $\Theta = AP_n^m(\mu) + BQ_n^m(\mu)$

or $\Theta = AP_n^m(\cos\theta) + BQ_n^m(\cos\theta)$, as $\mu = \cos\theta$. ...(17)

Part (a). Putting $\cos\theta = \mu$ and doing as before, (13A) or (13B) gives

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + n(n+1)\Theta = 0, \qquad ...(18)$$

which is Legendre equation and so solution of (13A) or (13B) is

$\Theta = AP_n(\mu) + BQ_n(\mu)$ or $\Theta = AP_n(\cos\theta) + BQ_n(\cos\theta)$.

9.23. Properties of the Associated Legendre Functions.

(*i*) $P_n^0(x) = P_n(x)$ (*ii*) $P_n^m(x) = 0$ if $m > n$.

Proof. We have $$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \qquad ... (1)$$

(*i*) Putting $m = 0$ in (1), $\qquad P_n^0(x) = P_n(x).$

(*ii*) Since $P_n(x)$ a polynomial of degree n, so when $m > n$, (1) gives $P_n^m(x) = (1-x^2)^{m/2} \times 0 = 0.$

9.24. Orthogonality relation for $P_n^m(x)$. *To show that*

$$\int_{-1}^{1} P_n^m(x) P_l^m(x)\, dx = \frac{2(n+m)!}{(2n+1)\,(n-m)!}\delta_{nl}, \qquad \text{where} \qquad \delta_{nl} = \begin{cases} 0, \text{ if } n \neq l \\ 1, \text{ if } n = l \end{cases}$$

Proof. Case I. Let $l \neq n$. Since $P_n^m(x)$ satisfy the corresponding associated equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\} y = 0. \quad i.e., \quad \frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\} y = 0.$$

$$\therefore \qquad \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n^m(x)\right\} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\} P_n^m(x) = 0. \qquad \ldots (1)$$

Similarly, for $P_l^m(x)$, $\dfrac{d}{dx}\left\{(1-x^2)\dfrac{d}{dx}P_l^m(x)\right\} + \left\{l(l+1) - \dfrac{m^2}{1-x^2}\right\} P_l^m(x) = 0. \qquad \ldots (2)$

Multiplying (1) by $P_l^m(x)$ and (2) by $P_n^m(x)$ and subtracting the resulting equations gives

$$P_l^m \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n^m(x)\right\} - P_n^m(x)\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_l^m(x)\right\} + [n(n+1) - l(l+1)]P_n^m(x)P_l^m(x) = 0.$$

Integrating between limits –1 to 1, we get

$$\int_{-1}^{1} P_l^m(x)\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n^m(x)\right\}dx - \int_{-1}^{1} P_n^m(x)\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_l^m(x)\right\}dx + [n(n+1) - l(l+1)]\int_{-1}^{1} P_n^m(x)P_l^m(x)dx = 0.$$

Integrating by parts, we now obtain

$$\left[P_l^m(x)(1-x^2)\frac{d}{dx}P_n^m(x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}P_l^m(1-x^2)\frac{d}{dx}P_n^m(x)dx - \left[P_n^m(x)(1-x^2)\frac{d}{dx}P_l^m(x)\right]_{-1}^{1}$$

$$+ \int_{-1}^{1}\frac{d}{dx}P_n^m(1-x^2)\frac{d}{dx}P_l^m(x)\,dx + [(n^2-l^2)+(n-l)]\int_{-1}^{1} P_n^m(x)P_l^m(x)dx = 0$$

or $\qquad (n-1)\,(n+l+1)\int_{-1}^{1} P_n^m(x)P_l^m(x)dx = 0 \qquad$ or $\qquad \int_{-1}^{1} P_n^m(n)P_l^m(x)dx = 0, \text{ if } n \neq 1 \quad \ldots (3)$

Case II. Let $l = n$, If $m > 0$, we have $\qquad P_n^m(x) = (1-x^2)^{m/2}\,\dfrac{d^m}{dx^m}P_n(x). \qquad \ldots (4)$

$$\therefore \quad \int_{-1}^{1} P_n^m(x)P_l^m(x)dx = \int_{-1}^{1}\{P_n^m(x)\}^2\,dx = \int_{-1}^{1}\left[(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right]\frac{d^m}{dx^m}P_n(x)dx, \text{ using (4)}$$

$$= \left[\left\{(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right\}\left\{\frac{d^{m-1}}{dx^{m-1}}P_l(x)\right\}\right]_{-1}^{1} - \int_{-1}^{1}\left\{\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}\frac{d}{dx}\left\{(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right\}dx$$

[On integrating by parts]

$$\therefore \qquad \int_{-1}^{1} \{P_n^m(x)\}^2 dx = -\int_{-1}^{1} \left\{\frac{d^{m-1}}{dx^{m-1}} P_n(x)\right\} \frac{d}{dx}\left\{(1-x^2)^m \frac{d^m}{dx^m} P_n(x)\right\} dx \qquad \ldots (5)$$

Since $P_n(x)$ satisfies the Legendre equations $(1-x^2)y'' - 2xy' + n(n+1)y = 0.$

$$\therefore \qquad (1-x^2)\frac{d^2 P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0 \qquad \ldots (6)$$

Differentiating both sides of (6) w.r.t. 'x' $(m-1)$ times with help of Leibnitz's theorems gives

$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}} P_n(x) - 2mx\frac{d^m}{dx^m} P_n(x) + [n(n+1) - m(m-1)]\frac{d^{m-1}}{dx^{m-1}} P_n(x) = 0$$

Multiplying by $(1-x^2)^{m-1}$, we get

$$(1-x^2)^m \frac{d^{m+1}}{dx^{m+1}} P_n(x) - 2xm(1-x^2)^{m-1}\frac{d^n}{dx^m} P_n(x) + [n(n+1) - m(m-1)](1-x^2)^{m-1}\frac{d^{m-1}}{dx^{m-1}} P_n(x) = 0$$

or $$\frac{d}{dx}\left\{(1-x^2)^m \frac{d^m}{dx^m} P_n(x)\right\} = -(n+m)\,(n-m+1)\,(1-x^2)^{m-1}\frac{d^{m-1}}{dx^{m-1}} P_n(x) \qquad \ldots (7)$$

Using (7), (5) reduces to

$$\int_{-1}^{1} \{P_n^m(x)\}^2 dx = \int_{-1}^{1} \left\{\frac{d^{m-1}}{dx^{m-1}} P_n(x)\right\} (n+m)\,(n-m+1)\,(1-x^2)^{m-1}\frac{d^{m-1}}{dx^{m-1}} P_n(x) dx$$

$$= (n+m)\,(n-m+1)\int_{-1}^{1} \left\{(1-x^2)^{(m-1)/2}\frac{d^{m-1}}{dx^{m-1}} P_n(x)\right\}^2 dx$$

$$= (n+m)\,(n-m+1)\int_{-1}^{1} \left\{P_n^{m-1}(x)\right\}^2 dx, \text{ using definition (4) for } P_n^m(x)$$

$$= \{(n+m)\,(n-m+1)\}\ \{(n+m-1)\,(n-m+2)\}\ \ldots\ldots \{(n+1)n\}\int_{-1}^{1} \{P_n^0(x)\}^2 dx$$

[on repeating the similar method $(m-1)$ times more]

$$= (n+m)\,(n+m-1)\,\ldots\,(n+1)\,n\,(n-1)\,\ldots\,\ldots (n-m+2)\,(n-m+1)\int_{-1}^{1} \{P_n(x)\}^2 dx,$$

$$= \frac{(n+m)!}{(n-m)!} \times \frac{2}{2n+1}, \text{ using Art. 9.8.} \qquad [\because \text{ From Art 9.23 } (i),\ P_n^0(x) = P_n(x)\ ]$$

Finally, let $m < 0$. Then if $k > 0$, we write $m = -k$. Then, we know that

$$P_n^{-k}(x) = (-1)^k \frac{(n-k)!}{(n+k)!} P_n^k(x) \qquad \ldots (9)$$

$$\therefore \qquad \int_{-1}^{1}\{P_n^m(x)\}^2\,dx = \int_{-1}^{1}\{P_n^{-k}(x)\}^2\,dx = (-1)^{2k}\left[\frac{(n-k)!}{(n+k)!}\right]^2\int_{-1}^{1}\{P_n^k(x)\}^2\,dx, \text{ using (9)}$$

$$= \left[\frac{(n-k)!}{(n+k)!}\right]^2 \times \frac{(n+k)!}{(n-k)!}\times\frac{2}{2n+1}, \quad = \frac{(n-k)!}{(n+k)!}\cdot\frac{2}{2n+1} = \frac{(n+m)!}{(n-m)!}\times\frac{2}{2n+1}, \text{ using (8) and } k = -m$$

From (8) and (10), we find $$\int_{-1}^{1}\{P_n^m(x)\}^2\,dx = \frac{(n+m)!}{(n-m)!}\times\frac{2}{2n+1} \qquad \ldots (11)$$

$$\therefore \qquad (3) \text{ and } (11) \qquad \Rightarrow \qquad \int_{-1}^{1} P_n^m(x)P_1^m(x)dx = \frac{2(n+m)!}{(2n+1)!(n-m)!}\delta_{nl},$$

9.24. Recurrence relations (Formulae) for $P_n^m(x)$. Prove that

$$(i)\ P_n^{m+1}(x) - \frac{2mx}{(1-x^2)^{1/2}}P_n^m(x) + \{n(n+1) - m(m-1)\}P_n^{m-1}(x) = 0.$$

$$(ii)\ (2n+1)xP_n^m(x) = (n+m)P_{n-1}^m(x) + (n-m-1)P_{n+1}^m(x).$$

$$(iii)\ (1-x^2)^{1/2}P_n^m(x) = \frac{1}{2n+1}\{P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)\}.$$

$$(iv)\ (1-x^2)^{1/2}P_n^m(x) = \frac{1}{2n+1}\{(n+m)\,(n+m-1)P_{n-1}^{m-1}(x)\ -(n-m+1)\,(n-m+2)P_{n+1}^{m-1}(x)\}.$$

Proof. (*i*) Since $P_n(x)$ is a solution of Legendre's equation, $(1-x^2)y'' - 2xy' + n(n+1)y = 0,$

hence $$(1-x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d}{dx}P_n(x) + n(n+1)P_n(x) = 0. \qquad \ldots (1)$$

Differentiating both sides of (1) w.r.t. 'x' (m – 1) times with help of Leibnitz's theorem, gives

$$\left\{(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_n(x) - 2x(m-1)\frac{d^m}{dx^m}P_n(x) - 2\frac{(m-1)\,(m-2)}{2!}\cdot\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}$$

$$-2\left\{x\frac{d^m}{dx^m}P_n(x) + (m-1)\cdot 1\cdot\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\} + n(n+1)\frac{d^{m-1}}{dx^{m-1}}P_n(x) = 0.$$

or $$(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_n(x) - 2xm\frac{d^m}{dx^m}P_n(x)\ +[n(n+1) - m(m-1)]\frac{d^{m-1}}{dx^{m-1}}P_n(x) = 0.$$

Multiplying both sides by $(1-x^2)^{(m-1)/2}$, we get

$$(1-x^2)^{(m+1)/2}\frac{d^{m+1}}{dx^{m+1}}P_n(x) - 2xm(1-x^2)^{-1/2}(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

$$+[n\,(n+1) - m(m-1)](1-x^2)^{(m-1)/2}\frac{d^{m-1}}{dx^{m-1}}P_n(x) = 0$$

Using the definition $P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$, the above result becomes

$$P_n^{m+1}(x) - \frac{2mx}{(1-x^2)^{1/2}}P_n^m(x) + [n(n+1) - m(m-1)]P_n^{m-1}(x) = 0.$$

(ii) From recurrence relations for Lengendre polynomial (Refer recurrence relations I and II of Art. 9.8), we have

$$(2n+1)\,x\,P_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x) \quad \ldots (1)$$

and

$$(2n+1)P_n(x) = \frac{d}{dx}P_{n+1}(x) - \frac{d}{dx}P_{n-1}(x) \quad \ldots (2)$$

Differentiating both sides of (1) w.r.t. 'x' m times with help of Leibnitz theorem, we get

$$(2n+1)\left\{x\frac{d^m}{dx^m}P_n(x) + m\cdot 1\cdot\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\} = (n+1)\frac{d^m}{dx^m}P_{n+1}(x) + n\frac{d^m}{dx^m}P_{n-1}(x) \quad \ldots (3)$$

Now, differentiating (2) w.r.t 'x' $(m-1)$ times, we get

$$(2n+1)\frac{d^{m-1}}{dx^{m-1}}P_n(x) = \frac{d^m}{dx^m}P_{n+1}(x) - \frac{d^m}{dx^m}P_{n-1}(x). \quad \ldots (4)$$

Putting the value of $(2n+1)\frac{d^{m-1}}{dx^{m-1}}P_n(x)$ given by (4) in (3), we get

$$(2n+1)x\frac{d^m}{dx^m}P_n(x) + m\left\{\frac{d^m}{dx^m}P_{n+1}(x) - \frac{d^m}{dx^m}P_{n-1}(x)\right\} = (n+1)\frac{d^m}{dx^m}P_n(x) + n\frac{d^m}{dx^m}P_{n-1}(x)$$

or

$$(2n+1)x\frac{d^m}{dx^m}P_m(x) = (n+m)\frac{d^m}{dx^m}P_{n-1}(x) + (n-m+1)\frac{d^m}{dx^m}P_{n+1}(x)$$

Multiplying both sides by $(1-x^2)^{m/2}$, we get

$$(2n+1)x(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) = (n+m)(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n-1}(x) + (n-m+1)(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n+1}(x).$$

Using the definition $P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$, the above result becomes

$$(2n+1)x\,P_n^m(x) = (n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x).$$

***(iii)*.** From recurrence relation for Legendre polynomial (Refer relation III of Art. 9.9), we have

$$(2n+1)P_n(x) = \frac{d}{dx}P_{n+1}(x) - \frac{d}{dx}P_{n-1}(x). \quad \ldots (1)$$

Differentiating (1) m times w.r.t. 'x', we get

$$(2n+1)\frac{d^m}{dx^m}P_n(x) = \frac{d^{m+1}}{x^{m+1}}P_{n+1}(x) - \frac{d^{m+1}}{dx^{m+1}}P_{n-1}(x).$$

Multiplying both sides by $(1-x^2)^{(m+1)/2}$, we get

$$(2n+1)(1-x^2)^{m/2}(1-x^2)^{1/2}\frac{d^m}{dx^m}P_n(x) = (1-x^2)^{(m+1)/2}\frac{d^{m+1}}{dx^{m+1}}P_{n+1}(x) - (1-x^2)^{(m+1)/2}\frac{d^{m+1}}{dx^{m+1}}P_{n-1}(x)$$

Using the definition $P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$, the above result becomes

$$(2n+1)(1-x^2)^{1/2}P_n^m(x) = P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x) \quad \text{or} \quad (1-x^2)^{1/2}P_n^m(x) = \frac{1}{2n+1}\left\{P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)\right\}.$$

(iv). From recurrence relation (*ii*), $xP_n^m(x) = \frac{1}{2n+1}\{(n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x)\}$

Putting this values of $xP_n^m(x)$ in recurrence relation (*i*), we get

$$P_n^{m+1}(x) - \frac{2m}{(1-x^2)^{1/2}} \cdot \frac{1}{2n+1}\{(n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x)\} + \{n(n+1) - m(m-1)\}P_n^{m-1}(x) = 0 \quad \ldots (1)$$

Replacing m by $(m-1)$ in recurrence relation (*iii*), we get

$$(1-x^2)^{1/2} P_n^{m-1}(x) = \frac{1}{2n+1}\left(P_{n+1}^m(x) - P_{n-1}^m(x)\right) \text{ or } P_n^{m-1}(x) = \frac{1}{(2n+1)(1-x^2)^{1/2}}\{P_{n+1}^m(x) - P_{n-1}^m(x)\}.$$

Putting this value of $P_n^{m-1}(x)$ in (1), we get

$$P_n^{m+1}(x) - \frac{2m}{(2n+1)(1-x^2)^{1/2}}\{(n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x)\} + \frac{1}{(2n+1)(1-x^2)^{1/2}}\{n(n+1)$$

$$-m\,(m-1)\}P_{n+1}^m(x) - P_{n-1}^m(x)\} = 0$$

or $$(1-x^2)^{1/2} P_n^{m+1}(x) = \frac{1}{2n+1}[\{2m(n+m) + n(n-1) - m(m-1)]P_{n-1}^m(x)$$

$$+\{2m\,(n-m+1) - n(n+1) + m(m-1)P_{n+1}^m(x)]$$

or $$(1-x^2)^{1/2} P_n^{m+1} = \frac{1}{2n+1}[(n+m)\,(n+m+1)P_{n-1}^m(x)\; -(n-m)\,(n-m+1)P_{n+1}^m(x)] \quad \ldots (2)$$

Replacing m by $(m-1)$ in (2), we get

$$(1-x^2)^{1/2} P_n^m(n) = \frac{1}{2n+1}[(n+m-1)\,(n+m)P_{n-1}^{m-1}(n)\; -(n-m+1)\,(n-m+2)P_{n+1}^{m-1}(x)].$$

EXERCISE

1. Show that $\sum_{n=0}^{\infty} P_{n+m}^m(x)t^n = \frac{(2m)!(1-x^2)^{m+1/2}}{2^m m!(1-2tx+t^2)^{m+1/2}}$.

2. Prove that (*a*) $P_n^m(x) = \frac{1}{2^n \cdot n!}(1-x^2)^{m/2}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n$ (*b*) $P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!}P_n^m(x)$.

3. Prove that $P_n^m(x) = \frac{(n+m)!(1-x^2)^{m/2}}{(n-m)!2^m m!}\,{}_2F_1\left(m-n, m+n+1; m+1; \frac{1-x}{2}\right)$.

4. Define associated Legendre's polynomials and prove their orthoganality condition.

5. Prove that $P_m^m(x) = \frac{(n+m)!}{(n-m)!} \times \frac{(1-x^2)^{m/2}}{2^m m!}\,{}_2F_1\left(m-n, m+n+1; m+1; \frac{1-x}{2}\right)$.

6. Prove that $P_n^m(-x) = (-1)^{m+n}P_n^m(x)$ **(Utkal 2003)**

10

Legendre Functions of the Second Kind – $Q_n(x)$

10.1. Some useful results: From Art. 9.2 of chapter 9, we have

$$Q_n(x) = \frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right]$$

Multiplying by $2\cdot 4\cdot 6\cdot \ldots \cdot 2n$ in the numerator and denominator, we get

$$Q_n(x) = \frac{n!\,(2\cdot 4\cdot 6\ldots 2n)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \ldots \cdot 2n\cdot (2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right]$$

or
$$Q_n(x) = \frac{n!\cdot 2^n\, n!}{(2n+1)!}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right]$$

or
$$Q_n(x) = \frac{2^n\,(n!)^2}{(2n+1)!}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right], \qquad \ldots(1)$$

which is a solution of Legendre's equation in descending powers of x, all the powers of x being negative. Re–writing (1), we have

$$Q_n(x) = \frac{2^n\,(n!)}{(2n+1)!}\left[(n!)x^{-(n+1)} + \frac{(n+2)!}{2(2n+3)}x^{-(n+3)} + \frac{(n+4)!}{2\cdot 4(2n+3)\,(2n+5)}x^{-(n+5)} + \ldots\right]$$

$$= \frac{2^n\,(n!)}{(2n+1)!}\left[(n!)x^{-(n+1)} + \frac{(n+2)!}{2(2n+3)}x^{-(n+2+1)} + \frac{(n+4)!}{2\cdot 4\,(2n+3)\,(2n+5)}x^{-(n+4+1)} + \ldots\right]$$

$$= \frac{2^n\,(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2\cdot 4\cdot \ldots \cdot 2r\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

$$\therefore \qquad Q_n(x) = \frac{2^n\,(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,(r!)(2n+3)\,(2n+5)\ldots(2n+2r+1)} \qquad \ldots(2)$$

Differentiating (2) w.r.t. 'x' we get

$$Q_n'(x) = -\frac{2^n.\,n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+2)}}{2^r\,r!(2n+3)\,(2n+5)\ldots(2n+2r+1)}. \qquad \ldots(3)$$

Putting $n-1$ for n in (3), we get

$$Q'_{n-1}(x) = -\frac{2^{n-1}(n-1)!}{(2n-1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!(2n+1)(2n+3)...(2n+2r-1)}$$

$$= -\frac{2n\cdot 2^{n-1}(n-1)!}{(2n-1)!\cdot 2n}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\cdot r!(2n+1)(2n+3)...(2n+2r-1)}$$

$$= -\frac{2^n\cdot n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\cdot r!(2n+1)(2n+3)...(2n+2r-1)}. \qquad ...(4)$$

Again, putting $n+1$ for n in (3), we get

$$Q'_{n+1}(x) = -\frac{2^{n+1}\cdot(n+1)!}{(2n+3)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r\cdot r!(2n+5)...(2n+2r+1)(2n+2r+3)}$$

$$= -\frac{2^n\cdot n!\cdot(2n+2)}{(2n+3)(2n+2)(2n+1)(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r\cdot r!(2n+5)...(2n+2r+3)}$$

$$= -\frac{2^n\cdot n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r\cdot r!(2n+1)(2n+3)...(2n+2r+3)}. \qquad ...(5)$$

10.2. Recurrence Relations (formulae) for $Q_n(x)$

I. $Q'_{n+1} - Q'_{n-1} = (2n+1)Q_n$. **[Bilaspur 1997, 98]**

Proof. We have, $Q'_{n-1} + (2n+1)Q_n$

$$= -\frac{2^n(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)...(2n+2r-1)} + (2n+1)\frac{2^n(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+3)(2n+5)...(2n+2r+1)}$$

[using results (4) and (2) Art. of 10.1.]

$$= \frac{2^n(n!)}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+3)(2n+5)...(2n+2r+1)} - \sum_{r=0}^{\infty}\frac{(n+2r)\,x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)...(2n+2r+1)}\right]$$

$$= \frac{2^n(n!)}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)...(2n+2r+1)}\times\{(2n+1)-(2n+2r+1\}\right]$$

$$= \frac{2^n(n!)}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{-2r(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)...(2n+2r+1)}\right] = -\frac{2^n(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!(2n+1)(2n+3)...(2n+2r+1)}$$

$$= -\frac{2^n(n!)}{(2n)!}\left[0+\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!(2n+1)(2n+3)...(2n+2r+1)}\right], \text{ as } \frac{1}{(-1)!}=0$$

$$= -\frac{2^n(n!)}{(2n)!}\sum_{s=0}^{\infty}\frac{(n-2s+2)!\,x^{-(n+2s+3)}}{2^s(s)!(2n+1)(2n+3)...(2n+2s+3)}, \text{ putting } r=s+1 \text{ so that } s=r-1$$

$= Q'_{n+1}$ by result (5) of Art. 10.1.

II. $nQ'_{n+1} + (n+1)Q'_{n-1} = (2n+1)xQ'_n$. **(Bilaspur 1998)**

Proof. We have, $(2n+1)xQ'_n - (n+1)Q'_{n-1}$

$$= -(2n+1)x \cdot \frac{2^n (n!)}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r+1)!\, x^{-(n+2r+2)}}{2^r \cdot r!\ (2n+3)\ (2n+5)...(2n+2r+1)}$$

$$-(n+1)\cdot(-1)\frac{2^n (n!)}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^r (r!)\ (2n+1)\ (2n+3)...(2n+2r-1)}$$,putting the values of Q'_n and Q'_{n-1}

$$= -\frac{2^n (n!)}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r+1)!\, x^{-(n+2r+1)}\ (2n+1)}{2^r (r!)\ (2n+1)\ (2n+3)...(2n+2r+1)} + \frac{2^n (n!)}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}\ (n+1)(2n+2r+1)}{2^r (r!)\ (2n+1)\ (2n+3)...(2n+2r+1)}$$

$$= -\frac{2^n (n!)}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^r (r!)\ (2n+1)\ (2n+3)...(2n+2r+1)} \times [(n+2r+1)\ (2n+1) - (n+1)\ (2n+2r+1)]$$

$$= -\frac{n \cdot 2^n (n!)}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^{r-1} (r-1)!\ (2n+1)\ (2n+3)...(2n+2r+1)}$$

$$= -\frac{n \cdot 2^n (n!)}{(2n)!} \left[\sum_{r=1}^{\infty} \left\{ \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^{r-1} (r-1)!\ (2n+1)\ (2n+3)...(2n+2r+1)} \right\} + 0 \right]$$

$$= -\frac{n \cdot 2^n (n!)}{(2n)!} \sum_{s=0}^{\infty} \frac{(n+2s+2)!\, x^{-(n+2s+3)}}{2^s (s!)\ (2n+1)\ (2n+3)...(2n+2s+3)}$$, putting $r = s+1$

$= nQ'_{n+1}$, by result (5) of Art. 10.1.

III. $(2n+1)\, xQ_n = (n+1)\, Q_{n+1} + nQ_{n-1}$.

Or $xQ_n = \frac{n+1}{2n+1} Q_{n+1} + \frac{n}{2n+1} Q_{n-1}$ *Or* $(n+1)\, Q_{n+1} - (2n+1)\, xQ_n + nQ_{n-1} = 0.$

Proof. We have, $nQ_{n-1} - (2n+1)\, xQ_n$

$$= n \times \frac{2^{n-1} (n-1)!}{(2n-1)!} \sum_{r=0}^{\infty} \frac{(n+2r-1)!\, x^{-(n+2r)}}{2^r (r!)\ (2n+3)\ (2n+5)...(2n+2r-1)}$$

$$-(2n+1) \times \frac{2^n (n!)}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^r (r!)\ (2n+3)\ (2n+5)...(2n+2r+1)},$$ putting values of Q_{n-1} and Q_n

$$= \frac{2^{n-1} (n!) \cdot 2n}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r-1)!\, x^{-(n+2r)} \cdot (2n+2r+1)}{2^r (r!)\ (2n+3)\ (2n+5)...(2n+2r-1)\ (2n+2r+1)}$$

$$-\frac{(2n+1)\, 2^n (n!)}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r)}}{2^r (r!)\ (2n+3)\ (2n+5)...(2n+2r+1)}$$

$$= \frac{2^n (n!)}{(2n+1)!} \left[\sum_{r=0}^{\infty} \frac{(n+2r-1)!\, x^{-(n+2r)}}{2^r (r!)\ (2n+3)\ (2n+5)...(2n+2r+1)} \times \{n(2n+2r+1) - (2n+1)\ (n+2r)\} \right]$$

$$= \frac{2^n(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}\,(-2r)\,(n+1)}{2^r(r!)\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

$$= -\frac{2^n(n+1)!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^{r-1}(r-1)!\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

$$= -(n+1)\frac{2^n(n!)}{(2n+1)!}\left[\sum_{r=1}^{\infty}\left\{\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^{r-1}(r-1)!\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}\right\}+0\right]$$

$$= -(n+1)\times\frac{2^n(n!)}{(2n+1)!}\left[\sum_{s=0}^{\infty}\frac{(n+2s+1)!\,x^{-(n+2s+2)}}{2^s(s!)\,(2n+3)\,(2n+5)\ldots(2n+2s+3)}\right],\ \text{taking } r=s+1$$

$$= -(n+1)\times\frac{2^n(n!)\,(2n+2)}{(2n+2)!}\sum_{s=0}^{\infty}\frac{(n+2s+1)!\,x^{-(n+2s+2)}}{2^s(s!)\,(2n+3)\,(2n+5)\ldots(2n+2s+3)}$$

$$= -(n+1)\times\frac{2^n(n+1)!}{(2n+3)!}\sum_{s=0}^{\infty}\frac{(n+2s+1)!\,x^{-(n+2s+2)}}{2^s(s!)\,(2n+5)\ldots(2n+2s+3)}$$

$= -(n+1)Q_{n+1}$, be definition (2).

IV. $(2n+1)(1-x^2)Q_n' = n(n+1)(Q_{n-1} - Q_{n+1})$.

Proof. Since Q_n is a solution of Legendre's equation, namely,

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+n(n+1)\,y=0 \qquad \text{so} \qquad \frac{d}{dx}[(1-x^2)Q'_n] = -n(n+1)Q_n. \qquad ...(1)$$

Integrating both sides of (1) between the limits ∞ to x, we get

$$\left[(1-x^2)Q_n'\right]_{\infty}^{x} = -n(n+1)\int_{\infty}^{x} Q_n\,dx$$

or $(1-x^2)\,Q'_n(x) = -n(n+1)\int_{\infty}^{x} Q_n\,dx$, as $\{Q_n'\}_{x=\infty}=0$ and $\{x^2\,Q_n'\}_{x=\infty}=0$.

But by recurrence relation I, we get $Q'_{n+1} - Q'_{n-1} = (2n+1)Q_n$. ...(3)

Integrating both sides of (3) between the limits ∞ to x, we get

$$\left[Q_{n+1}-Q_{n-1}\right]_{\infty}^{x} = \int_{\infty}^{x}(2n+1)Q_n\,dx \quad \text{or} \quad Q_{n+1}(x) - Q_{n-1}(x) = \int_{\infty}^{x}(2n+1)Q_n\,dx. \quad ...(4)$$

$[\because \{Q_{n+1}\}_{x=\infty} = 0 = \{Q_{n-1}\}_{x=\infty}]$

Now, from (4) and (2), $$(1-x^2)\,Q_n'(x) = -n(n+1)\,\frac{Q_{n+1}(x)-Q_{n-1}(x)}{2n+1}$$

or $$(2n+1)(1-x^2)\,Q_n'(x) = n(n+1)\,[Q_{n-1}(x) - Q_{n+1}(x)].$$

V. $xQ_n' - Q'_{n-1} = nQ_n$.

Proof. From recurrence relation III, $(n+1)\,Q_{n+1} - (2n+1)x\,Q_n + n\,Q_{n-1} = 0$. ...(1)

Differentiating (1) w.r.t. 'x', we have

$$(n+1)Q'_{n+1} - (2n+1)\,\{x\,Q_n' + 1.Q_n\} + nQ'_{n-1} = 0. \qquad ...(2)$$

Now, by recurrence relation I, $Q'_{n+1} = Q'_{n-1} + (2n+1)Q_n$. ...(3)

Putting the value of Q'_{n+1} given by (3) in (2), we get

$$(n+1)\{Q'_{n-1} + (2n+1)Q_n\} - (2n+1)(xQ_n' + Q_n) + nQ'_{n-1} = 0$$

or $$(2n+1)Q'_{n-1} - (2n+1)xQ_n' + (2n+1)Q_n(n+1-1) = 0$$

or $$Q'_{n-1} - xQ_n' + nQ_n = 0 \quad \textit{i.e.} \quad xQ_n' - Q'_{n-1} = nQ_n.$$

VI. $Q'_n - xQ'_{n-1} = nQ_{n-1}$.

Proof. From recurrence relation I, $$Q'_{n+1} - Q'_{n-1} = (2n+1)Q_n. \quad ...(1)$$

Multiplying both sides of (1) by x, $$xQ'_{n+1} - xQ'_{n-1} = (2n+1)xQ_n. \quad ...(2)$$

Now, from recurrence relation III, $$(n+1)Q_{n+1} - (2n+1)xQ_n + nQ_{n-1} = 0. \quad ...(3)$$

Putting the value of $(2n+1)xQ_n$ from (2) in (3), $$(n+1)Q_{n+1} - xQ'_{n+1} + xQ'_{n-1} + nQ_{n-1} = 0. \quad ...(4)$$

From recurrence relation V, $$nQ_n = xQ_n' - Q'_{n-1}. \quad ...(5)$$

Replacing n by $n+1$ in (5), we get $$(n+1)Q_{n+1} = xQ'_{n+1} - Q_n'. \quad ...(6)$$

Putting the value of $(n+1)Q_{n+1}$ from (6) in (4), we get

$$xQ'_{n+1} - Q'_n - xQ'_{n+1} + xQ'_{n-1} + nQ_{n-1} = 0 \quad \text{or} \quad Q_n' - xQ'_{n-1} = nQ_{n-1}.$$

VII. $(x^2-1)Q'_n = nxQ_n - nQ_{n-1}$.

Proof. Recurrence relation V, we have $$xQ_n' - Q'_{n-1} = nQ_n. \quad ...(1)$$

Multiplying both sides of (1) by x, we get $$x^2Q_n' - xQ'_{n-1} = nxQ_n. \quad ...(2)$$

Again, from recurrence relation VI, we get $$Q_n' - xQ'_{n-1} = nQ_{n-1}. \quad ...(3)$$

Subtracting (3) from (2), we have

$$x^2Q'_n - Q'_n = nxQ_n - nQ_{n-1} \quad \text{or} \quad (x^2-1)Q'_n = nxQ_n - nQ_{n-1}.$$

VIII. $(x^2-1)Q_n' = (n+1)Q_{n+1} - (n+1)xQ_n$.

Proof. Recurrence relations V and VI are $$xQ'_n - Q'_{n-1} = nQ_n \quad ...(1)$$

and $$Q'_n - xQ'_{n-1} = nQ_{n-1}. \quad ...(2)$$

Replacing n by $n+1$ in (1) and (2), $$xQ'_{n+1} - Q'_n = (n+1)Q_{n+1} \quad ...(3)$$

and $$Q_n'_{+1} - xQ_n' = (n+1)Q_n. \quad ...(4)$$

Multiplying both sides of (4) by x, $$xQ'_{n+1} - x^2Q'_n = (n+1)xQ_n. \quad ...(5)$$

Subtracting (5) from (3), we have $$-Q'_n + x^2Q'_n = (n+1)Q_{n+1} - (n+1)xQ_n$$

or $$(x^2-1)Q_n' = (n+1)Q_{n+1} - (n+1)xQ_n.$$

10.3. Theorem. *The associated Legendre function of the second kind defined by*

$$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x)$$

satisfy Legendre's associated equation $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0.$$

Proof. Proceed as in chapter 9.

10.4. Theorem. *Assuming P_n as a solution of Legendre's equation, show that the complete solution of this equation is given by $aP_n + bQ_n$, where $Q_n = cP_n\int\frac{dx}{(1-x^2)P_n^2}$, c being a constant.*

Proof. The Legendre's equation is given by $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0. \quad ...(1)$$

Let $y = uP_n$ be the complete solution of (1), where u is a function of x.

Now, $$\frac{dy}{dx} = u\frac{dP_n}{dx} + P_n\frac{du}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = u\frac{d^2P_n}{dx^2} + 2\frac{dP_n}{dx}\frac{du}{dx} + P_n\frac{d^2u}{dx^2}.$$

Putting these values of (1), we have

$$(1-x^2)\left(u\frac{d^2P_n}{dx^2}+2\frac{dP_n}{dx}\frac{du}{dx}+P_n\frac{d^2u}{dx^2}\right)-2x\left(u\frac{dP_n}{dx}+P_n\frac{du}{dx}\right)+n(n+1)uP_n=0$$

or $$(1-x^2)\left(P_n\frac{d^2u}{dx^2}+2\frac{dP_n}{dx}\frac{du}{dx}\right)-(1-x^2)\,u\frac{d^2P_n}{dx^2}-2xu\frac{dP_n}{dx}-2xP_n\frac{du}{dx}+n(n+1)uP_n=0$$

or $$(1-x^2)\left(P_n\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dP_n}{dx}\right)+u\left\{(1-x^2)\frac{d^2P_n}{dx^2}-2x\frac{dP_n}{dx}+n(n+1)P_n\right\}-2xP_n\frac{du}{dx}=0$$

or $$(1-x^2)\left(P_n\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dP_n}{dx}\right)-2xP_n\frac{du}{dx}=0$$

$$\left[\because\ P_n \text{ is a solution of Legendre's equation} \ \Rightarrow\ (1-x^2)\frac{d^2P_n}{dx^2}-2x\frac{dP_n}{dx}+n(n+1)P_n=0.\right]$$

Now dividing throughout by $(1-x^2)P_n\,(du/dx)$, we have

$$\frac{d^2u/dx^2}{du/dx}+2\frac{dP_n/dx}{P_n}-\frac{2x}{1-x^2}=0. \qquad ...(2)$$

Integrating (2), $\log(du/dx)+2\log P_n+\log(1-x^2)=\log k$

or $\log\{(du/dx)\,.\,P_n^{\,2}(1-x^2)\}=\log k$, k being an arbitrary constant.

or $\frac{du}{dx}\,.\,P_n^{\,2}(1-x^2)=k$ or $\frac{du}{dx}=\frac{k}{(1-x^2)P_n^2}$.

Integrating, $u=k\int\frac{1}{(1-x^2)\,P_n^2}dx+a$, where a and k are arbitrary constants of integration.

Hence the complete solution of Legendre's equation is

$$y=uP_n=\left[k\int\frac{dx}{(1-x^2)\,P_n^2}+a\right]P_n=aP_n+kP_n\int\frac{dx}{(1-x^2)\,P_n^2}=aP_n+\frac{k}{c}\times cP_n\int\frac{dx}{(1-x^2)\,P_n^2}$$

or $$y=aP_n+bQ_n, \quad \text{where} \quad b=k/c.$$

10.5. Christoffel's second summation formula

From recurrence relations of $P_n(x)$ and $Q_n(x)$, we have

$$(2n+1)\,xP_n(x)=(n+1)\,P_{n+1}(x)+nP_{n-1}(x), \qquad ...(1)$$

and $$(2n+1)\,yQ_n(y)=(n+1)\,Q_{n+1}(y)+nQ_{n-1}(y), \qquad ...(2)$$

Multiplying both sides of (1) by $Q_n(y)$ and (2) by $P_n(x)$ and then subtracting, we have

$$(2n+1)\,(x-y)\,P_n(x)\,Q_n(y)=(n+1)\,\{P_{n+1}(x)\,Q_n(y)-P_n(x)\,Q_{n+1}(y)\}+n\{P_{n-1}(x)\,Q_n(y)-Q_{n-1}(y)\,P_n(x)\}$$

or $$(2n+1)\,P_n(x)\,Q_n(y)\,(x-y)+n\{P_n(x)\,Q_{n-1}(y)-P_{n-1}(x)\,Q_n(y)\}=(n+1)\,\{P_{n+1}(x)\,Q_n(y)-P_n(x)Q_{n+1}(y)\}. \qquad ...(3)$$

Putting $n=1, 2, 3, ...n$ in succession in (3), we have

$$3P_1(x)\,Q_1(y)\,(x-y)+\{P_1(x)\,Q_0(y)-Q_1(y)\,P_0(x)\}=2\{P_2(x)\,Q_1(y)-P_1(x)\,Q_2(y)\}, \qquad ...(A_1)$$

$$5P_2(x)\,Q_2(y)\,(x-y)+2\{P_2(x)\,Q_1(y)-P_1(x)\,Q_2(y)\}=3\{P_3(x)\,Q_2(y)-P_2(x)\,Q_3(y)\}, \qquad ...(A_2)$$

$$7P_3(x)\,Q_3(y)\,(x-y)+3\{P_3(x)\,Q_2(y)-P_2(x)\,Q_3(y)\}=4\{P_4(x)\,Q_3(y)-P_3(x)\,Q_4(y)\}, \qquad ...(A_3)$$

... ...

$$(2n+1)P_n(x)Q_n(y)\,(x-y)+n\{P_n(x)Q_{n-1}(y)-P_{n-1}(x)Q_n(y)\}=(n+1)\{P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)\}. \quad ...(A_n)$$

Adding (A_1), (A_2), (A_3), (A_4) etc. upto (A_n), we have

$$(x-y)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)+\{P_1(x)\,Q_0(y)-Q_1(y)\,P_0(x)\}=(n+1)\{P_{n+1}(x)\,Q_n(y)-P_n(x)\,Q_{n+1}(y)\}$$

or $$(y-x)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)+\{Q_1(y)\,P_0(x)-P_1(x)\,Q_0(y)\}=-(n+1)\{P_{n+1}(x)\,Q_n(y)-P_n(x)\,Q_{n+1}(y)\}$$

or $$(y-x)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)+[\{yQ_0(y)-1\}\times P_0(x)-xQ_0(y)\,P_0(x)]$$
$$=-(n+1)\{P_{n+1}(x)\,Q_n(y)-Q_{n+1}(y)\,P_n(x)\} \qquad [\because\ Q_1(y)=yQ_0(y)-1,\ P_1(x)=x,\ P_0(x)=1]$$

or $$(y-x)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)+(y-x)\,P_0(x)\,Q_0(y)-P_0(x)=-(n+1)\{P_{n+1}(x)\,Q_n(y)-Q_{n+1}(y)\,P_n(x)\}$$

or $$\mathbf{(y-x)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)=1-(n+1)[P_{n+1}(x)\,Q_n(y)-Q_{n+1}(y)\,P_n(x)],}$$

which is called *Christoffel's second summation formula.*

10.6. A relation connecting $P_n(x)$ and $Q_n(x)$: $\dfrac{1}{y-x}=\sum_{m=0}^{\infty}(2m+1)\,P_m(x)\,Q_m(y)$.

Proof. Let $f(x)=1/(y-x)$.

Now, $$f(x)=\frac{1}{y(1-x/y)}=\frac{1}{y}\left(1-\frac{x}{y}\right)^{-1}=y^{-1}\left(1+\frac{x}{y}+\frac{x^2}{y^2}+\ldots+\frac{x^m}{y^m}+\ldots\right)$$
$$=y^{-1}+y^{-2}x+y^{-3}x^2+\ldots+y^{-m-1}x^m+y^{-m-2}x^{m+1}+y^{-m-3}x^{m+2}+y^{-m-4}x^{m+3}+\ldots \quad ...(1)$$

Let $$f(x)=A_0+A_1x+A_2x^2+\ldots, \quad ...(2)$$

where A's are constant. Further suppose that $f(x)$ is also expressed as

$$f(x)=\sum_{m=0}^{\infty}B_mP_m(x), \quad ...(3)$$

then, we know that

$$B_m=\frac{1\cdot2\cdot3\ldots m}{1\cdot3\cdot5\ldots(2m-1)}\left[A_m+\frac{(m+1)(m+2)}{2(2m+3)}A_{m+2}+\frac{(m+1)(m+2)(m+3)(m+4)}{2\cdot4\cdot(2m+3)(2m+5)}A_{m+4}+\ldots\right].$$

Comparing (1) and (2), $A_0=y^{-1}$, $A_1=y^{-2}$, $\ldots A_m=y^{-(m+1)}$, $A_{m+1}=y^{-(m+2)}$,...

$$\therefore B_m=\frac{m!}{1\cdot3\cdot5\ldots(2m-1)}\left[y^{-(m+1)}+\frac{(m+1)(m+2)}{2(2m+3)}y^{-(m+3)}+\ldots\right]=(2m+1)\,Q_n(y),\ \text{by}$$

definition

$$\therefore \qquad f(x)=\frac{1}{y-x}=\sum_{m=0}^{\infty}B_m\,P_m(x)=\sum_{m=0}^{\infty}(2m+1)P_m(x)Q_m(y).$$

10.7. Neumann's Integral for $Q_m(y)$.

To show that $\quad Q_m(y)=\dfrac{1}{2}\int_{-1}^{1}\dfrac{P_m(x)}{y-x}\,dx, \quad (y>1).$ **(Bilaspur 1996, 98)**

Proof. From Art. 10.6 we have $\quad \dfrac{1}{y-x}=\sum_{m=0}^{\infty}(2m+1)P_m(x)Q_m(y).$

$$\therefore \frac{1}{2}\int_{-1}^{1} P_m(x)\times\frac{1}{y-x}dx = \int_{-1}^{1} P_m(x)\times\left\{\sum_{m=0}^{\infty}(2m+1)P_m(x)\,Q_m(y)\right\}dx = Q_m(y)\int_{-1}^{1}\{P_m(x)\}^2.(2m+1)dx$$

$$\left(\because \int_{-1}^{1} P_m(x)\,P_n(x)dx = 0, \text{ if } m \neq n\right)$$

$$= (2m+1)\,Q_m(y)\times\frac{2}{2m+1} = 2Q_m(y). \qquad \left[\because \int_{-1}^{1}(P_m)^2\,dx = \frac{2}{2m+1}\right]$$

Hence $$Q_m(y) = \frac{1}{2}\int_{-1}^{1}\frac{P_m(x)}{y-x}\,dx.$$

10.8 SOLVED EXAMPLES ON CHAPTER 8

Ex. 1. *Prove that* (*i*) $(x^2-1)\,(Q_nP'_n - P_nQ'_n) = c$. **[Agra 2006]** (*ii*) $\dfrac{Q_n}{P_n} = \displaystyle\int_x^{\infty}\frac{dx}{(x^2-1)P_n^2}$.

(*iii*)*From* (*ii*) *deduce that* (*a*) $Q_0(x) = \dfrac{1}{2}\log\dfrac{x+1}{x-1}$. (*b*) $Q_1(x) = \dfrac{x}{2}\log\dfrac{x+1}{x-1} - 1$.

Sol. (*i*) Legendre's equation is $(1-x^2)y'' - 2xy' + n(n+1)y = 0$. ...(1)

Since P_n and Q_n are both solutions of (1), $(1-x^2)P''_n - 2xP'_n + n(n+1)P_n = 0$ ...(2)

and $(1-x^2)Q''_n - 2xQ'_n + n(n+1)Q_n = 0$. ...(3)

Multiplying (2) by Q_n, (3) by P_n and then subtracting, we get

$$(1-x^2)\,(P''_n\,Q_n - Q''_n\,P_n) - 2x(P'_n\,Q_n - Q'_n\,P_n) = 0$$

or $$(1-x^2)\frac{d}{dx}\,(P'_n\,Q_n - Q'_n\,P_n) - 2x(P'_n\,Q_n - Q'_n\,P_n) = 0$$

or $$\frac{d}{dx}\,\{(1-x^2)\,(P'_n\,Q_n - Q'_n\,P_n)\} = 0 \qquad ...(4)$$

Integrating w.r.t. 'x', (4) gives $(1-x^2)\,(Q_n\,P_n' - P_n\,Q_n') = -c$

or $(x^2-1)\,(Q_n\,P_n' - P_n\,Q_n') = c$, c being an arbitrary constant. ...(5)

Part (*ii*) From part (*i*) above, we have

$$Q_n\,P_n' - P_n\,Q_n' = \frac{c}{x^2-1} = \frac{c}{x^2}\left(1-\frac{1}{x^2}\right)^{-1} \quad \text{or} \quad Q_n\,P_n' - P_n\,Q_n' = \frac{c}{x^2}\left(1+\frac{1}{x^2}+\frac{1}{x^4}+\ldots\right). \quad ...(6)$$

From Art. 9.2, $$Q_n = \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)\,(n+2)}{2\cdot(2n+3)}x^{-(n+3)} + \ldots\right] \qquad ...(7)$$

and $$P_n = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{(2n-1)\cdot2}x^{n-2} + \ldots\right]. \qquad ...(8)$$

Using (7) and (8), L.H.S. of (6) $= \dfrac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left\{x^{-(n+1)} + \dfrac{(n+1)\,(n+2)}{2\cdot(2n+3)}x^{-(n+3)} + \ldots\right\}$

$$\times\left[\frac{1\cdot3\cdot5\ldots(2n-1)}{n!}\left\{nx^{n-1} - \frac{n(n-1)\,(n-2)}{(2n-1)\cdot2}x^{n-3} + \ldots\right\}\right]$$

$$-\left[\frac{1\cdot3\cdot5...(2n-1)}{n!}\left\{x^n-\frac{n(n-1)}{(2n-1)\cdot2}x^{n-2}+...\right\}\right]\times\left[\frac{n!}{1\cdot3\cdot5...(2n+1)}\left\{-(n+1)x^{-(n+2)}\right.\right.$$

$$\left.\left.-\frac{(n+1)\,(n+2)\,(n+3)\,x^{-(n+4)}}{2\cdot(2n+3)}+...\right\}\right]$$

$\therefore$ the coefficient of $1/x^2$ in L.H.S. of (6)

$$=\frac{n!}{1\cdot3\cdot5...(2n+1)}\times\frac{1\cdot3\cdot5...(2n-1)}{n!}n-\frac{1\cdot3\cdot5...(2n-1)}{n!}\times\frac{n!}{1\cdot3\cdot5...(2n+1)}(-n-1)$$

$$=\frac{n}{2n+1}+\frac{n+1}{2n+1}=\frac{2n+1}{2n+1}=1.$$

Also the coefficient of $1/x^2$ in R.H.S. of (6) = 1

Hence, by equating the coefficients of $1/x^2$ on both sides of (6), we have $c = 1$. With this value of c, (5) becomes

$$(x^2-1)\,(P_n'\,Q_n-P_n\,Q_n')=1 \qquad \text{or} \qquad -(x^2-1)\,(Q_n'\,P_n-P_n'\,Q_n)=1 \ldots (*)$$

or $$\frac{Q_n'\,P_n-P_n'\,Q_n}{P_n^2}=-\frac{1}{(x^2-1)P_n^2} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{Q_n}{P_n}\right)=-\frac{1}{(x^2-1)P_n^2}.$$

Integrating both sides w.r.t. x from ∞ to x, we get

$$\left[\frac{Q_n}{P_n}\right]_\infty^x=-\int_\infty^x\frac{dx}{(x^2-1)P_n^2}=\int_x^\infty\frac{dx}{(x^2-1)P_n^2} \qquad \text{or} \qquad \frac{Q_n(x)}{P_n(x)}-\lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)}=\int_x^\infty\frac{dx}{(x^2-1)P_n^2} \qquad ...(9)$$

Now, $$\lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)}=\lim_{x\to\infty}\frac{\frac{d^n}{dx^n}Q_n(x)}{\frac{d^n}{dx^n}P_n(x)}, \text{ by L'Hospital's rule}$$

$$=\lim_{x\to\infty}\frac{\frac{n!}{1\cdot3\cdot5\,(2n+1)}\{(-1)^n(n+1)\,(n+2)...2n\,x^{-(2n+1)}+...\}}{\frac{1\cdot3\cdot5...(2n-1)}{n!}n!}, \text{ using (7) and (8)}$$

= 0, on taking limit as $x\to\infty$.

$\therefore$ (9) reduces to $$\frac{Q_n(x)}{P_n(x)}=\int_x^\infty\frac{dx}{(x^2-1)P_n^2}. \qquad ...(10)$$

Part (iii). Deductions from part (ii).

(*a*)Replacing n by 0 in (10), we get $$\frac{Q_0(x)}{P_0(x)}=\int_x^\infty\frac{dx}{(x^2-1)P_0^2(x)}.$$

$\therefore$ $$Q_0(x)=\int_x^\infty\frac{dx}{(x^2-1)}, \text{ as } P_0(x)=1$$

$$=\frac{1}{2}\left[\log\frac{x-1}{x+1}\right]_x^\infty=-\frac{1}{2}\log\frac{x-1}{x+1}=\frac{1}{2}\log\frac{x+1}{x-1}, \text{ as } \lim_{x\to\infty}\log\left(\frac{x-1}{x+1}\right)=\lim_{x\to\infty}\left(\frac{1-1/x}{1+1/x}\right)=0$$

(*b*) Replacing n by 1 in (10), we get $\dfrac{Q_1(x)}{P_1(x)} = \displaystyle\int_x^\infty \frac{dx}{(x^2-1)P_1^2(x)}$.

$\therefore \quad Q_1(x) = x\displaystyle\int_x^\infty \frac{dx}{(x^2-1)x^2}$, as $P_1(x) = x$

$$= x\int_x^\infty \left[\frac{1}{x^2-1} - \frac{1}{x^2}\right]dx = x\left[\frac{1}{2}\log\frac{x-1}{x+1} + \frac{1}{x}\right]_x^\infty = -x\,.\left[\frac{1}{2}\log\frac{x-1}{x+1} + \frac{1}{x}\right], \text{ as before}$$

$$= -\frac{x}{2}\log\frac{x-1}{x+1} - 1 = \frac{x}{2}\log\frac{x+1}{x-1} - 1$$

Ex. 2. *Prove that*

(*i*) $n(Q_nP_{n-1} - P_{n-1}Q_n) = (n-1)\ (Q_{n-1}P_{n-2} - Q_{n-2}P_{n-1})$ *and deduce that*

(*ii*) $n(Q_nP_{n-1} - Q_{n-1}P_n) = -1$ *or* $P_nQ_{n-1} - Q_nP_{n-1} = 1/n$. **(Bilaspur 1997)**

(*iii*) $P_nQ_{n-2} - Q_nP_{n-2} = \{(2n-1)x/n(n-1)\}$.

Sol. (*i*) From recurrence relations of P_n and Q_n, we have

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1} \qquad ...(1)$$

and $$(2n+1)xQ_n = (n+1)Q_{n+1} + nQ_{n-1}. \qquad ...(2)$$

Replacing n by $n-1$ in (1) and (2), we get

$$(2n-1)xP_{n-1} = nP_n + (n-1)P_{n-2} \qquad ...(3)$$

and $$(2n-1)xQ_{n-1} = nQ_n + (n-1)Q_{n-2}. \qquad ...(4)$$

Multiplying (3) by Q_{n-1} and (4) by P_{n-1} and then subtracting, we have

$$0 = n(P_n\,Q_{n-1} - P_{n-1}\,Q_n) + (n-1)\ (P_{n-2}\,Q_{n-1} - P_{n-1}\,Q_{n-2})$$

or $$n(Q_n\,P_{n-1} - Q_{n-1}\,P_n) = (n-1)\ (Q_{n-1}\,P_{n-2} - Q_{n-2}\,P_{n-1}). \qquad ...(5)$$

Part (ii) Deduction. Let $U_n = n(Q_n\,P_{n-1} - Q_{n-1}\,P_n)$. ...(6)

Then (5) may be written as $U_n = U_{n-1}$, which gives $U_{n-1} = U_{n-2} = U_{n-3} = ... = U_3 = U_2 = U_1$.

Thus, we have $U_n = U_1$. or $n(Q_n\,P_{n-1} - Q_{n-1}\,P_n) = Q_1P_0 - Q_0P_1$, by (5)

or $n(Q_n\,P_{n-1} - Q_{n-1}\,P_n) = Q_1 - xQ_0$, as $P_0 = 1$ and $P_1 = x$ ...(7)

But $Q_1 - xQ_0 = \dfrac{x}{2}\log\dfrac{x+1}{x-1} - 1 - x\times\dfrac{1}{2}\log\dfrac{x+1}{x-1} = -1$. [Do as in part (*iii*) of Ex. 1]

With this value of $Q_1 - xQ_0$, (7) gives the required result.

Part (iii). Multiplying (3) by Q_{n-2} and (4) by P_{n-2} and then subtracting, we get

$$(2n-1)x\ (P_{n-1}\,Q_{n-2} - Q_{n-2}\,P_{n-1}) = n(P_n\,Q_{n-2} - Q_n\,P_{n-2})$$

or $$P_n\,Q_{n-2} - Q_n\,P_{n-2} = \frac{(2n-1)x}{n}\ (P_{n-1}\,Q_{n-2} - Q_{n-1}\,P_{n-2}). \qquad ...(9)$$

Replacing n by $n-1$ in (1), $P_{n-1}\,Q_{n-2} - Q_{n-1}\,P_{n-2} = 1/(n-1)$. ...(10)

Using (10), (9) reduces to $P_n\,Q_{n-2} - Q_n\,P_{n-2} = \{(2n-1)x/n(n-1)\}$.

Ex. 3. *Prove that* : $Q_2(x) = \dfrac{1}{2}P_2(x)\log\dfrac{x+1}{x-1}x - \dfrac{3}{2}x$.

Sol. From recurrence relation III for $Q_n(x)$, $(n+1)Q_{n-1} = (2n+1)xQ_n - nQ_{n-1}$. ...(1)

Replacing n by 1 in (1), we get

$$2Q_2 = 3xQ_1 - Q_0 = 3x\left[\frac{x}{2}\log\frac{x+1}{x-1} - 1\right] - \frac{1}{2}\log\frac{x+1}{x-1} \qquad \text{[Do as in part (}iii\text{) of Ex. 1]}$$

$$= \frac{3x^2-1}{2}\log\frac{x+1}{x-1} - 3x = P_2(x)\log\frac{x+1}{x-1} - 3x \qquad \left[\because P_2(x) = \frac{3x^2-1}{2}\right]$$

$$\therefore \qquad Q_2(x) = \frac{1}{2}P_2(x)\log\frac{x+1}{x-1} - \frac{3}{2}x.$$

Ex. 4. *Prove that* $\dfrac{d^{n+1}}{dx^{n+1}}[Q_n(x)] = -\dfrac{(-2)^n\, n!}{(x^2-1)^{n+1}}$.

Sol. We know that

$$Q_n(x) = \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-(n+5)} + \ldots\right] \quad \ldots(1)$$

Differentiating both sides of (1) $(n+1)$ times w.r.t. x, we get

$$\frac{d^{n+1}}{dx^{n+1}}[Q_n(x)] = \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left[\frac{d^{n+1}}{dx^{n+1}}(x^{-n-1}) + \frac{(n+1)(n+2)}{2(2n+3)}\frac{d^{n+1}}{dx^{n+1}}(x^{-n-3})\right.$$

$$\left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5)}\frac{d^{n+1}}{dx^{n+1}}(x^{-n-5}) + \ldots\right]$$

$$= \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left[(-n-1)\,)(-n-2)\ldots(-n-1-n)x^{-n-1-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}\cdot(-n-3)(-n-4)\ldots\right.$$

$$\left.(-n-3-n)x^{-n-3-n-1} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5}(-n-5)(-n-6)\ldots(-n-5-n)x^{-n-5-n-1} + \ldots\right]$$

$$\left[\because \frac{d^n x^m}{dx^n} = m(m-1)(m-2)\ldots(m-n+1)x^{m-n}\right]$$

$$= \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left[(-1)^{n+1}(n+1)(n+2)\ldots(2n+1)x^{-2n-2}\right.$$

$$\left. + (-1)^{n+1}\frac{(n+1)(n+2)(n+3)\ldots(2n+3)}{2\cdot(2n+3)}x^{-2n-4} + (-1)^{n+1}\frac{(n+1)(n+2)\ldots(2n+5)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-2n-6} + \ldots\right]$$

$$= \frac{(-1)^{n+1}\, n!\,(n+1)(n+2)\ldots(2n+1)x^{-2n-2}}{1\cdot3\cdot5\ldots(2n+1)}\left[1 + \frac{(2n+2)(2n+3)}{2\cdot(2n+3)}x^{-2} + \frac{(2n+2)(2n+3)(2n+4)(2n+5)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-6} + \ldots\right]$$

$$= \frac{(-1)^{n+1}(2n+1)!\,x^{-2n-2}}{1\cdot3\cdot5\ldots(2n+1)}\left[1 + (n+1)x^{-2} + \frac{(n+1)(n+2)}{2!}x^{-4} + \ldots\right]$$

$$= (-1)^{n+1}\, 2\cdot4\cdot6\ldots(2n)\, x^{-2n-2}\left(1 - \frac{1}{x^2}\right)^{-n-1} = (-1)^{n+1}2^n\, n!\, x^{-2n-2}\left(\frac{x^2-1}{x^2}\right)^{-n-1} = -\frac{(-2)^n\, n!}{(x^2-1)^{n+1}}.$$

EXERCISE

1. Prove that $Q_n(x) = 2^n\, n! \int_x^\infty dx \int_x^\infty dx \ldots \int_x^\infty (x^2-1)^{-n-1} dx$. **(Bilaspur 1994, 96)**

2. Prove that $Q_n(x) = \int_0^\infty \frac{dt}{\{x + \Gamma(x^2-1)\cosh t\}^{n+1}}$.

3. By using the definition of $Q_n(x)$, find value of $Q_0(x)$ and $Q_1(x)$.

4. Show that the Wronskian $W(Q_n, P_x)$ of $Q_n(x)$ and $P_n(x)$ is $1/(x^2 - 1)$, *i.e.*, show that $W(Q_n, P_n) = 1/(x^2 - 1)$. **(Ravishankar 2004)**

[**Hint.** Refer equation (*) of part (*ii*) of Ex. 1 of Art. 10.8. Proceeding as indicated, get

$$(x^2-1)\,(Q_nP_n' - Q_n'P_n) = 1 \quad \text{or} \quad Q_nP_n' - Q_n'P_n = 1/(x^2-1) \quad \text{or} \quad W(Q_n, P_n) = 1/(x^2-1)$$

5. $(x^2-1)\,(Q_nP_n' - P_n\, Qn'') =$ (*a*) n^2 (*b*) 0 (*c*) k (*d*) None of these **[Agra 2006]**

[Hint: Ans (c) Refer Ex1, Art 10.8.]

11

Bessel Functions

11.1. Bessel's equation and its solution. [Garhwal 2004; Kanpur 2009]

The differential equation of the form $x^2y'' + xy' + (x^2 - n^2)y = 0$ ...(1)

or $y'' + (1/x) \times y' + (1 - n^2/x^2)y = 0$ ...(1)′

is called *Bessel's equation of order n*, *n* being a non–negative constant. We now solve (1) in series by using the well known method of Frobenius.

Let the series solution (1) be $y = \sum_{m=0}^{\infty} c_m x^{k+m}$, $c_0 \neq 0$. ...(2)

$\therefore\ y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$ and $y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}$.

Substitution for y, y', y'' in (1) now gives

$$x^2 \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} + x \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} + (x^2 - n^2) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m \{(k+m)(k+m-1) + (k+m) - n^2\} x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0.$$

But the bracketed expression in the above identity

$$= (k+m)^2 - (k+m) + (k+m) - n^2 = (k+m)^2 - n^2 = (k+m+n)(k+m-n).$$

So the above identity becomes

$$\sum_{m=0}^{\infty} c_m (k+m+n)(k+m-n) x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} \equiv 0. \quad ...(3)$$

Equating to zero the smallest power of x, namely x^k, (3) gives the indicial equation

$c_0(k+n)(k-n) = 0$ *i.e.*, $(k+n)(k-n) = 0$ as $c_0 \neq 0$. Its roots are $k = n, -n$.

Next equating to zero the coefficient of x^{k+1} in (3) gives

$c_1(k+1+n)(k+1-n) = 0$, so that $c_1 = 0$ for $k = n$ and $k = -n$.

Finally equating to zero the coefficient of x^{k+m} in (3) gives

$c_m(k+m+n)(k+m-n) + c_{m-2} = 0$ or $c_m = \dfrac{1}{(k+m+n)(n-k-m)} c_{m-2}$. ...(4)

Putting $m = 3, 5, 7, \ldots$ in (4) and using $c_1 = 0$, we find

$$c_1 = c_3 = c_5 = c_7 = \ldots = 0. \quad ...(5)$$

Putting $m = 2, 4, 6, \ldots$ in (4) gives $c_2 = \dfrac{1}{(k+2+n)(n-k-2)} c_0$,

$$c_4 = \frac{1}{(k+4+n)(n-k-4)} c_2 = \frac{1}{(k+4+n)(n-k-4)(k+2+n)(n-k-2)} c_0$$

and so on. Putting these values in (2), we get

$$y = c_0 x^k \left[1 + \frac{x^2}{(n+k+2)(n-k-2)} + \frac{x^4}{(n+k+2)(n-k-2)(n+k+4)(n-k-4)} + \ldots \right]$$

Replacing k by n and $-n$ and also replacing c_0 by a and b in the above equation gives

$$y = ax^n \left\{ 1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4 \cdot 8(1+n)(2+n)} - \ldots \right\} \quad \ldots(6)$$

and

$$y = bx^{-n} \left\{ 1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4 \cdot 8(1-n)(2-n)} - \ldots \right\}. \quad \ldots(7)$$

The particular solution of (1) obtained from (6) above by taking the arbitrary constant $a = 1/\{2^n \Gamma(n+1)\}$, is called the *Bessel function of the first kind of order n*. It will be denoted by $J_n(x)$. Thus, we have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8(n+1)(n+2)} - \ldots \right] \quad \ldots(8)$$

or

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r! \Gamma(n+r+1)} \left(\frac{x}{2} \right)^{2r+n} \quad \ldots(9)$$

Replacing b by $1/\{2^n \Gamma(n+1)\}$ in (7) and proceeding as above gives

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r! \Gamma(-n+r+1)} \left(\frac{x}{2} \right)^{2r-n} \quad \ldots(10)$$

Let n be non–integral. We know that $\Gamma(m) = \infty$ if m is zero or a negative integer and $\Gamma(m)$ is finite otherwise. Since n is not an integer and r is always integral, the factor $\Gamma(-n+r+1)$ in (10) is always finite and non–zero. For $2r < n$, (10) shows that $J_{-n}(x)$ contains negative powers of x. On the other hand, (9) shows that $J_n(x)$ is not containing negative powers of x at all. Therefore, we find that at $x = 0$, $J_n(x)$ is finite while $J_{-n}(x)$ is infinite, and so one cannot be expressed as a constant multiple of the other. From these arguments we conclude that $J_n(x)$ and $J_{-n}(x)$ are two independent solutions of (1) when n is not an integer (this condition being stronger than $2n$ non–integral which was assumed earlier). Thus, the general solution of Bessel equation (1) when n is not an integer is

$y = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constant. ... (11)

11.2. Bessel's functions of the first kind of order n. Definition [Nagpur 2003]

Bessel's function of the first kind and of order n is denoted by $J_n(x)$ and is defined as

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r! \Gamma(n+r+1)} \left(\frac{x}{2} \right)^{2r+n}, \text{ where } n \text{ is any non–negative constant.} \quad \ldots(1)$$

Remark 1. When n is an integer, $\Gamma(n+r+1) = (n+r)!$ and so (1) may be rewritten as

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(n+r)!} \left(\frac{x}{2} \right)^{2r+n}. \quad \ldots(2)$$

Replacing n by 0 and 1 in turn in (2), Bessel's functions of orders 0 and 1 are given by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots. \quad \textbf{Kanpur 2005]}$$

and $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 3} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \ldots.$$

Remark 2. When there is no confusion regarding the variable, we shall write J_n for $J_n(x)$ and J'_n for $d\,J_n(x)/dx$ etc.

11.3. List of important results of Gamma function Γ(n) and Beta function B(m, n).

For more details, refer Chapter 6.

(*i*) $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$, $n > 0$ (*ii*) $\Gamma(n) = 2\int_0^\infty e^{-x^2}\, x^{2n-1}\, dx$

(*iii*) $\Gamma(1) = 1$ (*iv*) $\Gamma(1/2) = \sqrt{\pi}$

(*v*) $\Gamma(n+1) = n\,\Gamma(n)$, $n > 0$ (*vi*) $\Gamma(n+1) = n\,!$, if n is +ve integer

(*vii*) $B(m, n) = B(n, m)$ (*viii*) $\Gamma(n)\,\Gamma(1-n) = \pi/\sin n\pi$.

(*ix*) $\Gamma(m) = \infty$, so that $1/\Gamma(m) = 0$ if $m = 0$ or –ve integer

(*x*) $B(m, n) = \int_0^i x^{m-1}(1-x)^{n-1}\, dx = \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$, where $m > 0$, $n > 0$

(*xi*) $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ (*xii*) $\Gamma(2n) = \dfrac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\,\Gamma(n+\frac{1}{2})$

11.4. Relation between $J_n(x)$ and $J_{-n}(x)$, n being an integer

Theorem. I Show that when n is

(*i*) positive integer, $J_{-n}(x) = (-1)^n J_n(x)$. **[Agra 2006; Kanpur 2006, 07, 08 MDU Rohtak 2004; Purvanchal 2006; Meerut 2006; Nagpur 1995;]**

(*ii*) any integer, $J_{-n}(x) = (-1)^n J_n(x)$ **[Kanpur 2004, 08; Ranchi 2010; Meerut 1993]**

Proof. Part (*i*). Let *n* be a +ve integer. We know that

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}. \qquad \ldots(1)$$

Since $n > 0$, so $\Gamma(-n+r+1)$ is infinite (and so $1/\Gamma(-n+r+1)$ is zero) for $r = 0, 1, 2, \ldots, (n-1)$. Keeping this in mind we see that the sum over r in (1) must be taken from n to infinity. Thus,

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \qquad \ldots(2)$$

From (2), $J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^{m+n} \frac{1}{(m+n)\,!\,\Gamma(m+1)} \left(\frac{x}{2}\right)^{2(m+n)-n}$, (on changing the variable of summation to $m = r - n$ so that $r = m + n$ and so $m = 0$ when $r = n$ and $m = \infty$ when $r = \infty$)

$$\therefore \quad J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m (-1)^n \frac{1}{\Gamma(m+n+1)\,m!} \left(\frac{x}{2}\right)^{2m+n} = (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma(r+n+1)\,r!} \left(\frac{x}{2}\right)^{2m+n},$$

(on changing the variable of summation from m to r while keeping the limits of summation unchanged.)

Thus, for $n > 0$ $J_{-n}(x) = (-1)^n J_n(x)$, by the definition of $J_n(x)$. ... (3)

Part (*ii*). Let n < 0. Let p be a positive integer such that $n = -p$. Since $p > 0$, from part (*i*) above, we have $J_{-p}(x) = (-1)^p J_p(x)$ so that $J_p(x) = (-1)^{-p} J_{-p}(x)$.

But $p = -n$ hence the above result becomes $J_{-n}(x) = (-1)^n J_n(x)$, ...(4)

which is of the same form as (3). Hence the required result holds for any integer.

Remark. When n is an integer $J_{-n}(x)$ is not independent of $J_n(x)$, because $J_{-n}(x)$ is a constant multiple of $J_n(x)$ as shown above. Hence $y = AJ_n(x) + BJ_{-n}(x)$ is not the general solution of Bessel equation when n is an integer. Of course, when n is not an integer, the most general solution of Bessel equation is given by $y = AJ_n(x) + BJ_{-n}(x)$. When n is an integer, the nature of general solution is indicated by the following theorem.

Theorem II. The two independent solutions of Bessel's equation may be taken to be $J_n(x)$ and

$$Y_n(x) = \frac{\cos n\pi\, J_n(x) - J_{-n}(x)}{\sin n\pi}, \text{ for all values of } n. \qquad ...(5)$$

Proof. Case I. Let n be not an integer. Since n is not an integer, $\sin n\pi \neq 0$. Hence (5) shows that $Y_n(x)$ is a linear combination of $J_n(x)$ and $J_{-n}(x)$. But we know that $J_n(x)$ and $J_{-n}(x)$ are independent solutions if n is not an integer. Hence $J_n(x)$ and a linear combination of $J_n(x)$ and $J_{-n}(x)$ will also be independent solutions. Thus we find that $J_n(x)$ and $Y_n(x)$ are two independent solutions of Bessel's equation.

Case II. Let n be an integer. Then we have $\cos n\pi = (-1)^n$, $\sin n\pi = 0$ and $J_{-n}(x) = (-1)^n J_n(x)$.

Using these values in (5), we find that $Y_n(x)$ has the form 0/0 and so $Y_n(x)$ is undefined. To make $Y_n(x)$ meaningful, we define it as

$$Y_n(x) = \lim_{v \to n} Y_v(x) = \lim_{v \to n} \frac{\cos v\pi\, J_v(x) - J_{-v}(x)}{\sin v\pi} \qquad ...(6)$$

$$= \frac{[(\partial / \partial v)\{(\cos v\pi\, J_v(x) - J_{-v}(x)\}]_{v=n}}{[(\partial / \partial v)\cos v\pi)]_{v=n}}, \text{ by L' Hospital's rule}$$

$$= \frac{\left[-\pi \sin v\pi\, J_v(x) + \cos v\pi \frac{\partial}{\partial v} J_v(x) - \frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{[\pi \cos v\pi]_{v=n}} = \frac{\cos n\pi \left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - \left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{\pi \cos n\pi}$$

$$= \frac{(-1)^n \left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - (-1)^{2n} \left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{\pi (-1)^n} = \frac{1}{\pi}\left[\frac{\partial}{\partial v} J_v(x) - (-1)^n \frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n} \qquad ...(7)$$

We now establish the following two results about $Y_n(x)$ as given by (6).

(*i*) $Y_n(x)$ is a solution of Bessel's equation. (*ii*) $Y_n(x)$ is a solution independent of $J_n(x)$.

Proof of (i). Since $J_v(x)$ and $J_{-v}(x)$ are solutions of Bessel's equation of order v, we must have

$$x^2 \frac{d^2 J_v}{dx^2} + x \frac{dJ_v}{dx} + (x^2 - v^2) J_v = 0 \qquad ...(8)$$

and

$$x^2 \frac{d^2 J_{-v}}{dx^2} + x \frac{dJ_{-v}}{dx} + (x^2 - v^2) J_{-v} = 0. \qquad ...(9)$$

Differentiating (8) and (9) w.r.t. 'v', we obtain

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right) + x \frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right) + (x^2 - v^2) \frac{\partial J_v}{\partial v} - 2vJ_v = 0 \qquad ...(10)$$

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right) + x \frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right) + (x^2 - v^2) \frac{\partial J_{-v}}{\partial v} - 2vJ_{-v} = 0. \qquad ...(11)$$

Multiplying (11) by $(-1)^v$ and subtracting from (10) gives

$$x^2 \frac{d^2}{dx^2}\left\{\frac{\partial J_v}{\partial v} - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\} + x \frac{d}{dx}\left\{\frac{\partial}{\partial v} J_v - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\} + (x^2 - v^2)\left\{\frac{\partial}{\partial v} J_v - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\}$$
$$- 2v\{J_v - (-1)^v J_{-v}\} = 0 \qquad ...(12)$$

Taking $v = n$ in (12) and using (7), we have

$$x^2\frac{d^2}{dx^2}\{\pi\, Y_n(x)\} + x\frac{d}{dx}\{\pi\, Y_n(x)\} + (x^2 - n^2)\,\pi Y_n(x) - 2n\{J_n(x) - (-1)^n\, J_{-n}(x)\} = 0$$

Since n is an integer, $J_{-n}(x) = (-1)^n J_n(x)$ by theorem I and hence the last term in the above equation vanishes. So the above equation reduces to

$$x^2 Y_n'' + xY_n' + (x^2 - n^2)Y_n = 0, \qquad ...(13)$$

showing that $Y_n(x)$ is also a solution of Bessel's equation of order n.

Proof of (ii). We know that an explicit expression $Y_n(x)$ for n integral is given by

$$Y_n(x) = \frac{2}{\pi}\left\{\log\frac{x}{2} + \gamma - \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r}\right\}J_n(x) - \frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{1}{m!\,(n+m)!}\left(\frac{x}{2}\right)^{n+2m}\sum_{r=1}^{m}\left\{\frac{1}{r} + \frac{1}{r+n}\right\}$$

$$-\frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{-n+2m}, \qquad ...(14)$$

where γ is Euler's constant. From (14) we find that $Y_n(x)$ is infinite when $x = 0$, whereas $J_n(x)$ is infinite when $x = 0$. So $Y_n(x)$ as given by (6) and $J_n(x)$ are two independent solutions of Bessel's equation of order n.

Remark 1. General solution of Bessel's equation when n is an integer is

$$y = AJ_n(x) + BY_n(x), \text{ A and B being arbitrary constants.} \qquad ...(15)$$

where $Y_n(x)$ is given by (6). $Y_n(x)$ is known as *Bessel's function of order n of the second kind.* $Y_n(x)$ is also called the *Neumann function of order n* and is denoted by $N_n(x)$.

Remark 2. Equations reducible to Bessel's equation

Consider $$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \qquad ...(16)$$

Let $z = \lambda x$ so that $\dfrac{dy}{dx} = \lambda\dfrac{dy}{dz}$ and $\dfrac{d^2y}{dx^2} = \lambda^2\dfrac{d^2y}{dz^2}$. Then (16) becomes

$$z^2(d^2y/dz^2) + z(dy/dz) + (z^2 - n^2)y = 0, \qquad ...(17)$$

which is Bessel's equation of order n. As explained in remark 1 above, the general solution of (17) is

$y = AJ_n(z) + BY_n(z)$ or $y = AJ_n(\lambda x) + BY_n(\lambda x)$. ...(18)

Thus $J_n(\lambda x)$ and $Y_n(\lambda x)$ are solutions of (16), which is called the *modified Bessel's equation.*

11.5. Bessel's function of the second kind of order n. Definition

This is denoted by $Y_n(x)$ and is defined by

$$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi},\ n \neq \text{integer}$$

and

$$Y_n(x) = \lim_{v\to n}\frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi},\ n \text{ is an integer.}$$

11.6. Integration of Bessel's equation xy″ + y′ + xy = 0 in series for n = 0. Bessel's function of zeroth order, i.e. $J_0(x)$. [Kakitiya 1997]

Bessel's equation for $n = 0$ is given by $$xy'' + y' + xy = 0 \qquad ...(1)$$

Let its series solution be $$y = \sum_{m=0}^{\infty} c_m x^{k+m},\ c_0 \neq 0. \qquad ...(2)$$

$$\therefore\ y' = \sum_{m=0}^{\infty}(k+m)\,c_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty}(k+m)\,(k+m-1)\,c_m x^{k+m-2}. \qquad ...(3)$$

Substituting for y, y', y'' in (1), we obtain

$$x\sum_{m=0}^{\infty}(k+m)(k+m-1)\,c_m x^{k+m-2}+\sum_{m=0}^{\infty}(k+m)\,c_m x^{k+m-1}+x\sum_{m=0}^{\infty}c_m x^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-1}+\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1}+\sum_{m=0}^{\infty}c_m x^{k+m+1}=0$$

or $$\sum_{m=0}^{\infty}\{(k+m)(k+m-1)+(k+m)\}\,c_m x^{k+m-1}+\sum_{m=0}^{\infty}c_m x^{k+m+1}=0$$

or $$\sum_{m=0}^{\infty}(k+m)^2\,c_m x^{k+m-1}+\sum_{m=0}^{\infty}c_m x^{k+m+1}=0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the lowest power of x, namely x^{k-1}, we have

$k^2c_0 = 0$ so that $k = 0, 0$ (as $c_0 \neq 0$)

Now equating to zero the coefficient of next higher power of x, namely x^k, in (4), we have

$c_1(k+1)^2 = 0$ so that $c_1 = 0$ as $(k+1)^2 \neq 0$ for $k = 0$.

Finally, equating to zero the coefficient of x^{k+m-1} in (4), we get

$$(k+m)^2c_m + c_{m-2} = 0 \quad \text{or} \quad c_m = -\frac{c_{m-2}}{(k+m)^2}, \text{ for all } m \geq 2.$$

When $k = 0$, we have $c_m = -(1/m^2)c_{m-2}$ for all $m \geq 2$ ...(5)

Putting $m = 3, 5, 7, \ldots$ in (5) and noting that $c_1 = 0$, we obtain

$$c_1 = c_3 = c_5 = c_7 = \ldots = 0. \qquad ...(6)$$

Next, putting = 2, 4, 6, ... in (5), we have

$$c_2 = -\frac{c_0}{2^2}, \qquad c_4 = -\frac{c_2}{4^2} = \frac{c_0}{2^2\cdot 4^2}, \qquad c_6 = -\frac{c_4}{6^2} = -\frac{c_0}{2^2\cdot 4^2\cdot 6^2}, \text{ ... and so on}$$

Putting the above values in (2) for $k = 0$, $y = \sum_{m=0}^{\infty}c_m x^m = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots.$

or $$y = c_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\ldots\ldots \text{ad. inf}\right)$$

If $c_0 = 1$, the above solution is denoted by $J_0(x)$ so that

$$J_0(x) = 1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\ldots\ldots \text{ad. inf}, \qquad ...(7)$$

where $J_0(x)$ is known as *Bessel's function of zeroeth order.*

Note 1. Replacing n by 0 in Art. 11.2, we can deduce (7).

Note 2. From (7), we have $J_0(0) = 1$. **[Nagpur 1995]**

11.6.A. Solved examples based on Art. 11.1 to 11.6

Ex. 1. *Prove that*

(*i*) $J_{-1/2}(x) = \sqrt{(2/\pi x)}\cos x$. **[Garhwal 2005; Nagpur 2005; Kanpur 2009, 10; Agara 2010; Bhopal 2010; Ranchi 2010]**

(*ii*) $J_{1/2}(x) = \sqrt{(2/\pi x)}\sin x$. **[Nagpur 2003, 05; Garhwal 2004; Kanpur 2004; 07]**

(*iii*) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = 2/(\pi x)$. **[Lucknow 2010; Meerut 1992, 93]**

Sol. By the definition of $J_n(x)$, we have

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1-\frac{x^2}{2\cdot 2(n+1)}+\frac{x^4}{2\cdot 4\cdot 2^2(n+1)(n+2)}-\ldots\right] \qquad ...(1)$$

Part (i). Replacing n by – (1/2) in (1) and simplifying, we get

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \ldots\right] = \sqrt{\frac{2}{\pi x}}\cos x, \text{ as } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Part (ii). Replacing n by 1/2 in (1) and simplifying, we get

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\Gamma(3/2)}\cdot\left[1 - \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \ldots\right] = \sqrt{\left(\frac{x}{2}\right)}\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\cdot\frac{1}{x}\left[x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \ldots\right]$$

$[\because\ \Gamma(p+1) = p\Gamma(p)]$

$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

Part (iii). Squaring and adding the results of (*i*) and (*ii*), we get

$$[J_{1/2}(x)]^2 + J_{-1/2}(x)]^2 = (2/\pi x)\ (\sin^2 x + \cos^2 x) = 2/\pi x.$$

Ex. 2. *Prove that* $\lim_{z\to 0}\frac{J_n(z)}{z^n} = \frac{1}{2^n\Gamma(n+1)^n}$, *where* $n > -1$. **(Kanpur 2005, 07)**

Sol. By defination, $$J_n(z) = \frac{z^n}{2^n\ \Gamma(n+1)}\left[1 - \frac{z^2}{4(n+1)} + \frac{z^4}{4\ .\ 8\ .\ (n+1)\ .\ (n+2)} - \ldots\right]$$

$$\therefore\quad \lim_{z\to 0}\frac{J_n(z)}{z^n} = \frac{1}{2^n\ \Gamma(n+1)}\left[1 - \frac{z^2}{4(n+1)} + \frac{z^4}{4\ .\ 8\ .\ (n+1)\ .\ (n+2)} - \ldots\right] = \frac{1}{2^n\Gamma(n+1)}$$

Ex. 3. *Write the general solution of the following equations:*

(*i*) $x^2(d^2y/dx^2) + x(dy/dx) + (x^2 - 25)y = 0$ **(MDU Rohtak 2005)**

(*ii*) $x^2(d^2y/dx^2) + x(dy/dx) + (x^2 - 9/16)y = 0$

(*iii*) $d^2y/dx^2 + (1/x) \times (dy/dx) + (1 - 1/6.25x^2)\ y = 0$

(*iv*) $x^2\ (d^2z/dx^2) + x(dz/dx) + (x^2 - 64)\ z = 0$

(*v*) $z(d^2y/dz^2) + (dy/dz) + zy = 0$

Sol. In what follows, we shall use the following solutions of Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Result I: $y = A\ J_n\ (x) + B\ J_{-n}(x)$, where n is not an integer. A, B being arbitrary constants.

Result II: $y = A\ J_n(x) + B\ Y_n(x)$, where n is an integer, A, B being arbitrary constants.

(*i*) Given $x^2y'' + xy' + (x^2 - 5^2)y = 0$, which is Bessel's equation of order 5, which is an integer. Its general solution is $y = A\ J_5(x) + B\ Y_5(x)$, where A and B are arbitrary constants.

(*ii*) Given $x^2y'' + xy' + \{x^2 - (3/4)^2\} = 0$, which is Bessel's equation of order 3/4, which is not an integer. Its solution is $y = A\ J_{3/4}(x) + B\ J_{-3/4}\ (x)$, where A and B are arbitrary constants

(*iii*) Re-writing, given equating becomes $x^2y'' + xy' + \{x^2 - (2/5)^2\}y = 0$

As in part (*ii*), solution is $y = A\ J_{2/5}(x) + B\ J_{-2/5}(x)$, A, B being arbitrary constants

(*iv*) Given $x^2(d^2z/dx^2) + x(dz/dx) + (x^2 - 8^2)\ z = 0$

As in part (*i*), solution is $z = A\ J_8\ (x) + B\ Y_8(x)$, A, B being arbitrary constants

(*v*) Re-writing the given equation, $z^2(d^2y/dz^2) + z(dy/dz) + z^2y = 0$

which is a Bessel equation of order 0, which is an integer. Its solution is $y = A\ J_0(z) + BY_0(z)$.

Ex. 4. *Solve the following differential equation*:

(*i*) $x^2(d^2y/dx^2) + x(dy/dx) + (4x^4 - 1/4)y = 0$

(*ii*) $x(d^2y/dx^2) + dy/dx + (y/4) = 0$ by using the subtitution $z = \sqrt{x}$

Sol. (*i*) Suppose that $z = x^2$ ... (1)

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2x\frac{dy}{dz}, \text{ by (1)} \qquad \text{... (2)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dx}\left(\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = 2\frac{dy}{dz} + (2x)^2\frac{d^2y}{dz^2}, \text{ by (1)}$$

Substituting the above values in the given equation, we get

$$x^2\left(2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}\right) + 2x^2\frac{dy}{dz} + \left(4x^4 - \frac{1}{4}\right)y = 0 \quad \text{or} \quad 4z^2\frac{d^2y}{dz^2} + 4z\frac{dy}{dz} + \left(4z^2 - \frac{1}{4}\right)y = 0$$

or $$z^2(d^2y/dz^2) + z(dy/dz) + \{z^2 - (1/4)^2\}y = 0 \qquad \text{... (3)}$$

(3) is a Bessel's equation of order 1/4. Since 1/4 is a positive non-integral real number, hence solution of (3) is $y = A\,J_{1/4}(z) + B\,J_{-1/4}(z)$ or $y = A\,J_{1/4}(x^2) + B\,J_{-1/4}(x^2)$.

(*ii*) Given $z = \sqrt{x}$ ... (1)

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{2\sqrt{x}}\frac{dy}{dz}, \text{ using (1)} \qquad \text{... (2)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\frac{dy}{dz}\right) = -\frac{x^{-3/2}}{4}\frac{dy}{dz} + \frac{1}{2\sqrt{x}}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

or $$\frac{d^2y}{dx^2} = -\frac{1}{4x^{3/2}}\frac{dy}{dz} + \frac{1}{2\sqrt{x}}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = -\frac{1}{4x^{3/2}}\frac{dy}{dx} + \frac{1}{4x}\frac{d^2y}{dz^2} \qquad \text{... (3)}$$

Using (1), (2) and (3), the given equation, reduces to

$$x\left(-\frac{1}{4x^{3/2}}\frac{dy}{dz} + \frac{1}{4x}\frac{d^2y}{dz^2}\right) + \frac{1}{2\sqrt{x}}\frac{dy}{dz} + \frac{1}{4}y = 0 \quad \text{or} \quad -\frac{1}{4z}\frac{dy}{dz} + \frac{1}{4}\frac{d^2y}{dz^2} + \frac{1}{2z}\frac{dy}{dz} + \frac{1}{4}y = 0$$

or $$z^2(d^2y/dz^2) + z(dy/dz) + (z^2 - 0^2)y = 0,$$

which is a Bessel's equation of order 0 and so its solution is

$$y = A\,J_0(z) + B\,Y_0(z) \quad \text{or} \quad y = A\,J_0(\sqrt{x}) + B\,Y_0(\sqrt{x}),$$

which is the general solution of the given equation, A, B being arbitrary constants.

Ex. 5 (a) *Solve* $x(d^2y/dx^2) + 2(dy/dx) + (xy)/2 = 0$ *in terms of Bessel's functions.*

(KU Kurukshetra 2005)

(b) Solve $x(d^2z/dx^2) - 2(dz/dx) + xz = 0$ by using the substitution $y = z/x^{3/2}$.

Sol. (a) Given $x(d^2y/dx^2) + 2(dy/dx) + (xy)/2 = 0$ ... (1)

Assume that $z = y\sqrt{x}$ so that $y = z/\sqrt{x}$ ... (2)

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(x^{-1/2}z) = x^{-1/2}\frac{dz}{dx} - \frac{x^{-3/2}}{2}z$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(x^{-1/2}\frac{dz}{dx}\right) - \frac{1}{2}\frac{d}{dx}(x^{-3/2}z) = x^{-1/2}\frac{d^2z}{dx^2} - \frac{x^{-3/2}}{2}\frac{dz}{dx} - \frac{1}{2}\left(x^{-3/2}\frac{dz}{dx} - \frac{3x^{-5/2}}{2}z\right)$$

or $$\frac{d^2y}{dx^2} = \frac{1}{x^{1/2}}\frac{d^2z}{dx^2} - \frac{1}{x^{3/2}}\frac{dz}{dx} + \frac{3}{4x^{5/2}}z$$

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x\left(\frac{1}{x^{1/2}}\frac{d^2z}{dx^2} - \frac{1}{x^{3/2}}\frac{dz}{dx} + \frac{3z}{4x^{5/2}}\right) + 2\left(\frac{1}{x^{1/2}}\frac{dz}{dx} - \frac{z}{2x^{3/2}}\right) + \frac{x}{2}\times\frac{z}{x^{1/2}} = 0$$

or $$x^{1/2}\frac{d^2z}{dx^2} + \frac{1}{x^{1/2}}\frac{dz}{dx} - \frac{z}{4x^{3/2}} + \frac{1}{2}x^{1/2}z = 0 \quad \text{... (3)}$$

Multiplying both sides of (3) by $x^{3/2}$, $$x^2(d^2z/dx^2) + x(dz/dx) + (x^2/2 - 1/4)\,z = 0 \quad \text{... (4)}$$

Let $$u = x/\sqrt{2} \quad \text{... (5)}$$

$\therefore$ $$\frac{dz}{dx} = \frac{dz}{du}\frac{du}{dx} = \frac{dz}{du}\times\frac{1}{\sqrt{2}}, \text{ by (5)} \quad \text{... (6)}$$

From (*6*), we have $$\frac{d}{dx} \equiv \frac{1}{\sqrt{2}}\frac{d}{du} \quad \text{... (7)}$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\sqrt{2}}\frac{d}{du}\left(\frac{1}{\sqrt{2}}\frac{dz}{du}\right), \text{ by (6) and (7)}$$

Thus, $$d^2y/dx^2 = (1/2) \times (d^2z/du^2) \quad \text{... (8)}$$

Substituting the above values in (4), we get

$$2u^2\times\frac{1}{2}\frac{d^2z}{du^2} + u\sqrt{2}\times\frac{1}{\sqrt{2}}\frac{dz}{du} + \left(u^2 - \frac{1}{4}\right)z = 0 \quad \text{or} \quad u^2\frac{d^2z}{du^2} + u\frac{dz}{du} + \left\{u^2 - \left(\frac{1}{2}\right)^2\right\}z = 0 \quad \text{... (9)}$$

which is a Bessel equation of order 1/2. Since 1/2 a positive non-negative integer, hence the required solution is given by

$$z = A\,J_{1/2}(u) + B\,J_{1/2}(u) \quad \text{or} \quad y\sqrt{x} = A\,J_{1/2}(x/\sqrt{2}) + BJ_{-1/2}(x/\sqrt{2}), \text{ by (2) and (5)}$$

(b) Ans. $z = c_1\,x^{3/2}J_{3/2}(x) + c_2x^{3/2}J_{-3/2}(x)$, c_1, c_2 being arbitrary constants

Ex. 6. *Verify that the Bessel function* $J_{1/2}(x) = (\sin x)\times(2/\pi x)^{1/2}$ *satisfies the Bessel equation of order* 1/2. **(MDU Rohtak 2006)**

Sol. Bessel equation of order 1/2 is given by

$$x^2(d^2y/dx^2)\;\; x(dy/dx) + (x^2 - 1/4)y = 0 \quad \text{... (1)}$$

Let $$y = J_{1/2}(x) = (2/\pi)^{1/2}\times(x^{-1/2}\sin x) \quad \text{... (2)}$$

$\therefore$ $$dy/dx = (2/\pi)^{1/2}\times\left\{x^{-1/2}\cos x + (-1/2)\times x^{-3/2}\sin x\right\}$$

$$d^2y/dx^2 = (2/\pi)^{1/2}\times\left\{-(1/2)\times x^{-3/2}\cos x - x^{-1/2}\sin x + (3/4)\times x^{-5/2}\sin x + (-1/2)\times x^{-3/2}\cos x\right\}$$

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x^2\times(2/\pi)^{1/2}\left\{-x^{-3/2}\cos x - x^{-1/2}\sin x + (3/4)\times x^{-5/2}\sin x\right\} + x\times(2/\pi)^{1/2}\left\{x^{-1/2}\cos x - (1/2)\times x^{-3/2}\sin x\right\} + (x^2 - 1/4)\times(2/\pi)^{1/2}\times(x^{-1/2}\sin x) = 0$$

or $$(2/\pi)^{1/2}\left\{-x^{1/2}\cos x - x^{3/2}\sin x + (3/4)\times x^{-1/2}\sin x + x^{1/2}\cos x - (1/2)\times x^{-1/2}\sin x + x^{3/2}\sin x - (1/4)\times x^{-1/2}\sin x\right\} = 0$$

or $0 = 0$, which is true. Hence $y = J_{1/2}(x)$ satisfies the Bessel equation (1) of order 1/2.

Ex. 7. *Show that* $d\,(x^n J_n(ax))/dx = a\,x^n J_{n-1}(ax)$ and hence deduce that $d(x J_1(x))/dx = x J_0(x)$.

Sol. We know that $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n} \quad \text{... (1)}$$

$$\therefore \quad x^n J_n(ax) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{ax}{2}\right)^{2r+n} \quad \text{or} \quad x^n J_n(ax) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+n} x^{2r+2n}}{2^{2r+n} r!\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}\left(x^n J_n(ax)\right) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+n} \times 2(r+n) x^{2r+2n-1}}{2^{2r+n} r!\,\Gamma\,(n+r+1)} \quad \text{... (2)}$$

From (1), $$J_{n-1}(ax) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\,(n+r)}\left(\frac{ax}{2}\right)^{2r+n-1}$$

$$\therefore \quad ax^n J_{n-1}(ax) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+n} x^{2r+2n-1} \times 2(r+n)}{2^{2r+n-1} r!\,\Gamma\,(n+r) \times 2(r+n)}$$

or $$ax^n J_{n-1}(ax) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+n} \times 2(r+n) x^{2r+2n-1}}{2^{2r+n} r!\,\Gamma\,(r+n+1)} \quad \text{... (3)}$$

From (2) and (3), $$\frac{d}{dx}\left(x^n J_n(ax)\right) = ax^n J_{n-1}(x) \quad \text{... (4)}$$

Putting $n = 1$ and $a = 1$ in (4), we have $$\frac{d}{dx}(xJ_1(x)) = x\,J_0(x)$$

Ex. 8. *Show that* $\displaystyle\int_0^1 \frac{u\,J_0(xu)}{(1-u^2)^{1/2}}\,du = \frac{\sin x}{x}$

Sol. We have (see Art. 11.2), $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots\ldots \quad \text{... (1)}$$

$$\therefore \quad \int_0^1 \frac{uJ_0(xu)}{(1-u^2)^{1/2}}\,du = \int_0^1 \frac{u}{(1-u^2)^{1/2}}\left(1 - \frac{x^2}{4}u^2 + \frac{x^4}{4 \cdot 16}u^4 \ldots\right)du$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\left(1 - \frac{x^2}{4}\sin^2\theta + \frac{x^4}{64}\sin^4\theta - \ldots\right)\cos\theta\,d\theta = \int_0^{\pi/2} \sin\theta\,d\theta - \frac{x^2}{4}\int_0^{\pi/2} \sin^3\theta\,d\theta + \frac{x^4}{64}\int_0^{\pi/2} \sin^5\theta\,d\theta \ldots.$$

[On putting $u = \sin\theta$ and $du = \cos\theta\,d\theta$]

$$= \left[-\cos\theta\right]_0^{\pi/2} - \frac{x^2}{4} \times \frac{2}{3} + \frac{x^4}{64} \times \frac{4 \cdot 2}{5 \cdot 3} \ldots.$$ [using Walli's formula of Integral calculus]

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) = \frac{\sin x}{x}$$

Ex. 9. *Show that* $\displaystyle\int_0^{\pi/2} J_1(z\cos\theta)\,d\theta = \frac{1-\cos z}{z}$ **[Bilaspur 1997]**

Sol. We have $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n} \quad \text{... (1)}$$

$$(1) \quad \Rightarrow \quad J_1(z\cos\theta) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(1+r+1)}\left(\frac{z\cos\theta}{2}\right)^{2r+1}$$

$$\therefore \quad \int_0^{\pi/2} J_1\left(z\cos\theta\right) d\theta = \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r!\,(r+1)!\,2^{2r+1}} \int_0^{\pi/2} \cos^{2r+1}\theta\, d\theta$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r!\,(r+1)!2^{2r+1}} \cdot \frac{2r(2r-2)...4\cdot 2}{(2r+1)(2r-1)...5\cdot 3} \qquad \text{[using a standard result of Integral calculus]}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r!\,(r+1)!2^{2r+1}} \cdot \frac{[2r(2r-2)...4\cdot 2]^2}{(2r+1)\,2r\,(2r-1)\,(2r-2)...5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r!\,(r+1)!2^{2r+1}} \cdot \frac{2^{2r}(r1)^2}{(2r+1)!} = \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{2(r+1)(2r+1)!} = \frac{1}{z} - \frac{1}{z} + \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - ... = \frac{1-\cos z}{z}$$

Ex. 10. *Prove that* $J_n(x) = \dfrac{x^n}{2^{n-1}\Gamma(n)} \displaystyle\int_0^{\pi/2} \sin\theta\cos^{2n-1}\theta\, J_0(x\sin\theta) d\theta$, *where* $n > -1/2$.

[Ravishankar 2002]

Sol. We have
$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n} \qquad \text{... (1)}$$

$$(1) \quad \Rightarrow \qquad J_0(x\sin\theta) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)}\left(\frac{x\sin\theta}{2}\right)^{2r} \qquad \text{... (2)}$$

Let
$$I = \int_0^{\pi/2} \sin\theta\cos^{2n-1}\theta\, J_0(x\sin\theta) d\theta \qquad \text{... (3)}$$

$$\therefore \quad I = \int_0^{\pi/2} \sin\theta\cos^{2n-1}\theta \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)} \frac{x^{2r}}{2^{2r}} \sin^{2r}\theta\right) d\theta, \text{ using (2)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(r!)^2 2^{2r}} \int_0^{\pi/2} \cos^{2n-1}\theta \sin^{2r+1}\theta\, d\theta = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(r!)^2 2^{2r}} \cdot \frac{\Gamma(n)\,\Gamma(r+1)}{\Gamma(n+r+1)} = \frac{\Gamma(n)}{2} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r}$$

or
$$\frac{x^n}{2^{n-1}\Gamma(n)} I = \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\,(n+r+1)}\left(\frac{x}{2}\right)^{2r} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r+1)!}\left(\frac{x}{2}\right)^{2r+n}$$

or
$$\frac{x^n}{2^{n-1}\Gamma(n)} \int_0^{\pi/2} \sin\theta\cos^{2n-1}\theta\, J_0(x\sin\theta)\, d\theta = J_n(x), \text{ by (1) and (3)}$$

Ex. 11. *Show that* $y = A\,J_n(x) \displaystyle\int_0^x \frac{dx}{xJ_n^2(x)} + B\,J_n(x)$ *is the complete solution of Bessel's equation.* **[Bilaspur 1994]**

Sol. The Bessel's equation is
$$y'' + (1/x)\times y + (1 - n^2/x^2)y = 0 \qquad \text{... (1)}$$

We know that a solution (1) is
$$u = J_n(x) \qquad \text{... (2)}$$

Let the complete solution of (1) be
$$y = u\,v \qquad \text{... (3)}$$

Comparing (1) with $y'' + Py' + Qy = R$, $P = 1/x$ and $Q = 1 - n^2/x^2$, $R = 0$

*Then, we know that v is given by

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u} \qquad \text{or} \qquad \frac{d^2v}{dx^2}+\left(\frac{1}{x}+\frac{2\,J_n'(x)}{J_n(x)}\right)\frac{dv}{dx}=0 \qquad \ldots (4)$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$ ... (5)

Then (4) yields $\dfrac{dq}{dx}+\left(\dfrac{1}{x}+\dfrac{2J_n'}{J_n}\right)q=0$ or $\dfrac{dq}{q}+\left(\dfrac{1}{x}+\dfrac{2J_n'}{J_n}\right)dx=0.$

Integrating, $\log q + \log x + 2 \log J_n = \log A$ or $q\,x\,J_n^2 = A$

or $q = dv/dx = A/(x\,J_n^2)$ or $dv = \left\{A/(x\,J_n^2)\right\}dx$

Integrating, $v = A\displaystyle\int_0^x \frac{dx}{x\,J_n^2} + B,$ ... (6)

where A and B are arbitrary constants. From (2), (3) and (6) the required complete integral is

$$y = J_n(x)\left(A\int_0^x \frac{dx}{x\,J_n^2}+B\right) \qquad \text{or} \qquad y = A\,J_n(x)\int_0^x \frac{dx}{x\,J_n^2(x)} + BJ_n(x)$$

Ex. 12. *Prove that if $n > m - 1$, then* $J_n(x) = \dfrac{2(x/2)^{n-m}}{\Gamma(n-m)}\displaystyle\int_0^1 (1-t^2)^{n-m-1}\,t^{m+1}\,J_m(xt)\,dt$.

Sol. Let $I = \displaystyle\int_0^1 (1-t^2)^{n-m-1}\,t^{m+1}\,J_m(xt)\,dt.$

Then using the definition of $J_m(xt)$, we have

$$I = \int_0^1 (1-t^2)^{n-m-1}\,t^{m+1}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r\,!\,\Gamma(m+r+1)}\left(\frac{xt}{2}\right)^{2r+m}dt$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r+m}}{r\,!\,\Gamma(m+r+1)}\int_0^1(1-t^2)^{n-m-1}\,t^{2m+2r+1}\,dt = \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r+m}}{r\,!\,\Gamma(m+r+1)}\int_0^1(1-t^2)^{n-m-1}\,(t^2)^{m+r}\,t\,dt$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r+m}}{r\,!\,\Gamma(m+r+1)}\cdot\frac{1}{2}\int_0^1(1-z)^{n-m-1}\,z^{(m+r+1)-1}\,dz, \text{ on putting } t^2 = z \text{ and } t\,dt = \frac{dz}{2}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r+m}}{r\,!\,\Gamma(m+r+1)}\cdot\frac{1}{2}\cdot\frac{\Gamma(n-m)\,\Gamma(m+r+1)}{\Gamma(n+r+1)}, \text{ provided } n-m>0 \text{ and } m+r+1>0 \textit{ i.e. } n>m>-1$$

$$\left[\because \int_0^1(1-z)^{\alpha-1}z^{\beta-1}dz = B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \text{ if } \alpha>0, \beta>0\right]$$

$$= \frac{\Gamma(n-m)}{2}\left(\frac{x}{2}\right)^{m-n}\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = \frac{\Gamma(n-m)}{2}\left(\frac{x}{2}\right)^{m-n}J_n(x), \text{ by def. of } J_n(x)$$

$$\therefore\ J_n(x) = 2\frac{(x/2)^{m-n}}{\Gamma(n-m)}\,I \qquad \text{or} \qquad J_n(x) = 2\frac{(x/2)^{m-n}}{\Gamma(n-m)}\int_0^1(1-t^2)^{n-m-1}t^{m+1}J_m(xt)\,dt, \text{ using (1)}$$

---

*Refer Chapter 10 in part I of ''Ordinary and Partial Differential Equations'' by Dr. M.D. Raisinghania published by S.Chand & Co., New Delhi

Ex. 13. *Prove that* (*i*) $J_n(x) = \dfrac{(x/2)^n}{\sqrt{\pi}\,\Gamma(n+1)} \int_{-1}^{1} (1-t^2)^{n-1/2} e^{ixt}\, dt$, if $n \geq -\frac{1}{2}$

(*ii*) $J_n(x) = \dfrac{x^n}{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n-1)} \int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt$.

Sol. Part (i). Let $I = \int_{-1}^{1} (1-t^2)^{n-1/2} e^{ixt}\, dt$.

But $e^A = 1 + A + \dfrac{A^2}{2!} + ... + \dfrac{A^r}{r!} + ... = \sum_{r=0}^{\infty} \dfrac{A^r}{r!}$

$\therefore \quad I = \int_{-1}^{1} (1-t^2)^{n-1/2} \left\{ \sum_{r=0}^{\infty} \dfrac{(ixt)^r}{r!} \right\} dt$ or $I = \sum_{r=0}^{\infty} \dfrac{(ix)^r}{r!} \int_{-1}^{1} (1-t^2)^{n-1/2}\, t^r dt$. ...(2)

Now, if r is odd (*i.e.* $r = 2m+1$), the integrand in the above integral is an odd function of t and hence it vanishes whereas if r is even (*i.e.* $r = 2m$), the integral is even function of t and so by a property of definite integral $\int_{-1}^{1} (1-t^2)^{n-1/2}\, t^{2m} dt = 2\int_0^1 (1-t^2)^{n-1/2}\, t^{2m} dt$. So (2) gives

$$I = \sum_{m=0}^{\infty} \frac{(ix)^m}{(2m)!} \cdot 2\int_0^1 (1-t^2)^{n-1/2}\, t^{2m}\, dt = \sum_{m=0}^{\infty} \frac{(i^2)^m\, x^{2m}}{(2m)!} \int_0^1 (1-t^2)^{n-1}\, t^{2m-1} \cdot 2t\, dt, \text{ by (1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{(2m)!} \int_0^1 (1-z)^{n-1/2}\, (z^{1/2})^{2m-1} \cdot dz, \quad \text{on putting } t^2 = z \text{ so that } 2t\, dt = dz$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{(2m)!} \int_0^1 (1-z)^{(n+1/2)-1}\, z^{(m+1/2)-1}\, dz = \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{(2m)!} \times \frac{\Gamma(n+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(n+m+1)},$$

provided $n+\frac{1}{2} > 0$, $m+\frac{1}{2} > 0$ *i.e.* $n > -\frac{1}{2}$, $m > -\frac{1}{2}$

$$\left[\because \int_0^1 (1-x)^{\alpha-1}\, x^{\beta-1}\, dx = B(\alpha,\beta) = \frac{\Gamma(a)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \text{ if } \alpha > 0, \beta > 0 \right]$$

$$= \Gamma(n+\tfrac{1}{2}) \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{(2m)!\,\Gamma(n+m+1)} \times \frac{(2m)!}{2^{2m}} \times \frac{\sqrt{\pi}}{m!}$$

$$\left[\because \text{ By duplication formula, when } m \text{ is a+ve integer, } \Gamma(m+\tfrac{1}{2}) = \frac{(2m)!\sqrt{\pi}}{2^{2m} m!} \right]$$

$$= \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n} = \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\left(\frac{x}{2}\right)^{-n} J_n(x), \text{ by definition of } J_n(x)$$

$$\therefore \qquad J_n(x) = \frac{(x/2)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}}\, I \qquad \text{...(3)}$$

or $$J_n(x) = \frac{(x/2)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{n-1/2}\, e^{ixt} dt, \text{ by (1).}$$

Part (ii). Since $e^{ixt} = \cos xt + i \sin xt$, (1) gives

$$I = \int_{-1}^{1} (1-t^2)^{n-1/2}\, (\cos xt + i \sin xt)\, dt = \int_{-1}^{1} (1-t^2)^{n-1/2} \cos xt + i \int_{-1}^{1} (1-t^2)^{n-1/2} \sin xt\, dt$$

or $$I = 2\int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt + 0 \qquad ...(4)$$

[$\because$ the integrand in the first integral is an even function of t, while the integrand in the second integral is an odd function of t and it is known that

$$\int_{-a}^{a} f(t)\, dt = \begin{cases} 2\int_0^a f(t)\, dt, \text{ if } f(t) \text{ is even function} \\ 0, \text{ if } f(t) \text{ is odd function} \end{cases}$$

Using (4) and noting that $\sqrt{\pi} = \Gamma(1/2)$, (3) gives

$$J_n(x) = \frac{(x/2)^n}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})} \cdot 2\int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt = \frac{x^n}{2^{n-1}\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})} \int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt.$$

Ex. 14. (*i*) *Prove that* $J_n J'_{-n} - J'_n J_{-n} = -\dfrac{2\sin n\pi}{x\pi}$. **[Vikram 2004]**

(*ii*) Prove that $\dfrac{d}{dx}\left(\dfrac{J_{-n}}{J_n}\right) = -\dfrac{2\sin n\pi}{\pi x\, J_n^2}$. **[Bilaspur 1998]**

(*iii*) Prove that $J'_n(x)\, J_{-n}(x) - J_n(x)\, J'_{-n}(x) = c/x$ where c is a constant. By considering the behaviour for the large values of x, show that $c = (2 \sin n\pi)/\pi$.

(*iv*) Show that the Wronskian $W(J_n, J_{-n})$ of J_n and J_{-n} is given by

$W(J_n, J_{-n}) = -(2/\pi x) \times \sin n\pi$. **[Ravishankar 1998, 2000]**

Sol. (i) We know that J_n and J_{-n} are solutions of Bessel's equation

$$y'' + (1/x)y' + (1 - n^2/x^2)y = 0.$$

$\therefore$ $$J_n'' + (1/x)J_n' + (1 - n^2/x^2)J_n = 0 \qquad ...(1)$$

and $$J_{-n}'' + (1/x)J_{-n}' + (1 - n^2/x^2)J_{-n} = 0 \qquad ...(2)$$

Multiplying (1) by J_{-n} and (2) by J_n and then subtracting, we have

$$J_n'' J_{-n} - J_{-n}'' J_n + (1/x) \times (J_n' J_{-n} - J_{-n}' J_n) = 0 \qquad ...(3)$$

Let $$J_n' J_{-n} - J_{-n}' J_n = v. \qquad ...(4)$$

Differentating w.r.t. 'x' (4) gives

$$J_n'' J_{-n} + J_n' J_{-n}' - (J_{-n}'' J_n + J_{-n}' J_n') = v' \quad \text{or} \quad J_n'' J_{-n} - J_{-n}'' J_n = v'. \qquad ...(5)$$

Using (4) and (5), (3) becomes

$v' + (1/x)v = 0$ or $(dv/dx) + (1/x)v = 0$ or $(1/v)dv + (1/x)dx = 0$.

Integrating, $\log v + \log x = \log c$ or $v = c/x$

or $$J'_n J_{-n} - J'_{-n} J_n = c/x, \text{ by (4)} \qquad ...(6)$$

Now, $$J_n(x) = \frac{1}{2^n\,\Gamma(n+1)}\left[x^n - \frac{x^{n+2}}{4(n+1)} + \frac{x^{n+4}}{4\cdot 8\cdot (n+1)(n+2)} - ...\right]$$

and $$J_{-n}(x) = \frac{1}{2^{-n}\,\Gamma(-n+1)}\left[x^{-n} - \frac{x^{2-n}}{4(1-n)} + \frac{x^{4-n}}{4\cdot 8\cdot (1-n)(2-n)} - ...\right]$$

$\therefore$ Using the above values of J_n and J_{-n}, (6) gives

$$\frac{1}{2\Gamma(n+1)}\left[nx^{n-1}-\frac{(n+2)\,x^{n+1}}{4(n+1)}+\frac{(n+4)\,x^{n+3}}{4\cdot 8\cdot(n+1)(n+2)}-\ldots\right]$$

$$\times\frac{1}{2^{-n}\,\Gamma(-n+1)}\left[x^{-n}-\frac{x^{2-n}}{4(1-n)}+\frac{x^{4-n}}{4\cdot 8\cdot(1-n)(2-n)}-\ldots\right]$$

$$-\frac{1}{2^{-n}\,\Gamma(-n+1)}\left[-n\,x^{-n-1}-\frac{(2-n)\,x^{1-n}}{4(1-n)}+\frac{(4-n)\,x^{3-n}}{4\cdot 8\cdot(1-n)(2-n)}-\ldots\right]$$

$$\times\frac{1}{2^{n}\,\Gamma(n+1)}\left[x^{n}-\frac{x^{n+2}}{4(n+1)}+\frac{x^{n+4}}{4\cdot 8\cdot(n+1)(n+2)}-\ldots\right]=\frac{c}{x}$$

Now comparing the coefficients of $1/x$ from both sides, we get

$$\frac{n}{2^{n}\Gamma(n+1)\cdot 2^{-n}\Gamma(-n+1)}+\frac{n}{2^{-n}\Gamma(-n+1)\cdot 2^{n}\Gamma(n+1)}=c \qquad \text{or} \qquad c=\frac{2n}{n\Gamma(n)\,\Gamma(1-n)}$$

or $$c=\frac{2}{(\pi/\sin n\pi)}=\frac{2\sin n\pi}{\pi} \qquad \left[\because\ \Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}\right]$$

Putting this value of c in (6) and multiplying both sides by (–1), we get

$$J_n\,J'_{-n}-J'_nJ_{-n}=-\frac{2\sin n\pi}{\pi x}. \qquad \ldots(7)$$

Part (ii). Dividing both sides of (7) by $J_n^{\,2}$, we get

$$\frac{J_nJ'_{-n}-J'_nJ_{-n}}{J_n^2}=-\frac{2\sin n\pi}{\pi x\,J_n^2} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right)=-\frac{2\sin n\pi}{\pi x\,J_n^2}.$$

Part (iii) Refer part (i).

(*iv*) By definition, $W(J_n,J_{-n})=\begin{vmatrix}J_n & J_{-n}\\ J'_n & J'_{-n}\end{vmatrix}=J_n\,J'_{-n}-J_n'\,J_{-n}=-(2/\pi x)\times\sin n\pi$, by (7)

Ex. 15(a). *Prove that* $J_n(x)=(-2)^n\,x^n\,\dfrac{d^{\,n}J_0(x)}{d(x^2)^n}$. **[Ranchi 2010]**

(*b*) *Show that if* $J_n(x)=kx^n\left(\dfrac{d}{dx^2}\right)^n J_0(x)$, *then* $k=-(2)^n$. **[Ravishankar 1998]**

Sol. (*a*) We know that $J_0(x)$ is a solution of Bessel's equation of order zero (*i.e.* $n=0$) namely,

$$d^2y/dx^2+(1/x)\times(dy/dx)+y=0. \qquad \ldots(1)$$

Let $x^2=X$ so that $x=\sqrt{X}$ and $dX/dx=2x=2\sqrt{X}$. ...(2)

$\therefore$ $$\frac{dy}{dx}=\frac{dy}{dX}\,\frac{dX}{dx}=2\sqrt{X}\,\frac{dy}{dX},\ \text{by (2)} \qquad \ldots(3)$$

From (3) $$\frac{d}{dx}\equiv 2\sqrt{X}\,\frac{d}{dX}. \qquad \ldots(3)'$$

$\therefore$ $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=2\sqrt{X}\,\frac{d}{dX}\left(2\sqrt{X}\,\frac{dy}{dx}\right),\ \text{by (3) and (3)}'$$

$$=4\sqrt{X}\,\frac{d}{dX}\left(\sqrt{X}\,\frac{dy}{dX}\right)=4\sqrt{X}\left[\frac{1}{2}(X)^{-1/2}\,\frac{dy}{dX}+\sqrt{X}\,\frac{d^2y}{dX^2}\right]=4X\,\frac{d^2y}{dX^2}+2\,\frac{dy}{dX} \quad \ldots(4)$$

Putting the values of x, dy/dx and d^2y/dx^2 from (2), (3) and (4) in (1), we get

$$4X\frac{d^2y}{dX^2}+2\frac{dy}{dX}+\frac{1}{\sqrt{X}}.2\sqrt{X}\frac{dy}{dX}+y=0 \qquad \text{or} \qquad 4\sqrt{X}\frac{d^2y}{dX^2}+4\frac{dy}{dX}+y=0. \quad ...(5)$$

Differentiation of (5) n times w.r.t. 'X' by Leibnitz theorem gives

$$4\left[X\frac{d^{n+2}y}{dX^{n+2}}+n.1\frac{d^{n+1}y}{dX^{n+1}}\right]+4\frac{d^{n+1}y}{dX^{n+1}}+\frac{d^ny}{dX^n}=0 \text{ or } 4X\frac{d^{n+2}y}{dX^{n+2}}+4(n+1)\frac{d^{n+1}y}{dX^{n+1}}+\frac{d^ny}{dX^n}=0 \quad ....(6)$$

Since $y=J_0(x)$ is a solution of (1), we now take

$$Y=\frac{d^ny}{dX^n}=\frac{d^nJ_0(x)}{d(x^2)^n} \qquad \text{so that} \qquad \frac{dY}{dX}=\frac{d^{n+1}y}{dX^{n+1}} \qquad \text{and} \qquad \frac{d^2y}{dX^2}=\frac{d^{n+2}y}{dX^{n+2}} \quad ...(7)$$

$\therefore$ (6) gives $$4X\frac{d^2y}{dX^2}+4(n+1)\frac{dy}{dX}+X=0. \quad ...(8)$$

But $J_n(x)$ is a solution of Bessel equation $$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0. \quad ...(9)$$

Let $y=x^nz$ so that $J_n(x)=x^nz.$ ...(10)

Differentiating (10), $\frac{dy}{dx}=x^n\frac{dz}{dx}+nx^{n-1}z$ and $\frac{d^2y}{dx^2}=x^n\frac{d^2z}{dx^2}+2nx^{n-1}\frac{dz}{dx}+n(n-1)z^{n-2}z$.

Using the above values of y, dy/dx and d^2y/dx^2, (9) gives

$$x^n\frac{d^2z}{dx^2}+2nx^{n-1}\frac{dz}{dx}+n(n-1)x^{n-2}z+\frac{1}{x}\left(x^n\frac{dz}{dx}+nx^{n-1}z\right)+\left(1-\frac{n^2}{x^2}\right)x^nz=0$$

or $$x^n\frac{d^2z}{dx^2}+(2n+1)\frac{x^n}{x}.\frac{dz}{dx}+x^nz=0 \qquad \text{or} \qquad \frac{d^2z}{dx^2}+(2n+1)\frac{1}{x}\frac{dz}{dx}+z=0. \quad ...(11)$$

Using (2) and also (3) and (4) [after replacing y by z here], (11) gives

$$\left(4X\frac{d^2z}{dX^2}+2\frac{dz}{dX}\right)+\frac{(2n+1)}{\sqrt{x}}.2\sqrt{X}\frac{dz}{dX}+z=0$$

or $$4X(d^2z/dX^2)+4(n+1)(dz/dX)+z=0. \quad ...(12)$$

Comparing (8) and (12), we have $z=kY$, where k is a constant to be determined. Using (7) and (10), we have $x^nz=kx^nY$ or $J_n(x)=kx^n\frac{d^nJ_0(x)}{d(x^2)^n}$. ...(13)

But $$J_0(x)=\sum_{r=0}^{\infty}\frac{(-1)^r\,x^{2r}}{(2^r\,r\,!)^2}=\sum_{r=0}^{\infty}\frac{(-1)^r\,X^r}{(2^r\,r\,!)^2} \qquad (\because\ x^2=X)$$

Differentiating both sides w.r.t. x^2 (*i.e.* X) n times, we have

$$\frac{d^nJ_0(x)}{d(x^2)^n}=\sum_{r=0}^{\infty}\frac{(-1)^r}{(2^r\,r\,!)^2}\frac{d^n}{dX^n}X^r \quad ...(14)$$

$$=\frac{(-1)^n}{(2^n\,n!)^2}\,n\,!+\text{terms involving } X \text{ (}i.e.\ x^2\text{)}$$

(Note that on differentiation all those terms in R.H.S. of (14) for which $r<n$ will vanish).

Using value of $J_n(x)$ and the above expression, (13) gives

$$\frac{x^n}{2^n\Gamma(n+1)}\left[1-\frac{x^2}{2\cdot 2(n+1)}+\ldots\right]=kx^n\left[\frac{(-1)^n\,n\,!}{(2^n\,n\,!)^2}+\text{terms contaning } x^2\right]$$

Equaing the coefficients of x^n from both sides, we get

$$\frac{1}{2^n\Gamma(n+1)}=k\frac{(-1)^n\,n\,!}{(2^n\,n\,!)^2} \qquad \text{or} \qquad k=(-1)^n\,2^n=(-2)^n \qquad \ldots(15)$$

$[\because\ \Gamma(n+1)=n\,!,\ n \text{ being +ve integer}]$

With this value of k, (13) gives $\quad J_n(x)=(-2)^n x^n\frac{d^nJ_0(x)}{d(x^2)^n}.$

(*b*) Proceed as in part (*a*). From (15), we have $k=(-2)^n$

EXERCISE 11 (A)

1. Show that $y=x^{-n/2}\,J_n(2\sqrt{x}\,)$ satisfies the equation $xy''+(n+1)y'+y=0$.

2. Show that $\int_0^{\pi/2}\sqrt{(\pi x)}\,J_{1/2}(2x)\,dx=1$.

3. (*a*) Prove : $\int_0^\infty J_n(bx)x^ne^{-ax}dx=\frac{2^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}\cdot\frac{b^n}{(a^2+b^2)^{n+1/2}}$

(*b*) Prove that $\int_0^\infty J_n(bx)\,x^{n+1}\,e^{-ax}dx=\frac{2^{n+1}\Gamma(n+\frac{1}{2})}{\pi}\cdot\frac{ab^n}{(a^2+b)^{n+3/2}},\ a>0.$

4. Prove : $\int_0^\infty J_n(bx)\,x^{n+1}e^{-ax^2}dx=\frac{b^n}{(2a)^{n+1}}\exp(-b^2/4a)$, where exp $(p)=e^p$

5. Prove that $\int_0^\infty J_n(bx)\,x^{n+1}e^{-ax^2}dx=\frac{b^n}{2^{n+1}\,a^{n+2}}\left(n+1-\frac{b^2}{4a}\right)\exp\left(-\frac{b^2}{4a}\right),a>0$.

6. For what value of n the general solution of Bessel's differential equation will be of the form $y=AJ_n(x)+BJ_n(x)$.

7. Write the differential equation satisfied by Bessel's function of order n. Express the following Bessel's functions in terms of trigonometric functions :

(*i*) $J_{1/2}(x)$, (*ii*) $J_{-1/2}(x)$, (*iii*) $J_{3/2}(x)$, (*iv*) $J_{-3/2}(x)$.

11.7. Recurrence Relations (Formulae) for $J_n(x)$. Prove that

I. $\frac{d}{dx}\{x^nJ_n(x)\}=x^nJ_{n-1}(x)$. **[Agra 2005; Guwahati 2007; Kanpur 2004, 09; Nagpur 1996]**

II. $\frac{d}{dx}\{x^{-n}J_n(x)\}=-x^{-n}J_{n+1}(x)$. **[Agra 2008; Gulbarga 2005; Kanpur 2009]**

III. $J'_n(x)=J_{n-1}(x)-(n/x)\,J_n$ or $xJ'_n=-nJ_n+xJ_{n-1}$. **[Agra 2010; Bilaspur 2004; KU Kurukshetra 2005; Agra 1997; Kanpur 2005, 09, 11; Meerut 2010; Nagpur 2010]**

IV. $J'_n(x)=(n/x)\,J_n(x)-J_{n+1}(x)$ or $xJ'_n=nJ_n-xJ_{n+1}$. **[Nagpur 2005; Bangalore 93, 94; Meerut 2005, 07, 11; Kanpur 1999; Bilaspur 1998]**

V. $J'_n(x)=\frac{1}{2}\{J_{n-1}(x)-J_{n+1}(x)\}$ or $J_{n-1}-J_{n+1}=2\,J'_n$. **Agra 2009; Jiwaji 2004, 07; Ravishankar 2004; Nagpur 1995; Kanpur 2007]**

VI. $J_{n-1}(x)+J_{n+1}(x)=(2n/x)J_n(x)$ or $xJ_{n+1}(x)+xJ_{n-1}(x)=2nJ_n(x)$ or $2J_n=x(J_{n-1}+J_{n+1})$. **[Agra 2008; Bangalore 2005; Meerut 2006; Kanpur 2006, 11; Nagpur 2010]**

Proof 1. Using the definition of $J_n(x)$, we have

$$\frac{d}{dx}\{x^n J_n(x)\} = \frac{d}{dx}\left\{x^n \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n}} \cdot \frac{d}{dx}(x^{2r+2n}) = \sum_{r=0}^{\infty} \frac{(-1)^r\,(2r+2n)\,x^{2r+2n-1}}{r!\Gamma(n+r+1)\,2^{2r+n}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2 \cdot (r+n)}{r\,!\,(n+r)\,\Gamma(n+r)} \cdot \frac{x^n \cdot x^{2r+n-1}}{2^{2r+n}} \qquad [\because\ \Gamma(n+1) = n\Gamma(n)]$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n\ -1+r+1)}\left(\frac{x}{2}\right)^{2r+n-1} = x^n J_{n-1}(x).$$ by the definition of $J_{n-1}(x)$.

II. Using the definition of $J_n(x)$, we have

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = \frac{d}{dx}\left\{x^{-n} \sum_{r=0}^{\infty}(-1)^r \frac{1}{r\,!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n}} \cdot \frac{d}{dx}(x^{2r}) = \sum_{r=0}^{\infty} \frac{(-1)^r\ 2rx^{2r-1}}{r(r-1)\,!\Gamma(n+r+1)\,2^{2r+n}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\ x^{2r-1}}{(r-1)\,!\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n-1}} = \sum_{r=1}^{\infty} \frac{(-1)^r\ x^{2r-1}}{(r-1)\,!\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n-1}}$$

(since $(r-1)\,! = \infty$ when $r = 0$ so the term corresponding to $r = 0$ vanishes)

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\ x^{2m+2-1}}{m\,!\Gamma(n+m+2)} \cdot \frac{x^n \cdot x^{-n}}{2^{2m+2+n-1}},$$ (on changing the variable of summation to $m = r-1$ so that $r = m+1$. Then $m = 0$ when $r = 1$ and $m = \infty$ when $r = \infty$)

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m\,!\Gamma(n+m+2)}\left(\frac{x}{2}\right)^{n+1+2m} = -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\Gamma(n+1+r+1)}\left(\frac{x}{2}\right)^{n+1+2r}$$

(on changing the variable of summation from m to r)

$= -x^{-n} J_{n+1}(x)$, by the definition of $J_{n+1}(x)$.

III. Recurrence relation I is

$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x) \quad \text{or} \quad n\,x^{n-1} J_n(x) + x^n J'_n(x) = x^n J_{n-1}(x).$$

Dividing both sides by x^{n-1},

$$n\,J_n(x) + x\,J'_n(x) = x\,J_{n-1}(x)$$

or $(n/x)\,J_n(x) + J_n'(x) = J_{n-1}(x)$ or $J'_n(x) = J_{n-1}(x) - (n/x)\,J_n(x)$

IV. Recurrence relation II is

$$\frac{d}{dx}\{x^{-n}\,J_n(x)\} = -x^{-n}\,J_{n+1}(x) \quad \text{or} \quad -n\,x^{-n-1} J_n(x) + x^{-n}\,J'_n(x) = -x^{-n} J_{n+1}(x)$$

Dividing both sides by x^{-n},

$$n\,x^{-1}\,J_n(x) + J'_n(x) = -J_{n+1}(x)$$

or $(-n/x)\,J_n + J'_n = J_{n+1}$ or $J'_n = (n/x)\,J_n - J_{n+1}$.

V. From recurrence relations III and IV, we have $J'_n(x) = J_{n-1}(x) - (n/x)J_n(x)$ ...(1)

and $J'_n(x) = (n/x)\,J_n(x) - J_{n+1}(x)$. ...(2)

Adding (1) and (2), $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ or $J'_n(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$.

VI. From recurrence relations III and IV, we have $J'_n(x) = J_{n-1}(x) - (n/x)J_n(x)$ ...(1)

and $J'_n(x) = (n/x)\,J_n(x) - J_{n+1}(x)$. ...(2)

Subtracting (2) from (1), we get

$0 = J_{n-1} + J_{n+1} - 2(n/x)\,J_n$ or $J_{n-1} + J_{n+1} = (2n/x)\,J_n$.

11.7. A. Solved examples based on recurrence relations

Ex. 1(*a*). *Show that* $x^nJ_n(x)$ is a solution of $x(d^2y/dx^2) + (1 - 2n) \times (dy/dx) + xy = 0$.

[CDLU 2004]

(*b*) *Show that* $x^{-n}J_n(x)$ is a solution of $x(d^2y/dx^2) + (1 + 2n) \times (dy/dx) + xy = 0$.

[MDU Rohtak 2005]

Sol. (*a*) Given $x(d^2y/dx^2) + (1 - 2n) \times (dy/dx) + xy = 0$ ... (1)

Let $y = x^n\,J_n(x)$ ... (2)

From recurrence relation I, $\dfrac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$

or $dy/dx = x^n\,J_{n-1}$, using (2) ... (3)

From (3), $d^2y/dx^2 = x^n\,J'_{n-1} + n\,x^{n-1}\,J_{n-1}$

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x(x^nJ'_{n-1} + nx^{n-1}J_{n-1}) + (1-2n)x^nJ_{n-1} + x^{n+1}J_n = 0 \quad \text{or} \quad x^{n+1}J'_{n-1} - (n-1)x^nJ_{n-1} + x^{n+1}J_n = 0$$

or $$x^{n+1}\left[J'_{n-1} - \{(n-1)/x\}\,J_{n-1}\right] + x^{n+1}J_n = 0 \quad \text{... (4)}$$

From recurrence relation VI, we have

$$J'_n(x) = \frac{n}{x}J_n - J_{n+1}(x) \quad \text{so that} \quad J'_{n-1} - \frac{n-1}{x}J_{n-1} = -J_n(x) \quad \text{... (5)}$$

Using (5), (4) reduces to $-x^{n+1}\,J_n + x^{n+1}\,J_n = 0$, *i.e.*, $0 = 0$, which is true.

Hence $x^n\,J_n$ is a solution of (1).

(*b*) Given $x(d^2y/dx^2) + (1 + 2n) \times (dy/dx) + xy = 0$ ... (1)

Let $y = x^{-n}J_n(x)$ ... (2)

From recurrence relation II, $\dfrac{d}{dx}\left\{x^{-n}J_n(x)\right\} = -x^{-n}J_{n+1}(x)$

or $dy/dx = -x^{-n}J_{n+1}$, using (2)

From (3), $d^2y/dx^2 = -x^{-n}J'_{n+1} + n\,x^{-n-1}J_{n+1}$ ... (3)

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x(-x^{-n}J'_{n+1} + nx^{-n-1}J_{n+1}) + (1+2n) \times (-x^{-n}J_{n+1}) + x^{-n+1}J_n = 0$$

or $-x^{-n+1}J'_{n+1} - (n+1)x^{-n}J_{n+1} + x^{-n+1}J_n = 0$ or $-x^{-n+1}\left[J'_{n+1} + \{(n+1)/x\}\,J_{n+1}\right] + x^{-n+1}J_n = 0$ ... (4)

From recurrence relation III, we have

$$J'_n(x) = J_{n-1}(x) - \frac{n}{x}J_n \quad \text{so that} \quad J'_{n+1} + \frac{n+1}{x}J_{n+1} = J_n(x) \quad \text{... (5)}$$

Using (5), (4) reduce to $-x^{-n+1}J_n + x^{-n+1}J_n = 0$, *i.e.*, $0 = 0$, which is true. Hence $x^{-n}J_n$ is a solution of (1).

Ex. 2. (Lommel theorems) Prove that

(*i*) $\left(\frac{1}{x}\frac{d}{dx}\right)^m \left(x^n J_n\right) = x^{n-m}J_{n-m}$, where m is positive integer and $m < n$.

(*ii*) $\left(\frac{1}{x}\frac{d}{dx}\right)^m (x^{-n}J_n) = (-1)^m x^{-n-m}J_{n+m}$

(*iii*) $J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$, n being positive integer.

Sol. (*i*) From recurrence relation I, we have

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1} \qquad \text{so that} \qquad \left(\frac{1}{x}\frac{d}{dx}\right)(x^n J_n) = x^{n-1}J_{n-1} \qquad \ldots (1)$$

Now, $\left(\frac{1}{x}\frac{d}{dx}\right)^m (x^n J_n) = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1}\left(\frac{1}{x}\frac{d}{dx}\right)(x^n J_n) = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1}(x^{n-1}J_{n-1})$, using (1)

.....

$= x^{n-m}J_{n-m}$, on preceeding as before m times more

(*ii*) From recurrence relation II, we have

$$\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1} \qquad \text{so that} \qquad \left(\frac{1}{x}\frac{d}{dx}\right)(x^{-n}J_{n+1}) = (-1)^1 x^{-n-1}J_{n+1} \qquad \ldots (2)$$

Now, $\left(\frac{1}{x}\frac{d}{dx}\right)^m (x^{-n}J_n) = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1}\left(\frac{1}{x}\frac{d}{dx}\right)(x^{-n}J_n) = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1}(-1)^1 x^{-n-1}J_{n+1}$, using (2)

.....

$= (-1)^m x^{-n-m}J_{n+m}$, on proceeding as before m times more ... (3)

(*iii*) Replacing n by 0 and m by n in part (*ii*), we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0 = (-1)^n x^{-n}J_n \qquad \text{or} \qquad J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$$

Ex. 3. *Prove that* $\int_0^1 t\{J_n(t)\}^2\, dt = \frac{1}{2}x^2\{J_n^2(x) - J_{n-1}(x)\,J_{n+1}(x)\}$.

Sol. We have, $\frac{d}{dt}\left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t)\,J_{n+1}(t)\}\right]$

$= t\{J_n^2(t) - J_{n-1}(t)\,J_{n+1}(t)\} + \frac{1}{2}t^2\{2J_n(t)\,J'_n(t) - J'_{n-1}(t)J_{n+1}(t) - J_{n-1}(t)\,J'_{n+1}(t)\}$

$= t\{J_n^2(t) - J_{n-1}(t)\,J_{n+1}(t)\} + \frac{1}{2}t^2\{2J_n(t) \times \frac{1}{2}\{J_{n-1}(t) - J_{n+1}(t)\} - J_{n+1}\left\{\frac{n-1}{t}J_{n-1}(t) - J_n(t)\right\}$

$$- J_{n-1}(t)\left\{J_n(t) - \frac{n+1}{t}J_{n+1}(t)\right\}, \text{ using recurrence relations III, IV and V}$$

$= t\,J_n^2(t)$, on simplification.

$$\therefore \qquad tJ_n^{\,2}(t) = \frac{d}{dt}\left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t)\,J_{n+1}(t)\}\right]. \qquad ...(1)$$

Integrating both sides of (1) w.r.t. 'x' from 0 to x, we get

$$\int_0^x t\,J_n^2(t)\,dt = \left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t)\,J_{n+1}(t)\}\right]_0^x = \frac{x^2}{2}\{J_n^{\,2}(x) - J_{n-1}(x)\,J_{n+1}(x)\}$$

$$[\because\ J_n(0) = J_{n+1}(0) = J_{n-1}(0) = 0]$$

Ex. 4. *Prove that* (*i*) $J_{-3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{-\cos x}{x} - \sin x\right)$ **[Kanput 2005, 10]**

(*ii*) $J_{3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)$ *i.e.* $\sqrt{\left(\frac{\pi x}{2}\right)}J_{3/2}(x) = \frac{\sin x}{x} - \cos x$.

[Purvanchal 2005, Agra 2005, Kakitiya 1997; Kanpur 2008, 11; Bangalore 1997; Meerut 2007; KU Kurukshetra 2004]

Sol. Proceed as in Ex. 1 of Art. 11.6A and prove that

$$J_{-1/2}(x) = \sqrt{(2/\pi x)}\ \cos x \qquad ...(1)$$

and
$$J_{1/2}(x) = \sqrt{(2/\pi x)}\ \sin x. \qquad ...(2)$$

Recurrence relation (VI) is $J_{n-1}(x) + J_{n+1}(x) = (2n/x) \times J_n(x)$. ...(3)

Part (i). Replacing n by $-\frac{1}{2}$ in (3), we have $J_{-3/2}(x) + J_{1/2}(x) = -(2/2x) \times J_{-1/2}(x)$

or $$J_{-3/2}(x) = -J_{1/2}(x) - (1/x)\times J_{-1/2}(x) = -\sqrt{\left(\frac{2}{\pi x}\right)}\ \sin x - \frac{1}{x}\sqrt{\left(\frac{2}{\pi x}\right)}\ \cos x, \text{ by (1) and (2)}$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left(-\frac{\cos x}{x} - \sin x\right).$$

Part (ii). Replacing n by $\frac{1}{2}$ in (3), we have $J_{-1/2}(x) + J_{3/2}(x) = (2/2x)\ \times J_{1/2}(x)$

or $$J_{3/2}(x) = -J_{-1/2}(x) + (1/x) \times J_{1/2}(x) = -\sqrt{\left(\frac{2}{\pi x}\right)}\ \cos x + \frac{1}{x}\sqrt{\left(\frac{2}{\pi x}\right)}\ \sin x, \text{ by (1) and (2)}$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right).$$

Ex. 5. *Prove that* (*i*) $J_{5/2}(x) = (2/\pi x)^{1/2}\left\{\left\{(3-x^2)/x^2\right\}\sin x - (3/x)\times\cos x\right\}$

[Kanpur 2006, 07]

(*ii*) $J_{-5/2}(x) = (2/\pi x)^{1/2}\left\{\left\{(3-x^2)/x^2\right\}\cos x + (3/x)\times\sin x\right\}$ **[KU Kurukshetra 2006]**

Sol. (*i*) From recurrence relation VI, we have

$J_n(x) = (x/2n)\ \{J_{n-1}(x) + J_{n-1}(x)\}$ or $J_{n+1} = (2n/x)J_n(x) - J_{n-1}(x)$ ... (1)

Putting $n = 3/2$ in (1), we have,

$$J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x) = \frac{3}{x}\times\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) - \sqrt{\frac{2}{\pi x}}\sin x$$

[using values of $J_{3/2}(x)$ and $J_{1/2}(x)$ as obtained in Ex. 4 (*ii*), Art. 11.7A and Ex. 1 (*ii*), Art 11.6A]

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2}\sin x - \frac{3\cos x}{x}\right)$$

(*ii*) Re-writing (1), $J_{n-1}(x) = (2n/x)\,J_n(x) - J_{n+1}(x)$ ... (2)

Putting $n = -3/2$ in (1), we have

$$J_{-5/2}(x) = -\frac{3}{x}J_{-3/2}(x) - J_{-1/2}(x) = -\frac{3}{x}\times\sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right) - \sqrt{\frac{2}{\pi x}}\cos x$$

[using values of $J_{-3/2}(x)$ and $J_{-1/2}(x)$ as obtained in Ex. 4 (*i*), Art. 11.7A and Ex. 1 (*i*), Art. 11.6A]

$$= \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\right) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{3-x^2}{x^2}\cos x + \frac{3\sin x}{x}\right)$$

Ex. 6. *Express* $J_4(x)$ in terms of J_0 and J_1.

Sol. Recurrence relation VI is $J_{n+1}(x) = (2n/x)J_n(x) - J_{n-1}(x)$. ...(1)

Replacing n by 3 in (1), we get $J_4(x) = (6/x)J_3(x) - J_2(x)$. ...(2)

Now replacing n by 2 in (1), we get $J_3(x) = (4/x)J_2(x) - J_1(x)$. ...(3)

Using (3), (2) becomes

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x}J_2(x) - J_1(x)\right] - J_2(x) \quad \text{or} \quad J_4(x) = \left(\frac{24}{x^2} - 1\right)J_2(x) - \frac{6}{x}J_1(x). \quad ...(4)$$

Next, replacing n by 1 in (1) gives $J_2(x) = (2/x)J_1(x) - J_0(x)$. ...(5)

Using (5), (4) becomes

$$J_4(x) = \left(\frac{24}{x^2} - 1\right)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \frac{6}{x}J_1(x) \quad \text{or} \quad J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) - \left(\frac{24}{x^2} - 1\right)J_0(x).$$

Ex. 7(a). *Prove that* (*i*) $\frac{1}{2}xJ_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} \ldots$.

(*ii*) $J_{n-1} = (2/x)\,[n\,J_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots]$.

Proof (i). Recurrence relation VI is $2nJ_n = x(J_{n-1} + J_{n+1})$.

Replacing n by $n+1$ in the above relation, we get

$2(n+1)J_{n+1} = x(J_n + J_{n+2})$ or $\frac{1}{2}xJ_n = (n+1)\,J_{n+1} - \frac{1}{2}xJ_{n+2}$. ...(1)

Replacing n by $n+2$ in (1), we get $\frac{1}{2}xJ_{n+2} = (n+3)J_{n+3} - \frac{1}{2}xJ_{n+4}$. ...(2)

Putting the value of $\frac{1}{2}xJ_{n+2}$ from (2) in (1), $\frac{1}{2}xJ_n = (n+1)J_{n+1} - (n+3)J_{n+3} + \frac{1}{2}xJ_{n+4}$. ...(3)

Repalcing n by $n+4$ in (1) gives $\frac{1}{2}xJ_{n+4} = (n+5)J_{n+5} - \frac{1}{2}xJ_{n+6}$. ...(4)

Putting the value of $\frac{1}{2}xJ_{n+4}$ from (4) in (3) gives

$$\frac{1}{2}xJ_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} - \frac{1}{2}xJ_{n+6}$$

Proceeding likewise and noting that $J_n \to 0$ as $n \to \infty$, we get

$$\frac{1}{2}xJ_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} - \ldots \quad ...(5)$$

(*ii*) Replacing n by $n-1$ in (5) and then multiplying both sides by $(2/x)$, we get

$$J_{n-1} = (2/x)\,[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots]$$

Ex. 7(b). *Prove that* $J_{n-1} = (2/x)[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots]$ and hence deduce that $(x/2)J_n = (n+1)J_{n+1} - (n+3)J_{n+3} - (n+5)J_{n+5} + \ldots$ **[Meerut 1993]**

Hint : Proceed as in Ex. 7(*a*).

Ex. 8. *Prove that* $J'_n = (2/x)\left[(n/2)J_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots\right]$

Sol. Recurrence relation III is $xJ'_n = -nJ_n + xJ_{n-1}$ or $J'_n = -(n/x)J_n + J_{n-1}$. ...(1)

From Ex. 7(a) part (*ii*), $J_{n-1} = (2/x)[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots]$ ... (2)

Putting the value of J_{n-1} from (2) in (1), we get

$$J_n' = -(n/x)J_n + (2/x)\,[nJ_n - (n+2)J_{n+2} + (n+4)\,J_{n+4} - \ldots]$$

or $$J'_n = (2/x)\left[(n/2)\,J_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots\right].$$

Ex. 9. *Prove* : $\dfrac{d}{dx}(J_n^{\,2} + J^2_{n+1}) = 2\left(\dfrac{n}{x}J_n^2 - \dfrac{n+1}{x}J^2_{n+1}\right)$ **[Agra 2005; Meerut 1998]**

Sol. From recurrence relation III and IV, we have

$$J'_n = -(n/x)J_n + J_{n-1} \qquad \ldots(1)$$

and $$J'_n = (n/x)J_n - J_{n+1}. \qquad \ldots(2)$$

Replacing n by $n+1$ in (1), we get $$J'_{n+1} = -\frac{n+1}{x}J_{n+1} + J_n. \qquad \ldots(3)$$

$$\frac{d}{dx}(J^2_n + J^2_{n+1}) = 2J_nJ'_n + 2J_{n+1}J'_{n+1} = 2J_n\left(\frac{n}{x}J_n - J_{n+1}\right) + 2J_{n+1}\left(-\frac{n+1}{x}J_{n+1} + J_n\right), \text{ by (2) and (3)}$$

$$= 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J^2_{n+1}\right), \text{ on simplification.}$$

Ex. 10. *Prove that* (*i*) $J_0^{\,2} + 2(J_1^{\,2} + J^2_{\,2} + J_3^{\,2} + \ldots) = 1$.

[Agra 2009, 10; Meerut 2008; Kanpur 2011]

(*ii*) $|\,J_0(x)\,| \le 1$ **[Agra 2009; Meerut 1996, 97, 98; Kanpur 2011]**

(*iii*) $|\,J_n(x)\,| \le 2^{-1/2}$, *when* $n \ge 1$. **[Meerut 1996, 97, 98; Kanpur 2011]**

Sol. (i) From Ex. 9 above, $$\frac{d}{dx}(J^2_n + J^2_{n+1}) = 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J^2_{n+1}\right). \qquad \ldots(1)$$

Replacing n by 0, 1, 2, 3 ... successively in (1), we get

$$\frac{d}{dx}(J_0^{\,2} + J_1^{\,2}) = 2\left(0 - \frac{1}{x}J_1^2\right)$$

$$\frac{d}{dx}(J_1^{\,2} + J_2^{\,2}) = 2\left(\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right)$$

$$\frac{d}{dx}(J_2^{\,2} + J_3^{\,2}) = 2\left(\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right)$$

... ...

Adding these columnwise and noting that $J_n \to 0$ as $n \to \infty$, we get

$$\frac{d}{dx}[J_0^{\,2} + 2(J_1^{\,2} + J_2^{\,2} + \ldots)] = 0$$

Integrating, $$J_0^{\,2}(x) + 2[J_1^{\,2}(x) + J_2^{\,2}(x) + \ldots] = C. \qquad \ldots(2)$$

Replacing x by 0 in (2) and noting that $J_0(0) = 1$ and $J_n(0) = 0$ for $n \ge 1$, we get

$1 + 2(0 + 0 + \ldots) = C$ or $C = 1$. Hence (2) becomes $J_0^{\,2} + 2(J_1^{\,2} + J_2^{\,2} + \ldots) = 1$ ...(3)

Part (ii). From (3), $J_0^{\,2} = 1 - 2(J_1^{\,2} + J_2^{\,2} + \ldots + J^2_{n-1} + J^2_n + J^2_{n+1} + \ldots.)$ ...(4)

Since $J_1^{\,2}, J_2^{\,2}, J_3^{\,2}$... are all positive or zero, (4) gives $J_0^{\,2} \le 1$ so that $|\,J_0(x)\,| \le 1$.

Part (iii). Solving (4) for $J_n^{\,2}$, we have

$$J_n^{\,2} = (1/2) \times (1 - J_0^{\,2}) - (J_1^{\,2} + J_2^{\,2} + \ldots + J^2_{n-1} + J^2_{n+1} + \ldots). \qquad \ldots(5)$$

Since $J_0^{\,2}, J_1^{\,2}, J_2^{\,2}$ ... are all positive or zero, (5) gives $J^2_n \le 1/2$ or $|\,J_n(x)\,| \le 2^{-1/2}$, where $n \ge 1$.

Ex. 11. *Prove that* (*i*) $\dfrac{d}{dx}\{xJ_n(x)\,J_{n+1}(x)\} = x\{J_n^{\,2}(x) - J^2_{n+1}(x)\}$

[Agra 2009; Sager 2004; Meerut 2005; Kanpur 2007]

(*ii*) $x = 2J_0J_1 + 6J_1J_2 + \ldots + 2(2n+1)J_nJ_{n+1} + \ldots.$ **[Bilaspur 1998]**

Sol. (i) $\frac{d}{dx}(xJ_n J_{n+1}) = J_n J_{n+1} + x(J'_n J_{n+1} + J_n J'_{n+1}) = J_n J_{n+1} + J_{n+1}(xJ'_n) + J_n(xJ'_{n+1})$. ...(1)

From recurrence relation III and IV, we have $xJ'_n = nJ_n - xJ_{n+1}$ ...(2)

and $xJ'_n = -nJ_n + xJ_{n-1}$ ...(3)

Replacing n by $n + 1$ in (3), $xJ'_{n+1} = -(n+1)J_{n+1} + xJ_n$. ...(4)

Putting the values of xJ'_n and xJ'_{n+1} from (2) and (4) in (1), we get

$\frac{d}{dx}(xJ_n J_{n+1}) = J_n J_{n+1} + J_{n+1}(nJ_n - xJ_{n+1}) + J_n[-(n+1)J_{n+1} + xJ_n] = x(J_n^2 - J^2_{n+1})$, other terms cancel.

Part (ii). From part (*i*) above we have $\frac{d}{dx}(xJ_n J_{n+1}) = x(J_n^2 - J^2_{n-1})$.

Replacing n by 0, 1, 2, ... successively, we get

$$\frac{d}{dx}(xJ_0 J_1) = x(J_0^2 - J_1^2) \quad ...(1)$$

$$\frac{d}{dx}(xJ_1 J_2) = x(J_1^2 - J_2^2) \quad ...(2)$$

$$\frac{d}{dx}(xJ_2 J_3) = x(J_2^2 - J^2_3) \quad ...(3)$$

....

Multiplying (1), (2), (3) ... by 1, 3, 5, ... respectively and adding, we have

$$\frac{d}{dx}[x(J_0 J_1 + 3J_1 J_2 + 5J_2 J_3 + ...)] = x[(J_0^2 - J_1^2) + 3(J_1^2 - J_2^2) + 5(J_2^2 - J^2_3) + ...]$$

$$= x[J_0^2 + 2(J_1^2 + J_2 + ...)] = x \times 1 = x \text{ , by Ex. 10. } (i)$$

Integrating, $x(J_0 J_1 + 3J_1 J_2 + 5J_2 J_3 + ...) = \int x dx + C = \frac{1}{2}x^2 + C$. ...(4)

Putting $x = 0$ in (4) gives $C = 0$. Putting $C = 0$ in (4) and simplifying, we get

$$2J_0 J_1 + 6J_1 J_2 + 10J_2 J_3 + ... = x.$$

Ex. 12. *Prove that* (*i*) $J_0' = -J_1$. **[Agra 2010; Kanpur 2008]** (*ii*) $J_2 - J_0 = 2J''_0$. **[Agra 2008]**

(*iii*) $J_2 = J_0'' - (1/x)J_0'$. **[Bilaspur 1997]** (*iv*) $J_2 + 3J'_0 + 4J'''_0 = 0$.

Sol. (*i*) Recurrence relation IV is $xJ'_n = nJ_n - xJ_{n+1}$. ...(1)

Replacing n by 0 in (1), we have $xJ'_0 = -xJ_1$ or $J_0' = -J_1$.

Part (ii). Recurrence relation V is $2J'_n = J_{n-1} - J_{n+1}$. ...(2)

Differentiating (2), w.r.t. 'x', we get $2J''_n = J'_{n-1} - J'_{n+1}$ ...(3)

Replacing n by $n - 1$ and $n + 1$ successively in (2), we have

$$2J'_{n-1} = J_{n-2} - J_n \quad ...(4)$$

and $$2J'_{n+1} = J_n - J_{n+2}. \quad ...(5)$$

Putting the values of J'_{n-1} and J'_{n+1} from (4) and (5) in (3), we have

$2J''_n = \frac{1}{2}(J_{n-2} - J_n) - \frac{1}{2}(J_n - J_{n+2})$ or $4J''_n = J_{n-2} - 2J_n + J_{n+2}$. ...(6)

Replacing n by 0 in (6), we have

$4J''_0 = J_{-2} - 2J_0 + J_2 = (-1)^2 J_2 - 2J_0 + J_2$ [$\because J_{-n} = (-1)^n J_n$]

$\therefore$ $4J''_0 = 2(J_2 - J_0)$ or $2J_0'' = J_2 - J_0$.

Part (iii). Replacing n by 1 in (1), we get

$xJ_1' = J_1 - xJ_2$ or $J_2 = x^{-1}J_1 - J_1'$. ...(7)

From part (*i*), $J_1 = -J'_0$ so that $J'_1 = -J''_0$.

$\therefore$ (7) gives $J_2 = x^{-1}(-J'_0) + J''_0 = J_0'' - x^{-1}J_0'$.

Part (iv). Differentating (6), we have $4J_n'' = J'_{n-2} - 2J'_n + J'_{n+2}$. ...(8)

Replacing n by $n - 2$ and $n + 2$ successively in (2), we get

$$2J'_{n-2} = J_{n-3} - J_{n-1} \quad ...(9)$$

and $$2J'_{n+2} = J_{n+1} - J_{n+3}. \quad ...(10)$$

Putting the values of J'_{n-2}, J'_{n+2} and J'_n from (9), (10) and (2) in (8), we get

$$J_n''' = \tfrac{1}{2}(J_{n-3} - J_{n-1}) - (J_{n-1} - J_{n+1}) + \tfrac{1}{2}(J_{n+1} - J_{n+3})$$

or $$8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}. \quad ...(11)$$

Replacing n by 0 in (11), we get

$$8J_0''' = J_{-3} - 3J_{-1} + 3J_1 - J_3 = -2J_3 + 6J_1 \qquad [\because J_{-n} = (-1)^n J_n]$$

$$= -2J_3 - 6J_0' \qquad [\because J_1 = -J'_0]$$

$$\therefore \quad 8J_0''' + 2J_3 + 6J_0' = 0 \quad \text{or} \quad J_3 + 3J_0' + 4J_0''' = 0.$$

Ex. 13. *Show that $J_n(x) = 0$ has no repeated roots except at $x = 0$.*

Sol. If possible suppose $J_n(x) = 0$ has repeated roots; then at least two roots must be equal (say α), that is, α is a double root of $J_n(x) = 0$. Then from the theory of equations, we have

$$J_n(\alpha) = 0 \quad \text{and} \quad J'_n(\alpha) = 0. \quad ...(1)$$

Recurrence relations III and IV, $J_{n-1}(x) = (n/x)J_n(x) + J'_n(x)$. ...(2)

and $$J_{n+1}(x) = (n/x)\,J_n(x) - J'_n(x). \quad ...(3)$$

Replacing x by α in (2) and (3) and using (1), we get

$J_{n+1}(\alpha) = 0$ and $J_{n-1}(\alpha) = 0$ except when $x = 0$.

Since two different power series have distinct sum functions, so $J_{n+1}(\alpha) = 0 = J_{n-1}(\alpha)$ must be absurd. Hence $J_n(x) = 0$ has no repeated roots except at $x = 0$.

Ex. 14. *From the recurrence formula $2J_n' = J_{n-1} - J_{n+1}$, deduce the result*

$$2^r J_n^{\,r}(x) = J_{n-r} - r\,J_{n-r+2} + \frac{r(r-1)}{2!}J_{n-r+4} + \ldots + (-1)^r J_{n+r}. \quad \ldots(1)$$

Sol. Given that $$2J'_n = J_{n-1} - J_{n+1}. \quad ...(2)$$

Clearly (2) shows that (1) is true for $r = 1$. We assume that (1) is true for some particular value of r, say $r = p$. Then we have

$$2^p J^p{}_n(x) = J_{n-p} - p\,J_{n-p+2} + \frac{p\,(p-1)}{2!}J_{n-p+4} + \ldots + (-1)^p J_{n+p}$$

or $$2^p J_n^{\,p}(x) = J_{n-p} - {}^pC_1 J_{n-p+2} + {}^pC_2 J_{n-p+4} + \ldots + (-1)^p J_{n+p}. \quad ...(3)$$

Differentiating (3) w.r.t. x and then multiplying by 2, we get

$$2^{p+1}\mathrm{J}_n^{\,p+1}(x) = 2J'_{n-p} - 2 \times {}^pC_1 J_{n-p+2} + 2 \times {}^pC_2 J'_{n-p+4} + \ldots + 2(-1)^p J'_{n+p}. \quad ...(4)$$

Replacing n by $n - p$, $n - p + 2$, $n - p + 4$ ..., $n + p$ successively in (2),

$$2J'_{n-p} = J_{n-p-1} - J_{n-p+1}$$

$$2J'_{n-p+2} = J_{n-p+1} - J_{n-p+2}$$

$$2J'_{n-p+4} = J_{n-p+3} - J_{n-p+5}$$

....

$$2J'_{n+p} = J_{n-p-1} - J_{n+p+1}$$

Substituting these values in (4), we have

$$2^{p+1}J_n^{\,p+1}(x) = J_{n-p-1} - J_{n-p+1} - {}^pC_1(J_{n-p+1} - J_{n-p-3}) + {}^pC_2(J_{n-p+4} - J_{n-p+5}) + \ldots + (-1)^p\,(J_{n-p-1} - J_{n+p+1})$$

$$= J_{n-p-1} - (1 + {}^pC_1)J_{n-p+1} + ({}^pC_1 + {}^pC_2)J_{n-p+2} + \ldots + (-1)^{p+1}J_{n+p+1}. \quad \ldots(5)$$

Since ${}^pC_r + {}^pC_{r-1} = {}^{p+1}C_r$, we have $1 + {}^pC_1 = {}^pC_0 + {}^pC_1 = {}^{p+1}C_1$, ${}^pC_1 + {}^pC_2 = {}^{p+1}C_2$ and so on. Then (5) becomes

$2^{p+1}J_n^{p+1}(x) = J_{n-p-1} - {}^{p+1}C_1 J_{n-p+1} + {}^{p+1}C_2 J_{n-p+2} + \ldots + (-1)^{p+1} J_{n+p+1}$, showing that (1) is true for $r = p + 1$ if it were true for $r = p$. So (1) is true for all natural numbers by mathematical induction.

Ex. 15. *Prove that* $x^2J''_n(x) = (n^2 - n - x^2)J_n(x) + xJ_{n+1}(x)$, where $n = 0, 1, 2, \ldots$.

Sol. Recurrence relation IV is $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$. ...(1)

Differentiating both sides of (1) w.r.t. 'x', we have

$$xJ''_n(x) + J'_n(x) = nJ'_n(x) - [xJ'_{n+1}(x) + J_{n+1}(x)]$$

or $$x^2J''_n(x) = (n-1)xJ'_n(x) - x[xJ'_{n+1}(x)] - xJ_{n+1}(x). \quad \ldots(2)$$

Recurrence relation III is $xJ'_n(x) = -nJ_n(x) + xJ_{n+1}(x)$

Replacing n by $(n + 1)$ in this relation, we obtain

$$xJ'_{n+1}(x) = -(n+1)J_{n+1}(x) + xJ_n(x). \quad \ldots(3)$$

Substituting for xJ'_n from (1) and for $xJ'_{n+1}(x)$ from (3) in (2), we get

$$x^2J''_n(x) = (n-1)[nJ_n(x) - xJ_{n+1}(x) - x[-(n+1)J_{n+1}(x) + xJ_n(x)] - xJ_{n+1}(x)$$

$$= [(n-1)n - x^2]J_n(x) + [-(n-1) + (n+1) - 1]xJ_{n+1}(x)$$

$\therefore$ $$x^2J''_n(x) = (n^2 - n - x^2)J_n(x) + xJ_{n+1}(x).$$

Ex. 16. *Show that** $\dfrac{J_{n+1}}{J_n} = \dfrac{(x/2)}{(n+1)-} \dfrac{(x/2)^2}{(n+2)-} \dfrac{(x/2)^3}{(n+3)-} \cdots$

Sol. Recurrence relation VI is $J_{n-1} + J_{n+1} = (2n/x) \times J_n$

or $$J_{n-1} = \frac{2n}{x}J_n - J_{n+1}$$ or $$\frac{J_{n-1}}{J_n} = \frac{2n}{x} - \frac{J_{n+1}}{J_n}. \quad \ldots(1)$$

Replacing n by $(n + 1)$ in (1), we get $$\frac{J_n}{J_{n+1}} = \frac{2(n+1)}{n} - \frac{J_{n+2}}{J_{n+1}} \quad \ldots(2)$$

$\therefore$ $$\frac{J_{n+1}}{J_n} = \frac{1}{\dfrac{J_n}{J_{n+1}}} = \frac{1}{\dfrac{2(n+1)}{x} - \dfrac{J_{n+2}}{J_{n+1}}}, \text{ using (2)}$$

$$= \frac{1}{\dfrac{2(n+1)}{x} - \dfrac{1}{(J_{n+1}/J_{n+2})}} = \frac{1}{\dfrac{2(n+1)}{x} - \dfrac{1}{\dfrac{2(n+2)}{x} - \dfrac{J_{n+3}}{J_{n+2}}}}$$

[With help of (2) by replacing n by $n + 1$]

---

*Student should read a chapter on continued fractions in some book on Algebra to understand the notations, namely, $\dfrac{a_1}{a_2 + \dfrac{a_3}{a_4 + \dfrac{a_5}{a_6 + \ldots}}} = \dfrac{a_1}{a_2 +} \dfrac{a_3}{a_4 +} \dfrac{a_5}{a_6 + \ldots}$

$$= \cfrac{1}{\cfrac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cfrac{1}{(J_{n+2}/J_{n+3})}}}$$

$$= \cfrac{1}{\cfrac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{2(n+3)}{x} - \cfrac{J_{n+4}}{J_{n+3}}}}} \qquad \left[\begin{array}{l}\text{With help of (2) by}\\ \text{replacing } n \text{ by } n+2\end{array}\right]$$

$$= \cfrac{x/2}{(n+1) - \cfrac{x/2}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{2(n+3)}{x} + \cdots}}} \qquad \left[\begin{array}{l}\text{Multiply numerator and}\\ \text{denominator by } x/2\end{array}\right]$$

$$= \cfrac{x/2}{(n+1) - \left[\cfrac{(x/2)^2}{(n+2) - \cfrac{(x/2)}{\cfrac{2(n+3)}{x} + \cdots}}\right]} \qquad \left[\begin{array}{l}\text{Repeating the}\\ \text{similar operations}\end{array}\right]$$

$$= \cfrac{(x/2)}{(n+1) -} \ \cfrac{(x/2)^2}{(n+2) -} \ \cfrac{(x/2)^3}{(n+3) - \cdots}.$$

EXERCISE 11 (B)

1. Show that all roots of $J_n(x)$ are real.

2. Prove that between any two zeros of $J_n(x)$ lie one and only one zero of $J_{n+1}(x)$ as well as $J_{n-1}(x)$.

3.(a) Prove that between any two consecutive positive roots of the equation $J_n(x) = 0$, there is one and only one root of the equation $J_{n+1}(x) = 0$.

(b) Show that between two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n-1}(x)$.

4. Prove (*i*) $J_{n+3} + J_{n+5} = (2/x)\,(n+4)J_{n+4}$. **[Kanpur 2009]**

(*ii*) $4J''_n = J_{n-2} - 2J_n + J_{n+2}$.

5. For Bessel's functions $J_n(x)$, find out a and b, where

$d\{J_n(x)\}/dx = aJ_{n-1}(x) + bJ_{n+1}(x)$. **Ans.** $a = 1/2$; $b = -(1/2)$

6. Show that $J_2 - J_0 = 2J_0''$ **[Kanpur 2006]**

7. Evaluate $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$. **Ans.** $J_3(x) = \{(8 - x^2)/x^2\}J_1(x) - (4/x)J_0(x)$

8. Prove that (*a*) $x^2J''_n(x) + J'_n(x) = (n^2/x)J_n(x) - xJ_n(x)$. **[Meerut 1998]**

(*b*) $\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2\,J_n(x)$. **[Kanpur 1998]**

11.7.B. Solved Example involving integration and recurrence relations

Ex. 1. *If* $n > -1$, show that $\int_0^x x^{n+1}J_n(x)\,dx = x^{n+1}J_{n+1}(x)$. **[Bilaspur 1998]**

Sol. From recurrence relations I, $\dfrac{d}{dx}\{x^nJ_n(x)\} = x^nJ_{n-1}(x)$. ...(1)

Replacing n by $n + 1$ in (1), $\dfrac{d}{dx}\{x^{n+1}J_{n+1}(x)\} = x^{n+1}J_n(x)$. ...(2)

Integrating (2) w.r.t. 'x' between the limits 0 and x, we get

$$[x^{n+1}J_{n+1}(x)]_0^x = \int_0^x x^{n+1}J_n(x)\,dx \qquad \text{or} \qquad \int_0^x x^{n+1}J_n(x)\,dx = x^{n+1}J_{n+1}(x).$$

Ex. 2. *Prove that* $J_{n+1}(x) = x\int_0^1 J_n(xy)\,y^{n+1}dy$.

Sol. Let $xy = t$ so that $x\,dy = dt$

$\therefore$ R.H.S. of (1) $= x\int_0^x J_n(t)\,(t/x)^{n-1}(dt/x) = x^{-n-1}\int_0^x t^{n+1}J_n(t)\,dt$

$= x^{-n-1}\int_0^x x^{n+1}J_n(x)\,dx = x^{-n-1}\,x^{n+1}J_{n+1}(x)$, by Ex. 1

$= J_{n+1}(x)$ = L.H.S. of (1).

Ex. 3. *Show that* (*a*) $\int_0^x x^{-n}J_{n+1}(x)\,dx = \dfrac{1}{2^n\,\Gamma(n+1)} - x^{-n}J_n,\ n > 1.$ **[Bilaspur 1997]**

(*b*) $\int_0^\infty x^{-n}J_{n+1}(x)\,dx = \dfrac{1}{2^n\,\Gamma(n+1)},\ n > -\dfrac{1}{2}.$

Sol. (a) From recurrence relation II, $\dfrac{d}{dx}\,[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x).$ ...(1)

Integrating (1) w.r.t. 'x' between the limits 0 and x, we get

$$\left[x^{-n}J_n(x)\right]_0^x = -\int_0^x x^{-n}\,J_{n+1}(x)\,dx \quad \text{or} \quad x^{-n}J_n(x) - \lim_{x\to 0}\frac{J_n(x)}{x^n} = -\int_0^x x^{-n}J_{n+1}(x)\,dx. \qquad ...(2)$$

But $$\lim_{x\to 0}\frac{J_n(x)}{x^n} = \lim_{x\to 0}\frac{1}{x^n}\,\frac{x^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{2\cdot 2(n+1)} + \ldots\right] = \frac{1}{2^n\,\Gamma(n+1)}$$

Hence (2) may be written as $$\int_0^x x^{-n}J_{n+1}(x)\,dx = \frac{1}{2^n\,\Gamma(n+1)} - x^{-n}J_n(x).$$

Part (b). Integrating (1) w.r.t. 'x' from 0 to ∞, we get

$$\left[x^{-n}J_n(x)\right]_0^\infty = -\int_0^\infty x^{-n}\,J_{n+1}(x)\,dx \quad \text{or} \quad \lim_{x\to\infty}\frac{J_n(x)}{x^n} - \lim_{x\to 0}\frac{J_n(x)}{x^n} = -\int_0^\infty x^{-n}J_{n+1}(x)\,dx. \qquad ...(3)$$

As in part (*a*), $$\lim_{x\to 0}\frac{J_n(x)}{x^n} = \frac{1}{2^n\Gamma(n+1)}. \qquad ...(4)$$

We know that for large values of x the approximate value of $J_n(x)$ is

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{1/2}\cos\left\{x - \left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\},\ n > \frac{1}{2} \qquad ...(5)$$

Using (5), $$\lim_{x\to\infty}\frac{J_n(x)}{x^n} = 0. \qquad ...(6)$$

Using (4) and (6), (3) reduces to $$\int_0^\infty x^{-n}J_{n+1}(x)\,dx = \frac{1}{2^n\,\Gamma(n+1)}.$$

Ex. 4. *Prove* (*i*) $\dfrac{d}{dx}\{xJ_1(x)\} = xJ_0(x).$ (*ii*) $\int_0^b xJ_0(ax)\,dx = \dfrac{b}{a}J_1(ab).$

Sol. (*i*) Recurrence relation I is $$\frac{d}{dx}\{x^nJ_n(x)\} = x^nJ_{n-1}(x). \qquad ...(1)$$

Replacing n by 1 in (1), we have $$\frac{d}{dx}\{xJ_1(x)\} = xJ_0(x).$$

Part (ii). Put $ax = t$, so that $adx = dt$. Then, we get

$$\int_0^b x\, J_0(ax)\, dx = \frac{1}{a^2}\int_0^{ab} t\, J_0(t)\, dt = \frac{1}{a^2}\int_0^{ab} \frac{d}{dt}\{t\, J_1(t)\}\, dt, \text{ by part } (i)$$

$$= \frac{1}{a^2}\left[tJ_1(t)\right]_0^{ab} = \frac{1}{a^2}\,[abJ_1(ab) - 0], \text{ as } J_1(0) = 0.$$

$$\therefore \qquad \int_0^b x\, J_0(ax)\, dx = \frac{b}{a} J_1(ab).$$

Ex. 5. *Prove that* (*i*) $\dfrac{d}{dx}\, J_0(x) = -J_1(x)$.

(*ii*) $\int_a^b J_0(x)\, J_1(x)\, dx = \frac{1}{2}[J_0^2(a) - J_0^2(b)]$. **[Nagpur 2005]**

Sol. (i) From recurrence relation II, $\dfrac{d}{dx}\{x^{-n}J_n(x)\} = -x^n J_{n+1}(x)$. ...(1)

Put $n = 0$ in (1). Then $\dfrac{d}{dx}J_0(x) = -J_1(x)$. ...(2)

Part (ii). Using (2), we have

$$\int_a^b J_0(x)\, J_1(x)\, dx = -\int_a^b J_0(x)\, J_0'(x)\, dx = -\left[\frac{[J_0(x)]^2}{2}\right]_a^b = \tfrac{1}{2}[J_0^2(a) - J_0^2(b)]$$

Ex. 6. *Evaluate* $\int J_3(x)\, dx$ *and express the result in terms of* J_0 *and* J_1.

Sol. From recurrence II, we have $x^{-n}J_{n+1} = \dfrac{d}{dx}\{x^{-n}J_n(x)\}$.

Integrating it, $\int x^{-n}J_{n+1}\, dx = -x^{-n}J_n$. ...(1)

Now, $\int J_3(x)\, dx = \int x^2\,(x^{-2}\, J_3)\, dx = x^2(-x^{-2}J_2) - \int 2x\,(-x^{-2}\, J_2)\, dx$.

[Integrating by parts and using (1) for $n = 2$]

$= -J_2 + 2\int x^{-1}\, J_2\, dx = -J_2 + 2(-x^{-1}J_1) + c$ [using (1) for $n = 1$]

$\therefore \qquad \int J_3(x)\, dx = -J_2 - 2x^{-1}J_1 + c.$ ...(2)

From recurrence relation VI, $(2n/x)J_n = J_{n-1} + J_{n+1}$ ...(3)

Put $n = 1$ in (3). Then $J_2 = 2J_1/x - J_0$ ...(4)

Using (4), (2) gives $\int J_3(x)\, dx = -(2J_1/x - J_0) - 2x^{-1}\, J_2 + c$

$\therefore \qquad \int J_3(x)\, dx = J_0 - 4J_1/x + c$, being an arbitrary constant.

Ex. 7. *Evaluate* $\int x^3 J_3(x)\, dx$. **[Gulbarga 2005]**

Sol. Since $\dfrac{d}{dx}\{x^{-n}J_n\} = -x^n J_{n+1}$ so $\int x^{-n}J_{n+1}\, dx = -x^{-n}J_n$. ...(1)

Now, $\int x^3 J_3(x)\, dx = \int x^5\,(x^{-2}J_3)\, dx = x^5(-x^{-2}J_2) - \int 5x^4\,(-x^{-2}\, J_2)\, dx$

(On integration by parts and using (1) for $n = 2$)

$$= -x^3J_2 + 5\int x^2 J_2\, dx = -x^3J_2 + 5\int x^3(x^{-1}\, J_2)\, dx = -x^3J_2 + 5\left[x^3\,(-x^{-1}J_1) - \int 3x^2(-x^{-1}J_1)\, dx\right]$$

(on integration by parts and using (1) for $n = 1$)

$= -x^3J_2 - 5x^2J_1 + 15\int x\,J_1\,dx = -x^3J_2 - 5x^2J_1 + 15\int x\,(-J_0')\,dx$ $[\because\ J_1 = -J_0']$

$= -x^3J_2 - 5x^2J_1 - 15\int x\,J_0'\,dx = -x^3J_2 - 5x^2J_1 - 15\left[x\,J_0 - \int 1\,.J_0\,dx\right]$, integrating by part,

$= -x^3J_2 - 5x^3J_1 - 15xJ_0 + 15x\int J_0\,dx$.

Remark. From Ex. 6. and 7 note that, in general, an integral of the form $\int x^m\,J_n(x)\,dx$, $m + n \geq 0$, can be completely integrated if $m + n$ is an odd integer, while if $m + n$ is even, then the integral can be put in terms of $\int J_0(x)\,dx$. Note that $\int J_0(x)\,dx$ cannot be expressed in closed form and so it must be left as such in final answer.

Ex. 8. *Evaluate* $\int x^4J_1(x)\,dx$.

Sol. Since $\dfrac{d}{dx}\{x^nJ_n\} = x^nJ_{n-1}$ so $\int x^nJ_{n-1}\,dx = x^nJ_n$. ...(1)

Now $\int x^4J_1\,dx = \int x^2\,(x^2J_1)\,dx = x^2(x^2J_2) - \int 2x(x^2J_2)\,dx$

[on integrating by parts and using (1) for $n = 2$]

$= x^4J_2 - 2\int x^3J_2\,dx = x^4J_2 - 2x^3J_3 + c$ [using (1) for $n = 3$]

Ex. 9. Express $\int x^{-3}\,J_4\,(x)\,dx$ in terms of J_0 and J_1.

Sol. Putting $n = 3$ in recurrence relation II $\dfrac{d}{dx}\{x^{-n}J_n\} = -x^{-n}J_{n+1}$ gives $\dfrac{d}{dx}\{x^{-3}J_3\} = -x^{-3}J_4$.

Integrating, $\int x^{-3}\,J_4\,(x)\,dx = -x^{-3}J_3 + c$, c being an arbitrary constant ...(1)

Recurrence relation VI is $J_{n+1} = (2n/x)J_n - J_{n-1}$. ...(2)

Replacing n by 2 and 1 successively in (2) gives $J_3 = (6/x)J_2 - J_1$ ...(3)

and $J_2 = (4/x)J_1 - J_0$. ...(4)

Using (4), (3) becomes $J_2 = (6/x)[(4/x)J_1 - J_0] - J_1 = (24x^{-2} - 1)J_1 - (6/x)J_0$ ...(5)

Using (5), (1) becomes $\int x^{-3}\,J_4\,(x)\,dx = -x^{-3}[(24x^{-2} - 1)J_1 - 6x^{-1}J_0] + c$.

Ex. 10. *Prove* $\int J_{n+1}\,dx = \int J_{n-1}(x)\,dx - 2J_n(x)$.

Sol. From recurrence relations, we have

$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ or $J_{n+1}(x) = J_{n-1}(x) - 2J'_n(x)$

Integrating $\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n(x)$.

Ex. 11(a). *Prove that* $\int x^{-1}J_4(x)\,dx = -x^{-1}J_3(x) - 2x^{-2}J_2(x) + c$.

Sol. Recurrence relation II is $\dfrac{d}{dx}\{x^{-n}J_n(x) = -x^{-n}J_{n+1}(x)$.

Integrating it, we get $\int x^{-n}\,J_{n+1}(x)\,dx = -\,x^{-n}J_n(x)$. ...(1)

$\therefore\ \int x^{-1}\,J_4(x)\,dx = \int x^2[x^{-3}J_4(x)]\,dx = x^2\left[-x^{-3}\,J_3(x)\right] - \int 2x \times [-x^{-3}J_3(x)]\,dx$

(Integrating by parts taking x^2 as first function and using result (1) for $n = 3$)

$= -x^{-1}J_3(x) + 2\int x^{-2}\,J_3(x)\,dx = -x^{-1}J_3(x) + 2[-x^{-2}J_2(x)] + c$, using result (1) for $n = 2$

$= -x^{-1}J_3(x) - 2x^{-2}J_2(x) + c$.

Ex. 11(b). *Prove that* $\int J_3(x)dx = -J_2(x) - (2/x) \times J_0(x) + C$ **[KU Kurukshetra 2005]**

Sol. From recurrence relation II, $\dfrac{d}{dx}\left\{x^{-n}J_n(x)\right\} = -x^{-n}J_{n+1}(x)$

Integrating, $\int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$ ... (1)

Now, $\int J_3(x)dx = \int x^2\left\{x^{-2}J_3(x)\right\}dx = x^2\int x^{-2}J_3(x)dx - \int 2x\left(\int x^{-2}J_3(x)dx\right)dx$

$= x^2\left(-x^{-2}J_2(x)\right) - \int 2x(-x^{-2}J_2(x))dx,$ using (1) for $n = 2$

$= -J_2(x) + 2\int x^{-1}J_2(x)dx = -J_2(x) - 2x^{-1}J_1(x) + C$, using (1) again for $n = 1$

Ex. 12. *Show that* (*i*) $\int_0^x x^3\, J_0(x)\, dx = x^3J_1(x) - 2x^2J_2(x)$. **[GATE 2003]**

(*ii*) $\int_0^1 x^3\, J_0(x)\, dx = 2J_0(1) - 3J_1(1)$.

Sol. (i) Since $\dfrac{d}{dx}\{x^nJ_n(x)] = x^nJ_{n-1}(x)$ so $\int x^nJ_{n-1}(x)\, dx = x^nJ_n(x)$ ...(1)

$\therefore\ \int_0^x x^3J_0(x)\, dx = \int_0^x x^2\,[x\, J_0(x)]\, dx = [x^2\{xJ_1(x)\}]_0^x - \int_0^x 2x\,\{x\, J_1(x)\}\, dx$

(Integrating by parts and using (1) for $n = 1$)

$= x^3J_1(x) - 2\int_0^x x^2J_1(x)\, dx = x^3J_1(x) - 2[x^2J_2(x)]^x{}_0$, using result (1) for $n = 2$

$= x^3\, J_1(x) - 2x^2\, J_2(x)$, as $J_2(0) = 0$

(ii) Proceed as in part (*i*) and prove that $\int_0^x x^3J_0(x)\, dx = x^3J_1(x) - 2x^2J_2(x)$. ...(2)

Putting $x = 1$ in (2), $\int_0^1 x^3J_0(x)\, dx = J_1(1) - 2J_2(1)$. ...(3)

Recurrence relation VI is $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$.

Putting $n = 1$ and re–writing it, we have $J_2(1) = 2J_1(1) - J_0(1)$ ...(4)

Substituting the above value of $J_2(1)$ in (3), we have

$$\int_0^1 x^3J_0(x)\, dx = J_1(1) - 2\{2J_1(1) - J_0(1)] = 2J_0(1) - 3J_1(1).$$

EXERCISE 11(C)

1. Prove that $\int x^2J_0(x)\, dx = x^2J_1(x) + xJ_0(x) - \int J_0(x)\, dx$.

2. Prove that $\int_0^x x^2J_0(x)\, J_1(x)\, dx = \frac{1}{2}x^2J_1{}^2(x)$.

3. Prove that $\int x^{-2}J_1(x)\, dx = -\frac{1}{3}x^{-1}J_2(x) - \frac{1}{3}J_1(x) + \frac{1}{2}\int J_0(x)\, dx$.

4. Prove that $\int J_0(x)\sin x\, dx = xJ_0(x)\sin x - xJ_1(x)\cos x + c$.

5. Show that $\int J_3dx = -3J_0 + 4J_1'$ **[Bilaspur 1997]**

11.8. Generating function for the Bessel's function $J_n(x)$

Prove : $\exp\left\{\frac{1}{2}x\left(z - \dfrac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} z^nJ_n(x)$ **[Kanpur 2007; Meerut 1996, 97]**

Or *Show that when n is a positive integer, $J_n(x)$ is the coefficient of z^n in the expansion of*

$\exp\{(x/2)\times(z-1/z)\}$ *i.e.,* $e^{(x/2)(z-1/z)}$ *in ascending and descending power of* z. *Also show that* J_n *is coefficient of* z^{-n} *multiplied by* $(-1)^n$ *in the expansion of the above expression.*

[Kanpur 2007; Kakitiya 1997; Kanpur 2005, 06]

Note. $\text{Exp}\{(x/2) \times (z - 1/z)\}$ is called the generating function for $J_n(x)$. Here $\exp A = e^A$.

Proof. We have $\exp\{(x/2) \times (z - 1/z)\} = e^{\frac{xz}{2} - \frac{x}{2z}} = e^{\frac{xz}{2}} \cdot e^{-\frac{x}{2z}}$

$$= \left[1 + \left(\frac{x}{2}\right)z + \left(\frac{x}{2}\right)^2 \frac{z^2}{2!} + \ldots + \left(\frac{x}{2}\right)^n \frac{z^n}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{z^{n+1}}{(n+1)} + \ldots\right] \times \left[1 - \left(\frac{x}{2}\right)z^{-1}\right.$$

$$\left. + \left(\frac{x}{2}\right)^2 \frac{z^{-2}}{2!} + \ldots + \left(\frac{x}{2}\right)^n \frac{(-1)^n z^{-n}}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{(-1)^{n+1} z^{-(n+1)}}{(n+1)!} + \ldots\right] \quad \ldots(1)$$

The coeff. of z^n in the product (1) is obtained by multiplying the coefficients of z^n, z^{n+1}, z^{n+2}, ... in the first bracket with the coeff. of z^0, z^{-1}, z^{-2}, ... in the second bracket respectively

$$\therefore \text{ coefficient of } z^n \text{ in product (1)} = \left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{(n+2)!\,2!} - \ldots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

[$\because$ $(n + r)! = \Gamma(n + r + 1)$, $n + r$ being positive integer].

The coeff. of z^{-n} in the product (1) is obtained by multiplying the coefficients of z^{-n}, z^{-n-1}, z^{-n-2}... of the second bracket with the coefficients of z^0, z^1, z^2, ... in the first bracket respectively

$$\therefore \text{ coeff. of } z^{-n} \text{ in product (1)} = \left(\frac{x}{2}\right)^n \frac{(-1)^n}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{(-1)^{n+1}}{(n+1)!} \frac{x}{2} + \left(\frac{x}{2}\right)^{n+2} \frac{(-1)^{n+2}}{(n+2)!\,2!} \left(\frac{x}{2}\right)^2 + \ldots$$

$$= (-1)^n \left[\left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{(n+2)\,2!} \cdots\right] = (-1)^n J_n(x), \text{ as before}$$

Thus the coefficient of $z^{-n} = (-1)^n J_n(x) \quad \Rightarrow \quad J_n(x) = (-1)^n \times$ the coefficient of z^{-n}.

Finally, in the product (1) the coefficient of z^0 is obtained by multiplying the coefficient of z^0, z^1, z^2, ... in the first bracket with the coefficients of z^0, z^{-1}, z^{-2}, ... in the second bracket and is thus

$$= 1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \left(\frac{1}{2!}\right)^2 - \left(\frac{x}{2}\right)^6 \left(\frac{1}{3!}\right)^2 + \ldots = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \ldots = J_0(x).$$

We observe that the coefficients of z^0, $(z - z^{-1})$, $(z^2 + z^{-2})$ ..., $[z^n + (-1)^n z^{-n}]$ ... are $J_0(x)$, $J_1(x)$, $J_2(x)$ ..., $J_n(x)$ ... respectively. Thus, (1) gives

$$\exp\{(x/2) \times (z - 1/z)\} = J_0(x) + (z - z^{-1})J_1(x) + (z^2 + z^{-2})J_2(x) + \ldots + [z^2 + (-1)^n z^{-n}]J_n(x) + \ldots$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x), \text{ as } J_{-n}(x) = (-1)^n J_n(x)$$

11.9. Trigonometric expansions involving Bessel's functions. *Show that*

(*i*) $\cos(x\sin\phi) = J_0 + 2\cos 2\phi \cdot J_2 + 2\cos 4\phi \cdot J_4 + \ldots$ **[Meerut 1995, KU Kurukshetra 2005]**

(*ii*) $\sin(x\sin\phi) = 2\sin\phi \cdot J_1 + 2\sin 3\phi \cdot J_2 + \ldots$ **[KU Kurukshreta 2006]**

(*iii*) $\cos(x\cos\phi) = J_0 - 2\cos 2\phi \cdot J_2 + 2\cos 4\phi \cdot J_4 - \ldots$

(*iv*) $\sin(x\cos\phi) = 2\cos\phi \cdot J_1 - 2\cos 3\phi \cdot J_3 + 2\cos 5\phi \cdot J_5 - \ldots$

(*v*) $\cos x = J_0 - 2J_4 + 2J_4 \ldots = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$ **[Kanpur 2011]**

(*vi*) $\sin x = 2J_1 - 2J_3 + 2J_5 \ldots = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$ **[Kanpur 2011]**

Proof. We know that

$$e^{(x/2)(z-1/z)} = J_0 + (z - z^{-1})J_1 + (z^2 + z^{-2})J_2 + (z^3 - z^{-3})J_3 + \ldots \quad \ldots(1)$$

Let $z = e^{i\phi}$ so that $z^n = e^{in\phi}$ and $z^{-n} = e^{-in\phi}$. Then (1) gives

$$e^{(x/2)(e^{i\phi} - e^{-i\phi})} = J_0 + (e^{i\phi} - e^{-i\phi})J_1 + (e^{2i\phi} + e^{-2i\phi})J_2 + (e^{3i\phi} - e^{-3i\phi})J_3 + \ldots \quad \ldots(2)$$

Since $\cos n\phi = (e^{ni\phi} + e^{-ni\phi})/2$ and $\sin n\phi = (e^{ni\phi} - e^{-ni\phi})/2i$, (2) gives

$$e^{xi\sin\phi} = J_0 + 2i\sin\phi \cdot J_1 + 2\cos 2\phi \cdot J_2 + 2i\sin 3\phi \cdot J_3 + \ldots$$

or $\cos(x\sin\phi) + i\sin(x\sin\phi) = (J_0 + 2\cos 2\phi \cdot J_2 + \ldots) + 2i(\sin\phi \cdot J_1 + \sin 3\phi \cdot J_3 + \ldots)$ ...(3)

Part (i). Equating real parts in (3), we get

$$\cos(x\sin\phi) = J_0 + 2\cos 2\phi \cdot J_2 + 2\cos 4\phi \cdot J_4 + \ldots \quad \ldots(4)$$

Part (ii). Equating imaginary parts in (3), we get

$$\sin(x\sin\phi) = 2\sin\phi \cdot J_1 + 2\sin 3\phi \cdot J_3 + 2\sin 5\phi \cdot J_5 + \ldots \quad \ldots(5)$$

Part (iii). Replacing ϕ by $\pi/2 - \phi$ in (4) and simplifying, we have

$$\cos(x\cos\phi) = J_0 - 2\cos 2\phi \cdot J_2 + 2\cos 4\phi \cdot J_4 - \ldots \quad \ldots(6)$$

Part (iv). Replacing ϕ by $\pi/2 - \phi$ in (5) and simplifying, we get

$$\sin(x\cos\phi) = 2\cos\phi \cdot J_1 - 2\cos 3\phi \cdot J_3 + 2\cos 5\phi \cdot J_5 - \ldots \quad \ldots(7)$$

Part (v) & (vi). Replacing ϕ by 0 in (6) and (7), we get

$$\cos x = J_0 - 2J_1 + 2J_4 - \ldots = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

and $$\sin x = 2J_1 - 2J_3 + 2J_5 - \ldots = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x).$$

11.9A. Solved examples based on Art 11.8 and 11.9.

Ex. 1. *Show that* (*i*) $x\sin x = 2(2^2J_2 - 4^2J_4 + 6^2J_6 - \ldots)$. **[Agra 2010]**

(*ii*) $x\cos x = 2(1^2J_1 - 3^2J_3 + 5^2J_5 - \ldots)$.

Sol. We know that $\cos(x\sin\phi) = J_0 + 2J_2\cos 2\phi + 2J_4\cos 4\phi + \ldots$ ...(1)

Differentiating (1) w.r.t. 'ϕ', $-\sin(x\sin\phi)\cdot x\cos\phi = 0 - 2\cdot 2J_2\sin 2\phi - 2\cdot 4J_4\sin 4\phi + \ldots$ ...(2)

Differentiating (2) w.r.t. 'ϕ', $-\cos(x\sin\phi)\cdot(x\cos\phi)^2 + \sin(x\sin\phi)\cdot(x\sin\phi)$

$$= -2\cdot 2^2J_2\cos 2\phi - 2\cdot 4^2J_4\cos 4\phi - 2\cdot 6^2J_6\cos 6\phi + \ldots \quad \ldots(3)$$

Replacing ϕ by $\pi/2$ in (3), we get $x\sin x = 2(2^2J_2 - 4^2J_4 + 6^2J_6 - \ldots)$

Part (ii). Start with $\sin(x \sin\phi) = 2J_1 \sin\phi + 2J_3 \sin 3\phi + 2J_5 \sin\phi + ...$

Differentiate this twice w.r.t. 'ϕ' as in part (*i*) and then replace ϕ by $\pi/2$. This will lead to the desired answer. Complete the solution yourself.

Ex. 2. ***Bessel's Integrals.*** *Show that*

(*i*) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x \sin\phi)\, d\phi$, where n is a positive integer

[Purvanchal 2004, 07; Punjab 2005]

(*ii*) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x \sin\phi)\, d\phi$, where n is any integer **[KU Kurukshetra 2004]**

(*iii*) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\phi)\, d\phi = \frac{1}{\pi}\int_0^{\pi} \cos(x \cos\phi)\, d\phi$. **[Agra 2006]**

(*iv*) Deduce that $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - ... = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2^r \cdot r!)^2}$.

Sol. (i). We shall use the following results :

$$\int_0^{\pi} \cos m\phi \cos n\phi\, d\phi = \int_0^{\pi} \sin m\phi \sin n\phi\, d\phi = \pi/2 \text{ when } m = n$$
$$= 0 \text{ when } m \neq n \qquad ...(1)$$

$$\cos(x \sin\phi) = J_0 + 2J_2 \cos 2\phi + 2J_4 \cos 4\phi + ... \qquad ...(2)$$

and $$\sin(x \sin\phi) = 2J_1 \sin\phi + 2J_3 \sin 3\phi + 2J_5 \sin 5\phi + ... \qquad ...(3)$$

Multiplying both sides of (2) by $\cos n\phi$ and then integrating w.r.t. 'ϕ' between limits 0 to π and using (1), we have

$$\int_0^{\pi} \cos(x \sin\phi) \cos n\phi\, d\phi = 0, \text{ if } n \text{ is odd} \qquad ...(4)$$

$$= \pi J_n, \text{ if } n \text{ is even.} \qquad ...(5)$$

Again, multiplying both sides of (3) by $\sin n\phi$ and then integrating w.r.t. 'ϕ' between limits 0 to π and using (1), we get

$$\int_0^{\pi} \sin(x \sin\phi) \sin n\phi\, d\phi = \pi J_n, \text{ if } n \text{ is odd} \qquad ...(6)$$

$$= 0, \text{ if } n \text{ is even.} \qquad ...(7)$$

Let n be odd. Adding (4) and (6), we get

$$\int_0^{\pi} [\cos(x \sin\phi) \cos n\phi + \sin(x \sin\phi) \sin n\phi]\, d\phi = \pi J_n.$$

or $$\int_0^{\pi} \cos(n\phi - x \sin\phi)\, d\phi = \pi J_n \quad \text{or} \quad J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x \sin\phi)\, d\phi. \qquad ...(8)$$

Next, let n be even. Then adding (5) and (7) as before, we again get (8). Thus (8) holds for each positive integer (even as well as odd).

Part (ii). Let n be any integer. Then as in part (*i*), if n is positive integer, we have

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x \sin\phi)\, d\phi \qquad ...(9)$$

Next, let n be a negative integer so that $n = -m$, where m is a positive interger. To prove the required result for a negative integer, we prove that

$$J_{-m}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(-m\phi - x \sin\phi)\, d\phi. \qquad ...(10)$$

Let $\phi = \pi - \theta$ so that $d\phi = -d\theta$. Then, we have

R.H.S. of (10)

$$= \frac{1}{\pi}\int_{\pi}^{0} \cos\{-m(\pi-\theta) - x\sin(\pi-\theta)\}(-d\theta) = \frac{1}{\pi}\int_{0}^{\pi} \cos[(m\theta - x\sin\theta) - m\pi]\, d\theta$$

$$= \frac{1}{\pi}\int_{0}^{\pi} [\cos(m\theta - x\sin\theta)\cos m\pi + \sin(m\theta - x\sin\theta)\sin m\pi]\, d\theta$$

$$= \frac{1}{\pi}\int_{0}^{\pi} (-1)^m \cos(m\theta - x\sin\theta)\, d\theta \qquad [\because \sin m\pi = 0 \text{ and } \cos m\pi = (-1)^m]$$

$$= \frac{1}{\pi}(-1)^m \int_{0}^{\pi} \cos(m\phi - x\sin\phi)\, d\phi = (-1)^m J_m(x) \qquad [\text{Using (9) as } m \text{ is + ve integer}]$$

$$= J_{-m}(x) = \text{L.H.S. of (10)} \qquad [\because J_{-m}(x) = (-1)^m J_m(x)]$$

Thus, (10) is true. (9) and (10), show that the required result holds for each integer.

Part (iii). Integrating (2) w.r.t. 'ϕ' between the limits 0 to π and using the result $\int_0^{\pi} \cos p\phi\, d\phi = 0$, if p is an even interger, we have ...(11)

$$\int_0^{\pi} \cos(x\sin\phi)\, d\phi = J_0(x)\int_0^{\pi} d\phi + 0 + 0 + \ldots = J_0(x).\ \pi.$$

$$\therefore \qquad J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi)\, d\phi. \qquad ...(12)$$

Replacing ϕ by $\pi/2 - \phi$ in (2) and simplifying, we get

$$\cos(x\cos\phi) = J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi - \ldots \qquad ...(13)$$

Integrating (13) w.r.t. 'ϕ' and using (11), we get

$$\int_0^{\pi} \cos(x\cos\phi)\, d\phi = J_0(x).\ \pi - 0 - 0\ldots \text{ or } \qquad J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\, d\phi. \qquad ...(14)$$

(iv) Deduction. From (14), $J_0(x) = \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{x^2\cos^2\phi}{2!} + \frac{x^4\cos^4\phi}{4!} - \ldots\right)d\phi$ ...(15)

But $$\int_0^{\pi} \cos^{2n}\phi\, d\phi = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\pi. \qquad ...(16)$$

Using (16), (14) becomes $$J_0(x) = \frac{1}{\pi}\left[x - \frac{x^2}{2!}\cdot\frac{1}{2}\pi + \frac{x^4}{4!}\cdot\frac{1\cdot 3}{2\cdot 4}\pi - \frac{x^6}{6!}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\pi + \ldots\right]$$

or $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \ldots = \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r\cdot r!)^2}.$$

Ex. 3. *Use the generating function to show that* $J_n(-x) = (-1)^n J_n(x)$.

[Agra 2008; Meerut 2006, 11; Kanpur 2008]

Sol. We have $$\sum_{n=-\infty}^{\infty} J_n(x)\, z^n = \exp\left\{\frac{x}{2}\left(z - \frac{1}{z}\right)\right\}. \qquad ...(1)$$

Replacing x by $-x$ in (1), we get

$$\sum_{n=-\infty}^{\infty} J_n(-x)\, z^n = \exp\left\{-\frac{x}{2}\left(z - \frac{1}{z}\right)\right\} = \exp\left\{\frac{x}{2}\left(-z - \frac{1}{-z}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x)\,.\,(-z)^n, \text{ by (1)}$$

$$\therefore \quad \sum_{n=-\infty}^{\infty} J_n(-x)\, z^n = \sum_{n=-\infty}^{\infty} J_n(x)\,(-1)^n\, z^n. \qquad ...(2)$$

Equating the coefficients of z^n from both sides of (2), we have $J_n(-x) = (-1)^n J_n(x)$.

Ex. 4. *Use the generating function to prove that* $J_n(x+y) = \sum_{n=-\infty}^{\infty} J_r(x)\, J_{n-r}(y)$.

Sol. By the generating function, $\exp\left\{\frac{1}{2}(x+y)\left(z-\frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x+y)\, z^n$. ...(1)

So we see that $J_n(x+y)$ is the coefficient of z^n on R.H.S. of (1). We next obtain the coefficient of z^n on L.H.S. of (1). Now, we have

L.H.S. of (1) $= \exp\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\} . \exp\left\{\frac{1}{2}y\left(z-\frac{1}{z}\right)\right\}$, as $\exp(A+B) = e^{A+B} = \exp A . \exp B$.

$$= \sum_{r=-\infty}^{\infty} J_r(x)\, z^r \,.\, \sum_{s=-\infty}^{\infty} J_s(y)\, z^s \qquad \text{[by definition of generating function]}$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_r(x)\, J_s(y)\, z^{r+s}. \qquad ...(2)$$

For a fixed value of r, we get z^n by taking $r + s = n$ *i.e.* $s = n - r$. Thus keeping r fixed, the coefficient of z^n in (1) is $J_r(x)\, J_{n-r}(y)$. So the total coefficient of z^n will be given by summing all such terms from $r = -\infty$ to $r = \infty$ and is given by $\sum_{r=-\infty}^{\infty} J_r(x)\, J_{n-r}(y)$.

Hence equating the coefficient of z^n from both sides of (1), $J_n(x) = \sum_{r=-\infty}^{\infty} J_r(x)\, J_{n-r}(y)$.

Ex. 5. *If* $a > 0$, *prove that* $\int_0^{\infty} e^{-ax} J_0(bx)\, dx = \frac{1}{\sqrt{(a^2+b^2)}}$.

[Ravishanker 1999; Purvanchal 2007; Lucknow 2010]

Sol. We know that (Refer Ex. 2. Part (*iii*)) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\phi)\, d\phi$.

$$\therefore \quad J_0(bx) = \frac{1}{\pi}\int_0^{\pi} \cos(bx \sin\phi)\, d\phi. \qquad ...(1)$$

$$\therefore \quad \int_0^{\infty} e^{-ax} J_0(bx)\, dx = \int_0^{\infty} e^{-ax}\left\{\frac{1}{\pi}\int_0^{\infty} \cos(bx \sin\phi)\, d\phi\right\} dx, \text{ using (1)}$$

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_0^{\pi} e^{-ax} \cos(bx \sin\phi)\, d\phi\right\} dx = \frac{1}{\pi}\int_0^{\pi}\left\{\int_0^{\infty} e^{-ax} \cos(bx \sin\phi)\, dx\right\} d\phi$$

(On interchanging the order of integration)

$$= \frac{1}{\pi}\int_0^{\pi}\left\{\int_0^{\infty} e^{-ax}\, \frac{e^{i\,bx\sin\phi} + e^{-i\,bx\sin\phi}}{2}\, dx\right\} d\phi = \frac{1}{2\pi}\int_0^{\pi}\left\{\int_0^{\infty} [e^{-(a-ib\sin\phi)x} + e^{-(a+ib\sin\phi)x}]\, dx\right\} d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\frac{e^{-(a-ib\sin\phi)x}}{-(a-ib\sin\phi)} + \frac{e^{-(a+ib\sin\phi)x}}{-(a+ib\sin\phi)}\right]_0^{\infty} d\phi = \frac{1}{2\pi}\int_0^{\pi}\left[\frac{1}{a-ib\sin\phi} + \frac{1}{a+ib\sin\phi}\right] d\phi$$

$$= \frac{a}{\pi}\int_0^{\pi}\frac{d\phi}{a^2+b^2\sin^2\phi} = \frac{a}{\pi}\int_0^{\pi}\frac{\operatorname{cosec}^2\phi\, d\phi}{b^2+a^2\operatorname{cosec}^2\phi} = \frac{a}{\pi}\int_0^{\pi}\frac{\operatorname{cosec}^2\phi\, d\phi}{b^2+a^2(1+\cot^2\phi)} = \frac{2a}{\pi}\int_0^{\pi/2}\frac{\operatorname{cosec}^2 d\phi}{(a^2+b^2)+a^2\cot^2\phi}$$

$$= \frac{2a}{\pi}\int_{\infty}^{0}\frac{(-dt)}{(a^2+b^2)+a^2t^2}, \text{ putting } \cos\phi = t \text{ so that } -\operatorname{cosec}^2\phi\, d\phi = dt$$

$$= -\frac{2a}{\pi a^2}\int_{\infty}^{0}\frac{dt}{t^2+(a^2+b^2)/a^2} = \frac{2}{a\pi}\int_0^{\infty}\frac{dt}{t^2+[\sqrt{(a^2+b^2)}/a]^2}$$

$$= \frac{2}{a\pi}\cdot\frac{1}{\sqrt{(a^2+b^2)}/a}\left[\tan^{-1}\frac{t}{\sqrt{(a^2+b^2)}/a}\right]_0^{\infty} = \frac{2}{\pi\sqrt{(a^2+b^2)}}\left(\frac{\pi}{2}-0\right) = \frac{1}{\sqrt{a^2+b^2}}.$$

Ex. 6. *Using the generating function, prove that*

(*i*) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ (*ii*) $2n\,J_n(x) = x\{J_{n+1}(x) + J_{n-1}(x)\}$

Sol. Generating function is given by $\sum_{n=-\infty}^{\infty} J_n(x)z^n = e^{(x/2)\times(z-1/z)}$ ... (1)

Diff. both sides of (1) w.r.t. 'x', $\sum_{n=-\infty}^{\infty} J_n'(x)z^n = e^{(x/2)\times(z-1/z)}\times(1/2)\times(z-1/z)$

or $2\sum_{n=-\infty}^{\infty} J_n'(x)z^n = (z-z^{-1})\sum_{n=-\infty}^{\infty} J_n(x)\,z^n = \sum_{n=-\infty}^{\infty} J_n(x)\,(z^{n+1}-z^{n-1})$, using (1)

or $$2\sum_{n=-\infty}^{\infty} J_n'(x)\,z^n = \sum_{n=-\infty}^{\infty} J_n(x)\,z^{n+1} - \sum_{n=-\infty}^{\infty} J_n(x)\,z^{n-1} \quad \text{... (2)}$$

Equating the coefficients of z^n on both sides of (2) yields $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(*ii*) Differentiating both sides of (1) w.r.t. 'z', we get

$$\sum_{n=-\infty}^{\infty} n\,J_n(x)z^{n-1} = e^{(x/2)\times(z-1/z)}\times(1/2)\times(1+z^{-2})$$

or $$2\sum_{n=-\infty}^{\infty} n\,J_{n-1}(x)z^{n-1} = x(1+z^{-2})\sum_{n=-\infty}^{\infty} J_n(x)z^n, \text{ using (1)}$$

or $$2\sum_{n=-\infty}^{\infty} n\,J_n(x)z^{n-1} = x\sum_{n=-\infty}^{\infty} J_n(x)z^n + x\sum_{n=-\infty}^{\infty} J_n(x)z^{n-2} \quad \text{... (3)}$$

Equating the coefficients of z^{n-1} on both sides of (3), $2n\,J_n(x) = x\{J_{n-1}(x) + J_{n+1}(x)\}$

Ex. 7. *Show that* $\int_0^y \frac{x\sin ax}{(y^2-x^2)^{1/2}}dx = \frac{\pi y}{2}J_1(ay)$

Sol. From result (ii) of Art. 11.9, $\sin(x\sin\phi) = 2\sin\phi\cdot J_1 + 2\sin 3\phi\cdot J_2 + \ldots$ ... (1)

Multiplying both sides of (1) by $\sin\phi$ and then integrating between the limits 0 and π, we have

$$\int_0^{\pi}\sin(x\sin\phi)\sin\phi\, d\phi = J_1\int_0^{\pi}(2\sin^2\phi)d\phi + J_2\int_0^{\pi}(2\sin\phi\sin 3\phi)\, d\phi + \ldots$$

$$= J_1\int_0^{\pi}(1-\cos 2\phi)\,d\phi + J_2\int_0^{\pi}(\cos 2\phi - \cos 4\phi)d\phi + \ldots = J_1\left[\phi - \frac{\sin 2\phi}{2}\right]_0^{\pi} + J_2\left[\frac{\sin 2\phi}{2} - \frac{\sin 4\phi}{2}\right]_0^{\pi} + \ldots.$$

$= J_1 \cdot \pi$, since the remaining terms vanish

Thus, $$\pi J_1(x) = \int_0^{\pi} \sin(x\sin\phi)\sin\phi\, d\phi \qquad \ldots (2)$$

Let $F(\phi) = \sin(x\sin\phi)\sin\phi$. Then, clearly $F(\pi-\phi) - F(\phi)$.

Hence, using a property of definite integrals, (2) yields $\pi J_1(x) = 2\int_0^{\pi/2} \sin(x\sin\phi)\sin\phi\, d\phi$

$$\Rightarrow \qquad \pi J_1(ay) = 2\int_0^{\pi/2} \sin(ay\sin\phi)\sin\phi\, d\phi \qquad \ldots (3)$$

Put $y\sin\phi = x$ so that $y\cos\phi\, d\phi = dx$ and hence

$$d\phi = \frac{dx}{y\cos\phi} = \frac{dx}{y(1-\sin^2\phi)^{1/2}} = \frac{dx}{y(1-x^2/y^2)^{1/2}} = \frac{dx}{(y^2-x^2)^{1/2}}$$

Hence (3) yields $\pi J_1(ay) = 2\int_0^{y}\left(\sin ax \times \frac{x}{y} \times \frac{1}{(y^2-x^2)^{1/2}}\right)dx \quad \Rightarrow \quad \int_0^{y} \frac{x\sin ax\, dx}{(y^2-x^2)^{1/2}} = \frac{\pi y}{2} J_1(ay)$

Ex. 8. *Prove that* (*i*) $J_n(x) = \dfrac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\dfrac{x}{2}\right)^n \displaystyle\int_0^{\pi} \cos(x\sin\phi)\cos^{2n}\phi\, d\phi$.

[Garhwal 2005, Ravishanker 2004; Ranchi 2010]

(*ii*) $J_n(x) = \dfrac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\dfrac{x}{2}\right)^n \displaystyle\int_0^{\pi} \cos(x\cos\phi)\sin^{2n}\phi\, d\phi$. **[Bilaspur 1998]**

Sol. (i). Let $$I = \int_0^{\pi} \cos(x\sin\phi)\cos^{2n}\phi\, d\phi. \qquad \ldots(1)$$

$\therefore\ I = 2\displaystyle\int_0^{\pi/2} \cos(x\sin\phi)\cos^{2n}\phi\, d\phi$, since $\cos(x\sin\phi)\cos^{2n}\phi$ is an even function.

or $$I = 2\int_0^{\pi/2} \cos^{2n}\phi\left[1 - \frac{x^2\sin^2\phi}{2!} + \frac{x^4\sin^4\phi}{4!} - \ldots\right]d\phi. \qquad \ldots(2)$$

But $$2\int_0^{\pi/2} \cos^p\phi\cos^q\phi\, d\phi = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}. \qquad \ldots(3)$$

Using (3), (2) may be written as

$$I = 2\left[\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(n+1)} - \frac{x^2}{2!}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(3/2)}{2\Gamma(n+2)} + \frac{x^4}{4!}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(5/2)}{2\Gamma(n+3)} - \ldots\right]$$

$$= \Gamma(n+\tfrac{1}{2})\left[\frac{\sqrt{\pi}}{\Gamma(n+1)} - \frac{x^2}{2!}\times\frac{\frac{1}{2}\sqrt{\pi}}{(n+1)\,\Gamma(n+1)} + \frac{x^4}{4!}\times\frac{(3/2)\times\frac{1}{2}\sqrt{\pi}}{(n+2)\,(n+1)\Gamma(n+1)} - \ldots\right]$$

$$= \frac{\Gamma(n+\frac{1}{2})\sqrt{\pi}}{\Gamma(n+1)}\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4\cdot 8(n+1)\,(n+2)} - \ldots\right] \qquad \ldots(4)$$

Also, $$\mathsf{J}_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8(n+1)(n+2)} - \ldots\right]. \quad \ldots(5)$$

Multiplying both sides of (4) by $\frac{x^n}{2^n \Gamma(n+\frac{1}{2})\sqrt{\pi}}$, we get

$$\frac{x^n I}{2^n \Gamma(n+\frac{1}{2})\sqrt{\pi}} = \frac{x^n}{2^n \Gamma(n+\frac{1}{2})}\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8 \cdot (n+1)(n+2)} - \ldots\right] \quad \ldots(6)$$

Using (1) and (5), (6) becomes $$\frac{1}{\sqrt{\pi}\ \Gamma(n+1/2)}\left(\frac{x}{2}\right)^n \int_0^{\pi} \cos(x \sin\phi) \cos^{2n}\phi\, d\phi = J_n(x).$$

Part (ii). Proceed as in part (*i*),

Ex. 9. *Prove that* $J_n(x) = \frac{x^n}{2^{n-1}\ \Gamma(n+\frac{1}{2})\ \sqrt{\pi}} \int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt$. **[Bilaspur 1994, 97]**

Sol. From part (*i*) of Ex. 8, $$J_n(x) = \frac{1}{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n \int_0^{\pi} \cos(x \sin\phi) \cos^{2n}\phi\, d\phi. \quad \ldots(1)$$

Let $$I = \int_0^{\pi} \cos(x \sin\phi) \cos^{2n}\phi\, d\phi = 2\int_0^{\pi/2} \cos(x \sin\phi) \cos^{2n}\phi\, d\phi \quad \ldots(2)$$

Let $\sin\phi = t$ so that $\cos\phi\, d\phi = dt$. Then (2) gives

$$I = 2\int_0^1 \cos xt \cos^{2n}\phi \frac{dt}{\cos\phi} = 2\int_0^1 \cos xt \cos^{2n-1}\phi\, d\,t$$

$$= 2\int_0^1 \cos xt\, (\cos^2\phi)^{(2n-1)/2} dt = 2\int_0^1 \cos xt\, (1-\sin^2\phi)^{n-1/2} d\,t\,.$$

$$\therefore \quad I = 2\int_0^1 \cos xt.\,(I-t^2)^{n-1/2}\, dt. \quad \ldots(3)$$

Using (2), (3) gives $$\int_0^{\pi} \cos(x \sin\phi) \cos^{2n} d\phi = 2\int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt\,. \quad \ldots(4)$$

Using (4), (1) $\Rightarrow$ $$J_n(x) = \frac{1}{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})2^n} \times \frac{x^n}{2^n} \times 2\int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt$$

$$\therefore \quad J_n(x) = \frac{x^n}{2^{n-1}\ \Gamma(n+\frac{1}{2})\ \sqrt{\pi}} \int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt\,.$$

Ex. 10. *Verify directly that representation* $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta)\, d\theta$ *satisfies Bessel's equation in which* $n = 0$. **[Indore 2004]**

Sol. Given $$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta)\, d\theta. \quad \ldots(1)$$

Bessel's equation of order n is $$x^2y'' + xy' + (x^2 - n^2)y = 0. \quad \ldots(2)$$

For $n = 0$, (2) reduces to $$x^2y'' + xy' + x^2y = 0. \quad \ldots(3)$$

In order to show that $J_0(x)$ satisfies (3), we must prove that

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0. \quad \ldots(4)$$

Differentiating both sides of (1) w.r.t. 'x', $J_0'(x) = -\frac{1}{\pi}\int_0^{\pi} \sin(x \sin\theta) \sin\theta \, d\theta$. ...(5)

Differentiating both sides of (5) w.r.t. 'x', $J_0''(x) = -\frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta) \sin^2\theta \, d\theta$. ...(6)

Integrating R.H.S. of (5) by parts taking sin θ as second function, we get

$$J_0'(x) = -\frac{1}{\pi}\left[\{-\sin(x \sin\theta)(\cos\theta)\}_0^{\pi} + \int_0^{\pi} \cos(x \sin\theta) \cdot x \cos\theta \cos\theta \, d\theta\right]$$

or $$J_0'(x) = -\frac{1}{\pi}\left[0 + x\int_0^{\pi} \cos(x \sin\theta) \cos^2\theta \, d\theta\right] = -\frac{x}{\pi}\int_0^{\pi} \cos(x \sin\theta) \cos^2\theta \, d\theta. \quad ...(7)$$

Using (1), (6) and (7), L.H.S. of (4)

$$= -\frac{x^2}{\pi}\int_0^{\pi} \cos(x \sin\theta) \sin^2\theta \, d\theta - \frac{x^2}{\pi}\int_0^{\pi} \cos(x \sin\theta) \cos^2\theta \, d\theta + \frac{x^2}{\pi}\int_0^{\pi} \cos(x \sin\theta) \, d\theta$$

$$= -\frac{x^2}{\pi}\int_0^{\pi} \cos(x \sin\theta)(\sin^2\theta + \cos^2\theta) \, d\theta + \frac{x^2}{\pi}\int_0^{\pi} \cos(x \sin\theta) \, d\theta = 0 = \text{R.H.S. of (4)}$$

Thus (4) is true. Hence we get the required result.

EXERCISE 11 (D)

1. Prove that (*i*) $J_0(x) = \frac{2}{x}\int_0^1 \frac{\cos xt \, dt}{\sqrt{(1-t^2)}}$ (*ii*) $\int_0^{\infty} \frac{J_n(x)}{x} dx = \frac{1}{n}$.

2. If n is non–negative, prove that $\int_0^{\infty} J_n(bx) \, dx = \frac{1}{b}$.

3. Prove that (*i*) $\int_0^{\infty} \sin ax \, J_0(bx) \, dx = \begin{cases} 0, & b > a \\ 1/(a^2+b^2)^{1/2}, & b < a \end{cases}$

(*ii*) $\int_0^{\infty} \cos ax \, J_0(bx) \, dx = \begin{cases} 1/(b^2-a^2)^{1/2}, & b > a \\ 0, & b < a \end{cases}$

4. Show that (*i*) $\int_0^{\pi/2} J_0(x \cos\theta) \cos\theta \, d\theta = \frac{\sin x}{x}$ (*ii*) $\int_0^{\pi/2} J_1(x \cos\theta) \, d\theta = \frac{1-\cos x}{x}$.

5. For Bessel function $J_n(x)$ prove that $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x \sin\theta) \, d\theta$ and hence show that

$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt \, dt}{\sqrt{(1-t^2)}}$. **[Purvanchal 2007]**

6. Show that the recurrence relation $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$, follows directly from differentiation of $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x \sin\theta) \, d\theta$.

7. Show that coefficient of t^n in the expansion of $e^{-(x/2)(t-1/t)}$ equals $\frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x \sin\phi) \, d\phi$.

11.10. Orthogonality of Bessel functions

If λ_i and λ_j are roots of the equation $J_n(\lambda a) = 0$, then

$$\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = \begin{cases} 0, \text{ if } i \neq j \text{ (different roots)} \\ \dfrac{a^2}{2} J_{n+1}^2(\lambda_i a), \text{ if } i = j \text{ (equal roots)} \end{cases}$$

i.e. $\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = \dfrac{a^2}{2} J_{n+1}^2(\lambda_i a)\, \delta_{ij}$. where δ_{ij} = Kronecker delta = $\begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

[Meerut 2010; Nagpur 2005; Purvanchal 2005, 06; Delhi Physics (H) 2002]

Proof. Case I. Let $i \neq j$, *i.e.*, let λ_i and λ_j be unequal roots of $J_n(\lambda a) = 0$.

$\therefore$ $J_n(\lambda_i a) = 0$ and $J_n(\lambda_j a) = 0$ ...(1)

Let $u(x) = J_n(\lambda_i x)$ and $v(x) = J_n(\lambda_j x)$. ...(2)

Then u and v are Bessel functions satisfying the modified Bessel's equation [Refer Art. 11.4]

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0. \quad ...(3)$$

$\therefore$ $x^2u'' + xu' + (\lambda_i^2x^2 - n^2)u = 0.$ ...(4)

and $x^2v'' + xv' + (\lambda_j^2x^2 - n^2)v = 0.$ ...(5)

Multiplying (4) by v and (5) by u and then subtracting, we get

$x^2(vu'' - uv'') + x(vu' - uv') + x^2(\lambda_i^2 - \lambda_j^2)uv = 0$ or $x(vu'' - uv'') + (vu' - uv') = x(\lambda_j^2 - \lambda_i^2)uv$

or $x\dfrac{d}{dx}(vu' - uv') + (vu' - uv') = x(\lambda_j^2 - \lambda_i^2)uv$

or $\dfrac{d}{dx}\{x(vu' - uv')\} = x(\lambda_j^2 - \lambda_i^2)uv.$ ...(6)

Integrating (6) w.r.t. x from 0 to a, $(\lambda_j^2 - \lambda_i^2)\int_0^a xuv\, dx = \left[x(vu' - uv')\right]_0^a$. ...(7)

Using (2), (7) reduces to

$$(\lambda_j^2 - \lambda_i^2)\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = \left[x\{J_n(\lambda_j x)J_n'(\lambda_i x) - J_n(\lambda_i x)\, J_n'(\lambda_j x)\}\right]_0^a$$

$$= a\{J_n(\lambda_j a)J'_n(\lambda_i a) - J_n(\lambda_i a)J'_n(\lambda_j a)\} = 0, \text{ using (1)}$$

Since $\lambda_i \neq \lambda_j$, the above equation gives $\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = 0$, when $i \neq j$. ...(8)

Case II. Let $i = j$ (equal roots). Multiplying (4) by $2u'$, we have

$2x^2u''u' + 2xu'^2 + 2(\lambda_i^2x^2 - n^2)uu' = 0$ or $\dfrac{d}{dx}\{x^2u'^2 - n^2u^2 + \lambda_i^2x^2u^2\} - 2\lambda_i^2xu^2 = 0$

$\therefore$ $2\lambda_i^2xu^2 = \dfrac{d}{dx}(x^2u'^2 - n^2u^2 + \lambda_i^2x^2u^2).$ ...(9)

Integrating (9) w.r.t. 'x' from 0 to a, $2\lambda_i^2\int_0^a xu^2\, dx = \left[x^2u'^2 - n^2u^2 + \lambda_i^2x^2u^2\right]_0^a$ ...(10)

Using (1) and (2) and noting that $J_n(0) = 0$, we have

$$2\lambda_i^2\int_0^a x\, J_n^2(\lambda_i x)\, dx = a^2\left[\{J_n'(\lambda_i x)\}^2\right]_{x=a} \quad ...(11)$$

From recurrence relation IV, we have $\dfrac{d}{dx}J_n(x) = \dfrac{n}{x}J_n(x) - J_{n+1}(x).$ ...(12)

Replacing x by $\lambda_i x$ in (12), we have

$\dfrac{d}{d(\lambda_i x)}J_n(\lambda_i x) = \dfrac{n}{\lambda_i x}J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$ or $\dfrac{1}{\lambda_i}\dfrac{d}{dx}J_n(\lambda_i x) = \dfrac{n}{\lambda_i x}J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$

or $$J'_n(\lambda_i x) = \frac{n}{x} J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x).$$

$$\therefore \quad \left[\{J_n'(\lambda_i x)\}^2\right]_{x=a} = \left[\left\{\frac{n}{x} J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)\right\}^2\right]_{x=a} = \{0 - \lambda_i J_{n+1}(\lambda_i a)\}^2, \text{ by (1)}$$

$$= \lambda_i^2 J^2_{n+1}(\lambda_i a).$$

Using this value in (11) and dividing both sides of the resulting equation by $2\lambda_i^2$, we get

$$\int_0^a x J_n^2(\lambda_i x)\, dx = \frac{a^2}{2} J_{n+1}^2(\lambda_i a). \qquad ...(13)$$

Combining (8) and (13), we have $$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx = \frac{a^2}{2} J_{n+1}^2(\lambda_i a)\, \delta_{ij}. \qquad ...(14)$$

11.11. Bessel–series or Fourier–Bessel expansion for f(x).

If $f(x)$ is defined in the region $0 \le x \le a$ and has an expansion of the form

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\lambda_i x), \qquad ...(1)$$

where the λ_i are the roots of the equation $$J_n(\lambda a) = 0, \qquad ...(2)$$

then $$c_i = \frac{2\int_0^a x f(x) J_n(\lambda_i x)\, dx}{a^2 J_{n+1}^2(\lambda_i a)}. \qquad ...(3)$$

Proof. Multiplying both sides of (1) by $xJ_n(\lambda_j x)$, $$xf(x) J_n(\lambda_j x) = \sum_{i=1}^{\infty} c_i x J_n(\lambda_i x) J_n(\lambda_j x) \qquad ...(4)$$

Integrating both sides of (4) w.r.t. 'x' from 0 to a, we get

$$\int_0^a x f(x) J_n(\lambda_j x)\, dx = \sum_{i=1}^{\infty} c_i \int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx. \qquad ...(5)$$

From the orthogonality property of Bessel functions, we have

$$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx = \begin{cases} 0, \text{ if } i \ne j \\ \frac{a^2}{2} J_{n+1}^2(\lambda_j a), \text{ if } i = j \end{cases} \qquad ...(6)$$

Using (6), (5) reduces to $$\int_0^a x f(x) J_n(\lambda_j x)\, dx = c_j \frac{a^2}{2} J^2_{n+1}(\lambda_j a). \qquad ...(7)$$

Replacing j by i in (7), we have

$$c_i \frac{a^2}{2} J^2_{n+1}(\lambda_i a) = \int_0^a x f(x) J_n(\lambda_i x)\, dx \qquad \text{or} \qquad c_i = \frac{2\int_0^a x f(x) J_n(\lambda_i x)\, dx}{a^2 J_{n+1}^2(\lambda_i a)}.$$

...(8)

11.11A. Solved Exaples based on Art. 11.11

Ex. 1. *Expand the function* $f(x) = 1$, $0 \le x \le a$ in a series of the form $\sum_{i=1}^{\infty} c_i J_0(\lambda_i x)$, where λ_i are the roots of the equation $J_0(\lambda a) = 0$.

Sol. Given $$f(x) = 1 = \sum_{i=1}^{\infty} c_i J_0(\lambda_i x), \qquad ...(1)$$

where $J_0(\lambda_i a) = 0.$ ...(2)

Then from Art. 11. 11 (with $n = 0$), we know that

$$c_i = \frac{2\int_0^a x\ f(x)\ J_0(\lambda_i x)\ dx}{a^2 J_1^2(\lambda_i a)} = \frac{2\int_0^a x\ J_0(\lambda_i x)\ dx}{a^2 J_1^2(\lambda_i a)}, \text{ as } f(x) = 1 \quad ...(3)$$

Let $\lambda_i x = t$ so that $dx = dt/\lambda_i$. Then, we have

$$\int_0^a x\ J_0(\lambda_i x)\ dx = \frac{1}{\lambda_i^2}\int_0^{a\lambda_i} t\ J_0(t)\ dt$$

$$= \frac{1}{\lambda_i^2}\int_0^{a\lambda_i} \frac{d}{dt}\{t\ J_1(t)\}\ dt, \text{ as } \frac{d}{dx}[x^n\ J_n(x)] = x^n J_{n-1}(x) \Rightarrow \frac{d}{dx}[x\ J_1(x)] = x\ J_0(x)$$

$$= \frac{1}{\lambda_i^2}\Big[t\ J_1(t)\Big]_0^{a\lambda_i} = \frac{1}{\lambda_i^2}[a\lambda_i\ J_1(a\lambda_i) - 0], \text{ as } J_1(0) = 0\ .$$

$$\therefore \quad \int_0^a x\ J_0(\lambda_i x)\ dx = (a/\lambda_i) \times J_1(a\lambda_i) \quad ...(4)$$

Using (4), (3) becomes $$c_i = \frac{2 \times (a/\lambda_i) \times J_1(a\lambda_i)}{a^2\ J_1^2\ (a\lambda_i)} = \frac{2}{a\lambda_i J_1(a\lambda_i)} \quad ...(5)$$

Using (5), (1) becomes $$1 = \frac{2}{a}\sum_{i=1}^{\infty} \frac{J_0(\lambda_i x)}{\lambda_i\ J_0(\lambda_i a)}.$$

Ex. 2. *Expand x in a series of the form $\sum_{r=1}^{\infty} C_r\ J_1(\lambda_r x)$ valid for the region $0 \le x \le 1$, where λ_r are the roots of the equation $J_1(\lambda) = 0$.*

Sol. Given $$f(x) = x = \sum_{r=1}^{\infty} C_r\ J_1(\lambda_r x), \quad ...(1)$$

where $$J_1(\lambda_r) = 0 \quad ...(2)$$

Then from Art. 11.11 (with $n = 1$, $i = r$ and $a = 1$), $$C_r = \frac{2\int_0^1 x^2\ J_1(\lambda_r x)\ dx}{J_2^2(\lambda_r)} \quad ...(3)$$

Let $\lambda_r x = t$, so that $dx = dt/\lambda_r$. Then we have

$$\int_0^1 x^2\ J_1(\lambda_r x)\ dx = \frac{1}{\lambda_r^3}\int_0^{\lambda_r} t^2\ J_1(t)\ dt$$

$$= \frac{1}{\lambda_r^3}\int_0^{\lambda_r} \frac{d}{dt}\{t^2\ J_2(t)\}\ dt, \quad \text{as} \quad \frac{d}{dx}[x^n\ J_n(x)] = x^n J_{n-1}(x) \Rightarrow \frac{d}{dx}[x^2\ J_2(x)] = x^2\ J_1(x)$$

$$= \frac{1}{\lambda_r^3}\Big[t^2\ J_2(t)\Big]_0^{\lambda_r} = \frac{1}{\lambda_r^3}[\lambda_r^2\ J_2(\lambda_r) - 0], \text{ as } J_2(0) = 0$$

$$\therefore \quad \int_0^1 x^2\ J_1(\lambda_r x)\ dx = \frac{1}{\lambda_r}\ J_2(\lambda_r). \quad ...(4)$$

Using (4), (3) becomes $$C_r = \frac{2}{\lambda_r\ J_2(\lambda_r)} \quad ...(5)$$

Using (5), (1) becomes $$x = 2\sum_{r=1}^{\infty} \frac{J_1(\lambda_r x)}{\lambda_r\ J_2(\lambda_r)},\ 0 \le x \le 1.$$

EXERCISE 11 (E)

1. Expand x^2 in a series of the form $\sum_{r=1}^{\infty} C_r\, J_0(\lambda_r x)$ valid for the region $0 \le x \le a$, where λ_r are the roots of the equation $J_0(\lambda a) = 0$. **Ans.** $x^2 = \dfrac{2}{a}\sum_{r=1}^{\infty} \dfrac{\{(\lambda_r a)^2 - 4\}\, J_0(\lambda_r x)}{\lambda_r^3\, J_0(\lambda_r a)}$

2. Prove that $1 = \sum_{n=1}^{\infty} \dfrac{2}{\alpha_n} \dfrac{J_0(\alpha_n x)}{J_1(\alpha_n)}$.

3. If λ_i are the solutions of $J_0(\lambda) = 0$, show that $\sum_{i=1}^{\infty} \dfrac{J_0(\lambda_i x)}{\{\lambda_i J_1(\lambda_n)\}^2} = -\dfrac{1}{2}\log x$, where $0 < x < 1$.

4. If $f(x) = \sum_{i=1}^{\infty} C_i\, J_0(\lambda_i x)$ where $J_0(\lambda_i) = 0$, $i = 1, 2, 3, \ldots$, show that $\int_0^1 x[f(x)]^2\, dx = \sum_{i=1}^{\infty} \lambda_i^2\, J_1^2(\lambda_i)$.

5. If λ_i are the positive roots of $J_0(\lambda) = 0$, show that $\dfrac{1 - x^2}{8} = \sum_{i=1}^{\infty} \dfrac{J_0(\lambda_i x)}{\lambda_i^3\, J_1(\lambda_i)}$, where $-1 < x < 1$

6. If λ_i are the positive roots of $J_1(\lambda) = 0$, show that $x^3 = 2\sum_{i=1}^{\infty} \dfrac{J_1(\lambda_i x)}{\lambda_i\, J_2(\lambda_i)}$, where $-1 < x < 1$

7. If λ_i are the positive roots of $J_1(\lambda) = 0$, show that $x^3 = 2\sum_{i=1}^{\infty} \dfrac{(8 - \lambda_i^2)\, J_1(\lambda_i x)}{\lambda_i^3\, J_1'(\lambda_i)}$, where $-1 < x < 1$

OBJECTIVE PROBLEMS ON CHAPTER 11

1. *Write* (*a*), (*b*), (*c*) *or* (*d*) *whichever is correct*

(*a*) $x\{J_{n-1}(x) + J_{n+1}(x)\}$ is equal to

(*a*) $2J_n(x)$ (*b*) $2J_n'(x)$ (*c*) $2n\, J_n(x)$ (*d*) None of these **[Agra 2005, 06]**

Sol. Ans. (*c*). Refer recurrence relation VI of Art. 11.7.

2. The Bessel's equation is (*a*) $z^2(d^2w/dz^2) - z\,(dw/dz) + (z^2 - n^2)\, w = 0$

(*b*) $z^2(d^2w/dz^2) + z\,(dw/dz) + (z^2 + n^2)\, w = 0$ (*c*) $z^2(d^2w/dz^2) + z(dw/dz) + (z^2 - n^2)w = 0$

(*d*) None of these. **[Agra 2005, 07]**

Sol. Ans. (c). Refer equation (1) of Art. 11.1

3. $d\,\{x^n\, J_n(x)\}/dx$ is equal to

(*a*) $x^n\, J_{n-1}(x)$ (*b*) $x^{n-1}\, J_n\,(x)$ (*c*) $x^n\, J_{n+1}(x)$ (*d*) $x^{n+1}\, J_n(x)$ **[Bhopal 2010]**

Sol. Ans. (*a*). Refer recurrence relation I of Art. 11.7.

4. *The Bessel's functions* $\{J_0(\alpha_k x)\}_{k=1}^{\infty}$ *with* α_k *denoting the* k^{th} *zero of* $J_0(x)$ *form an orthogonal system on* [0, 1] with respect to weight function (*a*) 1 (*b*) x^2 (*c*) x (*d*) $\sqrt{x}$.

Sol. Ans. (*c*) Refer Art. 11.10. **[GATE 2002]**

5. *If* $J_n(x)$ *and* $Y_n(x)$ *denote Bessel functions of order n of the first and second kind, then the general solution of the differential equation* $x(d^2y/dx^2) + (dy/dx) + xy = 0$ *is given by*

(*a*) $y(x) = \alpha x\, J_1(x) + \beta x Y_1(x)$ (*b*) $y(x) = \alpha\, J_1(x) + \beta Y_1(x)$

(*c*) $y(x) = \alpha J_0(x) + \beta Y_0(x)$ (*d*) $y(x) = \alpha x J_0(x) + \beta x Y_0(x)$. **[GATE 2005]**

Sol. Ans. (*c*) Refer remark of theorem I, Art. 11.4.

MISCELLANEOUS PROBLEMS ON CHAPTER 11

1. If $x > a$, show that $\int_0^{\pi} e^{a\cos\theta}\cos(x\sin\theta)\,d\theta = J_0\{(x^2-a^2)^{1/2}\}$

2. Prove that $J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)\,d\phi$ and deduce that (*i*) $\int_0^{\infty} e^{-ax}J_0(bx)dx = \frac{1}{(a^2+b^2)^{1/2}}$

(*ii*) $\int_0^{\infty}\cos(ax)J_0(bx)dx = 0$ or $\frac{1}{(b^2-a^2)^{1/2}}$, according $a^2 >$ or $< b^2$

(*iii*) $\int_0^{\infty}\sin(ax)J_0(bx)\,dx = -\frac{1}{(a^2-b^2)^{1/2}}$ or 0, according as $a^2 >$ or $< b^2$.

3. Show that (*i*) $\int_0^{\pi/2} J_{2n}(2x\cos\theta)\,d\theta = \frac{\pi}{2}\left[J_n(x)\right]^2$ (*ii*) $J_0(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin(x+y)}{x+y}J_0(y)\,dy$

(*iii*) $x^n = 2^n\sum_{r=0}^{\infty}\frac{(n+2r)\,(n+r-1)}{r\,!}J_{n+2r}(x)$

4. Show that $J_n(x)$ is even or odd function of x according as n is even or odd, respectively.

5. Show that $\int_0^1 J_0\left(\{x\,(t-x)\}^{1/2}\right) = 2\sin(t/2)$. **[Kanpur 2008]**

Sol. Setting $n = 0$ and $x = \{x\,(t-x)\}^{1/2}$ in result (1) of Art. 11.2, we have

$$j_0\left(\{x\,(t-x)\}^{1/2}\right) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(r+1)}\left[\frac{\{x(t-x\}^{1/2}}{2}\right]^{2r} = \sum_{r=0}^{\infty}\frac{(-1)^r x^r(t-x)^r}{2^{2r}r!\,\Gamma(r+1)}$$

$$\therefore\quad \int_0^1 J_0\left(\{x(t-x)\}^{1/2}\right)dx = \int_0^1\sum_{r=0}^{\infty}\frac{(-1)^r x^r(t-x)^r}{2^{2r}r!\,\Gamma(r+1)}dx = \sum_{r=0}^{\infty}\frac{(-1)^r}{2^{2r}r!\,\Gamma(r+1)}\int_0^1 x^r(t-x)^r\,dx$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r t^{2r+1}}{2^{2r}r!\,\Gamma(r+1)}\int_0^1 y^r(1-y)^r\,dy, \text{ putting } x = ty \text{ and } dx = tdy$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r t^{2r+1}}{2^{2r}r!\,\Gamma(r+1)}B(r+1,r+1), \text{ by definition 6.2, page 6.1}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r t^{2r+1}\,r!}{2^{2r}r!\,\Gamma(2r+1)}, \text{ since } \Gamma(r+1) = r!,\ r \text{ being a positive integer.}$$

$$= 2\sum_{r=0}^{\infty}(-1)^r\frac{(t/2)^{2r+1}}{(2r+1)!} = 2\left\{\frac{t}{2} - \frac{(t/2)^3}{3!} + \frac{(t/5)^5}{5!} - \ldots\ldots\right\}$$

$= 2\sin(t/2)$, as $\sin x = x - x^3/3! + x^3/5! - x^7/7! + \ldots$

6. Show that every non-trivial solution of Bessel's equation $x^2 y'' + xy' + (x^2 - n^2) y = 0$ has infinitely many zeros. **[Mumbai 2010]**

7. Find the normal form of Bessel's equation $x^2y'' + xy' + (x^2 - p^2) y = 0$ and use it to show that every non-trival solution has infinitely many positive zeros. **[Himachal 2010]**

8. Show that if x is real, between two consecutive zeros of $x^{-n} J_n (x)$, there lies one and only one zero of $x^{-n} J_{n+1} (x)$.

12

Hermite Polynomials

12.1. Hermite's equation and its solution. [Meerut 1993, 95, 96]

Hermite's equation is $(d^2y/dx^2) - 2x(dy/dx) + 2ny = 0,$ ...(1)

where n is a constant. We now solve (1) in series by using the method of Frobenius.

Let the series solution of (1) be $y = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad C_0 \neq 0$ ...(2)

Differentiating (2) and then putting the value of y, dy/dx and d^2y/dx^2 in (1), we get

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - 2xC_m(k+m) x^{k+m-1} + 2n \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - 2 \left[\sum_{m=0}^{\infty} C_m (k+m) x^{k+m} - \sum_{m=0}^{\infty} C_m n x^{k+m} \right] = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - 2 \sum_{m=0}^{\infty} C_m (k+m-n) x^{k+m} = 0.$$ ... (3)

(3) is an identity. To get the indicial equation, we equate to zero the coefficient of the smallest power of x, namely x^{k-2}, in (3) and obtain

$$C_0 k(k-1) = 0 \quad \text{or} \quad k(k-1) = 0, \quad \text{as} \quad C_0 \neq 0.$$ ...(4)

So the roots of indicial equation (4) are $k = 0, 1$. They are distinct and differ by an integer. The next smallest power of x is $k - 1$. So equating to zero the coefficient of x^{k-1} in (3), we get

$$C_1 (k+1) k = 0.$$ ... (5)

When $k = 0$ (one of the roots of the indicial equation), (5) shows that C_1 it indeterminate. Hence C_0 and C_1 may be taken as arbitrary constants. Equating to zero the coefficient of x^{k+m-2}, (3) gives $C_m (k+m)(k+m-1) - 2 C_{m-2} (k+m-2-n) = 0$

or $$C_m = \frac{2(k+m-2-n)}{(k+m)(k+m-1)} C_{m-2}.$$ ...(6)

Putting $k = 0$ in (6) gives $$C_m = \frac{2(m-2-n)}{m(m-1)} C_{m-2}.$$ ...(7)

Putting $m = 2, 4, 6, ..., 2m$, in (7), we have

$$C_2 = -\frac{2n}{2 \cdot 1} C_0 = -\frac{2n}{2!} C_0 = -\frac{(-1)^1 \cdot 2^1 \cdot n}{2!} C_0,$$

$$C_4 = \frac{2(2-n)}{4 \cdot 3} C_2 = \frac{(-1)^2 \cdot 2(2-n)}{4 \cdot 3} \times \frac{2n}{2!} C_0 = \frac{(-1)^2 2^2 n(n-2)}{4!} C_0,$$

... ...

and $$C_{2m} = \frac{(-1)^m 2^m \cdot n(n-2)...(n-2m+2)}{(2m)!} C_0$$

Next, putting $m = 3, 5, 7, ..., 2m + 1$, in (7), we get

$$C_3 = \frac{2(1-n)}{3\cdot 2} C_1 = -\frac{2(n-1)}{3!} C_1 = \frac{(-1)^1 2^1 (n-1)}{3!} C_1, C_5 = \frac{2(3-n)}{5\cdot 4} C_3 = \frac{(-1)^2 2^2 (n-1)(n-3)}{5!} C_1,$$

... … … … … …

and $$C_{2m+1} = \frac{(-1)^m 2^m (n-1)(n-3)\ldots(n-2m+1)}{(2m+1)!} C_1.$$

Putting the above values in (2) with $k = 0$, we get $y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + ...$

i.e. $y = (C_0 + C_2 x^2 + C_4 x^4 + ...) + (C_1 x + C_3 x^3 + C_5 x^5 + ...),$

i.e. $$y = C_0\left[1 - \frac{2n}{2!}x^2 + \frac{2^2 n(n-2)}{4!} x^4 \ldots + \frac{(-2)^m n(n-2)\ldots(n-2m+2)}{(2m)!} x^{2m} + \ldots\right]$$

$$+ C_1\left[x - \frac{2(n-1)}{3!} x^3 + \frac{2^2 (n-1)(n-3)}{5!} x^5 + \ldots + \frac{(-2)^m (n-1)(n-3)\ldots(n-2m+1)}{(2m+1)!} x^{2m+1} + \ldots\right] \quad ...(8)$$

or $$y = C_0 u + C_1 v, \text{ say,} \quad ...(9)$$

Since u/v is not merely a constant, u and v form a fundamental set (*i.e.* linearly independent) of solutions of (1). Hence (8) or (9) is the most general solution of (1) with C_0 and C_1 as two arbitrary constants.

Remarks. In practice we require a solution of (1) such that

(*i*) it is finite for all finite values of x and (*ii*) as $x \to \infty$, exp. $(1/2x^2)\, y(x) \to 0$.

The solution (8) in ascending powers of x does not satisfy the condition exp. $(1/2x^2)\ y(x) \to 0$ as $x \to \infty$. However, this requirement is easily seen to be satisfied provided the series terminate. Replacing m by $m + 2$ in (7), we have

$$C_{m+2} = \frac{2(m-n)}{(m+1)(m+2)} C_m. \quad ...(10)$$

Let n be a non–negative integer. Then (10) shows that C_{m+2} and all subsequent coefficients in (2) will vanish and so the corresponding series terminate. We shall now obtain the series solution of (1) in descending powers of x by assuming n to be a non–negative integer. Re–writing (2) for $k = 0$, we have as explained above

$$y = C_n x^n + C_{n-2} x^{n-2} + C_{n-4} x^{n-4} + \ldots \quad ...(11)$$

From (10), $$C_m = -\frac{(m+1)(m+2)}{2(n-m)} C_{m+2}. \quad ...(12)$$

Putting $m = n - 2, n - 4, ...,$ in (12), $$C_{n-2} = -\frac{(n-1)n}{2(n-n+2)} C_n = -\frac{n(n-1)}{2\cdot 2} C_n$$

$$C_{n-4} = -\frac{(n-3)(n-2)}{2(n-n+4)} C_{n-2} = -\frac{n(n-1)(n-2)(n-3)}{2^2 \cdot 2 \cdot 4} C_n$$

and so on. Putting these is (11), we have $y = a_n\{x^n$

$$-\frac{n(n-1)}{2\cdot 2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2 \cdot 2 \cdot 4} x^{n-4} + ... + (-1)^r \frac{n(n-1)(n-2r+1)}{2^r \cdot 2 \cdot 4 \ldots 2r} x^{n-2r} + \ldots\Big\}$$

or $$y = a_n \sum_{r=0}^{[n/2]} (-1)^r \frac{n(n-1)(n-2r+1)}{2^r \cdot 2 \cdot 4 \ldots 2r} x^{n-2r} = a_n \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{2^{2r} r!(n-2r)!} x^{n-2r},$$

where $[n/2] = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}$

Taking $a_n = 2^n$ and denoting the solution by $H_n(x)$, we obtain the standard solution of (1), known as *Hermite polynomial of order n*.

12.2. Hermite polynomial of order n. [Meerut 1994, 97, 98]

Hermite polynomial of order n is denoted and defined by

$$H_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!(n-2r)!} (2x)^{n-2r}, \quad \text{where,} \quad [n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases}$$

12.3. Generating function for Hermite polynomials.

Theorem. *Prove that* $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ **[Meerut 1992, 94]**

Proof. Using the well known expansion for exponential function, we get

$$e^{2tx-t^2} = e^{2tx} \cdot e^{-t^2} = \sum_{s=0}^{\infty} \frac{(2tx)^s}{s!} \sum_{r=0}^{\infty} \frac{(-t^2)^r}{r!} = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r \frac{(2x)^s}{r!\,s!} t^{s+2r} \quad \ldots(1)$$

Let $s + 2r = n$ so that $s = n - 2r$. So for a fixed value of r, the coefficient of t^n is given by

$$(-1)^r \frac{(2x)^{n-2r}}{r!(n-2r)!}. \quad \ldots(2)$$

Now, $s \geq 0 \Rightarrow n - 2r \geq 0 \Rightarrow n \geq 2r \Rightarrow r \leq n/2$, which gives all values of r for which (2) is the coefficient of t^n. If n is even, $r \leq n/2$ shows that r varies from 0 to $n/2$. Again, if n is odd, $r \leq n/2$ shows that r varies from 0 to $(n-1)/2$. Here note that r is an integer. Combining these results we see that r varies from 0 to $[n/2]$, where,

$$[n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases}$$

Hence the total coefficient of t^n in the expansion of e^{2tx-t^2} is given by

$$\sum_{r=0}^{[n/2]} (-1)^r \frac{1}{r!(n-2r)!} (2x)^{n-2r} \quad i.e. \quad \frac{1}{n!} \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!(n-2r)!} (2x)^{n-2r}, \quad i.e. \quad \frac{1}{n!} H_n(x).$$

Again the coefficient of t^n in $\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ is also $\frac{1}{n!} H_n(x)$.

This proves the required result.

12.4. Alternative expressions for the Hermite polynomials.

Theorem 1. *Prove that* $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$.

[Kanpur 2005, 07, 09; Garhwal 1996; Meerut 1997]

Proof. Using the generating function, we have $\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2tx-t^2}$. ...(1)

Expanding the function on the R.H.S. by Taylor's theorem, (1) gives

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \sum_{n=0}^{\infty} \left[\frac{\partial^n}{\partial t^n} e^{2tx-t^2} \right]_{t=0} \frac{t^n}{n!}. \quad \ldots(2)$$

Equating coefficient of t^n in (2) and cancelling $n\,!$ from both sides, we get

$$H_n(x) = \left[\frac{\partial^n}{\partial t^n} e^{2tx - t^2}\right]_{t=0} = \left[\frac{\partial^n}{\partial t^n} e^{x^2 - (x-t)^2}\right]_{t=0} = e^{x^2}\left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0} = e^{x^2}\left[(-1)^n \frac{\partial^n}{\partial x^n} e^{-(x-t)^2}\right]_{t=0}$$

$$\left\{\because \frac{\partial^n}{\partial t^n} f(x-t) = (-1)^n \frac{\partial^n}{\partial x^n} f(x-t)\right\}$$

$$= e^{x^2}\left[(-1)^n \frac{d^n}{dx^n} e^{-x^2}\right] = (-1)^n e^{x2} \frac{d^n}{dx^n} e^{-x^2}.$$

Theorem II. *Prove that* $H_n(x) = 2^n \left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\} x^n$. **[Meerut 1995, 96, 97]**

Proof. Since $\exp x = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n\,!}$, we have

$$\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right) e^{2tx} = \sum_{n=0}^{\infty} \frac{1}{n\,!}\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)^n e^{2tx} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!}\left(\frac{1}{2}\frac{d}{dx}\right)^{2n} e^{2tx} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!} \frac{1}{2^{2n}} \frac{d^{2n}}{dx^{2n}} e^{2tx}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!} \cdot \frac{1}{2^{2n}} (2t)^{2n} e^{2tx} = e^{2tx} \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!} t^{2n} = e^{2tx} \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n\,!} = e^{2tx} \cdot e^{-t^2}. \quad ...(1)$$

Using the generating function, we have $e^{2tx} \cdot e^{-t^2} = e^{2tx - t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n\,!}$. ...(2)

Also, $e^{2tx} = \sum_{n=0}^{\infty} \frac{(2t\,x)^n}{n\,!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n\,!} t^n$. ...(3)

Using (2) and (3), (1) becomes $\sum_{n=0}^{\infty} \exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right) \frac{2^n x^n}{n\,!} t^n = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n\,!}$. ...(4)

On equating the coefficients of t^n on both sides of (4), we have

$$\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right) \frac{2^n x^n}{n\,!} = \frac{H_n(x)}{n\,!} \quad \text{or} \quad \frac{2^n}{n\,!}\left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\} x^n = \frac{H_n(x)}{n\,!}$$

or $$H_n(x) = 2^n \left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\} x^n.$$

12.5. Hermite Polynomials for some special values of n.

By definition, $$H_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n\,!}{r\,!\,(n-2r)\,!} (2x)^{n-2r} \quad ...(1)$$

where, $$[n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases} \quad ...(2)$$

Putting $n = 0, 1, 2, 3, \ldots$ in (1), we have

$$H_0(x) = \sum_{r=0}^{0} (-1)^r \frac{0\,!}{r\,!\,(-2r)\,!} (2x)^{-2r} = (-1)^0 \frac{1}{0\,!\;0\,!} (2x)^0 = 1,$$

$$H_1(x) = \sum_{r=0}^{0} (-1)^r \frac{1!}{r!(1-2r)!}(2x)^{1-2r} = (-1)^0 \frac{1}{0!\,1!}(2x)^1 = 2x,$$

$$H_2(x) = \sum_{r=0}^{1} (-1)^r \frac{2!}{r!(2-2r)!}(2x)^{2-2r} = (-1)^0 \frac{2!}{0!\,2!}(2x)^2 + (-1)^1 \frac{2!}{1!\,0!}(2x)^0 = 4x^2 - 2,$$

$$H_3(x) = \sum_{r=0}^{1} (-1)^r \frac{3!}{r!(3-2r)!}(2x)^{3-2r} = (-1)^0 \frac{3!}{0!\,3!}(2x)^3 + (-1)^1 \frac{3!}{1!\,1!}(2x)^1 = 8x^3 - 12x,$$

$$H_4(x) = \sum_{r=0}^{2} (-1)^r \frac{4!}{r!(4-2r)!}(2x)^{4-2r} = (-1)^0 \frac{4!}{0!\,4!}(2x)^4 + (-1)^1 \frac{4!}{1!\,2!}(2x)^2 + (-1)^2 \frac{4!}{2!\,0!}(2x)^0$$

or $H_4(x) = 16x^4 - 48x^2 + 12$ and so on.

12.6. Evaluation values of $H_{2n}(0)$ and $H_{2n+1}(0) = 0$.

Theorem. *Prove that* $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$; $H_{2n+1}(0) = 0$.

[Meerut 1997, Kanpur 2006, 08, 09, 10]

Proof. Using the generating function, we have $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{2tx - t^2}$. ...(1)

Replacing x by 0 in (1), we have

$$\sum_{n=0}^{\infty} \frac{H_n(0)}{n!} t^n = e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{H_n(0)}{n!} t^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}. \quad ...(2)$$

Equating coefficients of t^{2n} on both sides of (2), we have

$$\frac{H_{2n}(0)}{(2n)!} = \frac{(-1)^n}{n!} \quad \text{or} \quad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}.$$

Since the R.H.S. of (2) does not contain odd powers of t. equating coefficients of t^{2n+1} on both sides of (2) gives $\frac{H_{2n+1}(0)}{(2n+1)!} = 0$ so that $H_{2n+1}(0) = 0$.

12.7. Orthogonality properties of the Hermite Polynomials.

Theorem. *Prove that* : $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n\, n!\, \sqrt{\pi}\, \delta_{nm}$

or $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \sqrt{\pi}\, 2^n\, n!, & \text{if } m = n \end{cases}$

or Show that Hermite polynomials are orthogonal over $(-\infty, \infty)$ with respect to the weight function e^{-x^2}.

Proof. Using the generating functions, we have

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2tx - t^2} \quad \text{and} \quad \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!} = e^{2sx - s^2}.$$

Multiplying their corresponding sides, gives $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{H_n(x) H_m(x)}{n!\, m!} t^n s^m = e^{2tx - t^2 + 2sx - s^2}$

Multiplying both sides by e^{-x^2} and then integerating both sides w.r.t. 'x' from $-\infty$ to ∞, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx\right]\frac{t^n s^m}{n!\, m!}$$

$$= \int_{-\infty}^{\infty} e^{-x^2 + 2x(t+s) - (t^2+s^2)}\, dx = \int_{-\infty}^{\infty} e^{-x^2 + 2x(t+s) - (t+s)^2} \times e^{(t+s)^2 - (t^2+s^2)}\, dx$$

$$= e^{2ts}\int_{-\infty}^{\infty} e^{-[x-(t+s)]^2} dx = e^{2ts}\int_{-\infty}^{\infty} e^{-y^2}\, dy, \text{ putting } x-(t+s) = y \text{ so that } dx = dy$$

$$= e^{2ts}\sqrt{\pi}, \quad \text{as} \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$$

$$= \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2ts)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n\sqrt{\pi}}{n!}\, t^n s^n. \qquad ...(1)$$

We note that powers of t and s are always equal in each term on R.H.S. of (1). Hence when $m \neq n$, equating coefficients of $t^n s^m$ on both sides of (1), we have

$$\frac{1}{n!\, m!}\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0 \;\Rightarrow\; \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0, \text{ when } n \neq m. \qquad ...(2)$$

Again equating coefficients of $t^n s^n$ on both sides (1), we have

$$\frac{1}{n!\, n!}\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2\, dx = \frac{2^n\sqrt{\pi}}{n!} \qquad \text{or} \qquad \int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2\, dx = 2^n\, n!\sqrt{\pi}. \qquad ...(3)$$

[Kanpur 2007; Utkal 2003]

Let $$\delta_{nm} = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m. \end{cases} \qquad ...(4)$$

Combining results (2) and (3) with help of (4), we get

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n\, n!\sqrt{\pi}\, \delta_{nm.}$$

12.8. Recurrence Relations (or formulae)

Theorem. (*i*) $H'_n(x) = 2n\, H_{n-1}(x)$ $(n \geq 1)$; $H'_0(x) = 0$. **[Kanpur 2007]**
(*ii*) $H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$ $(n \geq 1)$; $H_1(x) = 2x\, H_0(x)$. **[Kanpur 2007, 11]**
(*iii*) $H_n'(x) = 2x\, H_n(x) - H_{n+1}(x)$.
(*iv*) $H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0$.

Proof. (*i*) We know that $$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2tx - t^2} \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'x' we have

$$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} = 2t\, e^{2tx - t^2} = 2t\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \text{ by (1).}$$

Thus, $$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!}. \qquad ...(2)$$

Equating coefficients of t^n from both sides for $n = 0$, (2) gives $H_0'(x) = 0$.
Again equating coefficient of t^n from both sides for $n \geq 1$, (2) gives

$$\frac{H_n'(x)}{n!} = \frac{H_{n-1}(x)}{(n-1)!} \qquad \text{so that} \qquad H_n'(x) = 2n\, H_{n-1}(x) \qquad [\because\ n! = n(n-1)!]$$

(*ii*) We know that $$e^{2tx - t^2} = \sum_{n=0}^{\infty} H_n(x)\, \frac{t^n}{n!}. \qquad ...(3)$$

Differentiating both sides of (3) w.r.t. '*t*' gives $$(2x - 2t)\, e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{n t^{n-1}}{n!}\, H_n(x)$$

or $$(2x - 2t) \sum_{n=0}^{\infty} H_n(x)\, \frac{t^n}{n!} = \frac{0.t^{0-1}}{0!} H_0(x) + \sum_{n=1}^{\infty} \frac{n t^{n-1}}{n!} H_n(x)$$

or $$2x \sum_{n=0}^{\infty} H_n(x)\, \frac{t^n}{n!} - 2 \sum_{n=0}^{\infty} H_n(x) \frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(x) \qquad ...(4)$$

$[\because\ 0! = 1,\ H_n(x) = 1 \text{ and } n! = n(n-1)!]$

Equating coefficients of t^n from both sides for $n = 0$, (4) gives $2x\, H_0(x) = H_1(x)$.

Again equating coefficient of t^n from both sides for $n \geq 1$, (2) gives

$$2x \times \frac{H_n(x)}{n!} - 2\, \frac{H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}(x)}{n!}. \qquad ...(5)$$

On multiplying both sides of (5) by $n!$ and noting that $n! = n(n-1)!$, (5) gives

$$2x\, H_n(x) - 2n\, H_{n-1}(x) = H_{n+1}(x).$$

(*iii*) From recurrence relations (1) and (2), we get $H_n'(x) = 2n\, H_{n-1}(x)$ ...(1)

and $H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x).$ ...(2)

Adding (1) and (2), $H_n'(x) + H_{n+1}(x) = 2x\, H_n(x)$ or $H_n'(x) = 2x\, H_n(x) - H_{n+1}(x)$.

(*iv*) Since $H_n(x)$ is a solution of Hermite's differential equation $y'' - 2xy' + 2ny = 0$.

$\therefore$ $H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0.$

12.9. SOLVED EXAMPLES

Ex. 1. *Express* $H(x) = x^4 + 2x^3 + 2x^2 - x - 3$ *in terms of Hermite's polynomials.*

[Kanpur 2008, 10]

Sol. We know that $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$ and $H_4(x) = 16x^4 - 48x^2 + 12$. From these, we have

$$x^4 = (1/16) \times H_4(x) + 3x^2 - (3/4), \qquad ...(1)$$

$$x^3 = (1/8) \times H_3(x) + (3x/2) \qquad ...(2)$$

$$x^2 = (1/4) \times H_2(x) + 1/2 \qquad ...(3)$$

and $$x = (1/2) \times H_1(x), \qquad 1 = H_0(x). \qquad ...(4)$$

$\therefore$ $H(x) = x^4 + 2x^3 + 2x^2 - x - 3 = (1/16) \times H_4(x) + 3x^2 - (3/4) + 2x^3 + 2x^2 - x - 3$, by (1)

$= (1/16) \times H_4(x) + 2x^3 + 5x^2 - x - 15/4 = (1/16) \times H_4(x) + 2\,[(1/8) \times H_3(x) + 3x/2] + 5x^2 - x - 15/4$, by (2)

$= (1/16) \times H_4(x) + (1/4) \times H_3(x) + 5x^2 + 2x - (15/4)$

$= (1/16) \times H_4(x) + (1/4) \times H_3(x) + 5\,[(1/4) \times H_2(x) + (1/2)] + 2x - (15/4)$, by (3)

$= (1/16) \times H_4(x) + (1/4) \times H_3(x) + (5/4) \times H_2(x) + 2x - (5/4)$,

$= (1/16) \times H_4(x) + (1/4) \times H_3(x) + (5/4) \times H_2(x) + H_1(x) - (5/4) \times H_0(x)$, by (4)

Ex. 2. *Prove that, if* $m < n$, $\dfrac{d^m}{dx^m}\{H_n(x)\} = \dfrac{2^m . n!}{(n-m)!} H_{n-m}(x).$ **[Garhwal 2004, 05; Meerut 98]**

Sol. We know that $$\sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!} = e^{2tx - t^2}. \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'x' m times, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{d^m}{dx^m}\{H_n(x)\} = \frac{d^m}{dx^m} e^{2tx - t^2} = (2t)^m\, e^{2tx - t^2} = 2^m\, t^m \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}, \text{ by (1)}$$

$$\therefore \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{d^m}{dx^m}\{H_n(x)\} = 2^m \sum_{n=0}^{\infty} H_n(x)\,\frac{t^{n+m}}{n!}. \qquad ...(2)$$

Equating coefficients of t^n from both sides for $m < n$, (2) gives

$$\frac{1}{n!}\frac{d^m}{dx^m}\{H_n(x)\} = 2^m\,\frac{H_{n-m}(x)}{(n-m)!} \qquad \text{or} \qquad \frac{d^m}{dx^m}\{H_n(x)\} = \frac{2^m n!}{(n-m)!} H_{n-m}(x).$$

Ex. 3. *Prove that* $\displaystyle\int_{-\infty}^{\infty} x^2\, e^{-x^2}\,[H_n(x)]^2\,dx = (\sqrt{\pi})\,2^n\,n!\,(n+1/2).$

Sol. From the recurrence relations, we know that

$H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x)$ or $x\,H_n(x) = n\,H_{n-1}(x) + (1/2) \times H_{n+1}(x)$ ...(1)

or $$x^2\,H_n(x) = nx\,H_{n-1}(x) + (x/2) \times H_{n+1}(x) \qquad ...(2)$$

Replacing n by $n - 1$ and $n + 1$ successively in (1), we have

$$x\,H_{n-1}(x) = (n-1)\,H_{n-2}(x) + (1/2) \times H_n(x) \qquad ...(3)$$

and $$x\,H_{n+1}(x) = (n+1)\,H_n(x) + (1/2) \times H_{n+2}(x). \qquad ...(4)$$

Using (3) and (4), (2) becomes

$$x^2\,H_n(x) = n\,[(n-1)\,H_{n-2}(x) + (1/2) \times H_n(x)] + (1/2) \times [(n+1)\,H_n(x) + (1/2) \times H_{n+2}(x)]$$

or $$x^2\,H_n(x) = n\,(n-1)\,H_{n-2}(x) + (1/4) \times H_{n+2}(x) + (n+1/2)\,H_n(x) \qquad ...(5)$$

Multiplying both sides of (5) by $e^{-x^2} H_n(x)$ and then integrating w.r.t. 'x' from $-\infty$ to ∞, gives

$$\int_{-\infty}^{\infty} x^2\, e^{-x^2}\,\{H_n(x)\}^2\,dx = n\,(n-1)\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\,H_{n-2}(x)\,dx + \frac{1}{4}\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\,H_{n+2}(x) + \left(n+\frac{1}{2}\right)\int_{-\infty}^{\infty} e^{-x^2}\,\{H_n(x)\}^2\,dx$$

$= 0 + 0 + (n+1/2)\left(\sqrt{\pi}\right)\,2^n\,n!,$ as $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,H_n(x)\,H_m(x)\,dx = \left(\sqrt{\pi}\right)2^n\,n!\,\delta_{nm}$

$= (n+1/2)\left(\sqrt{\pi}\right)\,2^n\,n!.$

Ex. 4. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,H_n(x)\,H_m(x)\,dx.$ **[Meerut 1997]**

Sol. From the recurrence relations, we have $H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x)$

or $$xH_n(x) = n\,H_{n-1}(x) + (1/2) \times H_{n+1}(x). \qquad ...(1)$$

Multiplying both sides of (1) by $e^{-x^2}\,H_m(x)$ and then integrating w.r.t. 'x' from $-\infty$ to ∞, we have

$$\int_{-\infty}^{\infty} xe^{-x^2}\,H_m(x)\,H_n(x)\,dx = n\int_{-\infty}^{\infty} e^{-x^2}\,H_m(x)\,H_{n-1}(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\,H_{n+1}(x)\,dx$$

$$= n\sqrt{\pi}\,2^{n-1}(n-1)!\,\delta_{n-1,m} + (1/2)\times\sqrt{\pi}\,2^{n+1}(n+1)!\,\delta_{n+1,m}$$

$$\left[\because \int_{-\infty}^{\infty} e^{-x^2} H_n(x)\,H_m(x)\,dx = \sqrt{\pi}\,2^n\,n!\,\delta_{nm}\right]$$

$$= \sqrt{\pi}\,2^{n-1}\,n!\,\delta_{n-1,m} + \sqrt{\pi}\,2^n(n+1)!\,\delta_{n+1,m}.$$

Ex. 5. *Show that* $P_n(x) = \dfrac{2}{\sqrt{\pi}\,n!}\displaystyle\int_0^{\infty} t^n\,e^{-t^2}\,H_n(xt)\,dt$.

Sol. By the definition of Hermite polynomial (on replacing x by xt here), we have

$$H_n(xt) = \sum_{r=0}^{[n/2]}(-1)^r\,\frac{n!}{r!\,(n-2r)!}(2xt)^{n-2r}. \qquad ...(1)$$

Making use of (1), we have

$$\frac{2}{\sqrt{\pi}\,n!}\int_0^{\infty} t^n e^{-t^2} H_n(xt)\,dt$$

$$= \frac{2}{\sqrt{\pi}\,n!}\int_0^{\infty} t^n e^{-t^2}\left\{\sum_{r=0}^{[n/2]}(-1)^r\,\frac{n!\,2^{n-2r}\,x^{n-2r}}{r!\,(n-2r)!}\,t^{n-2r}\right\}dt = \sum_{r=0}^{[n/2]}\frac{2^{n-2r+1}\,(-1)^r\,x^{n-2r}}{\sqrt{\pi}\,r!\,(n-2r)!}\int_0^{\infty} e^{-t^2}\,t^{2n-2r}\,dt$$

$$= \sum_{r=0}^{[n/2]}\frac{2^{n-2r+1}\,(-1)^r\,x^{n-2r}}{\sqrt{\pi}\,r!\,(n-2r)!}\,\frac{1}{2}\,\Gamma(n-r+1/2)$$

$$\left[\because \int_0^{\infty} e^{-t^2}\,t^{2n-2r}\,dt = \int_0^{\infty} e^{-t^2}\,t^{2(n-r+1/2)-1}\,dt \;= \tfrac{1}{2}\Gamma\left(n-r+\tfrac{1}{2}\right), \text{as } \Gamma(x) = 2\int_0^{\infty} e^{-t^2}\,e^{2x-1}\,dt\right]$$

$$= \sum_{r=0}^{[n/2]}\frac{2^{n-2r}\,(-1)^r\,x^{n-2r}}{\sqrt{\pi}\,r!\,(n-2r)!}\times\frac{(2n-2r)!}{2^{2n-2r}\,(n-r)!}\sqrt{\pi} \qquad \left(\begin{array}{l}\because \text{ by using duplication formula, if n is}\\ \text{positive integer then } \Gamma\left(n+\dfrac{1}{2}\right) = \dfrac{(2n)!}{2^{2n}\,n!}\sqrt{\pi}\end{array}\right)$$

$$= \sum_{r=0}^{[n/2]}(-1)^r\,\frac{(2n-2r)!}{2^n\,r!\,(n-2r)!\,(n-r)!}\,x^{n-2r} = P_n(x), \text{ by the definition of Legendre polynomial.}$$

Ex. 6. *If* $\psi_n(x) = e^{-x^2/2}\,H_n(x)$, *where* $H_n(x)$ *in a Hermite's polynomial of degree n, then*

(*a*) $\displaystyle\int_{-\infty}^{\infty}\psi_m(x)\,\psi_n(x)\,dx = 2^n\,!\,\sqrt{\pi}\,\delta_{m,n}$

(*b*) $\displaystyle\int_{-\infty}^{\infty}\psi_m(x)\,\psi_n'(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \pm 1 \\ 2^{n-1}\,n!\,\sqrt{\pi}, & \text{if } m = n-1 \\ -2^n\,(n+1)!\,\sqrt{\pi}, & \text{if } m = n+1 \end{cases}$

Sol. Part (a) Given $\psi_n(x) = e^{-x^2/2}\,H_n(x).$ ...(1)

$$\therefore \int_{-\infty}^{\infty}\psi_m(x)\,\psi_n(x)\,dx = \int_{-\infty}^{\infty} e^{-x^2/2}\,H_m(x).e^{-x^2/2}\,H_n(x)\,dx, \text{ using (1)}$$

$$= \int_{-\infty}^{\infty} e^{-x^2}\,H_m(x)\,H_n(x)\,dx = 2^n\,n!\,\sqrt{\pi}\,\delta_{nm},$$

[using orthogonal properties of the Hermite polynomials]

Part (b) Here $\psi_n(x) = e^{-x^2/2}\, H_n(x)$ ...(2)

and $\psi_m(x) = e^{-x^2/2}\, H_m(x).$ ...(3)

From (2), $\psi_n'(x) = -xe^{-x^2/2}\, H_n(x) + e^{-x^2/2}\, H_n'(x)$ ...(4)

From recurrence relations, we have $xH_n(x) = n\, H_{n-1}(x) + (1/2) \times H_{n+1}(x)$ ...(5)

and $H_n'(x) = 2n\, H_{n-1}(x).$ ...(6)

Using (5) and (6), (4) reduces to

$$\psi_n'(x) = -e^{-x^2/2}\,[n\, H_{n-1}(x) + (1/2) \times H_{n+1}(x)] + e^{-x^2/2} \times 2n\, H_{n-1}(x)$$

or $\psi_n'(x) = e^{-x^2/2}\,[n\, H_{n-1}(x) - (1/2) \times H_{n+1}(x)]$ ...(7)

$$\therefore \int_{-\infty}^{\infty} \psi_m(x)\, \psi_n'(x)\, dx = \int_{-\infty}^{\infty} e^{-x^2/2}\, H_m(x).e^{-x^2/2}\,[n\, H_{n-1}(x) - \frac{1}{2}\, H_{n+1}(x)]\, dx, \text{ by (3) and (7)}$$

$$= n\int_{-\infty}^{\infty} e^{-x^2}\, H_{n-1}\, H_m(x)\, dx - \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\, H_{n+1}(x)\, H_m(x)\, dx$$

$$= n\, 2^{n-1}(n-1)!\, \sqrt{\pi}\, \delta_{n-1.m} - (1/2) \times 2^{n+1}\, (n+1)!\, \sqrt{\pi}\, \delta_{n+1,m},$$ using orthogonal properties

$$= 2^{n-1}\, n!\, \sqrt{\pi}\, \delta_{n-1,m} - 2^n\, (n+1)!\, \sqrt{\pi}\, \delta_{n+1,m}$$

$$= \begin{cases} 0, & \text{if } m \neq n \pm 1 \\ 2^{n-1}\, n!\, \sqrt{\pi}, & \text{if } m = n-1 \\ -2^n\, (n+1)!\, \sqrt{\pi}, & \text{if } m = n+1 \end{cases}$$

Ex. 7. *Using the Rodrigue's formula for $H_n(x)$ and integrating by parts iteratively, show that*

$$\psi = \int_{-\infty}^{\infty} \exp(-x^2)\, H_n(x)\, H_m(x)\, dx = 0, \text{ if } m \neq n$$

$$= 2^n\, n!\, \sqrt{\pi}, \text{ if } m = n$$

Sol. Rodrigue's formula for $H_n(x)$ is given by $H_n(x) = (-1)^n\, e^{x^2}\, \frac{d^n}{dx^n}\, e^{-x^2}.$

$$\therefore \exp(-x^2)\, H_n(x) = (-1)^n\, \frac{d^n}{dx^n}\, e^{-x^2}. \quad \text{...(1)}$$

$$\therefore \psi = \int_{-\infty}^{\infty} (-1)^n \exp(-x^2)\, H_n(x)\, H_m(x)\, dx = \int_{-\infty}^{\infty} (-1)^n \exp\left\{\frac{d^n}{dx^n}\, e^{-x^2}\right\} H_m(x)\, dx$$

$$= (-1)^n \left[\left\{\left(\frac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\right) H_m(x)\right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\frac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\right) \frac{d}{dx}\, H_m(x)\, dx\right],$$ integrating by parts

$$= 0 - \int_{-\infty}^{\infty} \left(\frac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\right) \frac{d}{dx}\, H_m(x)\, dx,$$ as first term is zero due to presence of e^{-x^2}

$$= (-1)^n\, (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{dx^n}\, H_m(x)\, .\, e^{-x^2}\, dx,$$ integrating by parts iteratively

$$\therefore \qquad \psi = \int_{-\infty}^{\infty} \frac{d^n}{dx^n} H_m(x) \, . e^{-x^2} \, dx. \qquad ...(2)$$

We know that $H_m(x)$ is a polynomial of degree m. Hence if $n > m$, $\dfrac{d^n}{dx^n} H_m(x) = 0.$

$\therefore$ If $n > m$, then from (2), $\psi = 0.$ ...(3)

Since ψ is symmetrical in m and n, it follows that if $m > n$, then $\psi = 0.$ ...(4)

From (3) and (4), we see that $\psi = 0$, for $m \neq n$ ...(5)

When, $m = n$, (2) gives

$$\psi = \int_{-\infty}^{\infty} \frac{d^n}{dx^n} H_n(x) \, . e^{-x^2} \, dx = \int_{-\infty}^{\infty} 2^n \, n \, ! \, e^{-x^2} \, dx, \quad \text{as} \quad \frac{d^n}{dx^n} H_n(x) = 2^n \, n \, !$$

$$= 2^n \, n \, ! \sqrt{\pi}, \quad \text{as} \quad \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

$\therefore$ $\psi = 2^n \, n \, ! \sqrt{\pi}$ when $m = n$ ...(6)

From (5) and (6), we get the required results.

Ex. 8. *Show that* $$\sum_{k=0}^{n} \frac{H_k(x) \, H_k \, (y)}{2^k \, k \, !} = \frac{H_{n+1} \, (y) \, H_n(x) - H_{n+1} \, (x) \, H_n(y)}{2^{n+1} \, n \, ! \, (y - x)}.$$

Sol. From the recurrence relations, we have $H_{n+1} \, (x) = 2x \, H_n(x) - 2n \, H_{n-1}(x)$

or $$x \, H_n(x) = n \, H_{n-1} \, (x) + (1/2) \times H_{n+1}(x). \qquad ...(1)$$

Replacing x by y in (1), $y \, H_n \, (y) = n \, H_{n-1} \, (y) + (1/2) \times H_{n+1} \, (y).$ ...(2)

Multiplying (2) by $H_n(x)$ and (1) by $H_n(y)$ and then subtracting, we have

$$(y - x) \, H_n \, (x) \, H_n(y) = (1/2) \times [H_{n+1} \, (y) \, H_n \, (x) - H_{n+1} \, (x) \, H_n \, (y)] - 2 \, [H_{n-1} \, (x) \, H_n \, (y) - H_{n-1} \, (y) \, H_n \, (x)]. \quad ...(3)$$

Putting $n = 0, 1, 2, 3, ..., (n - 1), n$ successively in (1), we have

$$(y - x) \, H_0 \, (x) \, H_0 \, (y) = (1/2) \times [H_1 \, (y) \, H_0 \, (x) - H_1 \, (x) \, H_0 \, (y)] - 0 \qquad ...(E_0)$$

$$(y - x) \, H_1 \, (x) \, H_1 \, (y) = (1/2) \times [H_2 \, (y) \, H_1 \, (x) - H_2 \, (x) \, H_1 \, (y)] - [H_0 \, (x) \, H_1(y) - H_0 \, (y) \, H_1(x)]. \quad ...(E_1)$$

$$(y - x) \, H_2 \, (x) \, H_2 \, (y) = (1/2) \times [H_3 \, (y) \, H_2 \, (x) - H_3 \, (x) \, H_2 \, (y)] - 2 \, [H_1 \, (x) \, H_2 \, (y) - H_1 \, (y) \, H_2 \, (x)]. \quad ...(E_2)$$

... ...

$$(y - x) \, H_{n-1} \, (x) \, H_{n-1} \, (y) = (1/2) \times [H_n \, (y) \, H_{n-1}(x) - H_n \, (x) \, H_{n-1}(y)] - 2 \, [H_{n-2}(x) \, H_{n-1}(y) - H_{n-2} \, (y) \, H_{n-1}(x)]. \quad ...(E_{n-1})$$

$$(y - x) \, H_n \, (x) \, H_n \, (y) = (1/2) \times [H_{n+1} \, (y) \, H_n \, (x) - H_{n+1} \, (x) \, H_n \, (y)] - 2 \, [H_{n-1}(x) \, H_n \, (y) - H_{n-1} \, (y) \, H_n \, (x)]. \quad ...(E_n)$$

Multiplying $(E_0), (E_1), (E_2), ..., (E_{n-1}), (E_n)$ by $1, \dfrac{1}{2 \, . \, 1!}, \dfrac{1}{2^2 . \, 2!}, \dfrac{1}{2^3 . \, 3!}, ..., \dfrac{1}{2^{n-1} \, (n-1)!}, \dfrac{1}{2^n . \, n!}$

respectively and adding (note that all terms, except the first term on R.H.S. of (E_n), cancel in pairs), we have

$$(y - x) \sum_{k=0}^{\infty} \frac{H_k(x) \, H_k \, (y)}{2^k \, k \, !} = \frac{H_{n+1} \, (y) \, H_n(x) - H_{n+1} \, (x) \, H_n(y)}{2^{n+1} \, n \, !}$$

or $$\sum_{k=0}^{\infty} \frac{H_k(x) \, H_k \, (y)}{2^k \, k \, !} = \frac{H_{n+1} \, (y) \, H_n(x) - H_{n+1} \, (x) \, H_n(y)}{2^{n+1} \, n \, ! \, (y - x)}.$$

EXERCISE

1. Show that $H_n(x) = 2^{n+1}\, e^{x^2} \int_x^\infty e^{-t^2}\, t^{n+1}\, P_n\left(\frac{x}{t}\right) dt$.

2. If $f(x)$ is a polynomial of degree m, show that $f(x)$ may be expressed in the form

$$f(x) = \sum_{r=0}^{m} C_r\, H_r(x), \qquad \text{where,} \qquad C_r = \frac{1}{2^r\, r!\, \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, f(x)\, H_r(x)\, dx.$$

Deduce that $\int_{-\infty}^{\infty} e^{-x^2} f(x)\, H_n(x) = 0$, if $f(x)$ is a polynomial of degree less than n.

3. Using the generating function for Hermite polynomials, evaluate the values of

(*i*) $H_0(x)$ (*ii*) $H_1(x)$ (*iii*) $H_2(x)$ (*iv*) $H_3(x)$.

4. Show that $H_n(x)$ defined by $e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, t^n$ satisfies the differential equation $H_n''(x) - 2x\, H_n'(x) + 2x\, H_n(x) = 0$.

5. The Hermite polynomial is defined for integral values of x by the identity $e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, t^n$. Show that $H_n(x)$ satisfies the differential equation $H_n''(x) - 2x\, H_n'(x) + 2x\, H_n(x) = 0$ and $H_n(x)$ is given by $H_n(x) = (-1)^n\, e^{x^2}\, \frac{d^n}{dx^n} e^{-x^2}$.

Further show that $\int_{-\infty}^{\infty} \{H_n(x)\}^2\, e^{-x^2}\, dx = \sqrt{\pi}\, 2^n\, n!$.

6. Show that $\int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2\, dx = 2^n\, n! \int_{-\infty}^{\infty} e^{-x^2}\, dx = 2^n\, n!\, \sqrt{\pi}$.

7. Show that (*a*) $H_5(x) = 32\, x^5 - 160\, x^3 + 120\, x$. (*b*) $H_6(x) = 64\, x^6 - 480\, x^4 + 720\, x^2 - 120$.

8. Show that for $n = 0, 1, 2$ ....

(*i*) $H_{2n}(x) = \frac{2^{n+1} \cdot (-1)^n\, e^{x^2}}{\sqrt{(\pi)}} \int_0^\infty e^{-t^3}\, t^{2n} \cos 2xt\, dt.$

(*ii*) $H_{2n}(x) = \frac{2^{2n+2} \cdot (-1)^n\, e^{x^2}}{\sqrt{(\pi)}} \int_0^\infty e^{-t^2}\, t^{2n+1} \sin 2xt\, dt.$ **[Kanpur-2004]**

9. Prove that $H_n(-x) = (-1)^n\, H_n(x)$. **[Kanpur-2004, 09]**

10. Prove that $x^n = \sum_{k=0}^{[n/2]} \frac{n!\, H_{n-2k}(x)}{2^n\, k!\, (n-2k)!}$ and express x^6 in terms of Hermite polynomials. **[Kanpur 2004, 09]**

11. Show that $d^n H_o(x)/dx^n = 2^n n!$ **(Kanpur 2008, 09)**

12. Show that $\int_0^x H_n(y)dy = \frac{1}{2(n+1)} \left[H_{n+1}(x) - H_{n-1}(o)\right]$ **(Kanpur 2010)**

13

Laguerre Polynomials

13.1. Laguerre's equation and its solution. [Meerut 1995, 96]

Laguerre's equation of order n is

$$x(d^2y/dx^2) + (1-x)(dy/dx) + ny = 0, \qquad ...(1)$$

where n is a positive integer. We obtain a solution of (1) which is finite for all values of x and which tends to infinity no faster than $e^{x/2}$ as $x \to \infty$. For this we use the well known method of Frobenius. Let the series solution of (1) be

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad C_0 \neq 0. \qquad ...(2)$$

Differentiating (2) and then putting the values of y, dy/dx and d^2y/dx^2 in (1), we have

$$x\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} + (1-x)\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} + n\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-1} + \sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} - \sum_{m=0}^{\infty} C_m(k+m)x^{k+m} + n\sum_{m=0}^{\infty} C_m x^{k+m-1} = 0$$

or $$\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1}\{(k+m-1)+1\} - \sum_{m=0}^{\infty} C_m x^{k+m}(k+m-n) = 0$$

or $$\sum_{m=0}^{\infty} C_m(k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} C_m(k+m-n)x^{k+m} = 0. \qquad ...(3)$$

(3) is an identity. To get the indicial equation, we equate to zero the coefficient of the smallest power of x, namely x^{k-1} in (3) and obtain

$$C_0k^2 = 0, \qquad \text{so that} \qquad k^2 = 0 \qquad (\because\ C_0 \neq 0) \qquad ...(4)$$

From (4) we see that the roots of indicial equation are equal. Next equating to zero the coefficient of x^{k+m-1}, we have

$$C_m(k+m)^2 - C_{m-1}(k+m-1-n) = 0 \qquad \text{or} \qquad C_m = \frac{k+m-1-n}{(k+m)^2}C_{m-1}. \qquad ...(5)$$

The two independent solutions in the present case are $(y)_{k=0}$ and $(\partial y/\partial k)_{k=0}$. But $(\partial y/\partial k)_{k=0}$ involves a term of the form $\log x$, and so is infinite when $x = 0$. Since we wish to obtain a solution finite for all finite values of x, we consider only the former solution, *i.e.*, $(y)_{k=0}$, as follows.

With $k = 0$, (5) and (2) reduce to $$C_m = \frac{m-1-n}{m^2}C_{m-1} \qquad ...(6)$$

and $$y = \sum_{m=0}^{\infty} C_m x^m = C_0 + C_1x + C_2x^2 + ... \qquad ...(7)$$

Putting $m = 1, 2, 3, \ldots$ in (6), we have

$$C_1 = \frac{-n}{1^2} C_0 = \frac{(-1)}{(1!)^2} nC_0, \qquad C_2 = \frac{1-n}{2^2} C_1 = -\frac{(n-1)}{2^2} \times (-1)nC_0 = (-1)^2 \frac{n(n-1)}{(2!)^2} C_0,$$

$$C_3 = \frac{2-n}{3^2} C_2 = -\frac{(n-2)}{3^2} \times (-1)^2 \frac{n(n-1)}{(2!)^2} C_0 = (-1)^3 \frac{n(n-1)(n-2)}{(3!)^2} C_0,$$

....

Thus, $$C_r = (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} C_0, \quad \text{for } r \le n.$$

Also, $$C_{n+1} = C_{n+2} = C_{n+3} = \ldots = 0.$$

With these values, (7) reduces to

$$y = C_0 \left[1 - \frac{n}{(1!)^2} x + \frac{n(n-1)}{(2!)^2} x^2 + \ldots + (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} x^r + \ldots. \right]$$

$$= C_0 \sum_{r=0}^{\infty} (-1)^r \frac{n(n-1)\ldots.(n-r+1)}{(r!)^2} x^r = C_0 \sum_{r=0}^{n} (-1)^r \frac{n(n-1)\ldots.(n-r+1)(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1.(r!)^2} x^r$$

Thus, $$y = C_0 \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r.$$

Taking $C_0 = 1$, we define the corresponding solution as the Laguerre polynomial of order n, and denote it by $L_n(x)$. Thus, we have $$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!\,(r!)^2} x^r.$$

13.2A. Laguerre polynomial of order (or degree) n. Definition. [Meerut 1997]

Laguerre polynomial of order n is denoted and defined by

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!\,(r!)^2} x^r.$$

13.2B. Alternative definition of Laguerre polynomial of order (or degree) n.

In Art. 13.1, we took $C_0 = 1$ to define $L_n(x)$ and the same is given in Art. 13.2A. However, some authors take $C_0 = n!$ to define $L_n(x)$. Thus, another definition of Laguerre polynomial is

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r (n!)^2 x^r}{(n-r)!\,(r!)^2}.$$

13.3. Generating function for Laguerre polynomials.

Theorem. *Prove that* $\dfrac{\exp\{-xt/(1-t)\}}{1-t} = \sum_{n=0}^{\infty} L_n(x)\, t^n$.

Proof. In order to prove the required result we must show that the coefficient of t^n in the expansion of L.H.S. (in ascending powers of t) is $L_n(x)$. Now,

$$\frac{\exp\{-xt/(1-t)\}}{1-t} = \frac{1}{1-t} \sum_{r=0}^{\infty} \left(\frac{-xt}{1-t} \right)^r \cdot \frac{1}{r!}, \qquad \text{as} \qquad \exp x = e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r (1-t)^{-(r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r \sum_{s=0}^{\infty} \frac{(r+s)!}{r!\,s!}, \text{ by the binomial theorem}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^r \frac{(r+s)!}{(r!)^2 s!} x^r t^{r+s} \quad .$$

Let r be fixed. Then the coefficient of t^n can be obtained by setting $r + s = n$ *i.e.* $s = n - r$. Hence, for the chosen fixed value of r, the coefficient of t^n is $(-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r$.

Now, $s \geq 0 \Rightarrow n - r \geq 0 \Rightarrow r \leq n$, which gives all allowed values of r for finding coefficient of t^n. Thus, the total coefficient of t^n is given by

$$\sum_{r=0}^{n} (-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r \qquad \textit{i.e.} \qquad L_n(x), \text{ by definition 13.2A}$$

This proves the desired result.

Remark. If we use definition 13.2B, then $\dfrac{\exp\{-xt/(1-t)\}}{1-t} = \sum_{n=0}^{\infty} \dfrac{L_n(x)\, t^n}{n!}$.

13.4. Alternative expression for the Laguerre polynomials

Prove that $L_n(x) = \dfrac{e^x}{n!} \dfrac{d^n}{dx^n} (x^n e^{-x})$. **[Kanpur 1992; Meerut 1992, 93; Garhwal 2005]**

Proof. By the Leibnitz's theorem, we have

$$D^n(uv) = d^n(uv)/dx^n = D^n u.v + {}^nC_1 D^{n-1} u.Dv + \ldots + {}^nC_r D^{n-r} u.D^r v + \ldots + uD^n v$$

i.e., $$D^n(uv) = \sum_{r=0}^{n} {}^nC_r \; D^{n-r} u \; D^r v \, .$$

$$\therefore \; \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) = \frac{e^x}{n!} \sum_{r=0}^{n} {}^nC_r D^{n-r} x^n \; D^r e^{-x}, \text{ . by (1)}$$

$$= \frac{e^x}{n!} \sum_{r=0}^{n} {}^nC_r \frac{n!}{\{n-(n-r)\}!} x^{n-(n-r)} . (-1)^r e^{-x}, \qquad \text{as} \qquad D^n x^m = \frac{m!}{(m-n)!} x^{m-n} \text{ and } D^n e^{ax} = a^n e^{ax}$$

$$= \sum_{r=0}^{n} \frac{e^x}{n!} \times \frac{n!}{r!(n-r)!} \frac{n!}{r!} x^r \times (-1)^r e^{-x} = \sum_{r=0}^{n} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r = L_n(x), \text{ by definition.}$$

Remark. If the use definition 13.2B, we get $L_n(x) = e^x \dfrac{d^n}{dx^n} (x^n e^{-x})$.

13.5. First few Laguerre polynomials.

We know that (Refer Art. 13.4) $$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}). \qquad \ldots(1)$$

Putting $n = 0, 1, 2, 3, 4, \ldots$ in succession in (1), we obtain

$$L_0(x) = \frac{e^x}{0!} (x^0 e^{-x}) = 1, \qquad L_1(x) = \frac{e^x}{1!} \frac{d}{dx} (xe^{-x}) = e^x (e^{-x} - xe^{-x}) = 1 - x,$$

$$L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2} (x^2 e^{-x}) = \frac{e^x}{2!} \frac{d}{dx} \left[\frac{d}{dx} (x^2 e^{-x}) \right] = \frac{e^x}{2!} \frac{d}{dx} (2xe^{-x} - x^2 e^{-x})$$

$$= \frac{e^x}{2!}[2e^{-x} + 2x(-e^{-x}) - \{2xe^{-x} + x^2(-e^{-x})\}] = \frac{1}{2!}(2 - 4x + x^2),$$

$$L_3(x) = \frac{e^x}{3!}\frac{d^3}{dx^3}(x^3e^{-x}) = \frac{e^x}{3!}\frac{d^2}{dx^2}\left[\frac{d}{dx}(x^3e^{-x})\right] = \frac{e^x}{3!}\frac{d^2}{dx^2}(3x^2e^{-3} - x^3e^{-x}) = \frac{e^x}{3!}\frac{d}{dx}\left[\frac{d}{dx}\{(3x^2 - x^3)e^{-x}\}\right]$$

$$= \frac{e^x}{3!}\frac{d}{dx}[(6x - 3x^2)e^{-x} - (3x^2 - x^3)e^{-x}] = \frac{e^x}{3!}\frac{d}{dx}\{(6x - 6x^2 + x^3)\, e^{-x}\}$$

$$= (e^x/3!)\times[(6 - 12x + 3x^2)e^{-x} - (6x - 6x^2 + x^3)e^{-x}] = (6 - 18x + 9x^2 - x^3)/3!,$$

$$L_4(x) = \frac{e^x}{4!}\frac{d^4}{dx^4}(x^4e^{-x}) = \frac{e^x}{4!}\frac{d^3}{dx^3}\left[\frac{d}{dx}(x^4e^{-x})\right] = \frac{e^x}{4!}\frac{d^3}{dx^3}[4x^3e^{-x} - x^4e^{-x}] = \frac{e^x}{4!}\frac{d^2}{dx^2}\left[\frac{d}{dx}\{(4x^3 - x^4)e^{-x}\}\right]$$

$$= \frac{e^x}{4!}\frac{d^2}{dx^2}[(12x^2 - 4x^3)e^{-x} - (4x^3 - x^4)e^{-x}] = \frac{e^x}{4!}\frac{d}{dx}\left[\frac{d}{dx}(12x^2 - 8x^3 + x^4)e^{-x}\right]$$

$$= \frac{e^x}{4!}\frac{d}{dx}[(24x - 24x^2 + 4x^3)e^{-x} - (12x^2 - 8x^3 + x^4)e^{-x}] = \frac{e^x}{4!}\frac{d}{dx}[(24x - 36x^2 + 12x^3 - x^4)e^{-x}]$$

$$= (e^x/4!)[(24 - 72x + 36x^2 - 4x^3)e^{-x} - (24x - 36x^2 + 12x^3 - x^4)e^{-x}] = (24 - 96x + 72x^2 - 16x^3 + x^4)/4!$$

Note : If we use the result given remark of Art. 13.4, namely, $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$

and proceed as before, then we have $L_0(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = 2 - 4x + x^2$, $L_3(x) = 6 - 18x + 9x^2 - x^3$, $L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4$ and so on.

13.6. Orthogonality properties of Laguerre's polynomials.

Theorem. Prove that $\int_0^\infty e^{-x} L_n(x)\, L_m(x)\, dx = 0$, *if* $m \neq n$

and $$\int_0^\infty e^{-x}\{L_n(x)\}^2 = 1$$ **(Kanpur 2004)**

or *prove that* $\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$

[Arga 1992, Rohilkhand 1996, Meerut 1996, 97]

Proof. Using the generating function, we get

$$\sum_{n=0}^{\infty} L_n(x)\, t^n = \frac{\exp\left(-\frac{xt}{1-t}\right)}{1-t} \quad \text{and} \quad \sum_{m=0}^{\infty} L_m(x)\, s^m = \frac{\exp\left(-\frac{xs}{1-s}\right)}{1-s}.$$

Multiplying the corresponding sides, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_n(x)\, L_m(x)\, t^n s^m = e^{-x\{t/(1-t) + s/(1-s)\}} \times \frac{1}{(1-t)(1-s)}. \quad \ldots(1)$$

$[\because \exp x.\exp y = e^x.e^y = e^{x+y}]$

Multiplying both sides of (1) by e^{-x} and then integrating both sides w.r.t. 'x', from 0 to ∞, gives

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_0^\infty e^{-x} L_n(x)\, L_m(x)\, dx\right] t^n s^m = \frac{1}{(1-t)(1-s)}\int_0^\infty e^{-x\{1 + t/(1-t) + s/(1-s)\}}\, dx$$

$$= \frac{1}{(1-t)(1-s)}\left[\frac{e^{-x\{1 + t/(1-t) + s/(1-s)\}}}{-\{1 + t/(1-t) + s/(1-s)\}}\right]_0^\infty = \frac{1}{(1-t)(1-s)} \times \frac{1}{1 + \frac{t}{1-t} + \frac{s}{1-s}}$$

$$= \frac{1}{(1-t)(1-s)+t(1-s)+s(1-t)} = \frac{1}{1-st} = (1-st)^{-1} = \sum_{n=0}^{\infty} s^n t^n \text{, by the binomial theorem}$$

$$\therefore \qquad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\, dx\right] t^n s^m = \sum_{n=0}^{\infty} s^n t^n . \qquad ...(2)$$

We note that the indices of t and s are always equal in each term on R.H.S. of (2). When $m \neq n$, equating coefficients of $t^n s^m$ on both sides of (2) gives

$$\int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\, dx = 0, \quad \text{if } m \neq n. \qquad ...(3)$$

Again equating coefficients of $t^n s^n$ on both sides of (2) gives $\int_0^{\infty} e^{-x}[L_n(x)]^2\, dx = 1.$ ...(4)

Using (5), (3) and (4) can be combined to give

$$\int_0^{\infty} e^{-x} L_m(x)\, L_n(x)\, dx = \delta_{mn} = \begin{cases} 0, \text{ if } m \neq n \\ 1, \text{ if } m = n \end{cases}.$$

Remark. If we use result of remark given in Art. 13.3, namely

$$\sum_{n=0}^{\infty} \frac{L_n(x) t^n}{n!} = \frac{\exp\{-xt/(1-t)\}}{1-t}, \qquad \text{we get} \qquad \int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\, dx = n!\, m!\, \delta_{mn}.$$

13.7. Expansion of a polynomial in a series of Laguerre polynomials

Theorem. *If $f(x)$ is polynomials of degree m, show that $f(x)$ may be expressed in the form*

$$f(x) = \sum_{r=0}^{m} C_r L_r(x), \qquad \text{where} \qquad C_r = \int_0^{\infty} e^{-x} L_r(x) f(x)\, dx .$$

Proof. Since $f(x)$ is polynomal of degree m, we write

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0. \qquad ...(1)$$

Again, we know that $L_m(x)$ is a polynomial of degree m of the form

$$L_m(x) = k_m x^m + k_{m-1} x^{m-1} + \ldots + k_1 x + k_0. \qquad ...(2)$$

Consider $f(x) - (a_m/k_m)L_m(x)$. Two cases may arise :

Case (i) Let $f(x) - (a_m/k_m)L_m(x) = 0$ so that $f(x) = (a_m/k_m)L_m(x)$, which proves the required result.

Case (ii) Let $f(x) - (a_m/k_m)L_m(x) = g_{m-1}(x)$, $g_{m-1}(x)$ being a polynomial of degree $m-1$. Taking $C_m = a_m/k_m$, we may write $f(x) = C_m L_m(x) + g_{m-1}(x).$ ...(3)

Taking $g_{m-1}(x)$ in place of $f(x)$ and proceeding as above, we have

$$g_{m-1}(x) = C_{m-1} L_{m-1}(x) + g_{m-2}(x). \qquad ...(4)$$

Making use of (4), (3) may be re–written as

$$f(x) = C_m L_m(x) + C_{m-1} L_{m-1}(x) + g_{m-2}(x). \qquad ...(5)$$

Noting that $L_0(x) = 1$ and proceeding as above, we finally obtain

$$f(x) = C_m L_m(x) + C_{m-1} L_{m-1}(x) + \ldots + C_1 L_1(x) + C_0 L_0(x) \quad \text{or} \quad f(x) = \sum_{r=0}^{m} C_r\, L_r(x). \qquad ...(6)$$

Since $\sum_{r=0}^{m} C_r\, L_r(x) = \sum_{s=0}^{m} C_s\, L_s(x)$, (6) gives $f(x) = \sum_{s=0}^{m} C_s\, L_s(x).$ ...(7)

Multiplying both sides of (7) by $e^{-x}L_r(x)$ and then integrating w.r.t. 'x' between the limits 0 to ∞, we have

$$\int_0^\infty e^{-x} f(x)\, L_r(x)\, dx = \sum_{s=0}^{m} C_s \left\{ \int_0^\infty e^{-x} L_r(x) L_s(x)\, dx \right\} \quad ...(8)$$

But

$$\int_0^\infty e^{-x} L_r(x) L_s(x)\, dx = \begin{cases} 0, & r = s \\ 1, & r \neq s. \end{cases} \quad ...(9)$$

Using (9), (8) gives

$$C_r = \int_0^\infty e^{-x} f(x) L_r(x)\, dx \quad ...(10)$$

(6) and (10) prove the required result.

13.8. Relations between Laguerre polynomials and their derivatives:

Recurrence relations (formulae). *Show that*

I. $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$. **[Kanpur 2005, 07–10; Meerut 1994]**

II. $xL_n'(x) = nL_n(x) - nL_{n-1}(x)$. **[Kanpur 2009; Meerut 1992, 95]**

III. $L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)$ **[Meerut 1997]**

Proof I. We know that

$$\sum_{n=0}^{\infty} L_n(x)\, t^n = \frac{\exp\{-xt/(1-t)\}}{1-t}. \quad ...(1)$$

Differentiating both sides of (1) w.r.t. 't', we get

$$\sum_{n=0}^{\infty} L_n(x).\, nt^{n-1} = \frac{1}{(1-t)^2} \exp\left\{-\frac{xt}{1-t}\right\} - \frac{1}{1-t} \times \exp\left\{-\frac{xt}{1-t}\right\} \times \frac{x}{(1-t)^2}$$

i.e.,

$$\sum_{n=0}^{\infty} L_n(x).\, nt^{n-1} = \frac{1}{1-t} \sum_{n=0}^{\infty} L_n(x)\, t^n - \frac{x}{(1-t)^2} \sum_{n=0}^{\infty} L_n(x)\, t^n \text{ , by (1)}$$

Multiplying both sides by $(1-t)^2$, the above equation becomes

$$(1-2t+t^2) \sum_{n=0}^{\infty} L_n(x).\, nt^{n-1} = (1-t) \sum_{n=0}^{\infty} L_n(x)\, t^n - x \sum_{n=0}^{\infty} L_n(x)\, t^n$$

or

$$\sum_{n=0}^{\infty} nL_n(x)t^{n-1} - 2\sum_{n=0}^{\infty} nL_n(x)t^n + \sum_{n=0}^{\infty} nL_n(x)t^{n+1} = \sum_{n=0}^{\infty} L_n(x)t^n - \sum_{n=0}^{\infty} L_n(x)t^{n+1} - x\sum_{n=0}^{\infty} L_n(x)t^n. \quad ...(2)$$

Equating the coefficients of t^n from both sides, (2) gives

$$(n+1)L_{n+1}(x) - 2nL_n(x) + (n-1)L_{n-1}(x) = L_n(x) - L_{n-1}(x) - xL_n(x)$$

or

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x). \quad ...(3)$$

Remark. If we use

$$\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{n!} = \frac{1}{1-t} e^{-xt/(1-t)}$$

and proceed as above, we have

$$L_{n+1}(x) = (2n+1-x)L_n(x) - n^2 L_{n-1}(x).$$

II. We know that

$$\sum_{n=0}^{\infty} L_n(x)\, t^n = \frac{\exp\{-xt/(1-t)\}}{1-t}. \quad ...(1)$$

Differentiating both sides of (1) w.r.t. 'x', we get

$$\sum_{n=0}^{\infty} L_n'(x)\, t^n = \frac{1}{1-t} \times \exp\left\{-\frac{xt}{1-t}\right\} \times \left(\frac{-t}{1-t}\right) \quad \text{or} \quad \sum_{n=0}^{\infty} L_n'(x)\, t^n = \frac{-t}{1-t} \sum_{n=0}^{\infty} L_n(x)\, t^n \text{, by (1)}$$

or $(1-t)\sum_{n=0}^{\infty} L_n'(x)\, t^n = -t \sum_{n=0}^{\infty} L_n(x)\, t^n$ or $\sum_{n=0}^{\infty} L_n'(x)\, t^n - \sum_{n=0}^{\infty} L_n'(x)\, t^{n+1} = -\sum_{n=0}^{\infty} L_n(x)\, t^{n+1}$ ...(2)

Equating the coefficients of t^n from both sides, (2) gives $L'_n(x) - L'_{n-1}(x) = -L_{n-1}(x)$. ...(3)

From recurrence relation (1), $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$. ...(4)

Differentiating (4) w.r.t. 'x', we have

$$(n+1)L'_{n+1}(x) = (2n+1-x)L'_n(x) - L_n(x) - nL'_{n-1}(x). \quad ...(5)$$

From (3), $L'_{n-1}(x) = L'_n(x) + L_{n-1}(x)$. ...(6)

Now, replacing n by $n+1$ in (3), we have

$L'_{n+1}(x) - L'_n(x) = -L_n(x)$ or $L'_{n+1}(x) = L'_n(x) - L_n(x)$. ...(7)

Putting the values of $L'_{n-1}(x)$ and $L'_{n+1}(x)$ from (6) and (7) in (5), we get

$$(n+1)[L'_n(x) - L_n(x)] = (2n+1-x)L'_n(x) - L_n(x) - n[L'_n(x) + L_{n-1}(x)]$$

or $xL'_n(x) = nL_n(x) - nL_{n-1}(x)$, on simplification

Remark. If we use $\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{n!} = \frac{1}{1-t} e^{-tx/(1-t)}$ and proceed as above, we have

$$L'_n(x) - xL'_{n-1} + nL_{n-1}(x) = 0.$$

III. We know that $\sum_{n=0}^{\infty} L_n(x) t^n = \frac{\exp\{-xt/(1-t)\}}{1-t}$. ...(1)

Differentiating both sides of (1) w.r.t. 'x', we get

$$\sum_{n=0}^{\infty} L_n'(x)\, t^n = \frac{1}{1-t} \exp\left\{-\frac{xt}{1-t}\right\} \times \left(-\frac{t}{1-t}\right)$$

$$= -\frac{t}{1-t} \sum_{r=0}^{\infty} L_r(x)\, t^r \text{, using (1) and writing } r \text{ in place of } n$$

$$= -t(1-t)^{-1} \sum_{r=0}^{\infty} L_r(x)\, t^r = -t \sum_{s=0}^{\infty} t^s \sum_{r=0}^{\infty} L_r(x)\, t^r \text{, by the binomial theorem}$$

$$\therefore \quad \sum_{n=0}^{\infty} L_n'(x)\, t^n = -\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} L_r(x)\, t^{r+s+1}. \quad ...(2)$$

Clearly the coefficient of t^n on L.H.S. of (2) is $L'_n(x)$. We now obtain the coefficient of t^n on R.H.S. of (2). Let $r + s + 1 = n$ so that $s = n - r - 1$. Hence, for a fixed value of r, the coefficient of t^n on R.H.S. of (2) is $-L_r(x)$.

But $s \geq 0 \Rightarrow n - r - 1 \geq 0 \Rightarrow r \leq n - 1$, which gives all the values of r for which $-L_r(x)$ is the coefficient of t^n. Hence the total coefficient of tn on R.H.S. of (2) is given by $-\sum_{r=0}^{n-1} L_r(x)$

Thus, equating the coefficients of t^n from both sides of (2), we get $L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)$.

13.9. SOLVED EXAMPLES

Ex. 1. *Prove that* (*i*) $L_n(0) = 1$. **[Meerut 1993]** (*ii*) $L_n(0) = n!$. **[Meerut 1996]**

Sol. (*i*) We know that (Refer Art. 13.3.) $$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)} \quad ...(1)$$

Putting $x = 0$ in (1), $\sum_{n=0}^{\infty} t^n L_n(0) = \frac{1}{1-t} = (1-t)^{-1} = 1 + t + t^2 + .,$ by the binomial theorem

or $$\sum_{n=0}^{\infty} t^n L_n(0) = \sum_{n=0}^{\infty} t^n, \quad ...(2)$$

which is an identity. Equating the coefficient of t^n on both sides of (2), we get $L_n(0) = 1$.

(*ii*) We know that (Refer remark of Art. 13.3) $$\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{n!} = \frac{1}{1-t} e^{-tx/(1-t)}. \quad ...(3)$$

Putting $x = 0$ in (3), $\sum_{n=0}^{\infty} \frac{t^n L_n(0)}{n!} = (1-t)^{-1} = \sum_{n=0}^{\infty} t^n$, by binomial theorem

Equating the coefficient of t^n on both sides, we get

$(1/n!)L_n(0) = 1$ or $L_n(0) = n!$.

Ex. 2. *Prove that* (*i*) $L'_n(0) = -n$. (*ii*) $L''_n(0) = \{n(n-1)\}/2$

Part (i). Since $L_n(x)$ is a solution of the Laguerre's equation

$$xy'' + (1-x)y' + ny = 0, \quad ...(1)$$

we get $$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0. \quad ...(2)$$

Putting $x = 0$ and using $L_n(0) = 1$, (3) gives

$0 + (1-0)L_n'(0) + n \times 1 = 0$ or $L_n'(0) = -n$.

Part (ii). We know that $$\frac{1}{1-t} \exp\{-xt/(1-t)\} = \sum_{n=0}^{\infty} t^n L_n(x). \quad ... (3)$$

Differentiating twice w.r.t. 'x', (3) gives $\frac{\exp\{-xt/(1-t)\}}{1-t}\left(-\frac{t}{1-t}\right)^2 = \sum_{n=0}^{\infty} L_n''(x)\, t^n$. ...(4)

Putting $x = 0$ in (4), we have $$\sum_{n=0}^{\infty} L_n''(0)\, t^n = t^2(1-t)^{-3}. \quad ...(5)$$

Equating the coefficients of t^n on both sides of (5), we get

$L_n''(0)$ = coefficients of t^n in the expansion of $t^2(1-t)^{-3}$ = coeff. of t^{n-2} in the explansion of $(1-t)^{-3}$

$$= \frac{(-3)(-3-1)\ldots\{-3-(n-2)+1\}}{(n-2)!}(-1)^{n-2} = \frac{(-3)(-4)\ldots(-n)}{(n-2)!}(-1)^{n-2} = \frac{3\cdot 4\cdot 5\ldots n}{(n-2)!}(-1)^{n-2}(-1)^{n-2}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdots n}{1\cdot 2\cdot (n-2)!} = \frac{n!}{2(n-2)!} = \frac{n(n-1)(n-2)!}{2(n-2)!} = \frac{n(n-1)}{2}. \quad [\because (-1)^{2n-2} = 1]$$

Ex. 3. *Prove that* $xL_n''(x) + (1-x)L_n'(x) + L_n(x) = 0$ *and hence deduce that* $L_n'(0) = -n$.

Sol. Since $L_n(x)$ satifies Laguerre's equation $x(x^2y/dx^2) + (1-x)(dy/dx) + ny = 0$,

therefore $$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0. \quad ...(1)$$

Replacing x by 0 in (1), we have

$L_n'(0) + nL_n(0) = 0$ or $L_n'(0) = -n$, as $L_n(0) = 1$.

Ex. 4. *Expand* $x^3 + x^2 - 3x + 2$ *in a series of Laguerre polynomials.* **[Meerut 1994]**

Sol. We know that $L_0(x) = 1, L_1(x) = 1-x, L_2(x) = (2-4x+x^2)/2$ and $L_3(x) = (6-18x+9x^2-x^3)/6$.

$\therefore$ $$x^3 = 6 - 18x + 9x^2 - 6L_3(x), \quad ...(1)$$

$$x^2 = 4x - 2 + 2L_2(x), \qquad ...(2)$$

$$x = 1 - L_1(x) \quad \text{and} \quad 1 = L_0(x). \qquad ...(3)$$

Now, $x^3 + x^2 - 3x + 2 = 6 - 18x + 9x^2 - 6L_3(x) + x^2 - 3x + 2$, by (1)

$= 8 - 21x + 10x^2 - 6L_3(x) = 8 - 21x + 10[4x - 2 + 2L_2(x)] - 6L_3(x)$, by (2)

$= -12 + 19x + 20L_2(x) - 6L_3(x) = -12 + 19[1 - L_1(x)] + 20L_2(x) - 6L_3(x)$, by (3)

$= 7 - 19L_1(x) + 20L_2(x) - 6L_3(x) = 7L_0(x) - 19L_1(x) + 20L_2(x) - 6L_3(x)$, by (3).

Ex. 5. *Find the values of* (*i*) $\int_0^\infty e^{-x} L_3(x) L_5(x)\, dx$ (*ii*) $\int_0^\infty e^{-x} \{L_4(x)\}^2\, dx$.

Sol. By results of Art 13.6, we have

(*i*) $\int_0^\infty e^{-x} L_3(x) L_5(x)\, dx = 0$ (*ii*) $\int_0^\infty e^{-x} \{L_4(x)\}^2\, dx = 1$.

Ex. 6. *Taking* $L_n(x)$ *to be the coefficient of* t^n *in the expansion of*

$\frac{1}{1-t}\exp\left(\frac{-xt}{1-t}\right)$, *prove that* $\int_0^{\pi/2} \frac{e^{-\tan\theta}}{\cos^2\theta} L_n(\tan\theta)\, L_m(\tan\theta)\, d\theta = \delta_{mn.}$

Sol. $\int_0^{\pi/2} \frac{e^{-\tan\theta}}{\cos^2\theta} L_n(\tan\theta)\, L_m(\tan\theta)\, d\theta = \int_0^\infty e^{-x} L_n(x)\, L_m(x)\, dx$

[on putting $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$]

$= \delta_{m,\,n,}$ by orthogonal property of $L_n(x)$.

Ex. 7. *Taking* $\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t}\exp\left\{\frac{-tx}{1-t}\right\}$, *prove that*

(*i*) $L_n'(x) = n[L'_{n-1}(x) - L_{n-1}(x)]$ (*ii*) $xL'_n(x) = nL_n(x) - n^2 L_{n-1}(x)$.

Sol. Part (*i*) Given $\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}$. ...(1)

Differentiating both sides of (1) w.r.t. 'x', we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n'(x) = \frac{1}{1-t} e^{-tx/(1-t)} \left(-\frac{t}{1-t}\right) \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n'(x) = -\frac{t}{1-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x), \text{ using (1)}$$

$$(1-t)\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n'(x) = -t\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n'(x) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} L_n'(x) - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} L_n(x). \quad ...(2)$$

Equating the coefficient of t^n on both sides of (2), we have

$$\frac{1}{n!} L_n'(x) = \frac{1}{(n-1)!} L'_{n-1}(x) - \frac{1}{(n-1)!} L_{n-1}(x) \quad \text{or} \quad L'_n(x) = n[L'_{n-1}(x) - L_{n-1}(x)].$$

Part (ii) : We have (From Remark of recurrence relation 1).

$$L_{n+1}(x) = (2n + 1 - x)L_n(x) - n^2 L_{n-1}(x). \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'x', we get

$$L'_{n+1}(x) = (2n + 1 - x)L'_n(x) - L_n(x) - n^2 L'_{n-1}(x). \qquad ...(2)$$

From part (*i*) above, we have $L'_n(x) = n[L'_{n-1}(x) - L_{n-1}(x)]$. ...(3)

Replacing n by $n + 1$ in (3), we get $L'_{n+1}(x) = (n + 1)[L'_n(x) - L_n(x)]$. ...(4)

Again, from (3), we get $nL'_{n-1}(x) = L'_n(x) + nL_{n-1}(x)$. ...(5)

Using (4) and (5), (2) reduces to

$$(n + 1)[L'_n(x) - L_n(x)] = (2n + 1 - x)L'_n(x) - L_n(x) - n[L'_n(x) + nL_{n-1}(x)]$$

or $$xL'_n(x) = nL_n(x) - n^2L_{n-1}(x).$$

Ex. 8. *Prove that* $\int_0^\infty e^{-st}L_n(t)\,dt = (1/s) \times (1 - 1/s)^n$. **[Kanpur 2011]**

Sol. By definition of $L_n(x)$, we have $$L_n(t) = \sum_{r=0}^{n} \frac{(-1)^r n!\, t^r}{(n-r)!\,(r!)^2}. \qquad ...(1)$$

Now, L.H.S. of the problem

$$= \int_0^\infty e^{-st}L_n(t)\,dt = \int_0^\infty e^{-st}\left(\sum_{r=0}^{n}\frac{(-1)^r n!\,t^r}{(n-r)!(r!)^2}\right)dt = \sum_{r=0}^{n}\frac{(-1)^r n!}{(n-r)!(r!)^2}\int_0^\infty e^{-st}t^{(r+1)-1}\,dt$$

$$= \sum_{r=0}^{n}\frac{(-1)^r n!}{(n-r)!(r!)^2}\cdot\frac{\Gamma(r+1)}{s^{r+1}} = \frac{1}{s}\sum_{r=0}^{n}\frac{(-1)^r n!}{(n-r)!(r!)}\left(\frac{1}{s}\right)^r, \qquad \text{as} \qquad \Gamma(r+1) = r!$$

$$= \frac{1}{s}\sum_{r=0}^{n} {}^nC_r\left(-\frac{1}{s}\right)^r = \frac{1}{s}\left(1-\frac{1}{s}\right)^n, \text{ by the binomial theorem.}$$

Ex. 9. *Prove that* $\int_x^\infty e^{-y}L_n(y)\,dy = e^{-x}[L_n(x) - L_{n-1}(x)]$. **[Kanpur 2007]**

Sol. $\int_x^\infty e^{-y}L_n(y)\,dy = [-e^{-y}L_n(y)]_x^\infty - \int_x^\infty (-e^{-y})L'_n(y)\,dy$, integrating by parts

$$= e^{-x}L_n(x) + \int_x^\infty e^{-y}L'_n(y)\,dy \quad = e^{-x}L_n(x) + \int_x^\infty e^{-y}\left(-\sum_{r=0}^{n-1}L_r(y)\right)dy, \quad \text{as} \quad L'_n(y) = -\sum_{r=0}^{n-1}L_r(y)$$

$$= e^{-x}L_n(x) - \sum_{r=0}^{n-1}\int_x^\infty e^{-y}L_r(y)\,dy.$$

$$\therefore \qquad \int_x^\infty e^{-y}L_n(y)\,dy + \sum_{r=0}^{n-1}\int_x^\infty e^{-y}L_r(y)\,dy = e^{-x}L_n(x). \qquad ...(1)$$

Re–writing (1), we have $$\sum_{r=0}^{n}\int_x^\infty e^{-y}L_r(y)\,dy = e^{-x}L_n(x). \qquad ...(2)$$

Subtracting (1) from (2), we have

$$\sum_{r=0}^{n}\int_x^\infty e^{-y}L_r(y)\,dy - \int_x^\infty e^{-y}L_n(y)\,dy - \sum_{r=0}^{n-1}\int_x^\infty e^{-y}L_r(y)\,dy = 0$$

or $$\int_x^\infty e^{-y}L_n(y)\,dy = \sum_{r=0}^{n}\int_x^\infty e^{-y}L_r(y)\,dy - \sum_{r=0}^{n-1}\int_x^\infty e^{-y}L_r(y)\,dy$$

$$= e^{-x}L_n(x) - e^{-x}L_{n-1}(x), \text{ using (2) for } n = n \text{ and } n = n-1$$

$$= e^{-x}[L_n(x) - L_{n-1}(x)].$$

Ex. 10. *Prove that* $\int_0^\infty e^{-y}x^k L_n(x)\,dx = \begin{cases} 0, \text{ if } k < n \\ (-1)^n n!, \text{ if } k = n. \end{cases}$ **[Kanpur 2004]**

Sol. From Art. 13.7, we know that if $f(x)$ is a polynomial of degree m, we have

$$f(x) = \sum_{r=0}^{m} C_r L_r(x), \qquad ...(1)$$

where $C_r = \int_0^\infty e^{-x} f(x) L_r(x)\, dx$. ...(2)

Let $f(x) = x^k$ so that $f(x)$ is a polynomial of degree k. Then (1) and (2) reduce to

$$x^k = \sum_{r=0}^{k} C_r L_r(x), \qquad ...(3)$$

where $C_r = \int_0^\infty e^{-x} x^k L_r(x)\, dx$. ...(4)

Case (i) Let $k < n$. Then expansion (3) shows that $C_n = 0$. Hence from (4), we have (with $r = n$)

$$\int_0^\infty e^{-x} x^k L_n(x)\, dx = 0. \qquad ...(5)$$

Case (ii). Let $k = n$. Then (3) and (4) give $x^n = \sum_{r=0}^{n} C_r L_r(x) = C_0 L_0(x) + \ldots + C_n L_n(x)$ ...(6)

and $C_r = \int_0^\infty e^{-x} x^n L_n(x)\, dx$. ...(7)

We have, by definition of $L_n(x)$ $L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r$. ...(8)

From (8), we see that, coefficient of x^n in $L_n(x) = (-1)^n \frac{n!}{0!(n!)^2} = \frac{(-1)^n}{n!}$. ...(9)

Equating the coefficients of x^n from both sides of (6) and using (9), we get

$1 = C_n(-1)^n/n!$ so that $C_n = (-1)^n n!$.

With this value of C_n, (7) (with $r = n$) reduces to $\int_0^\infty e^{-x} x^n L_n(x)\, dx = (-1)^n n!$. ...(10)

(5) and (10) together may be written as $\int_0^\infty e^{-x} x^k L_n(x)\, dx = \begin{cases} 0, \text{ if } k < n \\ (-1)^n n!, \text{ if } k = n. \end{cases}$

EXERCISE

1. Express $10 - 23x + 10x^2 - x^3$ in terms of Laguerre polynomials.

Ans. $L_0(x) + L_1(x) + 2L_1(x) - 6L_3(x)$

2. Prove that $L_n(2x) = n! \sum_{n=0}^{\infty} \frac{(-1)^m 2^{n-m}}{m!(n-m)!} L_{n-m}(x)$.

3. If $e^{-x} = \sum_{n=0}^{\infty} C_n L_n(x)$, then show that $C_n = 1/(2^{n+1} n!)$.

4. Show that $L_n(x)$ defined by $\exp\left(-\frac{xt}{1-t}\right) = (1-t) \sum_{n=0}^{\infty} \frac{L_n(x) t^n}{n!}$

satisfies the differential equation $xL''_n(x) + (1-x)L'_n(x) + nL_n(x) = 0$.

5. Laguerre polynomial $L_q(x)$ is defined by $e^{-xs/(1-s)} = \sum_{q=0}^{\infty} \frac{L_q(x)}{q!} s^q$, $s < 1$.

Show that $L'_q = qL'_{q-1} - qL_{q-1}$ and $L_{q+1} = (2q + 1 - x)L_q - q^2 L_{q-1}$.

6. State and prove that generating function for Laguerre polynomial. **[Meerut 1993]**

7. Show that $(i)\ L_n^{(n)}(0) = (-1)^n$ $(ii)\ L_n^{(n)}(0) = (-1)^n n!$

where $L_n^{(n)}$ stands for $d^n L_n(x)/dx^n$ **[Kanpur 2009]**

14

Hypergeometric Function

14.1. Pochhammer symbol. Definition.

Let n be a positive integer. Then Pochhammer symbol is denoted and defined by

$$(\alpha)_n = \alpha(\alpha + 1)...(\alpha + n - 1) \quad ...(1)$$

with $$(\alpha)_0 = 1. \quad ...(2)$$

Deductions. By definition, we have

I. $(\alpha)_n = \alpha(\alpha+1)...(\alpha+n-1) = \dfrac{1\cdot 2\cdot 3...(\alpha-1)\alpha(\alpha+1)...(\alpha+n-1)}{1\cdot 2\cdot 3...(\alpha-1)} = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ as $\Gamma(p) = (p-1)\Gamma(p-1)$

Thus, $$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \quad ...(3)$$

II. $(\alpha)_{n+1} = \alpha(\alpha+1)(\alpha+2)...[\alpha+(n+1)-1] = \alpha[(\alpha+1)(\alpha+2)...(\alpha+1+n-1)] = \alpha(\alpha+1)_n$

Thus, $$(\alpha)_{n+1} = \alpha(\alpha+1)_n. \quad ...(4)$$

III. $$(\alpha+n)(\alpha)_n = \alpha(\alpha+1)...(\alpha+n-1)(\alpha+n)$$
$$= \alpha(\alpha+1)...(\alpha+n-1)(\alpha+n+1-1) = (\alpha)_{n+1}$$

Thus, $$(\alpha+n)(\alpha)_n = (\alpha)_{n+1}. \quad ...(5)$$

14.2. General hypergeometric function. Definition.

The general hypergeometric function is denoted and defined by

$${}_mF_n(\alpha_1, \alpha_2,...\alpha_m\ ;\ \beta_1, \beta_2...\beta_n\ ;\ x) = \sum_{r=1}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_r...(\alpha_m)_r}{(\beta_1)_r(\beta_2)_r...(\beta_n)_r}.\frac{x^r}{r!} \quad ....(1)$$

The general hypergeometric function is also denoted by $${}_mF_n\left[\begin{matrix}\alpha_1, \alpha_2,...\alpha_m\ ;\\ \beta_1, \beta_2...\beta_n\ ;\end{matrix} x\right]. \quad ...(2)$$

Remark. We shall consider only two special cases of (1) in the present chapter. These are given in the following two articles for $m = n = 1$ and $m = 2$, $n = 1$ respectively.

14.3. Confluent hypergeometric (or Kummer) function. Definition.

Confluent hypergeometric function is denoted by ${}_2F_1(\alpha; \beta; x)$ or $F(\alpha; \beta; x)$ or $M(\alpha, \beta, x)$ and is defined by

$$F(\alpha; \beta; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r}.\frac{x^r}{r!}. \quad ...(1)$$

Remark. Sometimes we use the following modified definition of the confluent hypergeometric function

$$F(\alpha; \beta; x) = 1 + \frac{x}{1\cdot\beta}x + \frac{\alpha(\alpha+1)}{1\cdot 2\cdot\beta(\beta+1)}x^2 + ... \quad ...(2)$$

14.4. Hypergeometric function. Definition. [Ranchi 2010]

Hypergeometric function is denoted by ${}_2F_1(\alpha; \beta; \gamma; x)$ or simply $F(\alpha, \beta; \gamma; x)$ and is defined by

$$F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!}. \quad ...(1)$$

Remark 1. The series on the R.H.S. of (1) is $1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} \cdot x^2 + ...$ ...(2)

In particular, if $\alpha = 1$, $\beta = \gamma$, then the series (2) takes the form

$$1 + x + x^2 + x^3 + ...,$$

which is a geometric series. Since (2) reduces to a geometric series as a particular case, (2) is called *hypergeometric series*.

Remark 2. Sometimes we use the following modified definition of the hypergeometric series

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \cdot \beta}{\gamma} \frac{x}{1!} + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{x^3}{3!} + ... \quad ...(3)$$

Remark 3. Hypergeometric function $F(a, b; c; x)$ can be put in the following different forms:

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x) \quad \textbf{[Lucknow2010]} \quad ...(4)$$

$$= (1-x)^{-a} F\left(a, c-b; c; \frac{x}{x-1}\right) \quad \textbf{[Kanpur 2009]} \quad ...(5)$$

$$= (1-x)^{-b} F\left(b, c-a; c; \frac{x}{x-1}\right). \quad ...(6)$$

14.5. Gauss's hypergeometric equation or Gauss's equation or hypergeometric equation. Definition.

$x(1-x)(d^2y/dx^2) + \{\gamma - (\alpha + \beta + 1)x\}(dy/dx) - \alpha\beta y = 0$ is called *hypergeometric equation.*

14.6. Solution of the hypergeometric eqution. [Kanpur 2006]

For solution refer Ex. 7 of Art. 8.9 of chapter 8. Let α, β and γ be constants, then

$$x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\} y' - \alpha\beta y = 0 \quad ... (1)$$

is known the *hypergeometric equation*. Let $\gamma \neq 0, -1, -2, ...$. Then a solution of (1) is given by

$${}_2F_1(\alpha, \beta; \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2\gamma(\gamma+1)} x^2 + ... \quad ... (2)$$

${}_2F_1(\alpha; \beta; \gamma; x)$ called the *hypergeometric function*.

If γ is not an integer, then the other solution of (1) and linearly independent of the solution ${}_2F_1(\alpha, \beta; \gamma, x)$ is $x^{1-\gamma} {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$.

Thus, if γ is not an integer, the general solution of (1) is

$y = a\,{}_2F_1(\alpha, \beta; \gamma; x) + bx^{1-\gamma} {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$, where a and b are arbitrary constants. ... (3)

Note 1. The hypergeometric function $F(\alpha, \beta; \gamma; x)$ is defined only if (*i*) α and β are real numbers (*ii*) γ is any real number such that $\gamma \neq 0, -1, -2, ...$ (*iii*) the variable x satisfies $|x| < 1$.

Note 2. The general solution (3) of (1) exists if γ is not an integer.

Note 3. If either α or β is a negative integer, then $F(\alpha, \beta; \gamma; x)$ reduces to a polynomial, because after finite number of terms, the coefficient of each term will be zero. For examples, consider the following:

$$F(-2, b; \gamma; c) = 1 + \frac{(-2)b}{1 \cdot c}x + \frac{(-2)(-1)b(b+1)}{1 \cdot 2 \cdot c(c+1)}x^2 + 0 = 1 - \frac{2b}{c}x + \frac{b(b+1)}{c(c+1)}x^2 + \ldots$$

which is a polynomial of degree 2

Similarly, $F(a, -2; \gamma : c) = 1 - (2a/c)x + \{a(a+1)/c(c+1)\}x^2$.

14.7. Symmetric property of hypergeometric function.

Hypergeometric function does not change if the parameters α and β are interchanged, keeping γ fixed. Thus, $F(\alpha, \beta; \gamma; x) = F(\beta, \alpha; \gamma; x)$.

Proof. We have, by definition $$F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} \quad \ldots(1)$$

and $$F(\beta, \alpha; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\beta)_r (\alpha)_r}{(\gamma)_r} \cdot \frac{x^r}{r!}. \quad \ldots(2)$$

From (1) and (2), we have $F(\alpha, \beta; \gamma; x) = (\beta, \alpha; \gamma; x)$.

14.8. Differentiation of hypergeometric functions. Show that

$$\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x)$$

[Garhwal 2005; Kurukshreta 2006; Kanpur 2005, 08, 10]

and deduce that (*i*) $$\frac{d^n}{dx^n} F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha + n, \beta + n; \gamma + n; x).$$

(*ii*) $$\left[\frac{d^n}{dx^n} F(\alpha, \beta; \gamma; x)\right]_{x=0} = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n}.$$

Proof. By definition, we have $$F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r r!} x^r.$$

Differentiating both sides w.r.t. 'x', we have

$$\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r r!} \cdot r x^{r-1} = \sum_{r=1}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r (r-1)!} x^{r-1} \qquad (\because \text{ the term with } r = 0 \text{ vanishes})$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(\gamma)_{m+1} m!} x^m$$ (taking m as the new variable of summation such that $r = m + 1$ *i.e.* $m = r - 1$ so that when $r = 1$, $m = 0$, and $r = \infty$, $m = \infty$)

$$= \sum_{m=0}^{\infty} \frac{\alpha (\alpha + 1)_{m+1} \cdot \beta (\beta + 1)_m}{\gamma (\gamma + 1)_m m!} x^m$$, by Art. 14.1

$$= \frac{\alpha\beta}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha + 1)_m (\beta + 1)_m}{(\gamma + 1)_m m!} x^m = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x).$$

$$\therefore \qquad \frac{d}{dx}F(\alpha,\beta;\gamma;x) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;x). \qquad ...(1)$$

Deduction. (*i*) For each positive integer, we must show that

$$\frac{d^n}{dx^n}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+n;\gamma+n;x) \qquad ...(2)$$

Since $\alpha = (\alpha)_1$, $\beta = (\beta)_1$ and $\gamma = (\gamma)_1$, (1) shows that (2) is true for $n = 1$. We now assume that (2) is true for a particular value of n (say $n = m$) so that

$$\frac{d^m}{dx^m}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}F(\alpha+m,\beta+m;\gamma+m;x). \qquad ...(3)$$

Differentiating both sides of (3) w.r.t. 'x' we get

$$\frac{d^{m+1}}{dx^{m+1}}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{d}{dx}F(\alpha+m,\beta+m;\gamma+m;x) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{(\alpha+m)(\beta+m)}{(\gamma+m)}F(\alpha+m+1,\beta+m+1;\gamma+m+1;x)$$

[using (1) for $\alpha + m$, $\beta + m$, $\gamma + m$ in place of α, β, γ respectively]

$$\therefore \qquad \frac{d^{m+1}}{dx^{m+1}}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(\gamma)_{m+1}}F(\alpha+m+1,\beta+m+1;\gamma+m+1;x), \qquad ...(4)$$

where we have used relation (5) of Art. 14.1. (4) shows that (2) is true for $n = m + 1$. Thus if (2) is true for $n = m$, then (2) is also true $n = m + 1$. Hence by mathematical induction, (2) is true for each positive integer.

Deduction (*ii*). Putting $x = 0$ in (2), we have

$$\left[\frac{d^n}{dx^n}F(\alpha,\beta;\gamma;x)\right]_{x=0} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+n;\gamma+n;0) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\left[\sum_{r=0}^{\infty}\frac{(\alpha+n)_r\,(\beta+n)_r}{(\gamma+n)_r}\frac{x^r}{r!}\right]_{x=0} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.$$

14.9. Integral representation for the hypergeometric function

$$F(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}dt$$

[Purvanchal 2007; Garhwal 2004; Kanpur 2004, 07, 10]

or $$F(\alpha,\beta;\gamma;x) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}dt, \text{ if } \gamma > \beta > 0.$$

Proof. By definition, we have

$$F(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!} = \sum_{n=0}^{\infty}(\alpha)_n\frac{\Gamma(\beta+n)}{\Gamma(\beta)}.\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}.\frac{x^n}{n!}, \text{ by Art. 14.1}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)}\sum_{n=1}^{\infty}(\alpha)_n\frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\beta+n+\gamma-\beta)}.\frac{x^n}{n!}, \text{ multiplying and dividing by } \Gamma(\gamma-\beta)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}(\alpha)_n\left\{\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}\,dt\right\}\frac{x^n}{n!}, \text{ where } \gamma-\beta>0,\ \beta+n>0 \text{ so that } \gamma>\beta>0$$

$$\left[\because\ \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt\right]$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \left(\sum_{n=0}^{\infty} (\alpha)_n \frac{(xt)^n}{n!} \right) dt$$

Thus, $$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\, dt \qquad ...(1)$$

$$[\because \text{ the general term in the expansion of } (1 - xt)^{-\alpha} = \frac{(-\alpha)(-\alpha-1)...(-\alpha-n+1)}{n!}(-xt)^n$$

$$= (-1)^n \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!} \times (-1)^n x^n t^n = (\alpha)_n \frac{x^n t^n}{n!}, \text{ by } Art.14.1]$$

Also, $$B(\beta, \gamma-\beta) = \frac{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)} = \frac{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)} \Rightarrow \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} = \frac{1}{B(\beta, \gamma-\beta)}. \quad ...(2)$$

Using (2), (1) may be re–written as

$$F(\alpha, \beta; \gamma; x) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\, dt. \qquad ...(3)$$

Thus (1) and (3) are the required results.

14.10. Gauss Theorem. $$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$ **[Kanpur 2005, 06, 07 Ranchi 2010]**

Proof. From Art. 14.9, for $x = 1$ we have

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t)^{-\alpha}\, dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-\alpha-1}\, dt = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \frac{\Gamma(\beta)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\beta+\gamma-\beta-\alpha)}$$

$$\left(\because \int_0^1 t^{p-1}(1-t)^{q-1}\, dt = B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \right)$$

$$= \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}.$$

14.11. Vandermonde's theorem. $$F(-n, \beta; \gamma; 1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}.$$

Proof. From Art. 14.10, with $\alpha = -n$, we get

$$F(-n, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta+n)}{\Gamma(\gamma+n)\,\Gamma(\gamma-\beta)} = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta+n-1)\,(\gamma-\beta+n-2)...(\gamma-\beta)\,\Gamma(\gamma-\beta)}{(\gamma+n-1)(\gamma+n-2)...\gamma\Gamma(\gamma)\Gamma(\gamma-\beta)}$$

$$= \frac{(\gamma-\beta+n-1)\,(\gamma-\beta+n-2)...(\gamma-\beta)}{(\gamma+n-1)\,(\gamma+n-2)...\gamma} = \frac{(\gamma-\beta)_n}{(\gamma)_n}, \text{ by Art. 14.1}$$

14.12. Kummer's theorem.

$$F(\alpha, \beta; \beta - \alpha + 1; -1) = \frac{\Gamma(\beta - \alpha + 1)\,\Gamma(\beta/2 + 1)}{\Gamma(\beta + 1)\,\Gamma(\beta/2 - \alpha + 1)}. \qquad \textbf{[Purvanchal 2005]}$$

Proof. From Art. 14.9, with $x = -1$ and $\gamma = \beta - \alpha + 1$, we get

$F(\alpha, \beta; \beta - \alpha + 1; -1)$

$$= \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\alpha)\,\Gamma(\beta - \alpha + 1 - \beta)} \int_0^1 t^{\beta - 1}(1 - t)^{\beta - \alpha + 1 - \beta - 1}(1 - t)^{-\alpha}\, dt = \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta)\,\Gamma(1 - \alpha)} \int_0^1 t^{\beta - 1}(1 - t^2)^{-\alpha}\, dt$$

$$= \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta)\,\Gamma(1 - \alpha)} \int_0^1 (u^{1/2})^{\beta - 1}(1 - u)^{-\alpha}\, (du/2\sqrt{u}) \qquad \text{(putting } t^2 = u \text{ so that } dt = du/2\sqrt{u}\text{)}$$

$$= \frac{\Gamma(\beta - \alpha + 1)}{2\Gamma(\beta)\,\Gamma(1 - \alpha)} \int_0^1 u^{(\beta/2) - 1}(1 - u)^{1 - \alpha - 1}\, du = \frac{\Gamma(\beta - \alpha + 1)}{2\Gamma(\beta)\,\Gamma(1 - \alpha)} \cdot \frac{\Gamma(\beta/2)\,\Gamma(1 - \alpha)}{\Gamma(\beta/2 + 1 - \alpha)}$$

$$\left(\because \int_0^1 u^{p-1}(1 - u)^{q-1} = B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)} \right)$$

$$= \frac{\Gamma(\beta - \alpha + 1) \cdot (\beta/2) \cdot \Gamma(\beta/2)}{\Gamma(\beta/2 + 1 - \alpha)\beta\,\Gamma(\beta)} = \frac{\Gamma(\beta - \alpha + 1)\,\Gamma(\beta/2 + 1)}{\Gamma(\beta/2 + 1 - \alpha)\,\Gamma(\beta + 1)}$$

14.13. More about the confluent hypergeometric function and solution of confluent hypergeometric equation

The hypergeometric differential equation is

$$(x^2 - x)y'' + [(1 + \alpha + \beta)x - \gamma]y' + \alpha\beta y = 0. \qquad ...(1)$$

Replacing x by x/β in (1), we get

$$x\left(1 - \frac{x}{\beta}\right)y'' + \left\{\gamma - \left(1 + \frac{\alpha + 1}{\beta}\right)x\right\}y' - \alpha y = 0 \qquad ...(2)$$

Its solution is represented by the function $F(\alpha, \beta; \gamma; x/\beta)$

When $\beta \to \infty$, the equation (2) reduces to

$$xy'' + (\gamma - x)y' - \alpha y = 0 \qquad ...(3)$$

whose solution is given by

$$\lim_{\beta \to \infty} F(\alpha, \beta; \gamma; x/\beta). \qquad ...(4)$$

The equation (3) is known as the *confluent hypergeometric differential equation* or *Kummer's equation*.

$$\text{Now, } \lim_{\beta \to \infty} \frac{(\beta)_r}{\beta^r} = \lim_{\beta \to 0} \frac{\beta(\beta + 1)(\beta + 2)...(\beta + r - 1)}{\beta \cdot \beta \cdot \beta ... r \text{ times}} = \lim_{\beta \to \infty} \left(1 + \frac{1}{\beta}\right)\left(1 + \frac{2}{\beta}\right)...\left(1 + \frac{r - 1}{\beta}\right) = 1$$

Hence solution (4) may be written as

$$\lim_{\beta \to \infty} F\left(\alpha, \beta; \gamma; \frac{x}{\beta}\right) = \lim_{\beta \to \infty} \sum_{r=0}^{\infty} \frac{(\alpha)_r\,(\beta)_r}{r!\,(\gamma)_r}\left(\frac{x}{\beta}\right)^r = \lim_{\beta \to \infty} \sum_{r=0}^{\infty} \frac{(\alpha)_r\,(\beta)_r}{r!\,(\gamma)_r \beta^r} x^r = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!\,(\gamma)_r} x^r = F(\alpha; \gamma; x). \quad ...(5)$$

The function $F(a; \gamma; x)$ is called the *confluent hypergeometric function*.

Solution of differential equation (3) may also be obtained directly by the series integration method. Considering the equation (3), we find that $x = 0$ is a removable (non–essential) singularity and so the series representing the solution can be developed about the point $x = 0$.

Solution of the confluent hypergeometric differential equation when x = 0 and γ is not an integer. **[Kanpur 2010]**

Let us consider the solution of the hypergeometric differential equation (3) in the ascending powers of x as

$$y = x^k(a_0 + a_1x + a_2x^2 + ...) = \sum_{r=0}^{\infty} a_0 x^{k+r},\ a_0 \neq 0. \qquad ...(6)$$

$$\therefore \qquad y' = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} \qquad \text{and} \qquad y'' = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}.$$

Now putting the values of y, y' and y'' in (3) we get

$$x\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)\,x^{k+r-2} + (\gamma - x)\sum_{r=0}^{\infty} a_r(k+r)\,x^{k+r-1} - \alpha\sum_{r=0}^{\infty} a_r\,x^{k+r} = 0$$

or
$$\sum_{r=0}^{\infty} a_r[(k+r)(k+r-1) + \gamma(k+r)]\,x^{k+r-1} - \sum_{r=0}^{\infty}[a_r(k+r) + \alpha]\,x^{k+r} = 0. \qquad ...(7)$$

which is an identity and so coefficients of various powers of x must be zero.

Equating the coefficients of x^{k-1} (lowest powers of x) to zero, we get

$a_0[k(k-1) + \gamma k] = 0$ or $k(k-1) + \gamma k = 0$ as $a_0 \neq 0$.

Hence $k = 0$ and $k = 1 - \gamma$ are the roots of the indical equation.

Now equating to zero the coefficients of x^{k+i}, in (7) we get

$$[(k+i+1)(k+i) + \gamma(k+i+1)]a_{i+1} - [(k+i+\alpha)]a_i = 0.$$

$$\therefore \qquad a_{i+1} = \frac{(k+i)+\alpha}{(k+i+1)\,(k+i+\gamma)}a_i. \qquad ...(8)$$

Case I. When k = 0. Then (8) gives $\qquad a_{i+1} = \dfrac{(i+\alpha)}{(i+1)(i+\gamma)}a_i. \qquad ...(9)$

Putting $i = 0, 1, 2, 3, ...$ in (9), we get $\qquad a_1 = \dfrac{\alpha}{\gamma}a_0,$ $\qquad a_2 = \dfrac{1+\alpha}{2(1+\gamma)}a_1 = \dfrac{\alpha(\alpha+1)}{1.2.\gamma(\gamma+1)}a_0$

.....

$$a_m = \frac{\alpha(\alpha+1)(a+2)...(a+m-1)}{1\cdot 2\cdot 3...m\cdot\gamma(\gamma+1)...(\gamma+m-1)}a_0 = \frac{(\alpha)_m}{m!\,(\gamma)_m}a_0.$$

Substituting the values of $a_1.a_2...a_m$ in the series (6), we get

$$y = a_0\left[1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + ...\right] = a_0\sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!\,(\gamma)_m}x^m.$$

Taking $a_0 = 1$, we have $y = F(\alpha; \gamma; x)$ which is called the *confluent hypergeometric function of the first kind*.

Case II. When $k = 1 - \gamma$. Then (8) gives $\quad a_{i+1} = \dfrac{1-\gamma+i+\alpha}{(i+2-\gamma)(i+1)}a_i = \dfrac{(\alpha-\gamma+1)+i}{(2-\gamma+i)(i+1)}a_i$

$$\therefore \qquad a_{i+1} = \frac{\alpha'+1}{(\gamma'+i)(i+1)}a_i \qquad \text{where} \qquad \alpha - \gamma + 1 = \alpha' \qquad \text{and} \qquad 2 - \gamma = \gamma'.$$

Putting $i = 0, 1, 2, 3, ...$ in the above relation, we have

$$a_1 = \frac{\alpha'}{\gamma'}a_0, \quad a_2 = \frac{\alpha'+1}{(\gamma'+1)\cdot 2}a_1 = \frac{\alpha'(\alpha'+1)}{1\cdot 2\cdot\gamma'(\gamma'+1)}a_0 \qquad \text{Similarly,} \qquad a_m = \frac{(\alpha')_m}{m!\,(\gamma')_m}a_0.$$

Substituting the values of a_1, a_2, ... in (6), we have

$$y = a_0 x^{1-\gamma}\left[1 + \frac{\alpha'}{\gamma'}x + \frac{\alpha'(\alpha'+1)}{1\cdot 2\cdot \gamma'(\gamma'+1)}x^2 + \ldots\right] = a_0 x^{1-\gamma}\sum_{m=0}^{\infty}\frac{(\alpha')_m}{m!\,(\gamma')_m}x^m$$

or $$y = a_0 x^{1-\gamma} F(\alpha'; \gamma'; x) = a_0 x^{1-\gamma} F(\alpha - \gamma + 1;\ 2 - \gamma;\ x).$$

Putting $a_0 = 1$, we have $y = x^{1-\gamma}\, F(\alpha - \gamma + 1;\ 2 - \gamma,\ x)$ which is called the *confluent hypergeometric function of the second kind*. Thus the *general solution* of the confluent hypergeometric differential equation is given by

$$y = AF(\alpha;\ \gamma;\ x) + Bx^{1-\gamma}\, F(\alpha - \gamma + 1;\ 2 - \gamma;\ x), \qquad \text{where } \gamma > 0.$$

14.14. Differentiation of hypergeometric confluent functions.

Show that $\dfrac{d}{dx}F(\alpha;\beta;x) = \dfrac{\alpha}{\beta}F(\alpha+1;\beta+1;x)$ and deduce that

$$(i)\ \frac{d^n}{dx^n}F(\alpha;\beta;x) = \frac{(\alpha)_n}{(\beta)_n}F(\alpha+n;\beta+n;x) \qquad (ii)\left[\frac{d^n}{dx^n}F(\alpha;\beta;x)\right]_{x=0} = \frac{(\alpha)_n}{(\beta)_n}.$$

Proof. By definition, we have $$F(\alpha;\ \beta;\ x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\beta)_r}x^r.$$

Now the proof is similar to that of Art. 14.8.

14.15. Integral representation for confluent hypergeometric function.

$$F(\alpha;\ \beta;\ x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha-1}\,e^{xt}\,dt \qquad \textbf{[Purvanchal 2006]}$$

or $$F(\alpha;\ \beta;\ x) = \frac{1}{B(\alpha,\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha-1}\,e^{xt}\,dt\ , \text{ where } \beta > \alpha > 0.$$

Proof. By definition, we have

$$F(\alpha;\ \beta;\ x) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{\Gamma(\beta)}{\Gamma(\beta+n)}\frac{x^n}{n!}, \text{ by Art. 14.1}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\beta-\alpha)\,\Gamma(\alpha+n)}{\Gamma(\beta-\alpha+\alpha+n)}\frac{x^n}{n!} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\sum_{n=0}^{\infty}\left\{\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha+n-1}\,dt\right\}\frac{x^n}{n!}, \text{ if } \beta > \alpha$$

$$\left(\because \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = B(p,q) = \int_0^1 (1-t)^{p-1}t^{q-1}\,dt, \text{ if } p>0, q>0\right)$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha-1}\left(\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right)dt$$

$$\therefore \qquad F(\alpha;\ \beta;\ x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha-1}e^{xt}\,dt \qquad ...(1)$$

or $$F(\alpha;\ \beta;\ x) = \frac{1}{B(\alpha,\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha-1}e^{xt}\,dt \qquad ...(2)$$

$$\left(\because B(\alpha,\beta-\alpha) = \frac{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta-\alpha)} \Rightarrow \frac{1}{B(\alpha,\beta-\alpha)} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\right)$$

Thus (1) and (2) are the required results.

14.16. Theorem (Kummer's Relation). *Show that* $F(\alpha; \beta; x) = e^x F(\beta - \alpha; \beta; -x)$.

Proof. From Art. 14.15, we know that

$$F(\alpha\ ;\ \beta\ ;\ x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1} t^{\alpha-1}\, e^{xt}\ dt\ . \qquad ...(1)$$

Replacing α by $\beta - \alpha$ and x by $-x$ in (1), we get

$$F(\beta-\alpha;\ \beta;\ -x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\,\Gamma[\beta-(\beta-\alpha)]}\int_0^1 (1-t)^{\beta-(\beta-\alpha)-1} t^{\beta-\alpha-1}\, e^{-xt}\ dt$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} t^{\beta-\alpha-1}\, e^{-xt}\ dt = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha)}\int_1^0 u^{\alpha-1}(1-u)^{\beta-\alpha-1}\, e^{-x(1-u)}\ (-du)$$

(putting $1 - t = u$ so that $dt = -du$ and $t = 1 - u$)

$$= \frac{\Gamma(\beta)e^{-x}}{\Gamma(\beta-\alpha)\,\Gamma(\alpha)}\int_0^1 (1-u)^{\beta-\alpha-1}\, u^{\alpha-1}\, e^{xu}\ du = e^{-x}\,\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}\, t^{\alpha-1}\, e^{xt}\ dt$$

$= e^{-x}F(\alpha; \beta; x)$, by (1)

$\therefore\ F(\beta - \alpha; \beta; -x) = e^{-x}\, F(\alpha; \beta; x)$ so that $F(\alpha; \beta; x) = e^x(\beta - \alpha; \beta; -x)$.

14.17. Contiguous hypergeometric functions. Definitions.

According to Gauss, the function $F(\alpha', \beta'\ ;\ \gamma'\ ;\ x)$ is said to be contiguous to $F(\alpha, \beta\ ;\ \gamma\ ;\ x)$ when it is increased or decreased by one and only one of the parameters α, β, γ by unity.

According to above definition, there exist six hypergeometric functions contiguous to $F(\alpha, \beta\ ;\ \gamma\ ;\ x)$. These are denoted and defined as given below :

$F_{\alpha+} = F(\alpha + 1, \beta\ ;\ \gamma\ ;\ x)$, $F_{\beta+} = F(\alpha, \beta + 1\ ;\ \gamma\ ;\ x)$, $F_{\gamma+} = F(\alpha, \beta\ ;\ \gamma + 1\ ;\ x)$

$F_{\alpha-} = F(\alpha - 1, \beta\ ;\ \gamma\ ;\ x)$, $F_{\beta-} = F(\alpha, \beta - 1\ ;\ \gamma\ ;\ x)$, $F_{\gamma-} = F(\alpha, \beta\ ;\ \gamma - 1\ ;\ x)$.

14.18. To prove the contiguity relationship

$$(\alpha - \beta)F(\alpha, \beta\ ;\ \gamma\ ;\ x) = \alpha F(\alpha + 1, \beta\ ;\ \gamma\ ;\ x) - \beta F(\alpha, \beta + 1\ ;\ \gamma\ ;\ x)$$

or
$$(\alpha - \beta)F(\alpha, \beta, \gamma, x) = \alpha F_{\alpha+} - \beta F_{\beta+}.$$

Proof. We have, by definition 14.4, $\alpha F(\alpha + 1, \beta\ ;\ \gamma\ ;\ x) - \beta F(\alpha, \beta + 1\ ;\ \gamma\ ;\ x)$

$$= \alpha \sum_{r=0}^{\infty} \frac{(\alpha+1)_r (\beta)_r}{(\gamma)_r\ r!} x^r - \beta \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta+1)_r}{(\gamma)_r\ r!} x^r = \sum_{r=0}^{\infty} \frac{\alpha(\alpha+1)_r (\beta)_r}{(\gamma)_r\ r!} x^r - \sum_{r=0}^{\infty} \frac{(\alpha)_r \beta(\beta+1)_r}{(\gamma)_r\ r!}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha + r)\,(\alpha)_r (\beta)_r}{(\gamma)_r\ r!} x^r - \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r (\beta + r)_r}{(\gamma)_r\ r!} x^r$$

$[\because\ \alpha(\alpha + 1)_r = \alpha(\alpha + 1)(\alpha + 2)...(\alpha + 1 - r)(\alpha + 1 + r - 1)$, by Art. 14.1

$= [\alpha(\alpha + 1)(\alpha + 2)...(\alpha + r - 1)](\alpha + r) = (\alpha + r)(\alpha)_r$, by Art. 14.1 again

Similarly, $\beta(\beta + 1)_r = (\beta + r)(\beta)_r$]

$$= \sum_{r=0}^{\infty} [(\alpha + r) - (\beta + r)] \frac{(\alpha)_r (\beta)_r}{(\gamma)_r\ r!} x^r = (\alpha - \beta) \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r r!} x^r = (\alpha - \beta) F(\alpha, \beta\ ;\ \gamma\ ;\ x)$$

14.19. Contiguity relationship for confluent hypergeometric functions

$$(\alpha - \beta)xF(\alpha\ ;\ \beta + 1\ ;\ x) + \beta(\alpha + \beta - 1)F(\alpha\ ;\ \beta\ ;\ x) - \beta(\beta - 1)F(\alpha\ ;\ \beta - 1\ ;\ x) = 0.$$

Proof. Proceed as explained in Art. 14.18.

14.20. A SOLVED EXAMPLES

Ex. 1. *Prove that*

(*i*) $e^x = {}_1F_1(\alpha\ ;\ \alpha\ ;\ x)$.

(*ii*) $(1-x)^{-\alpha} = {}_2F_1(\alpha,\ \beta\ ;\ \beta\ ;\ x)$. **[Kanpur 2008, 09, Lucknow 2010]**

(*iii*) $(1-x)^{-1} = F(1,\ 1\ ;\ 1\ ;\ x),\ |\ x\ | < 1$. **[Kanpur 2004]**

(*iv*) $(1+x)^n = F(-n,\ 1\ ;\ 1\ ;\ -x)$.

(*v*) $ln(1+x) = \log_e(1+x) = x\ {}_2F_1(1,\ 1\ ;\ 2\ ;\ -x)$.

(*vi*) $\log(1-x) = -x\ {}_2F_1(1,\ 1\ ;\ 2\ ;\ x)$.

(*vii*) $\log\dfrac{1+x}{1-x} = 2x\ F\left(1/2, 1; 3/2; x^2\right) = \ . = F\begin{bmatrix} 1/2, 1; \\ 3/2 \end{bmatrix};\ x^2\Big]$ **[Kanpur 2005, 10]**

(*viii*) $\sin^{-1} x = xF(1/2;\ 1/2;\ 3/2;\ x^2)$.

(*ix*) $\tan^{-1} x = x\ F(1/2;\ 1;\ 3/2;\ -x^2)$. **[Kanpur 2006]**

Sol. (*i*) We have, by definition 14.3

$$ {}_1F_1(\alpha\ ;\ \beta\ ;\ x) = 1 + \frac{\alpha}{\beta}.\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}.\frac{x^2}{2!} + \ldots \text{ad. inf.} \qquad \ldots(1)$$

Replacing β by α in (1), we have

$$ {}_1F_1(\alpha\ ;\ \alpha\ ;\ x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \qquad \text{or} \qquad {}_1F_1(\alpha\ ;\ \alpha\ ;\ x) = e^x.$$

(*ii*) We have, by definition 14.4

$$ {}_2F_1(\alpha,\ \beta\ ;\ \gamma\ ;\ x) = 1 + \frac{\alpha\,\beta}{\beta}.\frac{x}{1!} + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \ldots \qquad \ldots(1)$$

Replacing γ by β in (1), we have

$$ {}_2F_1(\alpha,\ \beta\ ;\ \beta\ ;\ x) = 1 + \frac{\alpha\beta}{\beta}.\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \ldots = 1 + \alpha.\frac{x}{1!} + \alpha(\alpha+1)\frac{x^2}{2!} + \alpha(\alpha+1)(\alpha+2)\frac{x^2}{3!} + \ldots$$

$$= 1 + (-\alpha)(-x) + \frac{(-\alpha)(-\alpha-1)}{2!}(-x)^2 + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}(-x)^3 + \ldots$$

$= (1-x)^{-\alpha}$, by the binomial theorem.

(*iii*) and **(*iv*).** Procced like part (*ii*) above.

(*v*) We have, $$ {}_2F_1(\alpha,\ \beta\ ;\ \gamma\ ;\ x) = 1 + \frac{\alpha\cdot\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!} + \ldots \qquad \ldots(1)$$

Replacing α, β, γ, x by 1, 1, 2 and $-x$ respectively in (1), we get

$$ {}_2F_1(1,\ 1\ ;\ 2\ ;\ 1-x) = 1 + \frac{1\cdot1}{2}\cdot\frac{(-x)}{1!} + \frac{1\cdot2\cdot1\cdot2}{2\cdot3}\cdot\frac{(-x)^2}{2!} + \frac{1\cdot2\cdot3\cdot1\cdot2\cdot3}{2\cdot3\cdot4}\cdot\frac{(-x)^3}{3!} + \ldots$$

Multiplying both sides of the above equation by x, we get

$$ x\ {}_2F_1(1,\ 1\ ;\ 2\ ;\ -x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ad. inf.} = \log(1+x).$$

(*vi*) and **(*vii*).** Proceed like part (*v*) above

(*viii*) We have, $$ F(\alpha,\ \beta\ ;\ \gamma\ ;\ x) = 1 + \frac{\alpha\cdot\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \ldots \qquad \ldots(1)$$

Replacing α, β, γ, x by 1/2, 1/2, 3/2 and x^2 respectively in (1), we get

$$F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right)=1+\frac{(1/2)\times(1/2)}{(3/2)}\cdot\frac{x^2}{1!}+\frac{(1/2)\times(3/2)\times(1/2)\times(3/2)}{(3/2)\times(5/2)}\cdot\frac{x^4}{2!}+\ldots$$

$$\therefore\qquad xF\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x\right)=x+1^2\cdot\frac{x^3}{3!}+1^2\cdot3^2\cdot\frac{x^5}{5!}+1^2\cdot3^3\cdot5^2\cdot\frac{x^7}{7!}+\ldots=\sin^{-1}x.$$

(*ix*) We have, $$F(\alpha,\beta;\gamma;x)=1+\frac{\alpha\cdot\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\ldots \qquad \ldots(1)$$

Replacing α, β, γ and x by 1/2, 1, 3/2 and $-x^2$ respectively in (1), we have

$$F\left(\frac{1}{2},1;\frac{3}{2};-x^2\right)=1+\frac{(1/2)\times1}{(3/2)}\cdot\frac{(-x^2)}{1!}+\frac{(1/2)\times(3/2)\times1\times2}{(3/2)\times(5/2)}\cdot\frac{(-x^2)^2}{2!}+\ldots$$ so that

$$F\left(\frac{1}{2},1;\frac{3}{2};-x^2\right)=1-\frac{x^2}{3}+\frac{x^4}{5}-\ldots\infty\ \Rightarrow\ xF\left(\frac{1}{2},1;\frac{3}{2};-x^2\right)=x-\frac{x^3}{3}+\frac{x^5}{5}+\ldots\infty=\tan^{-1}x.$$

Ex. 2. *Show that* $\lim\limits_{a\to\infty}{}_2F_1(1,a;1;x/a)=e^x$.

Sol. By definition, $${}_2F_1(1,a;1;x/a)=1+\frac{1\cdot a}{1\cdot1}\left(\frac{x}{a}\right)+\frac{1\cdot2\cdot a\,(a+1)}{1\cdot2\cdot1\cdot2}\left(\frac{x}{a}\right)^2+\ldots.$$

or $${}_2F_1(1,a;1;x/a)=1+\frac{x}{1!}+\left(1+\frac{1}{a}\right)\frac{x^2}{2!}+\left(1+\frac{1}{a}\right)\left(1+\frac{2}{a}\right)\frac{x^3}{3!}+\ldots$$

$$\therefore\qquad \lim_{a\to\infty}{}_2F_1(1,a;1;x/a)=1+x/1!+x^2/2!+x^3/3!+\ldots=e^x$$

Ex. 3. *Show than* ${}_2F_1(a,1;a;x)=(1-x)^{-1}$.

Sol. $$F(a,1;a;x)=1+\frac{a\cdot1}{1\cdot a}x+\frac{a\,(a+1)\cdot1\cdot2}{1\cdot2\,a(a+1)}x^2+\frac{a\,(a+1)\,(a+2)\cdot1\cdot2\cdot3}{1\cdot2\cdot3\cdot a\,(a+1)\,(a+2)}x^3+\ldots$$
$$=1+x+x^2+x^3+\ldots.=(1-x)^{-1}$$

Ex. 4. *Show that* $\left[\frac{d}{dx}\,{}_2F_1(a,b;c;x)\right]_{x=c}=\frac{ab}{c}$.

Sol. By definition, $${}_2F_1(a,b;c;x)=1+\frac{ab}{1\cdot c}x+\frac{a\,(a+1)\,b\,(b+1)}{1\cdot2\cdot c\,(c+1)}x^2+\ldots$$

$$\therefore\qquad \frac{d}{dx}\,{}_2F_1(a,b;c;x)=0+\frac{ab}{c}\cdot1+\frac{a\,(a+1)\,b\,(b+1)}{2\cdot c\,(c+1)}\cdot2x+\ldots\ \Rightarrow\left[\frac{d}{dx}\,{}_2F_1(a,b;c;x)\right]_{x=0}=\frac{ab}{c}$$

Ex. 5. *Show that* $1-n\,x\,{}_2F_1(1-n,\ 1;\ 2;\ x)=(1-x)^n$, where n is any natural number.

Sol. Using formula $F(\alpha,\beta;\gamma;x)=1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\cdot2\cdot\gamma(y+1)}x^2+\ldots,$ we get

$$1-n\,x\,{}_2F_1\,(1-n,\ 1;2;x)$$

$$= 1 - nx\left\{1 + \frac{(1-n).1}{1\cdot 2}x + \frac{(1-n)(2-n)\cdot 1\cdot 2}{1\cdot 2\cdot 2\cdot 3}x^2 + \frac{(1-n)(2-n)\ldots(n-1-n)\cdot 1\cdot 2\ldots(n-1)}{1\cdot 2\ldots.(n-1)\cdot 2\cdot 3\ldots.n}x^{n-1} + 0\right\}$$

$$= 1 - nx\left\{1 + \frac{n-1}{2!}x + \frac{(n-1)(n-2)}{3!}x^2 + \ldots + \frac{(-1)^{n-1}(n-1)(n-2)\cdots 2\cdot 1}{n!}x^{n-1}\right\}$$

$$= 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{(-1)^n n!}{n!}x^n$$

$$= 1 + {}^nc_1(-x) + {}^nc_2(-x)^2 + {}^nc_3(-x)^3 + \ldots + {}^nc_n(-x)^n = (1-x)^n$$

Ex. 6. *Show that* $\lim_{a,b\to\infty} {}_2F_1(a, b; 1/2; x^2/4ab) = \cosh x$

Sol. Using formula ${}_2F_1(\alpha, \beta; \gamma; x) + 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\,\gamma(\gamma+1)}x^2 + \ldots.$, we get

$${}_2F_1(a, b; 1/2; x^2/4ab) = 1 + \frac{ab}{1\times(1/2)}\times\frac{x^2}{4ab} + \frac{a(a+1)b(b+1)}{1\times 2\times(1/2)\times(3/2)}\times\left(\frac{x^2}{4ab}\right)^2 + \ldots$$

$$= 1 + x^2/2 + (1+1/a)(1+1/b)\times(x^4/24) + \ldots.$$

$$\therefore \quad \lim_{a,b\to\infty} {}_2F_1(a, b; 1/2; x^2/4ab) = 1 + x^2/2! + x^4/4! + \ldots = \cosh x$$

Ex. 7. *Show that* ${}_2F_1(a-1, b-1, c; x) - F(a, b-1); c; x) = (x/c)\times(1-b)\ {}_2F_1(a, b; c+1; x)$

Sol. Using formula ${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\,\gamma(\gamma+1)}x^2 + \ldots,$ we get

$$F(a-1, b-1; c; x) = 1 + (a-1)(b-1)\times(x/c)$$

$$+ \frac{(a-1)a(b-1)b}{1\cdot 2\cdot c(c+1)}x^2 + \frac{(a-1)a(a+1)(b-1)b(b+1)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3 + \ldots. \qquad \ldots(1)$$

and $F(a, b-1; c; x) = 1 + a(b-1)\times(x/c)$

$$+ \frac{a(a+1)(b-1)b}{1\cdot 2\cdot c(c+1)}x^2 + \frac{a(a+1)(a+2)(b-1)b(b+1)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3 + \ldots \qquad \ldots(2)$$

Subtracting (2) from (1), $F(a-1, b-1; c; x) - F(a, b-1; c; x)$

$$= \frac{(a-1-a)(b-1)}{c}x + \frac{(a-1-a-1)a(b-1)b}{1\cdot 2\cdot c(c+1)}x^2 + \frac{(a-1-a-2)a(a+1)(b-1)b(b+1)}{1\cdot 2\cdot 3\,c(c+1)(c+2)}x^3 + \ldots$$

$$= \frac{1-b}{c}x\left\{1 + \frac{2ab}{1\cdot 2\cdot(c+1)}x + \frac{3a(a+1)b(b+1)}{1\cdot 2\cdot 3(c+1)(c+2)}x^2 + \ldots\right\} = (x/c)\times(1-b)\ {}_2F_1(a, b; c+1; x)$$

Ex. 8. *Find the third derivative of* ${}_2F_1(2, 3; 1; x)$ *w.r.t. 'x'*

(*b*) *Find the fourth derivative of the following hypergeonetre functions w.r.t 'x':*

(*i*) $2F_1(2, 1; 4; x)$ (*ii*) ${}_2F_1(2, -2; 5; x)$.

Sol. (*a*) By Art. 14.8, $$\frac{d^n}{dx^n}\,{}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha + n, \beta + n; \gamma + n; x) \quad \ldots (1)$$

$$\therefore \quad \frac{d^3}{dx^3}\,{}_2F_1(2, 3; 1; x) = \frac{(2)_3 (3)_3}{(1)_3} F(2 + 3, 3 + 3; 1 + 3; x) \quad \ldots (2)$$

But $(\alpha)_n = \alpha\,(\alpha + 1)\,(\alpha + 2)...(\alpha + n - 1)$, by Art. 14.1

$\therefore\ (1)_3 = 1 \cdot 2 \cdot 3, \quad (2)_3 = 2 \times (2 + 1) \times (2 + 2) = 24; \quad (3)_3 = 3 \times (3 + 1) \times (3 + 2) = 60$

Hence, (2) yields $$\frac{d^3}{dx^3}\,{}_2F_1(2, 3; 1; x) = \frac{24 \times 60}{6}\,{}_2F_1(5, 6; 4; x) = 240\,{}_2F_1(5, 6; 4; x)$$

(*b*) Proceed as in part (*a*) **Ans.** (*i*) (24/7) × *F*(6, 5; 8; *x*) (*ii*) 0

Ex. 9. *Find the solutions of the following equations :*

(*i*) $x(1-x)\,y'' + (3/2 - 2x)\,y' + 2y = 0$ about $x = 0$ **(KU Kurukshetra 2004)**

(*ii*) $(x - x^2)y'' + (3/2 - 2x)y' - (y/4) = 0$ about $x = 0$ **(KU Kurukshetra 2004)**

(*iii*) $8x(1-x)y'' + (4 - 14x)y' - y = 0$ about $x = 0$

(*iv*) $4x(1-x)y'' + y' + 8y = 0$ about $x = 0$

Sol. In what follows, we shall use the following results:

I. Hypergeometric equation is given by $x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\}\,y' - \alpha\beta y = 0$

II. General solution of hypergeometric equation is

$y = a\,{}_2F_1(\alpha, \beta; \gamma; x) + b\,x^{\gamma-1}\,{}_2F_1(1 - \gamma + \alpha, 1 - \gamma + \beta; 2 - \gamma; x)$, where *a* and *b* are arbitrary constants

III. ${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha + 1)\,\beta(\beta + 1)}{1 \cdot 2\,\gamma\,(\gamma + 1)} x^2 + \ldots.$

Part (*i*) Re-writing the given equation, we have, $x(1-x)y'' + (3/2 - 2x)y' + 2y = 0 \quad \ldots (1)$

Comparing (1) with $x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0$, we have $\gamma = 3/2$, $\alpha + \beta + 1 = 2$ and $\alpha\beta = -2$. Solving these, $\alpha = 2$, $\beta = -1$, $\gamma = 3/2$.

Here γ is not an integer. The general solution of (1) is $y = au + bv$, where

$$u = {}_2F_1(\alpha, \beta; \gamma; x) = F(2, -1; 3/2; x) = 1 + \frac{2 \times (-1)}{1 \times (3/2)} x + \frac{2 \times 3 \times (-1) \times 0}{1 \times 2 \times (3/2) \times (5/2)} x^2 + \ldots = 1 - \frac{4x}{3}$$

and $v = x^{1-\gamma}\,{}_2F_1(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma; x)$

$= x^{1-(3/2)}\,{}_2F_1(2 + 1 - 3/2, -1 + 1 - 3/2; 2 - 3/2; x) = x^{-1/2}\,{}_2F_1(3/2, -3/2; 1/2; x)$

Hence the general solution of (1) is given by $y = a(1 - 4x/3) + b\,x^{-1/2}\,{}_2F_1(3/2, -3/2; 1/2; x)$

(*ii*) Given $x(1-x)y'' + (3/2 - 2x)\,y' - (1/4) \times y = 0 \quad \ldots (1)$

Comparing (1) with $x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0$, we have

$\alpha = 1/2, \beta = 1/2, \gamma = 3/2.$ Hence γ is not an integer. The general solution of (1) is $y = au = bv$,

where $u = {}_2F_1(\alpha, \beta; \gamma; x) = {}_2F_1(1/2, 1/2; 3/2; x)$ and

$$v = x^{1-\gamma}{}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x) = x^{1-3/2}{}_2F_1(1/2+1-3/2, 1/2+1-3/2; 2-3/2; x)$$

$$= x^{-1/2}{}_2F_1(0; 0; 1/2; x) = x^{-1/2}\left(1 + \frac{0\times 0}{1\times(1/2)}x + \ldots\right) = \frac{1}{\sqrt{x}}$$

General solution of (1) is $y = a\ {}_2F_1(1/2, 1/2; 3/2; x) + b/\sqrt{x}$, a, b being arbitrary constants

(*iii*) Ans. $y = a\ (1 - x)^{-1/4} + bx^{1/2}\ {}_2F_1(1, 3/4; 3/2; x)$

(*iv*) Ans. $y = a\ (1 - 8x + 32x^2/5) + bx^{3/4}\ 2F_1(7/4 - 5/4; 7/4; x)$

Ex. 10. *Solve the Legendre equation* $(1 - x^2)\ (d^2y/dx^2) - 2x\ (dy/dx) + n(n + 1)y = 0$ by changing it to a hypergeometre equation. **(MDU Rohtak 2004)**

Sol. Given $(1 - x^2)\ (d^2y/dx^2) - 2x\ (dy/dx) + n(n + 1)y = 0$ ... (1)

Let $z = x^2$ ... (2)

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\ \frac{dz}{dx} = 2x\frac{dy}{dz}$$ ... (3)

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dx}\left(\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = 2\frac{dy}{dz} + (2x)^2\frac{d^2y}{dz^2}$$

Substituing the above values in (1), $$(1-x^2)\left(2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}\right) - (2x)^2\frac{dy}{dz} + n(n+1)y = 0$$

or $$(1-z)\left(2\frac{dy}{dz} + 4z\frac{d^2z}{dz^2}\right) - 4z\frac{dy}{dz} + n(n+1)y = 0$$

or $$4z(1-z)(d^2y/dz^2) + (2-6z)(dy/dz) + n(n+1)y = 0$$

or $$z(1-z)\frac{d^2y}{dz^2} + \left(\frac{1}{2} - \frac{3}{2}z\right)\frac{dy}{dz} + \frac{n(n+1)}{4}y = 0$$ ... (4)

Refer Ex. 9 for hypergeometric equation and its solution. Comparing (4) with hypergeometre equation $$z(1-z)\ (d^2y/dz^2) + \{\gamma - (\alpha+\beta+1)z\}\ (dy/dz) - \alpha\beta\ y = 0,$$ ... (5)

we have, $\gamma = 1/2$, $\alpha + \beta + 1 = 3/2$ and $\alpha\beta = -(1/4)\times n(n+1)$

Solving these, $\alpha = (n+1)/2$, $\beta = -(n/2)$ and $\gamma = 1/2$

Note that here γ is not an integer. The general solution of (5) is given by

$$y = a\ {}_2F_1(\alpha, \beta; \gamma; z) + b\,z^{\gamma-1}{}_2F_1(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; z)$$

Hence the required solution of (1) is given by

$$y = a\ {}_2F_1\left(\frac{n+1}{2}, -\frac{n}{2}; \frac{1}{2}; z\right) + b\,z^{1/2}{}_2F_1\left(\frac{n+1}{2}+1-\frac{1}{2}, -\frac{n}{2}+1-\frac{1}{2}; 2-\frac{1}{2}; z\right)$$

or $y = a\ {}_2F_1\left(\frac{n+1}{2}, -\frac{n}{2}; \frac{1}{2}; x^2\right) + bx\ {}_2F_1\left(\frac{n+2}{2}, -\frac{n-1}{2}; \frac{3}{2}; x^2\right)$, where a and b are arbitrary constants.

Ex. 11. *Show that* (*i*) $\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{(1-x^2\sin^2\theta)^{1/2}} = {}_2F_1(1/2, 1/2; 1; x^2), |x|<1$

(*ii*) $\frac{2}{\pi}\int_0^{\pi/2} (1-x^2\sin\theta)^{1/2}\, d\theta = {}_2F_1(-1/2, 1/2; 1; x^2), |x|<1$

Sol. By Art. 14.9, $${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\, dt$$

so that $$\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\, dt = \frac{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)} \times {}_2F_1(\alpha, \beta; \gamma; x) \quad \text{... (A)}$$

Part (*i*) Putting $t = \sin^2\theta$ and $dt = 2\sin\theta\cos\theta\, d\theta$, we have

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{(1-x^2\sin^2\theta)^{1/2}} = \frac{2}{\pi}\int_0^1 \frac{1}{(1-x^2t)^{1/2}} \frac{dt}{2\sin\theta\cos\theta} = \frac{2}{\pi}\int_0^1 \frac{1}{(1-x^2t)^{1/2}} \frac{dt}{2\,t^{1/2}(1-t)^{1/2}}$$

$$= \frac{1}{\pi}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-x^2t)^{-1/2}\, dt = \frac{1}{\pi}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{1-\frac{1}{2}-1}(1-x^2t)^{-\frac{1}{2}}\, dt$$

$$= \frac{1}{\pi} \times \frac{\Gamma(1/2)\Gamma(1-1/2)}{\Gamma(1)}\, {}_2F_1(1/2; 1/2; 1; x^2),$$ using result (A) taking $\beta = 1/2, \gamma = 1, \alpha = 1/2$, $x = x^2$

$$= \frac{1}{\pi} \times \frac{\sqrt{\pi}\times\sqrt{\pi}}{1} \times {}_2F_1(1/2, 1/2, 1; x^2) = {}_2F_1(1/2, 1/2; 1; x^2)$$

(*ii*) Putting $t = \sin^2\theta$ and $dt = 2\sin\theta\cos\theta\, d\theta$, we have

$$\frac{2}{\pi}\int_0^{\pi/2} (1-x^2\sin^2\theta)^{1/2}\, d\theta = \frac{2}{\pi}\int_0^1 (1-x^2t)^{1/2} \frac{dt}{2\sin\theta\cos\theta} = \frac{1}{\pi}\int_0^1 (1-x^2t)^{1/2} \frac{dt}{t^{1/2}(1-t)^{1/2}}$$

$$= \frac{1}{\pi}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-x^2t)^{1/2}\, dt = \frac{1}{\pi}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{1-\frac{1}{2}-1}(1-x^2t)^{1/2}\, dt$$

$$= \frac{1}{\pi} \times \frac{\Gamma(1/2)\Gamma(1-1/2)}{\Gamma(1)}\, {}_2F_1(-1/2, 1/2; 1; x^2) = {}_2F_1(-1/2, 1/2; 1; x^2)$$

[using result (A), taking $\beta = 1/2, \gamma = 1, \alpha = -1/2, x = x^2$]

Ex. 12. *Show that if* $|x| < 1$ *and* $|x/(1-x)| < 1$, *then*

$${}_2F_1(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha}\, {}_2F_1\left(\alpha, \gamma-\beta; \gamma; \frac{x}{x-1}\right)$$

or $$F(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha}\, F\left[\alpha, \gamma-\beta; \gamma; \frac{-x}{1-x}\right].$$

Sol. By integral representation for the hypergeometric function (refer Art. 14.9), we have

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\, dt. \quad \text{...(1)}$$

Putting $u = 1 - t$ so that $dt = -du$ and $t = 1 - u$, (1) gives

$$ {}_2F_1(\alpha, \beta\ ;\ \gamma\ ;\ x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_1^0 (1-u)^{\beta-1} u^{\alpha-\beta-1} (1-x+xu)^{-\alpha}\ (-du) $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}(1-x)^{-\alpha}\left\{1+\frac{xu}{1-x}\right\}^{-\alpha} du $$

$$ = \frac{(1-x)^{-\alpha}\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}\left\{1-\frac{x}{x-1}u\right\}^{-\alpha} du. \qquad ...(2) $$

Replacing β and x by $\gamma - \beta$ and $x/(x-1)$ in (1), we get

$$ {}_2F_1\left(\alpha, \gamma-\beta\,;\gamma\,;\frac{x}{x-1}\right) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma[\gamma-(\gamma-\beta)]} \int_0^1 t^{\gamma-\beta-1}(1-t)^{\gamma-(\gamma-\beta)-1}\left\{1-\frac{xt}{x-1}\right\}^{-\alpha} dt $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)} \int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}\left\{1-\frac{x}{x-1}u\right\}^{-\alpha} du. \qquad ...(3) $$

Using (3), (2) reduces to $ {}_2F_1(\alpha, \beta\ ;\ \gamma\ ;\ x) = (1-x)^{-\alpha}\ {}_2F_1\left(\alpha, \gamma-\beta\,;\gamma\,;\frac{x}{x-1}\right)$

Ex. 13. *Prove that* $P_n(x) = {}_2F_1\left(-n, n+1\,;1\,;\frac{1-x}{2}\right)$

[Punjab 2005, Purvanchal 2005,06, Kanpur 2004, 06]

Sol. $P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n$, by Rodrigue's formula

$$ = \frac{(-1)^n}{n!\,2^n}\frac{d^n}{dx^n}(1-x^2)^n = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left\{(1-x)^n.\frac{(1+x)^n}{2^n}\right\} = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n.\left\{1-\frac{1-x}{2}\right\}^n\right] $$

$$ = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n.\left\{1-n\left(\frac{1-x}{2}\right)+\frac{n(n-1)}{2!}\left(\frac{1-x}{2}\right)^2-\ldots\right\}\right], \text{ by the binomial theorem} $$

$$ = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n-\frac{n}{2}(1-x)^{n+1}+\frac{n(n-1)}{2!\,2^2}(1-x)^{n+2}+\ldots\right] $$

$$ = \frac{(-1)^n}{n!}\left[(-1)^n n!-(-1)^n\frac{n}{2}\frac{(n+1)!}{1!}(1-x)+(-1)^n\times\frac{n(n-1)}{2!\,2^2}\times\frac{(n+2)}{2!}(1-x)^2+\ldots\right] $$

$$ \left[\because \frac{d^n}{dx^n}(a-bx)^m = (-1)^n b^n.\frac{m!}{(m-n)!}(a-bx)^{m-n}\right] $$

$$ = \frac{(-1)^{2n}}{n!}\left[n!-\frac{n(n+1)}{2}n!(1-x)+\frac{n(n-1)}{2!\,2^2}(n+2)(n+1)n!(1-x)^2+\ldots\right] $$

$$ = 1+\frac{(-n)(n+1)}{1\cdot 1!}\left(\frac{1-x}{2}\right)+\frac{(-n)(-n+1)(n+1)(n+2)}{2\cdot 1\cdot 2!}\left(\frac{1-x}{2}\right)^2+\ldots = {}_2F_1\left(-n, n+1\,;1\,;\frac{1-x}{2}\right), \text{ by definition.} $$

Ex. 14. *Show that* $P_n(\cos\theta) = \cos^n\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; -\tan^2\theta\right)$.

Sol. From Laplace's first integral for $P_n(x)$, (Refer Art. 9.6 in chapter 9), we have

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x \pm (x^2-1)^{1/2}\cos\phi]^n\, d\phi. \qquad ...(1)$$

Let $x = \cos\theta$. Then, we have $(x^2-1)^{1/2} = (\cos^2\theta - 1)^{1/2} = \{(-1)\times(1-\cos^2\theta)\}^{1/2} = i\sin\theta$. With these values and taking positive sign in (1), we get

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n\, d\phi = \frac{\cos^n\theta}{\pi}\int_0^{1}(1 + i\tan\theta\cos\phi)^n\, d\phi$$

$$= \frac{\cos^n\theta}{\pi}\int_0^{\pi}\left\{1 + i\tan\theta\cos\phi + \frac{n(n-1)}{2!}i^2\tan^2\theta\cos^2\phi + ...\right\}d\phi, \text{ by the binomial theorem}$$

$$= \frac{\cos^n\theta}{\pi}\left[\int_0^{\pi} d\phi + 0 + \frac{n(n-1)}{2}i^2\tan^2\theta\left(2\int_0^{\pi/2}\cos^2\phi\, d\phi\right) + 0\right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}i^4\tan^4\theta\left(2\int_0^{\pi/2}\cos^4\phi\, d\phi\right) + ...\right], \text{ as } \int_0^{2a} f(x)\, dx = \begin{cases} 0, \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\, dx, \text{ if } f(2a-x) = f(x) \end{cases}$$

$$= \frac{\cos^n\theta}{\pi}\left[\pi - \frac{n(n-1)}{2}\tan^2\theta\times 2\times\frac{1}{2}\times\frac{\pi}{2} + \frac{n(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}\tan^2\theta\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} + ...\right]$$

$$= \cos^n\theta\left[1 - \frac{\left(-\frac{n}{2}\right)\left(-\frac{n-1}{2}\right)}{1}\frac{(-\tan^2\theta)}{1!} + \frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}+1\right)\left(-\frac{n-1}{2}\right)\left(-\frac{n-1}{2}+1\right)}{1\cdot 2}\frac{(-\tan^2\theta)^2}{2!} + ...\right]$$

$$= \cos^n\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; \tan^2\theta\right), \text{ by definition.}$$

EXERCISE

1. Show that ${}_1F_1(\beta;\gamma;x) = \lim_{\beta\to\infty}{}_2F_1(\alpha,\beta;\gamma;x/\beta)$.

2. Show that ${}_2F_1(\alpha, \beta; \beta-\alpha+1; -1) = \frac{\Gamma(1+\beta-\alpha)\Gamma(1+\beta/2)}{\Gamma(1+\beta)\Gamma(1+\beta/2-\alpha)}$ and deduce that

$${}_2F_1(\alpha, 1-\alpha; \gamma; 1/2) = \frac{\Gamma(\gamma/2)\Gamma(\gamma/2+1/2)}{\Gamma(\alpha/2+\gamma/2)\Gamma(1/2-\alpha/2+\gamma/2)}.$$

3. Evaluate the integral $\int_0^{\infty} e^{-sx}{}_1F_1(\alpha;\beta;x)\, dx$. [**Ans.** $(1/s)_2 \times F_1(\alpha, 1; \beta; s]$

[**Hint.** Use Art. 14.15]

4. Prove that $F(\alpha, \beta+1; \gamma+1; x) - F(\alpha, \beta; \gamma; x) = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}xF(\alpha+1, \beta+1; \gamma+2; x)$.

5. The complete elliptic integral of the first kind is

$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}}$. Show that $K = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$. **[Kanpur 2011]**

6. The complete elliptic integral of the second kind is

$E = \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \phi)}\, d\phi$. Show that $E = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$, $| k | < 1$.

7. Prove the following relations :

(*i*) $F(\alpha - 1, \beta - 1 ; \gamma ; x) - F(\alpha, \beta - 1 ; \gamma ; x) = \frac{(1 - \beta)x}{\gamma} F(\alpha, \beta ; \gamma + 1 ; x)$

(*ii*) $\alpha F(\alpha + 1, \beta ; \gamma ; x) - (\gamma - 1)F(\alpha, \beta ; \gamma - 1 ; x) = (\alpha + 1 - \gamma)F(\alpha, \beta ; \gamma ; x)$.

8. Prove that $F(\alpha, \beta; \gamma; 1/2) = 2^{\alpha} F(\alpha, \gamma - \beta ; \gamma ; -1)$.

9. Show that (*i*) $e^x - 1 = xF(1 ; 2 ; x)$.

(*ii*) $(1 + x/\alpha)e^x = F(\alpha + 1 ; \alpha ; x)$.

10. The incomplete Gamma function is defined by the equation $\gamma(\alpha, x) = \int_0^{\pi} e^{-t} t^{\alpha - t}\, dt$, $\alpha > 0$.

Prove that $\gamma(\alpha, x) = \alpha^{-1} x^{\alpha} F(\alpha ; \alpha + 1 ; -x)$.

11. Prove that following relations : (*i*) $\beta F(\alpha ; \beta ; x) = \beta F(\alpha - 1 ; \beta ; x) + xF(\alpha ; \beta + 1 ; x)$.
(*ii*) $\alpha F(\alpha + 1 ; \beta ; x) - (\beta - 1)F(\alpha ; \beta - 1 ; x) = (\alpha - \beta + 1)F(\alpha ; \beta ; x)$.

12. Prove the following relations :

(*i*) $F(\alpha, \beta ; \gamma ; x) - F(\alpha, \beta ; \gamma - 1 ; x) = -\frac{\alpha\beta x}{\gamma(\gamma - 1)} F(\alpha + 1, \beta + 1 ; \gamma + 1 ; x)$

(*ii*) $F(\alpha + 1; \gamma ; x) - F(\alpha ; \gamma ; x) = \left(x/\gamma\right) \times F(\alpha + 1, \gamma + 1 ; x)$.

13. Hypergeometric function ${}_2F_1(\alpha, \beta ; \gamma ; x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha + 1).\beta(\beta + 1)}{1 \cdot 2 \cdot \gamma(\gamma + 1)} x^2 + \ldots$ is the solution of the differential equation $x(1 - x)y'' + [\gamma - (\alpha + \beta + 1)x]y - \alpha\beta y = 0$. Show that

(*a*) $\left[\frac{d}{dx} {}_2F_1(\alpha, \beta ; \gamma ; x)\right]_{x = 0} = \frac{\alpha\beta}{\gamma}$. (*b*) ${}_2F_1\left(\alpha, \beta ; \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2} ; \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\beta\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\beta\right)}$.

(*c*) ${}_2F_1\left(\alpha, \beta ; \gamma ; \frac{1}{2}\right) = 2^{\alpha}\, {}_2F_1(\alpha, \gamma - \beta ; \gamma ; -1)$.

14. Prove that (*a*) $P_n(\cos \theta) = {}_2F_1(-n ; n + 1 ; 1 ; \sin^2 \theta/2)$.
(*b*) $P_n(\cos \theta) = (-1)^n\, {}_2F_1(n + 1, -n ; 1 ; \cos^2 \theta/2)$.

(*c*) $P_n(x) = \left(\frac{x - 1}{2}\right)^n {}_2F_1\left(-n, -n ; 1 ; \frac{x + 1}{x - 1}\right)$. **[Purvanchal 2007]**

15. Show that

(*a*) $H_{2n}(x) = (-1)^n \frac{(2n!)}{n!} {}_1F_1\left(-n ; \frac{1}{2} ; x^2\right)$. (*b*) $H_{2n + 1}(x) = (-1)^n \frac{2(2n + 1)!}{n!} x\ {}_1F_1\left(-n ; \frac{3}{2} ; x^2\right)$.

16. Show that $L_n(x) = n!\ {}_1F_1(-n ; 1 ; x)$.

17. Prove that, for $|x| < a$, ${}_2F_1\,(a, b, c, x) = (1 - x)^{c - a - b}\, {}_2F_1\,(c - a, c - b, c_1, x)$. **[Lucknow 2010]**

18. State the confluent hyper geometric equation and explain its solution. **[Lucknow 2010]**

15

Orthogonal Sets of Functions And Strum Liouville Problem

15.1. Orthogonality. Two functions $f(x)$ and $g(x)$ defined on some interval $a \le x \le b$ are said to be *orthogonal* on $a \le x \le b$ if $\int_a^b f(x)\, g(x)\, dx = 0.$

The norm $\| f(x) \|$ of $f(x)$ is defined by $\| f(x) \| = \left\{ \int_a^b f^2(x)\, dx \right\}^{1/2}$

15.2. Orthogonal set of functions. Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be an *orthogonal set of functions on the interval* $a \le x \le b$ *if* $\int_a^b f_m(x)\, f_n(x)\, dx = 0$, whenever $m \ne n$..

15.3. Orthonormal set of functions. Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be *orthonormal* on $a \le x \le b$ if they are orthogonal on $a \le x \le b$ and all have norm 1. Thus, set $\{f_n(x)\}$ is orthonormal on $a \le x \le b$, if $\int_a^b f_m(x)\, f_n(x)\, dx = \begin{cases} 0, & \text{when } m \ne n \\ 1, & \text{when } m = n \end{cases}$ *i.e.,* $\int_a^b f_m(x)\, f_n(x)\, dx = \delta_{mn}$,

where δ_{mn} = Kronecker delta $= \begin{cases} 0, & \text{when } m \ne n \\ 1, & \text{when } m = n \end{cases}$

15.4. Orthogonality with respect to a weight function. Let $p(x) > 0$. Then two functions $f(x)$ and $g(x)$ defined on some interval $a \le x \le b$ are said to be *orthogonal* on $a \le x \le b$ with respect to weight function $p(x)$, if $\int_a^b p(x)\, f(x)\, g(x)\, dx = 0$.

Then, the norm $\| f(x) \|$ of $f(x)$ is defined by $\| f(x) \| = \left\{ \int_a^b p(x)\, f^2(x)\, dx \right\}^{1/2}$

15.5. Orthogonal set of functions with respect to a weight function. Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be orthogonal on $a \le x \le b$ with respect to weight function $p(x) > 0$, if

$$\int_a^b p(x)\, f_m(x)\, f_n(x)\, dx = 0, \text{ whenever } m \ne n.$$

15.6. Orthonormal set of functions with respect to a weight function. Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be orthonormal with respect to a weight function $p(x) > 0$ if they are orthogonal with respect to weight function $p(x)$ on $a \le x \le b$ and all have norm 1. Thus, set $\{f_n(x)\}$ is orthonormal with respect to weight function $p(x)$, if

$$\int_a^b p(x)\, f_m(x)\, f_n(x)\, dx = \delta_{mn} = \begin{cases} 0, & \text{when } m \neq n \\ 1, & \text{when } m = n \end{cases}$$

Remark 1. Terms defined in Art. 15.1, 15.2 and 15.3 are particular cases of Art. 15.4, 15.5 and 15.6 respectively for $p(x) = 1$.

15.7. Working rule for getting orthonormal set $\{f_n(x)\}$ of functions corresponding to a known orthogonal set $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$, where none of the functions $f_n(x)$ have zero norm. Divide each function $f_n(x)$ by its norm $\| f_n(x) \|$ and get a new function $\phi_n(x) = f_n(x)/\| f_n(x) \|$. Then, we have

$$\| \phi_n(x) \| = \left\{ \int_a^b \phi_n^2(x)\, dx \right\}^{1/2} = \left[\int_a^b \left\{ \frac{f_n(x)}{\| f_n(x) \|} \right\}^2 dx \right]^{1/2} = \frac{1}{\| f_n(x) \|} \left\{ \int_a^b f_n^2(x)\, dx \right\}^{1/2} = \frac{1}{\| f_n(x) \|} \cdot \| f_n(x) \| = 1,$$

showing that norm of $\phi_n(x) = 1$. Hence the set $\{\phi_n(x)\}$ *i.e.,* $\{f_n(x)/\| f_n(x) \|\}$ is an orthonormal set of functions.

15.8. Gram–Schmidt process of Orthonormalization. **[Kanpur 2010]**

Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of a linearly independent functions for each of which norm $\| f_n(x) \|$ exists and is non–zero. Then we wish to obtain an orthonormal set $\{\phi_n(x)\}$, where $n = 1, 2, 3, \ldots$ such that

$$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = \begin{cases} 0, & \text{when } m \neq n \\ 1, & \text{when } m = n \end{cases} \qquad \ldots(1)$$

We select $f_1(x)$ and obtain $\phi_1(x) = f_1(x)/\| f_1(x) \|$ ...(2)

We next choose $f_2(x)$ and let $F_2(x) = f_2 + c\, \phi_1$, ...(3)

where c is chosen in such a manner so that F_2 may be orthogonal to ϕ_1,

i.e., $\int_a^b F_2\, \phi_1\, dx = 0$ or $\int_a^b (f_2 + c\, \phi_1)\, \phi_1\, dx = 0$, using (3)

or $\int_a^b f_2 \phi_1\, dx + c \int_a^b \phi_1^2 dx = 0$ or $\int_a^b f_2 \phi_1\, dx + c = 0$, by (1)

$\therefore\ c = -\int_a^b f_2 \phi_1\, dx$. With this value of c, (3) gives $F_2 = f_2 - \phi_1 \int_a^b f_2 \phi_1\, dx$ ...(4)

We now take $\phi_2(x) = F_2 / \| F_2 \|$ ...(5)

Now choose f_3 and let $F_3(x) = f_3 + c_1 \phi_1 + c_2 \phi_2$, ...(6)

where c_1 and c_2 are chosen in such a manner so that F_2 may be orthogonal to ϕ_1 and ϕ_2, *i.e.,*

$\int_a^b (f_3 + c_1 \phi_1 + c_2 \phi_2) \phi_1\, dx$ and $\int_a^b (f_3 + c_1 \phi_1 + c_2 \phi_2) \phi_2\, dx = 0$

or $\int_a^b f_3 \phi_1\, dx + c_1 = 0$ and $\int_a^b f_3\, \phi_2\, dx + c_2 = 0$, using (1)

$\therefore\quad c_1 = -\int_a^b f_3 \phi_1\, dx$ and $c_2 = -\int_a^b f_3 \phi_2\, dx$ ...(7)

Using (7), (6) gives $F_3 = f_3 - \phi_1 \int_a^b f_3 \phi_1\, dx - \phi_2 \int_a^b f_3 \phi_2\, dx$ ...(8)

So we take $\phi_3(x) = F_3 / \| F_3 \|$ ...(9)

By continuing the above process, the nth normalized function ϕ_n is given by

$$\phi_n = F_n / \| F_n \|, \qquad \ldots(10)$$

where $F_n = f_n - \phi_1 \int_a^b f_n \phi_1\, dx - \phi_2 \int_a^b f_n \phi_2\, dx - \ldots - \phi_{n-1} \int_a^b f_n\, \phi_{n-1}\, dx$ ...(11)

Above process will fail if and only if at some stage $F_r = 0$ for $1 \le r \le n$.

But, $F_r = 0 \Rightarrow f_r - \phi_1 \int_a^b f_r \phi_1 \, dx - \phi_2 \int_a^b f_r \phi_2 \, dx - \ldots - \phi_{r-1} \int_a^b f_r \phi_{r-1} \, dx = 0$

$\Rightarrow$ f_r is a linear combination of $\phi_1, \phi_2, \ldots, \phi_{r-1} \Rightarrow$ f_r is a linear combination of $f_1, f_2, \ldots, f_{r-1}$ (by our construction of new functions $\phi_1, \phi_2, \ldots, \phi_{r-1}$)

$\Rightarrow$ $f_1, f_2, \ldots, f_r$ is a linearly dependent set for $1 \le r \le n$.

But this a contradiction because $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$, is linearly independent set and its every subset would also be linearly independent. Hence $F_r \ne 0$ for $1 \le r \le n$ and so we would always get a set $\{\phi_n(x)\}$, where $n = 1, 2, 3, \ldots$ which would be orthonormal.

Remark. Sometimes orthonormalization is required with respect to a weight function $p(x) > 0$. Then (1) takes the form $\int_a^b p(x)\, \phi_m(x)\, \phi_n(x)\, dx = \begin{cases} 0, & \text{when } m \ne n \\ 1, & \text{when } m = n \end{cases}$

Also, $\int_a^b F_2 \phi_1 \, dx = 0$ would take the form $\int_a^b p(x)\, F_2 \phi_2 \, dx = 0$ etc. Rest of the procedure is similar.

15.9. Illustrative Solved Examples

Ex. 1. *Show that the set of functions {sin (nπx/c)}, n = 1, 2, 3, ... is orthogonal on the interval (0, c) and find the corresponding orthonormal set.*

Sol. Here the given functions are $f_n(x) = \sin(n\pi x/c)$, $n = 1, 2, 3, \ldots$ For $m \ne n$, we have

$$\int_0^c f_m(x)\, f_n(x)\, dx = \int_0^c \sin\frac{m\pi x}{c} \sin\frac{n\pi x}{c}\, dx = \frac{1}{2}\int_0^c \left\{\cos\frac{(m-n)\pi x}{c} - \cos\frac{(m+n)\pi x}{c}\right\} dx$$

$$= \frac{1}{2}\left[\frac{c}{\pi(m-n)}\sin\frac{(m-n)\pi x}{c} - \frac{c}{\pi(m+n)}\sin\frac{(m+n)\pi x}{c}\right]_0^c = 0,$$

showing that the given set of functions is orthogonal.

$$\text{Norm of } f_n(x) = \| f_n(x) \| = \left\{\int_0^c \sin^2(n\pi x/c)\, dx\right\}^{1/2} = \left[\frac{1}{2}\int_0^c \{1 - \cos(2n\pi x/c)\}\right]^{1/2}$$

$$= \left[\{x/2 - (c/4n\pi)\sin(2n\pi x/c)\}^{1/2}\right]_0^c = (c/2)^{1/2}.$$

Let $\phi_n(x) = f_n(x)/\| f_n(x) \| = (2/c)^{1/2} \sin(n\pi x/c)$ Hence the required orthonormal set is given by $\{\phi_n(x)\}$ *i.e.,* $\{(2/c)^{1/2} \sin(n\pi x/c)\}$ where $n = 1, 2, 3, \ldots$

Ex. 2.(*a*) *Show that the set of functions {cos nx}, n = 0, 1, 2, 3, ... is orthogonal on the interval $-\pi \le x \le \pi$, and find the corresponding orthonormal set of functions.*

(*b*) *Show that the set of functions {cos nx}, n = 0, 1, 2, 3, ... is an orthogonal set of functions on $0 \le x \le \pi$, and find the corresponding orthonormal set of functions.*

Sol. (*a*) Here the given functions are $f_n(x) = \cos nx$, $n = 0, 1, 2, \ldots$ For $m \ne n$, we have

$$\int_{-\pi}^{\pi} f_m(x)\, f_n(x)\, dx = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 2\int_0^{\pi} \cos mx \cos nx\, dx$$

$$= \int_0^{\pi} \{\cos(m+n)x + \cos(m-n)x\}\, dx = \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{\pi} = 0,$$

showing that the given set of functions is orthogonal.

Norm of $f_n(x)$, *i.e.*, $\| f_n(x) \|$, for $n \neq 0$

$$= \left\{\int_{-\pi}^{\pi} f_n^2(x)\,dx\right\}^{1/2} = \left\{\int_{-\pi}^{\pi} \cos^2 nx\,dx\right\}^{1/2} = \left\{2\int_0^{\pi} \cos^2 nx\,dx\right\}^{1/2} = \left\{\int_0^{\pi} (1+\cos 2nx)\,dx\right\}^{1/2} = \left[\{x+(1/2n)\sin 2nx\}^{1/2}\right]_0^{\pi} = \sqrt{\pi}$$

Again, for $n = 0$, $\quad \| f_n(x) \| = \| 1 \| = \left[\int_{-\pi}^{\pi} (1)^2\,dx\right]^{1/2} = \sqrt{2\pi}$

Thus, $\quad \phi_n(x) = \dfrac{f_n(x)}{\| f_n(x) \|} = \begin{cases} 1/\sqrt{2\pi}, \text{ if } n = 0 \\ (\cos nx)/\sqrt{\pi}\ , \text{ if } n \neq 0 \end{cases}$

Hence, the required orthonormal set is $1/(2\pi)^{1/2}, (\cos x)/\sqrt{\pi}, (\cos 2x)/\sqrt{\pi}, (\cos 3x)/\sqrt{\pi}, \ldots$

(*b*) Proceed as in part (*a*), **Ans.** $1/\sqrt{\pi}, (2/\pi)^{1/2}\cos x, (2/\pi)^{1/2}\cos 2x, (2/\pi)^{1/2}\cos 3x, \ldots$

Ex. 3. (*a*) *Show that the functions sin x, sin 2x, sin 3x, ..., 1, cos x, cos 2x, cos 3x, ... constitute an orthogonal set on the interval (–π, π). Normalize the set.* **[Lucknow 2010]**

Sol. For $m \neq n$, we have the following results :

(*i*) $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 2\int_0^{\pi} \sin mx \sin nx\,dx$, as $\sin mx \sin nx$ is an even function

$$= \int_0^{\pi} \{\cos(m-n)x - \cos(m+n)x\}\,dx = \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi} = 0$$

(*ii*) $\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$, as $\sin mx \cos nx$ is an odd function of x.

(*iii*) $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 2\int_0^{\pi} \cos mx \cos nx\,dx$, as $\cos mx \cos nx$ is an even function of x

$$= \int_0^{\pi} \{\cos(m+n)x + \cos(m-n)x\}\,dx = \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{\pi} = 0$$

(*iv*) $\int_{-\pi}^{\pi} 1.\sin mx\,dx = 0$, as $\sin mx$ is an odd function.

(*v*) $\int_{-\pi}^{\pi} 1.\cos nx\,dx = 2\int_0^{\pi} \cos nx\,dx = (2/n)\times\left[\sin nx\right]_0^{\pi} = 0$

Relations (*i*), (*ii*), (*iii*), (*iv*) and (*v*) together show that the given set of functions is orthogonal.

Now, $\| \sin nx \| = \left\{\int_{-\pi}^{\pi} \sin^2 nx\,dx\right\}^{1/2} = \left\{2\int_0^{\pi} \sin^2 nx\,dx\right\}^{1/2} = \left\{\int_0^{\pi} (1-\cos 2nx)\,dx\right\}^{1/2} = \left\{\left[x - \frac{1}{2n}\sin 2nx\right]_0^{\pi}\right\}^{1/2} = \sqrt{\pi}$

$\| \cos nx \| = \left\{\int_{-\pi}^{\pi} \cos^2 nx\,dx\right\}^{1/2} = \left\{2\int_0^{\pi} \cos^2 nx\,dx\right\}^{1/2} = \left\{\int_0^{\pi} (1+\cos 2nx)\,dx\right\}^{1/2} = \left\{\left[x + \frac{1}{2n}\sin 2nx\right]_0^{\pi}\right\}^{1/2} = \sqrt{\pi}$

and $\quad \| 1 \| = \left\{\int_{-\pi}^{\pi} 1^2\,dx\right\}^{1/2} = \left\{\left[x\right]_{-\pi}^{\pi}\right\}^{1/2} = \sqrt{2\pi}$

Hence the required orthonormal set is

$$(\sin x)/\sqrt{\pi}, (\sin 2x)/\sqrt{\pi}, \ldots, 1/\sqrt{2\pi}, \quad (\cos x)/\sqrt{\pi}, (\cos 2x)/\sqrt{\pi}, \ldots.$$

Ex. 3. (*b*) *Show that the functions 1, cos (2nπx/T), sin (2nπx/T), n = 1, 2, 3, ... are orthogonal on the interval –T/2 ≤ x ≤ T/2, and find the corresponding orthonormal set.*

Sol. Do as in Ex. 3(*a*). **Ans.** $\left\{1/\sqrt{T}, (2/T)^{1/2}\cos(2n\pi x/T), (2/T)^{1/2}\sin(2n\pi x/T)\right\}$, $n = 1, 2, 3, \ldots$

Ex. 4. (*a*) *Show that the functions $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on the interval (–1, 1) and determine the constants A and B so that the function $f_3(x) = 1 + Ax + Bx^2$ is orthogonal to both f_1 and f_2 on the interval (–1, 1).* **[Meerut 2007; Kanpur 2011]**

(*b*) *Show that the functions $f_1(x) = 4$ and $f_2(x) = x^3$ are orthogonal on the interval (–2, 2) and determine constants A and B so that the function $f_3(x) = 1 + Ax + Bx^2$ is orthogonal to both f_1 and f_2.*

Sol. (*a*) Here $$\int_{-1}^{1} f_1(x)\, f_2(x)\, dx = \int_{-1}^{1} x\, dx = \left[x^2/2 \right]_{-1}^{1} = 0,$$

showing that $f_1(x)$ and $f_2(x)$ are orthgonal on the interval (–1, 1).

If $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ on the interval (–1, 1), then by definition,

$$\int_{-1}^{1} f_1(x)\, f_3(x)\, dx = 0 \quad \text{and} \quad \int_{-1}^{1} f_2(x)\, f_3(x)\, dx = 0$$

or $$\int_{-1}^{1} (1 + Ax + Bx^2)\, dx = 0 \quad \text{and} \quad \int_{-1}^{1} x(1 + Ax + Bx^2)\, dx = 0$$

or $$\left[x + \frac{A}{2}x^2 + \frac{B}{3}x^3 \right]_{-1}^{1} = 0 \quad \text{and} \quad \left[\frac{1}{2}x^2 + \frac{1}{3}Ax^3 + \frac{1}{4}Bx^4 \right]_{-1}^{1} = 0$$

or $2 + (2/3) \times B = 0$ and $(2/3) \times A = 0$ so that $A = 0$ and $B = -3$.

So f_3 is orthogonal to both f_1 and f_2 if $A = 0$ and $B = -3$.

(*b*) Proceed as in part (*a*). **Ans.** $A = 0$ and $B = -3/4$.

Ex. 5. *Given that $f_1(x) = a_0$, $f_2(x) = b_0 + b_1 x$ and $f_3(x) = c_0 + c_1 x + c_2 x^2$. Determine the constants a_0, b_0, c_0, b_1, c_1 and c_2 so that the given functions form an orthonormal set on the interval $-1 \le x \le 1$.*

Sol. Since f_1, f_2 and f_3 form an orthonormal set on $-1 \le x \le 1$, we have

$$(i)\ \int_{-1}^{1} f_1 f_2\, dx = 0 \qquad (ii)\ \int_{-1}^{1} f_2 f_3\, dx = 0 \qquad (iii)\ \int_{-1}^{1} f_3 f_1\, dx = 0,$$

$$(iv)\ \| f_1 \| \neq 0 \qquad (v)\ \| f_2 \| \neq 0 \qquad (vi)\ \| f_3 \| \neq 0.$$

Now, $$(i) \Rightarrow \int_{-1}^{1} a_0(b_0 + b_1 x)\, dx = 0 \ \Rightarrow\ \left[a_0 b_0 x + \frac{1}{2} a_0 b_1 x^2 \right]_{-1}^{1} = 0 \ \Rightarrow\ a_0\, b_0 = 1 \quad ...(1)$$

$$(ii) \Rightarrow \int_{-1}^{1} (b_0 + b_1 x)(c_0 + c_1 x + c_2 x^2)\, dx = 0$$

or $$\left[b_0 c_0 x + \frac{1}{2} b_0 c_1 x^2 + \frac{1}{3} b_0 c_2 x^3 + \frac{1}{2} b_1 c_0 x^2 + \frac{1}{3} b_1 c_1 x^3 + \frac{1}{4} b_1 c_2 x^4 \right]_{-1}^{1} = 0$$

or $2b_0 c_0 + (2/3) \times b_0 c_2 + (2/3) \times b_1 c_1 = 0$ or $b_0(3c_0 + c_2) + b_1 c_1 = 0$ ...(2)

$$(iii) \Rightarrow \int_{-1}^{1} a_0(c_0 + c_1 x + c_2 x^2)\, dx = 0 \Rightarrow \left[a_0 c_0 x + \frac{1}{2} a_0 c_1 x^2 + \frac{1}{3} a_0 c_2 x^3 \right]_{-1}^{1} = 0$$

or $2a_0 c_0 + (2/3) \times a_0 c_2 = 0$ or $a_0(3c_0 + c_2) = 0$ ...(3)

$$(iv)\ \Rightarrow \left[\int_{-1}^{1} a_0^2\, dx \right]^{1/2} \neq 0 \ \Rightarrow\ (2a_0^2)^{1/2} \neq 0 \ \Rightarrow\ a_0 \neq 0 \quad ...(4)$$

Now, (1) and (4) give $b_0 = 0.$ ...(5)

Again, since $b_0 \neq 0$, so by (2), $b_1 c_1 = 0$ ...(6)

$\therefore$ $(v) \Rightarrow \left[\int_{-1}^{1} (b_0 + b_1 x)^2 dx\right]^{1/2} \neq 0 \Rightarrow \left[\int_{-1}^{1} b_1^2 x^2 dx\right]^{1/2} \neq 0$, using (5)

Thus, $[(2/3)\times b_1^2]^{1/2} \neq 0$ so that $b_1 \neq 0$ ...(7)

$\therefore$ from (6), we get $c_1 = 0$, as $b_1 \neq 0$ ...(8)

Finally, $(vi) \Rightarrow \left[\int_{-1}^{1} (c_0 + c_1 x + c_1 x^2)^2 dx\right]^{1/2} \neq 0 \Rightarrow \left[\int_{-1}^{1} (c_0 + c_2 x^2)^2 dx\right]^{1/2} \neq 0$,

$\therefore$ $\left\{\left[c_0^2 x + (2/3)\times c_0 c_2 x^3 + (1/5)\times c_2^2 x^5\right]_{-1}^{1}\right\}^{1/2} \neq 0$

or $\{2c_0{}^2 + (4/3)\times c_0 c_2 + (2/5)\times c_2{}^2\}^{1/2} \neq 0$ or $\{c_0{}^2 + (2/3)\times c_0 c_2 + (1/5)\times c_2{}^2\}^{1/2} \neq 0$ ...(9)

From (3) and (5), we get $3c_0 + c_2 = 0$ or $c_2 = -3c_0$ ...(10)

Using (10), (9) can be re–written as

$\{c_0{}^2 - 2c_0{}^2 + (9/5)\times c_0{}^2\}^{1/2} \neq 0$ so that $c_0 \neq 0$ ...(11)

From (4), (5), (7), (8), (10) and (11), we find that $b_0 = 0$, $c_1 = 0$, $c_2 = -3c_0$ and a_0, c_0 can take arbitrary real values.

Ex. 6. *Show that the functions $1 - x$, $1 - 2x + x^2/2$ and $1 - 3x + 3x^2/2 - x^3/6$ are orthogonal with respect to e^{-x} on $0 \le x < \infty$. Determine the corresponding orthonormal functions.*

Sol. Let $f_1(x) = 1 - x$, $f_2(x) = 1 - 2x + x^2/2$ and $f_3(x) = 1 - 3x + 3x^2/2 - x^3/6$

Here the given weight function $= p(x) = e^{-x}$.

Now, $\int_0^\infty p(x) f_1 f_2\, dx = \int_0^\infty e^{-x}(1-x)(1-2x+x^2/2)\, dx = \int_0^\infty e^{-x}(1-3x+5x^2/2-x^3/2)\, dx$

$$= \int_0^\infty e^{-x} x^0\, dx - 3\int_0^\infty e^{-x} x\, dx + \frac{5}{2}\int_0^\infty e^{-x} x^2\, dx - \frac{1}{2}\int_0^\infty e^{-x} x^3\, dx \qquad ...(1)$$

To evaluate integrals involved in (1), we make use of the following result of Gamma function,

$$\int_0^\infty e^{-x} x^n\, dx = n!, \text{ if } n \text{ is a non–negative integer} \qquad ...(2)$$

Using (2), (1) $\Rightarrow$ $\int_0^\infty p(x) f_1 f_2\, dx = 0! - 3\cdot 1! + (5/2)\cdot 2! - (1/2)\cdot 3! = 1 - 3 + 5 - 3 = 0$ ...(3)

Next, $\int_0^\infty p(x) f_2 f_3\, dx = \int_0^\infty e^{-x}(1-2x+x^2/2)(1-3x+3x^2/2-x^3/6)\, dx$

$= \int_0^\infty e^{-x}(1-5x+8x^2-14x^3/3+13x^4/12-x^5/12)\, dx = 0! - 5\cdot 1! + 8\cdot 2! - (14/3)\cdot 3! + (13/12)\cdot 4! - (1/12)\cdot 5!$, using (2)

Thus, $\int_0^\infty p(x) f_2 f_3\, dx = 1 - 5 + 16 - 28 + 26 - 10 = 0$ ...(4)

Again, $\int_0^\infty p(x) f_3 f_1\, dx = \int_0^\infty e^{-x}(1-3x+3x^2/2-x^3/6)(1-x)\, dx$

$$= \int_0^\infty e^{-x}(1-4x+9x^2/2-5x^3/3+x^4/6)\,dx = 0! - 4\cdot 1! + (9/2)\cdot 2! - (5/3)\cdot 3! + (1/6)\cdot 4!, \text{ using (2)}$$

Thus, $$\int_0^\infty p(x)\, f_1 f_3\, dx = 1 - 4 + 9 - 10 + 4 = 0 \qquad ...(5)$$

From (3), (4) and (5) we find that f_1, f_2, f_3 are orthogonal with respect to weight function $p(x)$ on $0 \le x < \infty$. Also, we have

$$\| f_1 \| = \left[\int_0^\infty e^{-x}(1-x)^2 dx\right]^{1/2} = \left[\int_0^\infty e^{-x}(1-2x+x^2)\,dx\right]^{1/2} = [0! - 2\cdot 1! + 2!]^{1/2} = 1$$

$$\| f_2 \| = \left[\int_0^\infty e^{-x}(1-2x+x^2/2)^2\,dx\right]^{1/2} = \left[\int_0^\infty e^{-x}(1-4x+5x^2-2x^3+x^4/4)\,dx\right]^{1/2}$$

$$= [0! - 4\cdot 1! + 5\cdot 2! - 2\cdot 3! + (1/4)\cdot 4!]^{1/2} = (1 - 4 + 10 - 12 + 6)^{1/2} = 1$$

$$\| f_3 \| = \left[\int_0^\infty e^{-x}(1-3x+3x^2/2-x^3/6)^2 dx\right]^{1/2}$$

$$= \left[\int_0^\infty e^{-x}(1-6x+12x^2-28x^3/3+13x^4/4-x^5/2+x^6/36)\,dx\right]^{1/2}$$

$$= [0! - 6\cdot 1! + 12\cdot 2! - (28/3)\cdot 3! + (13/4)\cdot 4! - (1/2)\cdot 5! + (1/36)\cdot 6!]^{1/2} = (1-6+24-56+78-60+20)^{1/2} = 1$$

Since norm of each of the functions f_1, f_2 and f_3 is unity, it follows that the given set of functions is an orthonormal set.

Ex. 7. *With help of 1, x, x^2 construct three functions ϕ_0, ϕ_1 and ϕ_2 which are orthogonal over $-1 \le x \le 1$.*

Sol. We take $$\phi_0(x) = 1 \qquad ...(1)$$

Next, choose $$\phi_1 = x + c \quad \phi_0(x) = x + c \qquad ...(2)$$

Let ϕ_1 be orthogonal to ϕ_0 so that

$$\int_{-1}^{1} \phi_0\phi_1\, dx = 0 \quad \text{or} \quad \int_{-1}^{1} (x+c)\,dx = 0 \quad \text{or} \quad \left[\frac{1}{2}x^2 + cx\right]_{-1}^{1} = 0$$

giving $c = 0$. Hence (2) gives $$\phi_1 = x. \qquad ...(3)$$

Next, we take $$\phi_2 = x^2 + c_1\phi_0 + c_2\phi_1 = x^2 + c_1 + c_2 x \qquad ...(4)$$

Let ϕ_2 be orthogonal to both ϕ_0 and ϕ_1 so that

$$\int_{-1}^{1} \phi_2\phi_0\, dx = 0 \quad \text{and} \quad \int_{-1}^{1} \phi_2\phi_1\, dx = 0$$

i.e., $$\int_{-1}^{1} (x^2 + c_1 + c_2 x)\,dx = 0 \quad \text{and} \quad \int_{-1}^{1} (x^2 + c_1 + c_2 x)x\,dx = 0$$

i.e., $$\left[(1/3)\times x^3 + c_1 x + (1/2)\times c_2 x^2\right]_{-1}^{1} = 0 \text{ and } \left[(1/4)\times x^4 + (1/2)\times c_1 x^2 + (1/3)\times c_2 x^3\right]_{-1}^{1} = 0$$

i.e., $(2/3) + 2c_1 = 0$ and $c_2 \times (2/3) = 0$ so that $c_1 = -1/3$, $c_2 = 0$

$\therefore$ (4) becomes $\phi_2(x) = x^2 - (1/3)$.

So the required functions are $\phi_0 = 1$, $\phi_1 = x$, $\phi_2 = x^2 - (1/3)$.

Ex. 8. *With the help of 1, x, x^2 construct three functions ϕ_0, ϕ_1 and ϕ_2 which are orthogonal with respect to e^{-x} over $0 \le x < \infty$.*

Sol. We take $\phi_0(x) = 1$ ...(1)

Next, choose $\phi_1 = x + c\ \phi_0(x) = x + c$ ...(2)

Let ϕ_1 be orthogonal to ϕ_0 with respect to e^{-x}. Then, we have

$$\int_0^\infty e^{-x}(x+c)\,dx = 0 \quad \text{or} \quad \int_0^\infty e^{-x}x\,dx + c\int_0^\infty e^{-x}dx = 0 \qquad ...(3)$$

We know that $\int_0^\infty e^{-x}x^n\,dx = n!$, where $n = 0, 1, 2, 3, ...$ ...(4)

Using (4), (3) gives $1! + c \times 0! = 0$ or $1 + c = 0$ or $c = -1$.

Hence (2) reduces to $\phi_1 = x - 1$ ...(5)

Finally, take $\phi_2 = x^2 + c_1\phi_0 + c_2\phi_1 = x^2 + c_1 + c_2(x - 1)$ ...(6)

Let ϕ_2 be orthogonal to ϕ_0 and ϕ_1 with respect to e^{-x}. Then, we have

$$\int_0^\infty e^{-x}\phi_2\phi_0\,dx = 0 \quad i.e., \quad \int_0^\infty e^{-x}(x^2 + c_2x + c_1 - c_2)\,dx = 0 \qquad ...(7)$$

and $$\int_0^\infty e^{-x}\phi_2\phi_1\,dx = 0 \quad i.e., \quad \int_0^\infty e^{-x}(x^2 + c_2x + c_1 - c_2)(x-1)\,dx = 0 \qquad ...(8)$$

Using (4), (7) gives $2! + c_2 \times 1! + (c_1 - c_2) \times 0! = 0$

or $2 + c_2 + c_1 - c_2 = 0$ so that $c_1 = -2$ ...(9)

Now, by (8), $$\int_0^\infty e^{-x}\left\{x^3 + (c_2 - 1)x^2 + (c_1 - 2c_2)x + (c_2 - c_1)\right\}dx = 0$$

or $3! + (c_2 - 1) \times 2! + (c_1 - 2c_2) \times 1! + (c_2 - c_1) \times 0! = 0$, using (4)

or $6 + 2(c_2 - 1) + c_1 - 2c_2 + c_2 - c_1 = 0$ so that $c_2 = -4$.

Since $c_1 = -2$, and $c_2 = -4$, so (6) gives $\phi_2 = x^2 - 4x + 2$.

So required functions are $\phi_0(x) = 1$, $\phi_1(x) = x - 1$ and $\phi_2(x) = x^2 - 4x + 2$.

Ex. 9. *Given the set of functions 1, x, x^2, x^3, ...,. Obtain from these a set of functions which are mutually orthonormal in (–1, 1).* **[Kanpur 2009]**

Sol. Let $\{\phi_n(x)\}$ be the required orthonormal set of function so that

$$\int_{-1}^{1}\phi_n\phi_m\,dx = \begin{cases}0, & \text{if } m \ne n\\ 1, & \text{if } m = n\end{cases} \qquad ...(1)$$

Step 1. Choose $f_1(x) = 1$ and take $\phi_1(x) = f_1(x)/\| f_1(x) \|$.

Now, $\| f_1(x) \| = \left[\int_{-1}^{1} f_1^2(x)\,dx\right]^{1/2} = \left[\int_{-1}^{1} 1^2\,dx\right]^{1/2} = \sqrt{2}$. So $\phi_1(x) = \dfrac{1}{\sqrt{2}}$ ...(2)

Step 2. Choose $f_2(x) = x$ and take a function $g_2(x) = f_2(x) + c\ \phi_1(x) = x + (c/\sqrt{2})$ ...(3)

Let $g_2(x)$ and $\phi_1(x)$ be orthogonal on the interval (–1, 1)

$$\therefore \int_{-1}^{1} g_2(x)\,\phi_1(x)\,dx = 0 \quad \text{or} \quad \int_{-1}^{1}(x + c/\sqrt{2})\cdot(1/\sqrt{2})\,dx = 0$$

or $\left[x^2/2 + cx/\sqrt{2}\right]_{-1}^{1} = 0$ so that $c = 0$

Hence, by (3), $g_2(x) = x$

Now, $$\| g_2(x) \| = \left[\int_{-1}^{1} g_2^2(x)\,dx\right]^{1/2} = \left[\int_{-1}^{1} x^2\,dx\right]^{1/2} = \left(\frac{2}{3}\right)^{1/2}$$

$\therefore$ $$\phi_2(x) = g_2(x)/\| g_2(x) \| = (3/2)^{1/2}\cdot x \qquad ...(4)$$

Step 3. Choose $f_3(x) = x^2$ and take a function

$$g_3(x) = f_3(x) + c_1\,\phi_1(x) + c_2\,\phi_2(x) = x^2 + c_1\,\phi_1 + c_2\,\phi_2 \qquad ...(5)$$

Let $g_3(x)$ be orthgonal to $\phi_1(x)$ and $\phi_2(x)$ so that

(i) $\int_{-1}^{1} g_3(x)\phi_1(x)\,dx = 0$ $\qquad$ (ii) $\int_{-1}^{1} g_3(x)\phi_2(x)\,dx = 0$

$(i) \Rightarrow$ $\int_{-1}^{1} (x^2 + c_1\phi_1 + c_2\phi_2)\,\phi_1\,dx = 0$ or $\int_{-1}^{1} x^2\phi_1\,dx + c_1\int_{-1}^{1}\phi_1^2\,dx + c_2\int_{-1}^{1}\phi_2\phi_1\,dx = 0$

or $$\int_{-1}^{1} x^2\cdot(1/\sqrt{2})\,dx + c_1\cdot 1 + c_2\cdot 0 = 0\text{, using (1) and (2)}$$

or $$(2/3)\times(1/\sqrt{2}) + c_1 = 0 \quad \text{or} \quad c_1 = -\left(\sqrt{2}/3\right) \qquad ...(6)$$

Next, from (ii), $\int_{-1}^{1} (x^2 + c_1\phi_1 + c_2\phi_2)\,\phi_2\,dx = 0$ or $\int_{-1}^{1} x^2\phi_2\,dx + c_1\int_{-1}^{1}\phi_1\phi_2\,dx + c_2\int_{-1}^{1}\phi_2^2\,dx = 0$

or $$\int_{-1}^{1} x^2\times(3/2)^{1/2}\times x\,dx + 0 + c_2\times 1 = 0\text{, using (1) and (4)}$$

or $(3/2)^{1/2}\left[x^4/4\right]_{-1}^{1} + c_2 = 0$ or $c_2 = 0$

$\therefore$ (5) gives $$g_3(x) = x^2 - (\sqrt{2}/3)\times(1/\sqrt{2}) = (3x^2 - 1)/3$$

Now, $$\| g_3(x) \| = \left[\int_{-1}^{1} g_3^2(x)\,dx\right]^{1/2} = \left[\frac{1}{9}\int_{-1}^{1}(3x^2-1)^2\,dx\right]^{1/2}$$

$$= \left[\frac{1}{9}\int_{-1}^{1}(9x^4 - 6x^2 + 1)\,dx\right]^{1/2} = \frac{1}{3}\left\{\left[\frac{9}{5}x^5 - 2x^3 + x\right]_{-1}^{1}\right\}^{1/2} = \frac{2\sqrt{2}}{3\sqrt{5}}$$

$\therefore$ $$\phi_3(x) = g_3(x)/\| g_3(x) \| = (1/2)\times(5/2)^{1/2}\times(3x^2 - 1) \qquad ...(7)$$

Proceeding like wise, we obtain

$$\phi_4(x) = \left(\frac{7}{2}\right)^{1/2}\cdot\left(\frac{5x^3 - 3x}{2}\right), \qquad \phi_5(x) = \left(\frac{9}{2}\right)^{1/2}\cdot\left(\frac{5x^4 - 30x^2 + 3}{8}\right) \qquad ...(8)$$

and so on. The required orthonormal set of functions $\{\phi_n(x)\}$ is given by (2), (4), (7), (8) and so on.

EXERCISE 15(A)

1. Show that the functions $\sin x$, $\sin 2x$, $\sin 3x$, ... are orthogonal on the interval $(0, \pi)$.

2. Show that the functions 1, $\cos 2x$, $\cos 4x$, $\cos 6x$, ... are orthogonal on interval $0 \le x \le \pi$, and find the corresponding orthonormal set. **Ans.** $\{1/\sqrt{\pi}\,,\ (2/\pi)^{1/2}\cos 2nx\}$, $n = 1, 2, 3, ...$

3. Show that the functions $\sin \pi x$, $\sin 2\pi x$, $\sin 3\pi x$, ... form an orthogonal set on the interval $-1 \le x < 1$ and obtain the corresponding orthonormal set.

Ans. $\sin \pi x$, $\sin 2\pi x$, $\sin 3\pi x$, ... *i.e.,* the given set itself.

4. Show that the functions 1, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$, ... form an orthogonal set on $-2 \le x \le 2$ and find the corresponding orthonormal set.

Ans. $1/2, (\cos \pi x)/2, (\sin \pi x)/2, (\cos 2\pi x)/2, (\sin 2\pi x)/2, \ldots$

5. Show that each of the following set is orthogonal on the given interval and find the corresponding orthonormal sets :

(*i*) $1, \cos(2\pi x/c), \cos(4\pi x/c), \cos(6\pi x/c), \ldots, 0 \le x \le c$ **Ans.** $\{1/\sqrt{c}, (2/c)^{1/2} \cos(2n\pi x/c)\}$, $n = 1, 2, 3, \ldots$

(*ii*) $1, \cos 2x, \sin 2x, \cos 4x, \sin 4x, \ldots, -\pi/2 \le x \le \pi/2$ **Ans.** $1/\sqrt{\pi}, (2/\pi)^{1/2} \cos 2x, (2/\pi)^{1/2} \sin 2x, \ldots$

15.10. Strum–Liouville equation. Strum–Liouville problem. Eigen (or characteristic) functions and eigen (or characteristic) values.

[Meerut 2010; Ravishankar 1998; Himanchal 2010]

Definitions. A differential equation of the form $[r(x)\, y']' + [q(x) + \lambda\, p(x)]y = 0$ ... (1) is known as *Strum–Liouville equation.*

We assume that the functions p, q, r and r' in (1) are continuous in $a \le x \le b$ and $p(x) > 0$. Here λ is a parameter independent of x.

Equation (1) is considered on some interval $a \le x \le b$, satisfying boundary conditions at the two end points a and b,

$a_1 y(a) + a_2 y'(a) = 0$ and $b_1 y(b) + b_2 y'(b) = 0$ ...(2)

with the real constants a_1, a_2, b_1, b_2. Suppose that a_1, a_2 in (2) are not both zero and so are b_1, b_2. The boundary value problem consisting of (1) and (2) is called a *Strum–Liouville problem.*

Clearly $y = 0$ is always a solution of Strum–Liouville problem for any value of the parameter λ. $y = 0$ known as a *trivial solution* is of no practical use. The non–zero solutions of the Strum–Liouville problem given by (1) and (2) are called the *eigenfunctions* of the problem and the values of λ for which such solutions exist, are called *eigenvalues* of the problem.

Remark. A special case of (1) and (2). Let $p = r = 1$ and $q = 0$ in (1). Also, let $a_1 = b_1 = 1$ and $a_2 = b_2 = 0$ in (2). Then (1) and (2) reduce to $y'' + \lambda y = 0$ with $y(a) = 0$, $y(b) = 0$. This is the simplest form of Strum–Liouville problem.

15.11. Orthogonality of eigenfunctions

Theorem. *Suppose that the functions p(x), q(x), r(x) and r′(x) in the Strum–Liouville equation [r(x) y′]′ + {q(x) + λp(x)}y = 0 are real valued and continuous and p(x) > 0 on the interval a ≤ x ≤ b. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the Strum–Liouville problem (given by the above Strum Liouville equation and boundary conditions $a_1 y(a) + a_2 y'(a) = 0$ and $b_1 y(b) + b_2 y'(b) = 0$) that correspond to different eigenvalues λ_m and λ_n respectively. Then y_m, y_n are orthogonal on that interval with respect to the weight function p(x).*

Prove that eigenfunctions corresponding to different eigenvalues are orthogonal with respect to some weight function. **[Himanchal 2009]**

Proof. Consider the following Strum–Lioville problem :

$$[r(x)\, y']' + [q(x) + \lambda\, p(x)]y = 0 \quad \ldots(1)$$

$$a_1 y(a) + a_2 y'(a) = 0 \quad \ldots(2a)$$

$$b_1 y(b) + b_2 y'(b) = 0, \quad \ldots(2b)$$

where p, q, r and r' are real valued and continuous and $p(x) > 0$ on $a \le x \le b$. Let a_1, a_2 in (2a) be given constants, not both zero and so be b_1, b_2 in (2b).

Let y_m and y_n be eigenfunctions of the above Strum–Liouville problem that correspond to different eigenvalues λ_m and λ_n. Then, by definition of eigen functions, y_m and y_n both satisfy (1).

Hence $$(ry'_m)' + (q + \lambda_m p)\, y_m = 0 \quad ...(3)$$

and $$(ry'_n)' + (q + \lambda_n p)\, y_n = 0 \quad ...(4)$$

Multiplying (3) by y_n and (4) by y_m and subtracting, we get

$$(ry'_m)'\, y_n - (ry'_n)'\, y_m + (\lambda_m - \lambda_n) p\, y_m y_n = 0 \quad \text{or} \quad (\lambda_m - \lambda_n) p\, y_m y_n = (ry'_n)'\, y_m - (ry'_m)'\, y_n$$

or $$(\lambda_m - \lambda_n) p\, y_m y_n = \frac{d}{dx}\{(ry'_n) y_m - (ry'_m) y_n\}, \quad ...(5)$$

which can be verified by performing the indicated differentiation of the expression in brackets on R.H.S. of (5). Since $r(x)$ and $r'(x)$ are continuous by assumption and y_m, y_n are solutions of (1), it follows that the expression within brackets on R.H.S. of (5) is continuous on $a \le x \le b$. Integrating both sides of (5) over x from a to b, we thus obtain

$$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = \left[r(y'_n y_m - y'_m y_n) \right]_a^b$$

or $$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = r(b)\, \{y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)\} - r(a)\, \{y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)\} \quad ...(6)$$

We now have to consider several cases depending on whether $r(x)$ vanishes or does not vanish at a or b.

Case I. Let $r(a) = r(b) = 0$. Then (6) reduces to $$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = 0 \quad ...(7)$$

Cae II. Let $r(b) = 0$ but $r(a) \neq 0$. Then (6) reduces to

$$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = -r(a) \{y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)\} \quad ...(8)$$

Since y_m and y_n both satisfy ($2a$), we have

$$a_1 y_m(a) + a_2 y'_m(a) = 0 \quad ...(9)$$

and $$a_1 y_n(a) + a_2 y'_n(a) = 0 \quad ...(10)$$

Let $a_2 \neq 0$. Multiplying (10) by $y_m(a)$ and (9) by $y_n(a)$ and then subtracting, we get

$$a_2 \{y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)\} = 0$$

Since $a_2 \neq 0$, so $$y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a) = 0 \quad ...(11)$$

Using (11), (8) reduces to (7). If $a_2 = 0$, then let $a_1 \neq 0$. Now, multiplying (9) by $y'_n(a)$ and (10) by $y'_m(a)$ and then subtracting, we get

$$a_1 \{y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)\} = 0$$

Since $a_1 \neq 0$, so $$y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a) = 0$$

Hence as before (8) reduces to (7).

Case III. Let $r(a) = 0$ but $r(b) \neq 0$. Then (6) reduces to

$$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = r(b) \{y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)\} \quad ...(12)$$

Since y_m and y_n both satisfy ($2b$), we have

$$b_1 y_m(b) + b_2 y'_m(b) = 0 \quad ...(13)$$

and $$b_1 y_n(b) + b_2 y'_n(b) = 0 \quad ...(14)$$

Let $b_2 \neq 0$. Multiplying (14) by $y_m(b)$ and (13) by $y_n(b)$ and then subtracting, we get

$$b_2 \{y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)\} = 0$$

Since $b_2 \neq 0$, so $$y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b) = 0 \quad ...(15)$$

Using (15), (12) reduces to (7). If $b_2 = 0$, then let $b_1 \neq 0$. Now, multiplying (13) by $y'_n(b)$ and (14) by $y'_m(b)$ and then subtracting, we get

$$b_1 \{y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)\} = 0$$

Since $b_1 \neq 0$, so $\quad y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b) = 0$

Hence, as before, (12) reduces to (7).

Case IV. Let $r(a) \neq 0$ and $r(b) \neq 0$. There is no loss of generality by assuming that $a_2 \neq 0$ and $b_2 \neq 0$. Then, proceeding as in cases II and III, relations (11) and (15) can be proved. Then, using (11) and (15), (6) reduces to (7).

Case V. Let $r(a) = r(b)$. Proceed as in case IV to show that (6) reduces to (7).

From the above discussion, we see that in all situations, we get (7). Since λ_m and λ_n are different, (7) reduces to $\quad \int_a^b p(x)\, y_m(x)\, y_n(x)\, dx = 0,$

showing that $y_m(x)$ and $y_n(x)$ are orthogonal with respect to weight function $p(x)$.

15.12. Reality of eigenvalues [Meerut 2010; Purvanchal 2005; Lucknow 2010]

Theorem. *To prove that all eigenvalues of Strum Liouville problem are real.*

Proof. Consider the following Strum–Liouville problem :

$$[r(x)\, y']' + [q(x) + \lambda\, p(x)]y = 0 \qquad ...(1)$$

$$a_1 y(a) + a_2 y'(a) = 0 \qquad ...(2a)$$

$$b_1 y(b) + b_2 y'(b) = 0, \qquad ...(2b)$$

where p, q, r and r' are real valued and continuous and $p(x) > 0$ on $a \leq x \leq b$. Let a_1, a_2 in (2a) be given constants, not both zero, and so be b_1, b_2 in (2b).

Let $y(x)$ be an eigenfunction corresponding to an eigenvalue $\lambda = \alpha + i\beta$, where α, β are real constants. This eigenfunction $y(x)$ satisfies (1), (2a) and (2b) and may be a complex valued function.

Taking the complex conjugates of all the terms in (1), (2a) and (2b), we get

$$[r(x)\, \bar{y}']' + [q(x) + \bar{\lambda}\, p(x)]\, \bar{y} = 0 \qquad ...(3)$$

$$a_1 \bar{y}(a) + a_2 \bar{y}'(a) = 0 \qquad ...(4a)$$

$$b_1 \bar{y}(b) + b_2 \bar{y}'(b) = 0 \qquad ...(4b)$$

The above equations (3), (4a) and (4b) show that $\bar{y}(x)$ is the eigenfunction corresponding to the eigenvalue $\bar{\lambda} = \alpha - i\beta$. Multiplying (1) by $\bar{y}$ and (3) by y and subtracting, we get

$$(r\, y')'\, \bar{y} - \left(r\, \bar{y}'\right)' y + (\lambda - \bar{\lambda}) p\, y\, \bar{y} = 0 \qquad \text{or} \qquad (\lambda - \bar{\lambda}) p\, y\, \bar{y} = \left(r\, \bar{y}'\right)' y - (r\, y')'\, \bar{y}$$

or $$(\lambda - \bar{\lambda}) p\, y\, \bar{y} = \frac{d}{dx}\{(r\, \bar{y}')y - (r\, y')\bar{y}\}, \qquad ...(5)$$

which can be verified by performing the indicated differentiation of the expression in brackets on R.H.S. of (5). Integrating both sides of (5) w.r.t. 'x' from a to b, we thus obtain

$$(\lambda - \bar{\lambda}) \int_a^b p\, y\, \bar{y}\, dx = [r(\bar{y}'\, y - y'\, \bar{y})]_a^b$$

or $$(\lambda - \bar{\lambda}) \int_a^b p\, y\, \bar{y}\, dx = r(b)\{\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b)\} - r(a)\{\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a)\} \qquad ...(6)$$

We now have to consider several cases depending on whether $r(x)$ vanishes or does not vanish at a or b.

Case I. Let $r(a) = r(b) = 0$. Then (6) reduces to $\quad (\lambda - \bar{\lambda}) \int_a^b p\, y\, \bar{y}\, dx = 0 \qquad ...(7)$

Case II. Let $r(b) = 0$ but $r(a) \neq 0$. Then (6) reduces to

$$(\lambda - \bar{\lambda})\int_a^b p\, y\, \bar{y}\, dx = -r(a)\{\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a)\} \quad ...(8)$$

Consider relations (2a) and (4a). Let $a_2 \neq 0$. Multiplying (4a) by $y(a)$ and (2a) by $\bar{y}(a)$ and then subtracting, we get $a_2\{\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a)\} = 0$

Since $a_2 \neq 0$, so $\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a) = 0$. ...(9)

Using (9), (8) reduces to (7). If $a_2 = 0$, then assume that $a_1 \neq 0$. Now, multiplying (2a) by $\bar{y}'(a)$ and (4a) by $y'(a)$ and then subtracting, we get

$$a_1\{\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a)\} = 0$$

Since $a_1 \neq 0$, so $\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a) = 0$

Hence as before (8) reduces to (7).

Case III. Let $r(a) = 0$ but $r(b) \neq 0$. Then (6) reduces to

$$(\lambda - \bar{\lambda})\int_a^b p\, y\, \bar{y}\, dx = r(b)\{\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b)\} \quad ...(10)$$

Consider relations (2b) and (4b). Let $b_2 \neq 0$. Multiplying (4b) by $y(b)$ and (2b) by $\bar{y}(b)$ and then subtracting, we get $b_2\{\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b)\} = 0$.

Since $b_2 \neq 0$, so $\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b) = 0$ ...(11)

Using (11), (10) reduces to (7). If $b_2 = 0$, then assume that $b_1 \neq 0$. Now, multiplying (2b) by $\bar{y}'(b)$ and (4b) by $y'(b)$ and then subtracting, we get

$$b_1\{\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b)\} = 0$$

Since $b_1 \neq 0$, so $\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b) = 0$

Hence as before (10) reduces to (7).

Case IV. Let $r(a) \neq 0$ and $r(b) \neq 0$. There is no loss of generality by assuming that $a_2 \neq 0$ and $b_2 \neq 0$. Then proceeding as in cases II and III, relations (9) and (11) can be proved. Then, using (9) and (11), (6) reduces to (7).

Case V. Let $r(a) = r(b)$. Proceed as in case IV to show that (6) reduces to (7).

From the above discussion, we see that in all situations we get (7).

Now, $\lambda = \alpha + i\beta \Rightarrow \bar{\lambda} = \alpha - i\beta$ and hence $\lambda - \bar{\lambda} = (\alpha + i\beta) - (\alpha - i\beta) = 2i\beta$.

Again $y\,\bar{y} = |y|^2$, where $|y|$ stands for modulus of y. Then, (7) reduces to

$$2i\beta\int_a^b p(x)\,|\,y(x)\,|^2\,dx = 0 \quad \text{or} \quad \beta\int_a^b p(x)\,|\,y(x)\,|^2\,dx = 0 \quad ...(12)$$

Since $\int_a^b p(x)\,|\,y(x)\,|^2\,dx$ has a positive value in the given interval $a \le x \le b$, (12) reduces to $\beta = 0$ and hence $\lambda = \alpha + \imath\beta = \alpha$, which is real. Since λ is an arbitrary eigenvalue, it follows that eigenvalues of Strum–Liouville problem are all real.

15.13. SOLVED EXAMPLES

Ex. 1. *Find the eigenvalues and the corresponding eigenfunctions of* $X'' + \lambda X = 0$, $X(0) = 0$ *and* $X'(L) = 0$. **[Himanchal 2009; Jiwaji 2004; Meerut 2006; Ravishakar 2005]**

Sol. Given $X'' + \lambda X = 0$ ...(1)

with boundary conditions $X(0) = 0$ and $X'(L) = 0$ ...(2)

Case I. Let $\lambda = 0$. Then solution of (1) is $X(x) = Ax + B.$ ...(3)

From (3), $X'(x) = A$ ...(4)

Replacing x by 0 in (3) and using (2), we get $B = 0$. Again, replacing x by L in (4) and using (2), we get $A = 0$. With $A = 0$, $B = 0$, (3) reduces to $X(x) = 0$. Since $X(x) \neq 0$, so there is no eigen function corresponding to $\lambda = 0$.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$ (*i.e.,* λ is negative). So (1) gives $X'' - \mu^2 X = 0$

whose solution is $X(x) = Ae^{\mu x} + Be^{-\mu x}$ ...(5)

From (5), $X'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ ...(6)

Using (2), (5) and (6) reduces to

$0 = A + B$ and $A\mu e^{\mu L} - B\mu e^{-\mu L} = 0$ ...(7)

Solving (7), we find $A = B = 0$. Hence (5) gives $X(x) = 0$. Since $X(x) \neq 0$, so there is no eigen function corresponding to $\lambda = -\mu^2$.

Case III. Let $\lambda = \mu^2$ where $\mu \neq 0$ (*i.e.,* λ is positive). So (1) gives $X'' + \mu^2 X = 0$

whose solution is $X(x) = A \cos \mu x + B \sin \mu x$ ...(8)

From (8), $X'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$ ...(9)

Using (2), (8) and (9) reduce to $0 = A$ and $0 = B\mu \cos \mu L$

Thus $A = 0$ and $B \cos \mu L = 0,$ as $\mu \neq 0$ ...(10)

Now consider $B \cos \mu L = 0$ ...(11)

If $B = 0$, then with $A = 0$, (8) reduces to $X(x) = 0$, which is not an eigen function. So $B \neq 0$ for the existence of eigen functions. Since $B \neq 0$, (11) gives

$\cos \mu L = 0$ so that $\mu L = (2n - 1)\pi/2,$ $n = 1, 2, 3, ...$

$\therefore$ $\mu = (2n - 1)\pi/2L$ ...(12)

Using $A = 0$ and (12), (8) reduces to $X(x) = B \sin \{(2n - 1)\pi/2L\}$, $n = 1, 2, 3, ...$ and then $\lambda = \mu^2 = (2n - 1)^2\pi^2/4L^2$, $n = 1, 2, 3, ...$

So required eigenfunctions $X_n(x)$ with corresponding eigenvalues λ_n are

$X_n(x) = B_n \sin \{(2n - 1)\pi/2L\},$ and $\lambda_n = (2n - 1)^2\pi^2/4L^2,$ $n = 1, 2, 3, ...$

Note. We can take $B_n = 1$ while writing eigenfunctions.

Ex. 2. *Find the eigenvalues and eigenfunctions of the Strum–Liouville problem*

$X'' + \lambda X = 0,\ X'(0) = 0,\ X'(L) = 0.$ **(Kanpur 2009; Meerut 1995)**

Sol. Given $X'' + \lambda X = 0$ ...(1)

with boundary conditions $X'(0) = 0$ and $X'(L) = 0$ ...(2)

Case I. Let $\lambda = 0$. Then solution of (1) is $X(x) = Ax + B$ ...(3)

From (3), $X'(x) = A$ ...(4)

Using (2), (4) gives $0 = A$ and $0 = A$

These gives $A = 0$, while B is arbitrary. Hence (3) reduces to $X(x) = B$ which is non–zero. Hence taking $B = 1$, $X(x) = 1$ is an eigenfunction with $\lambda = 0$ as the corresponding eigenvalue.

Case II. Let $X = -\mu^2$, where $\mu \neq 0$. Then (11) reduces to $X'' - \mu^2 X = 0$ whose solution is

$X(x) = Ae^{\mu x} + Be^{-\mu x}$ ...(5)

From (5), $X'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ ...(6)

Using (2), (6) reduces to $0 = A\mu - B\mu$ and $A\mu e^{\mu L} - B\mu e^{-\mu L} = 0$

i.e., $A - B = 0$ and $Ae^{\mu L} - Be^{-\mu L} = 0,$ as $\mu \neq 0.$

Solving these equations, $A = B = 0$. So (5) gives $X(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) reduces to $X'' + \mu^2 X = 0$ whose solution is

$$X(x) = A \cos \mu x + B \sin \mu x \quad ...(7)$$

From (7), $X'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$ ...(8)

Using (2), (8) reduces to $0 = B\mu$ and $0 = -A\mu \sin \mu L + B\mu \cos \mu L$

i.e., $B = 0$ and $A \sin \mu L = 0$, as $\mu \neq 0$ ...(9)

We now consider $A \sin \mu L = 0$. ...(10)

If $A = 0$, then with $B = 0$, (8) reduces to $X(x) = 0$, which is not an eigenfunction. So we take $A \neq 0$ for the existence of eigenfunctions. Since $A \neq 0$, (10) reduce to

$\sin \mu L = 0$ so that $\mu L = n\pi$, $n = 1, 2, 3, ...$

$\therefore$ $\mu = (n\pi/L)$, $n = 1, 2, 3, ...$ ...(11)

($n = 0$ is omitted as $n = 0 \Rightarrow \mu = 0$ which is contrary to our assumption $\mu \neq 0$).

Using $B = 0$ and (11), (7) reduces to $X(x) = A \cos (n\pi x/L)$, $n = 1, 2, 3, ...$

and then $\lambda = \mu^2 = n^2\pi^2/L^2$, $n = 1, 2, 3, ...$

Hence the required eigenfunctions $X_n(x)$ with the corresponding eigenvalues λ_n are given by (taking $A = 1$). $X_n(x) = \cos (n\pi x/L)$, $\lambda_n = n^2\pi^2/L^2$, $n = 0, 1, 2, 3, ...$

Ex. 3. *Find all the eigenvalues and eigenfunctions of the Strum–Liouville problem $y'' + \lambda y = 0$ with $y(0) + y'(0) = 0$ and $y(1) + y'(1) = 0$.*

Sol. Given $y'' + \lambda y = 0$ ...(1)

with boundary conditions $y(0) + y'(0) = 0$ ...(2)

$\therefore$ $y(1) + y'(1) = 0$ ...(3)

Case I. Let $\lambda = 0$. Then solution of (1) is $y(x) = Ax + B$ ...(4)

From (4), $y'(x) = A$ ...(5)

From (4) and (5), $y(0) = B$, $y'(0) = A$. With these values, (2) gives

$$B + A = 0 \quad ...(6)$$

From (4) and (5), $y(1) = A + B$ and $y'(1) = A$. With these values (3) gives

$$2A + B = 0 \quad ...(7)$$

Solving (6) and (7), $A = B = 0$. Hence (4) reduces to $X(x) = 0$, which is not an eigenfunction and so $\lambda = 0$ is not an eigenvalue.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (1) reduces to $y'' - \mu^2 y = 0$ whose solution is

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \quad ...(8)$$

From (8), $y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ ...(9)

From (8) and (9), $y(0) = A + B$ and $y'(0) = \mu(A - B)$ ...(10)

Using (10), (2) reduces to $A + B + \mu(A - B) = 0$

i.e., $A(1 + \mu) + B(1 - \mu) = 0$ ...(11)

Again, from (8) and (9), we have

$y(1) = Ae^{\mu} + Be^{-\mu}$ and $y'(1) = \mu(Ae^{\mu} - Be^{-\mu})$ ...(12)

Using (12), (3) reduces to $Ae^{\mu} + Be^{-\mu} + \mu(Ae^{\mu} - Be^{-\mu}) = 0$

i.e., $Ae^{\mu}(1 + \mu) + Be^{-\mu}(1 - \mu) = 0$ ...(13)

We now use the theory of determinants for solving (11) and (13). For non–trivial solution of these equations, we must have

$$\begin{vmatrix} 1+\mu & 1-\mu \\ e^{\mu}(1+\mu) & e^{-\mu}(1-\mu) \end{vmatrix} = 0 \Rightarrow (1 + \mu)(1 - \mu)(e^{-\mu} - e^{\mu}) = 0, \quad \text{giving} \quad \mu = -1 \text{ and } \mu = 1.$$

When $\mu = -1$, (11) and (13) give $B = 0$, while A will be arbitrary. So (8) reduces to $y(x) = Ae^{-x}$ and the corresponding eigenvalue is given by $\lambda = -\mu^2 = -(-1)^2 = -1$.

Next when $\mu = 1$, (11) and (13) give $A = 0$, while B will be arbitrary. So (8) reduces to $y(x) = Be^{-x}$ and the corresponding eigenvalue is given by $\lambda = -\mu^2 = -1^2 = -1$.

Taking $A = B = 1$, $y(x) = e^{-x}$ is an eigen function and $\lambda = -1$ is the corresponding eigenvalue.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) reduces to $y'' + \mu^2 y = 0$ whose solution is

$$y(x) = A \cos \mu x + B \sin \mu x. \quad ...(14)$$

From (14), $y'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$ ...(15)

From (14) and (15), $y(0) = A$ and $y'(0) = B\mu$. ...(16)

Using (16), (2) reduces to $A + B\mu = 0$ ...(17)

Again, from (14) and (15), we have

$y(1) = A \cos \mu + B \sin \mu$ and $y'(1) = -A\mu \sin \mu + B\mu \cos \mu$ ...(18)

Using (18), (3) reduces to $A \cos \mu + B \sin \mu - A\mu \sin \mu + B\mu \cos \mu = 0$. ...(19)

From (17), $A = -B\mu$. With this value of A, (19) gives

$$-B\mu \cos \mu + B \sin \mu + B\mu^2 \sin \mu + B\mu \cos \mu = 0$$

or $B(1 + \mu^2) \sin \mu = 0$ or $B \sin \mu = 0$, as $(\mu^2 + 1) \neq 0$. ...(20)

If $B = 0$, then (17) gives $A = 0$. Hence (14) reduces to $y(x) = 0$, which is not an eigen function. So take $B \neq 0$. Then (20) gives

$\sin \mu = 0$ so that $\mu = n\pi$, $n = 1, 2, 3, ...$ ...(21)

(Here we omit $n = 0$, for $n = 0$ gives $\mu = 0$ so that $\lambda = \mu^2 = 0$ which has been considered in case I)

Using $A = -B\mu = -Bn\pi$ and (21), (14) reduces to

$y(x) = B(\sin n\pi x - n\pi \cos n\pi x)$ $n = 1, 2, 3, ...$ and then $\lambda = \mu^2 = n^2\pi^2$, $n = 1, 2, 3, ...$

Hence the required eigenfunctions $y_n(x)$ with the corresponding eigenvalues λ_n are given by (taking $B = 1$) $y_n(x) = \sin n\pi x - n\pi \cos n\pi x$, $n = 1, 2, 3, ...$ and $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, ...$

Ex. 4. *Find all the eigenvalues and eigenfunctions of* $4(e^{-x}y')' + (1 + \lambda)e^{-x} y = 0$, $y(0) = 0$, $y(1) = 0$.

Sol. Re–writing the given equation, we have

$4(e^{-x}y'' - e^{-x}y') + (1 + \lambda)e^{-x}y = 0$ or $4y'' - 4y' + (1 + \lambda)y = 0$ ...(1)

Also, given $y(0) = 0$ and $y(1) = 0$ ...(2)

Case I. Let $\lambda = 0$. Then (1) reduces to

$4y'' - 4y' + y = 0$ *i.e.,* $(4D^2 - 4D + 1)y = 0$, where $D = d/dx$. ...(3)

Here auxiliary equation is $4D^2 - 4D + 1 = 0$ *i.e.,* $(2D - 1)^2 = 0$. This gives $D = 1/2, 1/2$.

Hence solution of (3) is $y(x) = (A + Bx)e^{x/2}$ ...(4)

Using (2), (4) reduces to $0 = A$ and $0 = (A + B)e^{1/2}$

These give $A = B = 0$. So (4) reduces to $y(x) = 0$, which is not an eigenfunction.

Case II. Let $\lambda = -\mu^2$ where $\mu \neq 0$. Then (1) reduces to

$4y'' - 4y' + (1 - \mu^2)y = 0$ *i.e.,* $(4D^2 - 4D + 1 - \mu^2)y = 0$. ...(5)

Here auxiliary equation is $4D^2 - 4D + 1 - \mu^2 = 0$ giving

$$D = [4 \pm \{16 - 16(1 - \mu^2)\}^{1/2}] \big/ 8 = 1/2 \pm \mu/2$$

Hence solution of (5) is $y = Ae^{(1/2 + \mu/2)x} + Be^{(1/2 - \mu/2)x}$ ...(6)

Using (2), (6) reduces to $0 = A + B$, $0 = Ae^{1/2 + \mu/2} + Be^{1/2 - \mu/2}$

These give $A = B = 0$. So (6) reduces to $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) reduces to

$4y'' - 4y' + (1 + \mu^2)y = 0$ *i.e.,* $(4D^2 - 4D + 1 + \mu^2)y = 0$ ...(7)

Here auxiliary equation is $4D^2 - 4D + 1 + \mu^2 = 0$

giving $D = [4 \pm \{16 - 16(1+\mu^2)\}^{1/2}]/8 = 1/2 \pm i(\mu/2)$

Hence solution of (7) is given by $y(x) = e^{x/2}\{A \cos(\mu x/2) + B \sin(\mu x/2)\}$ ...(8)

Using (2), (8) reduces to

$0 = A$ and $0 = e^{1/2}\{\cos(\mu/2) + B \sin(\mu/2)\}$ *i.e.,* $A = 0$ and $B \sin(\mu/2) = 0$ ...(9)

Consider $B \sin(\mu/2) = 0$ ...(10)

If $B = 0$, then with $A = 0$, (8) reduces to $y(x) = 0$, which is not an eigen function. So we take $B \neq 0$ for the existence of eigen function. As $B \neq 0$, (10) gives

$\sin(\mu/2) = 0$ so that $\mu/2 = n\pi$ or $\mu = 2n\pi$, $n = 1, 2, 3, \ldots$ ...(11)

($n = 0$ is not being considered here as $n = 0 \Rightarrow \mu = 0$ which is contrary to our assumption.)

Using $A = 0$ and (11), (8) reduces to

$y(x) = Be^{x/2} \sin n\pi x$ and $\lambda = \mu^2 = 4n^2\pi^2$, $n = 1, 2, 3, \ldots$.

Hence the required eigenfunctions $y_n(x)$ with the corresponding eigenvalues λ_n are given by (taking $B = 1$) $y_n(x) = e^{x/2} \sin n\pi x$, $\lambda_n = 4n^2\pi^2$, $n = 1, 2, 3, \ldots$

Ex. 5. *For the eigen–value problem given below, obtain the set of orthogonal eigenfunctions in the interval (0, 2c) : $X'' + \lambda X = 0$, $X(0) = X(2c)$, $X'(0) = X'(2c)$.*

Sol. Given $X'' + \lambda X = 0$ ...(1)

with boundary conditions $X(0) = X(2c)$ ...(2)

and $X'(0) = X'(2c)$ ...(3)

Case I. Let $\lambda = 0$. Then solution of (1) is $X(x) = Ax + B$ ...(4)

From (4), $X'(x) = A$ ...(5)

From (4), $X(0) = B$ and $X(2c) = 2cA + B$. So (2) reduces to

$B = 2cA + B$ and hence $A = 0$.

Next, from (5), $X'(0) = X'(2c) = A$. So (3) gives $A = A$.

Hence corresponding to the eigen value $\lambda = 0$, the eigenfunction is

$X(x) = B$ or $X(x) = 1$, taking $B = 1$.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (1) becomes $X'' - \mu^2 X = 0$ whose solution is

$X(x) = Ae^{\mu x} + Be^{-\mu x}$ ...(6)

From (6), $X'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ ...(7)

From (6), $X(0) = A + B$ and $X'(2c) = Ae^{2\mu c} + Be^{-2\mu c}$. So (2) gives

$A + B = Ae^{2\mu c} + Be^{-2\mu c}$ or $A(1 - e^{2\mu c}) + B(1 - e^{-2\mu c}) = 0$ ...(8)

From (7), $X'(0) = \mu(A - B)$ and $X'(2c) = \mu(Ae^{2\mu c} - Be^{-2\mu c})$

$\therefore$ (3) gives $\mu(A - B) = \mu(Ae^{2\mu c} - Be^{-2\mu c})$ or $A(1 - e^{2\mu c}) - B(1 - e^{-2\mu c}) = 0$ ...(9)

Solving (8) and (9), $A = B = 0$. So (6) reduces to $X(x) = 0$, which is not an eigen function. So there is no eigenfunction corresponding to $\lambda = -\mu^2$.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) becomes $X'' + \mu^2 X = 0$ whose solution is

$X(x) = A \cos \mu x + B \sin \mu x$ ...(10)

From (10), $X'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$ ...(11)

From (10), $X(0) = A$ and $X(2c) = A \cos 2\mu c + B \sin 2\mu c$

$\therefore$ (2) gives $A = A \cos 2\mu c + B \sin 2\mu c$ or $A(1 - \cos 2\mu c) - B \sin 2\mu c = 0$ ...(12)

From (11), $X'(0) = B\mu$ and $X'(2c) = \mu(-A \sin 2\mu c + B \cos 2\mu c)$

$\therefore$ (3) gives $B\mu = \mu(-A \sin 2\mu c + B \cos 2\mu c)$ or $A \sin 2c\mu + B(1 - \cos 2c\mu) = 0$ ...(13)

For non–trival solution of (12) and (13), we must have

$$\begin{vmatrix} 1 - \cos 2\mu c & -\sin 2\mu c \\ \sin 2\mu c & 1 - \cos 2\mu c \end{vmatrix} = 0$$

or $(1 - \cos 2\mu c)^2 + \sin^2 2\mu c = 0$ or $\cos 2\mu c = 1 = \cos 0$

$\therefore \quad 2\mu c = 2n\pi$ or $\mu = n\pi/c,\ n = 1, 2, 3, \ldots$ ...(14)

($n = 0$ is omitted here as $n = 0 \Rightarrow \mu = 0$, which is contrary to our assumption $\mu \neq 0$.)

With this value of μ, (10) becomes $X(x) = A \cos (n\pi x/c) + B \sin (n\pi x/c)$ ...(15)

Taking $A = 1$ and $B = 0$ in (15), the eigenfunctions are given by $X(x) = \cos (n\pi x/c)$. Again taking $A = 0$ and $B = 1$ in (15), the eigenfunctions are given by $X(x) = \sin (n\pi x/c)$.

Note that $\int_0^{2c} 1 \cdot \cos \frac{n\pi x}{c} dx = 0, \int_0^{2c} 1 \cdot \sin \frac{n\pi x}{c} dx = 0\ , \ \int_0^{2c} \sin \frac{n\pi x}{c} \cos \frac{m\pi x}{c} dx = 0,$ for $m \neq n$ ... (16)

In view of (16) the required eigenfunctions which are orthogonal on $(0, 2c)$ are given by

$\{1, \cos (n\pi x/c), \sin (n\pi x/c)\}, (n = 1, 2, 3, \ldots)$

Ex. 6. *Find the eigenvalues and eigenfunctions of* $[xy'(x)]' + (\lambda/x)\, y(x) = 0, y'(1) = y'(e^{2\pi}) = 0$ **[Kanpur 2009; Himanchal 2009]**

Sol. Re–writing the given equation, $xy'' + y' + (\lambda/x)y = 0$

or $x^2y'' + xy' + \lambda y = 0$ *i.e.,* $(x^2D^2 + xD + \lambda)y = 0,$ where $D \equiv d/dx$ ...(1)

This is a homogeneous differential equation. To solve it, we take

$x = e^z$ so that $z = \log x$ ...(2)

We know that, if $D_1 \equiv d/dz \equiv x(d/dx)$, then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$ ...(3)

Using (3), (1) reduces to $\{D_1(D_1 - 1) + D_1 + \lambda\}y = 0$ or $(D_1^{\,2} + \lambda)y = 0$ ...(4)

Also, given that $y'(1) = 0$ and $y'(e^{2\pi}) = 0,$ ...(5)

Case I. Let $\lambda = 0$. Then solution of (4) *i.e.,* $D_1^{\,2}y = 0$ is

$y = Az + B$ or $y = A \log x + B$, using (2) ...(6)

From (6), $y'(x) = A/x.$ ...(7)

Using (5), (7) reduces to $0 = A$ and $0 = A/e^{2\pi}$

These give $A = 0$, while B may be taken as arbitrary. With these values and taking $B = 1$, (6) gives $y(x) = 1$ as the eigenfunction corresponding to eigenvalue $\lambda = 0$.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then solution of (4) is

$y = Ae^{\mu z} + Be^{-\mu z} = A(e^z)^{\mu} + B(e^z)^{-\mu}$ or $y(x) = Ax^{\mu} + Bx^{-\mu},$ as $x = e^z$ ...(8)

From (8), $y'(x) = A\mu x^{\mu - 1} - B\mu x^{-\mu - 1}$ ...(9)

Using (5), (9) reduces to $0 = A\mu - B\mu$ and $0 = A\mu e^{2\pi(\mu - 1)} - B\mu e^{-2\pi(\mu + 1)}$

i.e., $A - B = 0$ and $Ae^{2\pi\mu} - Be^{-2\pi\mu} = 0,$ as $\mu \neq 0,$

giving $A = B = 0$. So (8) reduces to $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then solution of (4) is $y = A \cos \mu z + B \sin \mu z,$

i.e., $y(x) = A \cos (\mu \log x) + B \sin (\mu \log x),$ by (2) ...(10)

From (10), $y'(x) = -(A\mu/x) \sin (\mu \log x) + (B\mu/x) \cos (\mu \log x).$ ...(11)

Using (5), (11) reduces to (noting that $\log 1 = 0$ and $\log e^{2\pi} = 2\pi$)

$0 = B\mu$ and $0 = -(A\mu/e^{2\pi}) \sin 2\pi\mu + (B\mu/e^{2\pi}) \cos 2\pi\mu,$

i.e., $B = 0$ and $A \sin 2\pi\mu = 0,$ as $\mu \neq 0.$ ...(12)

Consider $A \sin 2\pi\mu = 0$ ...(13)

If $A = 0$, then with $B = 0$, (10) reduces to $y(x) = 0$, which is not an eigenfunction. So we take $A \neq 0$ for the existence of eigenfunctions. Since $A \neq 0$, (13) gives

$\sin 2\pi\mu = 0$ so that $2\pi\mu = n\pi,$ $n = 1, 2, 3, \ldots$

Thus, $\mu = n/2,$ $n = 1, 2, 3, ..$ ...(14)

$n = 0$ is not being considered because $n = 0 \Rightarrow \mu = 0 \Rightarrow \lambda = 0$, which has already been considered in case I. Using $B = 0$ and (14), (10) reduces to

$y(x) = A \cos \{(n/2) \times \log x\},$ with $\lambda = \mu^2 = n^2/4,$ $n = 1, 2, 3, \ldots.$

So the required eigenfunctions $y_n(x)$ (taking A=1) with the corresponding eigenvalues λ_n are $y_n(x) = \cos\{(n/2) \times \log x\}$, $n = 1, 2, 3, ...$, $\lambda_n = n^2/4$, $n = 1, 2, 3, ...$; and $y(x) = 1$ with $\lambda = 0$.

Ex. 7. *Find all the eigenvalues and eigenfunctions of the Strum–Liouville problem* $(x^3y')' + \lambda xy = 0$, $y(1) = 0$, $y(e) = 0$.

Sol. Re–writing the given equation $x^3y'' + 3x^2y' + \lambda xy = 0$

or $x^2y'' + 3xy' + \lambda y = 0$ or $(x^2D^2 + 3xD + \lambda)y = 0$, $D \equiv d/dx$ ...(1)

This is a homogeneous differential equation. To solve it, we take

$x = e^z$ so that $z = \log x$ ...(2)

We know that, if $D_1 \equiv d/dz \equiv x(d/dx)$, then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$ ...(3)

Using (3), (1) reduces to $\{D_1(D_1 - 1) + 3D_1 + \lambda\}y = 0$ or $(D_1^2 + 2D_1 + \lambda)y = 0$ ...(4)

Also, given that $y(1) = 0$ and $y(e) = 0$ ...(5)

Case I. Let $\lambda = 0$. Then (4) reduces to $D_1(D_1 + 2)y = 0$ whose solution is

$y = Ae^{0.z} + Be^{-2z}$ or $y(x) = A + B/x^2$ ...(6)

Putting $x = 1$ in (6) and using (5), $A + B = 0$ ...(7)

Putting $x = e$ in (6) and using (5), $A + B/e^2 = 0$ ...(8)

Solving (7) and (8), we get $A = B = 0$ and so (6) gives $y(x) = 0$, which is not an eigenfunction.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (4) reduces to $(D_1^2 + 2D_1 - \mu^2)y = 0$ ...(9)

whose auxiliary equation is $D_1^2 + 2D_1 - \mu^2 = 0$

Solving it, $D_1 = \{-2 \pm (4 + 4\mu^2)^{1/2}\}/2 = -1 \pm (1 + \mu^2)^{1/2}$

or $D_1 = -1 \pm k$, where $k = (1 + \mu^2)^{1/2}$ ...(10)

Hence solution of (9) is $y(x) = Ae^{(-1+k)z} + Be^{(-1-k)z}$

or $y(x) = Ax^{(-1+k)} + Bx^{(-1-k)}$ or $y(x) = x^{-1}(Ax^k + B/x^k)$ ...(11)

Putting $x = 1$ in (11) and using (5), $A + B = 0$ ...(12)

Putting $x = e$ in (11) and using (5), $Ae^{k-1} + Be^{-k-1} = 0$ ...(13)

Solving (12) and (13), we get $A = B = 0$. Hence (11) gives $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (4) reduces to $(D_1^2 + 2D_1 + \mu^2)y = 0$ ...(14)

whose auxiliary equation is $D_1^2 + 2D_1 + \mu^2 = 0$ ...(15)

Solving it, $D_1 = \{-2 \pm (4 - 4\mu^2)^{1/2}\}/2 = -1 \pm i(\mu^2 - 1)^{1/2}$

or $D_1 = -1 \pm ip$, where $p = (\mu^2 - 1)^{1/2}$ and $\mu^2 > 1$ ...(16)

Hence solution of (14) is $y(x) = e^{-z}(A \cos pz + B \sin pz)$

i.e., $y(x) = (1/x) \times \{A \cos(p \log x) + B \sin(p \log x)\}$ ...(17)

Putting $x = 1$ in (17) and using (5), we get $A = 0$.

Putting $x = e$ and $A = 0$ in (17) and using (5), we get $(B/e) \sin(p) = 0$. But we must take $B \neq 0$ for non–trivial solution (and hence for getting eigenfunction).

$\therefore \sin p = 0$ so that $p = n\pi$ or $(\mu^2 - 1)^{1/2} = n\pi$, by (16)

or $\mu^2 - 1 = n^2\pi^2$ or $\mu^2 = 1 + n^2\pi^2$ or $\lambda = 1 + n^2\pi^2$

Hence the required eigenfunctions are $y_n(x) = (1/x) \times \sin(n\pi \log x)$ (taking $B = 1$) and the corresponding eigenvalues are $\lambda_n = n^2\pi^2$, where $n = 1, 2, 3, ...$

Ex. 8. *Find the eigenvalues and the corresponding eigenfunctions of the boundary value problem :* $y'' + 2y' + (1 + \lambda)y = 0$, $y(0) = 0$, $y'(a) = 0$

Sol. Given $(D^2 + 2D + 1 + \lambda)y = 0$, $D \equiv d/dx$ ...(1)

with $y(0) = 0$, $y'(a) = 0$ ...(2)

Case I. Let $\lambda = 0$. Then solution of (1) is $y(x) = (A + Bx)e^{-x}$ ...(3)

From (3), $y'(x) = Be^{-x} - (A + Bx)e^{-x}$ ...(4)

Using (2), (3) and (4) reduce to $0 = A$ and $Be^{-a} - (A + Ba)e^{-a} = 0$,

i.e., $A = 0$ and $Be^{-a}(1 - a) = 0$. These give $A = B = 0$.

So (3) reduces to $y(x) = 0$, which is not an eigenfunction.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (1) reduces to

$$[D^2 + 2D + 1 - \mu^2]y = 0 \quad ...(5)$$

whose auxiliary equation is $D^2 + 2D + (1 - \mu^2) = 0$.

This gives $D = [-2 \pm \{4 - 4(1 - \mu^2)\}^{1/2}]/2 = -1 \pm \mu$

Hence solution of (5) is $y(x) = Ae^{(-1+\mu)x} + Be^{-(1+\mu)x}$ ...(6)

so that $y'(x) = A(-1 + \mu)\, e^{(-1+\mu)x} - B(1 + \mu)e^{-(1+\mu)x}$ ...(7)

Using (2), (6) and (7) reduce to

$0 = A + B$ and $0 = A(-1 + \mu)\, e^{(-1+\mu)a} - B(1 + \mu)e^{-(1+\mu)a}$,

i.e., $A + B = 0$ and $A(\mu - 1)e^{\mu a} - B(\mu + 1)e^{-\mu a} = 0$.

These give $A = B = 0$. So (6) reduces to $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) reduces to

$$(D^2 + 2D + 1 + \mu^2)y = 0 \quad ...(8)$$

whose auxiliary equation is $D^2 + 2D + 1 + \mu^2 = 0$.

This gives $D = [-2 \pm \{4 - 4(1 + \mu^2)\}^{1/2}]/2 = -1 \pm i\mu$.

Hence solution of (8) is $y(x) = e^{-x}\,(A \cos \mu x + B \sin \mu x)$ ...(9)

From (9), $y'(x) = -e^{-x}\,(A \cos \mu x + B \sin \mu x) + e^{-x}\,(-A\mu \sin \mu x + B\mu \cos \mu x)$ ...(10)

Using (2), (9) and (10) reduce to

$0 = A$, $0 = -e^{-a}\,(A \cos \mu a + B \sin \mu a) + \mu e^{-a}\,(-A \sin \mu a + B \cos \mu a)$

$\therefore \quad A = 0$ and $B(\mu \cos \mu a - \sin \mu a) = 0$ ...(11)

Consider $B(\mu \cos \mu a - \sin \mu a) = 0$ ...(12)

If $B = 0$, then with $A = 0$, (9) reduces to $y(x) = 0$, which is not an eigenfunction. So we take $B \neq 0$ for the existence of eigenfunctions. Since $B \neq 0$, (12) reduces to

$$\mu \cos \mu a - \sin \mu a = 0 \quad \text{or} \quad \tan \mu a = \mu, \quad ...(13)$$

which is a trigonometrical equation in μ. Let μ_n ($n = 1, 2, 3, ...$) be positive roots of (13). With $A = 0$, (9) reduces to $y(x) = Be^{-x} \sin \mu x$

Hence the required eigenfunctions $y_n(x)$ with the corresponding eigenvalues λ_n are given by (taking $B = 1$) $y_n(x) = e^{-x} \sin \mu_n x$ and $\lambda_n = \mu_n^2$, $n = 1, 2, 3, ...$, where μ_n ($n = 1, 2, 3, ...$) are positive roots of (13).

EXERCISE 15(B)

1. For the Strum–Liouville problem $X'' + \lambda X = 0$, $X(0) = 0$, $X(\pi) = 0$, obtain the eigenfunctions and the corresponding eigenvalues. **[Nagpur 2005; Bilaspur 2004; Bhopal 2004; Meerut 2005, 11; Ravishaker 2004; Vikram 2004; Lucknow]** **Ans.** $X_n(x) = \sin nx$, $\lambda_n = n^2$, $n = 1, 2, 3, ...$

2. Find all eigenvalues and eigenfunctions of the Strum–Liouville problem $X'' + \lambda X = 0$, $X(0) = 0$, $X'(\pi/2) = 0$. **[Jabalpur 2004]** **Ans.** $X_n(x) = \sin (2n - 1)x$, $\lambda_n = (2n - 1)^2$, $n = 1, 2, 3, ...$

3. Find all eigenvalues and eigenfunctions of the problem $X'' + \lambda X = 0$, $X'(-\pi) = 0$, $X'(\pi) = 0$

Ans. $X_n = \cos[(n/2) \times (\pi + x)]$, $n = 0, 1, 2, ...$, $\lambda_n = n^2/4$, $n = 1, 2, 3, ...$

4. Find the eigenvalues and the corresponding eigenfunctions of the boundary value problem $y'' + \lambda y = 0$, $y(0) + \pi y'(0) = 0$, $y(\pi) = 0$.

Ans. $y_n(x) = \sin \mu_n x - \pi\mu_n \cos \mu_n x$, $\lambda_n = \mu_n^2$, $n = 1, 2, 3, ...$ where μ_n are +ve roots of $\tan \mu\pi = \mu\pi$; $y(x) = x - \pi$ is eigenfunction corresponding to eigenvalue $\lambda = 0$.

5. Find all the eigenvalues and eigenfunctions of Strum-Liouville problem :

$y'' + \lambda^2 y = 0,\ y'(0) = y'(l) = 0,\ 0 \le x \le l$ **[Nagpur 2005]**

Hint. Refer case III of Ex. 2 Art 15.13. Here take $X = y,\ \mu = \lambda$ and L = l. Then eigenvalues $= \lambda_n = n\pi/l$ and the corresponding eigenfunctions $= y_n = \cos(n\pi x/l),\ n = 1, 2, 3, ...$

6. Find the eigenvalues and eigenfunctions of the Strum-Liouville problem $y'' + \lambda y = 0,\ y'(0) = 0,\ y'(\pi) = 0.$ **[CDVU 2004; M.D.U. Rohtak 2005]**

[Hint : This is a particular case of Ex. 2 of Art 15.13. Here $X = y$ and $L = \pi.$

Ans. Eigenfunctions: $y_n(x) = \cos nx$ and eigenvalues $\lambda_n = n^2,$ where n = 0, 1, 2, ...]

7. Find the eigenvalues and the corresponding eigenfunctions of the eigen value problem $X'' + \lambda X = 0,\ X(0) = 0,\ X(1) - X'(1) = 0.$

Ans. $X_n = \sin \mu_n x,\ \lambda_n = \mu_n^2,\ n = 1, 2, 3, ...$ where μ_n are positive roots of $\tan \mu = \mu$. Again $X(x) = x$ is eigen function corresponding to the eigen value $\lambda = 0$.

8. Find the eigenvalues and eigenfunctions of the Strum Liouville problem $y'' + \lambda y = 0,\ y(0) = y(\pi) = 0.$ **Ans.** $\lambda_n = n^2,\ y_n = \sin n\pi,\ n = 1, 2, ..., ...$

9. Solve the Strum–Liouville problem $(xX')' + \lambda(1/x)X = 0,\ X'(1) = 0,\ X(b) = 0\ (b > 1)$ and normalize the eigen functions. **Ans.** $\lambda_n = \dfrac{(2n-1)\pi}{2\log b},\ X_n = \cos\left\{\dfrac{(2n-1)\pi \log x}{2\log b}\right\},\ n = 1, 2, 3, ...$

10. Find the eigenvalues and eigenfunctions of $y'' - 4\lambda y' + 4\lambda^2 y = 0,\ y'(0) = 0,\ y(2) + 2y'(2) = 0.$ **[Purvanchal 2006]**

15.14. Orthogonality of Legendre polynomials

We know that Legendre's differential equation is $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ which can be re–written in the form $[(1 - x^2)y']' + \lambda y = 0,$ where $\lambda = n(n + 1)$ ...(1) and is therefore in the form of Strum–Liouville equation $[r(x)\, y']' + [q(x) + \lambda\, p(x)]y = 0$ ...(2)

Comparing (1) with (2), here $r(x) = 1 - x^2,$ $q(x) = 0$ and $p(x) = 1.$

Since $r(-1) = r(1) = 0$, we need no boundary conditions to form a Strum–Liouville problem. Further, we know that $P_n(x)$ for n = 0, 1, 2, 3, ... are solutions of (1) and so they are eigenfunctions. Again they have continuous derivatives and hence it follows that Legendre polynomials $P_n(x)$, n = 0, 1, 2, ... are orthogonal on the interval $-1 \le x \le 1$ with respect to the weight function $p(x) = 1$,

i.e., $$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0\ ,\ \text{when}\quad m \ne n$$

Remark. For alternative proof, refer Art. 9.8 of chapter 9.

15.15. Orthogonality of Bessel functions

The Bessel functions $J_n(u)$ with fixed integer $n \ge 0$ satisfies Bessel's equation (refer equation (1) of Art. 11.1 of chapter 11, by taking u as independent variable in place of x)

$$u^2(d^2y/du^2) + u(dy/du) + (u^2 - n^2)y = 0$$

$$\therefore \qquad u^2 \frac{d^2 J_n(u)}{du^2} + u \frac{d\, J_n(u)}{du} + (u^2 - n^2)\, J_n(u) = 0 \qquad ...(1)$$

Let $u = \lambda x,$ where λ is a constant. ...(2)

Then, $$\frac{d\, J_n(u)}{du} = \frac{d\, J_n(\lambda x)}{dx} \cdot \frac{dx}{du} = \frac{1}{\lambda} J_n'(\lambda x)\ ,\ \text{using (2)} \qquad ...(3)$$

and $$\frac{d^2 J_n(u)}{du^2} = \frac{d}{du}\left(\frac{d\, J_n(u)}{du}\right) = \frac{d}{du}\left(\frac{J_n'(\lambda x)}{\lambda}\right),\ \text{by (3)}$$

$$= \frac{d}{dx}\left(\frac{J_n'(\lambda x)}{\lambda}\right)\cdot\frac{dx}{du} = \frac{1}{\lambda^2}J_n''(\lambda x)\text{, by (2)} \qquad ...(4)$$

Using (2), (3) and (4), (1) reduces to

$$x^2\ J_n''\ (\lambda x) + x\ J_n'(\lambda x) + (\lambda^2 x^2 - n^2)\,J_n(\lambda x) = 0$$

which can be re–written in the form $[x\ J_n'(\lambda x)]' + (-n^2/x + \lambda^2 x)\,J_n(\lambda x) = 0$ ...(5)

and is therefore in the form of Strum–Liouville equation for each fixed n

$$[r(x)\,y']' + [q(x) + \lambda\,p(x)]y = 0 \qquad ...(6)$$

Comparing (5) with (6), here $r(x) = x$, $q(x) = -n^2/x$ and $p(x) = x$ and the parameter is λ^2 in place of λ.

Since $r(x) = 0$ for $x = 0$, it follows from Art. 15.11 that solutions of (5) on an interval $0 \le x \le a$ satisfying the boundary condition $J_n(\lambda a) = 0$, (n fixed) ...(7) form an orthogonal set with respect to the weight function $p(x) = x$.

We know that $J_n(u)$ has infinitely many real zeros, say, $u = \alpha_1 < \alpha_2 < ....$ Hence (7) gives

$\lambda a = \alpha_i$ and so $\lambda = \lambda_i = \alpha_i/a$, $i = 1, 2, 3, ...$ ...(8)

Further $J_n'(\lambda x)$ is continuous at $x = 0$. Hence for each fixed non–negative integer n the sequence of Bessel functions of the first kind $J_n(\lambda_1 x), J_n(\lambda_2 x), J_n(\lambda_3 x), ...$, with λ_i as in (8), forms an orthogonal set on the interval $0 \le x \le a$ with respect to the weight function $p(x) = x$, that is

$$\int_0^a x\,J_n(\lambda_i x)\,J_n(\lambda_j x)\,dx = 0\text{, for } i \ne j.$$

Remark. From the above discussion, it follows that we obtain infinitely many orthogonal sets, each corresponding to one of the fixed values of n.

15.16. Orthogonality on an infinite interval

(*i*) Orthogonality of Hermite Polynomial $H_n(x)$. *Hermite polynomials are orthogonal over* $]-\infty, \infty[$ *with respect to the weight function* $e^{-x^2/2}$, that is,

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\,H_n(x)\,dx = 0\,, \quad \text{for} \quad m \ne n.$$

For proof, refer Art. 12.7 of chapter 12.

(*ii*) Orthogonality of Laguerre's polynomial $L_n(x)$. *Laguerre polynomials* are orthogonal over $[0, \infty[$ with respect to the weight function e^{-x}, that is,

$$\int_0^{\infty} e^{-x} L_m(x)\,L_n(x)\,dx = 0, \quad \text{for} \quad m \ne n.$$

For proof, refer Art. 13.6 of chapter 13.

15.17. Orthogonal expansion or genralized Fourier series

Let $\{y_n(x)\}$, $n = 0, 1, 2, 3, ...$ be an orthogonal set of functions with respect to weight function $p(x)$ on the interval $a \le x \le b$ and $f(x)$ a function that can be represented by a convergent series

$$f(x) = C_0 y_0(x) + C_1 y_1(x) + C_2 y_2(x) + ... = \sum_{n=0}^{\infty} C_n\ y_n(x) \qquad ...(1)$$

This is called an *orthogonal expansion or generalised Fourier series.* Coefficients C_0, C_1, C_2, C_3, ... are called *Fourier Constants.* Since $\{y_n(x)\}$ is an orthogonal set with respect to weight function $p(x)$, we get

$$\int_a^b p(x)\, y_n(x)\, y_m(x)\, dx = 0, \text{ when } m \neq n. \qquad ...(2)$$

Again, $$\| y_n(x) \| = \text{Norm of } y_n(x) = \int_a^b p(x)\, y_n^2(x)\, dx \qquad ...(3)$$

Multiply both sides of (1) by $p(x)\, y_n(x)$ (n fixed). Then integrating over $a \le x \le b$ and assuming that term–by–term integration is permissible, we get

$$\int_a^b p(x)\, f(x)\, y_n(x)\, dx = C_0 \int_a^b p(x)\, y_0(x)\, y_n(x)\, dx + C_1 \int_a^b p(x)\, y_1(x)\, y_n(x)\, dx$$

$$+ \ldots C_{n-1} \int_a^b p(x)\, y_{n-1}(x)\, y_n(x) + C_n \int_a^b p(x)\, y_n^2(x)\, dx + C_{n+1} \int_a^b p(x)\, y_{n+1}(x) y_n(x)\, dx + \ldots$$

or $$\int_a^b p(x)\, f(x)\, y_n(x)\, dx = C_n \| y_n(x) \|^2, \text{ using (2) and (3)}$$

or $$C_n = \frac{1}{\| y_n(x) \|^2} \int_a^b p(x)\, f(x)\, y_n(x)\, dx, \quad n = 0, 1, 2, 3, \ldots \qquad ...(4)$$

We now discuss two particular cases of the generalised Fourier series.

Particular Case I. Fourier–Legendre Series. We know that the set $\{P_n(x)\}$, $n = 0, 1, 2, 3, \ldots$ of Legendre polynomials is an orthogonal set on the interval $-1 \le x \le 1$. Hence, we get

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x), \qquad -1 \le x \le 1 \qquad ...(i)$$

where $$C_n = \frac{1}{\| P_n(x) \|^2} \int_{-1}^{1} f(x)\, P_n(x)\, dx, \qquad n = 0, 1, 2, \ldots \qquad ...(ii)$$

But, $$\| P_n(x) \|^2 = \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}, \text{ by Art 9.8, chapter 9.}$$

So (*ii*) becomes $$C_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x)\, P_n(x)\, dx \qquad ...(iii)$$

Expansion (*i*) of $f(x)$ in a series of Legendre's polynomials is known as *Fourier–Legendre series*. The constants C_n are given by (*iii*).

Note : For altenative proof refer, Art. 9.15, of chapter 9.

Particular Case II : Fourier–Bessel series. From Art. 15.15, we know that for a fixed n, the set of Bessel functions of first kind $J_n(\lambda_1 x)$, $J_n(\lambda_2 x)$, $J_n(\lambda_3 x)$, ..., with λ_i given by

$$\lambda_i = \alpha_i / a, \quad \text{where} \quad i = 1, 2, 3, \ldots, \quad (n \text{ fixed}) \qquad ...(i)$$

form an orthogonal set on the internal $0 \le x \le a$ with respect to the weight function $p(x) = x$.

$\therefore$ $$f(x) = \sum_{i=1}^{\infty} C_i J_n(\lambda_i x) \qquad ...(ii)$$

where $$C_i = \frac{1}{\| J_n(\lambda_i x) \|^2} \int_0^a x\, f(x)\, J_n(\lambda_i x)\, dx, \quad i = 1, 2, 3, \ldots \qquad ...(iii)$$

But, $$\| J_n(\lambda_i x) \|^2 = \int_0^a x\, J_n^2(\lambda_i x)\, dx = \frac{a^2}{2} J_{n+1}^2(a\lambda_i), \text{ by Art. 11.10}$$

$\therefore$ (*iii*) becomes, $$C_i = \frac{2}{a^2 J_{n+1}^2(a\lambda_i)} \int_0^a x\, f(x)\, J_n(\lambda_i x)\, dx\,,\ n = 1, 2, 3, \ldots \quad \ldots(iv)$$

Expansion (*ii*) of $f(x)$ in a series of Bessel functions is known as *Fourier–Bessel series.* The constants C_i are given by (*iv*).

Note. For alternative proof refer, Art. 11.11, of chapter 11.

Ex. 1. *Obtain the formal expansion of the function $f(x) = \pi x - x^2$, $0 \le x \le \pi$, in the series of orthonormal characteristic functions $\{\phi_n\}$ of Strun–Liouville problem $y'' + \lambda y = 0$, $y(0) = y(\pi) = 0$.*

Sol. Proceed as usual to show that $\phi_n(x) = (2/\pi)^{1/2} \sin nx$. Then the orthogonal expansion or generalised Fourier series of $f(x)$ is given by (refer Art. 15.17)

$$f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x)\,, \qquad \text{where} \qquad C_n = \int_0^{\pi} f(x)\, \phi_n(x)\, dx\,, \qquad \ldots(1)$$

noting that $\| \phi_n(x) \| = 1$, $p(x) = 1$, $a = 0$, $b = \pi$.

Now, $$C_n = \int_0^{\pi} (\pi x - x^2)(2/\pi)^{1/2} \sin nx\, dx$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \left[(\pi x - x^2)\left(-\frac{1}{n}\cos nx\right) - (\pi - 2x)\left(-\frac{1}{n^2}\sin nx\right) + (-2)\left(\frac{1}{n^3}\cos nx\right) \right]_0^{\pi}$$

(By chain rule of integration by parts)

$$= \left(\frac{2}{\pi}\right)^{1/2} \cdot \left(\frac{2}{n^3}\right)(1 - \cos n\pi) = \begin{cases} (2/\pi)^{1/2} \times (4/n^3), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Hence the required expansion of $f(x)$ is given by

$$\pi x - x^2 = \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\right)^{1/2} \cdot \frac{4}{(2n-1)^3} \cdot \left(\frac{2}{\pi}\right)^{1/2} \sin(2n-1)x, \qquad i.e., \qquad \pi x - x^2 = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3},\ 0 \le x \le \pi$$

Ex. 2. *Obtain the formal expansion of $f(x) = \log x$, $1 \le x \le e^{2\pi}$ in a seies of orthogonal eigen functions of Strum–Liouville problem $[xy']' + (1/x)y = 0$, $y(0) = y(e^{2\pi}) = 0$.*

Sol. Do as in Ex. 1. **Ans.** $\log x = -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left\{\frac{2n-1}{2} \log|x|\right\}$

OBJECTIVE PROBLEMS ON CHAPTER 15

Write (a), (b), (c) or (d) whichever is correct

1. For the Strum-Liouville problem $(1+x^2)y'' + 2xy' + \lambda x^2 y = 0$ with $y'(1) = 0$ and $y'(10) = 0$, the eigenvalues, λ, satisfy

(*a*) $\lambda \ge 0$ (*b*) $\lambda < 0$ (*c*) $\lambda \ne 0$ (*d*) $\lambda \le 0$ **[GATE 2003]**

2. Let *n* be non-negative integer. The eigenvalues of the Strum-Liouville problem $y'' + \lambda y = 0$ with boundary conditions $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$ are

(*a*) n (*b*) $n^2\pi^2$ (*c*) $n\pi$ (*d*) n^2 **[GATE 2002]**

3. The eigenvalues of the Strum-Liouville problem $y'' + \lambda y = 0,\ 0 \le x \le \pi,\ y(0) = 0,\ y'(\pi) = 0$ are (*a*) $n^2/4$ (*b*) $(2n-1)^2\pi^2/4$ (*c*) $(2n-1)^2/4$ (*d*) $n^2\pi^2/4$ **[GATE 2001]**

4. The eigenvalues of the boundary value problem $x'' + \lambda x = 0,\ x(0) = 0,\ x(\pi) + x'(\pi) = 0$ satisfy

(*a*) $\lambda + \tan\lambda\pi = 0$ (*b*) $\sqrt{\lambda} + \tan\lambda\pi = 0$

(*c*) $\sqrt{\lambda} + \tan\sqrt{\lambda}\pi = 0$ (*d*) $\lambda + \tan\sqrt{\lambda}\pi = 0$. **[GATE 2000]**

ANSWERS

1. (*a*) 2. (*d*) 3. (*c*) 4. (*b*)

MISCELLANEOUS PROBLEM ON CHAPTER 15

1. Show that the set {1, cos $(2n\pi/T)x$, n = 1, 2, 3, ... in orthogonal set of functions on an interval $0 \le x \le T$. **[Lucknow 2010]**

2. Find eigenfunctions of the system $u'' + \lambda u = 0, -\pi \le x \le \pi$ with the boundary condition $u(-\pi) = u(\pi)$, $u'(-\pi) = u'(\pi)$.

Ans. {1, cos nx, sin nx}, n = 1, 2, 3 .... **[Nagpur 2010]**

3. Prove that the set of eigenfunctions of strum-Liouville problem $\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + \lambda q(x) = 0$, $y(0) = 0$, $y(1) = 0$, $q(x) > 0$, form a set of orthogonal functions **[Himanchal 2008]**

4. Define norm of a function. **[Meerut 2011]**

Hints. Refer Art. 15.1 and 15.4.

MISCELLANEOUS PROBLEMS BASED ON THIS PART OF THE BOOK

Ex. 1. *The value of $P_n(-1)$ is*

(a) 1 *(b) 0* *(c) – 1* *(d) $(-1)^n$* **[Agra 2008]**

Sol. Ans. (*d*). Refer part (*ii*) of Ex. 2, page 9.6

Ex. 2. *The value of $P_0(x)$ is*

(a) 0 *(b) ∞* *(c) 1* *(d) None of these* **[Agra 2007]**

Sol. Ans. (*c*). Refer Art. 9.2

Ex. 3. *Rodrigue's formula for $P_n(x)$ is*

(a) $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ *(b)* $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^{-n}$ *(c)* $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2+1)^n$

(d) None of these **[Agra 2007]**

Sol. Ans. (*a*). Refer Art-9.14

Ex. 4. *Statement for the following linked questions 14 (i) and 14 (ii).*

Let $n \geq 3$ be an integer. Let y be the polynomial solution of

$$(1 - x^2)\,y'' - 2xy' + n(n-1)\,y = 0 \qquad \text{satisfying} \qquad y(1) = 1.$$

4. (*i*): *Then the degree of y is*

(a) n *(b) n – 1* *(c) less than n – 1* *(d) greater than n + 1*

4. (*ii*): *If* $I = \int_{-1}^{1} y(x)x^{n-3}dx$ *and* $J = \int_{-1}^{1} y(x)x^n dx$, *then*

(*a*) $I \neq 0,\ J \neq 0$ (*b*) $I \neq 0, J = 0$ (*c*) $I = 0, J \neq 0$ (*d*) $I = 0, J = 0$ **[GATE 2008]**

Sol: 4. (*i*): Ans. (*a*). Refer Ex. 2 (*i*), page 9.6

4. (*ii*): Ans. (*c*). Refer Ex. 5, page 9.29

5. *Using Rodrigue's formula, find value of $P_4(x)$ at $x = 1$.* **[Kanpur 2008]**

[Hint. Do as in Ex. 1, page 9.37 to get $P_4(x) = (35x^4 - 30x^2 + 3)/8$ and so the value of $P_4(x)$ *at* $x = 1$ is (35 – 30 + 3)/8, i.e. 1.

6. Show that the condition of integrability of (*i*) $Pdx + Qdy + Rdz = 0$ implies the orthoganality of any pair of intersecting curves of the families (*ii*) $(dx)/P = (dy)/Q = (dz)/R$ and (*iii*) $\dfrac{dx}{\partial Q/\partial z - \partial R/\partial y} = \dfrac{dy}{\partial R/\partial x - \partial P/\partial z} = \dfrac{dz}{\partial P/\partial y - \partial Q/\partial x}$. Hence show that the curves of (*iii*) all lie on the surfaces of (*i*). Verify this conclusion for $P = ny - mz, Q = lz - nx, R = mx - ly$. **[I.A.S. 2001]**

7. If f_1, f_2, f_3 are homogeneous functions of the same degree in x, y and z and if $xf_1 + yf_2 + zf_3 = 0$, then show that the equation $f_1dx + f_2dy + f_3dz = 0$ is integrable.

[Himanchal 2000; Kerla 2001, 07, Pune 2010]

8. If the equation $x_1\,dx_1 + x_2\,dx_2 + \ldots + x_n\,dx_n = 0$ has integrating factor, show that it has infinitely many. If $n = 2$, prove that the equation always has an integrating factor.

[Himanchal 2002; I.A.S. 2002; Calicut 2004, 05; Osmania 2000, 06; U.P.(P.C.S.) 2003]

9. $xdx + ydy + zdz = 0$ *is the first order differential equation of*

(a) sphere *(b) ellipsoid* *(c) right circular cone* *(d) hyperboloid.* **[I.A.S. (Prel) 2001]**

Sol. Ans. (*a*). Integrating the given differential equation, $x^2/2 + y^2/2 + z^2/2 = a^2/2$, where $a^2/2$ is an arbitrary constant of integration. Thus, we get $x^2 + y^2 + z^2 = a^2$, which is a sphere.

10. Show that $_1F_1(a\,;\,b;\,z) = \dfrac{1}{\Gamma(a)} \int_0^\infty e^{-t}t^{a-1}\ {}_0F_1(-;\,b;\,zt)\ dt$ **[Kanpur 2009]**

Sol. Using Art. 14.3 for the value of ${}_0F_1(-;\,b;\,zt)$, we get

$$\text{R.H.S.} = \frac{1}{\Gamma(a)}\int_0^\infty e^{-t}\ t^{a-1}\left(\sum_{r=0}^{\infty}\frac{(zt)^r}{(b)_r\,r!}\right)dt$$

$$= \frac{1}{\Gamma(a)}\sum_{r=0}^{\infty}\frac{z^r}{(b)_r\,r!}\int_0^\infty e^{-t}\ t^{a+r-1}\,dt = \frac{1}{\Gamma(a)}\sum_{r=0}^{\infty}\frac{z^r}{(b)_r\,r!}\,\Gamma(a+r)$$

[Using definition of Gamma function 6.2 page 6.1]

$$= \sum_{r=0}^{\infty}\frac{(a)_r}{(b)_r}\,\frac{z^r}{r!} = F\ (a;\,b;\,z), \text{ using art-14.3}$$

11. Apply the method of Frobenius to the equation $xy'' + 2y' + xy = 0$ to derive its general solution $y = c_0\ \{(\cos x)/x\} + c_1\ \{(\sin x)/x\}$ **[Delhi B.Sc. (Hons.) II 2011, Nagpur 1996]**

Sol. Given $\qquad xy'' + 2y' + xy = 0 \qquad \ldots (1)$

Let the series solution of (1) be $\qquad y = \sum_{m=0}^{\infty} c_m\,x^{kem}, \text{ where } c_0 \neq 0 \qquad \ldots (2)$

From (2), $\qquad y' = \sum_{m=0}^{\infty}(k+m)c_m\,x^{k+m-1}$ and $y'' = \sum_{m=0}^{\infty} c_m\,(k+m)(k+m-1)x^{k+m-2} \qquad \ldots (3)$

Substituting the values of y, y' and y'' given by (2) and (3) in (1), we get

$$x\sum_{m=0}^{\infty} c_m\,(k+m)(k+m-1)x^{k+m-2} + 2\sum_{m=0}^{\infty} c_m\,(k+m)\,x^{k+m-1} + x\sum_{m=0}^{\infty} c_m\,x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m\,(k+m)(k+m-1)x^{k+m-1} + 2\sum_{m=0}^{\infty} c_m\,(k+m)\,x^{k+m-1} + \sum_{m=0}^{\infty} c_m\,x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty} c_m\,(k+m)\{(k+m-1)+2\}x^{k+m-1} + \sum_{m=0}^{\infty} c_m\,x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty} c_m\,(k+m)\,(k+m+1)\,x^{k+m-1} + \sum_{m=0}^{\infty} c_m\,x^{k+m+1} = 0 \qquad \ldots (4)$$

Equating to zero the hcoefficient of the smallest power of x, namely x^{k-1}, the above identity (4) yields the indicial equation

$c_0\,k\,(k+1) = 0$ or $k\,(k+1) = 0$, as $c_0 \neq 0$ so that $k = 0$ or $k = -1y$,

which are uequal and differ by an integer. Next, equations the coefficient of x^{k+m-1}, we arrive at the recurrence relation.

$$c_m(k+m)\,(k+m+1) + c_{m-2} = 0 \quad \text{so that } c_m = -\{1/(k+m)(k+m+1)\}c_{m-2} \qquad \ldots (5)$$

Finally, we equate to zero the coefficient of x^k in the identity (4) and get

$$c_1(k+1)(k+2)=0. \qquad \ldots (6)$$

If we take $k = -1$, (6) shows that c_1 is ideterminate. With $k = -1$ and using (5), we now proceed to express c_2, c_4, c_6 in terms of c_0 and $c_3, c_5, c_7, \ldots$ in terms of c_1 if c_1 is assume to be finite. Setting $k = -1$ in (5), we have

$$c_m=-\{1/m\,(m-1)\}\,c_{m-2} \qquad \ldots (7)$$

Putting $m = 3, 5, 7, \ldots$ in (7) by turn, we obtain

$$c_3=-\frac{1}{3\times 2}c_1=-\frac{c_1}{3!},\quad c_5=-\frac{1}{5\times 4}c_3=-\frac{1}{1\times 2\times 3\times 4\times 5}c_1=\frac{c_1}{5!}\ \text{and so on}$$

Putting m = 2, 4, 6, ... in (7) by turn, we obtain

$$c_2=-\frac{1}{2\times 1}c_0=-\frac{c_0}{2!},\ c_4=\frac{1}{4\times 3}c_2=\frac{1}{1\times 2\times 3\times 4}c_0=\frac{c_0}{4!}\ \text{and so on}$$

With $k = -1$ and substituting the above values of $c_2, c_3, c_4, c_5, \ldots$ in (1) the required solution is

$$y=x^{-1}(c_0+c_1x+c_2x^2+c_3x^3+\ldots)=x^{-1}(c_0+c_2x^2+c_4x^4+\ldots)+x^{-1}(c_1x+c_3x^3+c_5x^5+\ldots)$$

or $$y=c_0x^{-1}(1-x^2/2!+x^4/4!\ \ldots)+c_1x^{-1}(x-x^3/3!+x^5/5!-\ldots)$$

or $$y=c_0\{(\cos x)/x\}+c_1\{(\sin x)/x\},$$

$$[\because\ \cos x=1-x^2/2!+x^4/4!-\ldots,\ \sin x=x-x^3/3!+x^5/5-\ldots.]$$

Ex. 12. If $y=\sum_{m=0}^{\infty}c_mx^{r+m}$ is assumed to be a solution of the differential equation $x^2y''-xy'-3(1+x^2)y=0$, then the value of r are

(*a*) 1 and 3 (*b*) –1 and 3 (*c*) 1 and – 3 (*d*) –1 and –3 **[GATE 2012]**

Hint. Ans. (*b*). As usual find indicial equation $r^2-2r-3=0$ giving $r=-1,3$

PARTIAL DIFFERENTIAL EQUATIONS

WHERE IS WHAT

CONTENTS

1.	Origin of partial differential equations	1.3 – 1.20
2.	Linear partial differential equations of order one	2.1 – 2.40
3.	Non-linear partial differential equations of order one	3.1 – 3.84
4.	Homogeneous linear partial differential equations with constant coefficients	4.1 – 4.34
5.	Non-homogeneous partial differential equations with constant coefficients	5.1 – 5.30
6.	Partial differential equations reducible to equations with constant coefficients	6.1 – 6.11
7.	Partial differential equations of order two with variable coefficients	7.1 – 7.14
8.	Classification of partial differential equations Reduction to canonical (or normal) form. Riemann method	8.1 – 8.50
9.	Mong's methods	9.1 – 9.50
10.	Transport equations	10.1 – 10.5

1

Origin of Partial Differential Equations

1.1 INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. In the present part of the book, we propose to study various methods to solve partial differential equations.

1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E.) [Delhi Maths (H) 2001]

Definition. An equation containing one or more partial derivatives of an unknown frunction of two or more independent variables is known as a *partial differential equation.*

For examples of partial differential equations we list the following:

$$\partial z/\partial x + \partial z/\partial y = z + xy \qquad \ldots (1)$$

$$(\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x(\partial z/\partial x) \qquad \ldots (2)$$

$$z(\partial z/\partial x) + \partial z/\partial y = x \qquad \ldots (3)$$

$$\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz \qquad \ldots (4)$$

$$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2} \qquad \ldots (5)$$

$$y\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\right\} = z(\partial z/\partial y) \qquad \ldots (6)$$

1.3 ORDER OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

Definition. The *order* of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

In Art. 1.2, equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

1.4 DEGREE OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

In 1.2, equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

1.5 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

In Art. 1.2, equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

1.6 NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$$p = \partial z/\partial x, \qquad q = \partial z/\partial y, \qquad r = \partial^2 z/\partial x^2, \qquad s = \partial^2 z/\partial x \partial y \qquad \text{and} \qquad t = \partial^2 z/\partial y^2$$

In case there are n independent variables, we take them to be $x_1, x_2, ..., x_n$ and z is then regarded as the dependent variable. In this case we use the following notations :

$p_1 = \partial z/\partial x_1$, $p_2 = \partial z/\partial x_2$, $p_3 = \partial z/\partial x_3$, and $p_n = \partial z/\partial x_n$.

Sometimes the partial differntiations are also denoted by making use of suffixes. Thus we write $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xy} = \partial^2 u/\partial x \partial y$ and so on.

1.7 Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations with examples. [Delhi Maths (H) 2001; 2004]

Linear equation. *A first order equation* $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$.

For examples, $yx^2p + xy^2q = xyz + x^2y^3$ and $p + q = z + xy$ are both first order linear partial differential equations.

Semi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form* $P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$

For examples, $xyp + x^2yq = x^2y^2z^2$ and $yp + xq = (x^2z^2/y^2)$ are both first order semi-linear partial differential equations.

Quasi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form*

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

For examples, $x^2zp + y^2zp = xy$ and $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$ are first order quasi-linear partial differential equations.

Non-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *which does not come under the above three types, in known as a non-liner equation.*

For examples, $p^2 + q^2 = 1$, $p\, q = z$ and $x^2\, p^2 + y^2\, q^2 = z^2$ are all non-linear partial differential equations.

1.8 Origin of partial differential equations.

We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

1.9 Rule I. Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation $F(x, y, z, a, b) = 0$, ...(1)

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$\partial F/\partial x + p(\partial F/\partial z) = 0$ and $\partial F/\partial y + q(\partial F/\partial z) = 0$ ..(2)

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0, \quad ...(3)$$

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems: For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$F(x, y, z, a, b) = 0$, $\partial F / \partial x = 0$ and $\partial F / \partial y = 0$.

The equation free from a and b will be the required partial differential equation.

Three situations may arise :

Situation I. *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

For example, consider $z = ax + y$, ... (1)

where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $\partial z / \partial x = a$ ... (2)

Differentiating (1) partially w.r.t. 'y', we get $\partial z / \partial y = 1$ ... (3)

Eliminating a between (1) and (2) yields $z = x(\partial z / \partial x) + y$ ... (4)

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Situation II. *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.*

Example: *Eliminate a and b from* $az + b = a^2x + y$ ... (1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$a(\partial z / \partial x) = a^2$ ... (2) $a(\partial z / \partial y) = 1$ ... (3)

Eliminating a from (2) and (3), we have $(\partial z / \partial x)\,(\partial z / \partial y) = 1$,

which is the unique partial differential equation of order one.

Situation III. *When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.*

Example: *Eliminate a, b and c from* $z = ax + by + cxy$ ... (1)

Differentiating (1) partially w.r.t., 'x' and 'y', we have

$\partial z / \partial x = a + c\,y$ ... (2) $\partial z / \partial y = b + c\,x$ ... (3)

From (2) and (3), $\partial^2 z / \partial x^2 = 0$, $\partial^2 z / \partial y^2 = 0$ ... (4)

and $\partial^2 z / \partial x \partial y = c$ ... (5)

Now, (2) and (3) $\Rightarrow$ $x(\partial z / \partial x) = ax + cxy$ and $y(\partial z / \partial y) = by + cxy$

$\therefore$ $x(\partial z / \partial x) + y(\partial z / \partial y) = ax + by + cxy + cxy$

or $x(\partial z / \partial x) + y(\partial z / \partial y) = z + xy(\partial^2 z / \partial x \partial y)$, using (1) and (5) ... (6)

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

1.10 SOLVED EXAMPLES BASED ON RULE I OF ART 1.9

Ex. 1. *Find a partial differential equation by eliminating a and b from* $z = ax + by + a^2 + b^2$.

Sol. Given $z = ax + by + a^2 + b^2$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = b$.

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2,$$

which is the required partial differential equation.

Ex. 2. *Eliminate arbitrary constants a and b from* $z = (x - a)^2 + (y - b)^2$ *to form the partial differential equation.* **[Jiwaji 1999; Banglore 1995]**

Sol. Given $z = (x - a)^2 + (y - b)^2$. ...(1)

Differentiating (1) partially with respect to a and b, we get

$\partial z/\partial x = 2(x - a)$ and $\partial z/\partial y = 2(y - b)$.

Squatring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x - a)^2 + 4(y - b)^2 = 4\,[(x - a)^2 + (y - b)^2]$$

or $(\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z$, using (1).

Ex. 3. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations :*

(a) $z = a(x + y) + b$. (b) $z = ax + by + ab$. **[Bhopal 2010, Rewa 1996]**

(c) $z = ax + a^2y^2 + b$. **[Agra 2010]** (d) $z = (x + a)\,(y + b)$. **[Madurai Kamraj 2008]**

Sol. (a) Given $z = a(x + y) + b$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = a$.

Eliminating a between these, we get $\partial z/\partial x = \partial z/\partial y$,

which is the required partial differential equation.

(b) Given $z = ax + by + ab$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = b$ ....(2)

Substituting the values of a and b from (2) in (1), we get

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\mathrm{d}x)(\partial z/\partial y),$$

which is the required partial differential equation.

(c) Try yourself. **Ans.** $\partial z/\partial y = 2y(\partial z/\partial x)^2$.

(d) Try yourself. **Ans.** $z = (\partial z/\partial y)\,(\partial z/\partial x)$.

Ex. 4. *Eliminate a and b from* $z = axe^y + (1/2) \times a^2e^{2y} + b$. **[Meerut 2006]**

Sol. Given $z = axe^y + (1/2) \times a^2e^y + b$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = ae^y \quad ...(2)$$

and $$\partial z/\partial y = axe^y + a^2e^{2y} = x(ae^y) + (ae^y)^2. \quad ...(3)$$

Substituting the value of ae^y from (2) in (3), we get $\partial z/\partial y = x(\partial z/\partial x) + (\partial z/\partial x)^2$.

Ex. 5(*a*). *Form the partial differential equation by eliminating h and k from the equation* $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2$. **[Gulbarga 2005; I.A.S. 1996]**

Sol. Given $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$2(x - h) + 2z(\partial z/\partial x) = 0$ or $(x - h) = -z(\partial z/\partial x)$ ...(2)

and $2(y - k) + 2z(\partial z/\partial y) = 0$ or $(y - k) = -z(\partial z/\partial y)$. ...(3)

Substituting the values of $(x - h)$ and $(y - k)$ from (2) and (3) in (1) gives

$z^2(\partial z/\partial x)^2 + z^2(\partial z/\partial y)^2 + z^2 = \lambda^2$ or $z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = \lambda^2$,

which is the required partial differential equation.

Ex. 5(*b*). *Find the differential equation of all spheres of radius λ, having centre in the xy-plane.* **[M.D.U. Rohtak 2005; I.A.S. 1996, K.U. Kurukshetra 2005]**

Sol. From the coordinate geometry of three-dimensions, the equation of any sphere of radius λ, having centre $(h, k, 0)$ in the xy-plane is given by

$(x-h)^2 + (y-k)^2 + (z-0)^2 = \lambda^2$ or $(x-h)^2 + (y-k)^2 + z^2 = \lambda^2$, ...(1)

where h and k are arbitray constants. Now, proceed exactly in the same way as in Ex. 5(a).

Ex. 6. *Form the differential equation by eliminating a and b from* $z = (x^2 + a)(y^2 + b)$.

[Madras 2005; Sagar 1997, I.A.S. 1997]

Sol. Given $z = (x^2 + a)(y^2 + b)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = 2x(y^2 + b)$ or $(y^2 + b) = (1/2x) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = 2y(x^2 + a)$ or $(x^2 + a) = (1/2y) \times (\partial z/\partial y)$. ...(3)

Substituting the values of $(y^2 + b)$ and $(x^2 + a)$ from (2) and (3) in (1) gives

$z = (1/2y) \times (\partial z/\partial y) \times (1/2x) \times (\partial z/\partial x)$ or $4xyz = (\partial z/\partial x)(\partial z/\partial y)$,

which the required partial differential equation.

Ex. 7. *Form differential equation by eliminating constants A and p from* $z = A\, e^{pt} \sin px$.

Sol. Given $z = A\, e^{pt} \sin px$. ...(1)

Differentiating (1) partially with respect to x and t, we get

$\partial z/\partial x = Ap\, e^{pt} \cos px$ ...(2) $\partial z/\partial t = Ap\, e^{pt} \sin px$. ...(3)

Differentiating (2) and (3) partially with respect to x and t respectively gives

$\partial^2 z/\partial x^2 = -Ap^2 e^{pt} \sin px$. ...(4) $\partial^2 z/\partial t^2 = Ap^2 e^{pt} \sin px$. ...(5)

Adding (4) and (5), $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0$,

which is the required partial differential equation.

Ex. 8. *Find the differential equation of the set of all right circular cones whose axes coincide with z-axix.* **[I.A.S. 1998]**

Sol. The general equation of the set of all right circular cones whose axes coincide with z-axis, having semi-vertical angle α and vertex at (0, 0, c) is given by

$$x^2 + y^2 = (z-c)^2 \tan^2 \alpha, \qquad \text{... (1)}$$

in which both the constants c and α are arbitrary.

Differentiating (1) partially, w.r.t. x and y, we get

$2x = 2(z-c)(\partial z/\partial x)\tan^2\alpha$ and $2y = 2(z-c)(\partial z/\partial y)\tan^2\alpha$

$\Rightarrow\ y(z-c)(\partial z/\partial x)\tan^2\alpha = xy$ and $x(z-c)(\partial z/\partial y)\tan^2\alpha = xy$

$\Rightarrow\ y(z-c)(\partial z/\partial x)\tan^2\alpha = x(z-c)(\partial z/\partial y)\tan^2\alpha$

Thus, $y(\partial z/\partial x) = x(\partial z/\partial y)$, which is the required partial differential equation.

Ex. 9. *Show that the differential equation of all cones which have their vertex at the origin is* $px + qy = z$. *Verify that* $yz + zx + xy = 0$ *is a surface satisfying the above equation.*

[I.A.S. 1979, 2009]

Sol. The equation of any cone with vertex at origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad \text{...(1)}$$

where a, b, c, f, g, h are parameters. Differentiating (1) partially w.r.t. 'x' and 'y' by turn, we have (noting that $p = \partial z/\partial x$ and $q = \partial z/\partial y$)

$2ax + 2czp + 2fyp + 2g(px + z) + 2hy = 0$ or $ax + gz + hy + p(cz + gx + fy) = 0$ ...(2)

and $2by + 2czq + 2f(yq + z) + 2gxq + 2hx = 0$ or $by + fz + hx + q(cz + fy + gx) = 0$. ...(3)

Multiplying (2) by x and (3) by y and adding, we have

$$(ax^2 + by^2 + gzx + fyz + 2hxy) + (cz + fy + gx)(px + qy) = 0.$$

$$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0, \text{ using (1)}$$

or $(cz + fy + gx)(px + qy - z) = 0$ or $px + qy - z = 0$, ...(4)

which is required partial differential equation.

Second Part : Given surface is $yz + zx + xy = 0$ ...(5)

Differentiating (5) partially w.r.t. 'x' and 'y' by turn, we get

$yp + px + z + y = 0$ and $z + qy + xq + x = 0$. ...(6)

Solving (6) for p and q, $p = -(z + y)/(x + y)$ and $q = -(z + x)/(x + y)$.

$$\therefore \quad px + qy - z = -\frac{x(z+y)}{x+y} - \frac{y(z+x)}{x+y} - z = -\frac{2(xy+yz+zx)}{x+y} = 0, \text{ using (5)}$$

Hence (5) is a surface satisfying (4).

Ex. 10. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations:*

(*a*) $2z = x^2/a^2 + y^2/b^2$ **[Nagpur 1995; M.D.U. Rohtak 2006]**

(*b*) $2z = (ax + y)^2 + b$ **[Nagpur 1996; Delhi Maths (G) 2006; Pune 2010]**

Sol. (*a*) Given $2z = x^2/a^2 + y^2/b^2$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$2(\partial z/\partial x) = 2x/a^2$ ... (2) $2(\partial z/\partial y) = 2y/b^2$ ... (3)

From (2) and (3), $p = x/a^2$, $q = y/b^2$ $\Rightarrow$ $a^2 = x/p$, $b^2 = y/q$

Substituting these values of a^2 and b^2 in (1), we get

$2z = px + qy$, which is the required partial differential equation

(*b*) Given $2z = (ax + y)^2 + b$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$2p = 2a(ax + y)$ ... (2) $2q = 2(ax + y)$ ... (3)

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Dividing (2) by (3) yields $p/q = a$.

Substituting this value of a in (3), we get $q = (p/q)x + y$ or $px + qy = q^2$.

Ex. 11. *Eliminate a, b and c from* $z = a(x + y) + b(x - y) + abt + c$ **[I.A.S. 1998]**

Sol. Given $z = a(x + y) + b(x - y) + abt + c$ ... (1)

Differentiating (1) partially w.r.t. 'x', 'y' and 't', we get

$\partial z/\partial x = a + b$ ... (2) $\partial z/\partial y = a - b$ ... (3) $\partial z/\partial t = ab$ ... (4)

We have the identity: $(a + b)^2 - (a - b)^2 = 4ab$

$$\therefore \quad (\partial z/\partial x)^2 - (\partial z/\partial y)^2 = 4(\partial z/\partial t), \text{ using (2), (3) and (4)}$$

Ex. 12. *Form the partial differential equation by eliminating the arbitrary constants a and b from* $\log(az - 1) = x + ay + b$. **[I.A.S. 2002]**

Sol. (*a*) Given $\log(az - 1) = x + ay + b$ ... (1)

Differentiating (1) partially w.r.t. 'x', we get $\dfrac{a}{az-1}\dfrac{\partial z}{\partial x} = 1$ ... (2)

Differentiating (1) partially w.r.t. 'y', we get $\dfrac{a}{az-1}\dfrac{\partial z}{\partial y} = a$ ... (3)

From (3), $az - 1 = \dfrac{\partial z}{\partial y}$ so that $a = \dfrac{1 + (\partial z / \partial y)}{z}$ ... (4)

Putting the above values of $az - 1$ and a in (2), we have

$$\frac{1 + (\partial z / \partial y)}{z(\partial z / \partial y)} \frac{\partial z}{\partial x} = 1 \quad \text{or} \quad \left(1 + \frac{\partial z}{\partial y}\right) \frac{\partial z}{\partial x} = z \frac{\partial z}{\partial y}.$$

Ex. 13. *Find a partial differential equation by eliminating a, b, c, from* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

[Bhopal 2004; Jabalpur 2000, 03, Jiwaji 2000, Vikram 2002, 04; Ravishanker 2010]

Sol. Given $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} \frac{dz}{dx} = 0 \quad \text{or} \quad c^2 x + a^2 z \frac{dz}{dx} = 0 \quad \text{... (2)}$$

and $$\frac{2y}{b^2} + \frac{2x}{c^2} \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2 y + b^2 z \frac{\partial z}{\partial y} = 0. \quad \text{... (3)}$$

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2 + a^2 \left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{... (4)} \qquad c^2 + b^2 \left(\frac{\partial z}{\partial x}\right)^2 + b^2 z \frac{\partial^2 z}{\partial y^2} = 0. \quad \text{... (5)}$$

From (2), $c^2 = -(a^2 z / x) \times (\partial z / \partial x)$ ... (6)

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x} \frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad zx \frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial x}\right)^2 - z \frac{\partial z}{\partial x} = 0. \quad \text{... (7)}$$

Similarly, from (3) and (5), $$zy \frac{\partial^2 z}{\partial y^2} + y \left(\frac{\partial z}{\partial y}\right)^2 - z \frac{\partial z}{\partial y} = 0. \quad \text{... (8)}$$

Differentiating (2) partially w.r.t. y, $0 + a^2 \left\{(\partial z / \partial y)(\partial z / \partial x) + z(\partial^2 z / \partial x \partial y)\right\} = 0$

or $$(\partial z / \partial x)(\partial z / \partial y) + z(\partial^2 z / \partial x \partial y) = 0 \quad \text{... (9)}$$

(7), (8) and (9) are three possible forms of the required partial differential equations.

Ex. 14. *Find the partial differential equation of all planes which are at a constant distance 'a' from the origin.*

Sol. Let $lx + my + nz = a$ ... (1)

be the equation of the given plane where l, m, n are direction colines of the normal to the plane so that $l^2 + m^2 + n^2 = 1$, l, m, n being parameters ... (2)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$l + np = 0$ ...(3) $\qquad m + nq = 0,$ ... (4)

where $p = \partial z / \partial x$ and $q = \partial z / \partial y$. From (3) and (4), $l = -np$ and $m = -nq$. Substituting these values in (2), we have

$n^2(p^2 + q^2 + 1) = 1$ so that $n = (p^2 + q^2 + 1)^{-1/2}$ ... (5)

$\therefore \quad l = -np = -p(p^2 + q^2 + 1)^{-1/2}$ and $m = -nq = -q(p^2 + q^2 + 1)^{-1/2}$ ... (6)

Substituting the values of l, m, n given by (5) and (6) in (1), we get

$$-px\,(p^2+q^2+1)^{-1/2} - qy\,(p^2+q^2+1)^{-1/2} + z(p^2+q^2+1)^{-1/2} = a$$

or $z = px + qy + a\,(p^2+q^2+1)^{1/2}$, which is the required partial differential equation.

Ex. 15. *Show that the partial differential equation obtained by eliminating the arbitrary constants a and c from $z = ax + g(a)\,y + c$, where g (a) is an arbitrary function of a, is free of the variables x, y, z.*

Sol. Differentiating $z = ax + g(a)\,y + c$ partially w.r.t. 'x' and 'y' yields $p = a$ and $q = g(a)$. Eliminating a between them leads to $q = g(p)$ or $f(p, q) = 0$, where f is an arbitrary function of p and q. Clearly, the resulting partial differential equation contains p and q but none of the variables x, y, z.

Ex. 16. *Show that the partial differential equation obtained by eliminating the arbitrary constants a and b from $z = ax + by + f(a, b)$ is given by $z = px + qy + f(p, q)$.*

Sol. Differentiating $z = ax + by + f(a, b)$ ... (1)

partially with respect to 'x' and 'y', we get $p = a$ and $q = b$ ... (2)

Eliminating a and b from (1) and (2) yields $z = px + qy + f(p, q)$

Ex. 17. *Form a partial differential equation by eliminating a, b and c from the relation $ax^2 + by^2 + cz^2 = 1$.* **[Mysore 2004]**

Sol. Given $ax^2 + by^2 + cz^2 = 1.$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$2ax + 2cz(\partial z/\partial x) = 0$ ... (2) $\qquad 2by + 2cz\,(\partial z/\partial y) = 0$ ... (3)

Differentiating (2) partially w.r.t. 'y', we get

$$0 + 2c\left\{(\partial z/\partial y)\,(\partial z/\partial x) + z(\partial^2 z/\partial y\partial x\right\} = 0 \quad \text{or} \quad (\partial z/\partial x)\,(\partial z/\partial y) + z(\partial^2 z/\partial x\partial y) = 0\,, \ \ldots (4)$$

since c is an arbitrary constant. (4) is the desired partial differential equation.

Again, differentiating partially (2) w.r.t. x and (3) w.r.t. y, we get

$$2a + 2c\left\{(\partial z/\partial x)^2 + z(\partial^2 z/\partial x^2)\right\} = 0 \ \ldots (5) \qquad 2b + 2c\left\{(\partial z/\partial y)^2 + z(\partial^2 z/\partial y^2)\right\} = 0 \ \ldots (6)$$

From (2), $a = -(cz/x) \times (\partial z/\partial x)$. Putting this in (5), we get

$$-(cz/x) \times (\partial z/\partial x) + c\left\{(\partial z/\partial x)^2 + z(\partial^2 z/\partial x^2\right\} = 0 \ \text{ or } \ zx(\partial^2 z/\partial x^2) + x(\partial z/\partial x)^2 - z(\partial z/\partial x) = 0 \ \ldots (7)$$

Similarly, from (3) and (6), we get $zy(\partial^2 z/\partial y^2) + y(\partial z/\partial y)^2 - z(\partial z/\partial y) = 0$ . ... (8)

(4), (7) and (8) are three possible forms of the required partial differential equations.

EXERCISE 1 (A)

Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial differential equations.

1. $z = A\,e^{pt}\sin px$, (p and A). **Ans.** $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0.$

2. $z = A\,e^{-p^2 t}\cos px$, (p and A) **(Sagar 1999; Ranchi 2010)** **Ans.** $\partial^2 z/\partial x^2 = dz/\partial t$

3. $z = ax^3 + by^3$; (a, b) **Ans.** $x(\partial z/\partial x) + y(\partial z/\partial y) = 3z$

4. $4z = [ax + (y/a) + b]^2$; (a, b). **(Delhi B.A. (Prog) II 2011)** **Ans.** $z = (dz/\partial x)\,(\partial z/\partial y)$

5. $z = ax^2 + bxy + cy^2$, (a, b,c) **Ans.** $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = 2z$

6. $z^2 = ax^3 + by^3 + ab$, (a, b)

Ans. $9x^2y^2z = 6x^3y^2(\partial z/\partial x) + 6x^2y^3(\partial z/\partial y) + 4z(\partial z/\partial x)\,(\partial z/\partial y)$

7. $e^{1/\{z-(x^2/y)\}} = \dfrac{ax^2}{y^2} + \dfrac{b}{y}, (a, b)$ **Ans.** $x\dfrac{\partial z}{\partial x} + 2y\dfrac{\partial z}{\partial y} = 2(z - x^2/y)^2$

8. Find the differential equation of the family of spheres of radius 4 with centres on the *xy*-plane. **Ans.** $(x-y)^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = 16(\partial z/\partial x - \partial z/\partial y)^2$

9. Find the P.D.E of planes having equal x and y intercepts. **Ans.** $p - q = 0$

10. Find the partial differential equation of the family of spheres of radius 7 with centres on the plane $x - y = 0$. **Ans.** $(p^2 + q^2 + 1)\,(x - y)^2 = 49\,(p - q)$

11. Find the partial differential equation of all spheres whose centres lie on z-axis.

Ans. $x\,q - y\,p = 0$

1.11 Rule II. Derivation of partial differential equation by the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z.

[Meerut 1995]

Proof. Given $\phi(u, v) = 0.$ ...(1)

We treat z as dependent variable and x and y as independent variables so that

$\partial z/\partial x = p$, $\partial z/\partial y = q$, $\partial y/\partial x = 0$ and $\partial x/\partial y = 0$.

Differentiating (1) partially with respect to x, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

or $$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

or $$\frac{\partial \phi}{\partial u}\Big/\frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right). \quad ...(3)$$

Similarly, differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\Big/\frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) \quad ...(4)$$

Eliminating ϕ with the help of (3) and (4), we get

$$\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

or $$\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

or $$Pp + Qq = R, \quad ...(5)$$

where $P = \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}$, $Q = \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}$, $R = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}$.

Thus we obtain a linear partial differntial equation of first order and of first degree in p and q.

Note. If the given equation between x, y, z contains two arbitrary functions, then in general, their elimination gives rise to equations of higher order.

1.12 SOLVED EXAMPLES BASED ON RULE II OF ART. 1.11.

Ex. 1. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x + y + z, x^2 + y^2 - z^2) = 0$. *What is the order of this partial differential equation* ?

[Bilaspur 2003; Indore 2003; Jiwaji 2003; Vikram 2001]

Sol. Given $\phi(x + y + z, x^2 + y^2 - z^2) = 0.$ ...(1)

Let $u = x + y + z$ and $v = x^2 + y^2 - z^2.$ ...(2)

Then (1) becomes $\phi(u, v) = 0.$ ...(3)

Differentiating (3) w.r.t., 'x' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0. \quad ...(4)$$

From (2), $\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial z} = 1, \quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial z} = -2z, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial y} = 2y.$ ..(5)

From (4) and (5), $(\partial\phi/\partial u)(1 + p) + 2(\partial\phi/\partial v)(x - pz) = 0$

or $(\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(x - pz)/(1 + p).$ ...(6)

Again, differentiating (3) w.r.t., 'y' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = 0$$

or $(\partial\phi/\partial u)(1 + q) + 2(\partial\phi/\partial v)(y - zq) = 0,$ by (5)

or $(\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(y - qz)/(1 + q).$ ...(7)

Eliminating ϕ from (6) and (7), we obtain

$(x - pz)/(1 + p) = (y - qz)/(1 + q)$ or $(1 + q)(x - pz) = (1 + p)(y - qz)$

or $(y + z)p - (x + z)q = x - y$, which is the desired partial differential equation of first order.

Ex. 2. *Form a partial differential equation by eliminating the arbitrary function f from the equation* $x + y + z = f(x^2 + y^2 + z^2)$. **(Kanpur 2011)**

Sol. Given $x + y + z = f(x^2 + y^2 + z^2).$ ...(1)

Differentiating partially w.r.t. 'x' and 'y', (1) gives

$1 + p = f'(x^2 + y^2 + z^2).(2x + 2zp).$ ...(2)

and $1 + q = f'(x^2 + y^2 + z^2).(2y + 2zq).$ ...(3)

Eliminating $f'(x^2 + y^2 + z^2)$ from (2) and (3), we obtain

$(1 + p)/(2x + 2zp) = (1 + q)/(2y + 2zq)$ or $(1 + p)(y + zq) = (1 + q)(x + zp)$

or $(y - z)p + (z - x)q = x - y$, which is the required partial differential equations.

Ex. 3. *Eliminate the arbitrary functions f and F from* $y = f(x - at) + F(x + at)$.

(Sagar 1997; Vikram 1995; Jabalpur 2002)

Sol. Given $y = f(x - at) + F(x + at).$ ...(1)

From (1), $\partial y/\partial x = f'(x - at) + F'(x + at)$

and hence $\partial^2 y/\partial x^2 = f''(x - at) + F''(x + at).$ ...(2)

Also, $\partial y/\partial t = f'(x - at) \,.\, (-a) + F'(x + at) \,.\, (a)$

and hence $\partial^2 y/\partial t^2 = f''(x - at) \,.\, (-a)^2 + F''(x + at) \,.\, (a)^2$

or $\partial^2 y/\partial t^2 = a^2[f''(x - at) + F''(x + at)].$ ...(3)

Then, (2) and (3) $\Rightarrow$ $\partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2).$

Ex. 4. *Eliminate arbitrary function f from*

(*i*) $z = f(x^2 - y^2).$ **[Bilaspur 1996; Sagar 1996; Bangalore 1995]**

(*ii*) $z = f(x^2 + y^2).$ **[Meerut 1995; Pune 2010]**

Sol. (*i*) Given $z = f(x^2 - y^2).$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(x^2 - y^2) \times 2x$ so that $f'(x^2 - y^2) = (1/2x) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(x^2 - y^2) \times (-2y)$ so that $f'(x^2 - y^2) = -(1/2y) \times (\partial z/\partial y).$ ...(3)

Eliminating $f'(x^2 - y^2)$ between (2) and (3), we have

$$\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y} \quad \text{or} \quad y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

(*ii*) Proceed as in part (1). **Ans.** $y(\partial z/\partial x) - x(\partial z/\partial y) = 0$

Ex. 5. *Form a partial differential equation by eliminating the function f from*

(*i*) $z = f(y/x)$. **[Sagar 2000]** (*ii*) $z = x^n f(y/x)$.

Sol. Given $z = f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(y/x) \times (-y/x^2)$ or $f'(y/x) = -(x^2/y) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(y/x) \times (1/x)$ or $f'(y/x) = x(\partial z/\partial y)$. ...(3)

Eliminating $f'(y/x)$ between (2) and (3), we have

$$-\frac{x^2}{y}\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y} \quad \text{or} \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

which is the required partial differential equation.

(*ii*) Given $z = x^n f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = n\,x^{n-1} f(y/x) + x^n f'(y/x) \times (-y/x^2) \quad ...(2)$$

and $$\partial z/\partial y = x^n f'(y/x) \times (1/x). \quad ...(3)$$

Multiplying both sides of (2) by x, we have $x(\partial z/\partial x) = n\,x^n f(y/x) - yx^{n-1} f'(y/x)$. ...(4)

Multiplying both sides of (3) by y, we have $y(\partial z/\partial y) = y\,x^{n-1} f'(y/x)$. ...(5)

Adding (4) and (5), $x(\partial z/\partial x) + y(\partial z/\partial y) = n\,x^n f(y/x)$

or $x(\partial z/\partial x) + y(\partial z/\partial y) = nz$, by (1)

Ex. 6. *Form a partial differential equation by eliminating the function* ϕ from $lx + my + nz = \phi(x^2 + y^2 + z^2)$. **[Ravishankar 2003; Vikram 2003]**

Sol. Given $lx + my + nz = \phi(x^2 + y^2 + z^2)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$l + n(\partial z/\partial x) = \phi'(x^2 + y^2 + z^2) \times \{2x + 2z(\partial z/\partial x)\} \quad ...(2)$$

and $$m + n(\partial z/\partial y) = \phi'(x^2 + y^2 + z^2) \times \{2y + 2z(\partial z/\partial y)\} \quad ...(3)$$

Dividing (2) by (3), we get $$\frac{l + n(\partial z/\partial x)}{m + n(\partial z/\partial y)} = \frac{2\{x + z(\partial z/\partial x)\}}{2\{y + z(\partial z/\partial y)\}}$$

or $(ny - mz)(\partial z/\partial x) + (lz - nx)(\partial z/\partial y) = mx - ly$, which is the required partial differential equation.

Ex. 7. *Form partial differential eqn. by eliminating the function f from* $z = e^{ax+by} f(ax - by)$.

Sol. Given $z = e^{ax+by} f(ax - by)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = e^{ax+by}\, a f'(ax - by) + a\, e^{ax+by} f(ax - by) \quad ...(2)$$

and $$\partial z/\partial y = e^{ax+by}\{-b f'(ax - by)\} + b\, e^{ax+by} f(ax - by). \quad ...(3)$$

Multiplying (2) by b and (3) by a and adding, we get

$b(\partial z/\partial x) + a(\partial z/\partial y) = 2ab\, e^{ax+by} f(ax - by)$ or $b(\partial z/\partial x) + a(\partial z/\partial y) = 2abz$, by (1)

Ex. 8. *Form a partial differential equation by eliminating the arbitrary functions f and F from* $z = f(x + iy) + F(x - iy)$, *where* $i^2 = -1$. **[Bilaspur 2004; Jiwaji 1998; Meerut 2010]**

Sol. Given $z = f(x + iy) + F(x - iy)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = f'(x + iy) + F'(x - iy) \quad ...(2)$$

and $$\partial z/\partial y = if'(x + iy) - iF'(x - iy). \quad ...(3)$$

Differentiating (2) and (3) partial w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = f''(x + iy) + F''(x - iy) \quad ...(4)$$

and $$\partial^2 z/\partial y^2 = i^2 f''(x + iy) + i^2 F''(x - iy) = -\{f''(x + iy) + F''(x + iy)\}. \quad ...(5)$$

Adding (4) and (5), $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$, which is the required equation.

Ex. 9. *Form partial differential equation by eliminating arbitrary functions f and g from* $z = f(x^2 - y) + g(x^2 + y)$. **[Nagpur 1996 ; I.A.S. 1996; Kanpur 2011]**

Sol. Given $$z = f(x^2 - y) + g(x^2 + y). \quad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2xf'(x^2 - y) + 2xg'(x^2 + y) = 2x\{f'(x^2 - y) + g'(x^2 + y)\}. \quad ...(2)$$

and $$\partial z/\partial y = -f'(x^2 - y) + g'(x^2 + y). \quad ...(3)$$

Differentiating (2) and (3) w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = 2\{f'(x^2 - y) + g'(x^2 + y)\} + 4x^2\{f''(x^2 - y) + g''(x^2 + y)\} \quad ...(4)$$

and $$\partial^2 z/\partial y^2 = f''(x^2 - y) + g''(x^2 + y). \quad ...(5)$$

Again, (2) $\Rightarrow$ $f'(x^2 - y) + g'(x^2 + y) = (1/2x) \times (\partial z/\partial x)$. ...(6)

Substituting the values of $f''(x^2 - y) + g''(x^2 + y)$ and $f'(x^2 - y) + g'(x^2 + y)$ from (5) and (6) in (4), we have

$$\frac{\partial^2 z}{\partial x^2} = 2 \times \left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2},$$

which is the required partial differential equation.

Ex. 10. *Find the differential equation of all surfaces of revolution having z-axis as the axis of rotation*. **[I.A.S. 1997]**

Sol. From coordinate geometry of three dimensions, equation of any surface of revolution having z-axis as the axis of rotation may be taken as

$$z = \phi[(x^2 + y^2)^{1/2}], \text{ where } \phi \text{ is an arbitrary function.} \quad ...(1)$$

Differentating (1) partially with respect to x and y, we get

$$\partial z/\partial x = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2x \quad ...(2)$$

and $$\partial z/\partial y = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2y. \quad ...(3)$$

Dividing (2) by (3), $$\frac{\partial z/\partial x}{\partial z/\partial y} = \frac{x}{y} \quad \text{or} \quad y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}.$$

Ex. 11. *Form a partial differential equation by eliminating the arbitrary functions f and g from* $z = y\ f(x) + x\ g(y)$. (*Guwahati 2007*)

Sol. Given $$z = y\,f(x) + x\,g(y). \quad ...(1)$$

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\partial z/\partial x = y\,f'(x) + g(y)$ ...(2) $\qquad \partial z/\partial y = f(x) + x\,g'(y)$. ...(3)

Differentiating (3) with respect to x, $\partial^2 z/\partial x\partial y = f'(x) + g'(y)$. ...(4)

From (2) and (3), $$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] \quad \text{and} \quad g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right].$$

Substituting these values in (4), we have

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

or $$xy\frac{\partial^2 z}{\partial x\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \{x\,g(y) + y\,f(x)\} \quad \text{or} \quad xy\frac{\partial^2 z}{\partial x\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z, \text{ by (2)}$$

Ex. 12. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. **[Nagpur 1996; 2002]**

Sol. Given $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0.$ ...(1)

Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy.$ ...(2)

Then, (1) becomes $\phi(u, v) = 0.$ ...(3)

Differentiating (3) partially w.r.t. 'x', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0, \quad ...(4)$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Now, from (2), we have

$\partial u/\partial x = 2x, \quad \partial u/\partial y = 2y, \quad \partial u/\partial z = 2z, \quad \partial v/\partial x = -2y, \quad \partial v/\partial y = -2x, \quad \partial v/\partial z = 2z.$ ...(5)

Using (5), (4) reduces to $(\partial\phi/\partial u)\,(2x + 2pz) + (\partial\phi/\partial v)\,(-2y + 2pz) = 0$

or $(x + pz)\,(\partial\phi/\partial u) = (y - pz)\,(\partial\phi/\partial v).$ ...(6)

Again, differentiating (3) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial y}\right) = 0$$

or $(\partial\phi/\partial u)\,(2y + 2qz) + (\partial\phi/\partial v)\,(-2x + 2qz) = 0$, by (5)

or $(y + qz)\,(\partial\phi/\partial u) = (x - qz)\,(\partial\phi/\partial v).$ ...(7)

Dividing (6) by (7), $(x + pz)/(y + qz) = (y - pz)/(x - qz)$

or $pz(y + x) - qz(y + x) = y^2 - x^2$ or $(p - q)z = y - x.$

Ex. 13. *Eliminate the arbitrary function f and obtain the partial differential equation from* $z = e^y f(x + y)$ **[Madras 2005]**

Sol. Given $z = e^y f(x + y)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$$\partial z/\partial x = e^y f'(x+y) \quad \text{and} \quad \partial z/\partial y = e^y f(x+y) + e^y f'(x+y) \quad ... (2)$$

From (1) and (2), we have $\partial z/\partial y = z + \partial z/\partial x$

Ex. 14. *If* $z = f(x+ay) + \phi(x-ay)$, *prove that* $\partial^2 z/\partial y^2 = a^2(\partial^2 z/\partial x^2)$

Hint. Refer solved Ex. 3. **[Madurai Kamraj 2008; Jabalpur 2002]**

Ex. 15. *Equation of any cone with vertex at P* (a, b, c) *is of the form* $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0.$ *Find the differential equation of the cone.*

Sol. Let $(x - a)/(z - c) = u$ and $(y - b)/(z - c) = v$ ... (1)

Then, the equation of the given cone becomes $f(u, v) = 0$ ... (2)

Differentiating (2) partially with respect to 'x', we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}\left(\frac{1-0}{z-c} - \frac{x-a}{(z-c)^2}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial x}\right) = 0, \text{ using (1)}$$

or $$\frac{\partial f}{\partial u}\left(\frac{1}{z-c} - p\frac{x-a}{(z-c)^2}\right) - \frac{\partial f}{\partial v}\left(p\frac{y-b}{(z-c)^2}\right) = 0, \quad \text{where} \quad p = \frac{\partial z}{\partial x} \quad ... (3)$$

Differentiating (2) partially with respect to 'y', we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}\left(-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{1-0}{z-c} - \frac{y-b}{(z-c)^2}\frac{\partial z}{\partial y}\right) = 0, \text{ using (1)}$$

or $$-\frac{\partial f}{\partial u}\left(q\frac{x-a}{(z-c)^2}\right)+\frac{\partial f}{\partial v}\left(\frac{1}{z-c}-q\frac{y-b}{(z-c)^2}\right)=0, \quad \text{where} \quad q=\frac{\partial z}{\partial y} \qquad \ldots (4)$$

Eliminating $\partial f/\partial u$ and $\partial f/\partial v$ from (3) and (4), we have

$$\begin{vmatrix} \frac{1}{z-c}-p\frac{x-a}{(z-c)^2} & -p\frac{y-b}{(z-c)^2} \\ -q\frac{x-a}{(z-c)^2} & \frac{1}{z-c}-q\frac{y-b}{(z-c)^2} \end{vmatrix}=0$$

or $$\begin{vmatrix} z-c-p(x-a) & -p(y-b) \\ -q(x-a) & z-c-q(y-b) \end{vmatrix}=0$$

or $$\{z-c-p\,(x-a)\}\ \{z-c-q(y-b)\}-pq(x-a)\,(y-b)=0$$

or $(z-c)^2-p(x-a)\,(z-c)-q\,(y-b)\,(z-c)=0$ or $(x-a)p+(y-b)q=z-c.$

which in the required partial differential equation of the given cone.

EXERCISE 1 (B)

Eliminate the arbitrary functions and hence obtain the partial differential equations:

1. $z=e^{mx}\phi(x+y).$ **Ans.** $p-q=mz$

2. $z=f(x+ay)$ **[Bilaspur 1997; Jabalpur 1999]** **Ans.** $q=ap$

3. $z=xy+f(x^2+y^2)$ **[Delhi B.A./B.Sc. (Maths) (Prog.) 2007]** **Ans.** $py-qx=y^2-x^2$

4. $z=x+y+f(xy)$ **[Delhi B.A. (Prog) II 2010]** **Ans.** $px-qy=x-y$

5. $z=f(xy/z)$ **[Nagpur 1995 KU Kurukshetra 2004]** **Ans.** $px-qy=0$

6. $z=f(x-y)$ **[Delhi B.A. (Prog.) II 2011]** **Ans.** $p+q=0$

7. $z=(x-y)\,\phi(x^2+y^2)$ **Ans.** $(x-y)yp-(x-y)xq=(x+y)z$

8. $z=f(x^2+2y^2)$ **Ans.** $xq-yp=x^2-y^2$

9. $x=f(z)+g(y)$ **Ans.** $ps-qr=0$

10. $z=f(y+ax)+g(y+bx),\ a\neq b.$ **Ans.** $r-(a+b)s+abt=0$

11. $f(x+y+z)=xyz$ **Ans.** $x(y-z)p+y(z-x)q=z\,(x-y)$

12. $z=(x+y)\,f(x^2-y^2)$ **Ans.** $yp+xq=z$

13. $z=f(x)+e^y\,g(x)$ **Ans.** $t-q=0$

14. $f(x+y+z,\ x^2+y^2-z^2)=0$ **(CDLU 2004)** **Ans.** $p(y+z)-(x+z)q=x-y$

15. $z=f(xy)+g(x/y)$ **Ans.** $x^2(\partial^2z/\partial x^2)-y^2(\partial^2z/\partial y^2)+x(\partial z/\partial x)-y(\partial z/\partial y)=0$

16. $z=f(x-z)+g(x+y)$ **Ans.** $\dfrac{\partial z}{\partial y}\dfrac{\partial^2 z}{\partial x^2}+\left(1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)\dfrac{\partial^2 z}{\partial x\partial y}-\left(1-\dfrac{\partial z}{\partial x}\right)\dfrac{\partial^2 z}{\partial y^2}=0$

17. $z=f(x\cos\alpha+y\sin\alpha-at)+\phi(x\cos\alpha+y\sin\alpha+at).$

Ans. $\partial^2z/\partial x^2+\partial^2z/\partial y^2=(1/a^2)\times(\partial^2z/\partial t^2)$

18. $y=f(x+at)+xg(x+at)$ **Ans.** $a^2(\partial^2z/\partial x^2)-2a(\partial^2z/\partial x\partial t)+(\partial^2z/\partial t^2)=0$

19. $y=f(x-at)+xg(x-at)+x^2h(x-at).$ **(Jabalpur 1994)**

Ans. $\partial^3y/\partial t^3+3a(\partial^3y/\partial x\partial t^2)+3a^2(\partial^3y/\partial x^2\partial t)+a^3(\partial^3y/\partial x^3)=0$

20. $z=f(xy)+g(x+y)$ **Ans.** $x\,(y-x)r-(y^2-x^2)s+y\,(y-x)t+(p-q)\,(x+y)=0$

1.13 CAUCHY'S PROBLEM FOR FIRST ORDER EQUATIONS

The aim of an existence theorem is to establish conditions under which we can decide whether or not a given partial differential equation has a solution at all; the next step of proving that the solution, when it exists, is unique requires a uniqueness theorem. The conditions to be satisfied in the case of a first order partial differential equation are easily contained in the classic problem of Cauchy, which for the two independent variables can be stated as follows:

Cauchy's problem for first order partial differential equation

If (a) $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ *are functions which, together with their first derivatives, are continuous in the interval I defined by* $\mu_1 < \mu < \mu_2$.

(b) And if f (x, y, z, p, q) is a continuous function of x, y, z, p and q in a certain region U of the xyzpq space, then it is required to establish the existence of a function $\Phi(x, y)$ *with the following properties :*

(i) $\Phi(x, y)$ *and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.*

(ii) For all values of x and y lying in R, the point {x, y, $\Phi(x, y)$, $\Phi_x(x, y)$, $\Phi_y(x, y)$*} lies in* $\frac{U}{x}$ *and* $f[x, y, \Phi(x, y), \Phi_x(x, y), \Phi_y(x, y)] = 0$.

(iii) For all μ *belonging to the interval I, the point* $\{x_0(\mu), y_0(\mu)\}$ *belongs to the region R, and* $\Phi\{x_0(\mu), y_0(\mu)\} = z_0$

Stated geometrically, what we wish to prove is that there exists a surface $z = \Phi(x, y)$ *which passes through the curve C whose parametric equations are given by* $x = x_0(\mu)$, $y = y_0(\mu), z = z_0(\mu)$ *and at every point of which the *direction (p, q, – 1) of the normal is such that* $f(x, y, z, p, q) = 0$

Problem 1. *State the properties of* $\Phi(x, y)$ *if there exists a surface* $z = \Phi(x, y)$ *which passes through the curve C with parametric equations* $x = x_0(\mu)$, $y = y_0(\mu)$, $z = z_0(\mu)$ *and at every point of which the direction* $(p, q, -1)$ *of the normal is such that* $f(x, y, z, p, z) = 0$. **(Delhi B.Sc. (H) 2002)**

Sol. Hint. Refer conditions (*i*), (*ii*) and (*iii*) of the above Art. 1.13

Problem 2. *Solve the Cauchy's problem for* $zp + q = 1$, *when the initial data curve is* $x_0 = \mu$, $y_0 = \mu$, $z_0 = \mu/2$, $0 \le \mu \le 1$. **[Bangalore 2003; I.A.S. 2004]**

Sol. Given $f(x, y, z, p, q) = zp + q - 1 = 0$ ... (1)

Given inital data curve $x_0 = \mu, \quad y_0 = \mu, \quad z_0 = \mu/2, \quad 0 \le \mu \le 1$ ... (2)

From (1), $\partial f / \partial p = z, \qquad \partial f / \partial q = 1,$

and $\frac{\partial f}{\partial q}\frac{dx_0}{d\mu} - \frac{\partial f}{\partial p}\frac{dy_0}{d\mu} = 1 \times 1 - z \times 1 = 1 - \frac{1}{2}\mu \neq 0$, for $0 \le \mu \le 1$.

Now, we have the following ordinary differential equations :

---

*Let $z = \Phi(x, y)$ ... (1)
be the equation of the given surface

Let $F(x, y, z) = \Phi(x, y) - z$. ... (2)

From (1) and (2), $\frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} = \frac{\partial z}{\partial x} = p, \qquad \frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{\partial z}{\partial y} = q, \qquad \frac{\partial F}{\partial z} = -1$

Since ∇F is normal to the surface $F(x, y, z) = 0$, $\partial F / \partial x$, $\partial F / \partial y$, $\partial F / \partial z$ i.e., p, $q - 1$ are direction ratios of the normal to $F(x, y, z) = 0$ or $z = \Phi(x, y)$.

$$\frac{dx}{dt}=\frac{\partial f}{\partial p}, \qquad \frac{dy}{dt}=\frac{\partial f}{\partial q} \qquad \text{and} \qquad \frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

or $dx/dt = z, \qquad dy/dt = 1$ ... (3)

and $dz/dt = p(\partial f/\partial p) + q\ (\partial f/\partial q) = pz + q = 1,\ by\ (1)$ ... (4)

Integrating (3) and (4), $y = t + C_1$ and $z = t + C_2$ ... (5)

From (2), at $t = 0$, $x(\mu, 0) = \mu$, $y(\mu, 0) = \mu$ and $z(\mu, 0) = \mu/2$ ... (6)

Using (6), (5) reduces to $y = t + \mu$ and $z = t + \mu/2$ ... (7)

Then, from (3) and (7), $dx/dt = t + \mu/2$ so that $x = (1/2) \times t^2 + (1/2) \times \mu t + C_3$ ... (8)

Using (6), (8) reduces to $x = (1/2) \times t^2 + (1/2) \times \mu t + \mu$ ... (9)

Solving $y = t + \mu$ with (9) for μ and t in terms of x and y, we get

$$t = \frac{y - x}{1 - (y/2)} \qquad \text{and} \qquad \mu = \frac{x - (y^2/2)}{1 - (y/2)}$$

Putting these values in $z = t + \mu/2$, the required solution passing through the initial data curve is $z = \{2(y - x) + x - y^2/2\}/(2 - y)$.

OBJECTIVE PROBLEMS ON CHAPTER 1

Indicate the correct answer by writing (a), (b), (c) or (d)

1. Equation $p \tan y + q \tan x = \sec^2 z$ is of order
   (*a*) 1 (*b*) 2 (*c*) 0 (*d*) none of these **[Agra 2005, 2008]**
2. Equation $\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x \partial y) + (\partial z/\partial y)^2 = 0$ is of order
   (*a*) 1 (*b*) 2 (*c*) 3 (*d*) none of these **[Agra 2005, 2006]**
3. The equation $(2x + 3y)p + 4xq - 8pq = x + y$ is
   (*a*) linear (*b*) non-linear (*c*) quasi-linear (*d*) semi-linear **[Agra 2005, 06]**
4. $(x + y - z)(\partial z/\partial x) + (3x + 2y)(\partial z/\partial y) + 2z = x + y$ is
   (*a*) linear (*b*) quasi-linear (*c*) semi-linear (*d*) non-linear

Answers **1.** (*a*) **2.** (*b*) **3.** (*b*) **4.** (*b*)

MISCELLANEOUS EXAMPLES ON CHAPTER 1

Ex.1. *Formulate a partial differential equation by eliminating arbitrary constants a and b from the equation* $(x + a)^2 + (y + b)^2 + z^2 = 1$. *Examine whether the partial differential equation is linear or non-linear. Also,* find its order and degreee. **[Delhi Maths (H) 2008]**

Hint. Proceed as in Ex. 5(a), page 1.6 with $\lambda = 1$. *Thus we get the partial differential equation* $z^2\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1\right\} = 1$, *which is non-linear partial differential equation of order one and degree two.*

Ex. 2. *Eliminate arbitrary constants a and b from the following equations :*

(*i*) $ax^2 + by^2 + z^2 = 1$ **(Delhi B.A. (Prog.) II 2010)**

(*ii*) $z = ax + (1 - a)y + b$ **(Lucknow 2010)**

Ans. (*i*) $z(z - px - qy) = 1$ (*ii*) $p + q = 1$, where $p = \partial z/\partial x, q = \partial z/\partial y$

Ex. 3. (*i*) *Eliminate the arbitrary function* ϕ *from* $p + x - y = \phi(q - x + y)$ **(Ranchi 2010)**

(*ii*) *State true or false with justification. Eliminating arbitrary function f from* $z = f(x^2 + y^2)$*, we get first order non-linear partial differential equation.* **(Pune 2010)**

Ans. (*i*) $(1+\partial^2 z/\partial x^2)(1+\partial^2 z/\partial y^2) = (\partial^2 z/\partial x \partial y - 1)^2$ (*ii*) False. see Ex. 4 (ii), page 1.21.

Ex. 4. (*i*) *Obtain the partial differential equation by eliminating arbitrary function of f and g from the equation* $v = \{f(r-at) + g(r+at)\}/r$ **(Nagpur 2010)**

Ans. Given $v = (1/r) \times \{f(r-at) + g(r+at)\}$ ...(1)

$(1) \Rightarrow \partial v/\partial t = (1/r) \times \{-a f'(r-at) + a g'(r+at)\} = -(a/r) \times \{f'(r-at) - g'(r+at)\}$ ...(2)

$(2) \Rightarrow \partial^2 v/\partial t^2 = -(a/r) \times \{-a f''(r-at) - a g''(r+at)\} = (a^2/r) \times \{f''(r-at) + g''(r+at)\}$ ...(3)

$(1) \Rightarrow \partial v/\partial r = (1/r) \times \{f'(r-at) + g'(r+at)\} - (1/r^2) \times \{f(r-at) + g(r+at)\}$ ...(4)

$(4) \Rightarrow \partial^2 v/\partial r^2 = (1/r) \times \{f''(r-at) + g''(r+at)\} - (1/r^2) \times \{f'(r-at) + g'(r+at)\}$

$= -(1/r^2) \times \{f'(r-at) + g'(r+at)\} + (2/r^3) \times \{f(r-at) + g(r+at)\}$

$= (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r^2) \times \{f'(r-at)\} + g'(r+at)\} + (2/r^2) \times v$, using (1) and (3)

$= (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r) \times [\partial v/\partial r + (1/r^2) \times \{f(r-at)\} + g(r+at)\} + (2/r^2) \times v$

[Since from (4), $(1/r) \times \{f'(r-at) + g'(r+at)\} = \partial v/\partial r + (1/r^2) \times \{f(r-at) + g(r+at)\}$]

Thus, $\partial^2 v/\partial r^2 = (1/a^2) \times (\partial^2 v/\partial r^2) - (2/r) \times \{\partial v/\partial r + (1/r) \times v\} + (2/r^2) \times v$, using (1)

or $\partial^2 v/\partial r^2 = (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r) \times (\partial^2/\partial r)$, which is the required equation

2

Linear Partial differential equations of order one

2.1. LAGRANGE'S EQUATION

A quasi–linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z. Such a partial differential equation is known as *Lagrange equation.*

For Example $xyp + yzq = zx$ is a Lagrange equation.

2.2. Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are functions of x, y, z (Delhi Maths (H) 2009; Meerut 2003; Poona 2003, 10; Lucknow 2010)

Theorem. *The general solution of Lagrange equation*

$$Pp + Qq = R, \qquad \ldots (1)$$

is

$$\phi(u, v) = 0 \qquad \ldots (2)$$

where ϕ *is an arbitrary function and*

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \qquad \ldots (3)$$

are two independent solutions of

$$(dx)/P = (dy)/Q = (dz)/R \qquad \ldots (4)$$

Here, c_1 and c_2 are arbitrary constants and at least one of u, v must contain z. Also recall that u and v are said to be independent if u/v is not merely a constant.

Proof. Differentiating (2) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots (5)$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots (6)$$

Eliminating $\partial\phi/\partial u$ and $\partial\phi/\partial v$ between (5) and (6), we have

$$\begin{vmatrix} \partial u/\partial x + p(\partial u/\partial z) & \partial v/\partial x + p(\partial v/\partial z) \\ \partial u/\partial y + q(\partial u/\partial z) & \partial v/\partial y + q(\partial v/\partial z) \end{vmatrix} = 0$$

or

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

or

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0$$

$\therefore$

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \ldots (7)$$

Hence (2) is a solution of the equation (7)

Taking the differentials of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, we get

$$(\partial u/\partial x)dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz = 0 \qquad \ldots (8)$$

and $$(\partial v/\partial x)dx + (\partial v/\partial y)dy + (\partial v/\partial z)dz = 0 \quad \text{... (9)}$$

Since u and v are independent functions, solving (8) and (9) for the ratios $dx : dy : dz$, gives

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \quad \text{... (10)}$$

Comparing (4) and (10), we obtain

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P} = \frac{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q} = \frac{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}{R} = k, \text{ say}$$

$$\Rightarrow \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = kP, \qquad \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = kQ \qquad \text{and} \qquad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = kR$$

Substituting these values in (7), we get $k(Pp + Qq) = kR$ or $Pp + Qq = R$, which is the given equation (1).

Therefore, if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of the system of differential equations $(dx)/P = (dy)/Q = (dz)/R$, then $\phi(u, v) = 0$ is a solution of $Pp + Qq = R$, ϕ being an arbitrary function. This is what we wished to prove.

Note. Equations (4) are called *Lagrange's auxillary* (*or subsidiary*) equations for (1).

2.3. Working Rule for solving *Pp + Qq = R* by Lagrange's method.

[Delhi Maths Hons. 1998]

Step 1. Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R. \quad \text{...(1)}$$

Step 2. Write down Lagrange's auxiliary equations for (1) namely,

$$(dx)/P = (dy)/Q = (dz)/R \quad \text{...(2)}$$

Step 3. Solve (2) by using the well known methods (refer Art. 2.5, 2.7, 2.9 and 2.11). Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms :

$\phi(u, v) = 0$, $u = \phi(v)$ or $v = \phi(u)$, ϕ being an arbitrary function.

2.4. Examples based on working rule 2.3. In what follows we shall discuss four rules for getting two independent solutions of $(dx)/P = (dy)/Q = (dz)/R$. Accordingly, we have four types of problems based on $Pp + Qq = R$.

2.5. Type 1 based on Rule I for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of given equations (1).

2.6. SOLVED EXAMPLES BASED ON ART. 2.5

Ex. 1. *Solve* $(y^2z/x)p + xzq = y^2$. **[Indore 2004; Sagar 1994]**

Sol. Given $$(y^2z/x)p + xzq = y^2. \quad \text{...(1)}$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{(y^2z/x)} = \frac{dy}{xz} = \frac{dz}{y^2}. \quad \text{..(2)}$$

Taking the first two fractions of (2), we have

$$x^2zdx = y^2zdy \quad \text{or} \quad 3x^2dx - 3y^2dy = 0, \quad \text{...(3)}$$

Integrating (3), $x^3 - y^3 = c_1$, c_1 being an arbitrary constant ...(4)

Next, taking the first and the last fractions of (2), we get

$xy^2dx = y^2zdz$ or $2xdx - 2zdz = 0$. ...(5)

Integrating (5), $x^2 - z^2 = c_2$, c_2 being an arbitrary constant ...(6)

From (4) and (6), the required general integral is

$\phi(x^3 - y^3, x^2 - z^2) = 0$, ϕ being an arbitrary function.

Ex. 2. *Solve* (*i*) $a(p + q) = z$. **[Bangalore 1997]** (*ii*) $2p + 3q = 1$. **[Bangalore 1995]**

Sol. (*i*) Given $ap + aq = z$. ...(1)

The Lagrange's auxiliary equation for (1) are $(dx)/a = (dy)/a = (dz)/1$. ...(2)

Taking the first two members of (1), $dx - dy = 0$. ...(3)

Integrating (3), $x - y = c_1$, c_1 being an arbitrary constant ...(4)

Taking the last two members of (1), $dy - adz = 0$. ...(5)

Integrating (5), $y - az = c_2$, c_2 being an arbitrary constant. ...(6)

From (4) and (6), the required solution is given by

$\phi(x - y, y - az) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $p \tan x + q \tan y = \tan z$. **[Madras 2005 ; Kanpur 2007]**

Sol. Given $(\tan x)p + (\tan y)q = \tan z$. ...(1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$. ...(2)

Taking the first two fractions of (2), $\cot x\, dx - \cot y\, dy = 0$.

Integrating, $\log \sin x - \log \sin y = \log c_1$ or $(\sin x)/(\sin y) = c_1$. ...(3)

Taking the last two fractions of (2), $\cot y\, dy - \cot z\, dz = 0$.

Integrating, $\log \sin y - \log \sin z = \log c_2$ or $(\sin y)/(\sin z) = c_2$. ...(4)

From (3) and (4), the required general solution is

$\sin x/\sin y = \phi(\sin y/\sin z)$, ϕ being an arbitrary function.

Ex. 4. *Solve* $zp = -x$.

Sol. Given $zp + 0.q = -x$. ...(1)

The Lagrange's subsidiary equations for (1) are $(dx)/z = (dy)/0 = (dz)/(-x)$ ...(2)

Taking the first and the last members of (2), we get

$-xdx = zdz$ or $2xdx + 2zdz = 0$. ...(3)

Integrating (3), $x^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Next, the second fraction of (2) implies that $dy = 0$ giving $y = c_2$ ...(3)

From (4) and (5), the required solution is $x^2 + z^2 = \phi(y)$, ϕ being an arbitrary function.

Ex. 5. *Solve* $y^2p - xyq = x(z - 2y)$ **[Delhi Maths Hons. 1995, Delhi Meths(G) 2006]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{y^2} = \dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)}$. ...(1)

Taking the first two fractions of (1) and re–writing, we get

$2xdx + 2ydy = 0$ so that $x^2 + y^2 = c_1$. ...(2)

Now, taking the last two fractions of (1) and re–writing, we get

$\dfrac{dz}{dy} = -\dfrac{z-2y}{y}$ or $\dfrac{dz}{dy} + \dfrac{1}{y}z = 2$ ...(3)

which is linear in z and y. Its I.F. $= e^{\int (1/y)dy} = e^{\log y} = y$. Hence solution of (3) is

$z \cdot y = \int 2y dy + c_2$ or $zy - y^2 = c_2$. ...(4)

Hence $\phi(x^2 + y^2, zy - y^2) = 0$ is the desired solution, where ϕ is an arbitrary function.

Ex. 6. *Solve* $(x^2 + 2y^2)p - xyq = xz$ **[K.U. Kurukshetra 2005]**

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{x^2 + 2y^2} = \frac{dy}{-xy} = \frac{dz}{xz} \quad \text{... (1)}$$

Taking the last two fractions of (2) and re–writing, we get

$(1/y)\, dy + (1/z)dz = 0$ so that $\log y + \log z = \log c_1$ or $yz = c_1$ ... (2)

Taking the first two fractions of (1), we have

$$\frac{dx}{dy} = \frac{x^2 + 2y^2}{-xy} \quad \text{or} \quad 2x\frac{dx}{dy} + \left(\frac{2}{y^2}\right)x^2 = -4y \quad \text{... (3)}$$

Putting $x^2 = v$ and $2x(dx/dy) = dv/dx$, (3) yields

$dv/dx + (2/y)\, v = -4y$, which is a linear equation.

Its integrating factor $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and hence its solution is

$$yv^2 = \int \left\{(-4y)xy^2\right\} dy + c_2 \quad \text{or} \quad y^2x^2 + y^4 = c_2 \quad \text{... (4)}$$

From (2) and (4), the required solution is $\phi(yz, y^2x^2 + y^4) = 0$, ϕ being an arbitrary function.

EXERCISE 2 (A)

Solve the following partial differential equations

1. $(-a + x)p + (-b + y)q = (-c + z)$. **Ans.** $\phi\{(x-a)/(y-b), (y-b)/(z-c)\} = 0$

2. $xp + yq = z$ **(Kanpur 2011)** **Ans.** $\phi(x/z,\ y/z) = 0$

3. $p + q = 1$ **Ans.** $\phi(x-y, x-z) = 0$

4. $x^2p + y^2p = z^2$ **[Bilaspur 2001, Jabalpur 2000, Sagar 2000, Vikram 1999]**

Ans. $\phi(1/x - 1/y, 1/y - 1/z) = 0$

5. $x^2p + y^2q + z^2 = 0$ **Ans.** $\phi(1/x - 1/y, 1/y + 1/z) = 0$

6. $\partial z/\partial x + \partial z/\partial y = \sin x$ **[Meerut 1995]** **Ans.** $\phi(x-y, z + \cos x) = 0$

7. $yzp + 2xq = xy$ **[Nagpur 1996]** **Ans.** $\phi(x^2 - z^2, y^2 - 4z) = 0$

8. $xp + yq = z$ **[Bangalore 1995]** **Ans.** $\phi(x/y, x/z) = 0$

9. $yzp + zxq = xy$ **[M.S. Univ. T.N. 2007, Lucknow 2010, Revishankar 2004]**

Ans. $\phi(x^2 - y^2, x^2 - z^2) = 0$

10. $zp = x$ **Ans.** $\phi(y, x^2 - z^2) = 0$

11. $y^2p^2 + x^2q^2 = x^2y^2z^2$ **Ans.** $\phi(x^3 - y^3, y^3 + 3z^{-1}) = 0$

2.7. Type 2 based on Rule II for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one integral of (1) is known by using rule I explained in Art 2.5 and suppose also that another integral cannot be obtained by using rule I of Art. 2.5. Then one integral known to

us is used to find another integral as shown in the following solved examples. Note that in the second integral, the constant of integration of first integral should be removed later on.

2.8. SOLVED EXAMPLES BASED ON ART. 2.7

Ex. 1. *Solve* $p + 3q = 5z + \tan(y - 3x)$.

[Agra 2006; Meerut 2003; Indore 2002; Ravishankar 2003]

Sol. Given $p + 3q = 5z + \tan(y - 3x)$. ...(1)

The Lagrange's subsidary equations for (1) are $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$. ...(2)

Taking the first two fractions, $dy - 3dx = 0$. ...(3)

Integrating (3), $y - 3x = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), from (2) we get $\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$. ...(5)

Integrating (5), $x - (1/5) \times \log(5z + \tan c_1) = (1/5) \times c_2$, c_2 being an arbitrary constant.

or $5x - \log[5z + \tan(y - 3x)] = c_2$, using (4) ...(6)

From (4) and (6), the required general integral is

$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x)$, where ϕ is an arbitrary function.

Ex. 2. *Solve* $z(z^2 + xy)(px - qy) = x^4$.

Sol. Given $xz(z^2 + xy)p - yz(z^2 + xy)q = x^4$. ...(1)

The Lagrange's subsidiary equations for (1) are $\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4}$. ...(2)

Cancelling $z(z^2 + xy)$, the first two fractions give

$(1/x)\,dx = -(1/y)\,dy$ or $(1/x)\,dx + (1/y)\,dy = 0$. ...(3)

Integrating (3), $\log x + \log y = \log c_1$ or $xy = c_1$. ...(4)

Using (4), from (2) we get $\frac{dx}{xz(z^2 + c_1)} = \frac{dz}{x^4}$

or $x^3 dx = z(z^2 + c_1)dz$ or $x^3 dx - (z^3 + c_1 z)dz = 0$. ...(5)

Integrating (5), $x^4/4 - z^4/4 - (c_1 z^2)/2) = c_2/4$ or $x^4 - z^4 - 2c_1 z^2 = c_2$

or $x^4 - z^4 - 2xy\,z^2 = c_2$, using (4) ...(6)

From (4) and (6), the required general integral is

$\phi(xy, x^4 - z^4 - 2xy\,z^2) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $xyp + y^2 q = zxy - 2x^2$. **[Garhwal 2005]**

Sol. Given $xyp + y^2 q = zxy - 2x^2$. ...(1)

The Lagrange's subsidiary equations for (1) are $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}$. ...(2)

Taking the first two fractions of (2), we have

$(dx)/xy = (dy)/y^2$ or $(1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1 y$. Hence from second and third fractions of (2), we get

$\frac{dy}{y^2} = \frac{dz}{c_1 z y^2 - 2c_1^2 y^2}$ or $c_1 dy - \frac{dz}{z - 2c_1^2} = 0$. ...(5)

Integrating (5), $c_1 y - \log(z - 2c_1{}^2) = c_2$ or $x - \log[z - 2(x^2/y^2)] = c_2$, using (4). ...(6)

From (4) and (6), the required general solution is

$x - \log[z - 2(x^2/y^2)] = \phi(x/y)$, ϕ being an arbitrary function.

Ex. 4. *Solve* $xzp + yzq = xy$. **[Bhopal 1996; Jabalpur 1999; Jiwaji 2000; Punjab 2005; Agra 2007; Ravishanker 1996; Vikram 2000]**

Sol. Given $xzp + yzq = xy$. ...(1)

The Lagrange's subsidiary equations for (1) are $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$. ...(2)

Taking the first two fractions of (2), $(1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1 y$. Hence, from second and third fractions of (2), we get

$(1/yz)dy = (1/c_1 y^2)dz$ or $2c_1 y\, dy - 2z\, dz = 0$. ...(5)

Integrating (5), $c_1 y^2 - z^2 = c_2$ or $xy - z^2 = c_2$, using (4). ...(6)

From (4) and (6), the required solution is $\phi(xy - z^2, x/y) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $py + qx = xyz^2 (x^2 - y^2)$.

Sol. Given $py + qx = xyz^2 (x^2 - y^2)$. ...(1)

The Lagrange's auxiliary equations for (1) are $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$. ...(2)

Taking the first two fractions of (2), $2xdx - 2ydy = 0$. ...(3)

Integrating. $x^2 - y^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), the last two fractions of (2) give

$(dy)/x = (dz)/(xyz^2 c_1)$ or $2c_1 y\, dy - 2z^{-2}dz = 0$. ...(5)

Integrating (5), $c_1 y^2 + (2/z) = c_2$, c_2 being an arbitrary constant.

or $y^2 (x^2 - y^2) + (2/z) = c_2$, using (4). ...(6)

From (4) and (6), the required general solution is

$y^2 (x^2 - y^2) + (2/z) = \phi(x^2 - y^2)$, where ϕ is an arbitrary function.

Ex. 6. *Solve* $xp - yq = xy$ **[Madras 2005]**

Sol. The Lagrange's auxiliary equations for the given equation are

$(dx)/x = (dy)/(-y) = (dz)/(xy)$ ... (1)

Taking the first two fractions of (1), $(1/x)dx + (1/y)dy = 0$

Intergrating, $\log x + \log y = c_1$ so that $xy = c_1$ ... (2)

Using (2), (1) yields $(1/x)dx = (1/c_1)\, dz$ so that $\log x - \log c_2 = z/c_1$

or $\log (x/c_2) = z/c_1$ or $\log (x/c_2) = z/(xy)$, by (2)

Thus, $x/c_2 = e^{z/(xy)}$ or $xe^{-z/(xy)} = c_2$, c_2 being an arbitrary constant. ... (3)

From (2) and (3), the required solution is $x\, e^{-z/(xy)} = \phi(xy)$, ϕ being an arbitrary function

Ex. 7. *Solve* $p + 3q = z + \cot (y - 3x)$. **[M.D.U Rohtak 2006]**

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{z + \cot (y - 3x)} \quad \text{... (1)}$$

Taking the first two fractions of (1), $dy - 3\, dx = 0$ so that $y - 3x = c_1$ ... (2)

Taking the first and last fraction of (1), we have

$$dx = \frac{dz}{z + \cot (y - 3x)} \quad \text{or} \quad dx = \frac{dz}{z + \cot c_1}, \text{ using (2)}$$

Intergrating, $x = \log | z + \cot c_1| + c_2$, c_1 and c_2 being an arbitrary constants.

or $x - \log|z + \cot(y - 3x)| = c_2$, using (2) ... (3)

From (2) and (3), the required general solution is

$$x - \log|z + \cot(y - 3x)| = \phi(y - 3x), \quad \phi \text{ being an arbitrary function.}$$

Ex. 8. *Solve* $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$ **[Delhi B.Sc. II 2008; Delhi B.A. II 2010]**

Sol. Re–writing the given equation, we have

$$x(z - 2y^2)\,p + y(z - y^2 - 2x^3)\,q = z(z - y^2 - 2x^3) \quad \ldots (1)$$

The Lagrange's subsidiary equations for (1) are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \quad \ldots (2)$$

Taking the last two fraction, we get $(1/z)dz = (1/y)dy$

Integrating, $\log z = \log y + \log a$ or $z/y = a$ ... (3)

where a is an arbitrary constant. Using (3), (2) yields

$$\frac{dx}{x(ay-2y^2)} = \frac{dy}{y(ay-y^2-2x^3)} \quad \text{so that} \quad (ay - y^2 - 2x^3)\,dx + x(2y - a)\,dy = 0 \quad \ldots (4)$$

Comparing (4) with $Mdx + Ndy = 0$, here $M = ay - y^2 - 2x^3$ and $N = x(2y - a)$. Then $\partial M/\partial y = a - 2y$ and $\partial N/\partial x = 2y - a$. Now, we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x(2y-a)} \times 2(a - 2y) = -\frac{2}{x}, \text{ which is a function of } x \text{ alone.}$$

Hence, by usual rule, integrating factor of (1) $= e^{\int(-2/x)dx} = e^{-2\log x} = e^{x^{-2}} = x^{-2}$

Multiplying (4) by x^{-2}, we get exact equation $(ayx^{-2} - y^2x^{-2} - 2x)dx + x^{-1}(2y - a)\,dy = 0$

By the usual rule of solving an exact equation, its solution is

$$\int\left\{(ay - y^2)x^{-2} - 2x\right\}dx + \int x^{-1}(2y - a)dy = b$$

(Treating y as constant) (Integrating terms free from x)

or $(ay - y^2) \times (-1/x) - x^2 = b$ or $(y^2 - ax)/x - x^2 = b$

or $(y^2 - ax - x^3)/x = b$, where b is an arbitrary constant. ... (5)

From (3) and (5), required solution is $(y^2 - ax - x^3)/x = \phi(z/y)$, ϕ being an arbitrary function

EXERCISE 2 (B)

Solved the following differential equations:

1. $p - 2q = 3x^2 \sin(y + 2x)$. **Ans.** $x^2\sin(y+2x) - z = \phi(y+2x)$

2. $p - q = z/(x + y)$. **Ans.** $x - (x+y)\log z = \phi(x+y)$

3. $xy^2p - y^3q + axz = 0$. **Ans.** $\log z + (ax/3y^2) = \phi(xy)$

4. $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. **Ans.** $(x^2 + y^2 + z^2)/z = \phi(y/z)$

5. *(a)* $z(p - q) = z^2 + (x + y)^2$. **(Meerut 2011)** **Ans.** $e^{2y}[z^2 + (x+y)^2] = \phi(x+y)$

(b) $z(p + q) = z^2 + (x - y)^2$ **Ans.** $e^{2y}[z^2 + (x-y)^2] = \phi(x-y)$

6. $p - 2q = 3x^2 \sin(y + 2x)$. **Ans.** $x^3\sin(y+2x) - z = \phi(y+2x)$

7. $p - q = z/(x + y)$. **Ans.** $x - (x+y)\log z = \phi(x+y)$

8. $zp - zq = x + y$. **Ans.** $2x(x+y) - z^2 = \phi(x+y)$

9. $xyp + y^2q + 2x^2 - xyz = 0$. **Ans.** $x - \log|z - (2x/y)| = \phi(x/y)$

2.9. Type 3 based on Rule III for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction in (1) will be equal to $(P_1dx + Q_1dy + R_1dz)/(P_1P + Q_1Q + R_1R)$. ...(2)

If $P_1P + Q_1Q + R_1R = 0$, then we know that the numerator of (2) is also zero. This gives $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible by use of multipliers. In such cases second integral should be obtained by using rule I of Art. 2.5 or rule II of Art. 2.7 as the case may be.

2.10. SOLVED EXAMPLES BASED ON ART. 2.9

Ex.1. *Solve* $\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$.

Sol. Given $\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$. ...(1)

The Lagrange's subsidiary equations of (1) are $\dfrac{a\,dx}{(b-c)yz} = \dfrac{b\,dy}{(c-a)zx} = \dfrac{c\,dz}{(a-b)xy}$. ...(2)

Choosing x, y, z as multipliers, each fraction for (2)

$$= \frac{a\,x\,dx + b\,y\,dy + c\,z\,dz}{xyz[(b-c)+(c-a)+(a-b)]} = \frac{ax\,dx + by\,dy + cz\,dz}{0}.$$

$\therefore$ $ax\,dx + by\,dy + cz\,dz = 0$ or $2axdx + 2bydy + 2czdz = 0$.

Integrating, $ax^2 + by^2 + cz^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing ax, by, cz as multipliers, each fraction of (2)

$$= \frac{a^2x\,dx + b^2y\,dy + c^2\,z\,dz}{xyz[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2x\,dx + b^2y\,dy + c^2\,z\,dz}{0}.$$

$\therefore$ $a^2xdx + b^2ydy + c^2zdz = 0$ or $2a^2xdx + 2b^2ydy + 2c^2zdz = 0$.

Integrating, $a^2x^2 + b^2y^2 + c^2z^2 = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$\phi(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2) = 0$, where ϕ is an arbitrary function.

Ex. 2. *Solve* $z(x+y)p + z(x-y)q = x^2 + y^2$.

Sol. Given $z(x+y)p + z(x-y)q = x^2 + y^2$. ...(1)

The Langrange's subsidiary equations for (1) are $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$. ...(2)

Choosing x, $-y$, $-z$, as multipliers, each fraction

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}.$$

$\therefore$ $x\,dx - y\,dy - z\,dz$ or $2x\,dx - 2y\,dy - 2z\,dz = 0$.

Integrating, $x^2 - y^2 - z^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing y, x, $-z$ as multipliers, each fraction

$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}.$$

$\therefore$ $y\,dx + x\,dy - z\,dz = 0$ or $2d(xy) - 2zdz = 0$.

Integrating, $2xy - z^2 = c_2$, c2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$$\phi\,(x^2 - y^2 - z^2,\ 2xy - z^2) = 0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $(mz - ny)p + (nx - lz)q = ly - mx$. **[Patna 2003; Madras 2005; Delhi Maths Hons. 1989 91; Bhopal 2004; Meerut 2008, 10; Sagar 2002; I.A.S. 1977; Kanpur 2005, 06]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \qquad ...(1)$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0$

Integrating, $x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Again, choosing l, m, n as multipliers, each fraction of (1)

$$= \frac{ldx + mdy + n\,dz}{l(mx - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + n\,dz}{0}.$$

$\therefore\quad ldx + mdy + n\,dz = 0$ so that $l\,x + m\,y + n\,z = c_2$. ...(3)

From (2) and (3), the required general solution is given by

$$\phi\,(x^2 + y^2 + z^2,\ l\,x + m\,y + n\,z) = 0,\ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $x(y^2 - z^2)q - y(z^2 + x^2)q = z(x^2 + y^2)$.

Sol. The lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}. \qquad ...(1)$$

Choosing x, y, z, as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow\ xdx + ydy + zdz = 0$ so that $x^2 + y^2 + z^2 = c_1$. ...(2)

Choosing $1/x$, $-1/y$, $-1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx - (1/y)dy - (1/z)dz}{y^2 - z^2 + z^2 + x^2 - (x^2 + y^2)} = \frac{(1/x)dx - (1/y)dy - (1/z)dz}{0}$$

$\Rightarrow\ (1/x)dx - (1/y)dy - (1/z)dz = 0$ so that $\log x - \log y - \log z = \log c_2$

$\Rightarrow\quad \log\{x/(yz)\} = \log c_2 \quad \Rightarrow\quad x/yz = c_2.$ ...(3)

$\therefore$ The required solution is $\phi\,(x^2 + y^2 + z^2,\ x/yz) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $(y - zx)p + (x + yz)q = x^2 + y^2$.

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y - zx} = \frac{dy}{x + yz} = \frac{dz}{x^2 + y^2}. \qquad ...(1)$$

Choosing x, $-y$, z as multipliers, each fraction of (1)

$$= \frac{xdx - ydy + zdz}{x(y - zx) - y(x + yz) + z(x^2 + y^2)} = \frac{xdx - ydy + zdz}{0}$$

$\Rightarrow$ $2xdx - 2ydy + 2zdz = 0$ so that $x^2 - y^2 + z^2 = c_1$. ...(2)

Choosing $y, x, -1$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - dz}{y(y - zx) + x(x + yz) - (x^2 + y^2)} = \frac{d(xy) - dz}{0}$$

$\Rightarrow$ $d(xy) - dz = 0$ so that $xy - z = c_2$. ...(3)

$\therefore$ From (2) and (3) solution is $\phi(x^2 - y^2 + z^2, xy - z) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve* $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$. **[I.A.S. 2004; Agra 2005 ; Delhi Maths (H) 2006; M.S. Univ. T.N. 2007; Indore 2003; Meerut 2009; Purvanchal 2007]**

Sol. Here Lagrange's subsidiary equations for given equation are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}. \quad ...(1)$$

Choosing $1/x$, $1/y$, $1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2 + z - (x^2 + z) + x^2 - y^2} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\Rightarrow$ $\log(xyz) = \log c_1$ $\Rightarrow$ $xyz = c_1$. ...(2)

Choosing $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{xdx + ydy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{xdx + ydy - dz}{0}$$

$\Rightarrow$ $x\,dx + y\,dy - z\,dz = 0$ so that $x^2 + y^2 - 2z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi(x^2 + y^2 - 2z, xyz) = 0$, ϕ is being an arbitrary function.

Ex. 7. *Solve* $(x + 2z)q + (4zx - y)q = 2x^2 + y$. **[Meerut 2005]**

Sol. Here Lagrange's auxiliary equations are

$$\frac{dx}{x + 2z} = \frac{dy}{4zx - y} = \frac{dz}{2x^2 + y}. \quad ...(1)$$

Choosing $y, x, -2z$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - 2zdz}{y(x + 2z) + x(4zx - y) - 2z(2x^2 + y)} = \frac{d(xy) - 2zdz}{0}$$

$\Rightarrow$ $d(xy) - 2zdz = 0$ so that $xy - z^2 = c_1$. ...(2)

Choosing $2x, -1, -1$ as multipliers, each fraction of (1)

$$= \frac{2xdx - dy - dz}{2x(x + 2z) - (4zx - y) - (2x^2 + y)} = \frac{2xdx - dy - dz}{0}$$

$\Rightarrow$ $2xdx - dy - dz = 0$ so that $x^2 - y - z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi(xy - z^2, x^2 - y - z) = 0$, ϕ being an arbitrary function.

Ex. 8. *Solve* $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. **[Ranchi 2010; Meerut 1994]**

If the solution of the above equation represents a sphere, what will be the coordinates of its centre.

Sol. Here Lagrange's auxiliary equations for given equation are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y + z)} = \frac{dz}{x(y - z)}. \quad ...(1)$$

Taking the last two fractions of (1), we have

$(y - z)dy = (y + z)dz$ or $2ydy - 2zdz - 2(zdy + ydz) = 0.$

Integrating, $y^2 - z^2 - 2yz = c_1$, c_1 being an arbitrary constant. ...(2)

Choosing x, y, z as multipliers, each fractrion of (1)

$$= \frac{xdx + ydy + zdz}{x(z^2 - 2yz - y^2) + xy(y + z) + xz(y - z)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow \quad 2xdx + 2ydy + 2zdz = 0$ so that $x^2 + y^2 + z^2 = c_2.$ ...(3)

From (2) and (3), solution is $\phi(y^2 - z^2 - 2yz, x^2 + y^2 + z^2) = 0$, ϕ being an arbitrary function.

From the solution of the given equation, it follows that if it represents a sphere, then its centre must be at (0,0,0), *i.e.*, origin.

Ex. 9. *Solve* $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^2 - y^3)$. **[Jabalpur 2004; M.S. Univ. T.N. 2007]**

Sol. Here Lagrange's auxiliary equations for the given equation are given by

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}. \quad ...(1)$$

Taking first two fractions of (1), we have $(2y^4 - x^3y)dx = (y^3x - 2x^4)dy$

Dividing both sides by x^3y^3 gives $\left(\frac{2y}{x^3} - \frac{1}{y^2}\right)dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right)dy$

or $\left(\frac{1}{x^2}dy - \frac{2y}{x^3}dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0$ or $d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0.$

Integrating, $(y/x^2) + (x/y^2) = c_1$, c_1 being an arbitrary constant. ...(2)

Choosing $1/x$, $1/y$, $1/3z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)} = \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{0}$$

$\Rightarrow \quad (1/x)dx + (1/y)dy + (1/3z)dz = 0$ so that $\log x + \log y + (1/3) \times \log z = \log c_2$

$\Rightarrow \quad \log(xy\,z^{1/3}) = \log c_2 \quad \Rightarrow \quad xyz^{1/3} = c_2.$ ...(3)

From (2) and (3) solution is $\phi(xyz^{1/3}, y/x^2 + x/y^2) = 0$, ϕ being an arbitrary function.

Ex. 10. *Solve* $x^2p + y^2q = nxy$. **[Ravishankar 1998; Bhopal 1998; Jabalpur 2002]**

Sol. Here Lagrange's auxiliary equations are $(dx)/x^2 = (dy)/y^2 = (dz)/nxy$ ... (1)

Taking the first two fractions of (1), we get $x^{-2}dx - y^{-2}dy = 0.$

Integrating, $-1/x + 1/y = -c_1$ so that $(y - x)/xy = c_1.$ ...(2)

Choosing $1/x$, $-1/y$, c_1/n as multipliers, each fraction of (2)

$$= \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + c_1xy} = \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + y - x}, \text{ by (2)}$$

$$= \frac{(1/x)dx + (1/y)dy + (c_1/n)dz}{0} \quad \text{so that} \quad \frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz = 0.$$

Integrating, $\log x - \log y + (c_1/n)\,z = (c_1/n)c_2$, c_2 being an arbitrary constant.

or $z - (n/c_1)(\log y - \log x) = c_2$ or $z - (n/c_1)\log(y/x) = c_2$

or $z - \frac{nxy}{y - x}\log\frac{y}{x} = c_2$, using (2). ...(3)

From (2) and (3), the required general solution is

$$\phi\left(\frac{y-x}{xy}, z-\frac{nxy}{y-x}\log\frac{y}{x}\right)=0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 11. *Solve* $(x-y)p+(x+y)q=2xz$.

Sol. Here the Lagrange's subsidiary equations are $\dfrac{dx}{x-y}=\dfrac{dy}{x+y}=\dfrac{dz}{2xz}$. ...(1)

Taking the first two fractions of (1), $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}=\dfrac{1+(y/x)}{1-(y/x)}$. ...(2)

Let $y/x=v$ *i.e.*, $y=xv$. ...(3)

From (3), $(dy/dx)=v+x(dv/dx)$. ...(4)

Using (3) and (4), (2) gives $v+x\dfrac{dv}{dx}=\dfrac{1+v}{1-v}$

or $x\dfrac{dv}{dx}=\dfrac{1+v}{1-v}-v=\dfrac{1+v-v(1-v)}{1-v}=\dfrac{1+v^2}{1-v}$

or $\dfrac{1-v}{1+v^2}\,dv=\dfrac{dx}{x}$ or $\left(\dfrac{2}{1+v^2}-\dfrac{2v}{1+v^2}\right)dv=\dfrac{2dx}{x}$

Integrating, $2\tan^{-1}v-\log(1+v^2)=2\log x-\log c_1$

or $\log x^2-\log(1+v^2)-\log c_1=2\tan^{-1}v$

or $\log\{x^2(1+v^2)/c_1\}=2\tan^{-1}v$ or $x^2(1+v^2)=c_1e^{2\tan^{-1}v}$

or $x^2[1+(y^2/x^2)]=c_1e^{2\tan^{-1}(y/x)}$, as $v=y/x$ by (3)

or $(x^2+y^2)\,e^{-2\tan^{-1}(y/x)}=c_1$, c_1 being an arbitrary constant. ...(5)

Choosing 1, 1, $-1/z$ as multipliers, each fraction of (1)

$$=\frac{dx+dy-(1/z)dz}{(x-y)+(x+y)-(1/z)\times(2xz)}=\frac{dx+dy-(1/z)dz}{0}$$

$\Rightarrow dx+dy-(1/z)dz=0$ so that $x+y-\log z=c_2$. ...(6)

From (5) and (6), the required general solution is

$$\phi\,(x+y-\log z,\ (x^2+y^2)\ e^{-2\tan^{-1}(y/x)})=0, \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 12. *Solve* $y^2p+x^2q=x^2y^2z^2$.

Sol. Here Lagrange's auxiliary equations are $(dx)/y^2=(dy)/x^2=(dz)/x^2y^2z^2$. ...(1)

Taking the first two fractions of (1), we have

$3x^2dx-3y^2dy=0$ so that $x^3-y^3=c_1$. ...(2)

Choosing x^2, y^2, $-2/z^2$ as multipliers, each fraction of (1) $=\{x^2dx+y^2dy-(2/z^2)dz\}/0$

so that $3x^2dx+3y^2dy-(6/z^2)dz=0$.

Integrating, $x^3+y^3+(6/z)=c_2$, c_2 being an arbitrary constant. ...(3)

From (2) and (3), the required general solution is

$$\phi\,[x^3-y^3,\ x^3+y^3+(6/z)]=0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 13. *Solve* $(3x+y-z)p+(x+y-z)q=2(z-y)$. **[Bangalore 1992]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{3x+y-z}=\dfrac{dy}{x+y-z}=\dfrac{dz}{2(z-y)}$ ...(1)

Choosing 1, –3, 1 as multipliers, each ratio of (1) $= \{dx - 3dy - dz\}/0$

so that $dx - 3dy - dz = 0.$

Integrating, $x - 3y - z = c_1$, c_1 being an arbitrary constant. ...(2)

From (2), $z = c_1 - x + 3y.$ ...(3)

Substituting the above value of z, the first two fractions of (2) reduce to

$$\frac{dx}{3x+y-(c_1-x+3y)} = \frac{dy}{x+y-(c_1-x+3y)} \quad \text{or} \quad \frac{dx}{2x+4y+c_1} = \frac{dy}{4y+c_1}. \quad ...(3)$$

Let $u = 4y + c_1$ so that $dy = (1/4) \times du.$ ...(4)

Then, (3) $\Rightarrow \frac{dx}{2x+u} = \frac{(1/4)du}{u}$ or $\frac{dx}{du} = \frac{1}{4}\frac{2x+u}{u}$ or $\frac{dx}{du} - \frac{1}{2u}x = \frac{1}{4}$, which is linear. ...(5)

Integrating factor of (5) $= e^{-\int (1/2u)du} = e^{-(1/2)\log u} = e^{\log(u)^{-1/2}} = u^{-1/2} = 1/\sqrt{u}$.

Hence solution of (5) is $x \times \frac{1}{\sqrt{u}} = \int \frac{1}{4}\frac{1}{\sqrt{u}}du + c = \frac{1}{2}\sqrt{u} + c_2$

or $\frac{2x-u}{\sqrt{u}} = c_2$ or $\frac{2x-(4y+c_1)}{\sqrt{4y+c_1}} = c_2$, by (4)

or $\frac{2x-4y-(x-3y-z)}{\sqrt{4y+x-3y-z}} = c_2$, using (2) or $\frac{x-y+z}{\sqrt{x+y-z}} = c_2$ ...(6)

From (2) and (6), the required general solution is

$$\phi\left(x-3y-z, (x-y+z)/\sqrt{x+y-z}\right) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 14. *Solve* $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$. **[Delhi Maths Hons 95, 2000]**

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(x^2+3y^2)} = \frac{dy}{-y(3x^2+y^2)} = \frac{dz}{2z(y^2-x^2)}. \quad ...(1)$$

Choosing $1/x$, $1/y$, $-1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0} \quad \text{so that} \quad \frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0.$$

Integrating, $\log x + \log y - \log z = \log c_1$ so that $(xy)/z = c_1$. ...(2)

Taking the first two ratios of (1), $\frac{dy}{dx} = -\frac{y(3x^2+y^2)}{x(x^2+3y^2)} = -\left(\frac{y}{x}\right)\frac{3+(y/x)^2}{1+3(y/x)^2}$. ...(3)

Put $y/x = v$ or $y = xv$ so that $(dy/dx) = v + x(dv/dx)$. ...(4)

Using (4), (3) reduces to $v + x\frac{dv}{dx} = -v\frac{3+v^2}{1+3v^2}$ or $x\frac{dv}{dx} = -v\left[\frac{3+v^2}{1+3v^2} + 1\right]$

or $x\frac{dv}{dx} = -\frac{4(1+v^2)v}{1+3v^2}$ or $4\frac{dx}{x} + \frac{1+3v^2}{v(1+v^2)}dv = 0$

or $4\frac{dx}{x} + \left(\frac{1}{v} + \frac{2v}{1+v^2}\right)dv$, on resolving into partial fractions

Integrating, $4\log x + \log v + \log(1 + v^2)$ or $x^4 v(1 + v^2) = c_2'$

$x^4(y/x)[1+(y/x)^2]=c_2'$ or $xy(x^2+y^2)=c_2'$ or $c_1z(x^2+y^2)=c_2'$, by (2)

or $z(x^2+y^2)=c_2'/c_1$ or $z(x^2+y^2)=c_2$, where $c_2=c_2'/c_1$. ... (5)

$\therefore$ From (2) and (5) solution is $\phi(z(x^2+y^2), xy/z)=0$, ϕ being an arbitrary function.

Ex. 15. *Solve* $(y-z)p+(z-x)q=x-y$. **[Agra 2010; Delhi Maths Hons. 1992]**

Sol. Here the Lagrange's auxiliary equations are $\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}$. ...(1)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$=\frac{dx+dy+dz}{(y-z)+(z-x)+(x-y)}=\frac{dx+dy+dz}{0}.$$

$\therefore$ $dx+dy+dz=0$ so that $x+y+z=c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$=\frac{x\,dx+y\,dy+z\,dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\therefore$ $2x\,dx+2y\,dy+2z\,dz=0$ so that $x^2+y^2+z^2=c_2$ ...(3)

$\therefore$ From (2) and (3) solution is $\phi(x+y+z, x^2+y^2+z^2)=0$, ϕ being an arbitrary of function.

Ex. 16. *Solve the general solution of the equation* $(y+zx)p-(x+yz)q+y^2-x^2=0$.
[Delhi B.Sc. (Prog) II 2011; GATE 2001; Delhi Math Hons. 1997, 98]

Sol. Given $(y+zx)p-(x+yz)q=x^2-y^2$. ...(1)

Here the Lagrange's auxiliary equations are $\frac{dx}{y+zx}=\frac{dy}{-(x+yz)}=\frac{dz}{x^2-y^2}$. ...(2)

Choosing $x, y, -z$ as multipliers, each fraction of (2)

$$=\frac{xdx+ydy-zdz}{x(y+zx)-y(x+yz)-z(x^2-y^2)}=\frac{xdx+ydy-zdz}{0}$$

$\therefore$ $xdx+ydy-zdz=0$ so that $2xdx+2ydy-2zdz=0$.

Integrating, $x^2+y^2-z^2=c_1$, c_1 being an arbitrary constant. ...(3)

Choosing $y, x, 1$ as multipliers, each fraction of (2)

$$=\frac{ydx+xdy+dz}{y(y+zx)-x(x+yz)+x^2-y^2}=\frac{ydx+xdy+dz}{0}$$

$\therefore$ $ydx+xdy+dz=0$ or $d(xy)+dz=0$.

Integrating, $xy+z=c_2$, c_2 being an arbitrary constant. ...(4)

$\therefore$ The required solution is $\phi(x^2+y^2-z^2, xy+z)=0$, ϕ being an arbitrary function.

Ex. 17. *Solve* $x(y-z)\,p+y\,(z-x)q=z(x-y)$, *i.e.*, $\{(y-z)/(yz)\}p+\{(z-x)/(zx)\}q=(x-y)/(xy)$. **[Delhi B.A (Prog) II 2010; I.A.S. 2005, M.S. Univ. T.N. 2007; Vikram 2003]**

Sol. Given $x(y-z)p+y(z-x)q=z(x-y)$ ... (1)

The Lagrange's auxiliary equations for (1) are $\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$ ... (2)

Choosing $1/x$, $1/y$, $1/z$ as multipliers each fraction of (1)

$$=\frac{(1/x)dx+(1/y)dy+(1/z)dz}{(y-z)+(z-x)+(x-y)}=\frac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\therefore \log (xyz) = c_1$ or $xyz = c_1$ ... (3)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(xy - xz) + (yz - yx) + (zx - zy)} = \frac{dx + dy + dz}{0}$$

$\Rightarrow dx + dy + dz = 0$ so that $x + y + z = c_2$ ... (4)

From (3) and (4), solution is $\phi(x + y + z, xyz) = 0, \phi$ being an arbitrary function.

Ex. 18. *Solve* $2y(z - 3)p + (2x - z)q = y(2x - 3)$ **[Delhi Math (H) 1999]**

Sol. The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \quad \text{... (1)}$$

Taking the first and third fractions, $(2x - 3)dx = 2(z - 3)dz$.

Integrating, $x^2 - 3x = z^2 - 6z + C_1$ or $x^2 - 3x - z^2 + 6z = C_1$ ... (2)

Choosing 1, 2y, –2 as multipliers, each fraction of (1)

$$= \frac{dx + 2ydy - 2dz}{2y(z-3) + 2y(2x-z) - 2y(2x-3)} = \frac{dx + 2ydy - 2dz}{0}$$

$\therefore dx + 2ydy - 2dz = 0$ so that $x + y^2 - 2z = C_2$ ... (3)

From (2) and (3), solution is $\phi(x^2 - 3x - z^2 + 6z, x + y^2 - 2z) = 0, \phi$ being an arbitrary function.

Ex. 19. *Solve* $x^2(\partial z/\partial x) + y^2(\partial z/\partial y) = (x + y)z.$ **[Delhi Maths (H) 2001]**

Sol. Re-writing the given equation $x^2p + y^2q = (x + y)z$ ... (1)

The Lagrange's auxiliary equations for (1) are $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$ ... (2)

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0.$

Integrating, $-(1/x) + (1/y) = C_1$ or $1/y - 1/x = C_1$ ... (3)

Choosing $1/x, 1/y, -1/z$ as multipliers, each fraction of (2)

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{x + y - (x + y)} = \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0}$$

$\therefore (1/x)dx + (1/y)dy - (1/z)dz = 0$ so that $xy/z = C_2$ ... (4)

From (3) and (4), solution is $\Phi(1/y - 1/x, xy/z) = 0, \Phi$ being an arbitrary function.

Ex. 20. *Solve* $z(x + 2y)\, p - z(y + 2x)\, q = y^2 - x^2$ **[Vikram 1999]**

Sol. The Lagrange's subsidiary equations are $\frac{dx}{z(x+2y)} = \frac{dy}{-z(y+2x)} = \frac{dz}{y^2 - x^2}$ ... (1)

Taking the first two fraction of (1), we have

$(y + 2x)\, dx + (x + 2y)\, dy = 0$ or $2xdx + 2y\, dy + d\,(xy) = 0$

Integrating, $x^2 + y^2 + xy = C_1$, C_1 being an arbitrary constant ... (2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{(x^2z + 2xyz) - (y^2z + 2xyz) + (zy^2 - zx^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow$ $2x\,dx + 2y\,dy + 2z\,dz = 0$ so that $x^2 + y^2 + z^2 = C_2$ ... (3)

From (2) and (3), solution is $\phi(x^2 + y^2 + z^2, x^2 + y^2 + xy) = 0$, ϕ being an arbitrary function

EXERCISE 2(C)

Solve the following partial differential equations:

1. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ **Ans.** $\phi(x^2 + y^2 + z^2, xyz) = 0$

[Mysore 2004, Delhi B.Sc. (Prog). II 2007, M.S. Unit. T.N. 2007]

2. $z(xp - yq) = y^2 - x^2$ **Ans.** $\phi(x^2 + y^2 + z^2, xy) = 0$

3. $(y^2 + z^2)p - xyq + xz = 0$ **[I.A.S. 1990]** **Ans.** $\phi(x^2 + y^2 + z^2, y/z) = 0$

4. $yp - xq = 2x - 3y$ **[M.S. Univ. T.N. 2007]** **Ans.** $\phi(x^2 + y^2, 3x + 2y + z) = 0$

5. $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$ **Ans.** $\phi(xyz, 1/x + 1/y + 1/z) = 0$

[Meerut 2007, Bilaspur 2004, Rewa 2003]

2.11. Type 4 based on Rule IV for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction of (1) will be equal to $(P_1dx + Q_1dy + R_1dz)/(P_1P + Q_1Q + R_1R)$. ...(2)

Suppose the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$(P_2dx + Q_2dy + R_2dz)/(P_2P + Q_2Q + R_2R) \quad ...(3)$$

is such that its numerrator is exact differential of denominator. Fractions (2) and (3) are then combined to given an integral. This method may be repeated in some problems to get another integral. Sometimes only one integral is possible by using the above rule IV. In such cases second integral should be obtained by using rule 1 of Art. 2.5 or rule 2 of Art. 2.7 or rule 3 of Art. 2.9.

2.12. SOLVED EXAMPLES BASED IN ART. 2.11

Ex. 1. *Solve* $(y + z)p + (z + x)q = x + y$. **[Indore 2000; Jabalpur 2000, Jiwaji 2002, Kanpur 2008; Purvanchal 2007, Ravishankar 2002, 2005; Delhi BA (Prog.) II 2011]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$. ...(1)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx - dy}{(y+z)-(z+x)} = \dfrac{d(x-y)}{-(x-y)}$. ...(2)

Again, choosing 0, 1, –1 as multipliers, each fraction of (1) $= \dfrac{dy - dz}{(z+x)-(x+y)} = \dfrac{d(y-z)}{-(y-z)}$. ...(3)

Finally, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)}. \quad ...(4)$$

(2), (3) and (4) $\Rightarrow$ $\dfrac{d(x-y)}{-(x-y)} = \dfrac{d(y-z)}{-(y-z)} = \dfrac{d(x+y+z)}{2(x+y+z)}$. ...(5)

Taking the first two fractions of (5), $\dfrac{d(x-y)}{x-y} = \dfrac{d(y-z)}{y-z}$.

Integrating, $\log(x - y) = \log(y - z) + \log c_1$, c_1 being an arbitrary constant.

or $\log\{(x-y)/(y-z)\} = \log c_1$ or $(x-y)/(y-z) = c_1$. ...(6)

Taking the first and the third fractions of (5), $2\dfrac{d(x-y)}{(x-y)} + \dfrac{d(x+y+z)}{x+y+z} = 0$

Integrating, $2\log(x-y) + \log(x+y+z) = \log c_2$ or $(x-y)^2(x+y+z) = c_2$. ...(7)

From (6) and (7), the required general solution is

$$\phi[(x-y)^2(x+y+z), (x-y)/(y-z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$ **[Delhi Maths Hons 1997; Nagpur 2010]**

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}. \quad ...(1)$$

Taking the first two fractions of (1), $x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0$.

Integrating, $x^3 + y^3 = c_1$, c_1 being an arbitrary as constant. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)}. \quad ...(3)$$

Combining the third fraction of (1) with fraction (3), we get

$$\frac{dx-dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

Integrating, $\log(x-y) - \log z = \log c_2$ or $(x-y)/z = c_2$. ...(4)

From (3) and (4), solution is $\phi(x^3+y^3, (x-y)/z) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $(x^2-y^2-z^2)p + 2xyq = 2xz$ or $(y^2+z^2-x^2)p - 2xyq = -2xz$.

[Bangalore 1993, I.A.S. 1973; P.C.S. (U.P.) 1991; Bhopal 2010]

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}. \quad ...(1)$$

Taking the last two fractions of (1), we have

$(1/y)dy = (1/z)dz$ so that $(1/y)dy - (1/z)dz = 0$.

Integrating, $\log y - \log z = \log c_1$ or $y/z = c_1$. ...(2)

Choosing x, y, z as mnultipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}. \quad ...(3)$$

Combining the third fraction of (1) with fraction (3), we have

$$\frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)} = \frac{dz}{-2xz} \quad \text{or} \quad \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} - \frac{dz}{z} = 0.$$

Integrating, $\log(x^2+y^2+z^2) - \log z = \log c_2$ or $(x^2+y^2+z^2)/z = c_2$. ... (4)

From (2) and (4) solution is $\phi(y/z, (x^2+y^2+z^2)/z) = 0$, ϕ being an arbitrary function.

Ex. 4. *Solve* $(1+y)p + (1+x)q = z$. **[M.S. Univ. T.N. 2007; Kanpur 2011]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{1+y} = \dfrac{dy}{1+x} = \dfrac{dz}{z}$. ...(1)

Taking the first two fractions of (1), we have

$(1+x)dx = (1+y)dy$ or $2(1+x)dx - 2(1+y)dy = 0$.

Integrating, $(1 + x)^2 - (1 + y)^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Taking 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx+dy}{1+y+1+x} = \dfrac{d(2+x+y)}{2+x+y}$. ...(3)

Combining the last fraction of (1) with fraction (3), we get

$$\frac{d(2+x+y)}{2+x+y} = \frac{dz}{z} \quad \text{or} \quad \frac{d(2+x+y)}{2+x+y} - \frac{dz}{z} = 0.$$

Integrating, $\log (2 + x + y) - \log z = \log c_2$ or $(2 + x + y)/z = c_2$. ...(4)

From (2) and (4), the required general solution is given by

$$\phi [(1 + x)^2 - (1 + y^2), (2 + x + y)/z] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 5. *Find the general integral of* $xzp + yzq = xy$.

Sol. Here the Lagrange's auxiliary equations are $(dx)/xz = (dy)/yz = (dz)/xy$ ... (1)

From the first two fractions of (1), $(1/x)dx = (1/y)dy$.

Integrating, $\log x = \log y + \log c_1$ or $x/y = c_1$. ...(2)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1) $= \dfrac{(1/x)dx + (1/y)dy}{(1/x)xz + (1/y)yz} = \dfrac{ydx + xdy}{2xyz}$ ...(3)

Combining the last fraction of (1) with fraction (3), we have

$$\frac{ydx+xdy}{2xyz} = \frac{dz}{xy} \quad \text{or} \quad ydx + xdy = 2zdz \quad \text{or} \quad d(xy) = 2zdz \quad \text{or} \quad d(xy) - 2zdz = 0$$

Integrating, $xy - z^2 = c_2$, c_2 being an arbitrary constant. ...(4)

From (2) and (4) solution is $\phi (x/y, xy - z^2) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. **Delhi Math (H) 2005, 11, M.D.U. Rohtak 2005; Agra 2008, 09; Guwahati 2007; Meerut 2006; Sagar 2000; Ravishankar 2000; Lucknow 2010]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{x^2-yz} = \dfrac{dy}{y^2-zx} = \dfrac{dz}{z^2-xy}$. ...(1)

Choosing 1, –1, 0 and 0, 1, –1 as multipliers in turn, each fraction of (1)

$$= \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

so that $$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0.$$

Integrating, $\log (x - y) - \log (y - z) = \log c_2$ or $(x - y)/(y - z) = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}. \quad ...(3)$$

Again, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}. \quad ...(4)$$

From (3) and (4), $$\frac{xdx+ydy+zdz}{x+y+z} = dx + dy + dz$$

or $$2(x + y + z)\, d(x + y + z) - (2xdx + 2ydy + 2zdz) = 0.$$

Integrating, $$(x + y + z)^2 - (x^2 + y^2 + z^2) = 2c_2$$

or $$(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) - (x^2 + y^2 + z^2) = 2c_2$$

or $xy + yz + zx = c_2$, c_2 being an arbitrary constant. ...(5)

From (2) and (5), the required general solution is given by

$$\phi\,[xy + yz + zx,\ (x - y)/(y - z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 7. *Solve* $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$.

Sol. Here Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x - y)} \qquad ...(1)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 - y^2 - yz) - (x^2 - y^2 - zx)} = \frac{dx - dy}{z(x - y)}. \qquad ...(2)$$

Choosing x, $-y$, 0 as multipliers each fraction of (1)

$$= \frac{xdx - ydy}{x(x^2 - y^2 - yz) - y(x^2 - y^2 - zx)} = \frac{xdx - ydy}{(x - y)(x^2 - y^2)}. \qquad ...(3)$$

From (1), (2), (3) we have

$$\frac{dz}{z(x - y)} = \frac{dx - dy}{z(x - y)} = \frac{xdx - ydy}{(x - y)(x^2 - y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx - dy}{z} = \frac{2xdx - 2ydy}{2(x^2 - y^2)}. \qquad ...(4)$$

Taking the first two fractions of (4), we have

$dz = dx - dy$ so that $z - x + y = c_1$ ...(5)

Again , taking the first and third fractions of (4), $d(x^2 - y^2)/(x^2 - y^2) - (2/z)\,dz = 0$

Integrating, $\log (x^2 - y^2) - 2 \log z = c_2$ or $(x^2 - y^2)/z^2 = c_2$. ...(6)

From (5) and (6), solution is $\phi\,(z - x + y,\ (x^2 + y^2)/z^2) = 0$, ϕ being an arbitrary function.

Ex. 8. *Solve* $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)$.

Sol. Here the Lagrange's, auxiliary equations are $\frac{dx}{x^2 + y^2 + yz} = \frac{dy}{x^2 + y^2 - xz} = \frac{dz}{z(x + y)}$. ...(1)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 + y^2 + yz) - (x^2 + y^2 - xz)} = \frac{dx - dy}{z(x + y)}. \qquad ...(2)$$

Choosing x, y, 0 as multipliers, each fraction of (1)

$$= \frac{xdx + ydy}{x(x^2 + y^2 + yz) + y(x^2 + y^2 - xz)} = \frac{xdx + ydy}{(x + y)(x^2 + y^2)}. \qquad ...(3)$$

From (1), (2) and (3), we have

$$\frac{dz}{z(x + y)} = \frac{dx - dy}{z(x + y)} = \frac{xdx + ydy}{(x + y)(x^2 + y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx - dy}{z} = \frac{xdx + ydy}{x^2 + y^2}. \qquad ...(4)$$

Taking the first two fractions of (4), we have

$dz = dx - dy$ or $dz - dx + dy = 0$.

Integrating, $z - x + y = c_1$, c_1 being an arbitrary constant. ...(5)

Taking the first and third fractions of (4), we have

$$\frac{2xdx + 2ydy}{x^2 + y^2} = 2\frac{dz}{z} \quad \text{or} \quad \frac{d(x^2 + y^2)}{x^2 + y^2} - 2\frac{dz}{z} = 0.$$

Integrating, $\log (x^2 + y^2) - 2\log z = \log c_2$ or $(x^2 + y^2)/z^2 = c_2$. ...(6)

From (5) and (6), solution is $\phi\,(z - x + y,\ (x^2 + y^2)/z^2) = 0$, ϕ being an arbitrary function.

Ex. 9. *Solve* $\cos(x+y)p + \sin(x+y)q = z$. **[Garhwal 2010, Vikram 1998; Meerut 2007; Delhi Maths (H) 2007; Rajasthan 1994; Delhi B.A./B.Sc. (Prog.) Maths 2007]**

Sol. Here the Lagrange's auxiliary equations are $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$. ...(1)

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \quad ...(2)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$. ...(3)

From (1), (2) and (3), $\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$. ...(4)

Taking the first two fractions of (4), $\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}$. ...(5)

Putting $x + y = t$ so that $d(x + y) = dt$, (5) reduces to

$$\frac{dz}{z} = \frac{dt}{\cos t+\sin t} = \frac{dt}{\sqrt{2}\left\{\left(1/\sqrt{2}\right)\cos t+\left(1/\sqrt{2}\right)\sin t\right\}} = \frac{dt}{\sqrt{2}\left\{\sin(\pi/4)\cos t+\cos(\pi/4)\sin t\right\}} = \frac{dt}{\sqrt{2}\sin(t+\pi/4)}$$

Thus, $(\sqrt{2}/z)dz = \text{cosec}\ (t+\pi/4)\ dt$.

Integrating, $\sqrt{2}\ \log z = \log \tan \frac{1}{2}\left(t+\frac{\pi}{4}\right) + \log c_1$, or $z^{\sqrt{2}} = c_1 \tan\left(\frac{t}{2}+\frac{\pi}{8}\right)$

or $z^{\sqrt{2}}\ \cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = c_1$. as $t = x + y$ ...(6)

Taking the last two fraction of (4), $dx - dy = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y)$. ...(7)

On R.H.S. of (7), putting $x + y = t$, so that $d(x + y) = dt$, (7) reduces to

$dx - dy = \frac{\cos t-\sin t}{\cos t+\sin t}dt$. so that $x - y = \log(\sin t + \cos t) - \log c_2$

or $(\sin t + \cos t)/c_2 = e^{x-y}$ or $e^{-(x-y)}(\sin t + \cos t) = c_2$

or $e^{y-x}[\sin(x+y) + \cos(x+y)] = c_2$, as $t = x + y$. ...(8)

From (6) and (8), the required general solution is

$$\phi\left[z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right), e^{y-x}\left\{\sin(x+y)+\cos(x+y)\right\}\right] = 0, \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 10. *Solve* $\cos(x+y)p + \sin(x+y)q = z + (1/z)$. **[Delhi B.A. (Prog.) 2011]**

Sol. Do like Ex. 9. **Ans.** $\phi\left[(z^2+1)^{1/\sqrt{2}}\tan\left(\frac{3\pi}{8}-\frac{x+y}{2}\right), e^{y-x}\left\{\cos(x+y)+\sin(x+y)\right\}\right] = 0$

Ex. 11. *Solve* $xp + yq = z - a\sqrt{(x^2+y^2+z^2)}$. **[Meerut 1997; Jiwaji 1997; Rawa 1999]**

Sol. Here the lagrange's auxiliary equations are $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}$. ...(1)

Taking the first two fractions of (1), we have

$(1/x)dx = (1/y)dy$ or $(1/x)dx - (1/y)dy = 0.$

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1.$ ...(2)

Choosing x, y, z as multipliers, each fraction of (1) $= \dfrac{xdx + ydy + zdz}{x^2+y^2+z^2 - az\sqrt{(x^2+y^2+z^2)}}$ ...(3)

Combining first and third fractions of (1) with fraction (3), we get

$$\frac{dx}{x} = \frac{dz}{z - a\sqrt{(x^2+y^2+z^2)}} = \frac{xdx + ydy + zdz}{x^2+y^2+z^2 - az\sqrt{(x^2+y^2+z^2)}}. \quad ...(4)$$

Putting $x^2 + y^2 + z^2 = t^2$ so that $xdx + ydy + zdz = tdt$, (4) gives

$$\frac{dx}{x} = \frac{dz}{z-at} = \frac{tdt}{t^2 - azt} \quad \text{or} \quad \frac{dx}{x} = \frac{dz}{z-at} = \frac{dt}{t-az}. \quad ...(5)$$

Choosing 0, 1, 1 as multipliers, each fraction of (5) $= \dfrac{dz+dt}{(z+t)-a(t+z)} = \dfrac{d(z+t)}{(1-a)(z+t)}.$ ...(6)

Combining the first fraction of (5) with fraction (6), we get

$$\frac{dx}{x} = \frac{d(z+t)}{(1-a)(z+t)} \quad \text{or} \quad (1-a)\frac{dx}{x} - \frac{d(z+t)}{z+t} = 0.$$

Integrating, $(1 - a) \log x - \log (z + t) = \log c_2$, c_2 being an arbitrary constant.

or $\dfrac{x^{a-1}}{z+t} = c_2$ or $\dfrac{x^{a-1}}{z+\sqrt{(x^2+y^2+z^2)}} = c_2,$ as $t = (x^2 + y^2 + z^2)^{1/2}$ ...(7)

From (2) and (7), the required general solution is

$$\phi\,[x^{a-1}/\{z + \sqrt{(x^2+y^2+z^2)}\}, x/y] = 0,\ \phi \text{ being an arbitrary function.}$$

Ex. 12. *Solve* $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2 + y^2)$. **[I.A.S. 1993]**

Sol. Here the Lagrange's subsidiary equations are $\dfrac{dx}{x^3+3xy^2} = \dfrac{dy}{y^3+3x^2y} = \dfrac{dz}{2z(x^2+y^2)}.$ ...(1)

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx+dy}{x^3+3xy^2+3x^2y+y^3} = \dfrac{d(x+y)}{(x+y)^3}.$ ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx-dy}{x^3+3xy^2-y^3-3x^2y} = \dfrac{d(x-y)}{(x-y)^3}.$ ...(3)

From (2) and (3), $(x + y)^{-3}\, d(x + y) = (x - y)^{-3}\, d(x - y)$

or $u^{-3}du - v^{-3}dv = 0$, on putting $u = x + y$ and $v = x - y$.

Integrating, $u^{-2}/(-2) - v^{-2}/(-2) = c_1/2$ or $v^{-2} - u^{-2} = c_1$

or $(x - y)^{-2} - (x + y)^{-2} = c_1$, as $u = x + y$ and $v = x - y$. ...(4)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)\,dx + (1/y)\,dy}{(1/x)\times(x^3+3xy^2) + (1/y)\times(y^3+3x^2y)} = \frac{(1/x)dx + (1/y)dy}{4(x^2+y^2)}. \quad ...(5)$$

Combining the last fraction of (1) with fraction (5), we have

$$\frac{dz}{2z(x^2+y^2)} = \frac{(1/x)dx + (1/y)dy}{4(x^2+y^2)} \quad \text{or} \quad \frac{dx}{x} + \frac{dy}{y} - 2\frac{dz}{z} = 0.$$

Integrating, $\log x + \log y - 2 \log z = \log c_2$ or $(xy)/z^2 = c_2.$ ...(6)

From (4) and (6), the required general solution is given by

$$\phi\,[(x-y)^{-2}-(x+y)^{-2},\ (xy)/z^2]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 13. *Solve* $p+q=x+y+z$. **[Bhopal 2010, Bilaspur 2000, 02; I.A.S. 1975; Gulberge 2005]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{1}=\dfrac{dy}{1}=\dfrac{dz}{x+y+z}$. ...(1)

Taking the first two fractions of (1), $dx-dy=0$ so that $x-y=c_1$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction of (1) $=\dfrac{dx+dy+dz}{1+1+(x+y+z)}=\dfrac{d(2+x+y+z)}{2+x+y+z}$ ... (3)

Combining the first fraction of (1) with fraction (3), $d(2+x+y+z)/(2+x+y+z)=dx$.

Integrating, $\log(2+x+y+z)-\log c_2=x$ or $(2+x+y+z)/c_2=e^x$

or $e^{-x}(2+x+y+z)=c_2$, c_2 being arbitrary function ...(4)

From (2) and (4), the required general solution is

$$\phi[x-y,\ e^{-x}(2+x+y+z)]=0,\ \phi \text{ being an arbitrary function.}$$

Ex. 14. *Solve* $(2x^2+y^2+z^2-2yz-zx-xy)p+(x^2+2y^2+z^2-yz-2zx-xy)q=x^2+y^2+2z^2-yz-zx-2xy$. **[Meerut 1996 ; I.A.S. 1992]**

Sol. Here Lagrange's auxiliary equations are

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy}=\frac{dy}{x^2+2y^2+z^2-yz-2zx-xy}=\frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}. \quad ...(1)$$

Choosing 1, –1, 0 ; 0, 1, –1 and –1, 0, 1 as multipliers in turn, each fraction of (1)

$$=\frac{dx-dy}{x^2-y^2-yz+zx}=\frac{dy-dz}{y^2-z^2-zx+xy}=\frac{dz-dx}{z^2-x^2-xy+yz}$$

$$\therefore\quad \frac{dx-dy}{(x-y)\,(x+y+z)}=\frac{dy-dz}{(y-z)\,(x+y+z)}=\frac{dz-dx}{(z-x)\,(x+y+z)}. \quad ...(2)$$

Taking the first two fractions of (2), we have

$$(dx-dy)/(x-y)-(dy-dz)/(y-z)=0.$$

Integrating, $\log(x-y)-\log(y-z)=\log c_1$ or $(x-y)/(y-z)=c_1$. ...(3)

Taking the last two fractions of (2), $(dy-dz)/(y-z)-(dz-dx)/(z-x)=0$.

Integrating, $\log(y-z)-\log(z-x)=\log c_2$ or $(y-z)/(z-x)=c_2$. ...(4)

From (3) and (4), the required general solution is

$$\phi[(x-y)/(y-z),\ (y-z)/(z-x)]=0,\ \phi \text{ being an arbitrary function.}$$

Ex. 15. *Find the general solution of the partial differential equation* $px(x+y)-qy(x+y)+(x-y)(2x+2y+z)=0$. **[Delhi B.Sc. II (Prog) 2009; Delhi Maths Hons. 2006, 09, 11]**

Sol. Given $x(x+y)p-y(x+y)q=-(x-y)(2x+2y+z)$. ...(1)

Lagrange's auxiliary equations are $\dfrac{dx}{x(x+y)}=\dfrac{dy}{-y(x+y)}=\dfrac{dz}{-(x-y)(2x+2y+z)}$. ...(2)

Taking the first two fractions, $(1/x)dx=-(1/y)dy$ or $(1/x)dx+(1/y)dy=0$.

Integrating, $\log x+\log y=\log c_1$ or $xy=c_1$. ...(3)

Again, each fraction of (2)

$$=\frac{dx+dy}{x(x+y)-y(x+y)}=\frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)}$$

$$=\frac{dx+dy}{(x-y)(x+y)}=\frac{dx+dy+dz}{(x-y)(x+y)-(x-y)(2x+2y+z)}$$

Thus, $$\frac{dx+dy}{(x+y)} = \frac{dx+dy+dz}{x+y-(2x+2y+z)} = -\frac{dx+dy+dz}{x+y+z}$$

Thus, $\frac{dx+dy}{x+y} + \frac{dx+dy+dz}{x+y+z} = 0$, so that $\log(x+y) + \log(x+y+z) = \log c_2$

or $(x+y)(x+y+z) = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), solution is $\phi[xy, (x+y)(x+y+z)] = 0$, ϕ being an arbitrary function.

Ex. 16. *Solve* $\{my(x+y)-nz^2\}(\partial z/\partial x) - \{lx(x+y)-nz^2\}(\partial z/\partial y) = (lx-my)z$ **[I.A.S. 2001]**

Sol. Re-writing the given equation, $\{my(x+y)-nz^2\}\,p - \{lx(x+y)-nz^2\}q = (lx-my)z$ ...(1)

Lagrange's auxiliary equations for (1) are $\frac{dx}{my(x+y)-nz^2} = \frac{dy}{-lx(x+y)+nz^2} = \frac{dz}{(lx-my)z}$ ...(2)

Each fraction of (2) $= \frac{dx+dy}{(my-lx)(x+y)} = \frac{dz}{-(my-lx)z}$ so that $\frac{d(x+y)}{x+y} = -\frac{dz}{z}$

Integrating, $\log(x+y) = -\log z + \log C_1$ or $(x+y)z = C_1$ ... (3)

Taking lx, my, nz as multipliers, each fraction of (2)

$$= \frac{lx\,dx + my\,dy + nz\,dz}{lx\,my(x+y) - lx\,nz^2 - mylx(x+y) + mynz^2 + nz^2(lx-my)} = \frac{lx\,dx + my\,dy + nz\,dz}{0}$$

$\therefore\ 2lx\,dx + 2my\,dy + 2nz\,dz = 0$ so that $lx^2 + my^2 + nz^2 = C_2$ ...(4)

From (3) and (4), solution is $\Phi(xz+yz,\ lx^2+my^2+nz^2) = 0$, Φ being an arbitrary function.

Ex. 17. *Solve* $px(z-2y^2) = (z-qy)(z-y^2-2x^2)$. **[Delhi Maths (H) 2002]**

Sol. Re-writing the given equation $x(z-2y^2)p + y(z-y^2-2x^2)q = z(z-y^2-2x^2)$ ... (1)

Lagrange's auxiliary equations for (1) are $\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^2)} = \frac{dz}{z(z-y^2-2x^2)}$ ...(2)

Taking the last two fractions, $(1/y)\,dy - (1/z)\,dz = 0$ so that $y/z = C_1$ ...(3)

Taking 0, $-2y$, 1 as multipliers, each fraction of (2)

$$= \frac{-2y\,dy + dz}{-2y^2(z-y^2-2x^2) + z(z-y^2-2x^2)} = \frac{d(z-y^2)}{(z-2y^2)(z-y^2-2x^2)} \quad \text{... (4)}$$

Combining fraction (4) with first fraction of (2), we get

$$\frac{dx}{x(z-2y^2)} = \frac{d(z-y^2)}{(z-2y^2)(z-y^2-2x^2)} \quad \text{or} \quad \frac{d(z-y^2)}{dx} = \frac{z-y^2-2x^2}{x}$$

or $du/dx = (u-2x^2)/x$, taking $z-y^2 = u$ ... (5)

or $(du/dx) - (1/x)u = -2x$ which is an ordinary linear differential equation

whose I.F. $= e^{-\int (1/x)dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$ and solution is

$$u.\frac{1}{x} = \int (-2x)\left(\frac{1}{x}\right)dx + C_2 \quad \text{or} \quad \frac{z-y^2}{x} = -2x + C_2, \text{ using (5)}$$

or $(z-y^2)/x+2x=C_2$ or $(z-y^2+2x^2)/x=C_2$ ... (6)

From (3) and (6), the required general solution of (1)

$$\Phi(y/z,\ (z-y^2-2x^2)/x)=0\ ,\ \Phi \text{ being an arbitrary function.}$$

Ex. 18. *Solve* $px(z-2y^2)=(z-qy)\ (z-y^2-2x^3)$. **[I.A.S. 2006]**

Sol. Do like Ex. 17, **Ans.** $\Phi(y/z,\ (z-y^2+x^3)/x)=0$

For another method of solution, refer solved Ex. 8 of Art. 2.8.

Ex. 19. *Solve* $x(z+2a)p+(xz+2yz+2ay)q=z(z+a)$.

Sol. The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{x(z+2a)}=\frac{dy}{xz+2yz+2ay}=\frac{dz}{z(z+a)} \quad \text{... (1)}$$

Each fraction of (1) = $\dfrac{dx+dy}{2(x+y)(z+a)}=\dfrac{dz}{z(z+a)}$ or $\dfrac{d(x+y)}{x+y}=\dfrac{2}{z}dz$

Integrating, $\log(x+y)=2\log z+\log C_1$ or $(x+y)/z^2=C_1$ ...(2)

Taking the first and third ratios of (4), $\dfrac{dx}{x}=\dfrac{z+2a}{z(z+a)}dz$ or $\dfrac{dx}{x}=\left(\dfrac{2}{z}-\dfrac{1}{z+a}\right)dz$

Integrating, $\log x=2\log z-\log(z+a)+\log C_2$ or $x(z+a)/z^2=C_2$ ... (3)

From (2) and (3), solution is $\Phi\ \{(x+y)/z^2,\ x(z+a)/z^2)\}=0$. ϕ being an arbitrary function.

Ex. 20. *Solve* $2x\ (y+z^2)p+y(2y+z^2)q=z^3$ **[Delhi Maths (Hans.) 2007]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{2x(y+z^2)}=\frac{dy}{y(2y+z^2)}=\frac{dz}{z^3} \quad \text{...(1)}$$

Each fraction of (1) = $\dfrac{dx}{2x(y+z^2)}=\dfrac{z\,dy+y\,dz}{2yz(y+z^2)}=\dfrac{d(yz)}{2yz(y+z^2)}$

$\therefore\ (1/x)\ dx+(1/yz)\ d(yz)=0$ so that $x/(yz)=C_1$ ...(2)

From the last two fractions of (1), $\dfrac{dy}{dz}=\dfrac{y(2y+z^2)}{z^3}=\dfrac{2y^2}{z^3}+\dfrac{y}{z}$ or $y^{-2}\dfrac{dy}{dz}-\dfrac{1}{z}y^{-1}=\dfrac{2}{z^3}$ ...(3)

Putting $-y^{-1}=u$ and $(1/y^2)\times(dy/dz)=du/dz$ in (3), we get

$(du/dz)+(1/z)\ u=2/z^3$, which is an ordinary linear equation.

Its I.F. = $e^{\int(1/z)dz}=e^{\log z}=z$ and solution is $uz=\int(2/z^3)\ z\ dz-C_2=-2\ z^{-1}-C_2$

or $-y^{-1}z-2z^{-1}=-C_2$ or $z/y-2/z=C_2$ ... (4)

From (3) and (4), solution is $\Phi\ (x/yz,\ z/y-2/z)=0$, ϕ being arbitrary function.

Ex. 21. $xp+zq+y=0$. **[M.D.U. Rohtak 2004]**

Sol. Given equation is $xp+zq=-y$

Its Lagrange's auxiliary equation are $\frac{dx}{x}=\frac{dy}{z}=\frac{dz}{-y}$ ... (1)

Taking the last two fractions of (2), $2ydy + 2zdz = 0$ so that $y^2 + z^2 = C_1$ ... (2)

Choosing 0, z, $-y$ as multipliers, each fraction of (1)

$$=\frac{zdy-ydz}{z^2+y^2}=\frac{(1/z)dy-(y/z^2)dz}{1+(y/z)^2}=\frac{d(y/z)}{1+(y/z)^2} \quad \text{... (3)}$$

Combining the first faction of (1) with fraction (3), we get

$$\frac{dx}{x}=\frac{d(y/z)}{1+(y/z)^2} \quad \text{or} \quad \frac{dx}{x}-d\left(\tan^{-1}\frac{y}{z}\right)=0$$

Integrating, $\log|x| - \tan^{-1}(y/z) = C_2$, c_2 being an arbitrary constant. ... (4)

From (2) and (4), the required general solution is

$$\log|x|-\tan^{-1}(y/z)=\phi(y^2+z^2),\ \phi \text{ being an arbitrary function.}$$

Ex. 22. *Find the general solution of the differential equation* $x^2(\partial z/\partial x)+y^2(\partial z/\partial y)=(x+y)z.$ **[Delhi B.A./B.Sc. (Prog.) Maths 2007)**

Sol. Let $p=\partial z/\partial x$ and $q=\partial z/\partial y.$ Then, the given equation takes the form

$$x^2p + y^2q = z\,(x + y) \quad \text{... (1)}$$

The Lagrange's auxiliary equations for (1) are

$$(dx)/x^2 = (dy)/y^2 = (dz)/z\,(x + y) \quad \text{... (2)}$$

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0$

Integrating, $-(1/x) + (1/y) = c_1$ or $(x - y)/xy = c_1$ ... (3)

Chossing 1, –1, 0 as multipliers, each fraction of (2) $=\frac{dx-dy}{x^2-y^2}$ ... (4)

Combining the last fraction of (2) with fraction (4), we have

$$\frac{dx-dy}{(x-y)(x+y)}=\frac{dz}{z(x+y)} \quad \text{or} \quad \frac{dx-dy}{x-y}-\frac{dz}{z}=0$$

Integrating, $\log(x - y) - \log z = \sin c^2$ or $(x - y)/z = c^2$ ... (5)

From (5), $x - y = c_2\,z$ ... (6)

using (6), (3) becomes $(c_2z)/xy = a$ or $(xy)/z = c_2/c_1 = c_3$ say ... (7)

From (5) and (7), the required solution is $\phi\big((x,y)/z,(x-y)/z\big)=0.$

EXERCISE 2(D)

Solve the following partial differential equations:

1. $(x^2 + y^2)p + 2xy\,q = z(x + y)$ **Ans.** $(x+y)/z=\phi\left(y/(x^2-y^2)\right)$

2. $\{y\,(x + y) + az\}\,p + \{x\,(x + y) - az\}q = z(x + y)$ **Ans.** $(x+y)/z=\phi(x^2-y^2-2az)$

3. $(y^2 + yz + z^2)p + (z^2 + zx + x^2)q = x^2 + xy + y^2$ **Ans.** $\phi\left(\frac{y-z}{x-y},\frac{x-z}{x-y}\right)=0$

2.13. Miscellaneous Examples on *Pp + Qq = R*

Ex. 1. *Solve* $(x + y - z)(p - q) + a(px - qy + x - y) = 0$.

Sol. Let $u = x + y$ and $v = x - y$. ...(1)

Then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$, using (1) ...(2)

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$, using (1) ...(3)

From (2) and (3), we get $p - q = 2(\partial z / \partial v)$. ...(4)

and $px - qy = x\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} - y\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}$

or $px - qy = (x - y)\frac{\partial z}{\partial u} + (x + y)\frac{\partial z}{\partial v} = v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v}$, using (1) ...(5)

Using (1), (4) and (5), the given equation reduces to

$$2(u - z)\frac{\partial z}{\partial v} + a\left(v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v} + v\right) = 0$$

or $av(\partial z/\partial u) + (2u - 2z + au)(\partial z/\partial v) = -av$, ...(6)

which is Lagrange's linear equation. Its Lagrange's auxiliary equations are

$$\frac{du}{av} = \frac{dv}{2u - 2z + au} = \frac{dz}{-av}. \quad ...(7)$$

Taking the first and third fractions of (7), we have

$du + dz = 0$ so that $u + z = c_1$. ...(8)

Considering the first two fractions of (7) and eliminating z with help of (8), we have

$\frac{du}{av} = \frac{dv}{2u - 2(c_1 - u) + au}$ or $avdv = (4u - 2c_1 + au)du$.

Integrating, $(1/2) \times av^2 = 2u^2 - 2c_1u + (1/2) \times au^2 + c_2/2$

$av^2 = 4u^2 - 4u(u + z) + au^2 + c_2$, or $av^2 + 4uz - au^2 = c_2$ (using (8) ....(9)

From (8) and (9), the required general solution is given by

$\phi(u + z, av^2 + 4uz - au^2) = 0$, where ϕ is an arbitrary function and u and v are given by (1).

Ex. 2 (*a*). *Find the surface whose tangent planes cut off an intercept of constant length k from the axis of z.*

(*b*) *Formulate partial differential equation for surfaces whose tangent planes form a tetrahedrom of constant volume with the coordinate planes.* **[I.A.S. 2005]**

Sol. (*a*) We know that the equation of the tangent plane at point (x, y, z) to a surface is given by $p(X - x) + q(Y - y) = Z - z$, ...(1)

where X, Y, Z denote current coodrinates of any point on the plane (1). Since (1) cuts an intercept k on the z–axis, it follows that (1) must pass through the point $(0, 0, k)$. Hence putting $X = 0$, $Y = 0$ and $Z = k$ in (1), we obtain

$$px + qy = z - k, \quad ...(2)$$

which is well known Lagrange's linear equation. For (2), the Lagrange's auxiliary equations are

$$(dx)/x = (dy)/y = (dz)/(z - x). \quad ...(3)$$

Taking the first two fractions of (3), $(1/x)dx - (1/y)dy = 0$. so that $x/y = c_1$. ...(4)

Again, taking the first and third fraction of (3), $[1/(z - k)]dz - (1/x)dx = 0$

Integrating, $\log(z - k) - \log x = \log c_2$ or $(z - k)/x = c_2$. ...(5)

From (4) and (5), the required surface (solution) is given by

$\phi[y/x, (z - k)/x] = 0$, ϕ being an arbitrary function.

(*b*) Left as an exercise.

EXERCISE 2 (E)

Solve the following partial differential equations :

1. $p - qy \log y = z \log y$. **Ans.** $\phi(yz, e^x \log y) = 0$

2. $(p + q)(x + y) = 1$. **Ans.** $\phi(y - x, e^{-2z}y + x) = 0$

3. $x^2p + y^2q = x + y$. **Ans.** $\phi[(1/y) - (1/x), e^{-z}(x - y)] = 0$

4. $(x^2 + 2y^2)p - xyq = xz$. **Ans.** $\phi(x^2y^2 + y^4, yz) = 0$

5. $px - qy = (z - xy)^2$. **Ans.** $\phi[xy, xe^{1/(z-xy)}] = 0$

6. $zp + zq = z^2 + (x - y)^2$. **Ans.** $\log [z^2 + (x - y)^2] - 2x = \phi(x - y)$.

7. $x(y^n - z^n)p + y(z^n - x^n)q = z(x^n - y^n)$. **Ans.** $x^n + y^n + z^n = \phi(xyz)$.

8. $(xz + y^2)p + (yz - 2x^2)q + 2xy + z^2 = 0$. **Ans.** $\phi(yz + x^2, 2xz - y^2) = 0$.

9. $xyp + y(2x - y)q = 2xz$. **Ans.** $\phi(xy - x^2, z/xy) = 0$.

2.14. Integral surfaces passing through a given curve. In the last article we obtained general integral of $Pp + Qq = R$. We shall now present two methods of using such a general solution for getting the integral surface which passes through a given curve.

Method I. Let $Pp + Qq = R$ ...(1)

be the given equation. Let its auxiliary equations give the following two independent solutions

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. ...(2)

Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by

$x = x(t)$, $y = (t)$, $z = z(t)$, ...(3)

where t is a parameter. Then (2) may be expressed as

$u[x(t), y(t), z(t)] = c_1$ and $v[x(t), y(t), z(t)] = c_2$. ...(4)

We eliminate single parameter t from the equations of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

2.15. SOLVED EXAMPLES BASED ON ART. 2.14.

Ex. 1. *Find the integral surface of the linear partial differential equation* $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ *which contains the straight line* $x + y = 0$, $z = 1$. **[Delhi 2008; Pune 2010]**

Sol. Given $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$. ...(1)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{x(y^2+z)} = \dfrac{dy}{-y(x^2+z)} = \dfrac{dz}{(x^2-y^2)z}$. ...(2)

Proceed as in solved Ex. 6, Art. 2.10 and show that

$xyz = c_1$ and $x^2 + y^2 - 2z = c_2$. ...(3)

Taking t as parameter, the given equation of the straight line $x + y = 0$, $z = 1$ can be put in parametric form $x = t$, $y = -t$, $z = 1$. ...(4)

Using (4), (3) may be re–written as $-t^2 = c_1$ and $2t^2 - 2 = c_2$. ... (5)

Eliminating t from the equations of (5), we have

$2(-c_1) - 2 = c_2$ or $2c_1 + c_2 + 2 = 0$. ... (6)

Putting values of c_1 and c_2 from (3) in (6), the desired integral surface is

$$2xyz + x^2 + y^2 - 2z + 2 = 0.$$

Ex. 2. *Find the equation of the integral surface of the differential equation* $2y(z - 3)p + (2x - z)q = y(2x - 3)$, *which pass through the circle* $z = 0$, $x^2 + y^2 = 2x$. **[Meerut 2007]**

Sol. Given equation is $2y(z - 3)p + (2x - z)q = y(2x - 3)$. ...(1)

Given circle is $x^2 + y^2 = 2x$, $z = 0$. ...(2)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{2y(z-3)} = \dfrac{dy}{2x-z} = \dfrac{dz}{y(2x-3)}$. ...(3)

Taking the first and third fractions of (3), $(2x - 3)dx - 2(z - 3)dz = 0$.

Integrating, $x^2 - 3x - z^2 + 6z = c_1$, c_1 being an arbitrary constant. ...(4)

Choosing 1/2, y, -1 as multipliers, each fraction of (3)

$$= \frac{(1/2)dx + ydy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{(1/2)dx + ydy - dz}{0}$$

Hence $(1/2)dx + ydy - dz = 0$ or $dx + 2ydy - 2dz = 0$.

Integrating, $x + y^2 - 2z = c_2$, c2 being an arbitrary constant. ...(5)

Now, the parametric equations of given circle (2) are $x = t$, $y = (2t - t^2)^{1/2}$, $z = 0$. ...(6)

Substituting these values in (4) and (5), we have

$t^2 - 3t = c_1$ and $3t - t^2 = c_2$. ...(7)

Eliminating t from the above equations (7), we have $c_1 + c_2 = 0$. ...(8)

Substituting the values of c_1 and c_2 from (4) and (5) in (8), the desired integral surface is

$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$ or $x^2 + y^2 - z^2 - 2x + 4z = 0$.

Method II. Let $Pp + Qq = R$ ...(1)

be the given equation. Let is Lagrange's auxiliary equations give the following two independent integrals $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. ...(2)

Suppose we wish to obtain the integral surface passing though the curve which is determined by the following two equations

$\phi(x, y, z) = 0$ and $\psi(x, y, z) = 0$. ...(3)

We eliminate x, y, z from four equations of (2) and (3) and obtain a relation between c_1 and c_2. Finally, replace c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and obtain the desired integral surface.

Ex. 3. *Find the integral surface of the partial differential equation* $(x - y)p + (y - x - z)q = z$ through the circle $z = 1$, $x^2 + y^2 = 1$. **(Nagpur 2002)**

Sol. Given $(x - y)p + (y - x - z)q = z$. ...(1)

Lagrange's auxiliary equations for (1) are $\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction on (2) $= (dx + dy + dz)/0$

$\therefore\ dx + dy + dz = 0$ so that $x + y + z = c_1$. ...(3)

Taking the last two fractions of (2) and using (3) we get

$\frac{dy}{y-(c_1-y)} = \frac{dz}{z}$ or $\frac{2dy}{2y-c_1} - \frac{2dz}{z} = 0$.

Integrating it, $\log (2y - c_1) - 2 \log z = \log c_2$ or $(2y - c_1)/z^2 = c_2$

or $(2y - x - y - z)/z^2 = c_2$ or $(y - x - z)/z^2 = c_2$. ....(4)

The given curve is given by $z = 1$ and $x^2 + y^2 = 1$. ...(5)

Putting $z = 1$ in (3) and (4), we get $x + y = c_1 - 1$ and $y - x = c_2 + 1$. ...(6)

But $2(x^2 + y^2) = (x + y)^2 + (y - x)^2$. ...(7)

Using (5) and (6), (7) becomes

$2 = (c_1 - 1)^2 + (c_2 + 1)^2$ or $c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0$. ...(8)

Putting the values of c_1 and c_2 from (3) and (4) in (8), required integral surface is

$$(x + y + z)^2 + (y - x - z)^2/z^4 - 2(x + y + z) + 2(y - x - z)/z^2 = 0$$

or $$z^4(x + y + z)^2 + (y - x - z)^2 - 2z^4(x + y + z) + 2z^2(y - x - z) = 0.$$

Ex. 4. *Find the equation of the integral surface of the differential equation* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ *which passes through the line* $x = 1, y = 0$.

Sol. Given $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. ...(1)

Proceed as in solved Ex. 6, Art. 2.12 and show that

$$(x - y)/(y - z) = c_1 \quad ...(2)$$

and $$xy + yz + zx = c_2. \quad ...(3)$$

The given curve is represented by $x = 1$ and $y = 0$. ...(4)

Using (4) in (2) and (3), we obtain $-1/z = c_1$ and $z = c_2$

so that $(-1/z) \times z = c_1c_2$ or $c_1c_2 + 1 = 0$. ...(5)

Putting the values of c_1 and c_2 from (2) and (3) in (5), the required integral surface is

$[(x - y)/(y - z)]\ (xy + yz + zx) + 1 = 0$ or $(x - y)\ (xy + yz + zx) + y - z = 0$

Ex. 5. *Find the equation of surface satisfying* $4yzp + q + 2y = 0$ *and passing through* $y^2 + z^2 = 1$, $x + z = 2$. **[I.A.S. 1997]**

Sol. Given $4yzp + q = -2y$. ...(1)

Given curve is given by $y^2 + z^2 = 1$, and $x + z = 2$. ...(2)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{4yz} = \dfrac{dy}{1} = \dfrac{dz}{-2y}$. ...(3)

Taking the first and third fractions of (3), $dx + 2zdz = 0$ so that $x + z^2 = c_1$. ...(4)

Taking the last two fractions of (3), $dz + 2ydy = 0$ so that $z + y^2 = c_2$. ...(5)

Adding (4) and (5), $(y^2 + z^2) + (x + z) = c_1 + c_2$

or $1 + 2 = c_1 + c_2$, using (2) ...(6)

Putting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required suface is given by $3 = x + z^2 + z + y^2$ or $y^2 + z^2 + x + z - 3 = 0$.

Ex. 6. *Find the general integral of the partial differential equation* $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ *and also the particular integral which passes through the line* $x = 1$, $y = 0$. **[I.A.S. 2008]**

Sol. Given $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$. ...(1)

Given line is given by $x = 1$ and $y = 0$. ...(2)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{2xy-1} = \dfrac{dy}{z-2x^2} = \dfrac{dz}{2x-2yz}$. ...(3)

Taking z, 1, x as multipliers, each fraction of (3) $= (zdx + dy + x\,dz)/0$

so that $zdx + dy + xdz = 0$ or $d(xz) + dy = 0$

Integrating, $xz + y = c_1$. ...(4)

Again, taking x, y, 1/2 as multipliers, each fraction of (3) $= \{xdx + ydy + (1/2)dz\}/0$

so that $x\,dx + ydy + (1/2) \times dz = 0$ or $2xdx + 2ydy + dz = 0$

Integrating, $x^2 + y^2 + z = c_2$. ...(5)

Since the required curve given by (4) and (5) passes through the line (2), so putting $x = 1$ and $y = 0$ in (4) and (5), we get

$z = c_1$ and $1 + z = c_2$ so that $1 + c_1 = c_2$. ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), the eqution of the required surface is given by

$1 + xz + y = x^2 + y^2 + z$ or $x^2 + y^2 + z - xz - y = 1$.

Ex. 7. *Find the integral surface of* $x^2p + y^2q + z^2 = 0$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ *which passes through the hyperbola* $xy = x + y$, $z = 1$. **[I.A.S. 1994, 2009]**

Sol. Given $x^2p + y^2q + z^2 = 0$ or $x^2p + y^2q = -z^2$. ...(1)

Given curve is given by $xy = x + y$ and $z = 1$. ...(2)

Here Lagrange's auxiliary equations for (1) are $(dx)/x^2 = (dy)/y^2 = (dz)/(-z^2)$. ...(3)

Taking the first and third fractions of (1), $x^{-2}dx + z^{-2}dz = 0.$

Integrating, $-(1/x) - (1/z) = -c_1$ or $1/x + 1/z = c_1.$ ...(4)

Taking the second and third fractions of (1), $y^{-2}dy + z^{-2}dz = 0.$

Integrating, $-(1/y) - (1/z) = -c_2$ or $1/y + 1/z = c_2.$ ...(5)

Adding (4) and (5), $\frac{1}{x}+\frac{1}{y}+\frac{2}{z} = c_1 + c_2$ or $\frac{x+y}{xy}+\frac{2}{z} = c_1 + c_2$

or $(xy)/(xy) + 2 = c_1 + c_2,$ using (2) or $c_1 + c_2 = 3.$ ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we get

$1/x + 1/z + 1/y + 1/z = 3$ or $yz + 2xy + xz = 3xyz.$

Ex. 6. *Find the integral surface of the linear first order partial differential equation* $yp + xq = z - 1$ *which passes through the curve* $z = x^2 + y^2 + z,\ y = 2x$

Sol. Given equation is $yp + xq = z - 1$ ... (1)

and the given curve is given by $z = x^2 + y^2 + 1$ and $y = 2x$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}$ ... (3)

Taking the first two fractions, $2ydy - 2xdx = 0$

Integrating, it, $y^2 - x^2 = C_1$, C_1 being an arbitrary constant ... (4)

Taking the first and the last fractions of (3) and using (4), we get

$\frac{dx}{(x^2 + C_1)^{1/2}} = \frac{dz}{z-1}$ so that $\log(z-1) - \log\{x + (x^2 + C_1)^{1/2}\} = \log C_2$

or $\log(z - 1) - \log(x + y) = \log C_2,$ by (4) or $(z - 1)/(x + y) = C_2$ ... (5)

The parametric form of the given curve (2) is $x = t,\quad y = 2t,\quad z = 5t^2 + 1$ ... (6)

Substituting these values in (4) and (5), we get $3t^2 = C_1$ and $5t/3 = C_2$ ... (7)

Eliminating t from the above equations (7), we get $5\sqrt{C_1}/3\sqrt{3} = C_2$ ... (8)

Substituting the values of C_1 and C_2 from (4) and (5) in (8), the required surface is given by

$$5(y^2 - x^2)^{1/2}/3\sqrt{3} = (z-1)/(x+y).$$

Ex. 7. *Find the integral surface of the partial differential equation* $(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$ *passing through the curve* $xz = a^3,\ y = 0.$

Sol. Given equation is $(x-y)y^2p + (y-x)x^2q = (x^2 + y^2)z$ ... (1)

and the given curve is given by $xz = a^3$ and $y = 0$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$ ... (3)

Each fraction of (3) $= \frac{dx - dy}{(x-y)(y^2 + x^2)} = \frac{dz}{(x^2+y^2)z}$ so that $\frac{d(x-y)}{x-y} - \frac{dz}{z} = 0$

Integrating it, $(x-y)/z = C_1$, C_1 being an arbitrary contant ... (4)

Taking the first two fractions, $3x^2dx + 3y^2dy = 0$

Integrating it, $x^3 + y^3 = C_2$, C^2 being an arbitrary constant. ... (5)

The parameteric form of the given curve (2) is $z = t,\quad x = a^3/t,\quad y = 0$ ... (6)

Substituting these values in (4) and (5), we get

$$a^3/t^2 = C_1 \qquad \text{so that} \qquad t^2 = a^3/C_1 \qquad \ldots(7)$$

and $$(a^3/t)^3 = C_2 \qquad \text{so that} \qquad t^3 = a^9/C_2 \qquad \ldots(8)$$

Squaring both sides of (8), $t^6 = a^{18}/C_2^2$ or $(t^2)^3 = a^{18}/C_2^2$

or $(a^3/C_1)^3 = a^{18}/C_2^2$, since $t^2 = a^3/C_1$, by (7)

or $a^9/C_1^3 = a^{18}/C_2^2$, or $C_2^2 = a^9\ C_1^3$ ... (9)

Substituting the values of C_1 and C_2 from (4) and (5) in (9), the required integral surface of (1) is given by

$(x^3+y^3)^2 = a^9\ (x-y)^3/z^3$ or $z^3(x^3+y^3)^2 = a^9\ (x-y)^3$.

EXERCISE 2(F)

1. *Find particular integrals of the following partial differential equations to represent surfaces passing through the given curves :*

(*i*) $p + q = 1$; $x = 0$, $y^2 = z$. **Ans.** $(y-x)^2 = z - x$.

(*ii*) $xp + yq = z$; $x + y = 1$, $yz = 1$. **Ans.** $yz = (x+y)^2$.

(*iii*) $(y-z)p + (z-x)q = x - y$; $z = 0$, $y = 2x$ **Ans.** $5(x+y+z)^2 = 9(x^2+y^2+z^2)$.

(iv) $x(y-z)p + y(z-x)q = z(x-y)$; $x = y$; $x = y = z$. **Ans.** $(x+y+z)^3 = 27xyz$.

(*v*) $yp - 2xyq = 2xz$; $x = t$, $y = t^2$, $z = t^3$. **Ans.** $(x^2+y^2)^5 = 32y^2z^2$.

(*vi*) $(y-z)\,[2xyp + (x^2-y^2)q] + z(x^2-y^2) = 0$; $x = t^2$, $y = 0$, $z = t^3$.

Ans. $x^3 - 3xy^2 = z^2 - 2yz$.

2. Find the general solution of the equation $2x(y+z^2)p + y(2y+z^2)q = z^2$ and deduce that $yz(z^2 + yz - 2y) = x^2$ is a solution.

3. Find the general solution of $x(z+2a)p + (xz + 2yz + 2ay)q = z(z+a)$.

Find also the integral surfaces which pass through the curves :

(*i*) $y = 0$, $z^2 = 4ax$. (*ii*) $y = 0$, $z^3 + x(z+a)^2 = 0$.

4. Solve $xp + yq = z$. Find a solution representing a surface meeting the parabola $y^2 = 4x$, $z = 1$. **Ans.** General solution $\phi\,(x/2, y/2) = 0$; surface $y^2 = 4xz$.

2.16. SURFACES ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES

Let $$f(x, y, z) = C \qquad \ldots(1)$$

represents a system of surfaces where C is parameter. Suppose we wish to obtain a system of surfaces which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1) which passes through that point are $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

Let the surface $$z = \phi(x, y) \qquad \ldots(2)$$

cuts each surface of (1) at right angles. Then the normal at (x, y, z) to (2) has direction ratios $\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1. Since normals at (x, y, z) to (1) and (2) are at right angles, we have

$$p(\partial f/\partial x) + q(\partial f/\partial y) - (\partial f/\partial z) = 0 \qquad \text{or} \qquad p(\partial f/\partial x) + q(\partial f/\partial y) = \partial f/\partial z \qquad \ldots(3)$$

which is of the form $Pp + Qq = R$.

Conversely, we easily verify that any solution of (3) is orthogonal to every surface of (1).

2.17. SOLVED EXAMPLES BASED ON ART. 2.16.

Ex. 1. *Find the surface which intersects the surfaces of the system* $z(x+y) = c(3z+1)$ *orthogonally and which passes through the circle* $x^2 + y^2 = 1$, $z = 1$. **[I.A.S. 1999]**

Sol. The given system of surfaces is $$f(x, y, z) \equiv \{z(x+y\}/(3z+1) = C. \qquad \ldots(1)$$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{z}{3z+1}, \qquad \frac{\partial f}{\partial y} = \frac{z}{3z+1}, \qquad \frac{\partial f}{\partial z} = (x+y)\frac{(3z+1)-z\times 3}{(3z+1)^2} = \frac{x+y}{(3z+1)^2}.$$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \qquad \text{or} \qquad \frac{z}{3z+1}p + \frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

or $$z(3z+1)p + z(3z+1)q = x + y. \qquad ...(2)$$

Lagrange's auxiliary equations for (2) are $$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y}. \qquad ...(3)$$

Taking the first two fractions of (3), we get $dx - dy = 0$ so that $x - y = C_1$. ...(4)

Choosing $x, y, -z(3z+1)$ as multipliers, each fraction of (3) $= [xdx + ydy - z(3z+1)dz]/0$

$\therefore \quad xdx + ydy - 3z^2dz - zdz = 0$ or $2xdx + 2ydy - 6z^2dz - 2zdz = 0$

Integrating, $x^2 + y^2 - 2z^3 - z^2 = C_2$, C_2 being an arbitrary constant. ...(5)

Hence any surface which is orthogonal to (I) has equation of the form

$$x^2 + y^2 - 2z^3 - z^2 = \phi(x - y), \ \phi \text{ being an arbitrary function} \qquad ...(6)$$

In order to get the desired surface passing through the circle $x^2 + y^2 = 1$, $z = 1$ we must choose $\phi(x - y) = -2$. Thus, the required particular surface is $x^2 + y^2 - 2z^3 - z^2 = -2$.

Ex. 2. *Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by* $x(x^2 + y^2 + z^2) = C_1y^2$. **[I.A.S. 2001]**

Sol. Given family of surfaces is $x(x^2 + y^2 + z^2)/y^2 = C_1$

Let $$f(x,y,z) = x(x^2 + y^2 + z^2)/y^2 = C_1 \qquad ... (1)$$

Then the surfaces orthogonal to the system (1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z} \qquad \text{or} \qquad \frac{dx}{(3x^2 + y^2 + z^2)/y^2} = \frac{dy}{-2x(x^2 + z^2)/y^3} = \frac{dz}{2x/y^2z}$$

or $$\frac{dx}{y(3x^2 + y^2 + z^2)} = \frac{dy}{-2x(x^2 + z^2)} = \frac{dz}{2xyz} \qquad ... (2)$$

Taking x, y, z as multipliers, each fraction of (2)

$$= \frac{xdx + ydy + zdz}{xy(3x^2 + y^2 + z^2) - 2xy(x^2 + z^2) + 2xyz} = \frac{xdx + ydy + zdz}{xy(x^2 + y^2 + z^2)} \qquad ... (3)$$

Combining this fraction (3) with the last fraction of (2), we get

$$\frac{xdx + ydy + zdz}{xy(x^2 + y^2 + z^2)} = \frac{dz}{2xyz} \qquad \text{or} \qquad \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{\mathrm{dz}}{\mathrm{z}}$$

Integrating, $\log(x^2 + y^2 + z^2) = \log z + \log C_2$ or $(x^2 + y^2 + z^2)/z = C_2$ ...(4)

Taking $4x, 2y, 0$ as multipliers, each fraction of (2)

$$= \frac{4xdx + 2ydy}{4xy(3x^2 + y^2 + z^2) - 4xy(x^2 + y^2)} = \frac{4xdx + 2ydy}{4xy(2x^2 + y^2)} \qquad ... (5)$$

Combining this fraction (5) with the last fraction of (2), we get

$$\frac{4xdx + 2ydy}{4xy(2x^2 + y^2)} = \frac{dz}{2xyz} \qquad \text{or} \qquad \frac{4xdx + 2ydy}{2x^2 + y^2} = \frac{2dz}{z}$$

Integrating, $\log(2x^2 + y^2) = 2\log z + \log C_3$ or $(2x^2 + y^2)/y^2 = C_3$ ... (6)

From (4) and (5), the required general equation of the surfaces which are orthogonal to the given family of surfaces (1) is of the form $(x^2+y^2+z^2)/z = \phi\{(2x^2+y^2)/z^2\}$, *i.e.*,

or $x^2+y^2+z^2 = z\,\phi\{(2x^2+y^2)/z^2\}$, where ϕ is an arbitrary function.

Ex. 3. *Find the surface which is orthogonal to the one parameter system* $z = cxy(x^2+y^2)$ *which passes through the hyperbola* $x^2-y^2=a^2,\ z=0$

Sol. The given system of surfaces is $f(x,y,z) = z/(x^3y+xy^3) = C$ ... (1)

$$\frac{\partial f}{\partial x} = -\frac{z(3x^2y+y^3)}{(x^3y+xy^3)^2}, \qquad \frac{\partial f}{\partial y} = -\frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2}, \qquad \frac{\partial f}{\partial z} = \frac{1}{x^3y+xy^3}$$

The required orthogonal surface is solution of $p(\partial f/\partial x) + q(\partial f/\partial y) = \partial f/\partial z$

or $$-\frac{z(3x^2y+y^3)}{(x^3y+xy^3)^2}p - \frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2}q = \frac{1}{x^3y+xy^3}$$

or $$\{(3x^2+y^2)/x\}p + \{(3y^2+x^2)/y\}q = -(x^2+y^2)/z \qquad \text{... (2)}$$

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{(3x^2+y^2)/x} = \frac{dy}{(3y^2+x^2)/y} = \frac{dz}{-(x^2+y^2)/z} \qquad \text{... (3)}$$

Taking the first two fractions of (3), $2xdx - 2ydy = 0$ so that $x^2-y^2 = C_1$

Choosing x, y, $4z$ as multipliers, each fraction of (3) $= (xdx+ydy+4zdz)/0$

$\therefore$ $2xdx+2ydy+8zdz=0$ so that $x^2+y^2+4z^2 = C_2$

Hence any surface which is orthogonal to (1) is of the form

$$x^2+y^2+4z^2 = \Phi(x^2-y^2), \ \Phi \text{ being an arbitrary function.} \qquad \text{... (4)}$$

For the particular surface passing through the hyperbola $x^2-y^2=a^2,\ z=0$ we must take $\Phi(x^2-y^2) = a^4(x^2+y^2)/(x^2-y^2)^2$. Hence, the required surface is given by

$$(x^2+y^2+4z^2)^2\ (x^2-y^2)^2 = a^4(x^2+y^2)$$

Ex. 4. *Find the equation of the system of surfaces which cut orthogonally the cones of the system* $x^2+y^2+z^2 = cxy$.

2.18 (*a*). Geometrical description of the solutions of *Pp* + *Qq* = *R* and of the system of equations *dx*/*P* = *dy*/*Q* = *dz*/*R* and to establish relationship between the two.

[G.N.D.U. Amritsar 1998; Meerut 1997; Kanpur 1996]

Proof. Consider $Pp + Qq = R.$ ....(1)

and $(dx)/P = (dy)/Q = (dz)/R,$ ...(2)

where P, Q and R are functions of x.

Let $z = \phi(x, y)$ ...(3)

represent the solution of (1). Then (3) represents a surface whose normal at any point (x, y, z) has direction ratios $\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1. Also we know that the simultaneous equations (2) represent a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (1), we have

$$Pp + Qq + R(-1) = 0, \qquad \text{...(4)}$$

showing that the normal to surface (3) at any point is perpendicular to the member of family of curves (2) through that point. Hence the member must touch the surface at that point. Since this

holds for each point on (3), we conclude that the curves (2) lie completely on the surface (3) whose differential equation is (1).

2.18 (b). Another geomertical interpretation of Lagrange's equation Pp + Qq = R.

To show that the surfaces represented by Pp + Qq = R are orthogonal to the surfaces represented by Pdx + Qdy + Rdz = 0.

We know that the curves whose equations are solutions of

$$(dx)/P = (dy)/Q = (dz)/R \qquad ...(1)$$

are orthogonal to the system of the surfaces whose equation satisfies

$$Pdx + Qdy + Rdz = 0. \qquad ...(2)$$

Again from Art 2.18 (*a*) the curves of (1) lie completely on the surface represented by

$$Pp + Qq = R. \qquad ...(3)$$

Hence we conclude that surfaces represented by (2) and (3) are orthogonal.

2.19. SOLVED EXAMPLES BASED ON ART 2.18(*a*) AND ART. 2.18 (*b*)

Ex. 1. *Find the family orthogonal to* $\phi\,[z(x + y)^2, x^2 - y^2] = 0$.

Sol. Given $\phi[z(x + y)^2, x^2 - y^2] = 0.$ ...(1)

Let $u = z(x + y)^2$ and $v = x^2 - y^2$ ...(2)

Then (1) becomes $\phi(u, v) = 0.$ ...(3)

Differentiating (3) w.r.t. x and y partially by turn, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad ...(4)$$

and

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0. \qquad ...(5)$$

From (2), $(\partial u/\partial x) = 2z(x + y)$, $(\partial u/\partial y) = 2z(x + y)$, $(\partial u/\partial z) = (x + y)^2$,

$(\partial v/\partial x) = 2x$, $(\partial v/\partial y) = -2y$, $(\partial v/\partial z) = 0$.

Putting these values in (4) and (5), we get

$$(\partial\phi/\partial u)\left[2z(x+y) + p(x+y)^2\right] + (\partial\phi/\partial v)\,(2x+0) = 0 \qquad ...(6)$$

and

$$(\partial\phi/\partial u)\left[2z(x+y) + q(x+y)^2\right] + (\partial\phi/\partial v)\,(-2y+0) = 0 \qquad ...(7)$$

Evaluating the values of $-\dfrac{\partial\phi/\partial u}{\partial\phi/\partial v}$ from (6) and (7) and then equating these, we get

$$-\frac{\partial\phi/\partial u}{\partial\phi/\partial v} = \frac{2x}{2z(x+y) + p(x+y)^2} = \frac{-2y}{2z(x+y) + q(x+y)^2}$$

or $x(x + y)[2z + q(x + y)] = -y(x + y)[2z + p(x + y)]$ or $2xz + qx(x + y) + 2yz + py(x + y) = 0$

or $py(x + y) + qx(x + y) = -2z(x + y)$ or $py + qx = -2z$ ...(8)

which is differential equation of the family of surfaces given by (1). So the differential equation of the family of surfaces orthogonal to (8) is given by [use Art. 2.18 (b)]

$ydx + xdy - 2zdz = 0$ or $d(xy) - 2zdz = 0.$ ...(9)

Integrating (9), $xy - z^2 = C,$

which is the desired family of orthogonal surfaces, C being parameter

Ex. 2. *Find the family of surfaces orthogonal to the family of surfaces given by the differential equation* $(y + z)p + (z + x)q = x + y$.

Sol. Let $P = y + z,$ $Q = z + x$ and $R = x + y.$ ...(1)

Then, the given differential equation can be written as $Pp + Qq = R.$ ...(2)

Now, the differential equation of the family of surfaces orthogonal to the given family is

$Pdx + Qdy + Rdz = 0$ or $(y + z)dx + (z + x)dy + (x + y)dz = 0$

or $(ydx + xdy) + (ydz + zdy) + (zdx + xdz) = 0.$

Integrating, $xy + yz + zx = C,$

which is the required family of surfaces, C being a parameter.

2.20. The linear partial differential equation with n independent variables and its solution. Let $x_1, x_2 ..., x_n$ be the n independent variables and let $p_1 = \partial z/\partial x_1, p_2 = \partial z/\partial x_2, ..., p_n = \partial z/\partial x_n$, where z is the dependent variable. Consider the general linear partial differential equation with n independent variables $P_1 p_1 + P_2 p_2 + .. + P_n p_n = R,$ ...(1)

where $P_1, P_2,.., P_n$ are functions of $x_1, x_2..., x_n$. Let $u_1 = c_1, u_2 = c_2, ..., u_n = c_n$ be any n independent integrals of the auxiliary equations

$$(dx_1)/P_1 = (dx_2)/P_2 = ... = (dx_n)/P_n. \quad ...(2)$$

Then the general solution of (1) is given by $\phi(u_1, u_2, ...u_n) = 0.$ ...(3)

Note that the above procedure is generalization of Lagrange's method.

2.21. SOLVED EXAMPLES BASED ON ART. 2.20

Ex. 1. *Solve* $x_2x_3 p_1 + x_3x_1 p_2 + x_1x_2 p_3 + x_1x_2x_3 = 0.$

Sol. Re–writing the given equation in standard form, we have

$$x_2x_3 p_1 + x_3x_1 p_2 + x_1x_2 p_3 = -x_1x_2x_3. \quad ...(2)$$

The auxiliary equations for (2) are $\dfrac{dx_1}{x_2x_3} = \dfrac{dx_2}{x_3x_1} = \dfrac{dx_3}{x_1x_2} = \dfrac{dz}{-x_1x_2x_3}.$ ...(3)

Taking the first and the fourth fractions of (3), $x_1dx_1 + dz = 0$ so that $x_1^2 + 2z = c_1.$ ...(4)

Taking 1st and 2nd fractions of (3), $x_1dx_1 = x_2dx_2$ so that $x_1^2 - x_2^2 = c_2.$ ...(5)

Finally, 2nd and 3rd fractions of (3) give $x_2dx_2 = x_3dx_3$ so that $x_2^2 - x_3^2 = c_3.$ ...(6)

Hence the required general integral is

$\phi(x_1^2 + 2z, x_1^2 - x_2^2, x_2^2 - x_3^2) = 0,$ ϕ being an arbitrary function.

Ex. 2. *Solve* $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + t\dfrac{\partial z}{\partial t} = az + \dfrac{xz}{t}$

Sol. Here auxiliary equations for the given equation are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{az + xy/t}. \quad ...(1)$$

From the first two fractions of (1), $(1/x)dx - (1/y)dy = 0$ so that $x/y = C_1.$ ...(2)

From the first and third fractions of (1), $(1/x)dx - (1/t)dt = 0$ so that $x/t = C_2.$ ...(3)

Dividing (3) by (2), we have $y/t = C_2/C_1.$ ...(4)

Taking the first and third fractions of (1) and using (4), we get

$$\frac{dx}{x} = \frac{dz}{az + (C_2/C_1)x} \quad \text{or} \quad \frac{dz}{dx} = \frac{az + (C_2/C_1)x}{x}$$

or $$\frac{dz}{dx} - \left(\frac{a}{x}\right)z = \left(\frac{C_2}{C_1}\right), \text{ which is linear.} \quad ...(5)$$

I.F. of (5) $= e^{-\int (a/x)\,dx} = e^{-a\log x} = e^{\log x^{-a}} = x^{-a}$ and so solution of (5) is given by

$$zx^{-a} = C_3 + \frac{C_2}{C_1}\int x^{-a}dx = C_3 + \frac{C_2}{C_1}\,\frac{x^{1-a}}{1-a} = C_3 + \frac{y}{t}\,\frac{x^{1-a}}{1-a}, \text{ using (4)}$$

$$\therefore \quad zx^{-a} - \frac{y}{t}\,\frac{x^{1-a}}{1-a} = C_3, C_3 \text{ being an arbitrary constant.} \quad ...(6)$$

From (2), (3) and (6), the required general solution is

$$\phi\left(\frac{x}{y}, \frac{x}{t}, zx^{-a} - \frac{y}{t}\,\frac{x^{1-a}}{1-a}\right) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z) = xyz$. **[Bhopal 1995, 98; I.A.S. 1999]**

Sol. Here the auxiliary equations for the given equation are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{xyz}. \qquad ...(1)$$

Taking the first two fractions of (1), $(1/x)dx - (1/y)dy = 0$.

Integrating it, $\log x - \log y = \log C_1$ or $x/y = C_1$. ...(2)

Taking the first and third fractions of (1), $(1/x)dx - (1/z)dz = 0$

Integrating it, $\log x - \log z = \log C_2$ or $x/z = C_2$. ...(3)

Choosing yz, zx, xy as multipliers, each fraction of (1) $= \dfrac{yzdx + zxdy + xydz}{xyz + xyz + xyz} = \dfrac{d(xyz)}{3xyz}$. ...(4)

Combining the fourth fraction of (1) with fraction (4), we get

$$\frac{du}{xyz} = \frac{d(xyz)}{3xyz} \quad \text{or} \quad d(xyz) - 3du = 0 \quad \text{so that} \quad xyz - 3u = C_3. \quad ...(5)$$

From (2), (3) and (5), the required general solution is

$\phi(x/y, x/z, xyz - 3u) = 0$, ϕ being an arbitrary function.

Ex. 4. *Solve* $(y + z + w)(\partial w/\partial x) + (z + x + w)(\partial w/\partial y) + (x + y + w)(\partial w/\partial z) = (x + y + z)$.
[Ravishanker 2004 ; I.A.S. 1995; Indore 1998, Kanpur 2004]

Sol. Here the auxiliary eqautions of the given equation are

$$\frac{dx}{y+z+w} = \frac{dy}{z+x+w} = \frac{dz}{x+y+w} = \frac{dw}{x+y+z}. \qquad ...(1)$$

Each fraction of (1) $= \dfrac{dw-dx}{-(w-x)} = \dfrac{dw-dy}{-(w-y)} = \dfrac{dw-dz}{-(w-z)} = \dfrac{dw+dx+dy+dz}{3(w+x+y+z)}$. ...(2)

Taking the first and the fourth fractions of (2), $\dfrac{dw+dx+dy+dz}{3(w+x+y+z)} + \dfrac{dw-dz}{w-x} = 0$.

Integrating, $(1/3) \times \log (w + x + y + z) + \log (w - x) = \log C_1$

or $(w + x + y + z)^{1/3} (w - x) = C_1$. ...(3)

Similarly, $(w + x + y + z)^{1/3} (w - y) = C_2$. ...(4)

and $(w + x + y + z)^{1/3} (x - z) = C_3$. ...(5)

From (3), (4) and (5), the required general solution is

$\phi[(w + x + y + z)^{1/3} (w - x), (w + x + y + z)^{1/3} (w - y), (w + x + y + z)^{1/3} (w - z)] = 0$,

where ϕ is an arbitrary function.

Ex. 5. *Solve* $p_1 + p_2 + p_3 = 4z$. **Ans.** $\phi(ze^{-4x_1}, ze^{-4x_2}, ze^{-4x_3}) = 0$

Ex. 6. *Solve* $x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 + x_1x_2x_3 = 0$.

Sol. Putting the given equation in standard form, we have

$$x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 = -x_1x_2x_3. \qquad ...(1)$$

Here the auxiliary equations for (1) are $\dfrac{dx_1}{x_2x_3} = \dfrac{dx_2}{x_3x_1} = \dfrac{dx_3}{x_1x_2} = \dfrac{dz}{-x_1x_2x_3}$. ...(2)

Taking the first and second fractions of (2), we have

$2x_1dx_1 - 2x_2dx_2 = 0$ so that $x_1^2 - x_2^2 = C_1$. ...(3)

Taking the first and third fractions of (2), we have

$2x_1dx_1 - 2x_3dx_3 = 0$ so that $x_1^2 - x_3^2 = C_2$. ...(4)

Taking the first and fourth fractions of (2), we have

$2x_1dx_1 + 2dz = 0$ so that $x_1^2 + 2z = C_3$. ...(5)

From (3), (4) and (5), the required general solution is

$\phi(x_1^2 - x_2^2, x_1^2 - x_3^2, x_1^2 + 2z) = 0$, ϕ being an arbitrary function.

Ex. 7. *Solve* $p_1 + x_1p_2 + x_1x_2p_3 = x_1x_2x_3\sqrt{z}$. **Ans.** $\phi(x_1^2 - 2x_2, x_2^2 - 2x_3, x_3^2 - 4\sqrt{z}) = 0$

Ex. 8. *Solve* $(x_3 - x_2)p_1 + x_2p_2 - x_3p_3 + x_2^2 - (x_2x_1 + x_2x_3) = 0$.

Sol. Re–writing the given equation in the standard form, we get

$$(x_3 - x_2)p_1 + x_2p_2 - x_3p_3 = x_2x_1 + x_2x_3 - x_2^2. \quad ...(1)$$

Here the auxiliary equations for (1) are $\frac{dx_1}{x_3 - x_2} = \frac{dx_2}{x_2} = \frac{dx_3}{-x_3} = \frac{dz}{x_2x_1 + x_2x_3 - x_2^2}$. ...(2)

Taking the second and the third fractions of (2), we have

$(1/x_2)dx + (1/x_3)dx_2 = 0$ so that $\log x_2 + \log x_3 = \log C_1$ or $x_2x_3 = C_1$....(3)

Each fraction of (2) $= \frac{dx_1 + dx_2 + dx_3}{(x_3 - x_2) + x_2 - x_3} = \frac{dx_1 + dx_2 + dx_3}{0}$.

$\therefore\ dx_1 + dx_2 + dx_3 = 0$ so that $x_1 + x_2 + x_3 = C_2$. ...(4)

Each fraction of (2) $= \frac{x_2dx_1 + x_1dx_2}{x_2(x_3 - x_2) + x_1x_2} = \frac{d(x_1x_2)}{x_1x_2 + x_2x_3 - x_2^2}$. ...(5)

Combining the last fraction of (2) with fraction (5), we have

$\frac{dz}{x_1x_2 + x_2x_3 - x_2^2} = \frac{d(x_1x_2)}{x_1x_2 + x_2x_3 - x_2^2}$ or $dz - d(x_1x_2) = 0$.

Integrating, $z - x_1x_2 = C_3$, C_3 being an arbitrary constant. ...(6)

From (3), (4) and (5), the required general solution is

$\phi(x_2x_3, x_1 + x_2 + x_3, z - x_1x_2) = 0$, ϕ being an arbitrary function.

Ex. 9. *If u is a function of x, y and z which satisfies* $(y - z)(\partial u/\partial x) + (z - x)(\partial u/\partial y) + (x - y)(\partial u/\partial z) = 0$, *show that u contains x, y, z only in combinations of* $x + y + z$ and $x^2 + y^2 + z^2$. **(Nagpur 2002, 05)**

Sol. Here auxiliary equations for given equation are $\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y} = \frac{du}{0}$. ...(1)

Each fraction of (1) $= \frac{dx + dy + dz}{(y - z) + (z - x) + (x - y)} = \frac{xdx + ydy + zdz}{x(y - z) + y(z - x) + z(x - y)} = \frac{du}{0}$

$= \frac{dx + dy + dz}{0} = \frac{xdx + ydy + zdz}{0} = \frac{du}{0}$.

$\therefore\ dx + dy + dz = 0$, $2xdx + 2ydy + 2zdz = 0$ and $du = 0$

Integrating, $x + y + z = C_1$, $x^2 + y^2 + z^2 = C_2$ and $u = C_3$.

Hence the required general solution is

$u = f(x + y + z, x^2 + y^2 + z^2)$, f being an arbitrary function.

Ex. 10. *Prove that if* $x_1^3 + x_2^3 + x_3^3 = 1$ *when z = 0, the solution of the equation* $(s - x_1)p_1 + (s - x_2)p_2 + (s - x_2)p_2 + (s - x_3)p_3 = s - z$ can be given in the form $s^3\{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3\}^4 = (x_1 + x_2 + x_3 - 3z)^3$, where $s = x_1 + x_2 + x_3 + z$ and $p_i = \partial z / \partial x_i$, $i = 1, 2, 3$. **[I.A.S. 2000]**

Sol. Given $(s - x_1)p_1 + (s - x_2)p_2 + (s - x_3)p_3 = s - z$ ... (1)

where $s = x_1 + x_2 + x_3 + z$ ... (2)

The auxiliary equations for (2) are $\dfrac{dx_1}{s-x_1}=\dfrac{dx_2}{s-x_2}=\dfrac{dx_3}{s-x_3}=\dfrac{dz}{s-z}$

or $$\frac{dx_1}{x_2+x_3+z}=\frac{dx_2}{x_3+x_1+z}=\frac{dx_3}{x_1+x_2+z}=\frac{dz}{x_1+x_2+x_3}, \text{ using (2)} \quad \ldots (3)$$

Each fraction of (3) = $$\frac{dx_1+dx_2+dx_3-3dz}{2(x_1+x_2+x_3)+3z-3(x_1+x_2+x_3)}=\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} \quad \ldots (4)$$

Again, each fraction of (3) = $$\frac{dx_1+dx_2+dx_3+dz}{3(x_1+x_2+x_3+z)}=\frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \quad \ldots (5)$$

Then, (4) and (5) give $$\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)}=\frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \quad \ldots (5)$$

or $$\frac{d(x_1+x_2+x_3+z)}{x_1+x_2+x_3+z}+3\frac{d(x_1+x_2+x_3-3z)}{x_1+x_2+x_3-3z}=0$$

Integrating, $\log(x_1+x_2+x_3+z)+3\log(x_1+x_2+x_3-3z)=\log a$

or $(x_1+x_2+x_3+z)\,(x_1+x_2+x_3-3z)^3=a,$ where a is an arbitrary constant. ... (6)

Given that $x_1^3+x_2^3+x_3^3=1$ when $z=0$ ... (7)

Hence (6) gives $a=(x_1+x_2+x_3)^4$. Then (6) reduces to

$$(x_1+x_2+x_3+z)\,(x_1+x_2+x_3-3z)^3=(x_1+x_2+x_3)^4 \quad \ldots (8)$$

Now, each fraction of (3) = $$\frac{dx_1-dz}{-(x_1-z)}=\frac{3(x_1-z)^2\,d(x_1-z)}{-3(x_1-z)^3}=\frac{d(x_1-z)^3}{3(x_1-z)^3} \quad \ldots (9)$$

By symmetry, each fraction of (3) is also = $$\frac{d(x_2-z)^3}{-3(x_2-z)^3}=\frac{d(x_3-z)^3}{-3(x_3-z)^3} \quad \ldots(10)$$

Using (9) and (10), we find that each fraction of (3)

$$=\frac{d(x_1-z)^3}{-3(x_1-z)^3}=\frac{d(x_2-z)^3}{-3(x_2-z)^3}=\frac{d(x_3-z)^3}{-3(x_3-z)^3}=\frac{d[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}{-3[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]} \quad \ldots (11)$$

Then, from (4) and (11), we have

$$\frac{3d(x_1+x_2+x_3-3z)}{(x_1+x_2+x_3-3z)}=\frac{d[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}{[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}$$

Integrating it, $3\log(x_1+x_2+x_3-3z)+\log b=\log\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}$

or $(x_1-z)^3+(x_2-z)^3+(x_3-z)^3=b\,(x_1+x_2+x_3-3z)^3$ where b is an arbitrary constant. ...(12)

Putting $z=0$, (12) gives $x_1^3+x_2^3+x_3^3=b\,(x_1+x_2+x_3)^3$

or $1=b\,(x_1+x_2+x_3)^3$, using (7) so that $b=1/(x_1+x_2+x_3)^3$

$\therefore$ (12) $\Rightarrow$ $(x_1-z)^3+(x_2-z)^3+(x_3-z)^3=(x_1+x_2+x_3-3z)^3/(x_1+x_2+x_3)^3$ ... (13)

Raising both sides of (8) to power 3, we have

$$(x_1+x_2+x_3+z)^3\,(x_1+x_2+x_3-3z)^9=(x_1+x_2+x_3)^{12} \quad \ldots (14)$$

Raising both sides of (13) to power 4, we have

$$\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4=(x_1+x_2+x_3-3z)^{12}/(x_1+x_2+x_3)^{12} \quad \ldots (15)$$

Multiplying the corresponding sides of (14) and (15), we have

$$(x_1+x_2+x_3+z)^3\,\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4=(x_1+x_2+x_3-3z)^3$$

or $s^3\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4=(x_1+x_2+x_3-3z)^3$, using (2)

EXERCISE 2 (H)

Ex. 1. *Solve* $p_2 + p_3 = 1 + p_1$. **Ans.** $\phi(x_1 + x_2, x_1 + x_3, x_1 + z) = 0$

Ex. 2. *Solve* $zx_2x_3p_1 + zx_3x_1p_2 + zx_1x_2p_3 = x_1x_2x_3$. **Ans.** $\phi(x_1^2 - x_2^2, x_1^2 - x_2^2, x_1^2 - z^2) = 0$]

Ex. 3. *Solve* $x_1p_1 + 2x_2p_2 + 3x_3p_3 + 4x_4p_4 = 0$. **Ans.** $\phi(x_1^2/x_2, x_1^3/x_3, x_1^4/x_4, z) = 0$

Ex. 4. *Solve* $p_1 + p_2 + p_3\ \{1 - (z - x_1 - x_2 - x_3)^{1/2}\} = 3$.

Ans. $\phi[z - 3x_1, z - 3x_2, z + 6(z - x_1 - x_2 - x_3)^{1/2}] = 0$

Ex. 5. $p_1 + p_2 + p_3\ \{1 + (z + x_1 + x_2 + x_3)^{1/2}\} + 3 = 0$.

Ans. $\phi[z + 3x_1, z + 3x_2, z + 6(z + x_1 + x_2 + x_3)^{1/2}] = 0$

Ex. 6. $x_1p_1 + x_2p_2 + x_3p_3 = az + (x_1x_2)/x_3$. **[Delhi Maths (H) 1998]**

[**Hint.** This is same as Ex. 2 of Art. 2.21. Here $x = x_1$, $y = x_2$, $t = x_3$, $\partial z / \partial x = \partial z / \partial x = p_1$, $\partial z / \partial y = \partial z / \partial x_2 = p_2$, $\partial z / \partial t = \partial z / \partial x_3 = p_3$]

OBJECTIVE PROBLEMS ON CHAPTER 2

Select correct answer by writing (a), (b), (c) or (d).

1. The equation $Pp + Qq = R$ is known as (*a*) Charpit's equation

(*b*) Lagrange' equation (*c*) Bernoulli's equation (*d*) Clairaut's equation.

[Agra 2005, 06, 08]

2. The Lagrange's auxiliary equations for the partial differential equation $Pp + Qq = R$ are (*a*) $(dx)/P = (dy)/Q = (dz)/R$ (*b*) $(dx)/P = (dy)/Q$ (*c*) $(dx)/P = (dz)/R$. (*d*) none of these. **[Garhwal 2005]**

3. The general solution of $(y - z)\,p + (z - x)\,q = x - y$ is

(*a*) $\phi(x + y + z, x^2 + y^2 + z^2) = 0$, (*b*) $\phi(xyz, x + y + z) = 0$

(*c*) $\phi(xyz, x^2 + y^2 + z^2) = 0$ (*d*) $\phi(x^2 - y^2 - z^2, x - y - z) = 0$ **[M.S. Univ. T.N. 2007]**

[**Hint :** Refer Ex. 15, Art 2.10]

4. Subsidiary equations for equation $(y^2z/x) + zxy = y^2$ are

(*a*) $(dx)/y^2z = (dy)/(zx) = (dz)/y^2$ (*b*) $(dx)/x^2 = (dy)/y^2 = (dz)/zx$

(*c*) $(dx)/x^2 = (dy)/y^2 = (dz)/zx$ (*d*) $(dx)/(1/x^2) = (dy)/(1/y^2) = (dz)/(1/zx)$

[Kanpur 2004]

5. The general solution of the linear partial differential equation $Pp + Qq = R$ is

(*a*) $\phi(u, v) = 1$ (*b*) $\phi(u, v) = -1$ (*c*) $\phi(u, v) = 0$ (d) None of these **[Agra 2007]**

Answers. **1.** (*b*) **2.** (*a*) **3.** (*a*) **4.** (*d*) **5.** (c)

MISCELLANEOUS EXAMPLES ON CHAPTER 2

Ex. 1. *Tranform the equation* $yz_x - xz_y = 0$ *into one in polar coordinates and thereby show that the solution of the given equation represents surfaces of revolution.* **(I.A.S. 2007)**

Sol. Let $x = r\cos\theta$ and $y = r\sin\theta$ $\Rightarrow$ $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$ ... (1)

$\Rightarrow 2r\ (\partial r / \partial x) = 2x,\ 2r\ (\partial r / \partial y) = 2y$ $\Rightarrow$ $\partial r / \partial x = \cos\theta,\ \partial r / \partial y = \sin\theta$ ... (2)

$$\frac{\partial\theta}{\partial x} = \frac{1}{1 + y^2/x^2}\left(-\frac{y}{x^2}\right) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}, \quad \frac{\partial\theta}{\partial y} = \frac{1}{1 + y^2/x^3}\left(\frac{1}{x}\right) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \qquad \text{... (3)}$$

Now, $$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}$$, using (2) and (3) ...(4)

and $$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta \frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}$$ using (2) and (3) ...(5)

Using (1), (4) and (5), the given equation $yz_x - xz_y = 0$ reduce to

$$r\sin\theta\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right) - r\cos\theta\left(\sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right) = 0 \quad \text{or} \quad \frac{\partial z}{\partial \theta} = 0 \quad \text{... (6)}$$

Integrating (6) w.r.t. 'θ', $z = f(r)$, where f is an arbitrary function .... (7)

Clearly (7) represents surfaces of revolution, as required.

Ex.2. Solve $(y + z)\, p - (x + z)\, q = x - y$ **(Agra 2010)**

Hint. Do like Ex. 15, page 2.14. **Ans.** $\Phi(x + y + z,\ x^2 + y^2 - z^2) = 0$

Ex. 3. The integral surface satisfying equation $y(\partial z / \partial x) - x(\partial z / \partial y) = x^2 + y^2$ and passing through the curve $x = 1 - t$, $y = 1 + t$, $z = 1 + t^2$ is

(*a*) $z = xy + (x^2 - y^2)/2$ (*b*) $z = xy + (x^2 - y^2)^2/8$

(*c*) $z = xy + (x^2 - y^2)^2/4$ (*d*) $z = xy + (x^2 - y^2)^2/16$ **(GATE 2009]**

Ex. 4. Find the partial differential equation whose surfaces are orthogonal to the surface $z(x + y) = 3z + 1$ **[Pune 2010]** **Ans.** $z(p + q) = x + y - 3$

Ex. 5. if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are integral curves of $(dx)/P = (dy)/Q = (dz)/R$, then show that $F(u, v) = 0$ is general solution of $Pp + Qq = R$, where F is an arbitrary function.

[Pune 2010]

3

Non–linear partial differential equations of order one

3.1. Explanation of terms : complete integral (or complete solution), particular integral, singular integral (or singular solution), and general integral (or general solution) as applied to solutions of first order partial differential equations

[I.A.S. 1995; Meerut 1997; Delhi Maths Hons. 1995]

A *solution* or *integral* of a differential equation is a relation between the variables, by means of which and the derivatives obtained there from the equation is satisfied. Let us now discuss various classes of integrals of a partial differential equation of order one.

Complete Integral (C. I.). or complete solution (C.S.) **[Sagar 1995]**

Let us consider a relation $\phi(x, y, z, a, b) = 0$ ...(1)

in which x, y, z are variables such that z is dependent on x and y. Differentiating (1) partially w.r.t x and y respectively, we obtain

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q = 0, \qquad ...(2)$$

Since there are two arbitrary constants (namely a and b) connected by the above three equations, these can be eliminated and there will appear a relation of the form

$$f(x, y, z, p, q) = 0, \qquad ...(3)$$

which is a partial differential equation of order one.

Suppose, now, that (1) has been derived from (3), by using some method; then the integral (1), which has as many arbitrary constants as there are independent variables, is called the *complete integral* of (3).

Particular Integral : A particular integral of (3) is obtained by giving particular values to a and b in (1) which is the complete integral of (3).

Singular Integral (S.I.) or singular solution (S.S.) **[Delhi 2009; Sagar 1995]**

We know that the locus of all the points whose co–ordinates along with the values of p and q satisfy (3), represent the doubly infinite system of surfaces given by (1). The system is doubly infinite, since there are two constants a and b and each of these can take an infinite number of values. Since the envelope of all the surfaces given by (1) is touched at each of its points by some one of these surfaces, the coordinates of any point on the envelope along with the values of p and q belonging to the envelope at that point must also satisfy (3). Hence we conclude that the equation of the envelope is a solution of (3). The envelope of the surfaces given by (2) is obtained by eliminating a and b between the equations

$$\phi(x, y, z, a, b) = 0, \quad \partial \phi / \partial a = 0 \quad \text{and} \quad \partial \phi / \partial b = 0. \qquad ...(4)$$

The relation between x, y, and z so obtained is called the *singular integral*. In general, it is distinct from the complete integral. However, in exceptional cases it may be contained in the

complete integral, that is, singular integral may be obtained by giving particular values to the constants in the complete integral. Since other relations may appear in the process of getting the singular integral, it is necessary to test that the equation of singular integral satisfies the given differential equation.

General Integral (G.I.) or General Solution (G.S.).

Assume that in (1), one of the constants is a function of the other, say $b = F(a)$, then (1) becomes $$\phi(x, y, z, a, F(a)) = 0. \quad ...(5)$$

Now (5) represents one of the families of surfaces given by the system (1). As before, the equation of the envelope of the family of surfaces given by (5) must also satisfy (3). Again the equation so obtained will be distinct from that of the envelope of the surfaces, and it is not a particular integral. It is known as the *general integral* and is obtained by eliminating a between

$$\phi(x, y, z, a\ F(a)) = 0 \quad \text{and} \quad \partial\phi/\partial a = 0. \quad ...(6)$$

Since other relations may appear in the process of getting the singular integral, it is necessary to test that the equation of general integral satisfies the given differential equation.

Important Note. While solving a non–linear equation, we must not only obtain the complete integral but should also find the singular and general integrals. In absence of details of singular and general integrals, merely the complete solution is considered to be incomplete solution of the given partial differential equation. However, for reason of space, we have found complete integral only in some problems. The students are advised to find singular and general integrals also for such problems. Note that there is always a simple routine method for the same.

Also, if you are asked to find complete integral of a given equation, then you need not give singular and general integrals. Again, if examiner wants singular integral/general integral, then you must find them.

3.2. Geometrical interpretation of three types of integrals of f(x, y, z, p, q) = 0.

(*i*) **Complete integral.**

A complete integral, being a relation between x, y and z represents equation of a surface. Since it involves two arbitrary parameters, it belongs to a double infinite system of surfaces or to a single infinite system of family of surfaces.

(*ii*) **General integral.**

Let a complete solution of $f(x, y, z, p, q) = 0$ be

$$\phi(x, y, z, a, b) = 0. \quad ...(1)$$

A general integral is obtained by eliminating 'a' between (1) and the equations

$$b = \psi(a) \quad ...(2)$$

$$(\partial\phi/\partial a) + (\partial\phi/\partial b)\ \psi'(a) = 0. \quad ...(3)$$

where ψ is an arbitrary function.

The operation of elimination is equivalent to selecting from the system of families of surfaces a representative family and finding the envelope. Equations (1), (2) and (3) together represent a curve drawn on the surface of the family whose parameter is 'a' whereas the equation obtained by eliminating 'a' between them is the envelope of the family. It follows that the envelope touches the surface represented by (1) and (2) along the curve represented by (1), (2) and (3). This curve is known as *characteristic of the envelope* and the general integral thus represents the envelope of a family of surfaces considered as composed of its characteristics.

(*iii*) **Singular Integral.**

The singular integral is obtained by eliminating 'a' and 'b' between equation (1)

$$\partial\phi/\partial a = 0 \quad ...(4)$$

and $$\partial\phi/\partial b = 0. \quad ...(5)$$

The operation of elimination is equivalent to finding the envelope of all the surfaces included in the complete integral. (1), (4) and (5) give the point of contact of the particular surfaces represented by (1) with the general envelope. It follows that the singular integral represents the general envelope of all surfaces included in the complete integral.

3.3. Method of getting singular integral directly from the partial differential equation of first order.

Let the given partial differential equation be $f(x, y, z, p, q) = 0,$ ...(1)

whose complete integral is of the form $\phi(x, y, z, a, b) = 0,$ ...(2)

where 'a' and 'b' are arbitrary constants.

The singular integral of (1) is obtained by eliminating 'a' and 'b' between equation (2)

$\partial\phi/\partial a = 0$ ...(3) and $\partial\phi/\partial b = 0.$ ...(4)

The values of z, p, q derived from (2) when substituted in (1) will reduce it into an identity and the substitution of the values of p and q (but not of z) will in general render (1) equivalent to the integral equation. By using this substitution p and q are replaced by functions of x, y, z, a and b in (1). It follows that the singular integral is given by (1) and the equations obtained on differentiating (1) partially w.r.t. 'a' and 'b', namely the equations

$$\frac{\partial f}{\partial p}\frac{\partial p}{\partial a}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial a} = 0 \quad ...(5) \qquad \frac{\partial f}{\partial p}\frac{\partial p}{\partial b}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial b} = 0. \quad ...(6)$$

If $\frac{\partial f}{\partial p} \neq 0$ and $\frac{\partial f}{\partial q} \neq 0$, (5) and (6) hold if $\frac{\partial p}{\partial a}\frac{\partial q}{\partial b}-\frac{\partial p}{\partial b}\frac{\partial q}{\partial a} = 0,$

showing that there exists a functional relation between p and q which does not contain a and b explicitly. Let this functional relation be

$$\psi(p, q) = 0. \quad ...(7)$$

If both the constants a and b occur in p and q (which does nolt always happen), then (7) shows that one of them is a function of the other and the equations using them give general integral which is not now required.

Equations (5) and (6) are also true if

$\partial f/\partial p = 0$ ...(8) and $\partial f/\partial q = 0.$ ... (9)

Elimination of p and q from (1), (7) and (8) will yield a relation between x, y, z free from 'a' and 'b'. If this relation satisfies the given differential equation (1), it must be the singular integral.

3.4. COMPATIBLE SYSTEM OF FIRST–ORDER EQUATIONS

[Delhi Maths (H) 2007; Pune 2010]

Consider first order partial differential equations

$$f(x, y, z, p, q) = 0 \quad ...(1)$$

and $$g(x, y, z, p, q) = 0. \quad ...(2)$$

Equations (1) and (2) are known as compatible when every solution of one is also a solution of the other.

To find condition for (1) and (2) to be compatible. **[Delhi 2008; Pune 2011]**

Let J = Jacobian of f and $g \equiv \partial(f, g)/\partial(p, q) \neq 0.$ ...(3)

Then (1) and (2) can be solved to obtain the explicit expressions for p and q given by

$p = \phi(x, y, z)$ and $q = \psi(x, y, z).$ ...(4)

The condition that the pair of equations (1) and (2) should be compatible reduces then to the condition that the system of equations (4) should be completely integrable, *i.e.*, that the equation

$dz = pdx + qdy$ or $\phi\, dx + \psi dy - dz = 0$, using (4) ...(5)

should be integrable. (5) is integrable if*

$$\phi\left(\frac{\partial\psi}{\partial z}-0\right)+\psi\left(0-\frac{\partial\phi}{\partial z}\right)+(-1)\left(\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}\right)=0$$

which is equivalent to
$$\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}=\frac{\partial\phi}{\partial y}+\psi\frac{\partial\phi}{\partial z}. \qquad ...(6)$$

Substituting from equations (4) in (1) and differentiating *w.r.t.* 'x' and 'z' respectively, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial\psi}{\partial x}=0 \qquad ...(7)$$

and
$$\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial z}+\frac{\partial f}{\partial q}\frac{\partial\psi}{\partial z}=0. \qquad ...(8)$$

From (7) and (8),
$$\frac{\partial f}{\partial x}+\phi\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\left(\frac{\partial\phi}{\partial x}+\phi\frac{\partial\phi}{\partial z}\right)+\frac{\partial f}{\partial q}\left(\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}\right)=0. \qquad ...(9)$$

Similarly (2) yields
$$\frac{\partial g}{\partial x}+\phi\frac{\partial g}{\partial z}+\frac{\partial g}{\partial p}\left(\frac{\partial\phi}{\partial x}+\phi\frac{\partial\phi}{\partial z}\right)+\frac{\partial g}{\partial q}\left(\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}\right)=0. \qquad ...(10)$$

Solving (9) and (10),
$$\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}=\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+\phi\frac{\partial(f,g)}{\partial(z,p)}\right\}. \qquad ...(11)$$

Again, substituting from equations (4) in (1) and differentiating w.r.t. 'y' and 'z' and proceeding as before, we obtain
$$\frac{\partial\phi}{\partial y}+\psi\frac{\partial\phi}{\partial z}=-\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)}+\psi\frac{\partial(f,g)}{\partial(z,q)}\right\} \qquad ...(12)$$

Substituting from equations (11) and (12) in (1) and replacing ϕ, ψ by p, q respectively, we obtain

$$\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}\right\}=-\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}\right\} \quad \text{or} \quad [f,\,g]=0, \qquad ...(13)$$

where
$$[f,\,g]\equiv\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)} \qquad ...(14)$$

3.5. A PARTICULAR CASE OF ART. 3.4.

To show that first order partial differential equations $p = P(x, y)$ and $q = Q(x, y)$ are compatible if and only if $\partial P/\partial y = \partial Q/\partial x$. **[Delhi Maths (H) 2009; Pune 2010]**

Proof. Given $\partial z/\partial x = p = P(x, y)$ and $\partial z/\partial z = q = Q(x, y)$ ... (1)

Since $dz = (\partial z/\partial x)dx + (\partial z/\partial y)dy = pdx + qdy,$ ... (2)

it follows that the given partial differential equations (1) are compatible if and only if the single differential equation

$$dz = Pdx + Qdy \qquad ... (3)$$

is integrable.

Since P and Q are functions of two variables x and y, hence $Pdx + Qdy$ is an exact differential if and only if $\partial P/\partial y = \partial Q/\partial x$. Therefore (3) is integrable if and only if $\partial P/\partial y = \partial Q/\partial x$

Remark 1. If $\partial P/\partial y = \partial Q/\partial x,$ then the system of two given partial differential equations (1) is compatible and hence these will possess a common solution.

---

*$Pdx + Qdy + Rdz = 0$ is integrable if $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0.$

Remark 2. If $\partial P/\partial y \neq \partial Q/\partial x$, then the system of two given partial differential equations (1) is not compatible and hence these equations possess no solution.

3.6. SOLVED EXAMPLES BASED ON ART. 3.4 AND ART. 3.5.

Ex. 1. (*a*) *Show that the differential equations* $\partial z/\partial x = 5x - 7y$ *and* $\partial z/\partial y = 6x + 8y$ *are not compatible.*

(*b*) $\partial z/\partial x = 5x - 7y$, $\partial z/\partial y = 6x + 8y$ *possess* (*i*) *common solution* (*ii*) *No common solution* (*iii*) *No solution* (*iv*) *None of these. Point out correct choice.* **[Agra 2005, 06]**

(*c*) *Show that the differential equations* $p = x^2 - ay$, $q = y^2 - ax$ *are compactible and find their common solution.*

(*d*) *Show that the differential equations* $\partial z/\partial x = (x+y)^2$, $\partial z/\partial y = x^2 + 2xy - y^2$ *are compatible and solve them*

(*e*) *Show that* $p = x - y/(x^2 + y^2)$, $q = y + x/(x^2 + y^2)$ *are compatible and find their solution.*

(*f*) Show that $p = 1 + e^{x/y}$, $q = e^{x/y}(1 - x/y)$ *are compatible and find their solution.*

Sol. (a) Given $dz/dx = p = 5x - 7y$ and $dz/dy = q = 6x + 8y$ ...(1)

Comparing (1) with $p = P(x, y)$ and $q = Q(x, y)$ ... (2)

here $p = 5x - 7y$ and $Q = 6x + 8y$ ... (3)

We know that $p = P(x, y)$ and $q = Q(x, y)$ are compatible if $\partial P/\partial y = \partial Q/\partial x$. Hence the system (1) is compatible if $\partial P/\partial y = \partial Q/\partial x$.

From (3), $\partial P/\partial y = -7$ and $\partial Q/\partial x = 6$ and so $\partial P/\partial y \neq \partial Q/\partial x$

Therefore, the given system (1) is not compatible.

(*b*) **Ans.** (*iii*) As in part (a), the given system is not compatible. Hence the given equations have no solution (refer Art. 3.5).

(*c*) We known that the system of equations $p = P(x, y)$, $q = Q(x, y)$ ... (1)

is compatible if and only if $\partial P/\partial y = \partial Q/\partial x$.

Comparing $p = x^2 - ay$, and $q = y^2 - ax$ ... (2)

with (1), here $P = x^2 - ay$, and $Q = y^2 - ax$ ... (3)

From (3), $\partial P/\partial y = -a = \partial Q/dx$ and so equations (2) are compatible

To find the common solution of (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (x^2 - ay)\,dx + (y^2 - ax)\,dy = x^2dx + y^2dy - a\,d(xy)$$

Integrating, $z = (x^3 + y^3)/3 - axy + c$, c being an arbitrary contant ... (4)

(4) is the required common solution of the given equation (2).

(*d*) We know that the system of equations $p = P(x, y)$, $q = Q(x, y)$ ...(1)

is compatible if and only if $\partial P/\partial y = \partial Q/\partial x$.

Comparing $\partial z/\partial x = p = (x+y)^2$, and $\partial z/\partial y = q = x^2 + 2xy - y^2$ ... (2)

with (1), here $P = (x + y)^2 = x^2 + 2xy + y^2$ and $Q = x^2 + 2xy - y^2$ ... (3)

From (3), $\partial P/\partial y = 2x + 2y = \partial Q/\partial y$ and hence equations (2) are compatible.

The find the common solution of (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (x^2 + 2xy + y^2)dx + (x^2 + 2xy - y^2)\,dy \quad \text{... (4)}$$

Integrating (4) and noting that R.H.S. of (4) must an exact differential, we have, by method of solving an exact equation

$$z = \int_{\text{(Treating y as a constant)}} (x^2 + 2xy + y^2)dx + \int_{\text{(Integrating terms free from } x)} (x^2 + 2xy - y^2) \quad + c$$

or $z = x^3/3 + x^2y + y^2x - y^3/3 + c$, c being an arbitrary constant

(*e*) We know that the system of equation $p = P(x, y), \quad q = Q(x, y)$ ... (1)

is compatible if and only if $\partial P/\partial y = \partial Q/\partial x$.

Comparing $p = x - y/(x^2 + y^2)$, and $q = y + x/(x^2 + y^2)$ ... (2)

with (1), here $P = x - y/(x^2 + y^2)$ and $Q = y + x/(x^2 + y^2)$ ... (3)

From (3), $$\frac{\partial P}{\partial y} = 0 - \frac{1\cdot(x^2+y^2) - 2y\cdot y}{(x^2-y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \text{... (4)}$$

and $$\frac{\partial Q}{\partial x} = 0 + \frac{1\cdot(x^2+y^2) - 2x\cdot x}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2-y^2)^2} \quad \text{... (5)}$$

(4) and (5) $\Rightarrow \quad \partial P/\partial y = \partial Q/\partial x \quad \Rightarrow$ The system (2) is compatible.

To find the solution of the system (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = \{x - y/(x^2 + y^2)\}dx + \{y + x/(x^2 + y^2)\}dy \quad \text{... (6)}$$

Integrating (6) and noting that R.H.S. of (6) must be an exact differential, we obtain

$$z = \int_{\text{(Treating } y \text{ as a constant)}} \left\{x - y/(x^2 + y^2)\right\} dx + \int_{\text{(Integrating terms free from } x)} \left\{y + x/(x^2 + y^2)\right\} dy \; + c$$

or $z = x^2/2 - y \times (1/y) \times \tan^{-1}(x/y) + y^2/2 + c = (x^2 + y^2)/2 - \tan^{-1}(x/y) + c$,

which is the required solution, c being an arbitrary constant.

(*f*) We know that the system of equations $p = P(x, y), \quad$ and $\quad q = Q(x, y)$ ... (1)

is compatible if and only if $\partial P/\partial y = \partial Q/\partial x$.

Comparing $p = 1 + e^{x/y}$ and $q = e^{x/y}(1 - x/y)$ ... (2)

with (1), here $P = 1 + e^{x/y}$ and $Q = e^{x/y}(1 - x/y)$ ... (3)

$$(3) \;\Rightarrow\; \partial P/\partial y = 0 + e^{x/y}(-x/y^2) = -(x/y^2)\, e^{x/y} \quad \text{... (4)}$$

and $$\partial Q/\partial x = e^{x/y} \times (1/y) \times (1 - x/y) + e^{x/y} \times (-1/y) = -(x/y^2)e^{x/y} \quad \text{... (5)}$$

(4) and (5) $\Rightarrow \quad \partial P/\partial y = \partial Q/\partial x \quad \Rightarrow$ The system (2) is compatible.

To find the solution of (2). Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy \quad \text{... (6)}$$

Integrating (6) and noting that R.H.S. of (6) must be an exact differential, we obtain

$$z = \int_{\text{(Treating } y \text{ as constant)}} (1 + e^{x/y})dx \quad + \int_{\text{(Integrating terms free from } x)} e^{x/y}(1 - x/y)\, dy \quad + c$$

or $z = x + y\, e^{x/y} + c$, c being an arbitrary constant.

Ex. 2. *Show that the equations* $xp = yq$ *and* $z(xp + yq) = 2xy$ *are compatible and solve them.*
[Delhi Maths (Hons) 2005, 07, 11]

Sol. Let $f(x, y, z, p, q) = xp - yq = 0$ ...(1)

and $g(x, y, z, p, q) = z(xp + yq) - 2xy = 0$ ...(2)

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & xz \end{vmatrix} = 2xy,$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ \partial g/\partial z & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix} = -x^2p - xyq,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ \partial g/\partial y & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -2xy$$

and $$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ \partial g/\partial z & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = xyp + y^2q.$$

$$\therefore \quad [f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 2xy - x^2p^2 - xyqp - 2xy + xypq + y^2q^2$$

$$= -xp(xp + yq) + yq(xp + yq) = -(xp - yq)(xp + yq) = 0, \text{ using (1)}$$

Hence (1) and (2) are compatible.

Solving (1) and (2) for p and q, $p = y/z$ and $q = x/z$. ... (3)

Using (3) in $dz = pdx + qdy$, we have $dz = (y/z)dx + (x/z)dy$ or $z\,dz = d(xy)$.

Integrating, $z^2/2 = xy + c/2$ or $z^2 = 2xy + c$, where c is an arbitrary constant.

Ex. 3. *Show that the equations* $xp - yq = x$ *and* $x^2p + q = xz$ *are compatible and find their solution.* **[Delhi B.Sc. II (Prog) 2009; Delhi Maths Hons. 2007]**

Sol. Let $f(x, y, z, p, q) = xp - yq - x = 0.$ ...(1)

and $g(x, y, z, p, q) = x^2p + q - xz = 0.$ ...(2)

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix} = (p-1)x^2 - x(2xp - z).$$

Similarly, $\frac{\partial(f,g)}{\partial(z,p)} = x^2,$ $\frac{\partial(f,g)}{\partial(y,q)} = -q,$ $\frac{\partial(f,g)}{\partial(z,q)} = -xy.$

$$\therefore \quad [f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = (p-1)x^2 - x(2xp - z) - px^2 - q - xyq$$

$$= -x^2 + zx - q - xyq = -x^2 + x^2p - qxy, \text{ by (2)}$$

$$= x(-x + xp - yq) = 0, \text{ by (1)}$$

Hence (1) and (2) are compatible.

Solving (1) and (2) for p and q, $p = (1 + yz)/(1 + xy)$ and $q = x(z - x)/(1 + xy)$. ...(3)

Using (3) in $dz = pdx + qdy$, $dz = [(1 + yz)/(1 + xy)]\,dx + [x(z - x)/(1 + xy)]dy$

or $(1 + xy)dz = (1 + yz)dx + x(z - x)dy$ or $(1 + xy)dz - z(ydx + xdy) = dx - x^2dy$

or $$\frac{(1+xy)\,dz - z\,d(xy)}{(1+xy)^2} = \frac{dx - x^2dy}{(1+xy)^2} = \frac{(dx/x^2) - dy}{(y+1/x)^2}$$ or $$d\left(\frac{z}{1+xy}\right) = \frac{-d\,(y+1/x)}{(y+1/x)^2}.$$

Integrating it, $$\frac{z}{1+xy} = \frac{1}{(y+1/x)} + c$$ or $$\frac{z}{1+xy} = \frac{x}{1+xy} + c$$

or $z - x = c(1 + xy)$, c being an arbitrary constant.

Ex. 4. *Show that the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogeneous in x, y, z.* **[Delhi Maths, Hons. 2001, 06, 10]**

Sol. Given that differential equation $f(x, y, z, p, q) = 0$ ...(1)

is homogeneous in x, y, z. Then, clearly $f(x, y, z, p, q)$ will be a homogeneous function in variables x, y, z ; say of degree n. Then, by Euler's theorem on homogeneous function, we have

$$x(\partial f/\partial x) + y(\partial f/\partial y) + z(\partial f/\partial z) = nf \quad \text{so that} \quad x(\partial f/\partial x) + z(\partial f/\partial z) = 0, \text{ by (1)} \quad \ldots (2)$$

We take $g(x, y, z, p, q) = px + qy - z = 0$ ... (3)

Then, using (3), we have

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ p & x \end{vmatrix} = x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p},$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ \partial g/\partial z & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ -1 & x \end{vmatrix} = x\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p},$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ \partial g/\partial y & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ q & y \end{vmatrix} = y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q}$$

and

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ \partial g/\partial z & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ -1 & y \end{vmatrix} = y\frac{\partial f}{\partial z} + \frac{\partial f}{\partial q}$$

$$\therefore \quad [f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,q)}{\partial(z,q)}$$

$$= x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p} + p\left(x\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\right) + y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q} + q\left(y\frac{\partial f}{\partial z} + \frac{\partial f}{\partial q}\right)$$

$$= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + (px + qy)\frac{\partial f}{\partial y} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}, \text{ using (3)}$$

$= 0$, using (2)

Hence, the differential equation $z = px + qy$ is compatible with any differential equation $f(x, y, z, p, q)$ that is homogeneous in x, y, z.

Ex. 5. *If $u_1 = \partial u/\partial x$, $u_2 = \partial u/\partial y$, $u_3 = \partial u/\partial z$, show that the equations $f(x, y, z, u_1, u_2, u_3) = 0$ and $g(x, y, z, u_1, u_2, u_3) = 0$ are compatible if* $\dfrac{\partial(f,g)}{\partial(x,u_1)} + \dfrac{\partial(f,g)}{\partial(y,u_2)} + \dfrac{\partial(f,g)}{\partial(z,u_3)} = 0$. **(Delhi Maths (H) 2004)**

Sol. Treating z as constant, given equations are compatible if

$$\frac{\partial(f,g)}{\partial(x,u_1)} + u_1\frac{\partial(f,g)}{\partial(u,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + u_2\frac{\partial(f,g)}{\partial(u,u_2)} = 0. \quad \ldots(1)$$

Since f and g do not contain u, we have $\partial f/\partial u = 0$ and $\partial g/\partial u = 0$. ...(2)

$$\therefore \quad \frac{\partial(f,g)}{\partial(u,u_1)} = 0 \quad \text{and} \quad \frac{\partial(f,g)}{\partial(u,u_2)} = 0. \quad \ldots(3)$$

$$\therefore \quad \text{(1) reduces to} \quad \frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} = 0. \quad \ldots(4)$$

Similarly treating x and y constant respectively, given eqution are compatible if

$$\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0 \qquad ...(5) \qquad\qquad \frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0. \qquad ...(6)$$

We know that the given equations are compatible when they remain compatible even when any variable is taken as constant, *i.e.*, (4), (5) and (6) hold simultaneously. Hence adding (4), (5) and (6), the required condition for given equations to be compatible is

$$\frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0.$$

Ex. 6. *Show that the equations* $f(x,y,p,q)=0$, $g(x,y,p,q)=0$ *are compatible if* $\partial(f,\,g)/\partial(x,p)+\partial(f,g)/\partial(y,q)=0$

Verify that the equations $p=P(x,y),\ q=Q\,(x,y)$ *are compatible if* $\partial P/\partial y=\partial Q/\partial x$.

Sol. We know that $f(x,y,z,p,q)=0$ and $g(x,y,z,p,q)=0$ ... (1)

are compatible if $$\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0 \qquad ... (2)$$

First part: Comparing the given equations $f(x,y,p,q)=0$ and $g(x,y,p,q)=0$ with (1), we find that z is absent in given equations and so

$\partial f/\partial z=0$ and $\partial g/\partial z=0$ ... (3)

Now, $$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial p\\ \partial g/\partial z & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}0 & \partial f/\partial p\\ 0 & \partial g/\partial p\end{vmatrix}=0$$

and $$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial q\\ \partial g/\partial z & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}0 & \partial f/\partial q\\ 0 & \partial g/\partial q\end{vmatrix}=0$$

Substituting these values in (2), the required condition is

$$\partial(f,g)/\partial(x,p)+\partial(f,g)/\partial(y,q)=0$$

Second Part. Let $f=P(x,y)-p$ and $g=Q(x,y)-q$ ... (4)

Comparing (4) with (1), we find that z and q are absent in f and z and p are absent in g and

so $\partial f/\partial z=0,\quad \partial f/\partial q=0,\quad \partial g/\partial z=0$ and $\partial g/\partial p=0$ ... (5)

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix}\partial f/\partial x & \partial f/\partial p\\ \partial g/\partial x & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}\partial P/\partial x & -1\\ \partial Q/\partial x & 0\end{vmatrix}=\frac{\partial Q}{\partial x}$$

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial p\\ \partial g/\partial z & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}0 & -1\\ 0 & 0\end{vmatrix}=0$$

$$\frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix}\partial f/\partial y & \partial f/\partial q\\ \partial g/\partial y & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}\partial P/\partial y & 0\\ \partial Q/\partial y & -1\end{vmatrix}=-\frac{\partial P}{\partial y}$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial q\\ \partial g/\partial z & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}0 & 0\\ 0 & -1\end{vmatrix}=0$$

Substituting these values in (2), the required condition is

$\partial Q/\partial x-\partial P/\partial y=0$ or $\partial P/\partial y=\partial Q/\partial x$.

Ex. 7. *Show that* $p^2+q^2=1$ *and* $(p^2+q^2)x=pz$ *are compatible and solve them.*

Hint. Proceed as in solved Ex. 1 Ans. $z^2=x^2+(y+c)^2$.

Ex. 8. *Solve completey the simultaneous equations:* $z = px + qy$ *and* $2xy(p^2 + q^2) = z(yp + xq)$. **[Delhi Math (H) 2006, 10]**

Sol. Given $z = px + qy$ ... (1)

and $2xy(p^2 + q^2) - z(yp + xq) = 0$ ... (2)

Let $f(x, y, z, p, q) = 2xy\,(p^2 + q^2) - z\,(yp + xq) = z^2\{2(x/z)\,(y/z)\,(p^2 + q^2) - (y/z)p - (x/z)q\}$,

showing that $f(x, y, z, p, q)$ is homogeneous in x, y, z.

We know that (refer solved example 4) the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogenous in x, y, z. Hence (1) and (2) are compatible.

From (1), we have $q = (z - px)/y$ ... (3)

Using (3), (2) gives $2x(x^2 + y^2)p^2 - z(3x^2 + y^2)\,p + xz^2 = 0$

or $(2xp - z)\,\{(x^2 + y^2)p - xz\} = 0$

so that $p = z/2x,\ xz/(x^2 + y^2)$ ... (4)

Using (4), (3) gives $q = z/2y,\ yz/(x^2 + y^2)$ ... (5)

Using the corresponding values $p = z/2x,\ q = z/2y$ in $dz = px + qdy$, we get

$dz = (z/2x)dx + (z/2y)dy$ or $2(1/z)dz = (1/x)dx + (1/y)dy$

Integrating, $2\log z = \log x + \log y + \log C_1$ or $z^2 = C_1 xy$ ...(6)

Similarly, using the corresponding values $p = xz/(x^2 + y^2)$ and $q = yz/(x^2 + y^2)$ in $dz = pdx + qdy$, we get

$$dz = \frac{xzdx}{x^2 + y^2} + \frac{yzdy}{x^2 + y^2} \quad \text{or} \quad \frac{2dz}{z} = \frac{2(xdx + ydy)}{x^2 + y^2}$$

Integrating, $2\log z = \log(x^2 + y^2) + \log C_2$ or $z^2 = C_2(x^2 + y^2)$ ... (7)

(6) and (7) give two two common solutions of (1) and (2)

EXERCISE 3(A)

1. Show that $\partial z/\partial x = 7x + /8y - 1$ and $\partial z/\partial y = 9x + 11y - 2$ are not compatible.

2. Show that the partial differential equations $p = 6x - 4y + 1$ and $q = 4x + 6y + 1$ do not possess any common solution.

3. Show that the following system of partial differential equations are compatible and hence solve them

(*i*) $p = 6x + 3y,\ q = 3x - 4y$ **Ans.** $z = 3x^2 + 3xy - 2y^2 + c$

(*ii*) $p = ax + hy + g,\ q = hx + by + f$ **Ans.** $z = (ax^2 + by^2)/2 + hxy + gx + fy + c$

(*iii*) $\partial z/\partial x = y(2ax + by),\ \partial z/\partial y = x(ax + 2by)$ **Ans.** $z = ax^2y + bxy + c$

(*iv*) $p = x^4 - 2xy^2 + y^4,\ q = 4xy^3 - 2x^2y - \sin y$ **Ans.** $z = x^5/5 - x^2y^2 + xy^4 + \cos y + c$

(*v*) $p = (e^y + 1)\cos x,\ q = e^y \sin x$ **Ans.** $z = (e^y + 1)\sin x + c$

(*vi*) $p = y\,(1 + 1/x) + \cos y,\ q = x + \log x - x \sin y$ **Ans.** $z = y\,(x + \log x) + x\cos y + c$

(*vii*) $p = y^2 e^{xy^2} + 4x^3,\ q = 2xy\, e^{xy^2} - 3y^2$ **Ans.** $z = e^{xy^2} + x^4 - y^3 + c$

(*viii*) $p = \sin x \cos y + e^{3x},\ q = \cos x \sin y + \tan y$

Ans. $x = (1/3) \times e^{3x} - \cos x \cos y + \log \sec x + c$

3.7. Charpit's method.* (General method of solving partial differential equations of order one but of any degree. [Agra 2003; Delhi Maths (H) 2000, 05, 06, 08-11; Kanpur 1998; Meeerut 2003, 05; Nagpur 2002, 04, 06, 08; Rohilkhand 2001, 04]

Let the given partial equation differential of first order and non–linear in p and q be

$$f(x, y, z, p, q) = 0. \quad ...(1)$$

We know that $$dz = p\,dx + q\,dy. \quad ...(2)$$

The next step consists in finding another relation $F(x, y, z, p, q) = 0$ ...(3)

such that when the values of p and q obtained by solving (1) and (3), are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (3), differentiate partially (1) and (3) with respect to x and y and get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0, \quad ... (4)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0, \quad ... (5)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \quad ... (6)$$

and

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0. \quad ... (7)$$

Eliminating $\partial p/\partial x$ from (4) and (5), we get

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p} = 0$$

or

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0. \quad ...(8)$$

Similarly, eliminating $\partial q/\partial y$ from (6) and (7), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right) q + \left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0. \quad ...(9)$$

Since $\partial q/\partial x = \partial^2 z/\partial x \partial y = \partial p/\partial y$, the last term in (8) is the same as that in (9), except for a minus sign and hence they cancel on adding (8) and (9).

Therefore, adding (8) and (9) and rearranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0. \quad ...(10)$$

This is a linear equation of the first order to obtain the desired function F. As in Art 2.20 of chapter 2, integral of (10) is obtained by solving the auxiliary equations

$$\frac{dp}{(\partial f/\partial x) + p(\partial f/\partial z)} = \frac{dq}{(\partial f/\partial y) + q(\partial f/\partial z)} + \frac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dF}{0}. \quad ...(11)$$

---

*This is general method for solving equations with two independent variables. Since the solution by this method is generally more complicated, this method is applied to solve equations which cannot be reduced to any of the standard forms which will be discussed later on. Thus, Charpit's method is used in two situations (*i*) When you are asked to solve a problem by Charpit's method (*ii*) when the given equation is not of any four standard forms given in Articles 3.10, 3.12, 3.14 and 3.17.

Since any of the integrals of (11) will satisfy (10), an integral of (11) which involves p or q (or both) will serve along with the given equation to find p and q. In practice, however, we shall select the simplest integral.

Note. In what follows we shall use the following standard notations:

$$\partial f/\partial x = f_x, \qquad \partial f/\partial y = f_y, \qquad \partial f/\partial z = f_z, \qquad \partial f/\partial p = f_p, \qquad \partial f/\partial q = f_q.$$

Therefore, Charpit's auxiliary equations (11) may be re–written as

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0} \qquad \ldots (11)'$$

3.8A. WORKING RULE WHILE USING CHARPIT'S METHOD

Step 1. Transfer all terms of the given equation to L.H.S. and denote the entire expression by f.

Step 2. Write down the Charpit's auxiliary equations (11) or (11)′.

Step 3. Using the value of f in step 1 write down the values of $\partial f/\partial x$, $\partial f/\partial y$ ..., *i.e.*, f_x, f_y, ... etc. occuring in step 2 and put these in Charpit's equations (11) or (11)′.

Step 4. After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5. The simplest relation of step 4 is solved along with the given equation to determine p and q. Put these values of p and q in $dz = p\,dx + q\,dy$ which on integration gives the complete integral of the given equation.

The Singular and General integrals may be obtained in the usual manner.

Remark. Sometimes Charpit's equations give rise to $p = a$ and $q = b$, where a and b are constants. In such cases, putting $p = a$ and $q = b$ in the given equation will give the required complete integral.

3.8.B. SOLVED–EXAMPLES BASED ON ART. 3.8A.

Ex. 1. *Find a complete integral of* $z = px + qy + p^2 + q^2$.

[Bilaspur 2000l; Bhopal 1996, I.A.S. 1996; Indore 2000; Jabalpur 2000; K.U. Kurukshetra 2005; Ravishankar 2000; 04; Meerut 2010; Garhwal 2010]

Sol. Let $f(x, y, z, p, q) \equiv z - px - qy - p^2 - q^2 = 0$ ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$ ... (2)

From (1), $f_x = -p$, $f_y = -q$, $f_z = 0$, $f_p = -x - 2p$ and $f_q = -y - 2q$ ... (3)

Using (3), (2) reduces to

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} \qquad \ldots (4)$$

Taking the first fraction of (4), $dp = 0$ so that $p = a$ ... (5)

Taking the second fraction of (4), $dq = 0$ so that $q = b$ ... (6)

Putting $p = a$ and $q = b$ in (1), the required complete integral is

$z = ax + by + a^2 + b^2$, a, b being arbitrary constants.

Ex. 2. *Find a complete integral of* $q = 3p^2$. **[Agra 2006]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv 3p^2 - q = 0$. ...(1)

$\therefore$ Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or $$\frac{dp}{0+p.0}=\frac{dq}{0+q.0}=\frac{dz}{-6p^2+q}=\frac{dx}{-6p}=\frac{dy}{1}, \text{ using (1)} \quad ...(2)$$

Taking the first fraction of (1), $dp = 0$ so that $p = a$. ...(3)

Substituting this value of p in (1), we get $q = 3a^2$. ...(4)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = adx + 3a^2dy$ so that $z = ax + 3a^2y + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 3. *Find the complete integral of* $zpq = p + q$ **[Nagpur 2010; Meerut 2006]**

Sol. Let $f(x, y, z, p, q) = zpq - p - q = 0$ ... (1)

Here Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_z}=\frac{dz}{-p\,f_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

... (2)

From (1), $f_x = 0$, $f_y = 0$, $f_z = pq$, $f_p = zq - 1$ and $f_q = zp - 1$ ... (3)

Using (3), (2) reduces to

$$\frac{dp}{p^2q}=\frac{dq}{pq^2}=\text{.....} \quad \text{or} \quad \frac{dp}{p}=\frac{dq}{q} \quad \text{so that} \quad p = aq \quad ... (4)$$

Solving (1) and (2), $p = (1 + a)/z$ and $q = (1 + a)/az$.

$\therefore$ $dz = pdx + qdy = [(1 + a)/z]dx + [(1 + a)/az]dy$ or $2zdz = 2(1 + a)\,[dx + (1/a)dy]$

Integrating, $z^2 = 2(1 + a)\,[x + (1/a)y] + b$, a, b being arbitrary constants

Ex. 4. *Find a complete integral of* $p^2 - y^2q = y^2 - x^2$. **[M.D.U. Rohtak 2006]**

Sol. Here given equation is $f(x, y, z, p, q) = p^2 - y^2q - y^2 + x^2 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{2x}=\frac{dq}{-2qy-2y}=\frac{dz}{-p(2p)-q(-q^2)}=\frac{dx}{-2p}=\frac{dy}{y^2}, \text{ using (1)} \quad ...(2)$$

Taking the first and fourth fractions, $pdp + xdx = 0$ so that $p^2 + x^2 = a^2$ ... (3)

Solving (1) and (3) for p and q, $p = (a^2 - x^2)^{1/2}$, $q = a^2y^{-2} - 1$.

$\therefore$ $dz = pdx + qdy = (a^2 - x^2)^{1/2}dx + (a^2y^{-2} - 1)dy$.

Integrating, $z = (x/2) \times (a^2 - x^2)^{1/2} + (a^2/2) \times \sin^{-1}(x/a) - (a^2/y) - y + b$.

Ex. 5. *Find a complete integral of* $z^2(p^2z^2 + q^2) = 1$. **[I.A.S. 1997; Meerut 2007]**

Sol. Here given equation is $f(x, y, z, p, q) = p^2z^4 + q^2z^2 - 1 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_z}=\frac{dz}{-p\,f_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dq}{-f_q}$$

or $$\frac{dp}{p(4p^2z^3+2zq^2)}=\frac{dq}{q(4p^2z^3+2zq^2)}=\frac{dz}{-2p^2z^4-2q^2z^2}=\frac{dx}{-2pz^4}=\frac{dy}{-2qz^2}, \text{ by (1)} \quad ... (2)$$

Taking the first two fractions, $(1/p)dp = (1/q)dq$ so that $p = aq$.

Solving (1) and (2) for p and q, $p = \dfrac{a}{z(a^2z^2+1)^{1/2}}$, $q = \dfrac{1}{z(a^2z^2+1)^{1/2}}$.

$\therefore$ $dz = pdx + qdy = (a\,dx + dy)/z\,(a^2z^2 + 1)^{1/2}$ or $adx + dy = z(a^2z^2 + 1)^{1/2}dz$.

Integrating, $ax + y = \int (a^2z^2 + 1)^{1/2}.zdz.$ ...(3)

Putting $a^2z^2 + 1 = t^2$ so that $2a^2zdz = 2tdt$, (3) becomes

$ax + y = \int (1/a^2)t.t\,dt$ or $ax + y + b = (1/3a^2)t^3$, where $t = (a^2z^2 + 1)^{1/2}$

or $ax + y + b = (1/3a^2) \times (a^2z^2 + 1)^{3/2}$ or $9a^4(ax + y + b)^2 = (a^2z^2 + 1)^3$,

which is a complete integral, a and b being arbitrary constants.

Ex. 6. *Find a complete integral of* $px + qy = pq$. **[Kurukshetra 2006 Rajasthan 2000, 01, Gulbarga 2005; Meerut 2002; Kanpur 2004; Jiwaji 2004; Rewa 2001; Vikram 2000, 03, 04; Bhopal 2010]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv px + qy - pq = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

or $\dfrac{dp}{-(x-q)} = \dfrac{dy}{-(y-q)} = \dfrac{dz}{-p(x-q) - q(y-p)} = \dfrac{dp}{p + p.0} = \dfrac{dq}{q + q.0}$, by (1) ...(2)

Taking the last two fractions of (2), $(1/p)dp = (1/q)dq.$

Integrating, $\log p = \log q + \log a$ or $p = aq.$ ...(3)

Substituting this value of p in (1), we have

$aqx + qy - aq^2 = 0$ or $aq = ax + y,$ as $q \neq 0$ ...(4)

$\therefore$ From (3) and (4), $q = (ax + y)/a$ and $p = ax + y.$ ...(5)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (ax + y)dx + [(ax + y)/a]\,dy$ or $adz = (ax + y)\,(adx + dy)$

or $adz = (ax + y)\,d(ax + y) = udu$, where $u = ax + y$.

Integrating, $az = u^2/2 + b = (ax + y)^2/2 + b,$

which is a complete integral, a and b being arbitrary constants.

Ex. 7. *Find the complete integrals of following equations:*

(*i*) $q = (z + px)^2$ **[Indore 2004; Ravishanker 2005]**

(*ii*) $p = (z + qy)^2$ **[Meerut 2008, 09; Agra 2001; Delhi B.Sc. (Prog) 2008; Kurukshetra 2005]**

Sol. (*i*). Here given equations is $f(x, y, z, p, q) = (z + px)^2 - q = 0$ ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

or $\dfrac{dp}{2p(z+px) + 2p(z+px)} = \dfrac{dq}{2q(z+px)} = \dfrac{dz}{-2px(z+px) + q} = \dfrac{dx}{-2x(z+px)} = \dfrac{dy}{0}$, by (1)

Taking the second and fourth fractions, $(1/q)dq = -(1/x)dx.$

Integrating, $\log q = \log a - \log x$ so that $q = a/x.$ ...(2)

Substituting the above value of q in (1), we have

$(z + px)^2 = a/x$ or $px = \sqrt{a}/\sqrt{x} - z$ or $p = \sqrt{a}/x\sqrt{x} - z/x.$ ...(3)

$\therefore$ $dz = pdx + qdy = \left(\dfrac{\sqrt{a}}{x\sqrt{x}} - \dfrac{z}{x}\right) dx + \dfrac{a}{x}dy,$ by (2) and (3)

or $xdz = \sqrt{a}\,x^{-1/2}dx - zdx + ady$ or $xdz + zdx = \sqrt{a}\,x^{-1/2}dx + ady$

or $d(xz) = \sqrt{a}\,x^{-1/2}dx + ady.$

Integrating, $xz = 2\sqrt{a}\sqrt{x} + ay + b$, a, b being arbitrary constants

(*ii*) **Sol.** Do as in part (1). **Ans.** $yz = ax + \sqrt{ay} + b.$

Ex. 8. *Find a complete integral of* $yzp^2 - q = 0$.

Sol. Here $f(x, y, z, p, q) = yzp^2 - q = 0.$

Charpit's uxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{0 + p(yp^2)} = \frac{dq}{zp^2 + q(yp^2)} = \frac{dz}{-2yzp^2 + q} = \frac{dx}{-2yzp} = \frac{dy}{1}, \text{ by (1)} \quad ...(2)$$

Taking the first and fifth fractions, $(1/yp^3)\, dp = dy$

or $p^{-3}dp = ydy$ or $-2p^{-3}dp = -2ydy.$

Integrating, $p^{-2} = a^2 - y^2$ so that $p = 1/(a^2 - y^2)^{1/2}.$ ...(3)

Using (3), $(1) \Rightarrow q = yzp^2 \Rightarrow q = yz/(a^2 - y^2).$ ...(4)

$\therefore$ $$dz = pdx + qdy = \frac{dx}{(a^2 - y^2)^{1/2}} + \frac{yzdy}{(a^2 - y^2)}$$

or $$(a^2 - y^2)^{1/2}\, dz - \frac{yzdy}{(a^2 - y^2)^{1/2}} = dx \quad \text{or} \quad d\,[z\,(a^2 - y^2)^{1/2}] = dx.$$

Integrating, $z\,(a^2 - y^2)^{1/2} = x + b$ or $z^2(a^2 - y^2) = (x + b)^2$, a, b being arbitrary constants.

Ex. 9. *Find a complete integral of* $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0.$ **[I.A.S. 1994]**

Sol. Given equation is $f(x, y, z, p, q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0.$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-pf_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-p(32p^2z + 18q^2z + 8z)} = \frac{dq}{q(32p^2z + 18q^2z + 8z)} = \frac{dz}{-p(32pz^2) - q(18qz^2)} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2}.$$

Taking the first and second fractions, $(1/p)dp = (1/q)dq$ so that $p = aq$ ... (2)

Solving (1) and (2) for p and q, we have

$$q = \frac{2(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}} \quad \text{and} \quad p = \frac{2a(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}}. \quad ...(3)$$

Hence, $$dz = pdx + qdy = \frac{2(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}}\ (adx + dy), \text{ using (3)}$$

or $$(1/2) \times (16a^2 + 9)^{1/2}\, (1 - z^2)^{-1/2}\, (-2zdz) = -2(adx + dy). \quad ...(4)$$

Putting $1 - z^2 = t$ so that $-2zdz = dt$, (4) becomes

or $$(1/2) \times (16a^2 + 9)^{1/2}\, t^{-1/2}\, dt = -2\ (adx + dy).$$

Integrating, $(16a^2 + 9)^{1/2}\, t^{1/2} = -2(ax + y) + b$, a, b being arbitrary constants.

or $$(16a^2 + 9)^{1/2} \sqrt{(1 - z^2)} + 2(ax + y) = b, \text{ as } t = 1 - z^2.$$

Ex. 10(a). *Find a complete integral of* $(p^2 + q^2)x = pz$.

[Agra 2003; Rajasthan 2005; Ravishankar 2001; Delhi Maths (Hons) 2004, 05]

(b). *Find the complete integral of the partial differential equation* $(p^2 + q^2)x = pz$ *and deduce the solution which passes through the curve* $x = 0$, $z^2 = 4y$. **[Meerut 2007]**

Sol. Let $f(x, y, q, p, q) = (p^2 + q^2)x - pz = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

giving $dp/q^2 = dq/(-pq)$, by (1) or $2pdp + 2qdq = 0$.

Integrating, $p^2 + q^2 = a^2$, where a is an arbitrary constant. ...(2)

Solving (1) and (2), $p = a^2x/q$ and $q = (a/z) \times \sqrt{(z^2 - a^2x^2)}$. ...(3)

$\therefore$ $dz = pdx + qdy = \dfrac{a^2xdx}{z} + \dfrac{a\sqrt{(z^2 - a^2x^2)}\, dy}{z}$ or $\dfrac{zdz - a^2xdx}{\sqrt{(z^2 - a^2x^2)}} = ady.$

Putting $z^2 - a^2x^2 = t$ so that $2(zdz - a^2xdx) = dt$, we get

$(1/2\sqrt{t})dt = ady$ or $(1/2) \times t^{-1/2} = ady.$

Integrating, $t^{1/2} = ay + b$ or $\sqrt{(z^2 - a^2x^2)} = ay + b,$ as $t = \sqrt{z^2 - a^2x^2}$

or $z^2 - a^2x^2 = (ay + b)^2$ or $z^2 = a^2x^2 + (ay + b)^2$. ... (4)

(*b*) Proceeding as in part (a), (4) is the complete integral.

The parametric equations of the given curve $x = 0$, $z^2 = 4y$ are given by

$x = 0,$ $y = t^2,$ $z = 2t$ ... (5)

Therefore the intersections of (1) and (2) are determined by

$4t^2 = (at^2 + b)^2$ or $a^2t^4 + 2(ab - 2)t^2 + b^2 = 0$ ... (6)

Equation (6) has equal roots if its discriminant = 0, *i.e.*, if

$4(ab - 2)^2 - 4a^2b^2 = 0$ or $a^2b^2 = 1$ so that $b = 1/a$

Hence from (4), the appropriate one paremeter sub–system is given by

$z^2 = a^2x^2 + (ay + 1/a)^2$ or $a^4(x^2 + y^2) + a^2(2y - z^2) + 1 = 0,$

which is a quatratic equation in parameter 'a'. Therefore, this has for its envelope surface

$(2y - z^2)^2 - 4(x^2 + y^2) = 0$ or $(2y - z^2)^2 = 4(x^2 + y^2)$ ... (7)

The desired solution is given by the function z defined by equation (7).

Ex. 10(c). *Find a complete, singular and general integrals of* $(p^2 + q^2)y = qz$.

[Guwahati 2007; Agra 2001; Bilaspur 1998; Delhi Maths (H) 2003, 05; Garhwal 2005; Meerut 2010, 11; K.V. Kurukshetra 2004; Kanpur 2005; Rohilkhand 2001; Pune 2010]

Sol. Here the given equation is $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{-pq} = \dfrac{dq}{p^2} = \dfrac{dz}{-2p^2y + qz - 2q^2y} = \dfrac{dx}{-2py} = \dfrac{dy}{-2qy + z}$, by (1) ...(2)

Taking the first two fractions, we get $2pdp + 2qdq = 0$ so that $p^2 + q^2 = a$ ... (3)

Using (3), (1) gives $a^2y = qz$ or $q = a^2y/z.$

Putting this value of q in (3), we get

$$p = \sqrt{(a^2 - q^2)} = \sqrt{a^2 - (a^4y^2/z^2)} = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}\,.$$

Now putting these values of p and q in $dz = pdx + qdy$, we have

$dz = \dfrac{a}{z}\sqrt{(z^2 - a^2y^2)}\, dx + \dfrac{a^2y\, dy}{z}dy$ or $\dfrac{zdz - a^2y\, dy}{\sqrt{(z^2 - a^2y^2)}} = a\, dx.$

Integrating, $(z^2 - a^2y^2)^{1/2} = ax + b$ or $z^2 - a^2y^2 = (ax + b)^2$, ...(4)

which is a required complete integral, a, b being arbitrary constants.

Singular Integral. Differentiating (4) partially w.r.t. a and b, we have

$$0 = 2ay^2 + 2(ax + b)x \quad ...(5)$$

and $$0 = 2(ax + b). \quad ...(6)$$

Eliminating a and b between (4), (5) and (6), we get $z = 0$ which clearly satisfies (1) and hence it is the singular integral.

General Integral. Replacing b by $\phi(a)$ in (4), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2. \quad ...(7)$$

Differentiating (7) partially w.r.t. a, $-2ay^2 = 2[ax + \phi(a)] \,.\, [x + \phi'(a)]$. ...(8)

General integral is obtained by eliminating a from (7) and (8).

Ex. 11. *Find a complete integral of* $p(1 + q^2) + (b - z)q = 0$. **[Agra 1996]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv p(1 + q^2) + (b - z)q = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \dfrac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \dfrac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \dfrac{dx}{-\frac{\partial f}{\partial p}} = \dfrac{dy}{-\frac{\partial f}{\partial q}}$

or $$\frac{dp}{pq} = \frac{dq}{p^2} = \frac{dz}{-p(1+q^2) - (b-z)q} = \frac{dx}{-(q^2+1)} = \frac{dy}{-2pq - (b-z)}, \text{ by (1)}$$

First two fractions give $(1/p)dp = (1/q)dq$ so that $q = pc$.

Putting $q = pc$ in (1), we have $p = \sqrt{[c(z-b)-1]}\,/c$.

$\therefore$ $q = pc$ gives $q = \sqrt{[c(z-b)-1]}$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{[c(z-b)-1]}\left(\frac{dx}{c} + dy\right) \quad \text{or} \quad \frac{cdz}{\sqrt{[c(z-b)-1]}} = dx + c\,dy.$$

Integrating, $2\sqrt{[c(z-b)-1]} = x + cy + a$ or $4\{c(z - b) - 1\} = (x + cy + a)^2$

which is a complete integral, a and c being arbitrary constants.

Ex. 12. *Find a complete and singular integrals of* $2xz - px^2 - 2q\,xy + pq = 0$. **[I.A.S. 1991, 93, 2007, 2008; Delhi Hons. 2001, 01, 05; Kanpur 2001, 03; Meerut 2005; Bhopal 2004, 10; Indore 1999; M.D.U. Rohtak 2004, Ravishanker 2004; Rajasthan 2000, 03, 05, 10]**

Sol. Here given equation is $f(x, y, z, p, q) = 2xz - px - 2qxy + pq = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \dfrac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \dfrac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \dfrac{dx}{-\frac{\partial f}{\partial p}} = \dfrac{dy}{-\frac{\partial f}{\partial q}}$

or $$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 + 2xyq - 2pq}, \text{ by (1)}$$

The second fraction gives $dq = 0$ so that $q = a$

Putting $q = a$ in (1), we get $p = 2x(z - ay)/(x^2 - a)$

Putting values p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2x(z-ay)}{x^2 - a}\,dx + a\,dy \quad \text{or} \quad \frac{dz - ady}{z - ay} = \frac{2xdx}{x^2 - a}.$$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

or $z - ay = b(x^2 - a)$ or $z = ay + b(x^2 - a)$, ...(2)

which is the complete integral, a and b being arbitrary constants.

Differentiating (2) partially with respect to a and b, we get

$0 = y - b$ and $0 = x^2 - a$. ...(3)

Solving (3) for a and b, $a = x^2$ and $b = y$. ...(4)

Substituting the values of a and b given by (4) in (2), we get $z = x^2y$, which is the required singular integral.

Ex. 13. *Find a complete integrals of the following partial differential equations:*

(*i*) $q = px + p^2$. **[Sagar 2003; Meerut 1994]**

(*ii*) $q = -px + p^2$.

Sol. (*i*) Here given equation is $f(x, y, z, p, q) \equiv q - px - p^2 = 0$. ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{-p} = \dfrac{dq}{0} = \dfrac{dz}{-p(-x-2p)-q} = \dfrac{dx}{-(-x-2p)} = \dfrac{dy}{-1}$, by (1)

The 2nd fraction gives $dq = 0$ so that $q = a$.

Putting $q = a$ in (1) gives $p^2 + px - a = 0$ so that $p = (1/2) \times \left[-x \pm (x^2 + 4a)^{1/2}\right]$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = -(x/2) \times dx \pm (1/2) \times (x^2 + 4a)^{1/2}\, dx + a\, dy.$$

Integrating, the required complete integral is

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}(x^2 + 4a)^{1/2} + 2a \log \{x + (x^2 + 4a)^{1/2}\}\right] + ay + b,$$

Part (*ii*). Proceed like part (*i*) yourself. Complete integral is

$$z = \frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}(x^2 + 4a)^{1/2} + 2a \log \{x + (x^2 + 4a)^{1/2}\}\right] + ay + b.$$

Ex. 14. *Find a complete integral of* $pxy + pq + qy = yz$. **[Delhi B.A. (Prog) H 2010]**
[Garhwal 2001; Rohilkhand 1999; Meerut 2001, 02; Kanpur 2005]

Sol. Given $f(x, y, z, p, q) \equiv pxy + pq + qy - yz = 0$. ...(1)

Charpit's auxiliary equation are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{0} = \dfrac{dq}{(px+q)+qy} = \dfrac{dz}{-p(xy+q) - q(p+y)} = \dfrac{dx}{-(xy+q)} = \dfrac{dy}{-(p+y)}$, by (1)

The first fraction gives $dp = 0$ so that $p = a$.

Putting $p = a$ in (1) gives $axy + aq + qy = yz$ so that $q = y(z - ax)/(a + y)$.

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = adx + \frac{y(z-ax)}{a+y}dy \quad \text{or} \quad \frac{dz - adx}{z - ax} = \frac{y\, dy}{a+y} = \left(1 - \frac{a}{a+y}\right)dy.$$

Integrating, $\log (z - ax) = y - a \log (a + y) + \log b$, a, b, being arbitrary constants.

or $\log (z - ax) + \log (a + y)^a - \log b = y$ or $(z - ax)(y + a)^a = be^y$

Ex. 15. *Find a complete integral* $p^2 + q^2 - 2px - 2qy + 1 = 0$.

[Patna 2003; Meerut 99, 2003; Delhi Maths Hons 91; Ravishanker 2010]

Sol. Given $f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 1 = 0.$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-pf_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-p(2p-2x) - q(2q-2y)} = \frac{dx}{-(2p-2y)} = \frac{dy}{-(2q-2y)}, \text{ by (1)}$$

The first two fractions give $(1/p)dp = (1/q)dq$ so that $p = aq$.

Putting $p = aq$ in (1), $a^2q^2 + q^2 - 2aqx - 2qy + 1 = 0$ or $(a^2 + 1)q^2 - 2(ax - y)q + 1 = 0$.

$$\Rightarrow q = \frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}, \; p = aq = a\,\frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}.$$

Putting these values of p and q in $dz = p\,dx + y\,dy$, we get

$$dz = \frac{(ax+y) \pm \sqrt{\{(ax+y)^2 - (a^2+1)\}}}{(a^2+1)}(adx + dy). \qquad ...(2)$$

Put $ax + y = v$ so that $a\,dx + dy = dv$. Then (2) gives

$$(a^2 + 1)dz = [v \pm \sqrt{\{v^2 - (a^2+1)\}}]dv.$$

Integrating, $$(a^2 + 1)\,z = v^2/2 \pm [\,(v/2) \times \sqrt{\{v^2 - (a^2+1)\}}$$

$$- (1/2) \times (a^2 + 1) \log (v + \sqrt{\{v^2 - (a^2+1)\}})] + b$$

is the complete integral, where $v = ax + b$ and a, b are arbitrary constants.

Ex. 16. *Find a complete integral of* $p^2 + q^2 - 2px - 2qy + 2xy = 0$. **[PCS (U.P.) 2001; Garhwal 1993; Delhi 1997; Kanpur 1996; I.A.S. 1999; Meerut 2003; Rohitkhand 1998]**

Sol. Given equation is $f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0.$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}, \text{ by (1)}$$

which gives $$\frac{dp+dq}{2(x+y-p-q)} = \frac{dx+dy}{2(x+y-p-q)}$$

or $dp + dq = dx + dy$ *i.e.*, $dp - dx + dq - dy = 0$.

Integrating, $(p - x) + (q - y) = a$ ...(2)

Re–writing (1), $(p - x)^2 + (q - y)^2 = (x - y)^2$. ...(3)

Putting the value of $(q - y)$ from (2) in (3), we get

$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2$ or $2(p - x)^2 - 2a(p - x) + \{a^2 - (x - y)^2\} = 0.$

$$\therefore p - x = \frac{2a \pm \sqrt{[4a^2 - 4.2.\{a^2 - (x-y)^2\}]}}{4} \quad \Rightarrow \quad p = x + \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}],$$

$$\therefore \text{(2) gives} \quad q = a + y - p + x \quad \text{or} \quad q = y + (1/2) \times [a \mp \sqrt{\{2(x-y)^2 - a^2\}}].$$

Putting these value of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = x\,dx + y\,dy + (a/2) \times (dx + dy) \pm (1/2)\sqrt{\{2(x-y)^2 - a^2\}}(dx - dy)$$

or $$dz = x\,dx + y\,dy + \frac{a}{2}(dx + dy) \pm \frac{1}{\sqrt{2}}\sqrt{(x-y)^2 - a^2/2.}\,(dx - dy)\,.$$

Integrating, the desired complete integral is

$$z = \frac{x^2 + y^2}{2} + \frac{a(x+y)}{2} \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2 - a^2/2} \;-\frac{a^2}{4}\log\left[(x-y) + \sqrt{(x-y)^2 - a^2/2}\right]\right)$$

Ex. 17. *Find a complete integral of* $p^2x + q^2y = z$. **[Gujarat 2005; K.U. Kurukshetra 2001; Meerut 2008; Agra 2004; I.A.S. 2004, 06 ; Delhi Maths Hons. 1997; Punjab 2001]**

Sol. Given equation is $$f(x, y, z, p, q) = p^2x + q^2y - z = 0. \quad ...(1)$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} - \frac{dy}{-f_q}$$

or $$\frac{dp}{-p + p^2} = \frac{dq}{-q + q^2} = \frac{dz}{-2(p^2x + q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}, \text{ by (1)} \quad ...(2)$$

Now, each fraction in (2) $$= \frac{2px\,dp + p^2dx}{2px(-p + p^2) + p^2(-2px)} = \frac{2qy\,dq + q^2dy}{2qy(-q + q^2) + q^2(-2qy)}$$

or $$\frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2qy} \qquad i.e., \qquad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}.$$

Integrating it, $\log(p^2x) = \log(q^2y) + \log a$ or $p^2x = q^2ya$. ...(3)

Form (1) and (3), $aq^2y + q^2y = z$ or $q = [z/(1 + a)]^{1/2}$. ...(4)

Form (3) and (4), $$p = q\left(\frac{ya}{x}\right)^{1/2} = \left\{\frac{za}{(1+a)x}\right\}^{1/2}.$$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2}dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2}dy \quad \text{or} \quad (1 + a)^{1/2}z^{-1/2}\,dz = \sqrt{a}\,x^{-1/2}dx + y^{-1/2}dy.$$

Integrating, $(1 + a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$, a, b being arbitrary constants.

Ex. 18. *Find a complete integral of* $2z + p^2 + qy + 2y^2 = 0$. **[I.F.S. 2005; Meerut 2000; Rohilkhand 1993; Bilaspur 2004, M.D.U Rohtak 2005; Rawa 1999; Ranchi 2010]**

Sol. Given equation is $$f(x, y, z, p, q) = 2z + p^2 + qy^2 + 2y^2 = 0. \quad ... (1)$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dp}{-f_p} = \frac{dq}{-f_q}$$

or $$\frac{dp}{0 + 2p} = \frac{dq}{(q + 4y) + 2q} = \frac{dz}{-p \times (2p) - qy} = \frac{dx}{-2p} = \frac{dy}{-y}, \text{ by (1)}$$

Taking the first and fourth fractions, $dp = -\,dx$.

Integrating, $p = a - x$ or $p = -(x - a)$. ...(2)

Using (2), (1) becomes $2z + (a - x)^2 + qy + 2y^2 = 0$

$\therefore$ $q = -[2z + (x - a)^2 + 2y^2]/y.$ ...(3)

$\therefore$ $dz = p\,dx + q\,dy = -(x-a)\,dx - [\{2z + (x-a)^2 + 2y^2\}/y]dy$, by (2) and (3)

Multiplying both sides by $2y^2$ and re–writing, we have

$$2y^2dz = -2(x-a)y^2\,dx - 4zydy - 2y(x-a)^2dy - 4y^3dy$$

or $$2(y^2dz + 2zy\,dy) + [2(x-a)^2y^2dx + 2y(x-a)^2dy] + 4y^3dy = 0$$

or $$2d(y^2z) + d[y^2(x-a)^2] + 4y^3dy = 0.$$

Integrating, $2y^2z + y^2(x-a)^2 + y^4 = b$, a, b being arbitrary constants

Ex. 19(a). *Find a complete integral of* $2(z + px + qy) = yp^2$.

[Delhi B.A. (Prog.) II 2007, 10; CDLU 2004; Delhi Maths Hons. 1998, 2008]

Sol. Given equation is $f(x, y, z, p, q) = 2(z + px + qy) - yp^2 = 0$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dp}{-f_p} = \frac{dq}{-f_q}$$

or $$\frac{dp}{2p+2p} = \frac{dq}{2q - p^2 + 2q} = \frac{dz}{-p(2x - 2yp) - q \times 2y} = \frac{dx}{-(2x-2yp)} = \frac{dy}{-2y}, \text{ by (1)}$$

Taking the first and the last fractions, $\frac{dp}{4p} = \frac{dy}{-2y}$ or $\frac{dp}{p} + 2\frac{dy}{y} = 0.$

Integrating, $\log p + 2\log y = \log a$ or $py^2 = a.$ ...(2)

Solving (1) and (2) for p and q, $p = \frac{a}{y^2}$ and $q = -\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}$.

$\therefore$ $$dz = p\,dx + q\,dy = \frac{a}{y^2}dx + \left[-\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}\right]dy$$

Multiplying both sides by y and re–arranging, we get

$$(ydz + zdy) - a\left(\frac{y\,dx - x\,dy}{y^2}\right) - \frac{a^2}{2y^3}dy = 0 \quad \text{or} \quad d(yz) - ad\left(\frac{x}{y}\right) - \frac{a^2}{2}y^{-3}dy = 0.$$

Integrating, $yz - a(x/y) + (a^2/4y^2) = b$, a, b being arbitrary constants. ... (3)

Ex. 19(b). *Find the complete integral, general integral and the singular integral of* $2(z + xp + yq) = yp^2$ **[Delhi B.Sc. (H) 1998, 2008]**

Sol. Proceed as in solved Ex. 19(a) to get the complete integral (3).

General integral. Replacing b by $\phi(a)$ in (3), we get

$$yz - a(x/y) + (a^2/4y^2) = \phi(a) \quad \text{... (4)}$$

Differentiating (4) partially w.r.t. 'a', $-(x/y) + (a/2y^2) = \phi'(a)$ ... (5)

Then the general integral is obtained by eliminating a from (4) and (5).

Singular integral. Differentiating (3) partially w.r.t. 'a' and 'b' by turn, we get

$-(x/y) + (a/2y^2) = 0$ ... (6) $\qquad 0 = 1$ ... (7)

Relation (7) is absurd and hence there is no singular solution of the given equation.

Ex. 20. *Find a complete integral of* $z^2 = pqxy$. **[Delhi B.A. (Prog) II 2010]**

[Delhi Maths (H) 2004 Jabalpur 2004; Meerut 2006; Lucknow 2010]

Sol. The given equation is $f(x, y, z, p, q) = z^2 - pqxy = 0.$ ...(1)

Charpits's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-pqy+2pz}=\frac{dq}{-pqx+2qz}=\frac{dz}{-p(-qxy)-q(-pxy)}=\frac{dx}{qxy}=\frac{dy}{pxy}, \text{ by (1)} \quad ...(2)$$

Each fraction of (2) $= \dfrac{x\,dp+p\,dx}{x(-pqy+2pz)+pqxy} = \dfrac{y\,dq+q\,dy}{y(-pqx+2qz)+pqxy}$

or $$\frac{xdp+pdx}{2pxz}=\frac{ydq+qdy}{2qyz} \quad \text{or} \quad \frac{d(xp)}{xp}=\frac{d(yq)}{yq}.$$

Integrating, $\log(xp) = \log(yq) + \log a^2$ or $xp = a^2yq$. ...(3)

Solving (1) and (2) for p and q, $p = (az)/x$ and $q = z/(ay)$.

$\therefore\ dz = p\,dx + q\,dy = (az/x)\,dx + (z/ay)\,dy$ or $(1/z)\,dz = (a/x)\,dx + (1/ay)\,dy$.

Integrating, $\log z = a\log x + (1/a)\log y + \log b$ or $z = x^a y^{1/a}\, b$.

Ex. 21. *Using Charpit's method, find three complete integrals of* $pq = px + qy$.

(Kanpur 2004; Meerut 2002; Rajasthan 2001)

Sol. Here given equation is $f(x, y, z, p, q) = pq - px - qy = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_z}=\frac{dz}{-p\,f_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-p}=\frac{dq}{-q}=\frac{dz}{-p(q-x)-p(p-y)}=\frac{dx}{-(q-x)}=\frac{dy}{-(p-y)}, \text{ by (1)} \quad ...(2)$$

To find first complete integral. Taking the first two fractions of (2), we get

$(1/p)dp = (1/q)dq$ so that $\log p = \log q + \log a$ or $p = aq$. ...(3)

Using (3), $(1) \Rightarrow aq^2 = q(ax + y) \Rightarrow q = (ax + y)/a$. ...(4)

Hence, from (3), we have $p = ax + y$. ...(5)

$\therefore\ dz = p\,dx + q\,dy = (ax + y)dx + [(ax + y)/a]dy = (1/a)(ax + y)(a\,dx + y)$.

Putting $ax + y = t$ so that $a dx + dy = dt$, we get

$dz = (1/a) \times t\,dt$ so that $z = (1/2a) \times t^2 + b$ or $z = (1/2a) \times (ax + y)^2 + b$, as $t = ax + y$.

To find second complete integral. Taking the second and the fourth ratios in (2), we get

$dx/(q - x) = dq/q$ or $q\,dx + x\,dq = q\,dq$.

Integrating, $qx = q^2/2 + a/2$ or $q^2 - 2xq + a = 0$.

$\therefore\ q = [2x \pm 2(x^2 - a)^{1/2}]/2$ so that $q = x + (x^2 - a)^{1/2}$. ...(6)

Using (6), $(1) \Rightarrow p[x + (x^2 - a)^{1/2}] - px - y[x + (x^2 - a)^{1/2}] = 0$

so that $$p = \left\{1 + x/(x^2 - a)^{1/2}\right\} y. \quad ...(7)$$

$\therefore\ dz = p\,dx + q\,dy = \left\{1 + x/(x^2 - a)^{1/2}\right\} ydx + [x + (x^2 - a)^{1/2}]dy$

or $$dz = (y\,dx + x\,dy) + \left[\frac{xy\,dy}{(x^2-a)^{1/2}} + (x^2 - a)^{1/2}dy\right] \quad \text{or} \quad dz = d(xy) + d[y(x^2 - a)^{1/2}].$$

Integraing, $z = xy + y(x^2 - a)^{1/2} + b$, a, b being arbitrary constants.

To find third complete integral. Taking the first and the fifth ratios of (2) and proceeding as above third complete integral is $z = xy + x(y^2 - a)^{1/2} + b$.

Ex. 22. *Find complete integral of* $xp + 3yq = 2(z - x^2q^2)$. **[Delhi B.Sc. (Prog) II 2009; Delhi B.Sc. (Hons) II 2010;]**

Sol. Given equation is $f(x, y, z, p, q) = xp + 3yq - 2z + 2x^2q^2 = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{-p + 4xq^2} = \dfrac{dq}{q} = \dfrac{dz}{-p\,x - q(3y + 4x^2 q)} = \dfrac{dx}{-x} = \dfrac{dy}{-3y - 4x^2 q}$, by (1) ...(2)

(2) $\Rightarrow$ $\dfrac{dq}{q} = \dfrac{dx}{-x}$ $\Rightarrow$ $\log q = \log a - \log x$ $\Rightarrow$ $qx = a$ $\Rightarrow$ $q = \dfrac{a}{x}$. ...(3)

Using (3), (1) $\Rightarrow$ $xp + 3y(a/x) - 2z + 2x^2(a^2/x^2) = 0$ $\Rightarrow$ $p = \dfrac{2(z - a^2)}{x} - \dfrac{3ay}{x^2}$. ...(4)

$\therefore$ $dz = p\,dx + q\,dy = \left\{\dfrac{2(z - a^2)}{x} - \dfrac{3ay}{x^2}\right\} dx + \dfrac{a}{x} dy$

or $x^2 dz = 2x(z - a^2)dx - 3ay\,dx + ax\,dy$ or $x^2 dz - 2x(z - a^2)\,dx = -3ay\,dx + ax\,dy$

or $\dfrac{x^2 dz - 2x(z - a^2)dx}{x^4} = -\dfrac{3ay\,dx}{x^4} + \dfrac{a\,dy}{x^3}$ or $d\left(\dfrac{z - a^2}{x^2}\right) = d\left(\dfrac{ay}{x^3}\right)$

Integrating, $(z - a^2)/x^2 = (ay)/x^3 + b$ or $z = a(a + y/x) + bx^2$.

Ex. 23. *Find complete integrals of the following equations* :

(*i*) $(p^2 + q^2)^n (qx - py) = 1$.

(*ii*) $qx + py = (p^2 - q^2)^n$.

Sol. (*i*) Given equation is $f(x, y, z, p, q) = (p^2 + q^2)^n (qx - py) - 1 = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{q(p^2 + q^2)^n} = \dfrac{dq}{-p(p^2 + q^2)^n} = \ldots$ or $\dfrac{dp}{q} = \dfrac{dq}{-p}$ or $pdp + qdq = 0$.

Integrating, $p^2 + q^2 = \text{constant} = (1/a^2)$, say ...(2)

Using (2), (1) $\Rightarrow qx - py = a^{2n}$ or $qx = py + a^{2n}$. ...(3)

Using (3), (2) $\Rightarrow$ $p^2 + (p^2y^2 + a^{4n} + 2a^{2n}yp)/x^2 = 1/a^2$

or $p^2(x^2 + y^2) + 2a^{2n}yp + \{a^{4n} - (x^2/a^2)\} = 0$ so that

$$p = \frac{-ya^{2n} + \sqrt{\left\{a^{4n}y^2 - (x^2 + y^2)\left(a^{4n} - x^2/a^2\right)\right\}}}{x^2 + y^2} = \frac{-ya^{2n} + x\sqrt{\left[\left\{(x^2 + y^2)/a^2\right\} - a^{4n}\right]}}{x^2 + y^2} \quad \ldots(4)$$

$\therefore$ (3) $\Rightarrow$ $q = \dfrac{xa^{2n} + y\sqrt{\left[\left\{(x^2 + y^2)/a^2\right\} - a^{4n}\right]}}{x^2 + y^2}$. ...(5)

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = a^{2n}\left(\frac{x\,dy - y\,dx}{x^2 + y^2}\right) + \frac{x\,dx + y\,dy}{x^2 + y^2}\sqrt{\left\{\left(\frac{x^2 + y^2}{a^2}\right) - a^{4n}\right\}}.$$

Integrating, $z + b = a^{2n} \tan^{-1}\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\int \dfrac{1}{u}(ua^{-2} - a^{4n})^{1/2}\,du$, where $u = x^2 + y^2$.

Part (ii). Proceed as in part (*i*). If $u = x^2 + y^2$, then complete integral is

$$z + b = -\frac{1}{2}a^{2n}\log\frac{x-y}{x+y} - \frac{1}{2}\int\frac{1}{u}\sqrt{(a^{4n}+a^2u)}\,du.$$

Ex. 24. *Find complete integral of* $p^2 + q^2 - 2pq\ tanh\ 2y = sech^2\ 2y$.

Sol. Given $f(x, y, z, p, q) = p^2 + q^2 - 2pq \tanh 2y - \text{sech}^2\ 2y = 0.$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{0} = \frac{dq}{-4pq\,\text{sech}^2 2y + 4\,\text{sech}^2\,2y\,\tanh 2y} = \cdots, \text{by (1)}$$

Then, first fraction $\Rightarrow$ $dp = 0$ $\Rightarrow$ p = constant = a, say. ...(2)

Using (2), (1) $\Rightarrow$ $q^2 - (2a \tanh 2y)q + a^2 - \text{sech}^2\ 2y = 0$

$\Rightarrow$ $q = [2a \tanh 2y \pm 2\sqrt{(a^2\tanh^2 2y - a^2 + \text{sech}^2\ 2y)}\,]/2$

$\Rightarrow$ $q = a \tanh 2y + \sqrt{(1-a^2)}\ .\ \text{sech}\ 2y.$ ...(3)

[Note that $\text{sech}^2 2y = 1 - \tanh^2\ 2y$]

Using (2) and (3), $dz = p\,dx + q\,dy$ reduces to

$$dz = a\,dx + \{a \tanh 2y + \sqrt{(1-a^2)}\,\text{sech}\ 2y\}dy$$

Integrating, $$z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\int\frac{2dy}{e^{2y}+e^{-2y}}$$

or $$z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\int\frac{2e^{2y}dy}{1+(e^{2y})^2}$$

or $$z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\tan^{-1}(e^{2y}),$$

$$\left[\because \text{ on putting } e^{2y} = t \quad \text{and} \quad 2e^{2y}\,dy = dt, \quad \int\frac{2e^{2y}dy}{1+(e^{2y})^2} = \int\frac{dt}{1+t^2} = \tan^{-1}t = \tan^{-1}e^{2y}\right]$$

Ex. 25. *Find complete integral of the equation* $q = \{(1 + p^2)/(1 + y^2)\}x + yp(z - px)^2$.

Sol. Let $f(x, y, z, p, q) = \{(1 + p^2)/(1 + y^2)\}x + yp(z - px^2) - q = 0.$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{\{(1+p^2)/(1+y^2)\} - 2yp^2(z-px) + 2yp^2(z-px)} = \frac{dy}{1} = \ldots\ldots, \text{by (1)}$$

or $$\frac{dp}{1+p^2} = \frac{dy}{1+y^2} \quad \text{so that} \quad \tan^{-1}p - \tan^{-1}y = \text{constant} = \tan^{-1}a$$

$\Rightarrow$ $(p - y)/(1 + py) = a$ $\Rightarrow$ $p = (y + a)/(1 - ay).$ ...(2)

Using (2), (1) $\Rightarrow$ $q = \frac{1+a^2}{(1-ay)^2}x + \frac{y(y+a)}{(1-ay)^3}\{z\,(1-ay) - x(y+a)\}^2.$ ...(3)

Using (2) and (3), $dz = p\,dx + q\,dy$ reduces to

$$dz = \frac{y+a}{1-ay}dx + \left[\frac{1+a^2}{(1-ay)^2}x + \frac{y(y+a)}{(1-ay)^3}\{z(1-ay) - x(y+a)\}^2\right]dy$$

or $dz = d\left(\frac{y+a}{1-ay}x\right) + \frac{y(y+a)}{(1-ay)^3}\{z(1-ay) - x(y+a)\}^2 dy$ or $dz = du + \frac{y(y+a)}{1-ay}(z-u)^2 dy$, ...(4)

where $u = x(y + a)/(1 - ay)$. ...(5)

(4) $\Rightarrow$ $\frac{dz-du}{(z-u)^2} = \frac{y(y+a)}{1-ay}$dy. or $\frac{d(z-u)}{(z-u)^2} = \left\{-1 - \frac{1}{a^2}(1+ay) + \frac{a^2+1}{a^2}\frac{1}{1-ay}\right\} dy.$

Integrating, $b - \frac{1}{z-u} = -y - \frac{1}{a^2}\left(y + \frac{ay^2}{2}\right) - \frac{a^2+1}{a^3}\log(1 - ay)$, where u is given by (5).

Ex. 26. Find complete integral of $xp - yq = xq\, f(z - px - qy)$.

Sol. Let $F(x, y, z, p, q) = xp - yq - xq\, f(z - px - qy) = 0$. ...(2)

Charpit's auxiliary equations are

$$\frac{dp}{\partial F/\partial x + p(\partial F/\partial z)} = \frac{dq}{\partial F/\partial y + q(\partial F/\partial z)} = \frac{dz}{-p(\partial F/\partial p) - q(\partial F/\partial q)} = \frac{dx}{-(\partial F/\partial p)} = \frac{dy}{-(\partial F/\partial q)}$$

or $\frac{dp}{p - qf + xqpf' - pqxf'} = \frac{dq}{-q + xq^2 f' - xq^2 f'} = \cdots\cdots$, by (2) ...(3)

Each ratio of (3) $= \frac{x\,dp + y\,dq}{xp - yq - qxf} = \frac{x\,dp + y\,dq}{0}$, by (2)

$\Rightarrow$ $x\,dp + y\,dq = 0$ $\Rightarrow$ $x\,dp + y\,dq + p\,dx + q\,dy = p\,dx + q\,dy$

$\Rightarrow$ $dz - d(xp) - d(yq) = 0$, as $dz = pdx + qdy$

Integrating, $z - xp - yq = \text{constant} = a$, say ...(4)

$\therefore$ $xp + yq = z - a$. ...(5)

Using (4), (1) becomes $x\,p - y\,q = x\,q\,f(a)$. ...(6)

Subtracting (6) from (5), $2yq = z - a - xqf(a)$ $\Rightarrow$ $q = (z - a)/\{2y + xf(a)\}$ ...(7)

Using (7), (5) $\Rightarrow$ $p = \frac{(z-a)\{y + xf(a)\}}{x\{2y + xf(a)\}}$. ...(8)

Using (7) and (8), $dz = p\,dx + q\,dy$ reduces to

$$dz = (z - a)\left[\frac{\{y + xf(a)\}\,dx}{x\{2y + xf(a)\}} + \frac{dy}{2y + xf(a)}\right]$$

or $\frac{2dz}{z-a} = \frac{2y\,dx + 2xf(a)dx + 2x\,dy}{x\{2y + xf(a)\}} = \frac{2d(xy) + 2xf(a)\,dx}{2xy + x^2 f(a)}$.

Integrating, $2\log(z - a) = \log\{2xy + x^2 f(a)\} + \log b$ or $(z - a)^2 = b\{2xy + x^2 f(a)\}$.

Ex. 27. *Find a complete integral of* $px + qy = z(1 + pq)^{1/2}$

[Meerut 2001, 02; Kanpur 1995, I.A.S. 1992]

Sol. Given $f(x, y, z, p, q) = px + qy - z(1 + pq)^{1/2} = 0$. ...(1)

Charpit's auxiliary equation are $\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$

or $\frac{dp}{p - p(1+pq)^{1/2}} = \frac{dq}{q - q(1+pq)^{1/2}} = \ldots\ldots$ so that $\frac{dp}{p} = \frac{dq}{q}$, by (1)

$\Rightarrow$ $\log p = \log a + \log q$ $\Rightarrow$ $p = aq$. ...(2)

Using (2), (1) $\Rightarrow$ $q(ax + y) = z(1 + aq^2)^{1/2}$ or $q^2[(ax + y)^2 - az^2] = z^2$.

$\therefore\ q = \dfrac{z}{[(ax+y)^2 - az^2]^{1/2}}$ and $p = aq = \dfrac{az}{[(ax+y)^2 - az^2]^{1/2}}$.

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{z(a\,dx+dy)}{\sqrt{\{(ax+y)^2 - az^2\}}} \quad \text{or} \quad \frac{dz}{z} = \frac{a\,dx+dy}{\sqrt{\{(ax+y)^2 - az^2\}}}. \qquad \ldots(3)$$

Let $ax + y = \sqrt{a}\,u$ so that $a\,dx + dy = \sqrt{a}\,du$.

$$\therefore\ (3) \Rightarrow \frac{dz}{z} = \frac{\sqrt{a}\,du}{\sqrt{(au^2 - az^2)}} \quad \text{or} \quad \frac{du}{dz} = \frac{\sqrt{(u^2 - z^2)}}{z} = \sqrt{\left\{\left(\frac{u}{z}\right)^2 - 1\right\}}, \qquad \ldots(4)$$

which is linear homogeneus equation. To solve it, we put

$$\frac{u}{z} = v \quad \text{or} \quad u = vz \quad \text{so that} \quad \frac{du}{dz} = v + z\frac{dv}{dz}.$$

$\therefore$ (4) yields $v + z\dfrac{dv}{dz} = (v^2 - 1)^{1/2}$. or $\dfrac{dz}{z} = \dfrac{dv}{(v^2-1)^{1/2} - v}$

or $(1/z)dz = -\left[(v^2-1)^{1/2} + v\right]dv$, on rationalization.

Integrating, $\log z = -\left[\dfrac{v}{2}(v^2-1)^{1/2} - \dfrac{1}{2}\log\{v + (v^2-1)^{1/2}\right] - \dfrac{v^2}{2} + b$, where, $v = \dfrac{u}{z} = \dfrac{ax+y}{z\sqrt{a}}$

Ex. 28. *Find complete integral of* $(x^2 - y^2)\,pq - xy(p^2 - q^2) = 1$.

Sol. Let $f(x, y, z, p, q) = (x^2 - y^2)\,pq - xy(p^2 - q^2) - 1 = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{2pqx - z(p^2-q^2)} = \dfrac{dq}{-2pqy - x(p^2-q^2)} = \dfrac{dx}{-(x^2-y^2)y + 2pxy} = \dfrac{dy}{-(x^2-y^2)p - 2pxy}$, by (1)

Using x, y, p, q as multipliers, each fraction $= \dfrac{x\,dp + y\,dq + p\,dx + q\,dy}{0} = \dfrac{d(xp) + d(yq)}{0}$

$\Rightarrow\ d(xp + qy) = 0 \Rightarrow xp + yq = a \Rightarrow p = (a - qy)/x.$ ...(2)

Using (2), $(1) \Rightarrow (x^2 - y^2)\left(\dfrac{a - qx}{x}\right)q - xy\left[\left(\dfrac{a-qy}{x}\right)^2 - q^2\right] - 1 = 0$

or $\dfrac{a-qy}{x}\{(x^2-y^2)q - (a-qy)y\} + xyq^2 - 1 = 0$ or $\{(a-qy)/x\}(x^2q - ay) + xyq^2 - 1 = 0$

or $(a - qy)(x^2q - ay) + x^2yq^2 - x = 0$ or $aq(x^2 + y^2) = a^2y + x$

$\therefore\ q = \dfrac{a^2y + x}{a(x^2+y^2)}$ and $p = \dfrac{1}{x}\left[a - \dfrac{(a^2y+x)y}{a(x^2+y^2)}\right] = \dfrac{a^2x - y}{a(x^2+y^2)}$.

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{(a^2x - y)dx + (a^2y + x)dy}{a(x^2+y^2)} = a\frac{x\,dx + y\,dy}{x^2+y^2} + \frac{x\,dy - y\,dx}{a(x^2+y^2)}.$$

Integrating, $z = (a/2) \times \log(x^2 + y^2) + (1/a) \times \tan^{-1}(y/x) + b$.

Ex. 29. *Find a complete integral of* $2(pq + yp + qx) + x^2 + y^2 = 0$. **[Kanpur 1993]**

Sol. Given equation is $f(x, y, z, p, q) = 2(pq + yp + qx) + x^2 + y^2 = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\dfrac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\dfrac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\dfrac{dx}{-\frac{\partial f}{\partial p}}=\dfrac{dy}{-\frac{\partial f}{\partial q}}$

or $\dfrac{dp}{2q+2x}=\dfrac{dq}{2p+2y}=\dfrac{dz}{-p(2q+2y)-q(2p+2x)}=\dfrac{dx}{-(2q+2y)}=\dfrac{dy}{-(2p+2x)}$, by (1)

Each of these above fractions $= \dfrac{dp+dq+dx+dy}{(2q+2x)+(2p+2y)-(2q+2y)-(2p+2x)}$

$= (dp + dq + dx + dy)/0$

This $\Rightarrow$ $dp + dq + dx + dy = 0$ so that $(p + x) + (q + y) = a$. ...(2)

Re–writing (1), $2(p + x)(q + y) + (x - y)^2 = 0$ or $(p + x)(q + y) = -(x - y)^2/2$. ...(3)

Now, $(p + x) - (q + y) = \sqrt{\{(p+x)^2+(q+y)\}^2-4(p+x)(q+y)}$

$\therefore$ $(p + x) - (q + y) = \sqrt{a^2+2(x-y)^2}$, using (2) and (3) ...(4)

Adding (2) and (4), $2(p + x) = a + \sqrt{a^2+2(x-y)^2}$.

Substracting (4) from (2), $2(q + y) = a - \sqrt{a^2+2(x-y)^2}$.

These give $p = -x + \dfrac{a}{2}+\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}$, $q = -y + \dfrac{a}{2}-\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}$

Substituting the above values of p and q, $dz = p\,dx + q\,dy$ becomes

$$dz = -(x\,dx + y\,dy) + (a/2) \times (dx + dy) + (1/2)\times\sqrt{a^2+2(x-y)^2}\ (dx - dy)$$

or $$dz = -\frac{1}{2}d(x^2 + y^2) + \frac{a}{2}d(x + y) + \sqrt{2}\times\frac{1}{2}\sqrt{\frac{a^2}{2}+(x-y)^2}\ d(x - y) \quad ...(5)$$

Put $x - y = t$ so that $d(x - y) = dt$. Then (5) becomes

$$dz = -(1/2) \times d(x^2 + y^2) + (a/2) \times d(x + y) + (1/\sqrt{2}) \times \sqrt{\left(a/\sqrt{2}\right)^2 + t^2}\ dt.$$

$$\therefore\quad z = -\frac{x^2+y^2}{2} + a\frac{x+y}{2} + \frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{\left(a/\sqrt{2}\right)^2+t^2}\ + \frac{\left(a/\sqrt{2}\right)^2}{2}\log\left\{t+\sqrt{\left(a/\sqrt{2}\right)^2+t^2}\right\}\right] + b$$

Putting the value of t, the required complete integral is

$$z = -\frac{x^2+y^2}{2} + \frac{a(x+y)}{2} + \frac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\frac{a^2}{2}+(x-y)^2}\ + \frac{a^2}{2}\log\left\{x-y+\sqrt{\frac{a^2}{2}+(x-y)^2}\right\}\right] + b.$$

Ex. 30. *Solve* $z = (1/2) \times (p^2 + q^2) + (p - x)(q - y)$ **[I.A.S. 2002]**

Sol. Given $z = (1/2) \times (p^2 + q^2) + (p - x)(q - y)$

Re–writing (1), $f(x, y, z, p, q) = (1/2) \times (p^2 + q^2) + pq - xq - yp + xy - z = 0$ ... (2)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{-q+z-p}=\frac{dq}{-p+x-q}=\frac{dz}{-p(p+q-y)-q(p+q-x)}=\frac{dx}{-(p+q-y)}=\frac{dy}{-(p+q-x)}$$

Taking the first and the fourth fractions, we have

$dp = dx$ so that $p = x + a$, a being an arbitratry constant. ... (3)

Taking the second and the fifth fractions, we have

$dq = dy$ so that $q = y + b$, b being an arbitrary constant ... (4)

Putting $p = x + a$ and $q = y + b$ in (1), the required solution is

$z = (1/2) \times \{(x+a)^2 + (y+b)^2\} + ab$, a and b being arbitrary constants.

Ex. 31 *Find a complete integral of* $z = pq$. **[Sagar 2004, Ravishankar 2003, Rewa 2003]**

Sol. Here given equation is $f(x, y, z, p, q) = z - pq = 0$ ... (1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z}=\frac{dq}{f_y + q f_z}=\frac{dz}{-p f_p - q f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-p}=\frac{dq}{-q}=\frac{dz}{-2pq}=\frac{dx}{-q}=\frac{dy}{-p}, \text{ by (1)} \quad \text{... (2)}$$

Taking the first and last fractions of (2), $dp = dy$

Integrating $p = y + a$, a being an arbitrary constant ... (3)

Similarly, taking the second and fourth fractions of (2), we get

$dq = dx$ so that $q = x + b$, b being an arbitrary constant. ... (4)

Putting values of p and q given by (3) and (4) in (1), we get

$z = (x + b)(x + a)$, which is the required complete integral.

Ex. 32. *Use Charpit's method to find the complete integral of* $2x\{z^2(\partial z/\partial y)^2 + 1\} = z(\partial z/\partial x)$. **[I.A.S. 1998]**

Sol. Given $2x(z\partial z/\partial y)^2 + 2x - (z\partial z/\partial x) = 0$ ... (1)

Let $z\,dz = dZ$ so that $z^2 = 2Z$ ... (2)

Then (1) becomes $2x(\partial Z/\partial y)^2 + 2x - (\partial Z/\partial x) = 0$ or $2xQ^2 + 2x - P = 0$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$ ... (3)

Let $f(x, y, Z, P, Q) = 2x\,Q^2 + 2x - P = 0$ ... (4)

Charpit's auxiliary equations are $$\frac{dP}{f_x + P f_Z}=\frac{dQ}{f_y + Q f_Z}=\frac{dZ}{-Pf_P - Q f_Q}=\frac{dx}{-f_P}=\frac{dy}{-f_Q}$$

giving $$\frac{dP}{2Q^2+2}=\frac{dQ}{O} = \ldots, \text{ by (4)}$$ so that $dQ = 0$.

Integrating, $Q = a$, a being an arbitrars constant ... (5)

Using $Q = a$, (4) gives $P = 2x(a^2 + 1)$, $Q = a$ ... (6)

$\therefore$ $dZ = P\,dx + Q\,dy = 2x(a^2 + 1)dx + a\,dy$, by (5) and (6)

Integrating, $Z = x^2(a^2 + 1) + ay + b/2$, or $z^2/2 = x^2(a^2 + 1) + ay + b/2$, using (2)

or $z^2 = 2x^2(a^2+1) + 2ay + b$, which is complete integral of (1)

Ex. 33. *Solve by Charpit's method the partial differential equation.*

$$p^2x(x-1) + 2pqxy + q^2y(y-1) - 2pxz - 2qyz + z^2 = 0 .$$ **[I.A.S. 2000]**

Sol. Let $f(x, y, z, p, q) = p^2x(x-1) + 2pqxy + q^2y(y-1) - 2pxz - 2qyz + z^2 = 0$ ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$ ... (2)

From (1), $f_x = p^2(2x-1) + 2pqy - 2pz$, $f_y = 2pqx + q^2(2y-1) - 2qz$,

$f_z = -2px - 2qy + 2z$, $f_p = 2px(x-1) + 2qxy - 2xz$; $f_q = 2pxy + 2qy(y-1) - 2yz$

and so $f_x + p\,f_z = -p^2$, $f_y + q\,f_z = -q^2$. Then (2) becomes

$$\frac{dp}{-p^2} = \frac{dq}{-q^2} = \frac{dz}{-p\{2px(x-1) + 2qxy - 2xz\} - q\{2pxy + 2qy(y-1) - 2yz\}}$$

$$= \frac{dx}{-(2px^2 - 2px + 2qxy - 2xz)} = \frac{dy}{-(2pxy + 2qy^2 - 2qy - 2yz)} \quad \text{... (3)}$$

$$\text{Each fraction of (3)} = \frac{(1/p)\,dp}{-p} = \frac{(1/q)\,dq}{-q} = \frac{(1/p)\,dp - (1/q)dq}{-p+q} \quad \text{... (4)}$$

$$\text{Also, each fraction of (3)} = \frac{(1/x)\,dx - (1/y)\,dy}{-2px + 2p - 2qy + 2z + 2px + 2qy - 2q - 2z} \quad \text{... (5)}$$

$\therefore$ (4) and (5) $\Rightarrow$ $\dfrac{(1/p)\,dp - (1/q)\,dq}{-(p-q)} = \dfrac{(1/x)\,dx - (1/y)\,dy}{2(p-q)}$

or $(1/2) \times \{(1/x)\,dx - (1/y)dy\} = (1/q)\,dq - (1/p)\,dp$

Integrating, $(1/2) \times \{\log x - \log y\} = \log q - \log p + \log a$ or $(x/y)^{1/2} = aq/p$

or $p = (ay^{1/2}q)/x^{1/2}$, a being an arbitrary constant. ... (5)

Re–writing (1), $(px + qy - z)^2 = p^2x + q^2y$ or $px + qy - z = \pm(p^2x + qy^2)^{1/2}$ ... (6)

Taking + ive sign in (7), $px + qy - z = (p^2x + q^2y)^{1/2}$ ... (7)

[The case of – ve sign in (7) can be discussed similarly]

Substituting the value of p given by (6) in (8), $aqy^{1/2}x^{1/2} + qy - z = (a^2q^2y + q^2y)^{1/2}$

or $q\{y + a(xy)^{1/2} - (1+a^2)^{1/2}y^{1/2}\} = z$ so that $q = z/y^{1/2}\{y^{1/2} + a\,x^{1/2} - (1+a^2)^{1/2}\}$ ... (9)

Then (6) gives $p = az/x^{1/2}\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}$ ... (10)

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{az\,dx}{x^{1/2}\{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}} + \frac{z\,dy}{y^{1/2}\{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}}$$

or $$\frac{dz}{z} = \frac{a y^{1/2} dx + x^{1/2} dy}{(xy)^{1/2} \{y^{1/2} + a x^{1/2} - (1+a^2)^{1/2}\}}$$

Integrating, $$\log z = 2 \log \{y^{1/2} + a x^{1/2} - (1+a^2)^{1/2}\} + \log b$$

or $z = b \{y^{1/2} + a x^{1/2} - (1+a^2)^{1/2}\}^2$, a and b being an arbitrary constants.

Ex. 34. *Find the complete integral of* $(p+q)(px+qy) = 1$.

[Meerut 2007; Delhi Maths (H) 2007, Purvanchal 2007]

Sol. Let $$f(x, y, z, p, q) = (p+q)(px+qy) - 1 = 0 \quad \ldots (1)$$

Charpit's auxiliary equations $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

give $$\frac{dp}{p(p+q)} = \frac{dq}{q(p+q)} = \ldots$$ so that $$\frac{dp}{p} = \frac{dq}{q}$$, using (1)

Integrating, $p = aq$, a being an arbitrary constant ... (2)

Putting $p = aq$ in (2) gives $(aq+q)(aqx+qy) - 1 = 0$ or $q^2 (1+a)(ax+y) = 1$ ... (3)

$\therefore$ From (2) and (3), $q = 1/(1+a)^{1/2} (ax+y)^{1/2}$, $p = a/(1+a)^{1/2} (ax+y)^{1/2}$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{a\,dx}{(1+a)^{1/2}(ax+y)^{1/2}} + \frac{dy}{(1+a)^{1/2}(ax+y)^{1/2}} = \frac{d(ax+y)}{(1+a)^{1/2}(ax+y)^{1/2}}$$

Integrating, $z(1+a)^{1/2} = 2(ax+y)^{1/2} + b$, a, b being arbitrary constants.

Ex. 35. *Find the complete integral of the following partial differential equations*

(*a*) $px^5 - 4q^2x^2 + 6x^2z - 2 = 0$. **[Delhi B.Sc. (H) 2002; Delhi B.A. (Proj) II 2011]**

(*b*) $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$

Sol. (a) Let $$f(x, y, z, p, q) = px^5 - 4q^2x^2 + 6x^2z - 2 = 0 \quad \ldots (1)$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_y} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{5px^4 - 8q^2x + 12xz + 6px^2} = \frac{dq}{6qx^2} = \frac{dz}{-px^5 + 8q^2x^2} = \frac{dx}{-x^5} = \frac{dy}{8qx^2}$$, by (1)

Taking the second and the last fractions, $4\,dq = 3dy$

Integrating, $4q = 3y + 3a$ or $q = 3(y+a)/4$ ... (2)

Using (2), (1) gives $p = \{(9/4) \times (y+a)^2 - 6x^2z + 2\}/x^5$ ... (3)

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (9/4x^3)(y+a)^2\,dx - (6z/x^3)\,dx + (2/x^5)dx + (3/4)(y+a)dy$$

or $$(6z/x^3)dx + dz = \{(9/4x^3)(y+a)^2 dx + (3/4)(y+a)\,dy\} + (2/x^5)dx \quad \ldots (4)$$

The total differential equation (4) is always integrable. To solve (4), we first proceed to find the integrating factor of the L.H.S. of (4). Comparing L.H.S. of (4) with $M\,dx + N\,dz$ (here on L.H.S. we have variable x, z in place of usual variables x, y), we have $M = 6z/x^3$ and $N = 1$.

$\frac{1}{N}\left(\frac{\partial M}{\partial z} - \frac{\partial N}{\partial x}\right) = \frac{6}{x^3}$, which is function x alone and so I.F. = $e^{\int (6/x^3)dx} = e^{-3/x^2}$.

Multiplying both sides of (4) by I.F. e^{-3/x^2} , we get

$$(6z/x^3)\, e^{-3/x^2} dx + e^{-3/x^2} dz = (3/8) \times \{(6/x^3)\,(y+a)^2\, e^{-3/x^2} dx + 2(y+a)\, e^{-3/x^2} dy\} + (2/x^5)\, e^{-3/x^2} dx$$

or $$d(z\, e^{-3/x^2}) = (3/8) \times d\{(y+a)^2\, e^{-3/x^2}\} + (2/x^5) \times e^{-3/x^2} dx$$

Integrating, $$z\, e^{-3/x^2} = (3/8) \times (y+a)^2\, e^{-3/x^2} + 2\int (1/x^2)\, e^{-3/x^2} (1/x^3) dx$$

or $z\, e^{-3/x^2} = (3/8) \times (y+a)^2\, e^{-3/x^2} - (1/9) \times \int u\, e^u du$, putting $(-3/x^2) = u$ so that $(6/x^3)dx = du$

or $$z\, e^{-3/x^2} = (3/8) \times (y+a)^2\, e^{-3/x^2} - (1/9) \times (ue^u - e^u) + b$$

or $$z\, e^{-3/x^2} = (3/8) \times (y+a)^2\, e^{-3/x^2} - (1/9) \times (-3/x^2) e^{-3/x^2} + (1/9) \times e^{-3/x^2} + b$$

or $$z = (3/8) \times (y+a)^2 + (1/3x^2) + (1/9) + b e^{3/x^2}$$, a, b being arbitrary constants.

(b) Proceed exacts as in part (a) **Ans.** $z = (2/3) \times (y+a)^{3/2} + (1/3x^2) + (1/9) + b\, e^{3/x^2}$

Ex. 36. *Find the complete integral of* $(p+y)^2 + (q+x)^2 = 1$

Sol. Let $$f(x, y, z, p, q) = (p+y)^2 + (q+x)^2 - 1 = 0 \quad \ldots (1)$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\, f_z} = \frac{dq}{f_y + q\, f_z} = \frac{dz}{-pf_p - q\, f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{2(q+x)} = \frac{dq}{2(p+y)} = \frac{dz}{-2(p^2 + q^2 + py + qx)} = \frac{dx}{-2(p+y)} = \frac{dy}{-2(q+x)}$$, by (1)

Taking the first and the last fractions, $dp + dy = 0$ so that $p + y = a$ ... (2)

Using (2), (1) gives $a^2 + (q+x)^2 - 1 = 0$ or $q + x = (1 - a^2)^{1/2}$ ... (3)

Using (2) and (3) in $dz = p\,dx + q\,dy$, we get

$$dz = (a-y)dx + \{(1-a^2)^{1/2} - x\}dy = a\,dx - (1-a^2)^{1/2} dy - (y\,dx + x\,dy)$$

Integrating, $z = ax - (1-a^2)^{1/2} y - xy + b$, a, b being arbitrary constants.

Ex. 37. *Find the complete integral of* $2(y + zq) = q(xp + yq)$ **[Nagpur 2003, 06; Delhi B.Sc. (Prog) II 2011; Delhi B.Sc. (Hons) 2011]**

Sol. Let $$f(x, y, z, p, q) = 2y + 2zq - xpq - yq^2 = 0 \quad \ldots (1)$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\, f_z} = \frac{dq}{f_y + q\, f_z} = \frac{dz}{-p\, f_x - q\, f_y} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \ldots (2)$$

$$\frac{dp}{-pq+2pq}=\frac{dq}{2-q^2+2q^2}=\frac{dz}{2pqx+2qy-2qz}=\frac{dx}{qx}=\frac{dy}{xp+2yq-2z}, \text{ by (1)}$$

Taking the first and fourth fractions, $(1/pq)dp = (1/qx)dx$ or $(1/p)dp = (1/x)dx$

Integrating, $\log p = \log a + \log x$ or $p = ax$, ... (3)

where a is an arbitrary constant. Substituting the value of p given by (3) in (1), we have

$2y + 2zq - ax^2q - yq^2 = 0$ or $yq^2 + q(ax^2 - 2z) - 2y = 0.$

$$\Rightarrow \qquad q=[-(ax^2-2z)\pm\{(ax^2-2z)^2+8y^2\}^{1/2}]/(2y) \qquad \text{... (4)}$$

Substituting the values of p and q given by (3) and (4) in $dz = p\,dx + q\,dy$, we obtain

$$dz=ax\,dx+(1/2y)\times[2z-ax^2\pm\{(2z-ax^2)^2+8y^2\}^{1/2}]dy$$

or
$$\frac{2dz-2ax\,dx}{(2z-ax^2)\pm\{(2z-ax^2)^2+8y^2\}^{1/2}}=\frac{dy}{y} \qquad \text{... (5)}$$

Putting $2z - ax^2 = u$ and $2dz - 2\,ax\,dx = du$, (5) yields

$$\frac{du}{u\pm(u^2+8y^2)^{1/2}}=\frac{dy}{y} \quad \text{or} \quad \frac{du}{dy}=\frac{u}{y}\pm\left\{\left(\frac{u}{y}\right)^2+8\right\}^{1/2}, \qquad \text{... (6)}$$

which is linear homogeneous differential equation. To solve it, we put $u/y = v$, *i.e.*, $u = yv$ so that $du/dy = v + y(dv/dy)$ and so (6) reduces to

$$v+y\frac{dv}{dy}=v\pm(v^2+8)^{1/2} \quad \text{or} \quad \frac{dv}{(v^2+8)^{1/2}}=\frac{dy}{y},$$

taking positive sign. Integrating it, we have

$\log\{v+(v^2+8)^{1/2}\} = \log y + \log b$ or $v+(v^2+8)^{1/2} = by$

or $u/y+\left\{(u/y)^2+8\right\}^{1/2}=by$ or $u+(u^2+8y^2)^{1/2}=by^2$

or $2z-ax^2+\{(2z-ax^2)^2+8y^2\}^{1/2}=by^2$, as $u = 2z - ax^2$; a, b being arbitrary constants

EXERCISE 3(B)

Using Charpit's method, find a complete integral of the following equations :

1. $z = px + qy + pq$. **[Mysore 2004]** **Ans.** $z = ax + by + ab$

2. $pq = xz$. **Ans.** $z = (a + x^2/2)\,(b + y)$

3. $p^2 + px + q = z$. **Ans.** $z = ax + a^2 + be^y$

4. $(p + q)\,(z - px - qy) = 1$. **Ans.** $(a + b)\,(z - ax - by) = 1$

5. $px + qy + pq = 0$ **Ans.** $az = -(1/2) \times (y + ax)^2 + b$

6. $q = px + q^2$ **Ans.** $z = (a - a^2) \log x + ay + b$

7. $p - 3x^2 = q^2 - y$ **Ans.** $z = x^3 - (1/3) \times (a - x)^3 + ay - xy + b$

8. $x^2p^2 + y^2q^2 = 4$ **Ans.** $z = a \log x + (4 - a^2)^{1/2} \log y + b$

9. $xpq + yq^2 = 1$ **Ans.** $(z + b)^2 = 4(ax + b)$

10. $p + q = 3pq$ **Ans.** $az = b - (1/2) \times (y + ax)^2$

11. $pq + x\,(2y + 1)p + (y^2 + y)q - (2y + 1)z = 0$ **Ans.** $z = ax + b(a + y + y^2)$

12. $z^2(p^2 + q^2) = x^2 + e^{2y}$.

Ans. $$\frac{z^2}{2}=\frac{x\sqrt{(x^2+a)}}{2}+\frac{a}{2}\sinh^{-1}\frac{x}{\sqrt{a}}+\sqrt{(e^{2y}-a)}-\sqrt{a}\tan^{-1}\left(\frac{e^{2y}-a}{a}\right)+b$$

13. $p^2 - y^2q = x^2 - y^2$ **[Madurai Kamraj 2008]**

Ans. $z = \dfrac{x\sqrt{(x^2+b)}}{2} + \dfrac{b}{2}\sinh^{-1}\dfrac{x}{\sqrt{b}} - \dfrac{b}{2y^2} + \log y + c$

14. $p^2q\,(x^2 + y^2) = p^2 + q$ **Ans.** $z = \log[x + \sqrt{(x^2+a)}] + \dfrac{1}{2\sqrt{a}}\log\dfrac{y-\sqrt{a}}{y+\sqrt{a}} + b$

15. $yp = 2xy + \log q$. **[Lucknow 2010]** **[Ans.** $z = (a+2x)^2/4 + (1/a)\times e^{ay} + b$ **]**

3.9. Special methods of solutions applicable to certain standard forms:

We now consider equations in which p and q occur other than in the first degree, that is non–linear equations. We have already discussed the general method (*i.e.*, Charpit's method — see Art. 3.7). We now discuss four standard forms to which many equations can be reduced, and for which a complete integral can be obtained by inspection or by other shorter methods.

3.10. Standard Form I. Only p and q present. [Nagpur 2002; Bhopal 2010]

Under this standard form, we consider equations of the form $f(p, q) = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

giving $\dfrac{dp}{0} = \dfrac{dq}{0}$, by (1)

Taking the first ratio, $dp = 0$ so that p = constant = a, say ...(2)

Substituting in (1), we get $f(a, q) = 0$, giving q = constant = b, say, ... (3)

where b is such that $f(a, b) = 0$. ...(4)

Then, $dz = p\,dx + q\,dy = adx + bdy$, using (2) and (3).

Integrating, $z = ax + by + c$, ...(5)

where c is an arbitrary constant. (5) together with (4) give the required solution.

Now solving (4) for b, suppose we obtain $b = F(a)$, say.

Putting this value of b in (5), the *complete integral* of (1) is

$$z = ax + yF(a) + c, \qquad ...(6)$$

which contains two arbitrary constants a and c which are equal to the number of independent variables, namely x and y.

The *singular integral* of (1) is obtained by eliminating a and c between the complete integral (6) and the equations obtained by differentiating (6) partially w.r.t. a and c ; *i.e.*, between

$z = ax + yF(a) + c$, $0 = x + yF'(a)$ and $0 = 1$. ...(7)

Since the last equation in (7) is meaningless, we conclude that the equations of standard form I have no singular solution.

In order to find the general integral of (1), we first take $c = \phi(a)$ in (6), ϕ being an arbitrary function and obtain $z = ax + yF(a) + \phi(a)$. ...(8)

Now, we differentiate (8) partially with respect to a and get

$$0 = x + yF'(a) + \phi'(a). \qquad ....(9)$$

Eliminating a between (8) and (9), we get the general solution of (1).

Remark. Sometimes change of variables can be employed to transform a given equation to standard form I.

3.11. SOLVED EXAMPLES BASED ON ART. 3.10

Ex. 1. (a) *Solve* $pq = k$, *where k is a constant.* **[M.S. Univ. T.N. 2007; Meerut 1995]**

(b) Solve pq = 1 by standard from I **[Bhopal 2010]**

Sol. Given that $pq = k.$ ...(1)

Since (1) is of the form $f(p, q) = 0$, its solution is $z = ax + by + c,$ ...(2)

where $ab = k$ or $b = k/a$, on putting a for p and b for q in (1).

$\therefore$ From (2), the complete integral is $z = ax + (k/a)y + c,$ ...(3)

which contains two arbitrary constants a and c.

For singular solution, differentiating (3) partially with respect to a and c, we get $0 = x - (k/a^2)y$ and $0 = 1$. But $0 = 1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get

$$z = ax + (k/a)y + \phi(a). \quad ...(4)$$

Differentiating (4) partially with respect to 'a', we get $0 = x - (k/a^2)y + \phi'(a).$ ...(5)

Eliminating a from (4) and (5), we get the required general solution.

(b) Do like part (a) taking $k = 1$

Ex. 2. *Solve* (a) $p^2 + q^2 = m^2$, *where m is a constant.* **[Kanpur 1993]**

(b) $p^2 + q^2 = 1$ **[Meerut 2011]**

Sol. (a) Given that $p^2 + q^2 = m^2.$ ...(1)

Since (1) is of the form $f(b, q) = 0$, its solution is $z = ax + by + c,$ ...(2)

where $a^2 + b^2 = m^2$ or $b = (m^2 - a^2)^{1/2}$, on putting a for p and b for b in (1).

$\therefore$ From (2), the complete integral is $z = ax + y(m^2 - a^2)^{1/2} + c,$ ...(3)

which contains two arbitrary constants a and c.

For singular solution, differentiating (3) partially with respect to a and c, we get $0 = x - ay/(m^2 - a^2)^{1/2}$ and $0 = 1$. But $0 = 1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get

$$z = ax + y(m^2 - a^2)^{1/2} + \phi(a). \quad ...(4)$$

Differentiating (4) partially wit respect to 'a', we get

$$0 = a - ay/(m^2 - a^2)^{1/2} + \phi'(a). \quad ...(5)$$

Eliminating a from (4) and (5), we get the required general solution.

(b) **Hint.** Do like part (a) by taking $m = 1$

EQUATIONS REDUCIBLE TO STANDARD FORM I

Ex. 3. *Find the complete integral of* $z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0.$

Sol. The given equation can be rewritten as

$$z^2y\,(\partial z/\partial x)^2 + 6zxy(\partial z/\partial x) + 2zx^2(\partial z/\partial y) + 4x^2y = 0$$

or $$\left(\frac{z\,\partial z}{x\,\partial x}\right)^2 + 6\left(\frac{z\,\partial z}{x\,\partial x}\right) + 2\left(\frac{z\,\partial z}{y\,\partial y}\right) + 4 = 0, \text{ dividing by } x^2y \quad ...(1)$$

Put $x\,dx = dX,$ $y\,dy = dY$ and $z\,dz = dZ.$ ...(2)

so that $x^2/2 = X,$ $y^2/2 = Y$ and $z^2/2 = Z.$ ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + 6\,(\partial Z/\partial X) + 2(\partial Z/dY) + 4 = 0$

or $P^2 + 6P + 2Q + 4 = 0,$ where $P = \partial Z/\partial X,$ $Q = \partial Z/\partial Y.$ ...(4)

Equation (4) is of the form $f(P, Q) = 0$. Note that now we have P, Q, X, Y, Z in place of p, q, x, y, z in usual equations. Accordingly, solution of (4) is

$$Z = aX + bY + c, \qquad ...(5)$$

where $a^2 + 6a + 2b + 4 = 0$ or $b = -(a^2 + 6a + 4)/2$, on putting a for P and b for Q in (4). So, from (5), the required complete integral is

$Z = aX - \{(a^2 + 6a + 4)/2\}Y + c$, where a and c are arbitrary constants.

or $z^2/2 = a(x^2/2) - (a^2 + 6a + 4) \times (y^2/4) + c$, using (3)

or $z^2 = ax^2 - (2 + 3a + a^2/2)y^2 + c'$, where $c' = 2c$.

Ex. 4. *Find the complete integral of*

(*i*) $x^2p^2 + y^2q^2 = z$ **[Delhi Maths (H) 2004]**

(*ii*) $p^2x + q^2y = z$. **[Meerut 1994]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{x^2}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{x\,\partial z}{\sqrt{z}\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{\sqrt{z}\,\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $(1/x)dx = dX$, $(1/y)dy = dY$ and $(1/\sqrt{z})dz = dZ$ ...(2)

so that $\log x = X$, $\log y = Y$ and $2\sqrt{z} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P,Q) = 0$.

$\therefore$ solution of (4) is $Z = aX + bY + c$, ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $2\sqrt{z} = a \log x + \log y \,.\, \sqrt{1-a^2} + c$, by (3)

or $\log x^a + \log y^{\sqrt{1-a^2}} - \log c' = 2\sqrt{z}$, taking $c = -\log c'$

or $\log \{x^a y^{\sqrt{1-a^2}}/c'\} = 2\sqrt{z}$ or $x^a y^{\sqrt{1-a^2}} = c' e^{2\sqrt{z}}$

where a and c' are two arbitrary constants.

(*ii*) The given equation can be re–written as

$$\frac{x}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{\sqrt{x}}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\sqrt{y}}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $\left(1/\sqrt{x}\right)dx = dX$, $\left(1/\sqrt{y}\right)dy = dY$ and $\left(1/\sqrt{z}\right)dz = dZ$ ...(2)

so that $2\sqrt{x} = X$, $2\sqrt{y} = Y$ and $2\sqrt{z} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $z = aX + bY + c$, ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $2\sqrt{z} = 2a\sqrt{x} + 2\sqrt{y}\sqrt{1-a^2} + c$, by (3)

where a and c are two arbitrary constants.

Ex. 5. *Solve* $x^2p^2 + y^2q^2 = z^2$. **[Jabalpur 2000, 03; Gulbarga 2005; Bilaspur 1997; Meerut 2008, Sagar 2004, Vikram 1999; Ravi Shanker 1994, 96; Rohitkhand 2004]**

Sol. The given equation can be rewritten as

$$\frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{x\,\partial z}{z\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{z\,\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $(1/x)dx = dX$, $(1/y)dy = dY$ and $(1/z) = dZ$ ...(2)

so that $\log x = X$, $\log y = Y$ and $\log z = Z$. ...(3)

Using (2), (1) becomes $(dZ/dX)^2 + (dZ/dY)^2 = 1$ or $P^2 + Q^2 = 1$, ... (4)

where $P = dZ/dX$ and $Q = dZ/dy$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $Z = aX + bY + c$, ... (5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $Z = X\cos\alpha + Y\sin\alpha + \log c'$, taking $a = \cos\alpha$ and $c = \log c'$

or $\log z = \cos\alpha \log x + \sin\alpha \log y + \log c'$ or $z = c'\, x^{\cos\alpha}\, y^{\sin\alpha}$. ...(6)

To determine singular integral. Differentiating (6) partially w.r.t. α and c' successively, we obtain

$$0 = c'\cos\alpha \,.\, x^{\cos\alpha}\, y^{\sin\alpha} \log y - c'\sin\alpha \,.\, x^{\cos\alpha}\, y^{\sin\alpha} \log x \qquad ...(7)$$

and

$$0 = x^{\cos\alpha}\, y^{\sin\alpha}. \qquad ...(8)$$

Eliminating α and c' from (6), (7) and (8), the singlar solution is $z = 0$.

To determine general integral. Putting $c' = \phi(\alpha)$, where ϕ is an arbitrary function, (4) gives

$$z = \phi(\alpha)\sin\alpha\, x^{\cos\alpha}\, y^{\sin\alpha}. \qquad ...(9)$$

Differentiating (9), partially, w.r.t. 'α', we get

$$0 = \phi'(\alpha)\, x^{\cos\alpha}\, y^{\sin\alpha} + \phi(\alpha)\,\{x^{\cos\alpha}\, y^{\sin\alpha}\cos\alpha - y^{\sin\alpha}\, x^{\cos\alpha}\sin\alpha\}. \qquad ...(10)$$

The required general integral is obtained by eliminating α from (9) and (10).

Ex. 6. *Find a complete integral of* (*i*) $pq = x^m y^n z^{2l}$ **[Delhi B.Sc. (Prog) II 2007]**

(*ii*) $pq = x^m y^n z^l$ **[I.A.S. 1989, 94]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{z^{-l}z^{-l}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \quad \text{or} \quad \left(\frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}\right) = 1. \qquad ...(1)$$

Put $x^m dx = dX$, $y^n dy = dY$ and $z^{-l}dz = dZ$ ...(2)

so that $\frac{x^{m+1}}{m+1} = X$, $\frac{y^{n+1}}{n+1} = Y$ and $\frac{z^{1-l}}{1-l} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)\,(\partial Z/\partial Y) = 1$ or $PQ = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c$, ... (5)

where $ab = 1$ or $b = 1/a$, on putting a from P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \quad \text{or} \quad \frac{z^{1-l}}{1-l} = a\frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

(ii) The given equation can be rewritten as

$$\frac{z^{-l/2}z^{-l/2}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}=1 \quad \text{or} \quad \left(\frac{z^{-l/2}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l/2}}{y^n}\frac{\partial z}{\partial y}\right)=1. \qquad ...(1)$$

Put $x^m dx = dX$, $y^n dy = dY$ and $z^{-l/2}dz = dZ$ ...(2)

so that $\frac{x^{m+1}}{m+1} = X$, $\frac{y^{n+1}}{n+1} = Y$ and $\frac{z^{1-(l/2)}}{1-(l/2)} = Z.$ ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)\,(\partial Z/\partial Y) = 1$ or $PQ = 1,$ ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c,$ ... (5)

where $ab = 1$ or $b = 1/a,$ on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \quad \text{or} \quad \frac{z^{1-(l/2)}}{1-(l/2)} = a\frac{x^{m+1}}{m+1}+\frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

Ex. 7. *Find complete integral of* $p^m \sec^{2m}x + z^l q^n \operatorname{cosec}^{2n}y = z^{lm/(m-n)}$

Sol. The given equation can be re–written as

$$\frac{1}{z^{lm/(m-n)}}\left(\frac{1}{\cos^2 x}\frac{\partial z}{dx}\right)^m+\frac{z^l}{z^{lm/(m-n)}}\left(\frac{1}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n = 1 \text{ or } \left(\frac{z^{-l/(m-n)}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^m+\left(\frac{z^{-l/(m-n)}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n = 1. \quad ...(1)$$

Put $\cos^2 x\, dx = dX$, $\sin^2 y\, dy = dY$ and $z^{-l/(m-n)}dz = dZ$ ...(2)

i.e., $\{(1 + \cos 2x)/2\}dx = dX$, $\{(1 - \cos 2y)/2\}dy = dY$ and $z^{-l/(m-n)}\, dz = dZ$

so that $\frac{1}{2}(x + \frac{1}{2}\sin 2x) = X$, $\frac{1}{2}(y - \frac{1}{2}\sin 2y) = Y$ and $\frac{(m-n)z^{(m-n-l)/(m-n)}}{m-n-l} = Z.$ ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^m + (\partial Z/\partial Y)^n = 1$ or $P^m + Q^n = 1,$ ...(4)

where $P = \partial Z/dX$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $Z = aX + bY + c,$ ... (5)

where $a^m + b^n = 1$ or $b = (1 - a^m)^{1/n}$, on putting a for P and b for Q is (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + (1 - a^m)^{1/n}Y + c,$ a and c being two arbitrary constants.

or $$\frac{m-n}{m-n-l}z^{(m-n-l)/(m-n)} = \frac{a}{4}(2x + \sin 2x) + \frac{(1-a^m)^{1/n}}{4}(2y - \sin 2y) + c, \text{ by (3).}$$

Ex. 8. *Find the complete integral of* $(1 - x^2)\, yp^2 + x^2 q = 0$.

Sol. The given equation can be rewritten as

$$\frac{1-x^2}{x^2}\left(\frac{\partial z}{\partial x}\right)^2+\frac{1}{y}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \left(\frac{(1-x^2)^{1/2}}{x}\frac{\partial z}{\partial x}\right)^2+\left(\frac{1}{y}\frac{\partial z}{\partial y}\right) = 0. \qquad ...(1)$$

Put $\{x/(1 - x^2)^{1/2}\}dx = dX$ and $y\, dy = dY$ ...(2)

so that $X = \int \frac{x\,dx}{(1-x^2)^{1/2}} = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)dx = -(1 - x^2)^{1/2}$ and $Y = \frac{y^2}{2}$ ...(3)

Using (2), (1) becomes $(\partial z/\partial X)^2 + (\partial z/\partial Y) = 0$ or $P^2 + Q = 0,$ ...(4)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Note carefully that here the old variable z remains unchanged

even after transformation (2). Here (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c$, ... (5)

where $a^2 + b = 0$ or $b = -a^2$, on putting a for P and b for Q in (4),

$\therefore$ from (5), the required complete integral is

$z = aX - a^2Y + c$ or $z = -a(1 - x^2)^{1/2} - (a^2y^2)/2 + c$, by (3).

Ex. 9. *Find the complete integral of* $(y - x)(qy - px) = (p - q)^2$. **[Delhi Maths (H) 2005; Ravishankar 2010; Meerut 1995, 97; Agra 1999; Kanpur 2001, 04, 07, 08]**

Sol. Let X and Y be two new variables such that

$X = x + y$ and $Y = xy$. ...(1)

Given equation is $(y - x)(qy - px) = (p - q)^2$. ...(2)

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial x} + \dfrac{\partial z}{\partial Y}\dfrac{\partial Y}{\partial x} = \dfrac{\partial z}{\partial X} + y\dfrac{\partial z}{\partial Y}$ ...(3)

[$\because$ from (1), $\partial X/\partial x = 1$ and $\partial Y/\partial x = y$]

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial y} + \dfrac{\partial z}{\partial Y}\dfrac{\partial Y}{\partial y} = \dfrac{\partial z}{\partial X} + x\dfrac{\partial z}{\partial Y}$. ...(4)

[$\because$ from (1), $\partial X/\partial y = 1$ and $\partial Y/\partial y = x$]

Substituting the above values of p and q in (2), we have

$$(y - x)\left[y\left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right) - x\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right)\right] = \left[\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right) - \left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right)\right]^2$$

or $(y - x)^2\dfrac{\partial z}{\partial X} = (y - x)^2\left(\dfrac{\partial z}{\partial Y}\right)^2$ or $\dfrac{\partial z}{\partial X} = \left(\dfrac{\partial z}{\partial Y}\right)^2$ or $P = Q^2$, ...(5)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c$, ...(6)

where $a = b^2$, on putting a for P and b for Q in (5).

$\therefore$ from (6), the required complete integral is

$z = b^2X + bY + c$ or $z = b^2(x + y) + bxy + c$, by (1).

Ex. 10. *Find the complete integral of* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$. **[I.A.S. 1991; Kanpur 2006; Meerut 1997]**

Sol. Let X and Y be two new variables such that

$X^2 = x + y$ and $Y^2 = x - y$. ...(1)

Given equation is $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$. ...(2)

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial x} + \dfrac{\partial z}{\partial Y}\dfrac{\partial Y}{\partial x} = \dfrac{1}{2X}\dfrac{\partial z}{\partial X} + \dfrac{1}{2Y}\dfrac{\partial z}{\partial Y}$ ...(3)

[$\because$ from (1), $\partial X/\partial x = 1/2X$ and $\partial Y/\partial x = 1/2Y$]

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial y} + \dfrac{\partial z}{\partial Y}\dfrac{\partial Y}{\partial y} = \dfrac{1}{2X}\dfrac{\partial z}{\partial X} - \dfrac{1}{2Y}\dfrac{\partial z}{\partial Y}$. ...(4)

[$\because$ from (1), $\partial X/\partial y = 1/2X$ and $\partial Y/\partial y = -1/2Y$]

(3) and (4) $\Rightarrow$ $p + q = \dfrac{1}{X}\dfrac{\partial z}{\partial X}$ and $p - q = \dfrac{1}{Y}\dfrac{\partial z}{\partial Y}$. ...(5)

Using (1) and (5), (2) reduces to

$$X^2 \times \frac{1}{X^2}\left(\frac{\partial z}{\partial X}\right)^2 + Y^2 \times \frac{1}{Y^2}\left(\frac{\partial z}{\partial Y}\right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1, \quad ...(6)$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c,$ ...(7)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (6).

$\therefore$ from (7), the required complete integral is

$z = aX + Y\sqrt{1-a^2} + c$ or $z = a\sqrt{x+y} + \sqrt{x-y}\sqrt{1-a^2} + c$, by (1).

Ex. 11. *Find a complete integral of* $(x^2 + y^2)(p^2 + q^2) = 1$.

[Agra 2008; Indore 2004; Vikram 2000; Meerut 1995; Rohitkhand 1994]

Sol. Put $x = r\cos\theta$ and $y = r\sin\theta$. ...(1)

Then, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$. ...(2)

Differentiating (2) partially with respect to x and y, we get

$2r(\partial r/dx) = 2x$ and $2r(\partial r/\partial y) = 2y$

$\Rightarrow \quad \dfrac{\partial r}{\partial x} = \dfrac{r\cos\theta}{r} = \cos\theta$ and $\dfrac{\partial r}{\partial y} = \dfrac{r\sin\theta}{r} = \sin\theta$. ...(3)

and $$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\times\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \quad ...(4)$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\times\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \quad ...(5)$$

Given equation is $(x^2 + y^2)(p^2 + q^2) = 1.$ ....(6)

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial x} = \cos\theta\dfrac{\partial z}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial z}{\partial\theta}$, by (3) and (4)

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial y} = \sin\theta\dfrac{\partial z}{\partial r} + \dfrac{\cos 0}{r}\dfrac{\partial z}{\partial\theta}$, by (3) and (5).

Hence $p^2 + q^2 = (\partial z/\partial r)^2 + (1/r^2) \times (\partial z/\partial\theta)^2.$ ...(7)

$\therefore$ (6) becomes $r^2[(\partial z/\partial r)^2 + (1/r^2) \times (\partial z/\partial\theta)^2] = 1,$ using (2) and (7)

or $$\left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 = 1. \quad ...(8)$$

Let R be a new variable such that $(1/r)dr = dR$ so that $\log r = R.$ ...(9)

Then (8) becomes $(\partial z/\partial R)^2 + (\partial z/\partial\theta)^2 = 1$ or $P^2 + Q^2 = 1,$ ...(10)

where $P = \partial z/\partial R$ and $Q = \partial z/\partial\theta$. (10) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $z = aR + b\theta + c,$ ...(11)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (10)

$\therefore$ from (11), the required complete integral is

$z = aR + \theta\sqrt{1-a^2} + c$ or $z = a\log r + \theta\sqrt{1-a^2} + c,$

or $z = a\log(x^2 + y^2)^{1/2} + \tan^{-1}(y/x)\cdot\sqrt{1-a^2} + c,$ by (2)

or $z = (a/2) \times \log(x^2 + y^2) + \sqrt{1-a^2}\tan^{-1}(y/x) + c$, a and c being arbitrary constants.

Ex. 12. *Find the complete integral of* $z^2 = pqxy$ **[Meerut 2007; Punjab 2005]**

Sol. The given equation can be re–written as

$$\frac{xy}{z^2}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \quad \text{or} \quad \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)\left(\frac{y}{z}\frac{\partial z}{\partial y}\right) = 1 \quad ... (1)$$

Put $(1/x)dx = dX$, $(1/y)\,dy = dY$ and $(1/z)dz = dZ$

so that $\log x = X$, $\log y = Y$ and $\log z = Z$ ... (2)

Then (1) becomes $(\partial Z/\partial X)\,(\partial Z/\partial Y) = 1$ or $PQ = 1$ ... (3)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. Then, solution of (3) is

$$Z = aX + bY + C', \quad \text{where} \quad ab = 1 \quad \text{so that} \quad b = 1/a.$$

$\therefore$ $\log z = a\log x + (1/a)\log y + \log C$, taking $C' = \log C$ and using (2)

or $z = x^a\, y^{1/a} C$, a and C being arbitrary constants.

Ex. 13. *Find the complete integral of* $(x/p)^n + (y/q)^n = z^n$.

Sol. The given can be re–written as $(x/zp)^n + (y/zq)^n = 1$ ... (1)

Let $X = x^2/2$, $Y = y^2/2$, $Z = z^2/2$, $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$ ... (2)

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial Z}\,\dfrac{\partial Z}{\partial X}\,\dfrac{\partial X}{\partial x} = \dfrac{x}{z}P$. Similarly, $q = \dfrac{y}{z}Q$, using (2)

Hence $x/zp = 1/P$ and $y/zq = 1/Q$ and so (1) reduces to $P^{-n} + Q^{-n} = 1$, whose solution is

$Z = aX + bY + C'$, where $a^{-n} + b^{-n} = 1$ so that $b = (1 - a^{-n})^{-1/n}$ ... (3)

$\therefore$ (2) and (3) $\Rightarrow$ $z^2/2 = a(x^2/2) + (1 - a^{-n})^{-1/n}\,(y^2/2) + C/2$, taking $C' = C/2$

or $z^2 = ax^2 + (1 - a^{-n})^{-1/n} y^2 + C$, a and C being arbitrary constants.

Ex. 14. *Find the complete integral of* $p^3 \sec^6 x + z^2 q^2 \operatorname{cosec}^4 y = z^6$

Sol. The given equation can be re–written as

$$\frac{z^{-6}}{\cos^6 x}\left(\frac{\partial z}{\partial x}\right)^3 + \frac{z^{-4}}{\sin^4 y}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{z^{-2}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{z^{-2}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{... (1)}$$

Let $\cos^2 x\,dx = dX$, $\sin^2 y\,dy = dY$, $z^{-2}dz = dZ$ ... (2)

$\Rightarrow$ $X = (1/2)\int (1 + \cos 2x)dx$, $Y = (1/2)\int (1 - \cos 2y)dy$, $Z = -(1/z)$

$\Rightarrow$ $X = (1/2)\,\{x + (1/2)\sin 2x\}$, $y = (1/2)\,\{y - (1/2)\sin 2y\}$, $Z = -(1/z)$

$\Rightarrow$ $X = (1/2)\,(x + \sin x \cos x)$, $y = (1/2)\,(y - \sin y \cos y)$, $Z = -(1/z)$ ... (3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$ ... (4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. Now, solution of (4) is

$Z = aX + bY + C/2$, where $a^2 + b^2 = 1$ so that $b = (1 - a^2)^{1/2}$ ... (5)

$\therefore$ $-(1/z) = a(1/2)\,(x + \sin x \cos x) + (1 - a^2)^{1/2}(1/2)(y - \sin y \cos y) + C/2$, by (3) and (5)

or $(2/z) + a(x + \sin x \cos x) + (1 - a^2)^{1/2}(y - \sin y \cos y) + C = 0$.

Ex. 15. *Find the complete integral of* $yp + xq = pq$.

Sol. The given equation can be re–written as

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial z}{x\partial x} + \frac{\partial z}{y\partial y} = \left(\frac{\partial z}{x\partial x}\right)\left(\frac{\partial z}{y\partial y}\right) \quad \ldots (1)$$

Put $x\,dx = dX, \quad y\,dy = dY$ so that $x^2/2 = X, \quad y^2/2 = Y$ ... (2)

Then (1) becomes $\partial z/\partial X + \partial z/\partial Y = (\partial z/\partial X)\,(\partial z/\partial Y)$ or $P + Q = PQ$ ... (3)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Then solution of (3) is

$z = aX + bY + c$, where $a + b = ab$ so that $b = a/(a-1)$ ... (4)

or $z = a(x^2/2) + a(a-1)^{-1}\,(y^2/2) + c$, a and c being arbitrary constants, by (2) and (4)

Ex. 16. *Find the complete integral of* $p^2x^2 + px = q$

Sol. The given equation can be re–written as

$$x^2\left(\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x} = q \quad \text{or} \quad \left(x\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x} = q \quad \ldots (1)$$

Putting $(1/x)dx = dX$ so that $\log x = X$, (1) gives

$(\partial z/\partial X)^2 + \partial z/\partial X = q$ or $P^2 + P = q$, where $P = \partial z/\partial X$.

Its solution is $z = aX + by + c$ where $a^2 + a = b$

or $z = a\log x + (a^2 + a)y + c$, a and c being arbitrary constants. $[\because X = \log x]$

Ex. 17. *Find the complete integral, general integral and singular integral of* $pq = 4xy$. *Show that the equation is satisfied by* $z = 2xy + C$, *C being an arbitrary constant. What is the character of this integral.* **[Delhi Maths (H) 2007]**

Sol. The given equation can be re–written as

$$\frac{pq}{4xy} = 1 \quad \text{or} \quad \frac{1}{4xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \quad \text{or} \quad \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) = 1 \quad \ldots (1)$$

Putting $2x\,dx = dX, \quad 2y\,dy = dY$ so that $x^2 = X, \; y^2 = Y$, (1) gives

$(\partial z/\partial X)\,(\partial z/\partial Y) = 1$ or $PQ = 1$ whose solution is

$z = aX + bY + d$, where $ab = 1$ so that $b = 1/a$.

$\therefore \quad z = ax^2 + (1/a)y^2 + d$ ... (2)

is complete integral of (1) containing two arbitrary constants a and *d*.

General integral. Putting $d = \phi(a)$ in (2), we get

$$z = ax^2 + (1/a)\,y^2 + \phi(a) \quad \ldots (3)$$

Differenting (3) partially w.r.t. '*a*', $0 = x^2 - (1/a^2)\,y^2 + \phi'(a)$ ... (4)

Then general integral is obtained by eliminating *a* from (3) and (4).

Singular integral. Differenting (2) partially w.r.t. 'a' and 'd' by turn, we get

$0 = x^2 + (-1/a^2)\, y^2$ ... (5) $\qquad 0 = 1$ ... (6)

Since (6) is absurd, so (1) has no singular solution.

Discussion of the character of the given integral

$z = 2xy + C$, C being an arbitrary constant ... (7)

Differenting (7) partially w.r.t. x and y, we get $\partial z/\partial x = p = 2x$ and $\partial z/\partial y = q = 2y$. These values of p and q satisfy (1). Hence (1) is satisfied by (7).

Now, (7) can be derived from (2), if the values of p and q given by (7) and (2) are same, that is if $2ax = 2y$ and $2y/a = 2x$, *i.e.,* if we choose $a = y/x$. Putting $a = y/x$ and taking $d = C$ in (2), we have

$z = (y/x)\, x^2 + (x/y)\, y^2 + C$ or $z = 2xy + C$,

showing that (7) is a particular case of the complete integral (2)

We now show that (7) is a particular case of the general integral. To this end, replace $\phi(a)$ by C in (3) and write $z = ax^2 + (1/a)\, y^2 + C$ ... (8)

Differenting (8) partially w.r.t. 'a', we get

$0 = x^2 - (1/a^2)y^2$ or $a = y/x$ ... (9)

Eliminating a from (8) and (9), we get $z = 2xy + C$

Ex. 18. *Find the complete integral of* $z = p^2 - q^2$. **[Delhi Maths (G) 2006]**

Sol. Re–writing the given equation, we have

$$\frac{1}{z}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{1}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(z^{-1/2}\frac{\partial z}{\partial x}\right)^2 - \left(z^{-1/2}\frac{\partial z}{\partial y}\right)^{1/2} = 1 \quad \text{... (1)}$$

Let X, Y and Z be new variables such that

$dX = dx$, $\quad dY = dy$ and $\quad dZ = z^{-1/2}dz$ so that $X = x$, $\quad Y = y$, $\quad Z = 2z^{1/2}$ ... (2)

Let $P = \partial Z/\partial X$ and $Q = \partial Z/\partial y$. Using (2), (1) becomes

$$P^2 - Q^2 = 1, \quad \text{... (3)}$$

which is of the form $f(P, Q) = 0$. Hence a solution of (3) is $Z = aX + by + c$, ... (4)

where $a^2 - b^2 = 1$. Then $b = \pm(a^2 - 1)^{1/2}$ and so from (4), we have

$Z = aX \pm (a^2 - 1)^{1/2} Y + c$ or $2z^{1/2} = ax \pm (a^2 - 1)^{1/2} y + c$,

which is the complete integral, a and c being arbitrary constants and $|a| \geq 1$.

EXERCISE 3(C)

Solve the following partial differential equations (1 – 10)

1. $p^2 - q^2 = 1$ **Ans. C.I.** $z = ax + (a^2 - 1)^{1/2} + c$, a and c are arbitrary constants and $|a| \geq 1$; **S.I.** Does not exist, **G.I.** It is given by $z - ax - (a^2 - 1)^{1/2} y - \psi(a) = 0$, $-x - a(a^2 - 1)^{-1/2}\, y - \psi'(a)$, where ψ is an arbitrary function.

2. $p^2 - q^2 = \lambda$ **Ans. C.I.** $z = ax + (a^2 - \lambda)^{1/2} y + c,$ where a and c are arbitrary constants and $\lambda \le a^2$; **S.I.** Does not exist, **G.I.** It is given by $z - ax - (a^2 - \lambda)^{1/2} y - \psi(a) = 0,$

$$-x - a(a^2 - \lambda)^{1/2} y - \psi'(a) = 0$$

3. $p + q = pq$ **[Mysore 2004; Gulberga 2005; Kanpur 2011; Pune 2010]**

Ans. C.I. $z = ax + \{a/(a-1)\}y + c$, where a and c are arbitrary constants and $a \neq 1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - \{a/(a-1)\}y - \psi(a) = 0$ and $-x - y(a-1)^{-2} - \psi'(a) = 0,$ where ψ is an arbitrary function

4. $p + q + pq = 0.$ **Ans. C.I.** $z = ax - \{a/(a+1)\}y + c$, where a and c are arbitrary constants and $a \neq -1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax + \{a/(a+1)\}y - \psi(a) = 0,$

$$-x - \{(2a+1)/2(a^2+a)^{1/2}\}y - \psi'(a) = 0,$$ where ψ is an arbitrary function.

5. $p^2 + q^2 = npq.$ **(M.S. Univ. T.N. 2007), Ans. C.I.** $z = ax + (a/2) \times \{n + (n^2 - 4)^{1/2}y + c,$ where a and c are arbitrary constants and $n^2 \ge 4$; **S.S.** Does not exists; **G.S.** It is given by $z - ax - (a/2) \times \{n + (n^2 - 4)^{1/2}\} y - \psi(a) = 0,$ $-x - (1/2) \times \{n + (n^2 - 4)^{1/2\}} y - \psi'(a) = 0,$ where ψ is an arbitrary function.

6. $p = 2q^2 + 1.$ **Ans. C.I.** $z = ax + \{(a-1)/2\}^{1/2} y + c,$ where a and c are arbitrary constants and $a \ge 1;$ **S.S.** Does not exist; **G.S.** It is given by $z - ax - \{(a-1)/2\}^{1/2} y - \psi(a) = 0,$

$$-x - (2\sqrt{2}\sqrt{a-1})^{-1} y - \psi'(a) = 0,$$ where ψ is an arbitrary function.

7. $p = e^q$. **Ans. C.I.** $z = ax + y \log a + c$, where a and c are arbitrary constants and $a > 0$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - y \log a - \psi(a) = 0, - x - (y/a) - \psi'(a) = 0,$ where ψ is an arbitrary function.

8. $p^2q^3 = 1$ **Ans. C.I.** $z = ax + a^{-2/3}y + c$, where a and c are arbitrary constants and $a > 0$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - a^{-2/3} y - \psi(a) = 0,$ $-x + (2/3) \times a^{-5/3} y - \psi(a) = 0,$ where ψ is an arbitrary function.

9. $p^2 + p = q^2.$ **Ans. C.I.** $z = ax + (a^2 + a)^{1/2}y + c$, where a and c are arbitrary constants and $a \in \boldsymbol{R} - (-1, 0);$ **S.S.** Does not exist; **G.S.** It is given by $z - ax - (a^2 + a)^{1/2} y - \psi(a) = 0,$

$$-x - \{(2a+1)/2(a^2+a)^{1/2}\}y - \psi'(a) = 0,$$ where ψ is an arbitrary function.

10. $p^2 + 6p + 2q + 4 = 0.$ **C.I.** $z = ax - (2 + 3a + a^2/2)y + c$, where a and c are arbitrary constants; **S.S.** Does not exist; **G.S.** It is given by $z - ax + (2 + 3a + a^2/2)y - \psi(a) = 0,$

$$-x + (a+3)y - \psi'(a) = 0,$$ where ψ is an arbitrary function.

Find the complete integral (*solution*) *of the following equations* (*Ex. 11—18*).

11. $zy^2p = x(y^2 + z^2q^2).$ **Ans.** $z^2 = ax^2 \pm y^2(a-1)^{1/2} + c,$ where $a \ge 1$

12. $z^2(p^2/x^2 + q^2/y^2) = 1.$ **Ans.** $z^2 = ax^2 \pm y^2(1 - a^2)^{1/2} + c,$ where $-1 \le a \le 1$

13. $yp + x^2q^2 = 2x^2y.$ **Ans.** $(3z - ax^3 - b)^2 = 4(2 - a)y^2$

14. $(1 - y^2)\, xq^2 - y^2p = 0.$ **Ans.** $(2z - ax^2 - b)^2 = a(1 - y^2)$

15. $p^2y(1 + x^2) = qx^2.$ **Ans.** $z = a(1 + x^2)^{1/2} + (1/2) \times a^2y^2 + c$

16. $x^4p^2 + y^2zq - z^2 = 0.$ **Ans.** $xy \log z = ay + (a^2 - 1)x + bxy$

17. $p^2 + q^2 = z$. **[Bangalore 1995]** **Ans.** $2z^{1/2} = ax \pm (1 - a^2)^{1/2} y + c$, where $-1 \le a \le 1$

18. $x^2p^2 + y^2q^2 = 4z^2$. **Ans.** $\log z = a \log x + (4 - a^2)^{1/2} + c$, $-2 \le a \le 2$

3.12. Standard form II. Clairaut equation. [Meerut 2009; Nagpur 2002]

A first order partial differential equation is said to be of Clariaut form if it can be written in the form

$$z = px + qy + f(p, q). \qquad ...(1)$$

Let $$F(x, y, z, p, q) \equiv px + qy + f(p, q) - z. \qquad ...(2)$$

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or $$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px - qy - p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-x - (\partial f/\partial p)} = \frac{dy}{-y - (\partial f/\partial q)}, \text{ by (1)}$$

Then, first and second fractions $\Rightarrow$ $dp = 0$ and $dq = 0 \Rightarrow$ $p = a$ and $q = b$.

Substituting these values in (1), the complete integral is $z = ax + by + f(a, b)$

Remark 1. Observe that the complete integral of (1) is obtained by merely replacing p and q by a and b respectively. Singular and general integrals can be obtained by usual methods.

Remark 2. Sometimes change of variables can be employed to transform a given equation to standard form II.

3.13. SOLVED EXAMPLES BASED ON ART. 3.12

Ex. 1. *Solve* $z = px + qy + pq$. **[Ravishanker 1997; Bangalore 2005; Sagar 1995, 96]**

Sol. The complete integral is $z = ax + by + ab$, a, b being arbitrary constants ...(1)

Singular integral. Differentiating (1) partially w.r.t. a and b, we have

$$a = x + b \quad \text{and} \quad 0 = y + a. \qquad ...(2)$$

Eliminating a and b between (1) and (2), we get $z = -xy - xy + xy$ *i.e.*, $z = -xy$, which is the required singular solution, for it satisfies the given equation.

General Integral. Take $b = \phi(a)$, where ϕ denotes an arbitrary function.

Then (1) becomes $$z = a\,x + \phi(a)\,y + a\,\phi\,(a). \qquad ...(3)$$

Differentiating (3) partially w.r.t. a, $$0 = x + \phi'(a)y + \phi(a) - a\,\phi'(a). \qquad ...(4)$$

The general integral is obtained by eliminating a between (3) and (4).

Ex. 2. *Prove that complete integral of the equations* $(px + qy - z)^2 = 1 + p^2 + q^2$ is $ax + by + cz = (a^2 + b^2 + c^2)^{1/2}$. **[I.A.S. 1989]**

Sol. Re–writting the given equation, we have

$$px + qy - z = \pm\sqrt{(1 + p^2 + q^2)} \quad \text{or} \quad z = px + qy \pm\sqrt{(1 + p^2 + q^2)}$$

which is of standard form II and so its complete integral is

$$z = Ax + By \pm (1 + A^2 + B^2)^{1/2}. \qquad ...(1)$$

To get the desired form of solution we take +ve sign in (1) and set $A = -a/c$ and $B = -b/c$. Then (1) becomes $$z = -(ax + by)/c + (c^2 + a^2 + b^2)^{1/2}/c$$

or $$ax + by + cz = (a^2 + b^2 + c^2)^{1/2}.$$

Ex. 3. *Solve* $z = px + qy + c\sqrt{(1 + p^2 + q^2)}$. **[I.A.S. 1989; Meerut 1998]**

Sol. The complete integral of the given equation is

$$z = ax + by + c\sqrt{(1 + a^2 + b^2)}, \; a, \; b \text{ being arbitrary constants.} \qquad ...(1)$$

Singular Integral. Differentiating (1) partially w.r.t. a and b, we get

$0 = x + ac/\sqrt{(1+a^2+b^2)}$ ...(2) $0 = y + bc/\sqrt{(1+a^2+b^2)}$. ...(3)

$\therefore$ From (2) and (3), $x^2 + y^2 = (a^2c^2 + b^2c^2)/(1 + a^2 + b^2)$.

$\therefore$ $$c^2 - x^2 - y^2 = c^2 - \frac{a^2c^2 + b^2c^2}{1+a^2+b^2} = \frac{c^2}{1+a^2+b^2}$$

so that $1 + a^2 + b^2 = c^2/(c^2 - x^2 - y^2)$. ...(4)

From (2), $$a = -\frac{x\sqrt{(1+a^2+b^2)}}{c} = -\frac{x}{\sqrt{(c^2-x^2-y^2)}}, \text{ by (4)}$$

Similarly from (3) and (4), we obtain $b = -y/\sqrt{c^2 - x^2 - y^2}$.

Putting these values of a and b in (1), the singular solution is

$$z = -\frac{x^2}{\sqrt{(c^2-x^2-y^2)}} - \frac{y^2}{\sqrt{(c^2-x^2-y^2)}} + \frac{c^2}{\sqrt{(c^2-x^2-y^2)}} = \frac{c^2-x^2-y^2}{\sqrt{(c^2-x^2-y^2)}}$$

or $z = (c^2 - x^2 - y^2)^{1/2}$ or $z^2 = c^2 - x^2 - y^2$ or $x^2 + y^2 + z^2 = c^2$. ...(5)

We can easily verify that (1) is satisfied by (5).

General Integral. Take $b = \phi(a)$, where ϕ is an arbitrary function.

Then, (1) yeilds $z = ax + y\phi(a) + c[1 + a^2 + \{\phi(a)\}^2]^{1/2}$. ...(6)

Differentiating both sides of (6) partially w.r.t. 'a', we get

$$0 = x + y\phi'(a) + (c/2) \times [1 + a^2 + \{\phi(a)\}^2]^{-1/2} \times [2a + 2\phi(a)\ \phi'(a)]. \quad ...(7)$$

Eliminating a from (6) and (7), we get the general integral.

Ex. 4. *Find the complete and singular integrals of the following equations:*

(*i*) $z = px + qy + \log(pq)$ **[Indore 2004; K.U. Kurukshetra 2006]**

(*ii*) $z = px + qy - 2\sqrt{pq}$. **[Bangalore 1993; Lucknow 2010]**

Sol. (*i*) The complete integral is $z = ax + by + \log(ab)$

or $z = ax + by + \log a + \log b$, a, b being arbitrary constants ...(1)

Differentiating (1) partially with respect to a and b, we get

$0 = x + (1/a)$ and $0 = y + (1/b)$ so that $a = -1/x$ and $b = -1/y$. ...(2)

Eliminating a and b from (1) and (2), the required singular integral is

$z = -1 - 1 + \log(1/xy)$ or $z = -2 - \log(xy)$.

(*ii*) The complete integral is $z = ax + by - 2\sqrt{ab}$. ...(1)

Differentiating (1) partially with respect to a and b, we get

$$0 = x - \frac{2b}{2\sqrt{ab}} \quad \text{and} \quad 0 = y - \frac{2a}{2\sqrt{ab}} \quad \text{so that} \quad x = \sqrt{\frac{b}{a}} \quad \text{and} \quad y = \sqrt{\frac{a}{b}}. \quad ...(2)$$

Now, using (1) $x - z = x - (ax + by - 2\sqrt{ab}) = \sqrt{\frac{b}{a}} - a\sqrt{\frac{b}{a}} - b\sqrt{\frac{a}{b}} + 2\sqrt{ab}$, using (2)

$\therefore$ $x - z = \sqrt{(b/a)}$. ...(3)

Similarly, using (1) $y - z = y - (ax + by - 2\sqrt{ab}),\ = \sqrt{\frac{a}{b}} - a\sqrt{\frac{b}{a}} - b\sqrt{\frac{a}{b}} + 2\sqrt{ab}$

$\therefore$ $y - z = \sqrt{(a/b)}$. ...(4)

From (3) and (4), $(x - z)(y - z) = 1$,

which is singular integral as it satisfies the given equation.

Ex. 5. *Prove that the complete integral of $z = px + qy - 2p - 3q$ represents all possible planes through the point* (2, 3, 0). *Also find the envelope of all planes represented by the complete integral* (*i.e.*, *find the singular integral*). **(M.D.U. Rohtak 2006)**

Sol. Given that $z = px + qy - 2p - 3q,$ ...(1)

which is of the form $z = px + qy + f(p, q)$ and so its complete integral is

$z = ax + by - 2a - 3b,$ a, b being arbitrary constants ...(2)

Since (2) is a linear equation in x, y, z, it follows that (2) represents planes for various values of a and b. Again putting $x = 2, y = 3, z = 0$ in (2), we have

$0 = 2a + 3b - 2a - 3b$ *i.e.*, $0 = 0,$

showing that coordinates of the point (2, 3, 0) satisfy (2). Hence the complete integral (2) of (1) represents all possible planes passing through the point (2, 3,0).

Differentiating (2) partially with respect to a and b, we get

$0 = x - 2$ and $0 = y - 3$ so that $x = 2$ and $y = 3.$

Substituting these values in (2), we get $z = 0$ as the required envelope (*i.e.*, singular integral).

Ex. 6. *Prove that the complete integral of $z = px + qy + [pq/(pq - p - q)]$ represents all planes such that the algebraic sum of the intercepts on three coordinate axes is unity.*

Sol. Since the given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is

$z = ax + by + [ab/(ab - a - b)],$ a and b being arbitrary constants. ...(1)

Since (2) is a linear equation in x, y, z, it follows that (1) represents planes for various values of a and b. We now rewrite (1) in the intercept form of a plane as follows :

$$ax + by - z = ab/(a + b - ab)$$

or $$\frac{x}{[b/(a+b-ab)]} + \frac{y}{[a/(a+b-ab)]} + \frac{z}{[-ab/(a+b-ab)]} = 1.$$

$\therefore$ The algebic sum of the intercepts on three coordinate axes

$$= \frac{b}{a+b-ab} + \frac{a}{a+b-ab} + \frac{(-ab)}{a+b-ab} = \frac{b+a-ab}{a+b-ab} = 1, \text{ as required.}$$

Ex. 7. *Show that the complete integral of the equation $z = px + qy + (p^2 + q^2 + 1)^{1/2}$ represents all planes at unit distance from the origin.*

Sol. Given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is

$z = ax + by + (a^2 + b^2 + 1)^{1/2}$, a, b being an arbitrary constants.

or $$ax + by - z + (a^2 + b^2 + 1)^{1/2} = 0. \quad ...(1)$$

Since (2) is a linear equation in x, y, z, it follows that (1) represents planes for various values of a and b.

The perpendicular distance of (1) from origin (0, 0, 0)

$$= \frac{a.0 + b.0 - 0 + \sqrt{a^2 + b^2 + 1}}{\sqrt{\{a^2 + b^2 + (-1)^2\}}} = \frac{\sqrt{a^2 + b^2 + 1}}{\sqrt{a^2 + b^2 + 1}} = 1, \text{ as required}$$

Ex. 8. *Find the complete integral of the following equations:*

(*i*) $(p+q)(z-px-qy) = 1$ **[Pune 2010]**

(*ii*) $pqz = p^2(xq + p^2) + q^2(yp + q^2)$ **[Delhi B.A. (Prog) II 2008, 10]**

Sol. (*i*) Re–writing the given equation in the standard form $z = px + qy + f(p,q)$, we get

$z - px - qy = 1/(p+q)$ or $z = px + qy + 1/(p+q)$

$\therefore$ Its complete integral is $z = ax + by + 1/(a+b)$, where a and b are arbitrary constants.

(*ii*) Dividing both sides of the given equation by pq, $z = px + qy + (p^4 + q^4)/pq$,

Its complete integral is $z = ax + by + (a^4 + b^4)/ab$, a, b being arbitrary constants.

Ex. 9. (*a*) *Find the complete integral the equation* $2(y + zq) = q(xp + yq)$.

[Delhi Maths (H) 1999]

Sol. Re–writing the given equation, we have

$2zq = xpq + yq^2 - 2y$ or $z = (1/2)px + (1/2)qy - (y/q)$

or $$z = x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) - \frac{1}{2}\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)^{-1} \quad \ldots (1)$$

Putting $2x\,dx = dX$ and $2y\,dy = dY$ so that $x^2 = X$ and $y^2 = Y$, (1) gives

$z = X(\partial z/\partial X) + Y(\partial z/\partial Y) - 1/\{2(\partial z/\partial Y)\}$ or $z = PX + QY - (1/2Q)$,

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. The above equation is of the form $z = PX + Qy + f(P,Q)$ and hence its complete integral is

$z = aX + bY - (1/2b)$ or $z = ax^2 + by^2 - (1/2b)$, a and b being arbitrary constants.

Ex. 9. (*b*) *Find the complete integral of* $2q(z - px - qy) = 1 + q^2$.

Sol. Re–writing the given equation in the form $z = px + qy + f(p,q)$, we have

$z - px - qy = (1+q^2)/2q$ or $z = px + qy + (1+q^2)/2q$,

Its complete integral is $z = ax + by + (1+b^2)/2b$, a and b being arbitrary constants.

Ex. 10. *Find the compelte integral of* $p^2x + q^2y = (z - 2px - 2qy)^2$.

Sol. Taking positive root, the given equation reduces to

$z - 2px - 2qy = (p^2x + q^2y)^{1/2}$ or $z = 2px + 2qy + (p^2x + q^2y)^{1/2}$

or $$z = \sqrt{x}\,\frac{\partial z}{(1/2\sqrt{x})\partial x} + \sqrt{y}\,\frac{\partial z}{(1/2\sqrt{y})\partial y} + \frac{1}{2}\left[\left(\frac{\partial z}{(1/2\sqrt{x})\partial x}\right)^2 + \left(\frac{\partial z}{(1/2\sqrt{y})\partial y}\right)^2\right]^{1/2} \quad \ldots (1)$$

Put $(1/2\sqrt{x})\,dx = dX$ and $(1/2\sqrt{y})dy = dY$ so that $\sqrt{x} = X$ and $\sqrt{y} = Y$ ... (2)

Using (2), (1) gives $z = (\partial z/\partial X)X + (\partial z/\partial Y)Y + (1/2) \times \{(\partial z/\partial X)^2 + (\partial z/\partial Y)^2\}^{1/2}$

or $z = PX + QY + (1/2) \times (P^2 + Q^2)^{1/2}$, where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$.

It is of the Clairaut's form $z = Px + Qy + f(P,Q)$ and so its complete integral is given by

$z = aX + bY + (1/2) \times (a^2 + b^2)^{1/2}$ or $z = a\sqrt{x} + b\sqrt{y} + (1/2) \times (a^2 + b^2)^{1/2}$

Ex. 11. *Find a complete and the singular integral of* $4xyz = pq + 2px^2y + 2qxy^2$

Sol. The given equation can be rewritten as

$$z = \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) + x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right). \quad \ldots (1)$$

Put $2x\,dx = dX$ and $2y\,dy = dY$ ... (2)

so that $x^2 = X$ and $y^2 = Y$. ... (3)

Using (2), (1) becomes $z = (\partial z/\partial X)(\partial z/\partial Y) + X(\partial z/\partial X) + Y(\partial z/\partial Y)$

or $z = XP + YQ + PQ,$ ... (4)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $z = XP + YQ + f(P, Q)$.

$\therefore$ Solution of (4) is $z = aX + bY + ab$, a, b being arbitrary constants.

or $z = ax^2 + by^2 + ab$, which is complete integral. ... (5)

Differentiating (5) partially w.r.t a and b, we have

$0 = x^2 + b$ and $0 = y^2 + b$ so that $b = -x^2$ and $a = -y^2$ .... (6)

Eliminating a and b between (5) and (6), the required singular integral is

$z = -x^2y^2 - x^2y^2 + x^2y^2$ or $z = -x^2y^2$.

Ex. 12. *Find the complete and singular solutions of* $z = px + qy + p^2q^2$. **[Jabalpur 2000; Sagar 1995; Rewa 2003; Ravishankar 2004]**

Sol. Given $z = px + qy + p^2q^2$ ... (1)

Since (1) is in Clairaut's form, its complete solution is

$z = ax + by + a^2b^2$, a, b being arbitrary constants ... (2)

To find singular solution of (1). Differentiating (2) partially w.r.t. 'a' and 'b' successively,

$0 = x + 2ab^2$ and $0 = y + 2a^2b$ ... (3)

From (3), $a = -(y^2/2x)^{1/3}$ and $b = -(x^2/2y)^{1/3}$ ... (4)

Substituting the values of a and b given by (4) in (2), we get

$z = -x(y^2/2x)^{1/3} - y(x^2/2y)^{1/3} + (x^2y^2/16)^{1/3}$ or $z = -(3/4) \times 4^{1/3}\, x^{2/3}\, y^{2/3}$,

which is the required singular solution of (1)

EXERCISE 3 (D)

Solve the following partial differential equations : (1 – 9)

1. $z = px + qy - 2p - 3q$. **[M.D.U. Rohtak 2006]**

Ans. C.I. $z = ax + by - 2a - 3b$; **S.S.** $z = 0$; **G.S.** It is given by $z - ax - \psi(a)y + 2a + 3\psi(a) = 0$,

$x + (y-3)\psi'(a) - 2 = 0$

2. $z = px + qy + 5pq$. **Ans. S.I.** $z = ax + by + 5ab$; **S.S.** $5z + xy = 0$

G.S. $z - ax - \psi(a)y - 5a\,\psi(a) = 0$, $x + 5\psi(a) + (y + 5a)\,\psi'(a) = 0$

3. $z = px + qy + p^2 - q^2$ **[Purvanchal 2007]** **Ans. S.I.** $z = ax + by + a^2 - b^2$;

S.S. $x^2 - y^2 + 4z = 0$; **G.S.** $z - ax - \psi(a)y - a^2 + \{\psi(a)\}^2 = 0$; $x + 2a + \{y - 2\psi(a)\}\psi'(a) = 0$;

4. $z = px + qy + (q/p) - p$. **[Madras 2005]**

Ans. C.I. $z = ax + by + (b/a) - a$; **S.S.** $yz = 1 - x$; **G.S.** It is given by

$z - ax + \psi(a)y + (1/a) \times \psi(a) - a$, $-x + \psi'(a)y - (1/a^2)\,\psi(a) + (1/a) \times \psi'(a) = 0$

5. $z = px + qy + p/q$ **Ans. C.I.** $= ax + by + a/b$; **S.S.** $xz + 4 = 0$;

G.S. $z - ax - \psi(a)y - a/\psi(a) = 0$; $x + \psi'(a)y + 1/\psi(a) - \{a\psi'(a)\}/\{\psi(a)\}^2 = 0$

6. $z = px + qy + 2\sqrt{pq}$ **[Bangalore 1994]**

Ans. C.I. $z = ax + by + 2\sqrt{ab}$; **S.S.** $(x - z)(y - z) = 1$; **G.S.** $z - ax - \psi(a)y - 2\sqrt{a\psi(a)} = 0$,

$$x + \psi'(a) + \{\psi(a) + a\psi'(a)\}/2\sqrt{a\psi(a)} = 0$$

7. $z = px + qy - 2\sqrt{pq}$. **Ans. C.I.** $z = ax + by - 2\sqrt{ab}$; **S.S.** $(x - z)(y - z) = 1$;

G.S. $z - ax - \psi(a)y + 2\sqrt{a\psi(a)} = 0$, $x + \psi'(a)y - \{\psi(a) + a\psi'(a)\}/\sqrt{a\psi(a)} = 0$

8. $z = px + qy + p^2 + pq + q^2$. **[Ranchi 2010]**

Ans. C.I. $z = ax + by + a^2 + ab + b^2$; **S.S.** $x^2 + y^2 - xy + 3z = 0$, **G.S.** $z - ax - \psi(a)y - a^2 - a\psi(a) - \{\psi(a)\}^2 = 0$, $x + \{y + a + 2\psi(a)\}\psi'(a) + 2a + \psi(a) = 0$

9. $z = px + qy + (\alpha p^2 + \beta q^2 + 1)^{1/2}$. **Ans. C.I.** $z = ax + by + (\alpha a^2 + \beta b^2 + 1)^{1/2}$;

S.S. $x^2/\alpha + y^2/\beta + z^2 = 1$; **G.S.** $z - ax - \psi(a)y - [a\alpha^2 + \beta\{\psi(a)\}^2 + 1]^{1/2} = 0$; $x + \psi'(a)y$

$$+\{a\alpha + \beta\psi(a)\psi'(a)\}/[a\alpha^2 + \beta(\psi(a)\}^2 + 1]^{1/2} = 0$$

10. Find the complete integral of $z = px + qy - \sin(pq)$ **[GATE 2003]**

Ans. $z = ax + by - \sin(ab)$ a, b being arbitrary constants.

11. Find the complete integral and singular integral of the differential equation $z = px + qy + p^2 - q^2$. Find also a developable surface belonging to the general integral of this differential equation. **[I.A.S 1983]**

Ans. Complete integral is $z = ax + by + a^2 - b^2$; singular integral is $4z = 3(x^2 - y^2)$

3.14. Standard form III. Only p, q and z present. [Nagpur 2003; Delhi Maths (H) 2006]

Under this standard form we consider differential equation of the form

$$f(p, q, z) = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $\dfrac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \dfrac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \dfrac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \dfrac{dx}{-\frac{\partial f}{\partial p}} = \dfrac{dy}{-\frac{\partial f}{\partial q}}$

or $\dfrac{dp}{p(\partial f/\partial z)} = \dfrac{dq}{q(\partial f/\partial z)} = \dfrac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)} = \dfrac{dx}{-\partial f/\partial p} = \dfrac{dy}{-\partial f/\partial q}$, using (1)

Taking the first two ratios, $(1/p)dp = (1/q)dq$

Integrating, $q = ap$, a being an arbitrary constant. ...(2)

Now, $dz = p\,dx + q\,dy = p\,dx + ap\,dy$, using (2)

or $dz = p(dx + ady) = pd(x + ay) = p\,du$, ...(3)

where $u = x + ay$. ...(4)

Now, (3) $\Rightarrow$ $p = dz/du$ and so by (2) $q = ap = a(dz/du)$.

Substituting these values of p and q in (1), we get $f\left(\dfrac{dz}{du}, a\dfrac{dz}{du}, z\right) = 0$, ...(5)

which is an ordinary differential equation of first order. Solving (5), we get z as a function of u. Complete integral is then obtained by replacing u by $(x + ay)$.

3.15. Working rule for solving equations of the form $f(p,q,z) = 0$. ...(1)

Step I. Let $u = x + ay$, where a is an arbitrary constant. ...(2)

Step II. Replace p and q by dz/du and $a(dz/du)$ respectively in (1) and solve the resulting ordinary differential equation of first order by usual methods.

Step III. Replace u by $x + ay$ in the solution obtained in step II.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form III.

Remark 2. Singular and general integrals are obtined by well known methods.

3.16. SOLVED EXAMPLES BASED ON ART 3.15.

Ex. 1. *Find a complete integral of* $9(p^2z + q^2) = 4$.

[Delhi Maths (H) 2006; Banglore 1995; I.A.S. 1988; Meerut 1996; Rohilkhand 1995]

Sol. Given equation is $9(p^2z + q^2) = 4,$ ...(1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4 \quad \text{or} \quad \left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)}.$$

or $du = \pm (3/2) \times (z + a^2)^{1/2} dz$, separating variables u and z.

Integrating, $u + b = \pm (3/2) \times [(z + a^2)^{3/2}/(3/2)]$ or $u + b = \pm (z + a^2)^{3/2}$

or $(u + b)^2 = (z + a^2)^3$ or $(x + ay + b)^2 = (z + a^2)^3$, as $u = x + ay$

which is a complete integral contaning two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p^2 = qz$. **[Bilaspur 1996; Sagar 2004]**

Sol. Given equation is $p^2 = qz,$ ...(1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$\left(\frac{dz}{du}\right)^2 = \left(a\frac{dz}{du}\right)z \quad \text{or} \quad \frac{dz}{du} = az \quad \text{or} \quad \frac{dz}{z} = a\,du.$$

Integrating, $\log z - \log b = au$ or $z = be^{au}$ or $z = be^{a(x + ay)}$,

which is a complete integral containing two arbitrary constants a and b.

Ex. 3.(a) *Find a complete integral of* $z = pq$. **[Meerut 1994]**

Sol. Given equation is $z = pq,$ ... (1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$z = a\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad \frac{dz}{du} = \pm\frac{\sqrt{z}}{\sqrt{a}} \quad \text{or} \quad \pm\sqrt{a}\, z^{-1/2} dz = du.$$

Integrating, $\pm 2\sqrt{az} = u + b$ or $4(az) = (x + ay + b)^2$, as $u = x + ay$

Ex. 3.(b) *Find a complete integral of* $pq = 4z$.

Sol. Proceed as in Ex. 3.(a). **Ans.** $(x + ay + b)^2 = az$

Ex. 4.(a) *Find a complete integral of* $p(1 + q^2) = q(z - \alpha)$. **[Meerut 1999; Bilaspur 2002; Jiwaji 2003; Ravishanker 2005 Rewa 1998, Vikram 2004]**

(b) *Find a complete integral of* $p(1 + q^2) = q(z - 1)$. **[M.S. Univ. T.N. 2007]**

Sol. (*a*) Given equation is $p(1 + q^2) = q(z - \alpha),$ ...(1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$\frac{dz}{du}\left\{1 + \left(a\frac{dz}{du}\right)^2\right\} = a\frac{dz}{du}(z - \alpha) \quad \text{or} \quad 1 + a^2\left(\frac{dz}{du}\right)^2 = a(z - \alpha)$$

or $$\frac{dz}{du} = \pm\frac{\sqrt{\{a(z-\alpha)-1\}}}{a} \quad \text{or} \quad du = \pm\frac{a\,dz}{\sqrt{\{a(z-\alpha)-1\}}}.$$

Integrating, $u + b = \pm 2\sqrt{\{a(z-\alpha)-1\}}$ or $(u + b)^2 = 4\{a(z - \alpha) - 1\}^2$

or $(x + ay + b)^2 = 4\{a(z - \alpha) - 1\}^2$, a and b being arbitrary constants.

(*b*) Proceed as in part (a) by taking $\alpha = 1$.

Ex. 5.(a) *Find a complete integral of* $pz = 1 + q^2$. **[Meerut 1996]**

Sol. Given equation is $pz = 1 + q^2$, ...(1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$z\frac{dz}{du} = 1 + a^2\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad a^2\left(\frac{dz}{du}\right)^2 - z\frac{dz}{du} + 1 = 0.$$

$$\therefore \frac{dz}{du} = \frac{z \pm (z^2 - 4a^2)^{1/2}}{2a^2} \quad \text{or} \quad \frac{dz}{z \pm (z^2 - 4a^2)^{1/2}} = \frac{du}{2a^2}$$

or $$\frac{[z \mp (z^2 - 4a^2)^{1/2}]dz}{[z \pm (z^2 - 4a^2)^{1/2}][z \mp (z^2 - 4a^2)^{1/2}]} = \frac{du}{2a^2} \quad \text{or} \quad \frac{z \mp (z^2 - 4a^2)^{1/2}}{4a^2} = \frac{du}{2a^2}$$

or $$[z \mp (z^2 - 4a^2)^{1/2}]\, dz = 2du.$$

Integrating, $$\frac{z^2}{2} \mp \left[\frac{z}{2}(z^2 - 4a^2)^{1/2} - \frac{4a^2}{2}\log\left\{z + (z^2 - 4a^2)^{1/2}\right\}\right] = 2u + \frac{b}{2}$$

or $$z^2 \mp \left[z(z^2 - 4a^2)^{1/2} - 4a^2 \log\left\{z + (z^2 - 4a^2)^{1/2}\right\}\right] = 4(x + ay) + b.$$

Ex. 5. (b) *Find a complete integral of* $1 + p^2 = qz$.

Sol. Proceed as in Ex. 5.(*a*). The requied complete integral is

$$a^2z^2 \mp \left[az\sqrt{(a^2z^2 - 4)} - 4\log\left\{az + \sqrt{(a^2z^2 - 4)}\right\}\right] = 4(x + ay) + b.$$

Ex. 6. *Find complete integrals of the following partial differential equations*

(*i*) $p(z + p) + q = 0$

(*ii*) $p(1 + q) = qz$. **[Gulbarga 2005]**

Sol. (*i*) The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we get

$$\frac{dz}{du}\left(z + \frac{dz}{du}\right) + a\frac{dz}{du} = 0 \quad \text{or} \quad \frac{dz}{du} = -(z + a) \quad \text{or} \quad \frac{dz}{z + a} = -du.$$

Integrating, $\log(z + a) - \log b = -u$ or $z + a = be^{-u}$ or $z + a = be^{-(x + ay)}$.

(*ii*) Proceed as in part (*i*). **Ans.** $az - 1 = be^{x + ay}$.

Ex. 7. *Find a complete integral of* $p^3 + q^3 - 3pqz = 0$. **[I.A.S. 1991]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation,

$$\left(\frac{dz}{du}\right)^3 + a^3\left(\frac{dz}{du}\right)^3 - 3az\left(\frac{dz}{du}\right)^2 = 0 \quad \text{or} \quad (1 + a^3)\frac{dz}{du} = 3az \quad \text{or} \quad \frac{1 + a^3}{z}dz = 3au.$$

Integrating $(1 + a^3)\log z = 3au + b$ or $(1 + a^3)\log z = 3a(x + ay) + b$.

Ex. 8. *Find a complete integrals of* (*i*) $p + q = z/c$.

(*ii*) $p + q = z$.

Sol. (*i*) The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we get

$$\frac{dz}{du} + a\frac{dz}{du} = \frac{z}{c} \quad \text{or} \quad (1 + a)\frac{dz}{du} = \frac{z}{c} \quad \text{or} \quad \frac{c(1+a)}{z}dz = du.$$

Integrating, $c(1 + a) \log z = u + b$ or $c(1 + a) \log z = x + ay + b$.

(*ii*) Proceed as in part (*i*). **Ans.** $(1 + a) \log z = x + ay + b$.

Ex. 9. *Find a complete integral of* $p^2 = z^2(1 - pq)$. **[Jiwaji 1998; Meerut 2001]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(\frac{dz}{du}\right)^2 = z^2\left\{1 - a\left(\frac{dz}{du}\right)^2\right\} \quad \text{or} \quad \left(\frac{dz}{du}\right)^2 (1 + az^2) = z^2$$

or $$\frac{dz}{du} = \pm\frac{z}{\sqrt{(1+az^2)}} \quad \text{or} \quad \pm du = \frac{\sqrt{(1+az^2)}}{z}dz = \pm\frac{(1+az^2)dz}{z\sqrt{(1+az^2)}}.$$

or $$\pm\int du \pm b = \int\frac{dz}{z\sqrt{(1+az^2)}} + \frac{1}{2}\int\frac{2az\,dz}{\sqrt{(1+az^2)}}. \qquad ...(1)$$

Now, $$\int\frac{dz}{z\sqrt{(1+az^2)}} = \int\frac{(-1/t^2)dt}{(1/t)\times\sqrt{\{(1+(a/t^2)\}}}, \text{ putting } z = 1/t \text{ so that } dz = -(1/t^2)dt$$

$$= -\int\frac{dt}{(t^2+a)^{1/2}} = -\sinh^{-1}\frac{t}{\sqrt{a}} = -\sinh^{-1}\frac{1}{z\sqrt{a}}, \text{ as } t = \frac{1}{z} \qquad ...(2)$$

and $$\frac{1}{2}\int\frac{2az\,dz}{(1+az^2)^{1/2}} = \frac{1}{2}\int\frac{2v\,dv}{v}, \text{ putting } 1 + az^2 = v^2 \text{ and } 2az\,dz = 2v\,dv$$

$$= v = (1 + az^2)^{1/2}.$$

Using (2) and (3), (1) reduces to $\pm(u + b) = -\sinh^{-1}(1/z\sqrt{a}) + (1 + az^2)^{1/2}$

or $$\pm(x + ay + b) = -\sinh^{-1}(1/z\sqrt{a}) + (1 + az^2)^{1/2}.$$

Ex. 10. *Find complete and singular integrals of* $4(1 + z^3) = 9z^4pq$.

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$4(1 + z^3) = 9z^4a\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad \pm\frac{3\sqrt{a}z^2}{(1+z^3)^{1/2}}dz = 2\,du$$

or $$\pm\left(\sqrt{a}/t\right) \times 2t\,dt = 2\,du, \text{ putting } 1 + z^3 = t^2 \text{ so that } 3z^2dz = 2t\,dt$$

Integrating, $\pm\sqrt{a}\,t = u + b$ or $\pm\sqrt{a}\,(1 + z^3)^{1/2} = x + ay + b$

or $$a(1 + z^3) = (x + ay + b)^2, \qquad ...(1)$$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially w.r.t. a and b by turn, we get

$$1 + z^3 = 2y(x + ay + b) \qquad ...(2)$$

and $$0 = 2(x + ay + b). \qquad ...(3)$$

Eliminating a and b from (1), (2) and (3), the singular integral is $1 + z^3 = 0$. ...(4)

From (4), $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Thus these values of p and q together with $1 + z^3 = 0$ satisfy the given equation. Hence $1 + z^3 = 0$ is the required singular integral.

Ex. 11. *Find complete and singular integrals of* $q^2 = z^2p^2(1 - p^2)$.

[Madurai Kamraj 2008; CDLU 2004]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(a\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left[1-\left(\frac{dz}{du}\right)^2\right] \quad \text{or} \quad a^2 = z^2\left[1-\left(\frac{dz}{du}\right)^2\right]$$

or $$\left(\frac{dz}{du}\right)^2 = \frac{z^2 - a^2}{z^2} \quad \text{or} \quad du = \pm\frac{zdz}{(z^2 - a^2)^{1/2}}.$$

Integrating, $u + b = \pm (z^2 - a^2)^{1/2}$ or $(x + ay + b)^2 = z^2 - a^2$, ...(1)

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$-2a = 2y(x + ay + b) \quad ...(2)$$

and $$0 = 2(x + ay + b). \quad ...(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ satisfy the given equation and hence the required singular integral is $z = 0$.

Ex. 12. *Find complete, singular and general integral of* $p^3 + q^3 = 27z$. **[Ravishankar 2005]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(\frac{dz}{du}\right)^3 + \left(a\frac{dz}{du}\right)^3 = 27z \quad \text{or} \quad \frac{dz}{du}(1 + a^3)^{1/3} = 3z^{1/3}$$

or $$du = (1/3) \times (1 + a^3)^{1/3}\, z^{-1/3}\, dz.$$

Integrating, $u + b = (1/3) \times (1 + a^3)^{1/3} \times [z^{2/3}/(2/3)]$ or $2(u + b) = (1 + a^3)^{1/3}z^{2/3}$

or $8(u + b)^3 = (1 + a^3)z^2$ or $8(x + ay + b)^3 = (1 + a^3)z^2$, ...(1)

which is a complete integral containing two arbitrary contants a and b.

Singular Integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$24y(x + ay + b)^2 = 3a^2z^2 \quad ...(2)$$

and $$24(x + ay + b) = 0. \quad ...(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ satisfy the given equation and hence the required singular integral is $z = 0$.

General integral. Let $b = \phi(a)$, where ϕ is an arbitrary function. Then (1) becomes

$$8[x + ay + \phi(a)]^3 = z^2(1 + a^3). \quad ...(4)$$

Differentiating (4) partially w.r.t. 'a', $24[x + ay + \phi(a)]^2\,[y + \phi'(a)] = 3a^2z^2$. ...(5)

General integral is obtained by eliminating a from (4) and (5).

Ex. 13. *Find complete and singular integrals of* $z^2(p^2z^2 + q^2) = 1$.

[Delhi Maths Hons 2005; Meerut 2003]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$z^2\left[z^2\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 1 \quad \text{or} \quad z^2(z^2 + a^2)\left(\frac{dz}{du}\right)^2 = 1$$

or $$du = \pm z(z^2 + a^2)^{1/2}\, dz = \pm (1/2) \times (z^2 + a^2)^{1/2}\, (2zdz)$$

Integrating, $u + b = \pm (1/2) \times [(z^2 + a^2)^{3/2}/(3/2)]$

or $9(u + b)^2 = (z^2 + a^2)^3$ or $9(x + ay + b)^2 = (z^2 + a^2)^3$, ...(1)

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially, w.r.t. 'a' and 'b', we get

$$18(x + ay + b)y = 3(z^2 + a^2) \times 2a \qquad ...(2)$$

and $$18(x + ay + b) = 0. \qquad ...(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ do not satisfy the given equation. Hence $z = 0$ is not a singular solution of the given equation.

Ex. 14. (*i*) *Find a complete integral of* $z^2(p^2 + q^2 + 1) = k^2$.

[Jabalpur 2004; Banglore 1993; I.A.S. 1996; Meerut 1997]

(*ii*) *Find a complete and singular integral of* $z^2(p^2 + q^2 + 1) = 1$. **[I.A.S. 1979]**

Sol. (*i*) The given equation is of the form $f(p, q\, z) = 0$. Let $u = x + ay$ where a is an arbitrary constant. Replacing p by (dz/du) and q by $a(dz/du)$ in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = k^2 \qquad \text{or} \qquad (1 + a^2)\left(\frac{dz}{du}\right)^2 = \frac{k^2 - z^2}{z^2}$$

or $$\pm (1 + a^2)^{1/2} \frac{z}{(k^2 - z^2)^{1/2}}\, dz = du \qquad \text{or} \qquad \pm \frac{1}{2}(1 + a^2)(k^2 - z^2)^{-1/2}(-2z dz) = du.$$

Integrating, $$\pm (1 + a^2)^{1/2}\, (k^2 - z^2)^{1/2} = u + b \qquad \text{or} \qquad (1 + a^2)(k^2 - z^2) = (u + b)^2$$

or $$(1 + a^2)(k^2 - z^2) = (x + ay + b)^2.$$

(*ii*) Here $k = 1$. Proceed as in part (*i*) and get complete integral

$$(1 + a^2)(1 - z^2) = (x + ay + b)^2. \qquad ...(1)$$

Differentiating (1) partially w.r.t. a and b, we get

$$2a(1 - z^2) = 2(x + ay + b) \times y \qquad ...(2)$$

and $$0 = 2(x + ay + b). \qquad ...(3)$$

From (2) and (3), we get $x + ay + b = 0$ and $a = 0$. With these values (1) reduces to $z^2 = 1$, which is free from a and b. Again, from $z^2 = 1$, $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Now, $p = 0$, $q = 0$ and $z^2 = 1$, satisfy the given equation and hence singular integral of the given equation is $z^2 = 1$.

Ex. 15. *Find a complete integral of* (*i*) $q^2y^2 = z(z - px)$ **[Meerut 1997]**

(*ii*) $p^2x^2 = z(z - qy)$.

Sol. (*i*) Given equation can be rewritten as $$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right). \qquad ...(1)$$

We choose new variables X and Y such that $(1/x)dx = dX$ and $(1/y)dy = dY$. ...(2)

so that $$\log x = X \qquad \text{and} \qquad \log y = Y. \qquad ...(3)$$

Using (2), (1) becomes $$\left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right) \qquad \text{or} \qquad Q^2 = z\,(z - P), \qquad ...(4)$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q, z) = 0$. Let $u = X + aY$, where a is an arbitrary constant. Replacing P by dz/du and Q by $a(dz/du)$ in (4), we get

$$a^2\left(\frac{dz}{du}\right)^2 = z\left(z - \frac{dz}{du}\right) \qquad \text{or} \qquad a^2\left(\frac{dz}{du}\right)^2 + z\frac{dz}{du} - z^2 = 0.$$

$$\therefore \qquad \frac{dz}{du} = \frac{-z \pm (z^2 + 4a^2z^2)^{1/2}}{2a^2} = \frac{-1 \pm (1 + 4a^2)^{1/2}}{2a^2} z = kz, \qquad ...(5)$$

where $k = [-1 \pm (1 + 4a^2)^{1/2}]/2a^2$. ...(6)

From (5), $(1/kz)dz = du$ so that $(1/k) \log z = u + \log b$

or $\log z^{1/k} = X + aY + \log b = \log x + a \log y + \log b = \log (xby^a)$.

$\therefore\ z^{1/k} = xby^a$ is complete integral containing two arbitrary constants a and b and an absolute constant k given by (6).

(*ii*) Proceed as in part (*i*). **Ans.** $xby^a = z^{1/k}$, where $k = [-1 \pm (a^2 + 4)^{1/2}]/2$

Ex. 16. *Solve* $p^2 + q^2 = z$.

Sol. Given equation is $p^2 + q^2 = z$, ... (1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we have

$(du/dz)^2 + a^2(du/dz)^2 = z$ or $(dz/du)^2 = z/(1 + a^2)$

or $\dfrac{dz}{du} = \pm \dfrac{z^{1/2}}{(1+a^2)^{1/2}}$ or $\pm z^{-1/2}(1+a^2)^{1/2}\,dz = du$

Integrating, $\pm 2z^{1/2}(1+a^2)^{1/2} = u + b$ or $\pm 2z^{1/2}(1+a^2)^{1/2} = x + ay + b$

Thus, $4z(1 + a^2) = (x + ay + b)^2$, a, b being arbitrary constants ... (2)

(2) is the complete integral of the given equation (1).

Differentiating (2) partially w.r.t. 'a' and 'b', we get

$8az = 2y(x + ay + b)$ or $4az = y(x + ay + b)$ ... (3)

$0 = 2(x + ay + b)$ or $x + ay + b = 0$ ... (4)

Substituting the value of $x + ay + b$ from (4) in (3), we have

$4az = 0$ or $z = 0$, which is the singular solution.

In order to get the general solution, put $b = \psi(a)$ in (2) and get

$$4z(1+a^2) - \{x + ay + \psi(a)\}^2 = 0 \quad \text{... (5)}$$

Differentiating (5) partially w.r.t. 'a', $8az - 2\{x + ay + \psi(a)\} \times \{y + \psi'(a)\} = 0$ ... (6)

The required general solution is given by (5) and (6)

Ex. 17. *Find the complete integral of* $16p^2z^2 + 9q^2z^2 + 4(z^2 - 1) = 0$

Sol. Given equation is of the form $f(p, q, z) = 0$

Let $u = x + ay$, a being an arbitrary constant. ... (1)

Now replacing p and q by dz/du and $a(dz/du)$ respectively in the given equation, we have

$$16z^2\ (dz/du)^2 + 9a^2z^2\ (dz/du)^2 + 4(z^2 - 1) = 0$$

or $(16 + 9a^2)z^2 \left(\dfrac{dz}{du}\right)^2 = 4(1 - z^2)$ or $\dfrac{dz}{du} = \dfrac{2(1-z^2)^{1/2}}{z(16+9a^2)^{1/2}}$

or $(-1/2) \times (16 + 9a^2)^{1/2}\ (1 - z^2)^{-1/2}(-2z)dz = du$

Integrating, $-(16 + 9a^2)^{1/2}\ (1 - z^2)^{1/2} = u + b = x + ay + b$, by (1)

or $(16 + 9a^2)\ (1 - z^2) = (x + ay + b)^2$ is the complete integral, a, b being arbitrary constants

Ex. 18. *Find the complete integral of* $pq = x^4y^3z^2$

Sol. Re–writing the given equation, we get $(\partial z / x^4 \partial x)\ (\partial z / y^3 \partial y) = z^2$ ... (1)

Putting $x^4dx = dX$, $y^3dy = dY$ so that $x^5/5 = X$, $y^4/4 = Y$, (1) gives

$(\partial z/\partial X)\ (\partial z/\partial Y) = z^2$ or $PQ = z^2$ ... (2)

which is of the form $f(P,Q,z) = 0$. Let $u = X + aY$, a being an arbitrary constant. Replacing P and Q be dz/du and $a(dz/du)$ respectively in (2), we get

$a\,(dz/du)^2 = z^2$ giving $(\sqrt{a}/z)dz = du$

Integrating $\sqrt{a}\log z = u + b = X + aY + b$, as $u = X + aY$

or $\sqrt{a}\ \log z = x^5/5 + (ay^4)/4 + b$, a, b being arbitrars constants

EXERCISE 3(E)

Find the complete integral of the following equations (1–7)

1. $z = p^2 - q^2$. **Ans.** $x + ay + b = 4(1 - a^2)z$

2. $zpq = p + q$. **(Delhi 2007)** **Ans.** $x + ay + b = (4az)/(1 - a)$

3. $z^2p^2 + q^2 = p^2q$. **Ans.** $z = a\tan(x + ay + b)$

4. $p^2z^2 + q^2 = 1$. **[Delhi B.A. (Prog) II 2010, 11; M.S. Univ. T.N. 2007, Nagpur 2001; Meerut 2008]** **Ans.** $x + ay + b = \pm[(z/2)\times(z^2 + a^2)^{1/2} + (a^2/2)\times\sinh^{-1}(z/a)]$

5. $p^3 = qz$. **(Delhi B.Sc. (Prog) II 2007)** **Ans.** $4z = (x + ay + b)^2$

6. $16z^2p^2 + 25z^2a^2 + 9z^2 - 81 = 0$. **Ans.** $(16 + 25a^2)\,(9 - z^2) = 9(x + ay + b)^2$

7. $z^2 = 1 + p^2 + q^2$. **Ans.** $z = \cosh\{x + ay + b)/(1 + a^2)^{1/2}\}$

8. Using Charpit's method, discuss how to solve equations of the form $f(z, p, q) = 0$. Hence find complete integral of the equation $9(p^2z + q^2) = 4$. **[Delhi Maths (H) 2006]**

Hint : Refer Art. 3.14 and Ex. 1 of Art. 3.15

Solve the following partial differential equations (9 – 14)

9. $z^2(p^2 + q^2 + 2) = 1$ **Ans. C.I.** $(1 + a^2)\,(1 - 2z^2) = 4(x + ay + b)^2$; **S.S.** $2z^2 - 1 = 0$;

G.S. It is given by $(1 + a^2)\,(1 - 2z^2) - \{x + ay + \psi(a)\}^2 = 0$, $a(1 - 2z^2)$

$-4\{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$

10. $z = pq$. **Ans. C.I.** $4az = (x + ay + b)^2$; **S.S.** $z = 0$; **G.S.** It is given by

$4az - \{x + ay + \psi(a)\}^2$, $2z - \{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$

11. $p(1 - q^2) = q(1 - z)$. **Ans. C.I.** $4(1 - a + az) = (x + ay + b)^2$; **S.S.** Does not exist; **G.S.** It is given by $4(1 - a + az) - \{x + ay + \psi(a)\}^2 = 0$, $2z - 2 - \{x + ay + \psi(a)\}\{y + \psi'(a)\} = 0$

12. $p^2 + pq = 4z$. **Ans. C.I.** $(1 + a)z = (x + ay + b)^2$; **S.S.** $z = 0$; **G.S.** $(1 + a)z - \{x + ay + \psi(a)\}^2 = 0$,

$z - 2\{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$

13. $p^3 + q^3 = 3pqz$, $z > 0$. **Ans. C.S.** $(1 + a^3)\log z = 3a(x + ay) + b$; **S.S.** Does not exist. **G.S.** It is given by $(1 + a^3)\log z - 3a(x + ay) - \psi(a) = 0$, $3a^2\log z - 3x - 6ay - \psi'(a) = 0$

14. $p^2 + q^2 = 4z$. **Ans. C.I.** $4(1 + a^2)z - (x + ay + b)^2 = 0$; **S.S.** $z = 0$; **G.S.** It is given by

$(1 + a^2)z - \{x + ay + \psi(a)\}^2 = 0$, $az - \{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$.

3.17. Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$. *i.e.*, a form in which z does not appear and the terms containing x and p are on one side and those containing y and q on the other side. **[Bhopal 2010; Ravishankar 1999]**

Let $$F(x, y, z, p, q) = f_1(x, p) - f_2(y, q) = 0. \quad ...(1)$$

Then Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or $$\frac{dp}{\partial f_1/\partial x} = \frac{dq}{-\partial f_2/\partial y} = \frac{dz}{-p(\partial f_1/\partial p) + q(\partial f_2/\partial q)} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_2/\partial q}, \text{ by (1)}$$

Taking the first and the fourth ratios, we have

$(\partial f_1 / \partial p)\, dp + (\partial f_1 / \partial x) dx = 0$ or $df_1 = 0.$

Integrating, $f_1 = a$, a being an arbitrary constant.

$\therefore$ (1) $\Rightarrow$ $f_1(x, p) = f_2(y, q) = a.$ ...(2)

Now, (2) $\Rightarrow$ $f_1(x, p) = a$ and $f_2(y, q) = a.$ ...(3)

From (3), on solving for p and q respectively, we get

$p = F_1(x, a)$, say and $q = F_2(y, a)$, say ...(4)

Substituting these values in $dz = p\, dx + q\, dy$, we get $dz = F_1(x, a)\, dx + F_2(y, a)\, dy.$

Integrating, $$z = \int F_1(x, a)\, dx + \int F_2(y, a) dy + b,$$

which is a complete integral containing two arbitrary constants a and b.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form IV.

Remark 2. Singular and general integral are obtained by well known methods.

3.18. SOLVED EXAMPLES BASED ON ART 3.17

Ex. 1. *Find a complete integral of* $x(1 + y)p = y(1 + x)q$. **[Agra 1991]**

Sol. Separating p and x from q and y, the given equation reduces to

$$(xp)/(1 + x) = (yq)/(1 + y)$$

Equating each side to an arbitrary constant a, we have

$$\frac{xp}{1+x} = a \quad \text{and} \quad \frac{yq}{1+y} = a \quad \text{so that} \quad p = a\left(\frac{1+x}{x}\right) \quad \text{and} \quad q = a\left(\frac{1+y}{y}\right).$$

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = \frac{a(1+x)}{x}dx + \frac{a(1+y)}{y}dy \quad \text{or} \quad dz = a\left(\frac{1}{x}+1\right)dx + a\left(\frac{1}{y}+1\right)dy.$$

Integrating, $z = a(\log x + x) + a(\log y + y) + b = a(\log xy + x + y) + b,$

which is a complete integral containing two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p - 3x^2 = q^2 - y$. **[Meerut 1996]**

Sol. Equating each side to an arbitrary constant a, we get

$p - 3x^2 = a$ and $q^2 - y = a$ so that $p = a + 3x^2$ and $q = (a + y)^{1/2}.$

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (a + 3x^2)dx + (a + y)^{1/2}\, dy$ so that $z = ax + x^3 + (2/3) \times (a + y)^{3/2} + b.$

Ex. 3. *Find a complete integral of* $yp = 2yx + \log q$. **[Ravishankar 2005]**

Sol. Rewritting the given equation, $p = 2x + (1/y)\log q$ or $p - 2x = (1/y)\log q$.

Equating each side to an arbitrary constant a, we get

$p - 2x = a$ and $(1/y)\log q = a$ so that $p = a + 2x$ and $q = e^{ay}$.

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$dz = (a + 2x)dx + e^{ay}dy$ so that $z = (ax + x^2) + (1/a) \times e^{ay} + b$.

Ex. 4. *Find a complete integral of* $q = px + p^2$.

[Agra 1995; Meerut 1994; Bilaspur 2004; Jabalpur 1998]

Sol. Equating each side of the given equation to an arbitrary constant a, we have

$q = a$ and $px + p^2 = a$ or $q = a$ and $p^2 + px - a = 0$.

$\therefore$ $q = a$ and $p = [-x \pm (x^2 + 4a)^{1/2}]/2$.

Putting thse values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (1/2) \times [-x \pm (x^2 + 4a)^{1/2}]dx + a\,dy.$$

Integrating, $$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{(x^2+4a)} + 2a\log\left\{x + \sqrt{(x^2+4a)}\right\}\right] + ay + b.$$

Ex. 5. *Solve* $py + qx + pq = 0$. **[Kurukshetra 2004; I.A.S 1990]**

Sol. Given $py + q(x + p) = 0$. or $p/(p + x) = -q/y$.

Equating each side to an arbitrary constant a, we get

$p/(p + x) = a$ and $-q/y = a$ $\Rightarrow$ $p = (xa)/(1 - a)$ and $q = -ay$.

Putting thse values of p and q in $dz = p\,dx + q\,dy$, we get

$dz = \{a/(1 - a)\}\,x\,dx - ay\,dy$ so that $z = \{a/(1 - a)\} \times (x^2/2) - a \times (y^2/2) + b/2$

or $2z = \{a/(1 - a)\}x^2 - ay^2 + b$, a, b being arbitrary constants.

Ex. 6. *Find a complete integral of* $z^2(p^2 + q^2) = x^2 + y^2$, *i.e.*, $z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2] = x^2 + y^2$.

[Agra 2006; Jabalpur 2004; Rewa 2002 Sagar 1999; Vikram 1996 Delhi Maths Hons 1990; I.A.S. 1989; Kanpur 1994; Meerut 2003]

Sol. Given $$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 \quad \text{or} \quad \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \quad ...(1)$$

Let $z\,dz = dz$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$,

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side of the above equation to an arbitrary constant a^2, we get

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$.

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, we have

$$dZ = (a^2 + x^2)^{1/2}\,dx + (y^2 - a^2)^{1/2}\,dy.$$

Integrating, $$Z = (x/2) \times (a^2 + x^2)^{1/2} + (a^2/2) \times \log\{x + (a^2 + x^2)^{1/2}\}$$
$$+ (y/2) \times (y^2 - a^2)^{1/2} - (a^2/2) \times \log\{y + (y^2 - a^2)^{1/2}\} + (b/2)$$

or $$z^2 = x^2(a^2 + x^2)^{1/2} + a^2\log[x + (a^2 + x^2)^{1/2}] + y(y^2 - a^2)^{1/2} - a^2\log\{y + (y^2 - a^2)^{1/2}\} + b$$

[$\because$ From (2), $Z = z^2/2$]

Ex. 7. *Find a complete integral of* $z(p^2 - q^2) = x - y$.

[Bilaspur 2003; Indore 2002, 02; Jiwaji 2000; Bangalore 1995; I.A.S 1989]

Sol. Re–writting the given equation, $(\sqrt{z}\partial z/\partial x)^2 - (\sqrt{z}\partial z/\partial y)^2 = x - y.$ ...(1)

Let $\sqrt{z}\,dz = dZ$ so that $(2/3) \times z^{3/2} = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 - (\partial Z/\partial y)^2 = x - y$ or $P^2 - Q^2 = x - y,$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = Q^2 - y. \quad ...(3)$$

Equating each side to an arbitrary constant a, we get

$P^2 - x = a$ and $Q^2 - y = a$ so that $P = (x + a)^{1/2}$ and $Q = (y + a)^{1/2}$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (x + a)^{1/2}dx + (y + a)^{1/2}\,dy.$

Integrating, $Z = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$

or $(2/3) \times z^{3/2} = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3,$ as $Z = (2/3) \times z^{3/2}$

or $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b,$ a, b being arbitrary constants.

Ex. 8. *Find a complete integral of* $z(xp - yq) = y^2 - x^2.$

Sol. Re–writting the given equation, we have

$$xz\frac{\partial z}{\partial x} - yz\frac{\partial z}{\partial y} = y^2 - x^2 \quad \text{or} \quad x\left(z\frac{\partial z}{\partial x}\right) - y\left(z\frac{\partial z}{\partial y}\right) = y^2 - x^2. \quad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z.$ ...(2)

Using (2), (1) becomes $x(\partial Z/\partial x) - y(\partial Z/\partial y) = y^2 - x^2$ or $xP - yQ = y^2 - x^2,$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$xP + x^2 = yQ + y^2.$$

Equating each side to an arbitrary constant a, we have

$xP + x^2 = a$ and $yQ + y^2 = a$ so that $P = a/x - x$ and $Q = a/y - y.$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a/x - x)dx + (a/y - y)dy.$

Integrating, $Z = a \log x - (x^2/2) + a \log y - (y^2/2) + b/2$

or $z^2/2 = a(\log x + \log y) - (x^2 + y^2 - b)/2$ or $z^2 = 2a \log (xy) - x^2 - y^2 + b.$

Ex. 9. *Find a complete integral of* $p^2 + q^2 = z^2(x + y).$ **[Agra 2010; M.S. Univ. T.N. 2007]**

Sol. Given $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z^2(x + y)$ or $\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{z}\frac{\partial z}{\partial y}\right)^2 = x + y$ ....(1)

Let $(1/z)dz = dZ$ so that $\log z = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x + y$ or $P^2 + Q^2 = x + y,$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = y - Q^2.$$

Equating each side to an arbitrary constant a, we have

$P^2 - x = a$ and $y - Q^2 = a$ so that $P = (a + x)^{1/2}$ and $Q = (y - a)^{1/2}.$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a + x)^{1/2}dx + (y - a)^{1/2}dy.$

Integrating, $Z = (2/3) \times [(a + x)^{3/2} + (y - a)^{3/2}] + (2/3) \times b$

$\therefore$ $\log z = (2/3) \times [(a + x)^{3/2} + (y - a)^{3/2} + b]$ is a complete integral, using $Z = \log z$

Ex. 10. *Find a complete integral of* $p^2 + q^2 = (x^2 + y^2)z.$ **[Delhi Maths Hons. 1995]**

Sol. The given equation can be rewritten as

$$\frac{1}{z}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2 + y^2 \quad \text{or} \quad \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \quad ...(1)$$

Let $\left(1/\sqrt{z}\right)dz = dZ$ *i.e.*, $z^{-1/2}dz = dZ$ so that $2\sqrt{z} = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$,

where $P = \partial Z/\partial x$ *and* $Q = \partial Z/\partial y$. Separting P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side to an arbitrary constant a^2, we have

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a^2 + x^2)^{1/2}dx + (y^2 - a^2)^{1/2}dy$.

Integrating, $$Z = \frac{x}{2}(x^2 + a^2)^{1/2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}(y^2 - a^2)^{1/2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + \frac{b}{2}$$

or $4z^{1/2} = x(x^2 + a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2 - a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$, as $Z = 2\sqrt{z}$

Ex. 11. *Find a complete integral of* $(p^2/x) - (q^2/y) = (1/z) \times [(1/x) + (1/y)]$.

[Delhi B.Sc. Hons. 1996]

Sol. The given equation can be re–written as

$$\frac{z}{x}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{z}{y}\left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{or} \quad \frac{1}{x}\left(\sqrt{z}\,\frac{\partial z}{\partial x}\right)^2 - \frac{1}{y}\left(\sqrt{z}\,\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad ....(1)$$

Let $\sqrt{z}\,dz = dZ$ so that $(2/3) \times z^{3/2} = Z$. ...(2)

Using (2), (1) becomes

$$\frac{1}{x}\left(\frac{\partial Z}{\partial x}\right)^2 - \frac{1}{y}\left(\frac{\partial Z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{or} \quad \frac{P^2}{x} - \frac{Q^2}{y} = \frac{1}{x} + \frac{1}{y},$$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$(P^2 - 1)/x = (Q^2 + 1)/y.$$

Equating each side to an arbitrary constant a, we have

$(P^2 - 1)/x = a$ and $(Q^2 + 1)/y = a$ so that $P = (1 + ax)^{1/2}$ and $Q = (ay - 1)^{1/2}$.

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (1 + ax)^{1/2}dx + (ay - 1)^{1/2}dy$.

Integrating, $Z = (2/3a) \times (1 + ax)^{3/2} + (2/3a) \times (ay - 1)^{3/2} + (2/3a) \times b$

or $az^{3/2} = (1 + ax)^{3/2} + (ay - 1)^{3/2} + b$, as $Z = (2/3) \times z^{3/2}$.

Ex. 12. *Find a complete integral of* $yzp^2 = q$. **[M.S. Univ. T.N. 2007]**

Sol. Given $$yz^2\left(\frac{\partial z}{\partial x}\right)^2 = z\,\frac{\partial z}{\partial y} \quad \text{or} \quad y\left(z\frac{\partial z}{\partial x}\right)^2 = \left(z\frac{\partial z}{\partial y}\right). \quad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $y(\partial Z/\partial x)^2 = \partial Z/\partial y$ or $yP^2 = Q$, ...(3)

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P from y and Q, we get

$P^2 = Q/y = a^2$, (say) ; a being an arbitrary constant. Hence $P = a$ and $Q = ya^2$.

Then, $dZ = P\,dx + Q\,dy$ reduces to $dZ = a\,dx + ya^2dy$ so that $Z = ax + (a^2/y^2)/2 + b/2$

or $z^2/2 = ax + (a^2y^2)/2 + b/2$ or $z^2 = 2ax + a^2y + b$.

Ex. 13. *Find a complete integral of* $zpy^2 = x(y^2 + z^2q^2)$.

Sol. Given $$y^2\left(z\frac{\partial z}{\partial x}\right) = x\,y^2 + x\left(z\frac{\partial z}{\partial y}\right)^2 \quad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $y^2(\partial Z/\partial x) = xy^2 + x\,(\partial Z/\partial y)^2$ or $y^2P = x(y^2 + Q^2)$,

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from, Q and y, we get

$$P/x = (y^2 + Q^2)/y^2.$$

Equating each side to an arbitrary constant a, we get

$P/x = a$ and $1 + (Q^2/y^2) = a$ so that $P = ax$ and $Q = \pm (a-1)^{1/2} y$.

$\therefore \quad dZ = P\,dx + Q\,dy = ax\,dx \pm (a-1)^{1/2} y\,dy$

Integrating, $Z = (ax^2/2) \pm (a-1)^{1/2}(y^2/2) + b/2$ or $z^2 = ax^2 \pm (a-1)^{1/2} y^2 + b$, as $Z = z^2/2$.

Ex. 14. *Find the complete integral of the partial differential equation*

$$2p^2q^2 + 3x^2y^2 = 8x^2q^2(x^2 + y^2)$$ **[I.A.S. 2001]**

Sol. Re–writing the given equation, we have

$2q^2(p^2 - 4x^4) = x^2y^2(8q^2 - 3)$ or $(p^2 - 4x^4)/x^2 = y^2\ (8q^2 - 3)/2q^2 = 4a^2$, say

where a is an arbitrary constant. Then, $p^2 = 4x^2(a^2 + x^2)$ and $8q^2(y^2 - a^2) = 3y^2$

so that $p = 2x(a^2 + x^2)^{1/2}$ and $q = (3/2)^{1/2} \times (y/2) \times (y^2 - a^2)^{-1/2}$

Substituting these values in $dz = p\,dx + q\,dy$, we get

$$dz = 2x(a^2 + x^2)^{1/2}\,dx + (3/2)^{1/2} \times (y/2) \times (y^2 - a^2)^{-1/2}\,dy$$

Integrating, $z = 2\int x(a^2 + x^2)^{1/2}\,dx + (3/2)^{1/2} \times (1/2) \times \int y(y^2 - a^2)^{-1/2}\,dy + b$ ... (1)

Put $x^2 + a^2 = u$ and $y^2 - a^2 = v$ so that $2xdx = du$ and $2ydy = dv$ ... (2)

i.e., $xdx = (1/2) \times du$ and $ydy = (1/2) \times dv$. Then (1) reduces to

$$z = \int u^{1/2}\,du + (3/2)^{1/2} \times (1/4) \times \int v^{-1/2}\,dv + b$$

or $$z = (2/3) \times u^{3/2} + (3/2)^{1/2} \times (1/4) \times 2v^{1/2} + b$$

or $$z = (2/3) \times (x^2 + a^2)^{3/2} + (3/2)^{1/2} \times (1/2) \times (y^2 - a^2)^{1/2} + b,$$

which is the required complete integral containing a and b as arbitrary constants.

Ex. 15. *Find the complete integral of the partial differential equation* $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$ **[Delhi Maths (H) 2002; Agra 2005]**

Sol. Re–writing, $p^2/x^2 + y^2/q^2 = x^2 + y^2$ or $(p^2/x^2) - x^2 = y^2 - (y^2/q^2) = a^2$, say

$\Rightarrow \quad p = x(x^2 + a^2)^{1/2}$, and $q = y/(y^2 - a^2)^{1/2}$

$\therefore \quad dz = p\,dx + q\,dy$ becomes $dz = x(x^2 + a^2)^{1/2}\,dx + y(y^2 - a^2)^{-1/2}\,dy$

Integrating, $z = (1/3) \times (x^2 + a^2)^{1/2} + (y^2 - a^2)^{1/2} + b$,

which is complete integral with a and b as arbitrary constants.

Ex. 16. *Find the complete integral of* $(1 - x^2)yp^2 + x^2q = 0$

Sol. Re–writing, we have $(x^2 - 1)p^2/x^2 = q/y = a^2$, say

$\therefore \quad p = ax/(x^2 - 1)^{1/2}$ and $q = a^2y$. Hence $dz = p\,dx + q\,dy$ becomes

$dz = ax(x^2 - 1)^{-1/2}\,dx + a^2y\,dy$ so that $z = a(x^2 - 1)^{1/2} + (a^2y^2)/2 + b$.

Ex. 17. *Find the the complete integral of* $p + q - 2px - 2qy + 1 = 0$.

Sol. Re–writing, $p - 2px = 2qy - q - 1 = a$, say

$\therefore \quad p = \dfrac{a}{1 - 2x}, \quad q = \dfrac{a+1}{2y - 1}$ and so $dz = p\,dx + q\,dy = \dfrac{a\,dx}{1 - 2x} + \dfrac{(a+1)dy}{2y - 1}$

Integrating, $z = -(a/2) \times \log|1-2x| + (1/2) \times (a+1) \log|2y+1| + b$.

Ex. 18. *Find the complete integral of* $2x(z^2q^2 + 1) = pz$

Sol. Re–writing the given equation, we have $2x\{(z\partial z/\partial y)^2 + 1\} = (z\partial z/\partial x)$ ... (1)

Putting $z\,dz = dZ$ so that $z^2/2 = Z$, (1) reduces to

$2x\{(\partial Z/\partial y)^2 + 1\} = \partial Z/\partial x$ or $2x(Q^2 + 1) = P$, ... (2)

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Re–writing (2), we have

$(P/2x) - 1 = Q^2 = a^2$, say so that $P = 2x(1 + a^2)$, $Q = a$

$\therefore$ $dZ = P\,dx + Q\,dy$ becomes $dZ = 2x(1 + a^2)dx + a\,dy$

Integrating, $Z = (1 + a^2)x^2 + ay + b$ or $z^2/2 = (1 + a^2)x + ay + b$.

EXERCISE 3 (F)

Find a complete integral of the following equations (1 – 9)

1(a). $p^2 = q + x$. **Ans.** $z = (2/3) \times (a + x)^{3/2} + ay + b$.

(b). $p^2y\,(1 + x^2) = qx^2$. **[Delhi B.A (Prog) II 2011]** **Ans.** $z = a(1 + x^2)^{1/2} + (a^2y/2) + b$.

2. $p^2 + q^2 = x + y$. **[Agra 2009; Meerut 2007]** **Ans.** $3z = 2(x + a)^{3/2} + 2(y - a)^{3/2} + b$.

3. $p^2 + q^2 = x^2 + y^2$. **[Jiwaji 1999; Ravishankar 2003]**

Ans. $2z = x(x^2 + a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2 - a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$.

4. $pe^y = qe^x$. **[Jiwaji 1996]** **Ans.** $z = ae^x + ae^y + b$.

5. $p^{1/3} - q^{1/3} = 3x - 3y$. **Ans.** $z = 3x^3 - 3ax^2 + a^2x + 2y^4 - 4ay^3 + 3a^2y^2 - a^3y + b$.

6. $q = 2yp^2$. **Ans.** $z = ax + a^2y^2 + b$.

7. $p^2 - y^3q = x^2 - y^2$. **Ans.** $2z = x(x^2 + a^2)^{1/2} + a^2\sinh^{-1}(x/a) - (a^2/2) + \log y^2 + b$.

8. $z^2(p^2 + q^2) = x^2 + e^{2y}$. **[Delhi Maths (H) 2005]**

Ans. $z^2 = x(x^2 + a)^{1/2} + a\,\sinh^{-1}(x/\sqrt{a}) + 2(e^{2y} - a)^{1/2} - \sqrt{a}\,\tan^{-1}\{(e^{2y} - a)/a\}^{1/2} + b$

9. $p + q = px + qy$. **[Bangalore 1996]** **Ans.** $z = -a\log(1 - x) + a\log(y - 1) + b$.

Solve the following partial differential equations: (10 – 17)

10. $pq = xy$ **Ans. C.I.** $2z = ax^2 + y^2/a + b$; **S.S.** Does not exist **G.S.** $2z - ax^2 - y^2/a - \psi(a) = 0$, $x^2 - y^2/a^2 + \psi'(a) = 0$

11. $\sqrt{p} + \sqrt{q} = 2x$. **Ans. C.I.** $z = (2x - a)^3/6 + a^2y + b$; **S.S.** Does not exit; **G.S.** $z - (2x - a)^3/6 - a^2y - \psi(a) = 0$, $(2x - a)^2/2 - 2ay - \psi'(a) = 0$

12. $q\,(p - \cos x) = \cos y$. **Ans.** $z = ax + \sin x + (1/a) \times \sin y + b$; **S.S.** Does not exist **G.S.** $z - ax - (1/a) \times \sin y - \psi(a) = 0$, $-x - (1/a^2) \times \sin y + \psi'(a) = 0$

13. $q = xyp^2$ **Ans. C.I.** $2z = 4\sqrt{ax} + ay^2 + b$; **S.S.** Does not exist; **G.S.** $2z - 4\sqrt{ax} - ay^2 - \psi(a) = 0$, $2\sqrt{(x/a)} + y^2 + \psi'(a) = 0$

14. $x^2p^2 = q^2y$. **Ans. C.I.** $z = \sqrt{a}\log x + 2\sqrt{ay} + b$; **S.S.** Does not exist. **G.S.** $z - \sqrt{a}\log x - 2\sqrt{ay} - \psi(a) = 0$, $\log x + 2\sqrt{y} + 2\sqrt{a}\,\psi'(a) = 0$

15. $p - q = x^2 + y^2$. **Ans. C.I.** $z = (x^3 - y^3)/3 + a(x + y) + b$; **S.S.** Does not exit; **G.S.** $z - (x^3 - y^3)/3 - a(x + y) - \psi(a) = 0$, $x + y + \psi'(a) = 0$

16. $p^2 - x = q^2 - y$ **Ans. C.I.** $3z = 2(x + a)^{3/2} + 2(y + a)^{3/2} + b$; **S.S.** Does not exist

G.S. $3z-2(x+a)^{3/2}-2(y+a)^{3/2}-\psi(a)=0,\ 3(x+a)^{1/2}+3(y+a)^{1/2}+\psi'(a)=0$

17. $px+q=p^2$. **Ans. C.I.** $z=(1/4)\times\left\{x^2+x(x^2+4a)^{1/2}\right\}+a\log\{x+(x^2+4a^2)^{1/2}\}\ ay+b$; **S.S.** Does not exist **G.S.** $z-(1/4)\times\{x^2+x(x^2+4a)^{1/2}\}-a\log\{x+(x^2+4a)^{1/2}\}-ay-\psi(a)=0$, $(x/2)\times(x^2+4a)^{-1/2}+\log\{x+(x^2+4a)^{1/2}\}+(2a)/[\{x+(x^2+4a)^{1/2}\}\times(x^2+4a)]+y+\psi'(a)=0$

3.19. JACOBI'S METHOD [Himachel 2005; Meerut 2005, 06, 08; Pune 2010]

This method is used for solving partial differential equations involving three or more independent variables. The central idea of Jacobi's method is almost the same as that of Charpit's method for two independent variables. We begin with the case of three independent variables. The results arrived at are, however, general and will be used with suitable modification for the case of four independent variables and so on.

Let $p_1=\partial z/\partial x_1$, $p_2=\partial z/\partial x_2$ and $p_3=\partial z/\partial x_3$.

Consider a partial differential equation $f(x_1, x_2, x_3, p_1, p_2, p_3)=0$, ...(1)

where the dependent variable z does not occur except by its partial differential coefficients with respect to the three independent variables x_1, x_2, x_3.

The main idea in Jacobi's method is to get two additional partial differential equations of the first order

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3)=a_1 \quad ...(2)$$

and

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3)=a_2, \quad ...(3)$$

where a_1 and a_2 are two arbitrary constants such that (1), (2) and (3) can be solved for p_1, p_2, p_3 in terms of x_1, x_2, x_3 which when substituted in

$$dz=p_1dx_1+p_2dx_2+p_3dx_3, \quad ...(4)$$

makes it integrable, for which the conditions are

$$\partial p_2/\partial x_1=\partial p_1/\partial x_2,\quad \partial p_3/\partial x_2=\partial p_2/\partial x_3,\quad \text{and}\quad \partial p_1/\partial x_3=\partial p_3/\partial x_1 \quad ...(5)$$

Differentiating (1) and (2) partially, w.r.t. x_1, we have

$$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0 \quad ...(6)$$

and

$$\frac{\partial F_1}{\partial x_1}+\frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0. \quad ...(7)$$

Eliminating $\partial p_1/\partial x_1$ from (6) and (7), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_1}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_1}=0. \quad ...(8)$$

Similarly, differentiating (1) and (2) partially w.r.t. x_2 and then eliminating $\partial p_2/\partial x_2$ from the resulting equations, we have

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_2}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_2}=0. \quad ...(9)$$

Again, differentiating (1) and (2) partially w.r.t. x_3 and then eliminating $\partial p_3/\partial x_3$ from the resulting equation, we have

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_3}+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_3}=0. \quad ...(10)$$

Adding (8), (9) and (10) and using the relations (5), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)=0. \quad ...(11)$$

The L.H.S. of (11) is generally denoted by (f, F_1). Then, (11) becomes

$$(f, F_1)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right)=0. \quad ...(11)'$$

Starting with (1) and (3) in place of (1) and (2) and proceeding as above, we have a similar relation

$$(f, F_2)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(12)$$

Again, starting with (2) and (3) in place of (1) and (2) and proceeding as above, we again have a similar relation

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(13)$$

(11) [or (11)′] and (12) are linear equtions of first order with $x_1, x_2, x_3, p_1, p_2, p_3$ as independent variables and F_1, F_2 as dependent variables respectively. For both of these equations, Lagrange's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}, \quad ...(14)$$

which are known as *Jacobi's auxiliary equations*.

We try to find two independent integrals $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$ with help of (14). If these relations satisfy (13), these are the required two additional relations (2) and (3).

We now solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Substituting these values in (4) and then integrating the resulting equation, we shall obtain a complete integral of the given equation containing three arbitrary constants of integration.

3.20. Working rules for solving partial differential equations with three or more independent variable. Jacobi's method

Step 1 : Suppose the given equation with three independent variables is

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0. \quad ...(1)$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}.$$

Solving these equation we obtain two additional equations

$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ ...(2) $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$. ...(3)

where a_1 and a_2 are arbitrary constants.

While obtaining (2) and (3), try to select simple equations so that later on solutions of (1), (2) and (3) may be as easy as possible.

Step III. Verify that relations (2) and (3) satisfy the condition

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(4)$$

If (4) is satisfied then solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Their substitution in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$ and subsequent integration leads to a complete integral of the given equation.

Remark 1. Sometime, change of variables can be employed to reduce the given equation in

a form solvable by Jacobian method.

Remark 2. While solving a partial differential equation with four independent variables, we modify the above working rule as follows :

Step I. Suppose the given equation with four independent variables is

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0. \quad ...(1)$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_4}{\partial f/\partial x_4} = \frac{dx_4}{-\partial f/\partial p_4}$$

Solving these equations we obtain three additional equations

$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1$, ...(2) $F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2$, ...(3)

and $F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3$, ...(4)

where a_1, a_2 and a_3 are arbitrary constants.

Step IV. Verify that relations (2), (3) and (4) satisfy following three conditions:

$$(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0, \quad ...(4) \qquad (F_2, F_3) = \sum_{r=1}^{4}\left(\frac{\partial F_2}{\partial x_r}\frac{\partial F_3}{\partial p_r} - \frac{\partial F_2}{\partial p_r}\frac{\partial F_3}{\partial x_r}\right) = 0 \quad ...(5)$$

and $$(F_3, F_1) = \sum_{r=1}^{4}\left(\frac{\partial F_3}{\partial x_r}\frac{\partial F_1}{\partial p_r} - \frac{\partial F_3}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right) = 0. \quad ...(6)$$

If (4), (5) and (6) are satisfied, then solve (1), (2), (3) and (4) for p_1, p_2, p_3 and p_4 in terms of x_1, x_2, x_3 and x_4. Their substitution in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3 + p_4dx_4$ and subsequent integration leads to a complete integral of the given equation.

3.21 SOLVED EXAMPLES BASED ON ART 3.20.

Ex. 1. *Find a complete integral of* $p_1^3 + p_2^2 + p_3 = 1$. **[I.A.S. 1997; Meerut 2006]**

Sol. Let the given equation be rewritten as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0. \quad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1}, \text{ using (1)}$$

From first and third fractions, $dp_1 = 0$ and $dp_2 = 0$ so that $p_1 = a_1$ and $p_2 = a_2$.

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1$. ...(2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$. ...(3)

Now, $$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

or $$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

or $(F_1, F_2) = (0)(0) - (1)(0) + (0)(1) - (0)(0) + (0)(0) - (0)(0) = 0$, by (3) and (4).

Thus, we have verified that for relations (2) and (3), $(F_1, F_2) = 0$. Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, $p_1 = a_1$, $p_2 = a_2$, $p_3 = 1 - a_1^3 - a_2^2$.

Putting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we have

$$dz = a_1dx_1 + a_2dx_2 + (1 - a_1^3 - a_2^2)dx_3.$$

Integrating, $z = a_1x_1 + a_2x_2 + (1 - a_1^3 - a_2^2)\, x_3 + a_3$,

which is a complete integral of given equation containing three arbitrary constants a_1, a_2, and a_3.

Ex. 2. *Find a complete integral of* $x_3^2p_1^2p_2^2p_3^2 + p_1^2p_2^2 - p_3^2 = 0$. **[Delhi Maths (H) 2006]**

Sol. Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = x_3^2p_1^2p_2^2p_3^2 + p_1^2p_2^2 - p_3^2 = 0$. ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{0} = \frac{dx_1}{-(2p_1x_3^2p_2^2p_3^2 + 2p_1p_2^2)} = \frac{dp_2}{0} = \frac{dx_2}{-(2p_2x_3^2p_1^2p_3^2 + 2p_2p_1^2)} = \text{.....}, \text{ by (1)}$$

From first and third fractions, $dp_1 = 0$ and $dp_2 = 0$ so that $p_1 = a_1$ and $p_2 = a_2$.

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1$, ...(2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$. ...(3)

As in Ex. 1, verify that for relations (2) and (3), $(F_1, F_2) = 0$.

Hence (2) and (3) may be taken as the additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, we have $p_1 = a_1$, $p_2 = a_2$, $p_3 = \pm a_1a_2 \Big/ \sqrt{(1 - a_1^2a_2^2x_3^2)}$.

Putting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we get

$$dz = a_1dx_1 + a_2dx_2 \pm \left\{a_1a_2 \Big/ \sqrt{(1 - a_1^2a_2^2x_3^2)}\right\} dx_3, \text{ whose integration gives}$$

$z = a_1x_1 + a_2x_2 \pm \sin^{-1}(a_1a_2x_3) + a_3$, a_1, a_2, a_3 being arbitrary constants.

Ex. 3. *Find a complete integral of* $p_1x_1 + p_2x_2 = p_3^2$. **[Meerut 2007]**

Sol. Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1x_1 + p_2x_2 - p_3^2 = 0$. ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{p_1} = \frac{dx_1}{-x_1} = \frac{dp_2}{p_2} = \frac{dx_2}{-x_2} = \frac{dp_3}{0} = \frac{dx_3}{2p_3}, \text{ using (1)}$$ ...(2)

Taking the first two fractions of (2), $(1/x_1)dx + (1/p_1)dp_1 = 0$. $\Rightarrow \log x_1 + \log p_1 = \log a_1$.

$\therefore x_1p_1 = a_1$ and let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1p_1 = a_1$. ...(3)

Taking the third and fourth fractions of (2), $(1/x_2)dx_2 + (1/p_2)dp_2 = 0$.

$\therefore x_2p_2 = a_2$ and let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = x_2p_2 = a_2$. ...(4)

As in Ex. 1, verify that for relations (3) and (4), $(F_1, F_2) = 0$.

Solving (1), (3) and (4) for p_1, p_2, p_3, $p_1 = a_1/x_1$, $p_2 = a_2/x_2$ and $p_3 = (a_1 + a_2)^{1/2}$.

Putting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we have

$$dz = (a_1/x_1)dx_1 + (a_2/x_2)dx_2 + (a_1 + a_2)^{1/2}\, dx_3.$$

Integrating, $z = a_1 \log x_1 + a_2 \log x_2 + x_3\,(a_1 + a_2)^{1/2} + a_3$.

Ex. 4. *Find complete integral of* $2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0$.

[I.A.S. 1998, Meerut 1999]

Sol. Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0$. ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{2p_1x_3}=\frac{dx_1}{-2x_1x_3}=\frac{dp_2}{0}=\frac{dx_2}{-3x_3^2-2p_2p_3}=\frac{dp_3}{2p_1x_1+6p_2x_3}=\frac{dx_3}{-p_2^2}, \text{ by (1)} \quad ...(2)$$

Taking the first two fractions of (2), $(1/p_1)dp_1 + (1/x_1)dx_1 = 0$. so $p_1x_1 = a_1$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1x_1 = a_1$. ...(3)

From the third fraction of (2), $dp_2 = 0$ so that $p_2 = a_2$.

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$. ...(4)

As in Ex. 1, verify that for relations (3) and (4), $(F_1, F_2) = 0$.

Solving (1), (3) and (4) for p_1, p_2, p_3, $p_1 = a_1/x_1$, $p_2 = a_2$, $p_3 = -(2a_1x_3 + 3a_2x_3^2)/a_2^2$.

Putting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_2$, we have

$dz = (a_1/x_1)dx_1 + a_2dx_2 - \{(2a_1x_3 + 3a_2x_3^2)/a_2^2\}dx_3$, whose integration gives

$z = a_1 \log x_1 + a_2x_2 - (a_1x_3^2 + a_2x_3^3)/a_2^2 + a_3$. which is required complete integral

Ex. 5. *Find a complete integral of* $p_3x_3(p_1 + p_2) + x_1 + x_2 = 0$.

Sol. Given $f(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3(p_1 + p_2) + x_1 + x_2 = 0$. ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{1}=\frac{dx_1}{p_3x_3}=\frac{dp_2}{1}=\frac{dx_2}{-p_3x_3}=\frac{dp_3}{p_3(p_2+p_3)}=\frac{dx_3}{-x_3(p_1+p_2)}, \text{ by (1)} \quad ...(2)$$

Taking the two fractions of (2), $dp_1 - dp_2 = 0$ so $p_1 - p_2 = a_1$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 - p_2 = a_1$. ...(3)

Taking the fifth and sixth fractions of (2), $(1/p_3)dp_3 + (1/x_3)dx_3 = 0$ giving $p_3x_3 = a_3$

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3 = a_2$. ...(4)

Now, $(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$

$$=\left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1}-\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right)+\left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2}-\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)+\left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3}-\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)$$

$= (0)(0) - (1)(0) + (0)(0) - (-1)(0) + (0)(x_3) - (0)(p_3) = 0$ by (3) and (4)

Thus, we have verified that for the relations (3) and (4), $(F_1, F_2) = 0$.

From (1) and (4), $a_2(p_1 + p_2) + x_1 + x_2 = 0$ or $p_1 + p_2 = -(x_1 + x_2)/a_2$. ...(5)

Solving (3) and (5), $p_1 = \frac{a_1}{2} - \frac{x_1+x_2}{2a_2}$ and $p_2 = -\frac{a_1}{2} - \frac{x_1+x_2}{2a_2}$. ...(6)

Again, from (4), $p_3 = a_2/x_3$. ...(7)

Putting the values of p_1, p_2, p_3 given by (6) and (7) in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we have

$$dz = \frac{a_1}{2}(dx_1 - dx_2) - \frac{(x_1+x_2)}{2a_2}(dx_1 + dx_2) + \frac{a_2}{x_3}dx_3.$$

Integrating, $z = (a_1/2) \times (x_1 - x_2) - (1/4a_2) \times (x_1 + x_2)^2 + a_2 \log x_3 + a_3$.

Ex. 6. *Find a complete integral of* $(p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 = 3(x_1 + x_2 + x_3)$.

Sol. Let the given partial differential equation be re–written as

$$f(x_1, x_3, x_3, p_1, p_2, p_3) = (p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 - 3(x_1 + x_2 + x_3) = 0. \quad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}, \text{ giving}$$

$$\frac{dp_1}{2(p_1+x_1)-3} = \frac{dx_1}{-2(p_1+x_1)} = \frac{dp_2}{2(p_2+x_2)-3} = \frac{dx_2}{-2(p_2+x_2)} = \frac{dp_3}{2(p_3+x_3)-3} = \frac{dx_3}{-2(p_3+x_3)}. \quad ...(2)$$

$$\text{Each fraction of (2)} = \frac{dp_1+dx_1}{-3} = \frac{dp_2+dx_2}{-3} = \frac{dp_3+dx_3}{-3} \quad ...(3)$$

Then (3) $\Rightarrow$ $dp_1 + dx_1 = dp_2 + dx_2$ and $dp_3 + dx_3 = dp_2 + dx_2$

Integrating, $p_1 + x_1 = p_2 + x_2 + a_1$ and $p_3 + x_3 = p_2 + x_2 + a_2$,

where a_1 and a_2 are arbitrary constants

Let $$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 + p_1 - x_2 - p_2 = a_1. \quad ...(4)$$

and $$F_2(x_1, x_2, x_2, p_1, p_2, p_3) = x_3 + p_3 - x_2 - p_2 = a_2. \quad ...(5)$$

As in Ex. 1, verify that for relations (4) and (5), the condition $(F_1, F_2) = 0$ is satisfied. Hence (4) and (5) may be taken as two additional equations.

With help of (4) and (5), (1) reduces to

$$(x_2 + p_2 + a_1)^2 + (x_2 + p_2)^2 + (x_2 + p_2 + a_2)^2 = 3(x_1 + x_2 + x_3)$$

or $$3(p_2 + x_2)^2 + 2(p_2 + x_2)(a_1 + a_2) + a_1^2 + a_2^2 - 3(x_1 + x_2 + x_3) = 0.$$

$$\therefore \quad p_2 + x_2 = \left[-2(a_1 + a_2) \pm \sqrt{[4(a_1 + a_2)^2 - 12\{a_1^2 + a_2^2 - 3(x_1 + x_2 + x_3)\}}\right] \Big/ 6$$

$$\Rightarrow \quad p_2 = -x_2 + \left[-(a_1 + a_2) \pm \sqrt{\{9(x_1 + x_2 + x_3) - 2a_1^2 - 2a_2^2 + 2a_1a_2\}}\right] \Big/ 3$$

For sake of simplification, we take $a_1 = 3c_1$ and $a_2 = 3c_2$. Then, we get

$$p_2 = -x_2 - (c_1 + c_2) \pm \sqrt{\{(x_1 + x_2 + x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}. \quad ...(6)$$

$\therefore$ From (4), $$p_1 = x_2 + p_2 + 3c_1 - x_1$$

$\Rightarrow$ $$p_1 = -x_1 + 2c_1 - c_2 \pm \sqrt{\{(x_1 + x_2 + x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}, \text{by (6)}$$

Again, from (5), $$p_3 = x_2 + p_2 + 3c_2 - x_3$$

$\Rightarrow$ $$p_3 = -x_3 + 2c_2 - c_1 \pm \sqrt{\{(x_1 + x_2 + x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}, \text{ by (6)}$$

Substituting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we get

$$dz = -(x_1dx + x_2dx_2 + x_3dx_3) + [(2c_1 - c_2)dx_1 - (c_1 + c_2)dx_2 + (2c_2 - c_1)dx_3]$$
$$\pm (x_1 + x_2 + x_3 - 2c_1^2 - 2c_2^2 + 2c_1c_2)^{1/2} \ (dx_1 + dx_2 + dx_3).$$

Integrating, $z = -(1/2) \times (x_1^2 + x_2^2 + x_3^2) + (2c_1 - c_2)x_1 - (c_1 + c_2)x_2 + (2c_2 - c_1)x_3$
$$\pm (2/3) \times (x_1 + x_2 + x_3 - 2c_1^2 - 2c_2^2 + 2c_1c_2)^{3/2} + c_3,$$

which is a complete integral containing c_1, c_2, c_3 as arbitrary constants.

Ex. 7. *Find a complete integral of* $(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0$. **[Delhi B.Sc. (Hons) III 2011]**

Sol. Given $$(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0. \quad ...(1)$$

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Re–writting (1), we have

$$(x_2 + x_3)\left(\frac{1}{z}\frac{\partial z}{\partial x_2} + \frac{1}{z}\frac{\partial z}{\partial x_3}\right)^2 + \frac{1}{z}\frac{\partial z}{\partial x_1} = 0. \quad ...(2)$$

Let $(1/z)dz = dZ$ so that $Z = \log z$. ...(3)

Then, (2) $\Rightarrow$ $(x_2 + x_3)\,(\partial Z/\partial x_2 + \partial Z/\partial x_3)^2 + \partial Z/\partial x_1 = 0$. ...(4)

Let $P_1 = \partial Z/\partial x_1$, $P_2 = \partial Z/\partial x_2$, $P_3 = \partial Z/\partial x_3$. Then (4) becomes

$$(x_2 + x_3)(P_2 + P_3)^2 + P_1 = 0.$$

So here $f(x_1, x_2, x_3, P_1, P_2, P_3) \equiv (x_2 + x_3)(P_2 + P_3)^2 + P_1 = 0$. ...(5)

Jacobi's auxiliary equations take the form

$$\frac{dP_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dx_1}{-1} = \frac{dP_1}{0} = \frac{dx_2}{-2(x_2 + x_3)(P_2 + P_3)} = \frac{dP_2}{(P_2 + P_3)^2} = \frac{dx_3}{-2(x_2 + x_3)(P_2 + P_3)} = \frac{dP_3}{(P_2 + P_3)^2}. \quad ...(6)$$

Taking second ratio of (6), we have $dP_1 = 0 \Rightarrow P_1 = -a_1$.

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 = -a_1$. ...(7)

Taking the fourth and sixth ratios in (6), we get $dP_2 = dP_3 \Rightarrow P_2 - P_3 = a_2$.

Let $F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 - P_3 = a_2$. ...(8)

Using (7), (5) $\Rightarrow P_2 + P_3 = \pm \{a_1/(x_2 + x_3)\}^{1/2}$. ...(9)

Solving (8) and (9) for P_2 and P_3, we have

$$P_2 = \frac{1}{2}\left[a_2 \pm \left(\frac{a_1}{x_2 + x_3}\right)^{1/2}\right] \quad \text{and} \quad P_3 = \frac{1}{2}\left[\pm\left(\frac{a_1}{x_2 + x_3}\right)^{1/2} - a_2\right]. \quad ...(10)$$

Using (7) and (10), $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$ becomes

$$dZ = -a_1dx_1 + \frac{1}{2}\left[a_2 \pm \frac{\sqrt{a_1}}{(x_2 + x_3)^{1/2}}\right]dx_2 + \frac{1}{2}\left[\pm\frac{\sqrt{a_1}}{(x_2 + x_3)^{1/2}} - a_2\right]dx_3$$

or $$dZ = -a_1dx_1 + (1/2) \times a_2dx_2 - (1/2) \times a_2dx_3 \pm (1/2) \times \sqrt{a_1}\,(x_2 + x_3)^{-1/2}(dx_2 + dx_3).$$

Integrating and noting that $dZ = (1/z)dz$, complete integral is given by

$$\log z = -a_1x_1 + (a_2/2) \times (x_2 - x_3) \pm \sqrt{a_1}\,(x_2 + x_3)^{1/2} + a_3.$$

Ex. 8. *Find a complete integral of* $p_1p_2p_3 = z^3x_1x_2x_3$. **[Meerut 1998]**

i.e., $(\partial z/\partial x_1)(\partial z/\partial x_2)\,(\partial z/\partial x_3) = z^3x_1x_2x_3$. **[Delhi Maths (H) 2000, 10; I.A.S. 1995]**

Sol. Given $p_1p_2p_3 = z^3x_1x_2x_3$ or $(\partial z/\partial x_1)(\partial z/\partial x_2)\,(\partial z/\partial x_3) = z^3x_1x_2x_3$. ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Re–writting (1) we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = x_1x_2x_3. \quad ...(2)$$

Let $(1/z)dz = dZ$ so that $\log z = Z$. Then (2) becomes

$(\partial Z/\partial x_1)(\partial Z/\partial x_2)\,(\partial Z/\partial x_3) = x_1x_2x_3$ or $P_1P_2P_3 = x_1x_2x_3$.

$\therefore$ Here $f(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_1P_2P_3 - x_1x_2x_3 = 0$. ...(3)

$\therefore$ Jacobi's auxilliary equations are

$$\frac{dP_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dP_1}{-x_2x_3}=\frac{dx_1}{-P_2P_3}=\frac{dP_2}{-x_1x_3}=\frac{dx_2}{-P_1P_3}=\frac{dP_3}{-x_1x_2}=\frac{dx_3}{-P_1P_2}, \text{ by (3)}$$

Since from (3), $P_2P_3 = (x_1\ x_2\ x_3)/P_1$, hence first and second fractions give

$$\frac{dP_1}{-x_2x_3}=\frac{dx_1}{-(x_1x_2x_3/P_1)} \quad \text{or} \quad \frac{dP_1}{P_1}=\frac{dx_1}{x_1}.$$

Integrating, $\log P_1 = \log x_1 + \log a_1$ or $P_1 = a_1x_1$.

Thus, here we have $F_1(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_1 - a_1x_1 = 0.$ ...(4)

Similarly, $F_2(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_2 - a_2x_2 = 0.$ ...(5)

As in Ex. 1, verify that for (4) and (5) the condition $(F_1, F_2) = 0$ is satisfied. Hence (4) and (5) can be taken as two additional equations. Solving (3), (4) and (5) for P_1, P_2, P_3, we have

$P_1 = a_1x_1$, $P_2 = a_2x_2$ and $P_3 = x_3/(a_1a_2)$.

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$, we have

$$dZ = a_1x_1dx + a_2x_2dx_2 + \{x_3/(a_1a_2)\}dx_3.$$

Integrating, $Z = (1/2) \times a_1x_1^2 + (1/2) \times a_2x_2^2 + \{1/(2a_1a_2)\}x_3^2 + a_3/2$

or $2 \log z = a_1x_1^2 + a_2x_2^2 + \{1/(a_1a_2)\}x_3^2 + a_3$, as $Z = \log z$

Ex. 9. *Find a complete integral of* $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0$. **[Delhi Maths (H) 2009]**

Sol. Given equation is $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Dividing each term by z^2, (1) can be re–written as

$$\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)^2 + \left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) - \left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right) - \left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = 0. \quad ...(2)$$

Let $(1/z)dz = dZ$ so that $\log z = Z.$ ...(3)

Using (3), (2) becomes $P_1^2 + P_2P_3 - P_2 - P_3 = 0,$ ...(4)

Let us write $f(x_1, x_2, x_3, P_1, P_2, P_3) = P_1^2 + P_2P_3 - P_2 - P_3 = 0.$ ...(5)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dP_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial P_1}=\frac{dP_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial P_2}=\frac{dP_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dP_1}{0}=\frac{dx_1}{-2P_1}=\frac{dP_2}{0}=\frac{dx_2}{-P_3+1}=\frac{dP_3}{0}=\frac{dx_3}{-P_2+1}, \text{ by (5)}$$

Taking the third and fifth fractions, $dP_2 = 0$ and $dP_3 = 0$ so that $P_2 = a_1$ and $P_3 = a_2$.

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 = a_1.$ ...(6)

and $F_3(x_1, x_2, x_3, P_1, P_2, P_3) = P_3 = a_2.$ ...(7)

As in Ex. 1, verify that for (6) and (7), the condition $(F_1, F_2) = 0$ is satisfied. Hence (6) and (7) can be taken as two additional equations. Solving (4), (6) and (7) for P_1, P_2, P_3, we have

$P_2 = a_1$, $P_3 = a_2$, $P_1 = (a_1 + a_2 - a_1a_2)^{1/2}$.

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$, we have

$$dZ = (a_1 + a_2 - a_1a_2)^{1/2}dx_1 + a_1dx_2 + a_2dx_3.$$

Integrating, $Z = (a_1 + a_2 - a_1a_2)^{1/2}x_1 + a_1x_2 + a_2x_3 + a_3$. Then, the complete integral is

$$\log z = (a_1 + a_2 - a_1a_2)^{1/2}x_1 + a_1x_2 + a_2x_3 + a_3, \text{ using (3).}$$

Ex. 10. *Find a complete integral of* $2x_1x_3zp_1p_3 + x_2p_2 = 0$.

Sol. Given equation is $2x_1x_3zp_1p_3 + x_2p_2 = 0.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Multiplying each term by z, (1) can be re–written as

$$2x_1x_3\left(z\frac{\partial z}{\partial x_1}\right)\left(z\frac{\partial z}{\partial x_3}\right)+x_2\left(z\frac{\partial z}{\partial x_2}\right)=0. \qquad ...(2)$$

Let $zdz = dZ$ so that $z^2/2 = Z$. ...(3)

Using (3), (2) becomes $2x_1x_3P_1P_3 + x_2P_2 = 0,$ ...(4)

where $P_1 = \partial Z/\partial x_1$, $P_2 = \partial Z/\partial x_2$ and $P_3 = \partial Z/\partial x_3$. We re–write (4) as

$$f(x_1, x_2, x_3, P_1, P_2, P_3) = 2x_1x_3P_1P_3 + x_2P_2 = 0. \qquad ...(5)$$

$\therefore$ Jacobi's auxilairy equations are

$$\frac{dP_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial P_1}=\frac{dP_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial P_2}=\frac{dP_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial P_3}$$

or
$$\frac{dP_1}{2x_3P_1P_3}=\frac{dx_1}{-2x_1x_3P_3}=\frac{dP_2}{P_2}=\frac{dx_2}{-x_2}=\frac{dP_3}{2x_1P_1P_3}=\frac{dx_3}{-2x_1x_3P_2}, \text{ by (5)} \qquad ...(6)$$

Taking the first and second fractions of (6) and simplifying, we get

$(1/P_1)dP_1 + (1/x_1)dx_1 = 0$ so that $\log P_1 + \log x_1 = \log a_1$ or $P_1x_1 = a_1$

So here $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1x_1 = a_1.$ ...(7)

Taking the fifth and sixth fractions of (6) and simplifying, we get

$(1/P_3)dP_3 + (1/x_3)dx_3 = 0$ so that $\log P_3 + \log x_3 = \log a_3$ or $P_3x_3 = a_2$

So here $F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_3x_3 = a_2.$ ...(8)

As in Ex. 1, verify that for (7) and (8), the condition $(F_1, F_2) = 0$ is satisfied. Hence (7) and (8) can be taken as additional equations. Solving (5), (7) and (8) for P_1, P_2, P_3, we have

$$P_1 = a_1/x, \qquad P_3 = a_2/x_3, \qquad P_2 = -(2a, a_2)/x_2.$$

Putting these values in $dZ = P_1dx + P_2dx_2 + P_3dx_3$, we have

$$dZ = (a_1/x_1)dx_1 - \{(2a_1a_2)/x_2\}dx_2 + (a_2/x_3)dx_3.$$

Integrating, $Z = a_1\log x_1 - 2a_1a_2 \log x_2 + a_2 \log x_3 + a_3$

or $z^2/2 = a_1 \log x_1 - 2a_1a_2 \log x_2 + a_2 \log x_3 + a_3$, by (3).

Ex. 11. *Find a complete integral of* $p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0$.

Sol. [In the present problem we have four independent variables in places of three. According we shall use modified working as explained in remark 2 of Art 3.20]

The given equation can be written as

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0. \qquad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}=\frac{dp_4}{\partial f/\partial x_4}=\frac{dx_4}{-\partial f/\partial p_4}, \text{ giving}$$

$$\frac{dp_1}{p_4^3x_2x_3x_4^3}=\frac{dx_1}{-p_2p_3}=\frac{dp_2}{p_4^3x_1x_3x_4^3}=\frac{dx_2}{-p_1p_3}=\frac{dp_3}{p_4^3x_1x_2x_4^3}=\frac{dx_3}{-p_1p_2}=\frac{dp_4}{3p_4^3x_1x_2x_3x_4^2}=\frac{dx_4}{-3p_4^3x_1x_2x_3x_4^3}$$

Since from (1), $p_4^3x_2x_3x_4^3 = -p_1p_2p_3/x_1$, the first two fractions give

$$\frac{dp_1}{-(p_1p_2p_3/x_1)} = \frac{dx_1}{-p_2p_3} \qquad \text{or} \qquad \frac{dp_1}{p_1}=\frac{dx_1}{x_1}.$$

Integrating, $\log p_1 = \log x_1 + \log a_1$ or $p_1 = a_1 x_1$.

Let $F_1(x_1, x_2, x_3, x_4\, p_1, p_2, p_3, p_4) = p_1 - a_1 x_1 = 0.$ ...(2)

Similarly, $F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_2 - a_2 x_2 = 0$ ...(3)

and $F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_3 - a_3 x_3 = 0.$ ...(4)

With these values of F_1, F_2 and F_3, we can verify that

$$(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0.$$

Similarly, we see that $(F_2, F_3) = 0$ and $(F_3, F_1) = 0$. Hence (2), (3) and (4) can be taken as the three desired additional equations. Now solving (1), (2) (3) and (4) for p_1, p_2, p_3 and p_4, we get

$p_1 = a_1 x_1$, $p_2 = a_2 x_2$, $p_3 = a_3 x_3$ and $p_4 = (a_1 a_2 a_3)^{1/3} / x_4$.

Putting these in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$ and integrating the desired complete integral is $z = (1/2) \times (a_1 x_1^{\,2} + a_2 x_2^{\,2} + a_3 x_3^{\,2}) - (a_1 a_2 a_3)^{1/2} \log x_4 + a_4/2$

or $2z = a_1 x_1^{\,2} + a_2 x_2^{\,2} + a_3 x_3^{\,2} - 2(a_1 a_2 a_3)^{1/2} \log x_4 + a_4$,

Ex. 12. *Find a complete integral by Jacobi's method of the equation* $2x^2 y(\partial u/\partial x)^2 (\partial u/\partial z) = x^2(\partial u/\partial y) + 2y(\partial u/\partial x)^2$. **[Delhi Maths (H) 2001]**

Sol. Let $x = x_1$, $y = x_2$, $z = x_3$, $\partial u/\partial x = p_1$, $\partial u/\partial y = p_2$, and $\partial u/\partial z = p_3$

Then given equation becomes $2x_1^2 x_2 p_1^2 p_3 = x_1^2 p_2 + 2x_2 p_1^2$

Dividing by $x_1^2\ x_2$, $2p_1^2 p_3 = (p_2/x_2) + (2p_1^2/x_1^2)$, which can be written as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1^2(p_3 - 1/x_1^2) - p_2/x_2 = 0 \quad \text{... (1)}$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{4p_1^2/x_1^3} = \frac{dx_1}{-4p_1(p_3 - 1/x_1^2)} = \frac{dp_2}{p_2/x_2^2} = \frac{dx_2}{1/x_2} = \frac{dp_3}{0} = \frac{dx_3}{-2p_1^2}$$, by (1)

Taking the fifth fraction, $dp_3 = 0$ so that $p_3 = a_1$

Taking the second and fourth fractions, $(1/p_2)\, dp_2 = (1/x_2)\, dx_2$

Integrating, $\log p_2 = \log x_2 + \log(2a_2^2)$ or $p_2/x_2 = 2a_2^2$

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_3 = a_1$ ... (2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2/x_2 = 2a_2^2$ ... (3)

Now, $$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

or $$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

$$= (0)\ (0) - (0)\ (0) + (0)\ \ (0) - (0)\ \ (0) + (0)\ \ (0) - (1)\ (0) = 0$$

Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, $p_1 = a_2 x_1 /(a_1 x_1^2 - 1)^{1/2}$, $p_2 = 2a_2^2 x_2$, $p_3 = a_1$

Putting these in $du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 = a_2 x_1 (a_1 x_1^2 - 1)^{-1/2} dx_1 + 2a_2^2 x_2 dx_2 + a_1 dx_3$.

Integrating, $u = (a_2 / a_1) \times (a_1 x_1^2 - 1)^{1/2} + a_2^2 x_2^2 + a_1 x_3 + a_3$,

which is the complete integral with a_1, a_2, a_3 as arbitrary constants.

Ex. 13. *Show that a complete integral of the equation* $f(\partial u/\partial x, \partial u/\partial y, \partial u/\partial z) = 0$ *is* $u = ax + by + \theta(a, b)z + c$, *where a, b and c are arbitrary constants and* $f(a, b, \theta) = 0$

(*b*)*Find a complete integral of the equation* $\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = (\partial u/\partial x)\,(\partial u/\partial y)\,(\partial u/\partial z)$. **[Allahabad 2004, 06; Meerut 2004, 06; Purvanchal 2003]**

Sol. (*a*) Let $\partial u/\partial x = p_1$, $\partial u/\partial y = p_2$ and $\partial u/\partial z = p_3$.

Then gievn equation becomes $f(p_1, p_2, p_3) = 0$ ... (1)

We shall now proceed as in Ex. 1, Art. 3.21. Here Jacobi's auxiliary equations are given by

$$\frac{dp_1}{\partial f/\partial x} = \frac{dx}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial y} = \frac{dy}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial z} = \frac{dz}{-\partial f/\partial p_3}$$

$\Rightarrow$ $\frac{dp_1}{0} = \frac{dp_2}{0}$, using (1) $\Rightarrow$ $dp_1 = 0$ and $dp_2 = 0$

Integrating, $p_1 = a$, $p_2 = b$, a and b being arbitrary constants ... (2)

Putting $p_1 = a$ and $p_2 = b$ in (1), $f(a, b, p_3) = 0$ so that

p_3 = a function of $a, b = \theta(a, b)$, say ... (3)

Now, we have $du = (\partial u/\partial x)\,dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz = p_1 dx + p_2 dy + p_3 dz$

or $du = a\,dx + b\,dy + \theta(a, b)\,dz$, by (2) and (3)

Integrating, $u = ax + by + \theta\,(a,b)\,z + c$, ... (4)

where c is an arbitrary constant and a, b, θ are connected by relation

$f(a, b, \theta\,(a, b)) = 0$, by (1), (2) and (3) ... (5)

(*b*) Given $\partial u/\partial x + \partial u/\partial y + \partial u/\partial z - (\partial u/\partial x)\,(\partial u/\partial y)\,(\partial u/\partial z) = 0$ ... (*i*)

Let $p_1 = \partial u/\partial x$, $p_2 = \partial u/\partial y$ and $p_3 = \partial u/\partial z$. Then, (*i*) gives

$p_1 + p_2 + p_3 - p_1\,p_2\,p_3 = 0$ ... (*ii*)

Comparing (*ii*) with (1) of part (*a*), here

$f(p_1, p_2, p_3) = p_1 + p_2 + p_3 - p_1\,p_2\,p_3$ ... (*iii*)

Hence required complete integral is given by (4) and (5) of part (*a*) *i.e.*,

$u = a\,x + b\,y + \theta\,(a,b)\,z + c$, ... (*iv*)

where $a + b + \theta\,(a,b) - ab\,\theta(a, b) = 0$ ... (*v*)

From (*v*), $\theta(a, b) = (a + b)/(ab - 1)$ ... (*vi*)

From (*iv*) and (*vi*), $u = ax + by + \{(a + b)/(ab - 1)\} + c$,

which is the required complete integral of (*i*), *a*, *b*, *c* being arbitrary constants.

EXERCISE 3(G)

Find the complete integral of the following equation: (1 – 5)

1. $f(p_1, p_2, p_3) = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$, where $f(a_1, a_2, a_3) = 0$

2. $p_1 + p_2 + p_3 - p_1p_2p_3 = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$, where $a_1 + a_2 + a_3 - a_1a_2a_3 = 0$

3. $p_1x_1^2 - p_2^2 - ap_3^2 = 0$ **Ans.** $z = -(a_1^2 + a\,a_2^2)x_1^{-1} + a_1x_2 + a_2a_3x_3 + a_3$

4. $x_3(x_3 + p_3) = p_1^2 + p_2^2$ **Ans.** $z = a_1x_1 + a_2x_2 + (a_1^2 + a_2^2)\log x_3 - x_3^2/2 + a_3$

5. $x_3 + 2p_3 - (p_1 + p_3^2) = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + (a_1 + a_2)^2 \times (x_3/2) - (x_3^2/4) + a_3$

6. $x_1 + p_1^2 + x_2 + p_2^2 - x_3p_3^2 = 0$ **Ans.** $z = 2\,(a_1x_1)^{1/2} + 2\,(a_2x_2)^{1/2} + 2\,\{(a_1 + a_2)\,x_3\}^{1/2} + a_3$

7. *Show how to solve, by Jacobi method, a partial differential equation of the type* $f(x, \partial u/\partial x, \partial u/\partial z) = g(y, \partial u/\partial y, \partial u/\partial z)$ *and illustrate the method by finding a complete integral of equation* $2x^2y(\partial u/\partial x)^2(\partial u/\partial z) = x^2(\partial u/\partial y) + 2y(\partial u/\partial x)^2$. **[Meerut 2005]**

Sol. Try yourself **Ans.** $u = (ax^2 - b)^{1/2} + ay^2 + (z/b) + c$

8. *Prove that an equation of the "Clairaut" form* $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z) = f(\partial u/\partial x, \partial u/\partial y, \partial u/\partial z)$ *is always solvable by Jacobi's method. Hence solve*

$$(\partial u/\partial x + \partial u/\partial y + \partial u/\partial z)\,\{x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z)\} = 1$$

3.22. Jacobi's method for solving a non-linear first order partial differential equation in two independent variables.

[Delhi Maths (H) 1997; Amaravati 2001; Himanchal 2003, 05]

Let $$F(x, y, z, p, q) = 0 \quad \ldots (1)$$

be the non-linear first order equation in two independent variables x, y.

Then we know that a solution of (1) is of the form $$u(x, y, z) = 0 \quad \ldots (2)$$

showing that u can be treated as a dependent variable and x, y, z as three independent variables.

Differentiating (2) partially w.r.t. 'x' and 'y', respectively, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$$

or $$p_1 + p_3p = 0 \quad \text{and} \quad p_2 + p_3\ q = 0 \quad \ldots (3)$$

where $p = \partial z/\partial x,\ q = \partial z/\partial y,\ p_1 = \partial u/\partial x = \partial u/\partial x_1,\ p_2 = \partial u/\partial y\ = \partial u/\partial x_2,\ p_3 = \partial u/\partial z = \partial u/\partial x_3$

by taking $$x = x_1, \quad y = y_2 \quad \text{and} \quad z = x_3 \quad \ldots (4)$$

From (3), $$p = -(p_1/p_3) \quad \text{and} \quad q = -(p_2/p_3) \quad \ldots (5)$$

Using (4) and (5), (1) reduces to $$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \quad \ldots (6)$$

We now solve (6) by Jacobi's method as usual (refer Art. 3.20) to get the complete integral of (6). Finally, putting $x_1 = x,\ x_2 = y,\ x_3 = z$, we obtain solution of (6) containing original variables x, y, z and new dependent variable u. The solution so obtained will contain three arbitrary constants a_1, a_2, a_3 (say). However, for the given equation in the form (1), we need only two arbitrary constants in the final solution. The required solution $u = 0$ of (1) is obtained by making different choices of our third arbitrary constant.

Ex. 1. *Solve* $p^2x + q^2y = z$ *by Jacobi's method.* **[Nagpur 2002; Himanchal 2003, 05]**

Sol. Given $p^2x + q^2y = z$.

Let a solution of (1) be of the form $u(x, y, z) = 0$ ... (2)

So treating u as dependent variable and x, y, z as three independent variables, differentiation of (2) partially w.r.t 'x' and 'y' respectively gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0 \quad i.e. \quad p_1 + p_3 p = 0 \quad \text{and} \quad p_2 + p_3\, q = 0$$

so that $p = -p_1 / p_3$ and $q = -p_2 / p_3$ .... (3)

where $p_1 = \partial u/\partial x = \partial u/\partial x_1$, $p_2 = \partial u/\partial y = \partial u/\partial x_2$, $p_3 = \partial u/\partial z = \partial u/\partial x_3$, $p = \partial z/\partial x, q = \partial z/\partial y$

by taking $x = x_1$, $y = x_2$ and $z = x_3$ ... (4)

Using (3) and (4), (1) $\Rightarrow$ $x_1(p_1/p_3)^2 + x_2(p_2/p_3)^2 = x_3$. $\Rightarrow$ $x_1p_1^2 + x_2p_2^2 - x_3p_3^2 = 0$

Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = x_1p_1^2 + x_2p_2^2 - x_3\ p_3^2 = 0$ ... (5)

Now, the Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{p_1^2} = \frac{dx_1}{-2p_1x_1} = \frac{dp_2}{p_2^2} = \frac{dx_2}{-2p_2x_2} = \frac{dp_3}{-p_3^2} = \frac{dx_3}{2p_3x_3}, \text{ by (5)}$$

Taking the first two fractions, $(2/p_1)\ dp_1 + (1/x)\,dx = 0$.

Integrating, $2\log p_1 + \log x_1 = \log a_1$ so that $x_1p_1^2 = a_1$ or $p_1 = (a_1/x_1)^{1/2}$

Similarly, the third and fourth fractions give $p_2 = (a_2/x_2)^{1/2}$

Substituting these values of p_1 and p_2 in (5), we get $p_3 = \{(a_1 + a_2)/x_3\}^{1/2}$.

Putting the above values of p_1, p_2 and p_3 in $du = p_1dx_1 + p_2dx_2 + p_3dx_3$, we get

$$du = a_1^{1/2}\ x_1^{-1/2}dx_1 + a_2x_2^{-1/2}dx_2 + (a_1 + a_2)^{1/2}x_3^{-(1/2)}dx_3.$$

Integrating, $u = 2(a_1x_1)^{1/2} + (a_2x_2)^{1/2} + 2(a_1 + a_2)^{1/2}x_3^{1/2} + a_3$ ... (6)

Taking $a_2 = 1$ and using (4), the required solution $u = 0$ is given by

$$2(a_1x)^{1/2} + 2y^{1/2} + 2(a_1 + 1)^{1/2}z^{1/2} + a_3 = 0,$$

which is the complete integral containing two arbitrary constants a_1 and a_3.

Ex. 2. *Solve* $p^2 + q^2 = k^2$ by Jacobi's method **[Delhi B.A./B.Sc. (Prog) Maths 2007]**

Sol. Given $p^2 + q^2 = k^2$ ... (1)

Let a solution of (1) be of the form $u(x, y, z) = 0$ ... (2)

So treating u as dependent variable and x, y, z as three independent variable, differentiation of (2) partially w.r.t. 'x' and 'y' respectively gives

$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=0$ and $\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}=0$ *i.e.*, $p_1+p_3p=0$ and $p_2+p_3q=0$

so that $p=-(p_1/p_3)$ and $q=-(p_2/p_3)$ ... (3)

where $p_1=\partial u/\partial x=\partial u/\partial x_1,\ p_2=\partial u/\partial y=\partial u/\partial x_2,\ p_3=\partial u/\partial z=\partial u/\partial x_3,\ p=\partial z/\partial x,\ q=\partial z/\partial y$

by taking $x=x_1,\quad y=x_2$ and $z=x_3$ ... (4)

Using (3) and (4), (1) reduces to $p_1^2/p_3^2+p_2^2/p_3^2=k^2$ or $p_1^2+p_2^2=k^2p_3^2$

Let $f(x_1,x_2,x_3,p_1,p_2,p_3)=p_1^2+p_2^2-k^2p_3^2=0$ ... (5)

Now, the Jacobi auxilliary equations are given by

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{0}=\frac{dx_1}{-2p_1}=\frac{dp_2}{0}=\frac{dx_2}{-2p_2}=\frac{dp_3}{0}=\frac{dx_3}{2k^2p_3},\ \text{using (5)}$$

From the first and third fractions of (5), $dp_1=0$ and $dp_2=0$

Integrating, $p_1=a_1$ and $p_2=a_2$, a_1 and a_2 being arbitrary constants

With $p_1=a_1$ and $p_2=a_2$, (5) gives $p_3=(a_1^2+a_2^2)^{1/2}/k$

Putting the above values of p_1, p_2 and p_3 in $du=p_1dx_1+p_2dx_2+p_3dx_3$, we get

$$du=a_1dx_1+a_2dx_2+\{(a_1^2+a_2^2)^{1/2}/k\}dx_3$$

Integrating, $u=a_1x+a_2x_2+\{(a_1^2+a_2^2)^{1/2}/k\}x_3+a_3$ ... (6)

Taking $a_2=1$ and using (4), the required solution $u=0$ is given by

$$a_1x+x_2+\{(a_1^2+1)^{1/2}/k\}x_3+a_3=0,$$

which is the complete integral of (1) containing two arbitrary constants a_1 and a_3.

Ex. 3. *Solve the following partial differential equations by Jacobi's method:*

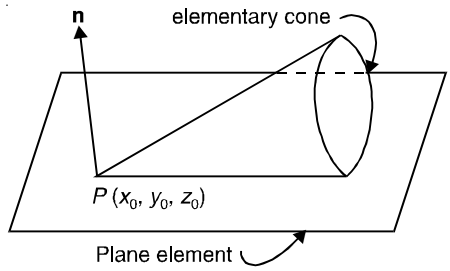
(*i*) $p=(z+qy)^2$ (*ii*) $(p^2+q^2)x=pz$ (*iii*) $xpq+yq^2=1$ **[Nagpur 2005]**

Hint. Proceed as in the above solved Ex. 1

3.23 Cauchy's method of characteristics for solving non-linear partial differential equation

$f(x,y,z,\ \partial z/\partial x,\partial z/\partial y)=0$ *i.e.*, $f(x,y,z,p,q)=0$ ... (1)

We know that the plane passing through the point $P(x_0,y_0z_0)$ with its normal parallel to the direction **n** whose direction ratios are $p_0, q_0, -1$ is uniquely given by the set of five numbers $D(x_0,y_0,z_0,p_0,q_0)$ and conversely any such set of five numbers defines a plane in three dimensional space. In view of this fact a set of five numbers $D(x,y,z,p,q)$ is known as a *plane element* of a three dimensional space. As a special case a plane element (x_0,y_0,z_0,p_0,q_0) whose components satisfy (1) is known as an *integral element* of (1) at *P*. Solving (1) for *q*, suppose we get

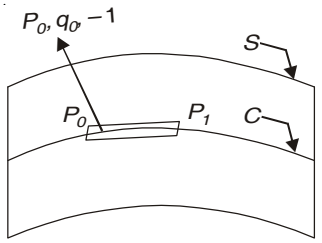


$$q = F(x, y, z, p).$$

which gives a value of q corresponding to known values of x, y, z and p. Then, keeping x_0, y_0 and z_0 fixed and varying p, we shall arrive at a set of plane elements $\{x_0, y_0, z_0, p,\ G(x_0, y_0, z_0, p)\}$ which depend on the single parameter p. As p varies, we get a set of plane elements all of which pass through the point P. Hence the above mentioned set of plane elements envelop a cone with vertex P. The cone thus obtained is known as the *elementary cone of* (1) at the point P.

Consider a surface S with equation $z = g(x, y)$ ... (2)

If the function $g(x, y)$ and its first partial derivatives $g_x(x, y)$ and $g_y(x, y)$ are continuous in a certain region R of the xy-plane, then the tangent plane at each point of S determines a plane element of the form $\{x_0, y_0, g(x_0, y_0),\ g_x(x_0, y_0),\ g_y(x_0, y_0)\}$ which will be referred as *the tangent element* of the surface S at the point $\{x_0, y_0, g(x_0, y_0)\}$.



Consider a curve C with parametric equations

$x = x(t)$, $y = y(t)$, $z = z(t)$, t being the parameter. ... (3)

Then curve C lies on (2) provided $z(t) = g\{x(t), y(t)\}$ ... (4)

holds good for all values of t in the appropriate interval I. Let P_0 be a point on curve C corresponding to $t = t_0$. Now, the direction ratios of the tangent line $P_0\ P_1$ are $x'(t_0), y'(t_0), z'(t_0)$ where $x'(t_0), y'(t_0)\ z'(t_0)$ denote the values of $dx/dt,\ dy/dt,\ dz/dt$ respectively at $t = t_0$

This direction will be perpendicular to direction of normal $\boldsymbol{n}$ (with direction ratios $p_0, q_0, -1$) if $p_0 x'(t_0) + q_0\ y'(t_0) + (-1)\ z'(t_0) = 0$ or $z'(t_0) = p_0\ x'(t_0) + q_0\ y'(t_0)$

It follows that any set $\{x(t), y(t), z(t), p(t), q(t)\}$ ...(5)

of five real functions satisfying the condition that $z'(t) = p(t)\ x'(t) + q(t)\ y'(t)$ ... (6)

defines a strip at the point (x, y, z) of the curve C. When such a strip is also an integral element of (1), then the strip under consideration is known as an *integral strip* of (1). In other words, the set of functions (5) is known as an integral strip of (1) provided these satisty (6) and the following additional condition

$$f\{x(t), y(t), z(t), p(t), q(t)\} = 0, \quad \text{for all } t \text{ in I.}$$

If at each point of the curve (3) touches a generator of the elementary cone, then the corresponding strip is known as a *characteristic strip*.

Derivation of the equations determining a *characteristic strip*

Clearly, the point $(x + dx,\ y + dy,\ z + dz)$ lies in the tangent plane to the elementary cone at P if

$$dz = p\,dx + q\ dy \quad \text{... (7)}$$

where p, q satisfy (1). Differentiation (7) w.r.t. 'p', we get

$$0 = dx + (dq/dp)\ dy \quad \text{... (8)}$$

Again, differentiating (1) partially w.r.t. 'p', we have

$\partial f/\partial p + (\partial f/\partial q)\ (dq/dp) = 0$ *i.e.,* $f_p + f_q (dq/dp) = 0$ ... (9)

Here, $\partial f/\partial p = f_p$ and $\partial f/\partial q = f_q$

Solving (7), (8) and (9) for the ratios of dy, dz to dx, we get

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p f_p + q f_q} \quad \ldots (10)$$

Hence along a characteristic strip $x'(t)$, $y'(t)$, $z'(t)$ will be proportional to f_p, f_q, $p f_p + q f_q$ respectively. If the parameter t be selected satisfying the relations

$$x'(t) = f_p \qquad \text{and} \qquad y'(t) = f_q .$$

then, we have $z'(t) = p f_p + q f_q$

Since along a characteristic strip p is a function of t, hence

$$p'(t) = (\partial p / \partial x)\ (dx/dt) + (\partial p / \partial y)\ (dy/dt) = (\partial p / \partial x)\ \ (\partial f / \partial p) + (\partial p / \partial y)\ (\partial f / \partial q), \text{ using (11)}$$

Thus, $$p'(t) = (\partial p / \partial x)\ (\partial f / \partial p) + (\partial q / \partial x)\ (\partial f / \partial q) \quad \ldots (12)$$

$$\left[\because \ \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x} \right]$$

Now, differentiating (1) partially w.r.t. 'x', gives $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$$

or $$f_x + p\ f_z + p'(t) = 0 \text{, using (12)}$$

Hence on a characteristic strip, $$p'(t) = -f_x - p f_z \quad \ldots (13)$$

Similarly, we have $$q'(t) = -f_y - q f_z \quad \ldots (14)$$

Here $f_x = \partial f / \partial x$, $f_y = \partial f / \partial y$, $f_z = \partial f / \partial z$

From (11), (13) and (14), we get the following system of five ordinary differential equations for the determination of the characteristic strip

$$x'(t) = f_p, \quad y'(t) = f_q, \quad z'(t) = p\ f_p + q f_q, \quad p'(t) = -f_x - p\ f_z \quad \text{and} \quad q'(t) = -f_y - q f_z \quad \ldots (15)$$

The above equations are called the *characteristic equations* of (1). In view of a well known result if the functions which are involved in (15) satisfy a Lipschitz condition, there exists a unique solution of (15) for given set of initial values of the variables. It follows that the characteristic strip is determined uniquely by any initial element $(x_0, y_0, z_0, p_0, q_0)$ and any initial value t_o of t.

Working rule for solving Cauchy's problem. **[Meerut 2005]**

Suppose we wish to find the integral surface of (1) which passes through a given curve with parametric equation $x = f_1(\lambda)$, $y = f_2(\lambda)$, $z = f_3(\lambda)$, λ being the parameter ... (16)

then in the solution $x = x\ (p_0, q_0, x_0, y_0, t_0, t)$ etc. ... (17)

of the characteristic equations (15), we shall assume that

$$x_0 = f_1(\lambda), \qquad y_0 = f_2(\lambda), \qquad z_0 = f_3(\lambda)$$

are the initial values of x, y, z respectively. Then the corresponding initial values of p_0, q_0 can be obtained by the following relations

$$f_3'(\lambda) = p_0\ f_1'(\lambda) + q_0\ f_3'(\lambda\ ) \qquad \text{and} \qquad f\{f_1(\lambda),\ f_2(\lambda),\ f_3(\lambda),\ p_0,\ q_0\} = 0$$

When the above values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 is substituted in (17), we shall be able to express x, y, z involving the two parameters t and λ of the form

$$x = \phi_1(t, \lambda), \qquad y = \phi_2(t, \lambda) \qquad \text{and} \qquad z = \phi_3(t, \lambda) \qquad \ldots (18)$$

which are known as characteristics of (1)

Finally, by eliminating λ and t from (18), we arrive at a relation of the form $G(x, y, z) = 0$, which is the required equation of the integral surface of (1) passing through the given curve (16).

3.24 Some Theorems:

Theorems 1. *A necessary and sufficient condition that a surface be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.*

Proof. Using geometrical considerations and Art 3.23, complete the proof yourself.

Theorem II. *Along every characteristic strip of the partial differential equation f (x, y, z, p, q) = 0 the function f (x, y, z, p, q) is a constant.*

Proof. Along a characteristic strip, we have

$$\frac{d}{dt} f\{x(t), y(t), z(t), p(t), q(t)\} = f_x x'(t) + f_y y'(t) + f_z z'(t) + f_p p'(t) + f_q q'(t)$$

$$= f_x f_p + f_y f_q + f_z (p f_p + q f_q) - f_p (f_x + p f_z) - f_q (f_y + q f_z)$$

$= 0$, using the characteristic equation (15) of Art. 3.23

showing that $f(x, y, z, p, q) = K$, a constant along the strip.

Corollary to theorem II. *If a characteristic strip contains at least one integral element of $f(x, y, z, p, q) = 0$ it is an integral strip of the equation $f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0$*

Proof. Left as an exercise.

3.25 SOLVED EXAMPLES BASED ON ART. 3.23

Ex. 1. *Find the characteristics of the equation $pq = z$, and determine the integral surface which passes through the parabola $x = 0, y^2 = z$.* **[Meerut 2005; I.A.S. 1999]**

Sol. Given equation is $pq = z$ ... (1)

We are to find its integral surface which passes through the given parabola given by

$$x = 0, \qquad \text{and} \qquad y^2 = z \qquad \ldots (2)$$

Re-writing (2) in parametric form, we have

$$x = 0, \qquad y = \lambda, \qquad z = \lambda^2, \qquad \lambda \text{ being a parameter} \qquad \ldots (3)$$

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = 0, \qquad y_0 = y_0(\lambda) = \lambda, \qquad z_0 = z_0(\lambda) = \lambda^2 \qquad \ldots (4A)$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values (x_0, y_0, z_0, p, q_0) satisfy (1), we have

$$p_0 q_0 = z_0, \qquad \text{or} \qquad p_0 q_0 = \lambda^2, \text{ by (4A)} \qquad \ldots (5)$$

Also, we have $z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

so that $2\lambda = p_0 \times 0 + q_0 \times 1$ or $q_0 = 2\lambda$, by (4A) ... (6)

Solving (5) and (6), $p_0 = \lambda/2$ and $q_0 = 2\lambda$ ... (4B)

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$x_0 = 0$, $y_0 = \lambda$, $z_0 = \lambda^2$, $p_0 = \lambda/2$, $q_0 = 2\lambda$ when $t = t_0 = 0$ ... (7)

Re-writing (1), let $f(x, y, z, p, q) = pq - z = 0$ ... (8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = q \quad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = p \quad \text{... (10)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = 2pq \quad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p \quad \text{... (12)}$$

and $$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = q \quad \text{... (13)}$$

From (9) and (13), $(dx/dt) - (dq/dt) = 0$, so that $x - q = C_1$, ... (14)

where C_1 is an arbitrary constant. Using initial values (7), (14) gives

$x_0 - q_0 = C_1$ or $0 - 2\lambda = C_1$ or $C_1 = -2\lambda$, Then (14) becomes

$x - q = -2\lambda$ or $x = q - 2\lambda$, ... (15)

From (10) and (12), $(dy/dt) - (dp/dt) = 0$ so that $y - p = C_2$, ... (16)

where C_2 is an arbitrary constant. Using initial values (7), (16) gives

$y_0 - p_0 = C_2$ or $\lambda - (\lambda/2) = C_2$ or $C_2 = \lambda/2$. Then (16) becomes

$y - p = \lambda/2$ or $y = p + (\lambda/2)$ ... (17)

From (12), $(1/p)\, dp = dt$ so that $\log p - \log C_3 = t$ or $p = C_3\, e^t$ ... (18)

Using initial values (7), (18) gives $p_0 = C_3 e^0$ or $\lambda/2 = C_3$

Hence (18) reduces to $p = (\lambda/2) \times e^t$ ... (19)

From (13), $(1/q)\, dq = dt$ so that $\log q - \log C_4 = t$ or $q = C_4\, e^t$ ... (20)

Using initial values (7), (20) gives $q_0 = C_4\, e^0$ or $2\lambda = C_4$

Hence (20) reduces to $q = 2\lambda e^t$ ... (21)

Using (21), (15) becomes $x = 2\lambda\, e^t - 2\lambda$ or $x = 2\lambda\, (e^t - 1)$ ... (22)

Using (19), (17) becomes $y = (\lambda/2)\, e^t + \lambda/2$ or $y = (\lambda/2) \times (e^t + 1)$ ... (23)

Substituting values of p and q from (19) and (21) in (11), we get

$dz/dt = 2\{(\lambda/2) \times e^t\} \times \{2\lambda\, e^t\}$ or $dz = 2\lambda^2\, e^{2t}\, dt$.

Integrating, $z = \lambda^2 e^{2t} + C_5$, C_5 being arbitrary constant .... (24)

Using initial values (7), (24) gives $z_0 = \lambda^2 e^0 + C_5$ or $\lambda^2 = \lambda^2 + C_5$ or $C_5 = 0$

Then, (24) gives $z = \lambda^2 e^{2t}$ or $z = \lambda^2 (e^t)^2$ ... (25)

The required characteristics of (1)are given by (22), (23) and (25)

To find the required integral surface of (1), we now proceed to eliminate two parameters t and λ from three equations (22), (23)and (25). Solving (22) and (23) for e^t and λ, we have

$e^t = (x+4y)/(4y-x)$ and $\lambda = (4y-x)/4$

Substituting these values of e^t and λ in (25), we have

$z = \{(4y-x)^2/16\} \times \{(x+4y)/(4y-x)\}^2$ or $16z = (4y+x)^2$,

which is the required integral surface of (1) passing through (2).

Ex. 2. *Find the solution of the equation* $z = (p^2+q^2)/2 + (p-x)(q-y)$ *which passes through the x-axis.* **[Himachal 1996; 2004; I.A.S. 2002]**

Sol. Given equation is $z = (p^2+q^2)/2 + (p-x)(q-y)$ ... (1)

We are to find its integral surface which passes through x-axis which is given by equations

$y = 0$ and $z = 0$ ... (2)

Re-writing (2) in parametric form, $x = \lambda, \quad y = 0, \quad z = 0$, λ being the parameter ... (3)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = x_0(\lambda) = \lambda$, $y_0 = y_0(\lambda) = 0$, $z = z_0(\lambda) = 0$ ... (4A)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy (1), we have

$z_0 = (p_0^2+q_0^2)/2 + (p_0 - x_0)(q_0 - x_0)$ or $0 = (p_0^2+q_0^2)/2 + q_0(p_0 - \lambda)$, by (4A)

or $p_0^2 + q_0^2 + 2q_0 p_0 - 2q_0\lambda = 0$ ... (5)

Also, we have $z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

so that $0 = p_0 \times 1 + q_0 \times 0$ or $p_0 = 0$, by (4A) ... (6)

Solving (5) and (6), $p_0 = 0$ and $q_0 = 2\lambda$ ... (4B)

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$x_0 = \lambda, \quad y_0 = 0, \quad z_0 = 0, \quad p_0 = 0, \quad q_0 = 2\lambda$ when $t = t_0 = 0$ ... (7)

Let $f(x, y, z, p, q) = (p^2+q^2)/2 + pq - py - qx + xy - z = 0$ ... (8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = p + q - y \quad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = q + p - x \quad \text{... (10)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = p(p+q-y) + q(q+p-x), \quad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p + q - y \quad \text{... (12)}$$

and $$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = p + q - x \quad \text{... (13)}$$

From (9) and (12), $(dx/dt) - (dp/dt) = 0$ so that $x - p = C_1$ ... (14)

where C_1 is an arbitrary constant. Using initial conditions (7), (14) gives $\lambda - 0 = C_1$ or $C_1 = \lambda$.

Hence (14) reduces to $x - p = \lambda$ or $x = p + \lambda$ ... (15)

From (10) and (13), $(dy/dt) - (dq/dt) = 0$ so that $y - q = C_2$, ... (16)

where C_2 is an arbitrary constant.

Using initial conditions (7), (16) gives $0 - 2\lambda = C_2$ or $C_2 = -2\lambda$.

Hence (16) reduces to $y - q = -2\lambda$ or $y = q - 2\lambda$ ... (17)

$\therefore \quad \dfrac{d(p+q-x)}{dt} = \dfrac{dp}{dt} + \dfrac{dq}{dt} - \dfrac{dx}{dt} = p + q - y + p + q - x - (p + q - y)$, using (9), (12) and (13)

or $\dfrac{d(p+q-x)}{dt} = p + q - x$ or $\dfrac{d(p+q-x)}{p+q-x} = dt$.

Integrating, $\log(p+q-x) - \log C_3 = t$ or $p + q - x = C_3\, e^t$, ... (18)

where C_3 is an arbitrary constant. Using initial conditions (7), (18) gives $0 + 2\lambda - \lambda = C_3$ or $C_3 = \lambda$.

Hence (18) reduces to $p + q - x = \lambda e^t$ ... (19)

Now, $\dfrac{d(p+q-y)}{dt} = \dfrac{dp}{dt} + \dfrac{dq}{dt} - \dfrac{dy}{dt} = p + q - y + p + q - x - (q + p - x)$, by (10), (12) and (13)

or $\dfrac{d(p+q-y)}{dt} = p + q - y$ or $\dfrac{d(p+q-y)}{b+q-y} = dt$.

Integrating, $\log(p+q-y) - \log C_4 = t$ or $p + q - y - C_4\, e^t$ ... (20)

where C_4 is an arbitrary constant. Using initial conditions (7), (20) gives $0 + 2\lambda - 0 = C_4$ or $C_4 = 2\lambda$.

Hence (20) reduces to $p + q - y = 2\lambda\, e^t$ ... (21)

From (9) and (21), $dx/dt = 2\lambda\, e^t$ so that $x = 2\lambda\, e^t + C_5$ ... (22)

where C_5 is an arbitary constant. Using initial conditions (7), (22) gives $\lambda = 2\lambda + C_5$ or $C_5 = -\lambda$.

Hence (22) reduces to $x = 2\lambda\, e^t - \lambda$ or $x = \lambda\,(2e^t - 1)$ ... (23)

From (10) and (19), $dy/dt = \lambda\, e^t$ so that $y = \lambda\, e^t + C_6$ ... (24)

where C_6 is an arbitrary constant. Using initial conditions (7), (24) gives $0 = \lambda + C_6$ or $C_6 = -\lambda$.

Hence (24) reduces to $y = \lambda e^t - \lambda$ or $y = \lambda\,(e^t - 1)$ ... (25)

Substituting value of y from (17) in (12), we get

$dp/dt = p + q - (q - 2\lambda)$ or $(dp/dt) - p = 2\lambda,$ ... (26)

which is a linear equation whose integrating factor = $e^{\int(-1)dt} = e^{-t}$ and solution is

$p\, e^{-t} = \int (2\lambda)\, e^{-t} dt + C_7 = -2\lambda e^{-t} + C_3$ or $p = -2\lambda + C_3\, e^t$ ... (27)

where C_7 is an arbitrary constant. Using initial condition (7), (27) gives $0 = -2\lambda + C_7$ or $C_7 = 2\lambda$.

Hence (27) reduces to $p = -2\lambda + 2\lambda e^t$ or $p = 2\lambda\,(e^t - 1)$ ... (28)

Substituting value of x from (15) in (13), we get

$$dq/dt = p+q-(p+\lambda) \qquad \text{or} \qquad dq/dt - q = -\lambda, \qquad \ldots (29)$$

which is a linear equation whose integrating factor = $e^{\int(-1)dt} = e^{-t}$ and solution is

$$q e^{-t} = \int(-\lambda)\, e^{-t} dt + C_8 = \lambda e^{-t} + C_8 \qquad \text{or} \qquad q = \lambda + C_8\, e^{t} \qquad \ldots (30)$$

where C_8 is an arbitrary constant. Using initial condition (7), (30) gives $2\lambda = \lambda + C_8$ or $C_8 = \lambda$.

Hence (30) reduces to $q = \lambda + \lambda\, e^{t}$ or $q = \lambda\,(1+e^{t})$ ... (31)

Substitutions the values of $p+q-x$ and $p+q-y$ from (13) and (24) respectively in (1) gives

$$dz/dt = p(2\lambda\, e^{t}) + q(\lambda\, e^{t}) = 2\lambda\,(e^{t}-1)\,(2\lambda\, e^{t}) + \lambda(1+e^{t})\,(\lambda\, e^{t})$$

[on putting values of p and q with help of (28) and (31)]

or $dz/dt = 5\lambda^2 e^{2t} - 3\lambda^2 e^{t}$ or $dz = (5\lambda^2 e^{2t} - 3\lambda^2 e^{t})dt$.

Integrating, $z = (5/2) \times \lambda^2 e^{2t} - 3\lambda^2 e^{t} + C_9$ ... (32)

where C_9 is an arbitrary constant. Using initial conditions (7), namely $z = 0$ where $t = 0$, (32) gives $0 = (5/2) \times \lambda^2 - 3\lambda^2 + C_9$ or $C_9 = 3\lambda^2 - (5/2)\lambda^2$. Hence (32) reduces to

$$z = (5/2) \times \lambda^2 (e^{2t} - 1) - 3\lambda^2 (e^{t} - 1) \qquad \ldots (33)$$

Solving (23) and (25) for λ and e^t, $\lambda = x - 2y$ and $e^{t} = (x-y)/(x-2y)$ ... (34)

Eliminating λ and e^t from (33) and (34), we have

$$z = \frac{5}{2}(x-2y)^2 \left\{ \left(\frac{x-y}{x-2y} \right)^2 - 1 \right\} - 3(x-2y)^2 \left(\frac{x-y}{x-2y} - 1 \right)$$

or $z = (5/2) \times \{(x-y)^2 - (x-2y)^2\} - 3\ \{(x-2y)\,(x-y) - (x-2y)^2\}$

or $z = (y/2) \times (4x - 3y)$, on simplification.

Ex. 3. *Determine the characteristics of the equation* $z = p^2 - q^2$ *and find the integral surface which passes through the parabola* $4z + x^2 = 0,\ y = 0$. **[Himachal 2000, 05]**

Sol. Do yourself, the required characteristics are $x = 2\lambda(2 - e^{-t})$, $y = 2\sqrt{2}\lambda\,(e^{-t} - 1)$, $z = -\lambda^2 e^{-2t}$, λ being parameter. Solution is $4z + (x + y\sqrt{2})^2 = 0$.

Ex. 4. *Determine the characteristics of the equation* $p^2 + q^2 = 4z$ *and find the solution of this equation which reduces to* $z = x^2 + 1$ *when* $y = 0$.

Miscellaneous Problem on Chapter 3

1. Show that the envelope of the family of surfaces touch each member of the family at all points of its characteristics. **[Meerut 2008]**

2. Find a complete integral of the partial differential equation $(p^2 + q^2)x = pz$ and deduce the surface solution which passes through the curve $x = 0,\ z^2 = 4y$. **[Meerut 2007]**

3. Solve $p^2 y + p^2 y x^2 = q x^2$ **[Pune 2010]**

Ans. Complete integral is $z = a(1+x^2)^{1/2} + (a^2 y^2)/2 + b$.

4. Given thta $(x-a)^2 + (y-b)^2 + z^2 = 1$ is complete integral of $z^2(1 + p^2 + q^2) = 1$. Find its singular integral. **[Pune 2010]**

Hint. Use definition on page 3.1. **Ans.** $z^2 = 1$

4

Homogenous Linear Partial Differential Equations with Constant Coefficients

4.1. Homogeneous and Non–homogeneous linear equations with constant coefficients. A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients being constants or functions of x and y, is known as a *linear partial differential equation*. The general form of such an equation is

$$\left(A_0\frac{\partial^n z}{\partial x^n}+A_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+\ldots+A_n\frac{\partial^n z}{\partial y^n}\right)+\left(B_0\frac{\partial^{n-1} z}{\partial x^{n-1}}+B_1\frac{\partial^{n-1} z}{\partial x^{n-2}\partial y}+\ldots\quad+B_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right)+\left(M_0\frac{\partial z}{\partial x}+M_1\frac{\partial z}{\partial y}\right)$$
$$+N_0 z=f(x,y),\quad\ldots(1)$$

where the coefficients $A_0, A_1, \ldots A_n, B_0, B_1, \ldots B_{n-1}, M_0, M_1$ and N_0 are constants or functions of x and y. If $A_0, A_1, \ldots A_n, B_0, B_1, \ldots B_{n-1}, M_0, M_1$ and N_0 are all constants, then (1) is called a *linear partial differential equation with constant coefficients*.

For convenience $\partial/\partial x$ and $\partial/\partial y$ will be denoted by D (or D_x) and D' (or D_y) respectively. Then (1) can be rewritten as

$$[(A_0D^n+A_1D^{n-1}D'+\ldots+A_nD'^n)+(B_0D^{n-1}+B_1D^{n-2}D'+\ldots+B_{n-1}D'^{n-1})$$
$$+(M_0D+M_1D')+N_0]z=f(x,y),\quad\ldots(2)$$

or, briefly,
$$F(D,D')z=f(x,y).\quad\ldots(3)$$

When all the derivatives appearing in (1) are of the same order, then the resulting equation is called a *linear homogeneous partial differential equation with constant coefficients* and it is then of the form

$$(A_0D^{n-1}+A_1D^{n-1}D'+\ldots+A_nD'^n)z=f(x,y).\quad\ldots(4)$$

On the other hand, when all the derivatives in (1) are not of the same order, then it is called a *non–homogeneous linear partial differential equation with constant coefficients*.

In this chapter we propose to study the various methods of solving homogeneous linear partial differential equation with constant coefficients, namely, (4)

4.2. Solution of a homogeneous linear partial differential equation with constant coefficients, namely,
$$(A_0D^{n-1}+A_1D^{n-1}D'+\ldots+A_nD'^n)z=f(x,y),\quad\ldots(1)$$

where $A_0, A_1, \ldots, A_n$ are constants. (1) may rewritten as

$$F(D,D')z=f(x,y),\quad\ldots(2)$$

where
$$F(D,D')=A_0D^{n-1}+A_1D^{n-1}D'+\ldots+A_nD'^n.\quad\ldots(3)$$

As in the case of linear ordinary differential equation with constant coefficients, we start with the following basic theorems.

Theorem I. *If u is the complementary function and z' a particular integral of a linear partial differential equation $F(D, D')z = f(x, y)$, then $u + z'$ is a general solution of the equation.*

Proof. Given
$$F(D,D')z=f(x,y).\quad\ldots(1)$$

The complementary function u of (1) is the most general solution of

$$F(D,D')z=0.\quad\ldots(2)$$

$$\therefore \qquad F(D, D')u = 0. \qquad ...(3)$$

Note that the complementary function must contain as many arbitrary constants as is the order of equation (2).

Any solution z' of (1) is called a particular integral of (1). Note that particular integral does not contain any arbitrary constant. Thus, by definition, we have

$$F(D, D')z' = f(x, y). \qquad ...(4)$$

Adding (3) and (4), $F(D, D')(u + z') = f(x, y)$,

showing that $u + z'$ is a solution of (1). Since (1) and (2) are of the same order, the general solution $u + z'$ will contain as many arbitrary constants as the general solution of (1) requires.

Theorem II. *If u_1, u_2, ..., u_n are solutions of the homogeneous linear partial differential equation $F(D, D')z = 0$, then $\sum_{r=1}^{n} c_r u_r$ is also a solution, where C_1, C_2, ..., C_r, ..., C_n are arbitrary constants.*

Proof. Given equation is $F(D, D')z = 0. \qquad ...(1)$

We have $F(D, D')(c_r u_r) = c_r F(D, D')u_r \qquad ...(2)$

and $F(D, D')\sum_{r=1}^{n} v_r = \sum_{r=1}^{n} F(D, D')v_r \qquad ...(3)$

for any set of functions v_r. Using results (2) and (3), we get

$$F(D, D')\sum_{r=1}^{n}(c_r u_r) = \sum_{r=1}^{n} F(D, D')(c_r u_r) = c_r \sum_{r=1}^{n} F(D, D')u_r \qquad ...(4)$$

Since u_r is solution of (1) for $r = 1, 2, ..., n$, so $F(D, D')u_r = 0$ for $r = 1, 2, 3, ..., n$.

$\therefore$ (4) gives $F(D, D')\sum_{r=1}^{n}(c_r u_r) = 0$, which proves the required result.

Note. For convenience we shall denote complementary function by C.F. and particular integral by P.I.

4.3. Method of finding the complementary function (C.F.) of the linear homogeneous partial differential equation with constant coefficients, namely, $F(D, D')z = f(x, y)$

i.e., $(A_0 D^n + A_1 D^{n-1} D' + ... + A_n D'^n)z = f(x, y), \qquad ...(1)$

where $A_0, A_1, ..., A_n$ are all constants.

The complementary function of (1) is the general solution of

$$(A_0 D^n + A_1 D^{n-1} D' + ... + A_n D'^n)z = 0. \qquad ...(2)$$

or $$[(D - m_1 D')(D - m_2 D') ... (D - m_n D')]z = 0. \qquad ...(3)$$

where $m_1, m_2, ..., m_n$ are some constants.

Clearly, the solution of any one of the equations

$$(D - m_1 D')z = 0, \quad (D - m_2 D')z = 0, ..., \quad (D - m_n D')z = 0 \qquad ...(4)$$

is also a solution of (3).

We now show that the general solution of $(D - mD')z = 0$ is $z = \phi(y + mx)$, where ϕ is an arbitrary function.

We have, $(D - mD')z = 0$ or $(\partial z/\partial x) - m(\partial z/\partial y) = 0$ or $p - mq = 0, \qquad ...(5)$

which is in Lagrange's form $Pp + Qq = R$. Here Lagrange's auxiliary equations for (5) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0}. \qquad ...(6)$$

Taking the first two fractions of (6), $dy + mdx = 0$ so that $y + mx = c_1$ ... (7)

From the third fraction of (6), $dz = 0$ so that $z = c_2$. ...(8)

Hence from (7) and (8), the general solution of (5) is $z = \phi(y + mx)$, where ϕ is an arbitrary function. So, we assume that a solution of (2) is of the form

$$z = \phi(y + mx). \qquad ...(9)$$

From (9), $Dz = \partial z/\partial x = m\phi'(y + mx)$,

$D^2z = \partial^2 z/\partial x^2 = m^2\phi''(y + mx)$,

.....

and $D^n z = \partial^n z/\partial x^n = m^n \phi^{(n)}(y + mx)$.

Again, $D'z = \partial z/\partial y = \phi'(y + mx)$,

$D'^2 z = \partial^2 z/\partial y^2 = \phi''(y + mx)$,

.....

and $D'^n z = \partial^n z/\partial y^n = \phi^{(n)}(y + mx)$.

Also, in general, $D^r D'^s z = \partial^{r+s} z/\partial x^r \partial y^s = m^r \phi^{(r+s)}(y + mx)$.

Substituting these values in (2) and simplifying, we get

$$(A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + ... + A_n)\phi^{(n)}(y + mx) = 0,$$

which is true if m is a root of the equation

$$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + ... + A_n = 0. \qquad ...(10)$$

The equation (10) is known as the *auxiliary equation* (*A.E.*) and is obtained by putting $D = m$ and $D' = 1$ in $F(D, D') = 0$.

Let $m_1, m_2, ..., m_n$ be n roots of A.E. (10). Two cases arise.

Case I. When $m_1, m_2, m_3, ... m_n$ are distinct. Then the part of C.F. corresponding to $m = m_r$ is $z = \phi_r(y + m_r x)$ for $r = 1, 2, 3, ..., n$. Since (2) is linear, the sum of the solutions is also a solution.

$\therefore$ C.F. of (2) $= \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + ... + \phi_n(y + m_n x)$, ...(11)

where $\phi_1, \phi_2, ..., \phi_n$ are arbitrary functions.

Case II. Repeated roots. Let m be repeated root of (10) and so consider

$$(D - mD')(D - mD')z = 0. \qquad ...(12)$$

Let $(D - mD')z = v$. ...(13)

Then, (12) $\Rightarrow$ $(D - mD')v = 0$ or $(\partial v/\partial x) - m(\partial v/\partial y) = 0$, ...(14)

which is in Lagrange's form. Hence Lagrange's auxiliary equations for (14) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dv}{0}. \qquad ...(15)$$

As before, two independent integrals of (15) are $y + mx = c_3$ and $v = c_4$.

$\therefore$ $v = \phi(y + mx)$ ...(16)

is a solution of (14), ϕ being as arbitrary function.

Using (16), (13) becomes $(\partial z/\partial x) - m(\partial z/\partial y) = \phi(y + mx)$ ...(17)

which is in Lagrange's form. Its Lagrange's auxiliary equations for (7) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y + mx)} \qquad ...(18)$$

Taking the first two fractions of (18), $dy + mdx = 0$ so that $y + mx = c_5$. ... (19)

Taking the first and third fractions of (18) and using (19), we get

$(dx)/1 = (dz)/\phi(c_5)$ so that $dz - \phi(c_5)dx = 0$.

Integrating, $z - x\,\phi(c_5) = c_6$ or $z - x\,\phi(y + mx) = c_6$, using (19). ...(20)

From (19) and (20), the general solution of (12) is

$z - x\phi(y + mx) = \psi(y + mx)$ or $z = \psi(y + mx) + x\phi(y + mx)$, ...(21)

where ϕ and ψ are arbitrary functions. (21) is a part of C.F. corresponding to the two times repeated root m. In general, if a root 'm' is repeated 'r' times, the corresponding part of C.F. is

$$\phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + \ldots + x^{r-1}\phi_r(y + mx).$$

4.4.A. Working rule for finding C.F. of linear homogeneous partial differential equation with constant coefficients

Step 1. Put the given equation in standard form $(A_0D^n + A_1D^{n-1}D' + \ldots A_nD'^n)z = f(x, y)$....(1)

Step 2. Replacing D by m and D' by 1 in the coefficients of z, we obtain auxiliary equation (A.E.) for (1) as $A_0m^n + A_1m^{n-1} + \ldots + A_n = 0$. ...(2)

Step 3. Solve (2) for m. Two cases will arise :

Case (*i*) Let $m = m_1, m_2, \ldots, m_n$ (different roots). Then

C.F. $= \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots + \phi_n(y + m_nx)$, where $\phi_1, \phi_2 \ldots, \phi_n$ are arbitrary functions.

If in the above case (*i*), $m = a_1/b_1, a_2/b_2, \ldots, a_n/b_n$, then

$$\text{C.F.} = \phi_1(b_1y + a_1x) + \phi_2(b_2y + a_2x) + \ldots + \phi_n(b_ny + a_nx)$$

Further if $m = -(a_1/b_1), -(a_2/b_2), \ldots, -(a_n/b_n)$, then

$$\text{C.F.} = \phi_1(b_1y - a_1x) + \phi_2(b_2y - a_2x) + \ldots + \phi_n(b_ny - a_nx)$$

Case (*ii*) Let $m = m'$ (repeated n times). Then corresponding to these roots

$$\text{C.F.} = \phi_1(y + m'x) + x\,\phi_2(y + m'x) + x^2\phi_2(y + m'x) + \ldots + x^{n-1}\phi_n(y + m'x).$$

In the above case (*ii*), if $m = a/b$ (repeated n times), Then corresponding to these n roots,

$$\text{C.F.} = \phi_1(by + ax) + x\,\phi_2(by + ax) + x^2\phi_2(by + ax) + \ldots + x^{n-1}\phi_n(by + ax)$$

And, if $m = -(a/b)$, (repeated n times), then

$$\text{C.F.} = \phi_1(by - ax) + x\,\phi_2(by - ax) + x^2\phi_2(by - ax) + \ldots + x^n\phi_n(by - ax)$$

Case (*iii*) Corresponding to a non–repeated factor D on L.H.S. of (1), the part of C.F. is taken as $\phi(y)$.

Case (*iv*) Corresponding to a repeated factor D^m on L.H.S. of (1), the part of C.F. is taken as

$$\phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + \ldots + x^{m-1}\phi_m(y).$$

Case (*v*) Corresponding to a non–repeated factor D' on L.H.S. of (1), the part of C.F. is taken as $\phi(x)$.

Case (*vi*) Corresponding to a repeated factor D'^m on L.H.S. of (1), the part of C.F. is taken as

$$\phi_1(x) + y\,\phi_2(x) + y^2\phi_3(x) + \ldots + y^{m-1}\phi_m(x).$$

4.4.B. Alternative working rule for finding C.F.

Let the given partial differential equation be $F(D, D')z = f(x, y)$. Factorize $F(D, D')$ into linear factors of the form $(bD - aD')$. Then we use the following results :

(*i*) Corresponding to each non–repeated factor $(bD - aD')$, the part of C.F. is taken as

$$\phi(by + ax).$$

(*ii*) Corresponding to a repeated factor $(bD - aD')^m$, the part of C.F. is taken as

$$\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + \ldots + x^{m-1}\phi_m(by + ax).$$

(*iii*) Corresponding to a non–repeated factor D, part of C.F. is taken as $\phi(y)$.

(*iv*) Corresponding to a repeated factor D^m, the part of C.F. is taken as

$$\phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + \ldots + x^{m-1}\phi_m(y).$$

(*v*) Corresponding to a non–repeated factor D', part of C.F. is taken as $\phi(x)$.

(*vi*) Corresponding to a repeated factor D'^m, the part of C.F. is taken as

$$\phi_1(x) + y\,\phi_2(x) + y^2\phi_3(x) + \ldots + y^{m-1}\phi_m(x).$$

4.5. Solved examples based on articles 4.4A and 4.4B

[Notations $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\partial y$ and $t = \partial^2 z/\partial y^2$ will be used]

Ex. 1. *Solve* (*a*) $r = a^2 t$. **[I.A.S. 1987; Meerut 1991]**

(*b*) $(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = 0$.

(*c*) $(D^2 - 3aDD' + 2a^2D'^2)z = 0$. **[Kanpur 2007; Meerut 2007]**

Sol. (*a*) Given equation is $\partial^2 z/\partial x^2 = a^2(\partial^2 z/\partial y^2)$ or $(D^2 - a^2D'^2)z = 0$. ...(1)

The auxiliary equation of (1) is $m^2 - a^2 = 0$ so that $m = a, -a$.

$\therefore$ The general solution of (1) is $z = \text{C.F.} = \phi_1(y + ax) + \phi_2(y - ax)$, where ϕ_1 and ϕ_2 are arbitrary functions.

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + x) + \phi_2(y - x)$

(*c*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + ax) + \phi_2(y + 2ax)$

Ex. 2. *Solve* (*a*) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$. **[Agra 2005]**

(*b*) $(\partial^3 z/\partial x^3) - 7(\partial^3 z/\partial x\partial y^2) + 6(\partial^3 z/\partial y^3) = 0$. **[Bhopal 2010]**

(*c*) $(D^3 - 3D^2D' + 2DD'^2)z = 0$. **[Meerut 2008; Lucknow 2010]**

Sol. (*a*) The auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$

or $(m - 1)(m - 2)(m - 3) = 0$ so that $m = 1, 2, 3$.

$\therefore$ The general solution of the given equation is

$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

(*b*) The given equation can be written as $(D^3 - 7DD'^2 + 6D'^3)z = 0$. ...(1)

Its auxiliary equation is $m^3 - 7m + 6 = 0$ or $(m - 1)(m - 2)(m + 3) = 0$.

Hence $m = 1, 2, -3$ and so the general solution of (1) is

$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y - 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

(*c*) Proceed as above. **Ans.** $z = \phi_1(y) + \phi_2(y + x) + \phi_3(y + 2x)$.

Ex. 3. (*a*) *Solve* $2r + 5s + 2t = 0$. **[Meerut 2011]**

(*b*) $2(\partial^2 z/\partial x^2) - 3(\partial^2 z/\partial x\partial y) - 2(\partial^2 z/\partial y^2) = 0$.

Sol. (*a*) Now, $r = \partial^2 z/\partial x^2 = D^2 z$, $s = \partial^2 z/Dx\partial y = DD'z$ and $t = \partial^2 z/\partial y^2 = D'^2 z$. Hence the given equation can be re–written as $(2D^2 + 5\,DD' + 2D'^2)y = 0$. ...(1)

Its auxiliary equation is $2m^2 + 5m + 2 = 0$ or $(2m + 1)(m + 2) = 0$.

So $m = -1/2, -2$ and hence the general solution of (1) is

$z = \phi_1(2y - x) + \phi_2(y - 2x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Alternative method : (1) can be re–written as $(2D + D')(D + 2D') = 0$.

So by using the alternative working rule 4.4B, the general solution of (1) is

$z = \phi_1(2y - x) + \phi_2(y - 2x)$, ϕ_1 ϕ_2 being arbitrary functions.

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(2y - x) + \phi_2(y + 2x)$

Ex. 4. *Solve* (*a*) $r + t + 2s = 0$ **[Kanpur 2009]**

(*b*) $25r - 40s + 16t = 0$ **[Bilaspur 1996; Jabalpur 2002; Sagar 2004];**

(*c*) $(4D^2 + 12DD' + 9D'^2)z = 0$. **[Kanpur 2008; Indore 2004]**

Sol. (*a*) Here $r = \partial^2 z/\partial x^2 = D^2 z$, $t = \partial^2 z/\partial y^2 = D'^2 z$, $s = \partial^2 z/\partial x\partial y = DD'z$.

So the given equation becomes $(D^2 + D'^2 + 2DD')z = 0$. ...(1)

Its auxiliary equation is $m^2 + 1 + 2m = 0$ or $(m + 1)^2 = 0$ or $m = -1, -1$.

So the general solution of (1) is $z = \phi_1(y - x) + x\phi_2(y - x)$, ϕ_1 and ϕ_2 being arbitrary functions.

(*b*) Here $r = \partial^2 z/\partial x^2 = D^2 z$, $s = \partial^2 z/\partial x\partial y = DD'z$, $t = \partial^2 z/\partial y^2 = D'^2 z$. So the given equation becomes $(25D^2 - 40DD' + 16D'^2)z = 0$. ...(1)

Its auxiliary equation is $25m^2 - 40m + 16 = 0$ or $(5m - 4)^2 = 0$ so that $m = 4/5, 4/5$.

Hence the general solution of (1) is $z = \phi_1(5y + 4x) + x\phi_2(5y + 4x)$.

Alternative method (1) may be written as $(5D - 4D')^2 z = 0$.

Using result (*ii*) of Art 4.4B, the solution of (1) is

$z = \phi_1(5y + 4x) + x\phi_2(5y + 4x)$, ϕ_1, ϕ_2 being arbitrary functions.

(*c*) Proceed as in part (b). **Ans.** $z = \phi_1(2y - 3x) + x\phi_2(2y - 3x)$

Ex. 5. *Solve* (*a*) $(D^3 - 4D^2D' + 4DD'^2)z = 0$. **[Bhopal 2000, 03]**

(*b*) $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$. **[Bilaspur 2004]**

(*c*) $(D^4 + D'^2 - 2D^2D'^2)z = 0$.

(*d*) $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$.

Sol. (*a*) The auxiliary equation of the given equation is

$m^3 - 4m^2 + 4m = 0$ or $m(m - 2)^2 = 0$ so that $m = 0, 2, 2$

Hence the general solution of the given equation is

$z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

(*b*) The auxiliary equation of the given equation is

$m^4 - 2m^3 + 2m - 1 = 0$ or $(m + 1)(m - 1)^3 = 0$ so that $m = -1, 1, 1, 1$.

Hence the general solution of the given equation is

$z = \phi_1(y - x) + \phi_2(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

(*c*) Try yourself. **Ans.** $z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y - x) + x\phi_4(y - x)$

(*d*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$

Ex. 6. *Solve* (*a*) $(D^3D'^2 + D^2D'^3)z = 0$. (*b*) $(D^3D' - 4D^2D'^2 + 4DD'^3)z = 0$.

Sol. (*a*) The given equation can be re–written as $D^2D'^2(D + D')z = 0$. ...(1)

Hence using the alternative method 4.4B for C.F., the general solution is

$z = \phi_1(y) + x\phi_2(y) + \phi_3(x) + y\phi_4(x) + \phi_5(y - x)$, where ϕ_1, ϕ_2, ϕ_3, ϕ_4 and ϕ_5 are arbitrary functions.

(*b*) The given equation can be re–written as

$DD'(D^2 - 4DD' + 4D'^2) = 0$ *Or* $DD'(D - 2D')^2 = 0$.

Using the working rule 4.4B, the required general solution is

$z = \phi_1(y) + \phi_2(x) + \phi_3(y + 2x) + x\phi_4(y + 2x)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

Ex. 7. *Solve* (*a*) $(\partial^4 z/\partial x^4) - (\partial^4 z/\partial y^4) = 0$ (*b*) $(D^4 + D'^4)z = 0$.

Sol. (*a*) Rewriting, the given equation is $(D^4 - D'^4)z = 0$. ...(1)

Its auxiliary equation is $m^4 - 1 = 0$ or $(m^2 - 1)(m^2 + 1) = 0 \Rightarrow m = 1, -1\ i, -i$.

Hence the general solution of (1) is $z = \phi_1(y - x) + \phi_2(y + x) + \phi_3(y + ix) + \phi_4(y - ix)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

(*b*) The auxiliary equation of the given equation is $m^4 + 1 = 0$ or $(m^2 + 1)^2 - 2m^2 = 0$

or $(m^2 + 1)^2 - (m\sqrt{2})^2 = 0$ or $(m^2 + 1 + \sqrt{2}m)(m^2 + 1 - \sqrt{2}m) = 0$

so that $m^2 + \sqrt{2}m + 1 = 0$ or $m^2 - \sqrt{2}m + 1 = 0 \Rightarrow m = (-1 \pm i)/\sqrt{2}, (1 \pm i)/\sqrt{2}$.

Let $z_1 = (-1 + i)/\sqrt{2}$ and $z_2 = (1 + i)/\sqrt{2}$, then, $m = z_1, \bar{z}_1, z_2, \bar{z}_2$,

where $\bar{z}_1$ and $\bar{z}_2$ denote complex conjugates of z_1 and z_2 respectively.

Hence the general solution of the given equation is

$z = \phi_1(y + z_1x) + \phi_2(y + \bar{z}_1x) + \phi_3(y + z_2x) + \phi_4(y + \bar{z}_2x)$, where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are arbitrary functions.

4.6. Particular integral (P.I.) of homogeneous linear partial differential equation given by

$$F(D, D')y = f(x, y) \quad \ldots (1)$$

The inverse operator $1/F(D,D')$ of the operator $F(D,D')$ is defined by the following identity

$$F(D,D')\left(\frac{1}{F(D,D')}f(x,y)\right)=f(x,y)$$

$\therefore$ Particular integral (P.I.) of (1) $=\dfrac{1}{F(D,D')}f(x,y)$

In what follows we shall treat the symbolic functions of D and D' as we do for the symbolic functions of D alone in ordinary differential equations. Thus it will be factorized and resolved into partial fractions or expanded in an infinite series as the case may be. The reader is advised to note carefully the following results :

(*i*) D, D^2, ... will stand for differentiating partially with respect to x once, twice and so on.

For example, $Dx^4y^5=\dfrac{\partial}{\partial x}x^4y^5=4x^3y^5$; $D^2x^4y^5=\dfrac{\partial^2}{\partial x^2}x^4y^5=12x^2y^5$.

(*ii*) D', D'^2... will stand for differentiating partially with respect to y once, twice and so on.

For example, $D'x^4y^5=\dfrac{\partial}{\partial y}x^4y^5=5x^4y^4$; $D'^2x^4y^5=\dfrac{\partial^2}{\partial y^2}x^4y^5=20x^4y^3$.

(*iii*) $1/D$, $1/D^2$, ... will stand for integrating partially with respect to x once, twice and so on.

For example, $\dfrac{1}{D}x^4y^5=\int x^4y^5\,dx=\dfrac{x^5y^5}{5}$; $\dfrac{1}{D^2}x^4y^5=\int\int x^4y^5\,dx\,dx=\dfrac{x^6y^5}{30}$

(*iv*) $1/D'$, $1/D'^2$, ... will stand for integrating partially with respect to y once, twice and so on.

For example, $\dfrac{1}{D'}x^4y^5=\int x^4y^5\,dy=\dfrac{x^4y^6}{6}$; $\dfrac{1}{D'^2}x^4y^5=\int\int x^4y^5\,dy\,dy=\dfrac{x^4y^7}{42}$.

4.7. Short methods of finding the P.I. in certain cases.

Before taking up the general method for finding P.I. of $F(D, D')z = f(x, y)$ we begin with cases when $f(x, y)$ is in two special forms. The methods corresponding to these forms are much shorter than the general methods to be discussed in Art. 4.12.

4.8. A Short Method I. When f(x, y) is of the form f(ax + by).

The method under consideration is based on the following theorem.

Theorem I. *If $F(D, D')$ be homogeneous function of D and D' of degree n, then*

$$\frac{1}{F(D, D')}\phi^{(n)}(ax + by) = \frac{1}{F(a, b)}\phi(ax + by),$$

provided $F(a, b) \neq 0$, $\phi^{(n)}$ being the nth derivative of ϕ w.r.t. $ax + by$ as a whole.

Proof. By direct differentiation, we have $D^r\phi(ax + by) = a^r\phi^{(r)}(ax + by)$,

$D'^s\phi(ax + by) = b^s\phi^{(s)}(ax + by)$ and $D^rD'^s\phi(ax + by) = a^rb^s\phi^{(r+s)}(ax + by)$.

Since $F(D, D')$ is homogeneous function of degree n, so we have

$$F(D, D')\phi(ax + by) = F(a, b)\phi^{(n)}(ax + by). \quad ...(1)$$

Operating both sides of (1) by $1/F(D, D')$, we have

$$\phi(ax + by) = F(a, b)\frac{1}{F(D, D')}\phi^{(n)}(ax + by). \quad ...(2)$$

Since $F(a, b) \neq 0$, dividing both sides of (2) by $F(a, b)$, we get

$$\frac{1}{F(D, D')}\phi^{(n)}(ax + by) = \frac{1}{F(a, b)}\phi(ax + by). \quad ...(3)$$

An important deduction from result (3) : Putting $ax + by = v$, (3) gives

$$\frac{1}{F(D, D')}\phi^{(n)}(v) = \frac{1}{F(a, b)}\phi(v)\,. \qquad ...(4)$$

Integrating both sides of (4) n times w.r.t. 'v', we have

$$\frac{1}{F(D, D')}\phi(v) = \frac{1}{F(a, b)}\int\int\ldots\int\phi(v)\,dv\,dv\ldots dv\,, \qquad \text{where} \qquad v = ax + by.$$

Exceptional case when F(a, b) = 0. When $F(a, b) = 0$, then the above theorem does not hold good. In such a case the new method is based on the following theorem. Note that $F(a, b) = 0$ if and only if $(bD - aD')$ is a factor $F(D, D')$.

Theorem II. $$\frac{1}{(bD - aD')^n}\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by).$$

Proof. Consider the equation $$(bD - aD')z = x^r\phi(ax + by) \qquad ...(1)$$

or $$bp - aq = x^r\phi(ax + by). \qquad ...(2)$$

Lagrange's subsidiary equations for (2) are $$\frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{x^r\phi(ax + by)}. \qquad ...(3)$$

Taking the first two fractions of (3), $adx + bdy = 0$ so that $ax + by = c_1$ ... (4)

Taking the first and third members of (4) and using (4), we get

$$\frac{dx}{b} = \frac{dz}{x^r\phi(c_1)} \qquad \text{or} \qquad dz = \frac{x^r\phi(c_1)}{b}dx\,.$$

Integrating, $$z = \frac{x^{r+1}\phi(c_1)}{b(r+1)} = \frac{x^{r+1}\phi(ax + by)}{b(r+1)}, \text{ by (4).} \qquad ...(5)$$

(5) is a solution of (1).

Now, from (1), $$z = \frac{1}{(bD - aD')}x^r\phi(ax + by). \qquad ...(6)$$

From (5) and (6), $$\frac{1}{(bD - aD')}x^r\phi(ax + by) = \frac{x^{r+1}}{b(r+1)}\phi(ax + by). \qquad ...(7)$$

Hence, if $z = \frac{1}{(bD - aD')^n}\phi(ax + by)$, then we have

$$z = \frac{1}{(bD - aD')^{n-1}}\left[\frac{1}{(bD - aD')}x^0\phi(ax + by)\right], \text{ as } x^0 = 1$$

$$= \frac{1}{(bD - aD')^{n-1}}\frac{x}{b}\phi(ax + by)\,, \text{ using (7) for } r = 0$$

$$= \frac{1}{b}\frac{1}{(bD - aD')^{n-2}}\left[\frac{1}{(bD - aD')}x\phi(ax + by)\right]$$

$$= \frac{1}{b}\frac{1}{(bD - aD')^{n-2}}\frac{x^2}{2b}\phi(ax + by)\,, \text{ using (7) for } r = 1$$

$$= \frac{1}{2!b^2}\frac{1}{(bD - aD')^{n-2}}x^2\phi(ax + by)$$

$$= \frac{1}{n!b^n}\frac{1}{(bD - aD')^{n-n}}x^n\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by)$$

[after repeated use of (7) for $n - 2$ times more]

Working rule for finding particular integral where f(x, y) = ϕ(ax + by).

The following rules will used depending upon the situation in hand.

Formula (i). When $F(a, b) \neq 0$ and $F(D, D')$ is a homogeneous function of degree n, then

$$\text{P.I.} = \frac{1}{F(D, D')}\phi(ax + by) = \frac{1}{F(a, b)}\int\int \ldots \int f(v)\, dv\, dv \ldots dv, \qquad \text{where} \qquad v = ax + by$$

Note that R.H.S. contains a multiple integral of nth order.

Formula (ii). When $F(a, b) = 0$, we have

$$\frac{1}{(bD - aD')^n}\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by).$$

4.9. Solved Examples based on Short Method I of Art. 4.8

Ex. 1. *Solve* $(D^2 + 3DD' + 2D'^2)z = x + y$. **[I.A.S. 1986, Meerut 2005, 07, 09, 10]**

Sol. The auxiliary equation of the given equation is $m^2 + 3m + 2 = 0$ giving $m = -1, -2$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$$

$$= \frac{1}{1^2 + 3 \times 1 \times 1 + 2 \times 1^2}\int\int v\, dv\, dv\,, \text{ where } v = x + y, \text{ using formula (i) of working rule}$$

$$= \int (v^2 / 2)\, dv = (1/6) \times (v^3/6) = (1/36) \times (x + y)^3.$$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$, *i.e.*,

$$z = \phi_1(y - x) + \phi_2(y - 2x) + (1/36) \times (x + y)^3.$$

Ex. 2. *Solve* (*a*) $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$. **[Vikram 2000]**

(*b*) $(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) = 12(x + y)$ **[Nagpur 2010]**

(*c*) $(D^2 + D'^2)z = 30(2x + y)$. **[Bhopal 1996, Sagar 2004]**

(*d*) $(\partial^2 z/\partial x^2) + 2(\partial^2 z/\partial x \partial y) + (\partial^2 z/\partial y^2) = 2x + 3y$. **[Kumaun 1992]**

(*e*) $(D^2 + 3DD' + 2D'^2)z = 2x + 3y$. **[Kurukshetra 2005]**

(*f*) $r + s - 2t = (2x + y)^{1/2}$. **[Lucknow 2010]**

Sol. (*a*) The auxiliary of the given equation is $2m^2 - 5m + 2 = 0$ giving $m = 1/2, 2$.

$\therefore$ C.F. $= \phi_1(2y + x) + \phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2}24(y - x) = 24\frac{1}{2D^2 - 5DD' + 2D'^2}(y - x)$$

$$= \frac{24}{2 \times (-1)^2 - 5 \times (-1) \times 2 + 2 \times 2^2}\int\int v\, dv\, dv\,, \text{ where } v = y - x, \text{ using formula } (i) \text{ of working rule}$$

$$= (24/20) \times \int (v^2 / 2)\, dv = (6/5) \times (v^3/6) = (1/5) \times (y - x)^3\,.$$

Hence the required general solution is $z = \phi_1(2y + x) + \phi_2(y + 2x) + (y - x)^3/5$.

(*b*) The given equation can be written as $(D^2 + D'^2)V = 12(x + y)$. ...(1)

Its auxiliary equation is $m^2 + 1 = 0$ so that $m = \pm i$.

$\therefore$ C.F. $= \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{D^2 + D'^2}12(x + y) = 12\frac{1}{D^2 + D'^2}(x + y) = \frac{12}{1^2 + 1^2}\int\int v\, dv\, dv$$

$$= 6\int (v^2/2)\,dv = v^3 = (x+y)^3 .$$

Hence the required general solution is $V = \phi_1(y+ix) + \phi_2(y-ix) + (x+y)^3$.

(*c*) Try yourself. **Ans.** $z = \phi_1(y+ix) + \phi_2(y-ix) + (2x+y)^3$

(*d*) Try yourself. **Ans.** $z = \phi_1(y-x) + x\phi_2(y-x) + (1/150) \times (2x+3y)^3$

(*e*) Proceed as in part (*d*). **Ans.** $y = \phi_1(y-x) + \phi_2(y-2x) + (1/240) \times (2x+3y)^3$

(*f*) Since $r = \partial^2 z/\partial x^2,\ s = \partial^2 z/\partial x \partial y,\ t = \partial^2 z/\partial y^2,$ the given equation can be re–written as

$$\partial^2 z/\partial x^2 + \partial^2 z/\partial x\,\partial y - 2(\partial^2 z/\partial y^2) = (2x+y)^{1/2} \quad \text{or} \quad (D^2 + DD' - 2D'^2)z = (2x+y)^{1/2}.$$

Its auxiliary equation is $m^2 + m - 2 = 0$ so that $m = 1, -2$.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 2D'^2}(2x+y)^{1/2} = \frac{1}{2^2 + 2\times 1 - 2\times 1^2}\iint v^{1/2}\,dv\,dv, \text{ where } v = 2x+y$$

$$= \frac{1}{4}\int \frac{2}{3}v^{3/2}\,dv = \frac{1}{4}\times\frac{2}{3}\times\frac{2}{5}v^{5/2} = \frac{1}{15}(2x+y)^{5/2}$$

Hence the required general solution is $z = \phi_1(y+x) + \phi_2(y-2x) + (1/15)(2x+y)^{5/2}$.

Ex. 3. *Solve* (*a*) $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$. **[Bhopal 2010; Indore 1998; Jabalpur 1998; Purvanchal 2007, Sagar 1999; K.V. Kurkshetra 2005]**

(*b*) $(D^2 - 2DD' + D'^2)z = e^{x+2y}$. **[Bhopal 1997, 98, Kanpur 2005]**

(*c*) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x+6y}$.

Sol. (*a*) Here auxiliary equation is $m^2 + 2m + 1 = 0$ so that $m = -1, -1$.

$\therefore$ C.F. $= \phi_1(y-x) + x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2}e^{2x+3y} = \frac{1}{(D+D')^2}e^{2x+3y} = \frac{1}{(2+3)^2}\iint e^v\,dv\,dv, \text{ where } v = 2x+3y$$

$$= (1/25)\times\int e^v\,dv = (1/25)\times e^v = (1/25)\times e^{2x+3y}$$

$\therefore$ Solution is $z = \text{C.F.} + \text{P.I.} = \phi_1(y-x) + x\phi_2(y-x) + (1/25) \times e^{2x+3y}$.

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+2y}$

(*c*) Here auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$ giving $m = 1, 2, 3$.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 6D^2D' + 11DD'^2 - 6D'^3}e^{5x+6y} = \frac{1}{(D-D')(D-2D')(D-3D')}e^{5x+6y}$$

$$= \frac{1}{(5-6)(5-12)(5-18)}\iiint e^v\,dv\,dv\,dv, \text{ where } v = 5x+6y$$

$$= \frac{1}{-91}\iint e^v\,dv\,dv = -\frac{1}{91}\int e^v\,dv = -\frac{1}{91}e^v = -\frac{1}{91}e^{5x+6y}$$

Hence the required solution is $z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+3x) - (1/91) \times e^{5x+6y}$.

Ex. 4. *Solve* (*a*) $r - 2s + t = \sin(2x+3y)$. **[Meerut 2007; Indore 2002; Vikram 1996;]**

(*b*) $(D^3 - 4D^2D' + 4DD'^2)z = 2\sin(3x+2y)$. **[Kanpur 2008; I.A.S. 2006]**

(*c*) $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x+3y)$.

(*d*) $(D^3 - 3DD'^2 - 2D'^3)z = \cos(x+2y)$. **[Delhi Maths Hons. 1992]**

Sol. (*a*) Since $r = \partial^2 z/\partial x^2,\ s = \partial^2 z/\partial x \partial y,\ t = \partial^2 z/\partial t^2,$ the given equation becomes

$\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x \partial y) + \partial^2 z/\partial y^2 = \sin(2x + 3y)$ or $(D^2 - 2DD' + D'^2)z = \sin(2x + 2y)$.

Its auxiliary equation is $m^2 - 2m + 1 = 0$ so that $m = 1, 1$.

$\therefore$ C.F. $= \phi_1(y + x) + x\,\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D - D')^2}\sin(2x + 3y) = \frac{1}{(2 - 3)^2}\int\int \sin v\, dv\, dv, \text{ where } v = 2x + 3y$$

$$= -\int \cos v\, dv = -\sin v = -\sin(2x + 3y).$$

Hence the required general solution is $z = \phi_1(y + x) + x\phi_2(y + x) - \sin(2x + 3y)$.

(*b*) The auxiliary equation of the given equation is $m^3 - 4m^2 + 4m = 0$.

or $m(m^2 - 4m + 4) = 0$ or $m(m - 2)^2 = 0$ so that $m = 0, 2, 2$.

$\therefore$ C.F. $= \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{D^3 - 4D^2D' + 4DD'^2}2\sin(3x + 2y)$$

$$= 2 \times \frac{1}{3^3 - 4 \times 3^2 \times 2 + 4 \times 3 \times 2^2}\int\int\int \sin v\, dv\, dv, \text{ where } v = 3x + 2y$$

$$= (2/3) \times \int\int (-\cos v)dv\, dv = -(2/3) \times \int \sin v\, dv = (2/3) \times \cos v = (2/3) \times \cos(3x + 2y)$$

The required general solution is $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + (2/3) \times \cos(3x + 2y)$.

(*c*) Proceed as in part (*b*). **Ans.** $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - (1/32) \times \sin(2x + 3y)$

(*d*) Proceed as in part (*b*). **Ans.** $z = \phi_1(y - x) + x\phi_2(y - x) + \phi_3(y + 2x) + (1/27) \times \sin(x + 2y)$

Ex. 5. *Solve* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \cos mx \cos ny$. **[Kanpur 2007; Nagpur 2010]**

Sol. Given equation can be written as $(D^2 + D'^2)z = \cos mx \cos ny$.

Its auxiliary equation is $m^2 + 1 = 0$ so that $m = \pm i$.

$\therefore$ C.F. $= \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + D'^2}\cos mx \cos ny = \frac{1}{D^2 + D'^2}\frac{\cos(mx + ny) + \cos(mx - ny)}{2}$$

$$= \frac{1}{2}\frac{1}{D^2 + D'^2}\cos(mx + ny) + \frac{1}{2}\frac{1}{D^2 + D'^2}\cos(mx - ny)$$

$$= \frac{1}{2}\frac{1}{m^2 + n^2}\int\int \cos v\, dv\, dv + \frac{1}{2}\frac{1}{m^2 + (-n)^2}\int\int \cos u\, du\, du,$$

where $v = mx + ny$ and $u = mx - ny$

$$= \frac{1}{2}\frac{1}{m^2 + n^2}\int \sin v\, dv + \frac{1}{2}\frac{1}{m^2 + n^2}\int \sin u\, du = \frac{1}{2}\frac{1}{m^2 + n^2}[-\cos v - \cos u]$$

$$= -\frac{1}{2(m^2 + n^2)}[\cos(mx + ny) + \cos(mx - ny)], \text{ as } v = mx + ny,\ u = mx - ny$$

$$= -\frac{1}{2(m^2 + n^2)} \times 2\cos mx \cos ny = -(m^2 + n^2)^{-1}\cos mx \cos ny.$$

Hence the required general solution is $z = \phi_1(y + ix) + \phi_2(y - ix) - (m^2 + n^2)^{-1}\cos mx \cos ny$.

Ex. 5. (*b*) *Solve* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \cos mx \sin ny$. **[Ravishankar 1999, 2001]**

Sol. Do like Ex. 5(*a*) **Ans.** $z = \phi_1(y + ix) + \phi_2(y - ix) + (\sin mx + \sin ny)/(m^2 + n^2)$

Ex. 6. *Solve the following partial differential equations :*

(*a*) $(D^2 - 2DD' + D'^2)z = \tan(y + x)$ or $(D - D')^2 z = \tan(y + x)$ **[Jiwaji 1996]**

(*b*) $(D^2 - 2aDD' + a^2D'^2)z = f(y + ax)$ or $(D - aD')^2 z = f(y + ax)$.

(*c*) $4r - 4s + t = 16 \log(x + 2y)$. **[Agra 2009; Meerut 2009; Ravishankar 2000]**

Sol. (*a*) Here auxiliary equation is $(m - 1)^2 = 0$ so that $m = 1, 1$.

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{(D - D')^2}\tan(y + x) = \frac{x^2}{1^2 \times 2!}\tan(y + x) = \frac{x^2}{2}\tan(y + x)$$

[Using formula (*ii*) of working rule with $a = 1$, $b = 1$, $m = 2$]

Hence the required general solution is $z = \phi_1(y + x) + x\phi_2(y + x) + (x^2/2) \times \tan(y + x)$.

(*b*) Here auxiliary equation is $(m - a)^2 = 0$ so that $m = a, a$.

$\therefore$ C.F. $= \phi_1(y + ax) + x\phi_2(y + ax)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D - aD')^2} f(y + ax) + \frac{x^2}{1^2 \times 2!} f(y + ax) = \frac{x^2}{2} f(y + ax).$$

[using formula (*ii*) of working rule with $a = a$, $b = 1$, $m = 2$]

$\therefore$ General solution is $z = \phi_1(y + x) + x\phi_2(y + x) + (x^2/2) \times f(y + ax)$.

(*c*) Since $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$, the given equation becomes

$4(\partial^2 z/\partial x^2) - 4(\partial^2 z/\partial x \partial y) + (\partial^2 z/\partial y^2) = 16 \log(x + 2y)$ or $(4D^2 - 4DD' + D'^2)z = 16 \log(x + 2y)$

Its auxiliary equation is $4m^2 - 4m + 1 = 0$ so that $m = 1/2.\ 1/2$.

$\therefore$ C.F. $= \phi_1(2y + x) + x\phi_2(2y + x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log(x + 2y) = 16 \times \frac{x^2}{2^2 \times 2!}\log(x + 2y) = 2x^2 \log(x + 2y)$$

[using formula (*ii*) of working rule with $a = 1$, $b = 2$, $m = 2$]

$\therefore$ The required solution is $z = \phi_1(2y + x) + x\phi_2(2y + x) + 2x^2 \log(x + 2y)$.

Ex. 7. *Solve the following partial differential equations :*

(*a*) $(2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x + y)$. **[M.D.U. Rohtak 2005]**

(*b*) $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$. **[Meerut 2006, 08]**

(*c*) $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$. **[Bhopal 2000, 03, Meerut 2007; Jabalpur 2004]**

(*d*) $r - t = x - y$.

(*e*) $2r - s - 3t = 5e^x/e^y$. **[Indore 2003; Jiwaji 2003, Vikram 1998]**

(*f*) $r + 5s + 6t = (y - 2x)^{-1}$ or $(\partial^2 z/\partial x^2) + 5(\partial^2 z/\partial x \partial y) + 6(\partial^2 z/\partial y^2) = 1/(y - 2x)$

[Agra 2009; Indore 2000; I.A.S. 1991; Garhwal 2005]

Sol. (*a*) Here auxiliary equation is $2m^2 - 5m + 2 = 0$ so that $m = 2, 1/2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(2y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2} 5 \sin(2x + y) = 5\frac{1}{(D - 2D')}\left[\frac{1}{(2D - D')}\sin(2x + y)\right]$$

$$= 5\frac{1}{D - 2D'}\frac{1}{(2 \times 2) - 1}\int \sin v\, dv,$$ where $v = 2x + y$, using formula (*i*) of working rule

$$= \frac{5}{3}\frac{1}{D-2D'}(-\cos v) = -\frac{5}{3}\frac{1}{D-2D'}\cos(2x+y) = -\frac{5}{3}\times\frac{x}{1^1\times 1!}\cos(2x+y)$$

[Using formula (*ii*) with $a = 2$, $b = 1$, $m = 2$]

$\therefore$ The required general solution is $z = \phi_1(y + 2x) + \phi_2(2y + x) - (5x/3) \times \cos(2x + y)$.

(*b*) Do as in part (*a*). **Ans.** $z = \phi_1(y + x) + \phi_2(y + 4x) - (x/3) \times \cos(4x + y)$

(*c*) Here auxiliary equation is $m^3 - 2m^2 - m + 2 = 0$ or $m^2(m - 2) - (m - 2) = 0$ or $(m^2 - 1)(m - 2) = 0$ so that $m = 2, 1, -1$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(y + x) + \phi_3(y - x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3}e^{x+y} = \frac{1}{(D-D')}\left\{\frac{1}{(D^2 - DD' - 2D'^2)}e^{x+y}\right\}$$

$$= \frac{1}{D-D'}\frac{1}{1^2 - (1\times 1) - (2\times 1)^2}\int e^v\, dv$$, where $v = x + y$, using formula (*i*) of working rule

$$= \frac{1}{D-D'}\left(-\frac{1}{2}\right)e^v = -\frac{1}{2}\frac{1}{D-D'}e^{x+y} = -\frac{1}{2}\times\frac{x}{1^1\times 1!}e^{x+y} = -\frac{x}{2}e^{x+y}$$

[Using formula (*ii*) with $a = 1$, $b = 1$, $m = 1$]

$\therefore$ Required solution is $z = \phi_1(y + 2x_1) + \phi_2(y + x) + \phi_3(y - x) - (x/2) \times e^{x+y}$.

(*d*) The given equation can be re–written as

$(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = x - y$ or $(D^2 - D'^2)z = x - y$. ...(1)

Its auxiliary equation is $m^2 - 1 = 0$ so that $m = 1, -1$.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. $= \frac{1}{D^2 - D'^2}(x-y) = \frac{1}{D+D'}\cdot\frac{1}{D-D'}(x-y) = \frac{1}{D+D'}\cdot\frac{1}{1-(-1)}\int v\, dv$, where $v = x - y$.

$$= \frac{1}{2}\frac{1}{D+D'}\frac{v^2}{2} = \frac{1}{4}\frac{1}{D+D'}(x-y)^2 = -\frac{1}{4}\frac{1}{[(-1)\times D - 1\times D']^1}(x-y)^2$$

$$= -\frac{1}{4}\frac{x}{(-1)^1\times 1!}(x-y)^2 = \frac{x}{4}(x-y)^2$$, by formula (*ii*) with $a = 1$, $b = -1$, $m = 1$.

$\therefore$ The required solution is $z = \phi_1(y + x) + \phi_2(y - x) + (x/4) \times (x - y)^2$.

(*e*) The given equation can be re–written as

$2(\partial^2 z/\partial x^2) - (\partial^2 z/\partial x\partial y) - 3(\partial^2 z/\partial y^2) = 5e^{x-y}$ or $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$ or $(D + D')(2D - 3D')z = 5e^{x-y}$.

C.F. $= \phi_1(y - x) + \phi_2(2y + 3x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. $= \frac{1}{(D+D')}\times\frac{1}{(2D-3D')}5e^{x-y} = 5\frac{1}{D+D'}\times\frac{1}{(2\times 1) - 3\times(-1)}\int e^v\, dv$, where $v = x - y$,

$$= \frac{1}{D+D'}e^v = -\frac{1}{[(-1)\times D - 1\times D']}e^{x-y} = -\frac{x}{(-1)^1\times 1!}e^{x-y} = xe^{x-y},$$

[using formula (*ii*) with $a = 1$, $b = -1$, $m = 1$]

$\therefore$ The required solution is $z = \phi_1(y - x) + \phi_2(2y + 3x) + xe^{x-y}$.

(*f*) Given equation can be rewritten as $(D^2 + 5DD' + 6D'^2)z = (y - 2x)^{-1}$. ...(1)

Its auxiliary equation is $m^2 + 5m + 6 = 0$ so that $m = -2, -3$.

$\therefore$ C.F. $= \phi_1(y - 2x) + \phi_2(y - 3x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. $= \dfrac{1}{D^2 + 5DD' + 6D'^2}(y - 2x)^{-1} = \dfrac{1}{D + 2D'}\left[\dfrac{1}{D + 3D'}(y - 2x)^{-1}\right]$

$= \dfrac{1}{D + 2D'} \times \dfrac{1}{-2 + (3 \times 1)} \int v^{-1}\, dv$, where $v = y - 2x$, by formula (i)

$= \dfrac{1}{D + 2D'} \log v = \dfrac{1}{D + 2D'} \log (y - 2x) = \dfrac{1}{[1 \times D - (-2) \times D']^1} \log (y - 2x)$

$= \dfrac{x}{1^1 \times 1!} \log (y - 2x)$, by formula (ii) with $a = -2$, $b = 1$, $m = 1$

$\therefore$ The general solution is $z = \phi_1(y - 2x) + \phi_2(y - 3x) + x \log (y - 2x)$.

Ex. 8. *Solve the following partial differential equation :*

(a) $(D^3 - 4D^2D' + 4DD'^2)z = 4 \sin (2x + y)$.

[Bilaspur 1995; Indore 2004;Rewa 2000, 01; MDU Rohtak 2004]

(b) $(D^3 - 4D^2D' + 4DD'^2)z = \cos (2x + y)$.

(c) $(D^3 - 4D^2D' + 4DD'^2)z = \sin (y + 2x)$.

Sol. (a) Here the auxiliary equation is $m^3 - 4m^2 + 4m = 0$ or $m(m^2 - 4m + 4) = 0$ or $m(m - 2)^2 = 0$ so that $m = 0, 2, 2$.

$\therefore$ C.F. $= \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, P.I. $= \dfrac{1}{D^3 - 4D^2D' + 4DD'^2} 4 \sin (2x + y) = 4 \dfrac{1}{(D - 2D')^2} \left\{\dfrac{1}{D} \sin (2x + y)\right\}$

$= 4 \dfrac{1}{(D - 2D')^2} \left\{-\dfrac{1}{2} \cos (2x + y)\right\}$, since $1/D$ stands for integration w.r.t. x treating y as constant.

$= -2 \dfrac{1}{(D - 2D')^2} \cos (2x + y) = -2 \dfrac{x^2}{1^2 \times 2!} \cos (2x + y)$,

[Using formula (ii) with $a = 2$, $b = 1$, $m = 2$]

So the required solution is $y = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - x^2 \cos (2x + y)$.

(b) Try yourself **Ans.** $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + (x^2/4) \times \sin (2x + y)$

(c) Try yourself **Ans.** $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - (x^2/4) \times \cos (2x + y)$

Ex. 9. *Solve the following partial differential equations :*

(a) $(D^2 - 3DD' + 2D'^2)z = e^{2x - y} + e^{x + y} + \cos (x + 2y)$.

(b) $(D^2 - 3DD' + 2D'^2)z = e^{2x - y} + \cos (x + 2y)$ **[Delhi Maths (H) 2006]**

(c) $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y + 2x} + (y + x)^{1/2}$.

(d) $(D_x^3 - 7D_x D_y^2 - 6D_y^3)z = \sin (x + 2y) + e^{3x + y}$. **[I.A.S. 1995]**

Sol. (a) Here auxiliary equation is $m^2 - 3m + 2 = 0$ so that $m = 1, 2$.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, P.I. corresponds to $e^{2x - y}$

$= \dfrac{1}{D^2 - 3DD' + 2D'^2} e^{2x - y} = \dfrac{1}{2^2 - 3 \times 2 \times (-1) + 2 \times (-1)^2} \int\int e^v\, dv\, dv$, where $v = 2x - y$

$= (1/12) \times \int e^v dv = (1/12) \times e^v = (1/12) \times e^{2x - y}$. ...(2)

P.I. corresponding to $e^{x+y} = \frac{1}{D^2 - 3DD' + 2D'^2} e^{x+y} = \frac{1}{D-D'}\left\{\frac{1}{D-2D'} e^{x+y}\right\}$

$= \frac{1}{D-D'}\left\{\frac{1}{1-(2\times 1)}\int e^v\, dv\right\}$, where $v = x + y$, using formula (i)

$= -\frac{1}{D-D'} e^v = -\frac{1}{(D-D')^1} e^{x+y} = -\frac{x}{1^1 \times 1!} e^{x+y} = -xe^{x+y}$. ...(3)

[Using formula (ii) with $a = b = 1$, $m = 1$]

Finally, P.I. corresponding to $\cos(x + 2y)$

$= \frac{1}{D^2 - 3DD' + 2D'^2} \cos(x+2y) = \frac{1}{1^2 - (3\times 1\times 2) + (2\times 2^2)} \iint \cos v\, dv\, dv$, where $v = x + 2y$

$= (1/3) \times \int \sin v dv = -(1/3) \times \cos v = -(1/3) \times \cos(x+2y)$ ...(4)

From (1), (2), (3) and (4), the required solution is z = C.F. + P.I.

or $z = \phi_1(y + x) + \phi_2(y + 2x) + (1/12) \times e^{2x-y} - xe^{x+y} - (1/3) \times \cos(x + 2y)$.

(b) This problem is same as part (a) except that the term e^{x+y} is missing on R.H.S. So, now you need not compute P.I. corresponding to e^{x+y}. Therefore the solution will take the form

$$y = \phi_1(y+x) + \phi_2(y+2x) + (1/12)\times e^{2x-y} - (1/3)\times\cos(x+2y)$$

(c) Here auxiliary equation is $m^3 - 4m^2 + 5m - 2 = 0$ giving $m = 1, 1, 2$.

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x)$., ϕ_1, ϕ_2, ϕ_3 being arbitrary function ...(1)

Now, P.I. corresponding to e^{y+2x}

$= \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} e^{y+2x} = \frac{1}{D-2D'}\left\{\frac{1}{(D-D')^2} e^{y+2x}\right\}$

$= \frac{1}{D-2D'} \frac{1}{(2-1)^2} \iint e^v\, dv\, dv$, where $v = y + x$, by formula (i)

$= \frac{1}{D-2D'} \int e^v\, dv = \frac{1}{D-2D'} e^v = \frac{1}{(1\times D - 2\times D')^1} e^{y+2x} = \frac{x}{1\times 1!} e^{y+x} = xe^{y+x}$, ...(2)

[using formula (ii) with $a = 2$, $b = 1$, $m = 1$]

Finally, P.I. corresponding to $(y + x)^{1/2}$

$= \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3}(y+x)^{1/2} = \frac{1}{(D-D')^2}\cdot\left\{\frac{1}{D-2D'}(y+x)^{1/2}\right\}$

$= \frac{1}{(D-D')^2} \times \frac{1}{1-(2\times 1)} \int v^{1/2}\, dv$, where $v = y + x$, using formula (i)

$= -\frac{1}{D-D'} \times \frac{2}{3} v^{3/2} = -\frac{2}{3} \frac{1}{(D-D')^2}(y+x)^{3/2} = -\frac{2}{3} \times \frac{x^2}{1^2 \times 2!}(y+x)^{3/2}$

$= -(x^2/3) \times (y + x)^{3/2}$, using formula (ii) with $a = b = 1$, $m = 2$ ...(3)

From (1), (2) and (3), the required general solution is

$$z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x) + xe^{y+x} - (x^2/3) \times (y + x)^{3/2}.$$

(*d*) Here note that D_x and D_y stand for D and D' respectively.

$\therefore$ Auxiliary equation is $m^3 - 7m - 6 = 0$ so that $m = -1, -2, 3$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, P.I. corresponding to $\sin(x + 2y)$

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}\sin(x+2y) = \frac{1}{1^3 - (7\times 1\times 2^2) - (6\times 2^3)}\iiint \sin v\, dv\, dv\, dv \text{, where } v = x + 2y$$

$$= -(1/75)\times\iint(-\cos v)\, dv\, dv = -(1/75)\times\int(-\sin v)\, dv = -(1/75)\times\cos v = -(1/75)\times\cos(x+2y)$$

and P.I. corresponding to e^{3x+y}

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}e^{3x+y} = \frac{1}{D_x - 3D_y}\left[\frac{1}{(D_x + D_y)(D_x + 2D_y)}e^{3x+y}\right]$$

$$= \frac{1}{D_x - 3D_y}\cdot\frac{1}{(3+1)(3+2)}\iint e^v\, dv\, dv \text{, where } v = 3x + y, \text{ by formula } (i)$$

$$= \frac{1}{20}\frac{1}{D_x - 3D_y}\int e^v\, dv = \frac{1}{20}\frac{1}{D_x - 3D_y}e^v = \frac{1}{20}\frac{1}{(D_x - 3D_y)^1}e^{3x+y}$$

$$= \frac{1}{20}\times\frac{x}{1^1\times 1!}e^{3x+y} = \frac{x}{20}e^{3x+y}\text{, using formula } (ii) \text{ with } a = 2,\ b = 1,\ m = 1.$$

Hence the required general solution is z = C.F. + P.I.

or $z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) - (1/75) \times \cos(x + 2y) + (1/20) \times x\, e^{3x+y}$.

Ex. 10. *Solve (i)* $(D^2 - 6DD' + 9D'^2)z = \tan(y + 3x)$ **[Delhi 2007; Ravishankar 2004]**

(*ii*) $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$

Sol (*i*) Here auxiliary equation is $(m - 3)^2 = 0$ so that $m = 3, 3$.

$\therefore$ C.F. $= \phi_1(y + 3x) + x\,\phi_2(y + 3x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D - 3D')^2}\tan(y+3x) = \frac{x^2}{1^2\times 2!}\tan(y+3x) = \frac{x^2}{2}\tan(y+3x)$$

$\therefore$ The required solution is $z = \phi_1(y + 3x) + x\,\phi_2(y + 3x) + (x^2/2)\times\tan(y + 3x)$.

(*ii*) Re-writing the given equation reduces to $(D - 3D')^2\, z = 2(3x + y)$

$\therefore$ C.F. $= \phi_1(y + 3x) + x\phi_1(y + 3x)$, ϕ_1, ϕ_2 being arbitrars constants.

Now, $$P.I. = \frac{1}{(D - 3D')^2}\, 2(3x + y) = 2\,\frac{x^2}{1^2\times 2!}(3x + y) = x^2(3x + y)$$

$\therefore$ The required solution is $z = \phi_1(y + 3x) + x\phi_2(y + 3x) + 3x^3 + x^2y$.

Ex. 11. *Solve (i)* $(D - 3D')^2\,(D + 3D')z = e^{3x+y}$ **[Delhi Maths (Hons) 2000, Agra 2005]**

(*ii*) $(D - 2D')\,(D + D')^2 z = \cos(2x + y)$

Sol. (*i*) C.F. $= \phi_1(y + 3x) + x\,\phi_2(y + 3x) + \phi_3(y - 3x)$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D - 3D')^2}\,\frac{1}{D + 3D'}\,e^{3x+y} = \frac{1}{(D - 3D')^2}\,\frac{1}{3 + (3\times 1)}\int e^y dv \text{, where } v = 3x + y$$

$$= \frac{1}{6}\frac{1}{(D-3D')^2}e^{v} = \frac{1}{6}\frac{1}{(D-3D')^2}e^{3x+y} = \frac{1}{6}\frac{x}{1\times 2!}e^{3x+y}$$

The required solution is $z = \phi_1(y+3x) + x\,\phi_2(y+3x) + \phi_3(y-3x) + (x^2/12)\times e^{3x+y}$

(*ii*) Try your self **Ans.** $z = \phi_1(y+2x) + \phi_2(y-x) + x\,\phi_3(y-x) - (x/9)\times\cos(2x+y)$

Ex. 12. Solve (*i*) $r + s - 2t = e^{x+y}$

(*ii*) $(D^3 - 7DD'^2 - 6D'^3)y = \sin(x+2y)$

(*iii*) $(D^3 - 3DD'^2 + 2D'^3)y = (x-2y)^{1/2}$ **[Delhi Maths (H) 2009]**

Sol. (*i*) Re-writing given equation becomes $(\partial^2 z/\partial x^2) + (\partial^2 z/\partial x\,\partial y) - 2(\partial^2 z/\partial y^2) = e^{x+y}$

or $(D^2 + DD' - 2D'^2)\,z = e^{x+y}$ or $(D-D')(D+2D')z = e^{x+y}$.

Its C.F. = $\phi(y+x) + \phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I. = $\frac{1}{D-D'}\frac{1}{D+2D'}e^{x+y} = \frac{1}{D-D'}\frac{1}{1+(2\times 1)}\int e^{v}dv$, where $v = x+y$

$= \frac{1}{3}\frac{1}{D-D'}e^{v} = \frac{1}{3}\frac{1}{D-D'}e^{x+y} = \frac{1}{3}\frac{x}{1!}e^{x+y}$

$\therefore$ The required solution is $z = \phi_1(y+x) + \phi_2(y-2x) + (x/3)\times e^{x+y}$.

(*ii*) Here the auxiliary equation is $m^3 - 7m - 6 = 0$ giving $m = -1, -2, 3$.

$\therefore$ C.F. = $\phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

P.I. = $\frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y) = \frac{1}{1^3 - (7\times 1\times 2^2) - (16\times 2^3)}\int\int\int \sin v\,(dv)^3$, where $v = x + 2y$

$= -\frac{1}{75}\iint(-\cos v)dv\,dv = -\frac{1}{75}\int(-\sin v)\,dv = -\frac{1}{75}\cos v = -\frac{1}{75}\cos(x+2y)$

$\therefore$ Required solution $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - (1/75)\times\cos(x+2y)$

(*iii*) The auxiliary equation is $m^3 - 3m + 2 = 0$ giving $m = 1, 1, 2$.

$\therefore$ C.F. = $\phi_1(y+x) + x\,\phi_2(y+x) + \phi_3(y+2x)$, ϕ_1, ϕ_2, ϕ_3 are arbitrary functions

P.I. = $\frac{1}{D^3 - 3DD'^2 + 2D'^3}(x-2y)^{1/2} = \frac{1}{1^3 - 3\times 1\times(-2)^2 + 2\times(-2)^3}\int\int\int v^{1/2}(dv)^3$, where $v = x - 2y$

$= -\frac{1}{27}\iint\frac{v^{3/2}}{(3/2)}\,dv\,dv = -\frac{1}{27}\int\frac{v^{5/2}}{(3/2)\times(5/2)}dv = -\frac{1}{27}\frac{v^{7/2}}{(3/2)\times(5/2)\times(7/2)}$

$= -(8/2835)\times v^{7/2} = -(8/2835)\times(x-2y)^{7/2}$

General solution is $z = \phi_1(y+x) + x\,\phi_2(y+x) + \phi_3(y+2x) - (8/2835)\times(x-2y)^{7/2}$.

Ex. 13. *Solve* $(D^2 - 3DD' + 2D'^2)z = \cos(x+2y)$

Sol. The auxiliary equation $m^2 - 3m + 2 = 0$ gives $m = 1, 2$.

$\therefore$ C.F. = $\phi_1(y+x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D^2 - 3DD' + 2D'^2}\cos(x+2y) = \frac{1}{1^2 - 3\cdot 1\cdot 2 + 2\cdot 2^2}\iint \cos v (dv)^2, \text{ where } v = x + 2y$$

$$= (1/3)\times \int \sin v\, dv = -(1/3)\times \cos v = -(1/3)\times \cos(x+2y)$$

$\therefore$ Solution is $z = \phi_1(y+x) + \phi_2(y+2x) - (1/3)\times\cos(x+2y)$

Ex. 14. *Solve* $(D^2 - DD' - 2D'^2)z = 2x + 3y + e^{3x+4y}$. **[I.A.S. 2000]**

Sol. The auxiliary equation $m^2 - m - 2 = 0$ giving $m = 2, -1$.

$\therefore$ C.F. = $\phi_1(y+2x) + \phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I. corresponding to $(2x + 3y)$

$$= \frac{1}{D^2 - DD' - 2D'^2}(2x+3y) = \frac{1}{2^2 - (2\times 3) - (2\times 3^2)}\iint v\,(dv)^2, \text{ where } v = 2x+3y$$

$$= -\frac{1}{20}\int \frac{v^2}{2}\, dv = -\frac{1}{20}\left(\frac{v^3}{2\times 3}\right) = -\frac{1}{60}(2x+3y)^3$$

P.I. corresponding to e^{3x+4y}

$$= \frac{1}{D^2 - DD' - 2D'^2}e^{3x+4y} = \frac{1}{3^2 - (3\times 4) - (2\times 4^2)}\iint e^v (dv)^2, \text{ where } v = 3x+4y$$

$$= -(1/35)\times e^v = -(1/35)\times e^{3x+4y}.$$

$\therefore$ General solution is $z = \phi_1(y+2x) + \phi_2(y-x) - (1/60)\times(2x+3y)^3 - (1/35)\times e^{3x+4y}$.

EXERCISE 4(A)

Solve the following partial differential equations:

1. $(D^2 - DD' - 6D'^2)z = \cos(2x+y)$ **[Agra 2009, 10]**

Ans. $z = \phi_1(y+3x) + \phi_2(y-2x) - (1/4)\times\cos(2x+y)$ ϕ_1, ϕ_2, being arbitrary functions.

2. $r - 4s + 4t = e^{2x+y}$ **[Agra 2010]**

Ans. $z = \phi_1(y+2x) + x\phi_2(y+2x) + (x^2/2)\times e^{2x+y}$ ϕ_1, ϕ_2 being arbitrary functions

3. $(D^3 - 4D^2D'^2 + 4DD'^2)z = 6\sin(3x+2y)$

Ans. $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + 2\cos(3x+2y)$, ϕ_1, ϕ_2, ϕ_3, being arbitrary functions.

4. $(D-3D')^2(D+3D')z = e^{3x+y}$ **[Agra 2005]**

Ans. $z = \phi_1(y+3x) + x\phi_2(y+3x) + \phi_3(y-3x) + (x/12)\times e^{3x+y}$, ϕ_1, ϕ_2, ϕ_3, being arbitrary functions.

4.10. Short Method II. When f(x, y) is of the form $x^m y^n$ or a rational integral algebraic function of *x* and *y*.

Then the particular integral (P.I.) is evaluated by expanding the symbolic function $1/f(D, D')$ in an infinite series of ascending powers of D or D'. In solved examples 1 and 2 of Art. 4.11, we have shown that P.I. obtained on expanding $1/f(D, D')$ in ascending powers of D is different from that obtained on expanding $1/f(D, D')$ in ascending powers of D'. Since to get the required general solution of given differential equation any P.I. is required, any of the two methods can be used. The difference in the two answers of P.I. is not material as it can be incorporated in the arbitrary functions occuring in C.F. of that given differential equation.

Remark : If $n < m$, $1/f(D, D')$ should be expanded in powers of D'/D whereas if $m < n$, $1/f(D, D')$ should be expanded in powers of D/D'.

4.11 SOLVED EXAMPLES BASED ON SHORT METHOD II

Ex. 1. *Solve* $(D^2 - a^2D'^2)z = x$ or $(\partial^2 z/\partial x^2) - a^2(\partial^2 z/\partial y^2) = x$.

Sol. Here auxiliary equation is $m^2 - a^2 = 0$ so that $m = a, -a$.

$\therefore$ C.F. $= \phi_1(y + ax) + \phi_2(y - ax)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, P.I. $= \dfrac{1}{D^2 - a^2D'^2}x = \dfrac{1}{D^2[1 - a^2(D'^2/D^2)]}x = \dfrac{1}{D^2}\left(1 - a^2\dfrac{D'^2}{D^2}\right)^{-1}x$

$= \dfrac{1}{D^2}\left(1 + a^2\dfrac{D'^2}{D^2} + \ldots\right)x = \dfrac{1}{D^2}x = \dfrac{x^3}{6}$. ...(2)

Alternatively, we can compute P.I. on follows :

P.I. $= \dfrac{1}{D^2 - a^2D'^2}x = \dfrac{1}{-a^2D'^2[1 - (D^2/a^2D'^2)]}x = -\dfrac{1}{a^2D'^2}\left(1 - \dfrac{D^2}{a^2D'^2}\right)^{-1}x$

$= -\dfrac{1}{a^2D'^2}\left(1 + \dfrac{D^2}{a^2D'^2} + \ldots\right)x = -\dfrac{1}{a^2D'^2}x = -\dfrac{1}{a^2} \times \dfrac{xy^2}{2}$. ...(3)

Hence the required general solution is $z =$ C.F. + P.I. that is,

$z = \phi_1(y + ax) + \phi_2(y - ax) + x^3/6$, using (1) and (2).

or $z = \phi_1(y + ax) + \phi_2(y - ax) - (xy^2)/(2a^2)$, using (1) and (3).

Ex. 2. *Solve* $(D^2 + 3DD' + 2D'^2)z = x + y$, *by expanding the particular integral in ascending powers of D as well as in ascending powers of D'.*

[Bhopal 2000, 03; Indore 1999; Jiwaji 1995; Rewa, 2002, 03; I.A.S. 1994]

Sol. Here auxiliary equation is $m^2 + 3m + 2 = 0$ so that $m = -2, -1$.

$\therefore$ C.F. $= \phi_1(y - 2x) + \phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, by expanding in ascending powers of D, we have

P.I. $= \dfrac{1}{D^2 + 3DD' + 2D'^2}(x + y) = \dfrac{1}{2D'^2\left[1 + \left(\dfrac{D^2}{2D'^2} + \dfrac{3}{2}\dfrac{D}{D'}\right)\right]}(x + y)$

$= \dfrac{1}{2D'^2}\left[1 + \left(\dfrac{D^2}{2D'^2} + \dfrac{3}{2}\dfrac{D}{D'}\right)\right]^{-1}(x + y) = \dfrac{1}{2D'^2}\left(1 - \dfrac{3}{2}\dfrac{D}{D'} + \ldots\right)(x + y)$

$= \dfrac{1}{2D'^2}\left(x + y - \dfrac{3}{2}y\right) = \dfrac{1}{2D'^2}\left(x - \dfrac{y}{2}\right) = \dfrac{xy^2}{4} - \dfrac{y^3}{24}$. ...(2)

Again, by expanding in ascending powers of D, P.I. of given equation is given by

P.I. $= \dfrac{1}{D^2 + 3DD' + 2D'^2}(x + y) = \dfrac{1}{D^2\left[1 + \left(\dfrac{3D'}{D} + \dfrac{2D'^2}{D^2}\right)\right]}(x + y) = \dfrac{1}{D^2}\left[1 + \left(\dfrac{3D'}{D} + \dfrac{2D'^2}{D^2}\right)\right]^{-1}(x + y)$

$= \dfrac{1}{D^2}\left(1 - \dfrac{3D'}{D} + \ldots\right)(x + y) = \dfrac{1}{D^2}(x + y - 3x) = \dfrac{yx^2}{2} - \dfrac{x^3}{3}$.

Hence the required general solution is given by $z =$ C.F. + P.I., *i.e.*,

$z = \phi_1(y - 2x) + \phi_2(y - x) + (1/4) \times xy^2 - (1/24) \times y^3$, using (1) and (2)

or $z = \phi_1(y - 2x) + \phi_2(y - x) + (1/2) \times yx^2 - (1/3) \times x^3$, using (1) and (3).

Ex. 3. *Solve* $(\partial^3 z/\partial x^3) - (\partial^3 z/\partial y^3) = x^3y^3$ or $(D^3 - D'^3)z = x^3y^3$. **[I.A.S. 1997]**

Sol. Here auxiliary equation is $m^3 - 1 = 0$ so that $m = 1, \omega, \omega^2$, where ω and ω^2 are complex cube roots of unity.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y + \omega x) + \phi_3(y + \omega^2 x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{D^3 - D'^3} x^3y^3 = \frac{1}{D^3[1 - (D'^3/D^3)]} x^3y^3 = \frac{1}{D^3}\left(1 - \frac{D'^3}{D^3}\right)^{-1} x^3y^3$$

$$= \frac{1}{D^3}\left(1 + \frac{D'^3}{D^3} + \ldots\right)x^3y^3 = \frac{1}{D^3}\left(x^3y^3 + \frac{1}{D^3}6x^3\right) = \frac{1}{D^3}\left(x^3y^3 + 6 \times \frac{x^6}{4 \times 5 \times 6}\right)$$

$$= (1/120) \times x^6y^3 + (1/10080) \times x^9.$$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$

or $z = \phi_1(y + x) + \phi_2(y + \omega x) + \phi_3(y + \omega^2 x) + (1/120) \times x^6y^3 + (1/10080) \times x^9$.

Ex. 4. *Solve* $r + (a + b)s + abt = xy$. **[Indore 1998; Vikram 1998, 2000; Rewa 1998]**

Sol. Given equation can be written as $[D^2 + (a + b)DD' + abD'^2] = xy$.

Its auxiliary equation is $m^2 + (a + b)m + ab = 0$ or $(m + a)(m + b) = 0$ so that $m = -a, -b$.

$\therefore$ C.F. $= \phi_1(y - ax) + \phi_2(y - bx)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{D^2 + (a + b)DD' + abD'^2} xy = \frac{1}{D^2\left[1 + (a + b)\frac{D'}{D} + ab\frac{D'^2}{D^2}\right]} xy$$

$$= \frac{1}{D^2}\left[1 + (a + b)\frac{D'}{D} + ab\frac{D'^2}{D^2}\right]^{-1} xy = \frac{1}{D^2}\left[1 - (a + b)\frac{D'}{D} + \ldots\right]xy$$

$$= \frac{1}{D^2}\left\{xy - \frac{a + b}{D} D'(xy)\right\} = \frac{1}{D^2}\left\{xy - \frac{a + b}{D} x\right\} = \frac{1}{D^2}\left\{xy - \frac{(a + b)x^2}{2}\right\}$$

$$= y \times \frac{x^3}{2 \times 3} - \frac{a + b}{2} \times \frac{x^4}{3 \times 4} = \frac{x^3y}{6} - \frac{(a + b)x^4}{24}.$$

Required general solution $z = \phi_1(y - ax) + \phi_2(y - bx) + (1/6) \times x^3y - (a + b) \times (x^4/24)$,

Ex. 5. *Solve* (*a*) $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$.

(*b*) $(D^2 + 3DD' + 2D'^2)z = x + y$. **[Meerut 1996]**

(*c*) $(\partial^2 z/\partial x^2) + 3(\partial^2 z/\partial x\,\partial y) + 2(\partial^2 z/\partial y^2) = 2x + 3y$.

(*d*) $(\partial^2 z/\partial x^2) + 3(\partial^2 z/\partial x\,\partial y) + 2(\partial^2 z/\partial y^2) = 6(x + y)$.

(*e*) $(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = x - y$.

Sol. (*a*) Here auxiliary equation is $2m^2 - 5m + 2 = 0$ so that $m = 2, 1/2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(2y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2}\ 24(y - x) = 24\ \frac{1}{2D^2\left(1 - \frac{5D'}{2D} + \frac{D'^2}{D^2}\right)}(y - x)$$

$$= \frac{12}{D^2}\left(1-\frac{5D'}{2D}+\frac{D'^2}{D^2}\right)^{-1}(y-x) = \frac{12}{D^2}\left(1+\frac{5D'}{2D}+\ldots\right)(y-x) = \frac{12}{D^2}\left\{(y-x)+\frac{5}{2D}D'(y-x)\right\}$$

$$= \frac{12}{D^2}\left(y-x+\frac{5}{2D}1\right) = \frac{12}{D^2}\left(y-x+\frac{5x}{2}\right) = \frac{12}{D^2}\left(y+\frac{3x}{2}\right) = \left(12y\times\frac{x^2}{2}\right)+18\times\left(\frac{x^3}{2\times 3}\right)$$

Hence the required general solution is $z = \phi_1(y+2x)+\psi_2(2y+x)+6x^2y+3x^3.$

(*b*)Try as in part (*a*). **Ans.** $z = \phi_1(y-2x)+\phi_2(y-x)-(1/3)\times x^3+(1/2)\times x^2y$

(*c*) Try yourself **Ans.** $z = \phi_1(y-x)+\phi_2(y-2x)+(3/2)\times x^2y-(7/6)\times x^3$

(*d*) Try as in part (*c*). [**Ans.** $z = \phi_1(y-x)+\phi_2(y-2x)+3x^2y-2x^3$

(*e*) Try yourself. **Ans.** $z = \phi_1(y+x)+\phi_2(y-x)+(1/6)\times x^3-(1/2)\times x^2y$

Ex. 6. *Solve* $(D^2-6DD'+9D'^2)z = 12x^2+36xy.$ **[Meerut 1994, Bhuj 1999, Jabalpur 2003]**

Sol. Re-writing the given equation, we get $(D-3D')^2 z = 12(x^2+3xy).$

Its auxiliary equation is $(m-3)^2 = 0$ so that $m = 3, 3.$

$\therefore$ C.F. = $\phi_1(y+3x)+x\phi_2(y+3x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I.$= \frac{1}{(D-3D')^2}12(x^2+3xy) = 12\frac{1}{D^2(1-3D'/D)^2}(x^2+3xy)$

[Take D common as power of y is less than that of x]

$$= \frac{12}{D^2}\left(1-\frac{3D'}{D}\right)^{-2}(x^2+3xy) = \frac{12}{D^2}\left(1+6\frac{D'}{D}+\ldots\right)(x^2+3xy)$$

[Retain upto D' as maximum power of y in (x^2+3xy) is one]

$$= \frac{12}{D^2}\left\{x^2+3xy+\frac{6}{D}D'(x^2+3xy)\right\} = \frac{12}{D^2}\left\{x^2+3xy+\frac{6}{D}(3x)\right\} = \frac{12}{D^2}\left\{x^2+3xy+18\times\frac{x^2}{2}\right\}$$

$$= \frac{12}{D^2}(10x^2+3xy) = 120\left(\frac{x^4}{3\times 4}\right)+36y\left(\frac{x^3}{2\times 3}\right) = 10x^4+6x^3y.$$

Hence the required general solution is $z = \phi_1(y+3x)+x\phi_2(y+3x)+10x^4+6x^3y.$

Ex. 7. (*a*) Solve $(\partial^2V/\partial x^2)+(\partial^2V/\partial y^2) = -4\pi(x^2+y^2)$.

(*b*) Find a real function V of x and y, satisfying $(\partial^2V/\partial x^2)+(\partial^2V/\partial y^2) = -4\pi(x^2+y^2)$ and reducing to zero, when $y = 0$. **[Nagpur 2005; I.A.S. 1998]**

Sol. (*a*) Given equation can be rewritten as $(D^2+D'^2)V = -4\pi(x^2+y^2).$ ...(1)

Its auxiliary equation is $m^2+1 = 0$ so that $m = i, -i.$

$\therefore$ C.F. $=\phi_1(y+ix)+\phi_2(y-ix)$, where ϕ_1, ϕ_2 are arbitrary functions.

P.I. $= \frac{1}{D^2+D'^2}[-4\pi(x^2+y^2)] = -4\pi\frac{1}{D^2+D'^2}(x^2+y^2) = -4\pi\frac{1}{D^2(1+D'^2/D^2)}(x^2+y^2)$

$$= -4\pi \frac{1}{D^2}\left(1+\frac{D'^2}{D^2}\right)^{-1}(x^2+y^2) = -\frac{4\pi}{D^2}\left(1-\frac{D'^2}{D^2}+\ldots\right)(x^2+y^2)$$

$$= -\frac{4\pi}{D^2}\left\{(x^2+y^2)-\frac{1}{D^2}D'^2(x^2+y^2)\right\} = -\frac{4\pi}{D^2}\left\{(x^2+y^2)-\frac{1}{D^2}.2\right\}$$

$$= -\frac{4\pi}{D^2}\left(x^2+y^2-2\times\frac{x^2}{2}\right) = -\frac{4\pi}{D^2}y^2 = -4\pi y^2\times\frac{x^2}{2} = -2\pi^2x^2y^2.$$

Hence the required general solution is $V = \phi_1(y+ix)+\phi_2(y-ix)-2\pi^2x^2y^2.$ ...(2)

(*b*)Proceed as in part (*a*) upto equation (2).Since we want real function $V(x, y)$ satisfying (1) and reducing to zero when $y = 0$, it follows $\phi_1(y+ix) = \phi_2(y-ix) = 0$ in (2) and hence the required solution is $V = -2\pi^2x^2y^2.$

Ex. 8. *Solve* $(D^2-2DD'+D'^2)z = e^{x+2y}+x^3.$ **[Bhopal 1995, 97, 98; Lucknow 2010]**

Sol. Given equation is $(D-D')^2z = e^{x+2y}+x^3.$ ...(1)

Its auxiliary equation is $(m-1)^2 = 0$ so that $m = 1, 1.$

$\therefore$ C.F. $= \phi_1(y+x)+x\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D-D')^2}e^{x+2y} = \frac{1}{(1-2)^2}\int\int e^v\,dv\,dv\,,\text{ where } v = x+2y$$

$= \int e^v\,dv = e^v = e^{x+2y}$, as $v = x + 2y$, using formula (*i*) of working rule of Art. 4.8 and P.I. corresponding to x^3

$$= \frac{1}{(D-D')^2}x^3 = \frac{1}{D^2(1-D'/D)^2}x^3 = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^3 = \frac{1}{D^2}(1+\ldots)x^3 = \frac{x^5}{20}$$

Hence the required general solution is $z = \phi_1(y+x)+x\phi_2(y+x)+e^{x+2y}+x^5/20.$

Ex. 9. *Solve* $\partial^2z/\partial x^2 - a^2(\partial^2z/\partial y^2) = x^2.$ **[Ravishanker 2000, 04; Vikram 1995]**

Sol. Let $D \equiv \partial/\partial x$ and $D' = \partial/\partial y.$ Then, the given equation becomes $(D^2-a^2D'^2)y = x^2$

The auxiliary equation is $m^2-a^2 = 0$ so that $m = a, -a.$

$\therefore$ $C.F. = \phi_1(y+ax)+\phi_2(y-ax)$, ϕ_1, ϕ_2 being arbitrary functions

Now, $$P.I. = \frac{1}{(D^2-a^2D'^2)}x^2 = \frac{1}{D^2\{1-a^2(D'^2/D^2)\}}x^2 = \frac{1}{D^2}\left(1-a^2\frac{D'^2}{D^2}\right)^{-1}x^2$$

$$= \frac{1}{D^2}\left(1+a^2\frac{D'^2}{D^2}+\ldots\right)x^2 = \frac{1}{D^2}x^2 = \frac{1}{D}\frac{x^3}{3} = \frac{x^4}{12}$$

Hence, the required general solution is $z = C.F. + P.I.$ i.e., $z = \phi_1(y+ax)+\phi_2(y-ax)+x^4/12.$

Ex. 10. *Solve* $\partial^3z/\partial x^2\partial y - 2(\partial^3z/\partial x\,\partial y^2)+\partial^3z/\partial y^3 = 1/x^2$ **(Vikram 1994)**

Sol. Let $D = \partial/\partial x$ and $D' = \partial/\partial y.$ Then the given equation becomes

$(D^2D'-2DD'^2+D'^3)z = 1/x^2$ or $(D-D')^2D'z = 1/x^2$ ...(1)

Corresponding to repeated factor $(D-D')^2$, the part of C.F. is $\phi_1(y+x)+x\phi_2(y+x).$ Again corresponding to factor D', the part of *C.F.* is $f_1(x)$.

$\therefore$ $\quad C.F.$ of (1) $= \phi_1(y+x)+x,\ \phi_2(y+x)+\phi_3(x)$

$$\text{P.I.} = \frac{1}{(D-D')^2 D'}\frac{1}{x^2} = \frac{1}{(D-D')^2}\int \frac{1}{x^2}\,dy = \frac{1}{(D-D')^2}\frac{y}{x^2} = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}\frac{y}{x^2}$$

$$= \frac{1}{D^2}\left(1+\frac{2D'}{D}+\frac{3D'^2}{D^2}+\ldots\right)\frac{y}{x^2} = \frac{1}{D^2}\left\{\frac{y}{x^2}+\frac{2}{D}\left(\frac{1}{x^2}\right)\right\} = y\frac{1}{D^2}\frac{1}{x^2}+\frac{2}{D^3}\frac{1}{x^2}$$

$$= y\frac{1}{D}\left(-\frac{1}{x}\right) = -y\log x,$$

where we have omitted a function of x as it can be included in the term $\phi_3(x)$ of C.F.

$\therefore$ Required general solution is $\quad z = \phi_1(y+x)+x\phi_2(y+x)+\phi_3(x)-y\log x,$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Ex. 11. *Solve* $(D^3-7DD'^2-6D'^3)z = x^2+xy^2+y^3+\cos(x-y)$ **[K.U. Kurukshetra, 2004]**

Sol. Given $\quad (D^3-7DD'^2-6D'^3)\,z = x^2+xy^2+y^3+\cos(x-y) \quad \ldots(1)$

Here auxiliary equation is $\quad m^3-7m-6=0 \quad$ so that $\quad m=-1,-2,3.$

$\therefore\ C.F. = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

P.I. Corresponding to $(x^2+xy^2+y^3)$

$$= \frac{1}{D^3-7DD'^2-6D'^3}(x^2+xy^2+y^3) = \frac{1}{D^3}\left\{1-\left(7\frac{D'^2}{D^2}+6\frac{D'^3}{D^3}\right)\right\}^{-1}(x^2+xy^2+y^3)$$

$$= \frac{1}{D^3}\left\{1+\left(7\frac{D'^2}{D^2}+6\frac{D'^3}{D^3}\right)+\ldots\right\}(x^2+xy^2+y^3) = \frac{1}{D^3}(x^2+xy^2+y^3)+\frac{7}{D^5}(2x+6y)+\frac{36}{D^6}1$$

$$= (x^5/60+x^4y^2/24+x^3y^3/6)+7(x^6/360+x^5y/20)+36\times(x^6/720)$$

$$= 5x^6/72+x^5/60+7x^5y/20+x^4y^2/24+x^3y^3/6$$

P.I. Corresponding to $\cos(x-y)$

$$= \frac{1}{D^3-7DD'^2-6D'^3}\cos(x-y) = \frac{1}{(D+D')}\frac{1}{(D^2-DD'-6D'^2)}\cos(x-y)$$

$$= \frac{1}{D+D'}\frac{1}{1^2-4\times1\times(-1)-6\times(-1)^2}\iint \cos v\,dv\,dv, \text{ where } v = x-y$$

$$= \frac{1}{D+D'}\times\frac{1}{(-4)}\times(-\cos v) = \frac{1}{4}\frac{1}{D+D'}\cos(x-y) - \frac{1}{4}\frac{1}{(-1)\times D-1\times D'}\cos(x-y) = -\frac{1}{4}\frac{x}{(-1)^1\times1!}\cos(x-y)$$

$$= (x/4)\times\cos(x-y)$$

Hence the required general solution is $\quad z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$

$$+(5/72)\times x^6+x^5/60+(7/20)\times x^5y+(1/24)\times x^4y^2+(1/6)\times(x^3y^3)+(x/4)\times\cos(x-y),$$

Ex 12. *Solve* $(D^2 - 2DD' - 15D'^2)z = 12xy$. **(K.U. Kurukshetra 2004; Meerut 2006, 2011)**

Sol. Here auxiliary equation is $m^2 - 2m - 15 = 0$ so that $m = 5, -3$.

$\therefore$ $C.F. = \phi_1(y + 5x) + \phi_2(y - 3x)$, ϕ_1, ϕ_2 being arbitrary functions

Now, $$\text{P.I.} = \frac{1}{D^2 - 2DD' - 15D'^2}(12xy) = \frac{1}{D^2(1 - 2D'/D - 15D'^2/D^2)}(12xy)$$

$$= \frac{12}{D^2}\left\{1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right\}^{-1}(xy) = \frac{12}{D^2}\left(1 + \frac{2D'}{D} + \frac{15D'^2}{D^2} + \ldots\right)(xy)$$

$$= \frac{12}{D^2}\left\{xy + \frac{2}{D}D'(xy) + \frac{15}{D^2}D'^2(xy) + \ldots\right\} = \frac{12}{D^2}\left(xy + \frac{2}{D}x\right)$$

$$= 12y\frac{1}{D^2}x + \frac{24}{D^3}x = 12y\left(\frac{x^3}{6}\right) + 24\left(\frac{x^4}{24}\right) = 2x^3y + x^4$$

Hence the required general solution is $z = \phi_1(y + 5x) + \phi_2(y - 3x) + 2x^3y + x^4$

Ex. 13. *Solve* $\partial^3u/\partial x^3 + \partial^3u/\partial y^3 + \partial^3u/\partial z^3 - 3(\partial^3u/\partial x\,\partial y\,\partial z) = x^3 + y^3 + z^3 - 3xyz$.

Sol. Let $D = \partial/\partial x$, $D' = \partial/\partial y$, $D'' = \partial/\partial z$. Then the given equation can be re-written as

$$(D^3 + D'^3 + D''^3 - 3DD'D'')u = x^3 + y^3 + z^3 - 3xyz.$$

or $$(D + D' + D'')(D + \omega D' + \omega^2 D'')(D + \omega^2 D' + \omega D'')u = x^3 + y^3 + z^3 - 3xyz,$$

where ω is a complex cube root of unity. ...(1)

For C.F., let us consider $$(D + \omega^2 D' + \omega D'')u = 0. \quad \ldots(2)$$

Subsidiary equations of (2) are $$\frac{dx}{1} = \frac{dy}{\omega^2} = \frac{dz}{\omega} = \frac{du}{0}. \quad \ldots(3)$$

Three independent integrals of (3) are

$y - \omega^2 x = \text{constant}$, $z - \omega x = \text{constant}$ and $u = \text{constant}$.

Hence, general solution of (2) is $u = \phi_1(y - \omega^2 x, z - \omega x)$.

Similarly, the contributions to complementary function corresponding to other factors in (1) are $\phi_2(y - \omega x, z - \omega^2 x)$ and $\phi_3(y - x, z - x)$ and hence

C.F. $= \phi_1(y - \omega^2 x, z - \omega x) + \phi_2(y - \omega x, z - \omega^2 x) + \phi_3(y - x, z - x)$,

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

P.I. corresponding to x^3

$$= \frac{1}{D^3 + D'^3 + D''^3 - 3DD'D''}x^3 = \frac{1}{D^3}\left\{1 + \left(\frac{D'^3}{D^3} + \ldots\right)\right\}^{-1}x^3 = \frac{1}{D^3}x^3 = \frac{x^6}{120}.$$

Similarly, P.I. corresponding to $y^3 = y^6/120$ and P.I. corresponding to $z^3 = z^6/120$.

Finally, P.I. corresponding to $(-3xyz)$

$$= -3\frac{1}{D^3 + D'^3 + D''^3 - 3DD'D''}xyz = \frac{1}{DD'D''}\left\{1 + \frac{D^3 + D'^3 + D''^3}{3DD'D''}\right\}^{-1}xyz$$

$$= \frac{1}{DD'D''}\left\{1 - \frac{D^3 + D'^3 + D''^3}{3DD'D''} + \ldots\ldots\right\}xyz = \frac{1}{DD'D''}xyz = \frac{x^2y^2z^2}{8}.$$

Hence the required general solution of (1) is $z = \phi_1(y - \omega^2 x, z - \omega x) + \phi_2(y - \omega x, z - \omega^2 x) + \phi_3(y - x, z - x) + (1/120) \times (x^6 + y^6 + z^6) + (1/8) \times x^2 y^2 z^2$.

Ex.14. *Solve* $\dfrac{\partial^3 u}{\partial x^3} + \dfrac{\partial^3 u}{\partial y^3} + \dfrac{\partial^3 u}{\partial z^3} - 3\dfrac{\partial^3 u}{\partial x \partial y \partial z} = x^3 - 3xyz.$

Sol. Proceed as in Ex. 13. Its general solution is $z = \phi(y - w^2 x, z - wx) + \phi_2(y - wx, z - w^2 x) + \phi_3(y - x, z - x) + (1/120)x^6 + (1/8)x^2 y^2 z^2$.

EXERCISE 4(B)

Solve the following partial differential equations:

1. $(D^2 - 2DD' + D'^2)z = 12xy$ **(Jiwaji 1998; Ravishanker 1999; Vikram 1995, 97)**

Ans. $z = \phi_1(y + x) + x\,\phi_2(y + x) + 2x^3 y + x^4$, ϕ_1, ϕ_2 being arbitrary functions

2. $(D^2 - DD' - 6D'^2)z = xy$ **(Sagar 2003, Vikram 1995)**

Ans. $z = \phi_1(y - 2x) + \phi_2(y + 3x) + (1/6) \times x^3 y + (1/24) \times x^4$

4.12. A general method of finding the particular integral of linear homogeneous equation with constant coefficients.

Let the given equation be $F(D, D')z = f(x, y)$, ...(1)

where $F(D, D')$ is a homogeneous function of D and D' of degree n, (say) so that

$$F(D, D') = (D - m_1 D')(D - m_2 D') \ldots (D - m_n D').$$

$$\therefore \quad \text{P.I. of (1)} = \frac{1}{F(D, D')} f(x, y) = \frac{1}{(D - m_1 D')(D - m_2 D') \ldots (D - m_n D')} f(x, y). \quad \ldots(2)$$

In order to evaluate P.I. given by (2), we consider a solution of the following equation :

$(D - mD')z = f(x, y)$ or $p - mq = f(x, y)$, ...(3)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equation for (3) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x, y)}. \quad \ldots(4)$$

Taking the first two fractions of (4), $dy + mdx = 0$ so that $y + mx = c$. ...(5)

Next, taking the first and the last fractions of (4), we have

$$dz = f(x, y)dx = f(x, c - mx)dx, \text{ as from (5), } y = c - mx$$

Integrating, $z = \int f(x, c - mx)\, dx$.

Thus, $$z = \frac{1}{D - mD'} f(x, y) = \int f(x, c - mx)\, dx, \quad \ldots(6)$$

where after integration the constant c must be replaced by $y + mx$ since the P.I. does not contain any arbitrary constant.

Hence the P.I. given by (2) can be obtained by applying the operation (6) by the factors, in succession, starting from the right.

Working rule for finding P.I. (General method) of F(D, D′)z = f(x, y).

$$\text{P.I.} = \frac{1}{(D - m_1 D')(D - m_2 D') \ldots (D - m_n D')} f(x, y) \quad \ldots(7)$$

We shall use one of the following formulas :

Formula I : $$\frac{1}{D - mD'} f(x, y) = \int f(x, c - mx)\, dx, \quad \text{where} \quad c = y + mx. \quad \ldots(8)$$

Formula II : $$\frac{1}{D + mD'} f(x, y) = \int f(x, c + mx)\, dx, \quad \text{where} \quad c = y - mx. \quad \ldots(9)$$

Hence in order to evaluate P.I. (7), we apply (8) or (9) depending on the factor $D - mD'$ and $D + mD'$. Note that result (9) can be obtained from (8) by replacing m by $-m$.

4.13 SOLVED EXAMPLES BASED ON GENERAL METHOD

Ex. 1. *Solve* $(\partial z/\partial x) + (\partial z/\partial y) = \sin x$.

Sol. Rewritting, the given equation is $(D + D')z = \sin x$. ...(1)

Its auxiliary equation is $m + 1 = 0$ so that $m = -1$.

$\therefore$ C.F. $= \phi(y - x)$, where ϕ is an arbitrary function.

and P.I. $= \dfrac{1}{D + D'} \sin x = \int \sin x \, dx = -\cos x$

Hence the required solution is z = C.F. + P.I. $= \phi(y - x) - \cos x$.

Ex. 2. *Solve* (*a*) $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$.

[Delhi Maths (H) 2004, 10; Bhopal 2004; Jiwaji 2000; Rewa 2003; Vikram 2002, 04]

(*b*) $(D - D')(D + 2D')z = (y + 1)e^x$. **[Delhi Maths (H) 1993; I.A.S. 2004]**

Sol. (*a*) Here given $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$ or $(D + D')(D - 2D')z = (y - 1)e^x$.

Its auxiliary equation is $(m + 1)(m - 2) = 0$ so that $m = -1, 2$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D + D')(D - 2D')}(y - 1)e^x = \frac{1}{D + D'}\left\{\frac{1}{D - 2D'}(y - 1)e^x\right\}$$

$$= \frac{1}{D + D'}\int (c - 2x - 1)e^x \, dx$$, by formula I of working rule of Art. 4.12. and taking $c = y + 2x$

$$= \frac{1}{D + D'}\left[(c - 2x - 1)e^x - \int (-2)e^x \, dx\right]$$, integrating by parts

$$= \frac{1}{D + D'}[(c - 2x - 1)e^x + 2e^x] = \frac{1}{D + D'}(c - 2x + 1)e^x$$

$$= \frac{1}{D + D'}\{(y + 2x) - 2x + 1\}e^x$$, replacing c by $y + 2x$ $= \dfrac{1}{D + D'}(y + 1)e^x$

$$= \int (c' + x + 1)e^x \, dx,$$ by formula II of working rule of Art. 4.12 and taking $c' = y - x$

$$= (c' + x + 1)\, e^x - \int (1.e^x)\, dx = (c' + x + 1)\, e^x - e^x = ye^x,$$ since $c' = y - x$

Hence the required general solution is z = C.F. + P.I. or $z = \phi_1(y - x) + \phi_2(y + 2x) + ye^x$.

(*b*) Here auxiliary equation is $(m - 1)(m + 2) = 0$ so that $m = 1, -2$.

$\therefore$ C.F.= $\phi_1(y + x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

and P.I. $= \dfrac{1}{(D - D')(D + 2D')}(y + 1)\, e^x = \dfrac{1}{D - D'}\left\{\dfrac{1}{D + 2D'}(y + 1)\, e^x\right\}$

$$= \frac{1}{D - D'}\int (c + 2x + 1)\, e^x \, dx = \frac{1}{(D - D')}\{(c + 2x + 1)\, e^x - 2e^x\}$$, where $c = y - 2x$

$$= \frac{1}{D - D'}(y - 1)\, e^x = \int (c' - x - 1)\, e^x \, dx$$, where $c' = y + x$

$= (c' - x - 1)e^x + e^x = ye^x$ as $c' = y + x$.

$\therefore$ General solution is $y = \phi_1(y + x) + \phi_2(y - 2x) + ye^x$.

Ex. 3. $(D^2 - 4D'^2)z = (4x/y^2) - (y/x^2)$. **[Delhi Maths (H) 2004, 08; Meerut 1992, Bhopal 2010]**

Sol. Here auxiliary equation is $m^2 - 4 = 0$ so that $m = 2, -2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I} = \frac{1}{(D + 2D')(D - 2D')}\left(\frac{4x}{y^2} - \frac{y}{x^2}\right) = \frac{1}{D + 2D'}\int \left\{\frac{4x}{(c - 2x)^2} - \frac{c - 2x}{x^2}\right\} dx$$, where $c = y + 2x$.

$$= \frac{1}{D+2D'}\int\left\{-\frac{2}{c-2x}+\frac{2c}{(c-2x)^2}-\frac{c}{x^2}+\frac{2}{x}\right\}dx = \frac{1}{D+2D'}\left\{\log(c-2x)+\frac{c}{c-2x}+\frac{c}{x}+2\log x\right\}$$

$$= \frac{1}{D+2D'}\left[\log y+\frac{y+2x}{y}+\frac{y+2x}{x}+2\log x\right]$$

$$= \int\left\{\log(c'+2x)+1+2\frac{x}{c'+2x}+\frac{c'+2x}{x}+2+2\log x\right\}dx, \text{ taking } c' = y-2x$$

$= x\log(c'+2x)+5x+c\log x+2x\log x-2x \quad = x\log y+y\log x+3x$, as $c' = y-2x$

$\therefore$ The required solution $z = \phi_1(y+2x)+\phi_2(y-2x)+x\log y+y\log x+3x$

Ex. 4. *Solve* $(D^2-DD'-2D'^2)z = (2x^2+xy-y^2)\sin xy-\cos xy$.

[I.A.S 2009; Meerut 1994; Delhi Maths (Hons.) 2007]

Sol. Here auxiliary equation is $m^2-m-2=0$ so that $m = 2, -1$.

$\therefore$ C.F. $= \phi_1(y+2x)+\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-2D')}\frac{1}{D+D'}\{(2x^2+xy-y)\sin xy-\cos xy\}$$

$$= \frac{1}{D-2D'}\frac{1}{D+D'}\{(2x-y)(x+y)\sin xy-\cos xy\}$$

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin x(c+x)-\cos x(c+x)\}\,dx, \text{ taking } c = y-x$$

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin(cx+x^2)-\cos(cx+x^2)\}\,dx$$

$$= \frac{1}{D-2D'}\left[-(x-c)\cos(cx+x^2)+\int\cos(cx+x^2)\,dx-\int\cos(cx+x^2)\,dx\right]$$

$$= \frac{1}{D-2D'}(y-2x)\cos xy, \text{ as } c = y-x$$

$$= \int(c'-4x)\cos(c'x-2x^2)\,dx, \text{ where } c' = y+2x$$

$= \int\cos t\,dt = \sin t$, putting $c'x-2x^2 = t$ so that $(c'-4x)dx = dt$

$= \sin(c'x-2x^2) = \sin xy$, as $c' = y+2x$.

$\therefore$ Required solution is $z = \phi_1(y+2x)+\phi_2(y-x)+\sin xy$.

Ex. 5. *Solve* (*a*) $r+s-6t = y\cos x$. or $(D^2+DD'-6D'^2)z = y\cos x$

[Bilaspur 2002, Indore 2002, Jabalpur 1999 Meerut 2000, 02, Ravishankar 1994, Jiwaji 1999, Garhwal 2005, 10; I.A.S.1992, 2008; Vikram 1999, Delhi Maths (Hons) 2007, Purvanchal 2007; Kanpur 2011]

(*b*) $(D^2+DD'-6D'^2)z = y\sin x$.

Sol. (*a*) Since $r = \partial^2z/\partial x^2$, $s = \partial^2z/\partial x\partial y$, $t = \partial^2z/\partial y^2$, the given equation becomes

$\partial^2z/\partial x^2+\partial^2z/\partial x\,\partial y-6(\partial^2z/\partial y^2) = y\cos x$ or $(D^2+DD'-6D'^2)z = y\cos x$....(1)

Its auxiliary equation is $m^2+m-6=0$ so that $m = 2, -3$.

$\therefore$ C.F. $= \phi_1(y+2x)+\phi_2(y-3x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2+DD'-6D'^2}y\cos x = \frac{1}{(D-2D')(D+3D')}y\cos x$$

$$= \frac{1}{D-2D'}\int(3x+c)\cos x\,dx, \text{ where } c = y-3x$$

$$= \frac{1}{D-2D'}\left[(3x+c)\sin x - \int 3\sin x\, dx\right], \text{ integrating by parts}$$

$$= \frac{1}{D-2D'}[y\sin x + 3\cos x], \text{ as } c = y - 3x$$

$$= \int [(c'-2x)\sin x + 3\cos x]\, dx, \text{ where } c' = y + 2x$$

$$= (c'-2x)(-\cos x) - \int (-2)(-\cos x)\, dx + 3\sin x, \text{ integrating by parts}$$

$$= y(-\cos x) - 2\sin x + 3\sin x, \text{ as } c' = y + 2x$$

$$= \sin x - y\cos x.$$

$\therefore$ General solution is $z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x.$

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y+2x) + \phi_2(y-3x) - y\sin x - \cos x$

Ex. 6. *Solve* $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$. **[Agra 2009; Meerut 1999; Bilaspur 2002; Indore 2004; Jabalpur 1999; Rewa 2002; Ranchi 2010]**

Sol. Given equation is $(D+D')^2 z = 2\cos y - x\sin y.$ ...(1)

Its auxiliary equation is $(m+1)^2 = 0$ so that $m = -1, -1.$

$\therefore$ C.F. $= \phi_1(y-x) = x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. $= \frac{1}{D+D'}\frac{1}{D+D'}(2\cos y - x\sin y) = \frac{1}{D+D'}\int [2\cos(x+c) - x\sin(x+c)]\, dx$, where $c = y - x$

$$= \frac{1}{D+D'}\left[2\int \cos(x+c)\, dx - \int x\sin(x+c)\, dx\right] = \frac{1}{D+D'}\left[2\sin(x+c) - \{-x\cos(x+c) + \int \cos(x+c)\, dx\}\right]$$

$$= \frac{1}{D+D'}[2\sin(x+c) + x\cos(x+c) - \sin(x+c)] = \frac{1}{D+D'}(\sin y + x\cos y), \text{ as } c = y - x$$

$$= \int [\sin(x+c') + x\cos(x+c')]\, dx = -\cos(x+c') + x\sin(x+c') - \int \{1.\sin(x+c')\}\, dx, \text{ where } c' = y - x$$

$$= -\cos(x+c') + x\sin(x+c') + \cos(x+c') = x\sin y, \text{ as } c' = y - x.$$

So the required solution is $z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y.$

Ex. 7. *Solve* $r - t = \tan^3 x\tan y - \tan x\tan^3 y$ or $(D^2 - D'^2)z = \tan^3 x\tan y - \tan x\tan^3 y$ **[Agra 2010; Delhi Maths (G) 2006]**

Sol. Given equation is $(D^2 - D'^2)z = \tan^3 x\tan y - \tan x\tan^3 y$

or $(D+D')(D-D')z = \tan^3 x\tan y - \tan x\tan^3 y.$ ...(1)

Its auxiliary equation is $(m+1)(m-1) = 0$ so that $m = -1, 1.$

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. $= \frac{1}{(D+D')(D-D')}(\tan^3 x\tan y - \tan x\tan^3 y)$

$$= \frac{1}{D+D'}\int [\tan^3 x\tan(c-x) - \tan x\tan^3(c-x)]\, dx, \text{ where } c = y + x$$

$$= \frac{1}{D+D'}\int [\tan x\tan(c-x)(\sec^2 x - 1) - \tan x\tan^2(c-x)\{\sec^2(c-x) - 1\}]\, dx$$

$$= \frac{1}{D+D'}\int [\tan x\sec^2 x\tan(c-x) - \tan(c-x)\sec^2(c-x)\tan x]\, dx$$

$$= \frac{1}{D+D'}\left[\frac{\tan^2 x}{2}\tan(c-x) - \int \frac{\tan^2 x}{2}\sec^2(c-x).(-1)\, dx \right.$$

$$\left. - \left\{\frac{\tan^2(c-x)}{2\times(-1)}\tan x - \int \frac{\tan^2(c-x)}{2\times(-1)}\sec^2 x\, dx\right\}\right]$$

$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x)]$$

$$+ \int(\sec^2 x - 1)\sec^2(c-x)\,dx - \int\{\sec^2(c-x) - 1\}\sec^2 x\,dx]$$

$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x) - \int\sec^2(c-x)\,dx + \int\sec^2 x\,dx]$$

$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x) + \tan(c-x) + \tan x]$$

$$= \frac{1}{2(D+D')}(\tan^2 x \tan y + \tan x \tan^2 y + \tan y + \tan x), \text{ as } c = y + x$$

$$= \frac{1}{2(D+D')}[\tan y(\tan^2 x + 1) + \tan x(\tan^2 y + 1)] = \frac{1}{2(D+D')}(\tan y \sec^2 x + \tan x \sec^2 y)$$

$$= \frac{1}{2}\left[\int \tan(c'+x)\sec^2 x\,dx + \int \tan x \sec^2(c'+x)\,dx\right], \text{ where } c' = y - x$$

$$= \frac{1}{2}\left[\tan(c'+x)\tan x - \int\sec^2(c'+x)\tan x\,dx + \int\tan x\sec^2(c'+x)\,dx\right]$$

[On integrating the first integral by parts keeping the second integral unchanged]

$= (1/2) \times \tan(c'+x)\tan x = (1/2) \times \tan y \tan x$, as $c' = y - x$.

$\therefore$ The required solution is $z = \phi_1(y-x) + \phi_2(y+x) + (1/2) \times \tan y \tan x$.

Ex. 8. *Solve* $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$. **[Meerut 1994]**

Sol. Re-writing the given equation is $(D+3D')(D-2D')z = x^2\sin(x+y)$. ...(1)

$\therefore$ C.F. $= \phi_1(y-3x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+3D')(D-2D')}x^2\sin(x+y) = \frac{1}{D+3D'}\cdot\left\{\frac{1}{D-2D'}, x^2\sin(x+y)\right\}$$

$$= \frac{1}{D+3D'}\int x^2\sin(x+c-2x)\,dx = \frac{1}{D+3D'}\int x^2\sin(c-x)\,dx, \text{ where } c = y + 2x$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \int 2x\cos(c-x)\,dx\right], \text{ integrating by parts}$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \left\{-2x\sin(c-x) + \int 2\sin(c-x)\,dx\right\}\right], \text{ integrating by parts}$$

$$= \frac{1}{D+3D'}[x^2\cos(c-x) + 2x\sin(c-x) - 2\cos(c-x)]$$

$$= \frac{1}{D+3D'}[(x^2-2)\cos(x+y) + 2x\sin(x+y)], \text{ as } c = y + 2x$$

$$= \int[(x^2-2)\cos(x+c'+3x) + 2x\sin(x+c'+3x)]\,dx, \text{ where } c' = y - 3x$$

$$= \int(x^2-2)\cos(4x+c')\,dx + 2\int x\sin(4x+c')\,dx$$

$$= (x^2-2)\frac{\sin(4x+c')}{4} - \int 2x\frac{\sin(4x+c')}{4}\,dx + 2\int x\sin(4x+c')\,dx$$

[Integrating by part 1st integral and keeping the second integral unchanged]

$$= \frac{1}{4}(x^2-2)\sin(4x+c') + \frac{3}{2}\int x\sin(4x+c')\,dx = \frac{x^2-2}{4}\sin(4x+c') + \frac{3}{2}\left[-\frac{x\cos(4x+c')}{4} + \int\frac{\cos(4x+c')}{4}\,dx\right]$$

$$= \frac{x^2-2}{4}\sin(4x+c') - \frac{3}{8}x\cos(4x+c') + \frac{3}{32}\sin(4x+c')$$

$$= \frac{1}{4}(x^2-2)\sin(4x+y-3x) - \frac{3}{8}x\cos(4x+y-3x) + \frac{3}{32}\sin(4x+y-3x), \text{ as } c' = y - 3x$$

$= \left(x^2/4 - 13/32\right) \sin(x+y) - (3x/8) \times \cos(x+y)$, on simplification

The solution is $z = \phi_1(y-3x) + \phi_2(y+2x) + [(x^2/4) - (13/32)] \sin(x+y) - (3x/8) \times \cos(x+y)$.

Ex. 9. *Solve* $(D^3 + D^2D' - DD'^2 - D'^3)\, z = e^y \cos 2x$

Sol. Here $D^3 + D^2D' - DD'^2 - D'^3 = D^2(D+D') - D'^2(D+D') = (D+D')^2(D-D')$

So the given equation reduces to $(D-D')\,(D+D')^2 z = e^y \cos 2x$

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-D')\,(D+D')} \frac{1}{D+D'} e^y \cos 2x = \frac{1}{(D-D')\,(D+D')} \int e^{a+x} \cos 2x\, dx, \text{ where } y - x = a$$

$$= \frac{1}{(D-D')(D+D')} e^a \int e^x \cos 2x\, dx = \frac{1}{(D-D')(D+D')} e^{y-x} \frac{1}{1^2+2^2} e^x (\cos 2x + 2\sin 2x) \,^*$$

$$= \frac{1}{5} \frac{1}{(D-D')\,(D+D')} e^y (\cos 2x + 2\sin 2x) = \frac{1}{5} \frac{1}{D-D'} \int e^{x+a} (\cos 2x + 2\sin 2x)\, dx, \text{ where } y - x = a$$

$$= \frac{1}{5} \frac{1}{D-D'} e^a \left\{ \int e^x \cos 2x\, dx + 2 \int e^x \sin 2x\, dx \right\}$$

$$= \frac{1}{5} \frac{1}{D-D'} e^{y-x} \left\{ \frac{e^x}{1^2+2^2} (\cos 2x + 2\sin 2x) + \frac{2e^x}{1^2+2^2} (\sin 2x - 2\cos 2x) \right\}$$

$$= \frac{1}{25} \frac{1}{D-D'} e^y (4\sin 2x - 3\cos 2x) = \frac{1}{25} \int e^{b-x} (4\sin 2x - 3\cos 2x)\, dx, \text{ where } b = y + x$$

$$= \frac{1}{25} e^b \left\{ 4 \int e^{-x} \sin 2x\, dx - 3 \int e^{-x} \cos 2x\, dx \right\}$$

$$= \frac{1}{25} e^{y+x} \left\{ \frac{4e^{-x}}{1^2+2^2} (-\sin 2x - 2\cos 2x) - \frac{3e^{-x}}{1^2+2^2} (-\cos 2x + 2\sin 2x) \right\}$$

$= -(1/25) \times e^y (\cos 2x + 2\sin 2x)$

$\therefore$ Required solution is $z = \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x) - (e^y/25) \times (\cos 2x + 2\sin 2x)$

Ex. 10. *Find the solution of the equation* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^{-x} \cos y$ *which* $\to 0$ *as* $x \to \infty$ *and has the value cos y when x = 0.* **[I.A.S. 1999]**

Sol. Given $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^{-x} \cos y$ or $(D^2 + D'^2) z = e^{-x} \cos y$ ... (1)

$$\text{P.I.} = \frac{1}{D^2 + D'^2} e^{-x} \cos y = \frac{1}{D^2 + D'^2} e^{(-1)x + 0.y} \cos y = e^{(-1)x+0.y} \frac{1}{(D-1)^2 + (D'+0)^2} \cos y$$

$$= e^{-x} \frac{1}{D^2 + D'^2 - 2D - 1} \cos y = e^{-x} \frac{1}{0^2 + (-1)^2 - 2D + 1} \cos y = e^{-x} \frac{1}{-2D} \cos y = -\frac{1}{2} x e^{-x} \cos y.$$

Now, the general solution of (1) is $z = \text{C.F.} + \text{P.I.}$

where C.F. is solution of $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ ... (2)

---

* We shall use the following results of Integral calculus directly

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2} (a\sin bx - b\cos bx), \text{ and } \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} (a\cos bx + b\sin bx)$$

Since $\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0,$

we observe that P.I. $\to$ 0 as $x \to \infty$. Also, we have P.I. = 0 when $x = 0$

Here we are to solve (1) satisfying the conditions $z \to 0$ as $x \to \infty$ and $z = \cos y$ when $x = 0$. It follows that C.F. of (1), that is solution of (2) must satisfy the conditions C.F. $\to 0$ as $x \to \infty$ and C.F. = cos y when $x = 0$. In other words, we now * solve (2) subject to conditions :

$$z(x,y) \to 0 \text{ as } x \to \infty \quad \text{... (3)}$$

and $$z(x,y) = \cos y \text{ when } x = 0 \quad \text{... (4)}$$

Let a solution of (2) be $$z(x,y) = X(x), Y(y) \quad \text{... (5)}$$

From (5), $\partial^2 z/\partial x^2 = X''(x)\, Y(y)$ and $\partial^2 z/\partial y^2 = X(x)\, Y''(y)$,

where prime denotes the derivative w.r.t. to the relevent variable. Substituting these in (2), we get

$$X''(x)\ Y(y) + X(x)\ Y''(y) = 0 \quad \text{or} \quad X''/X = -Y''/Y \quad \text{... (6)}$$

Since x and y are independent variables, (6) is true if each side is equal to a constant, say n^2. Since condition (4) involves trigonometric function cos y, we choose n as positive integer.

$\therefore$ (6) $\Rightarrow$ $X''/X = n^2$ and $-Y''/Y = n^2$

or $d^2X/dx^2 - n^2X = 0$ and $d^2Y/dy^2 + n^2Y = 0$

Solving these, $$X(x) = A\, e^{nx} + Be^{-nx} \quad \text{... (7)}$$

and $$Y(y) = C\cos ny + D\sin ny \quad \text{... (8)}$$

Take $A = 0$ in (7) for otherwise $X(x) \to \infty$ as $x \to \infty$ and so $z(x,y) \to \infty$ as $x \to \infty$ which contradicts (3) Using (7) and (8), (5) reduces to

$$z(x,y) = e^{-nx}(E\cos ny + F\sin ny) \quad \text{... (9)}$$

where E (= BC) and F (= BD) are new arbitrary constants.

Now, putting $x = 0$ in (9) and using (4), we get cos $y = E$ cos $ny + F$ sin ny which holds if we choose $n = 1$, $E = 1$ and $F = 0$. Hence from (9), the C.F. of (1) is given by $e^{-x}\cos y$. Keeping in mind this C.F. of (1) and P.I. already obtained, the required solution of (1) is

$$z = e^{-x}\cos y - (x/2) \times e^{-x}\cos y = (1/2)\ \times (2-x)\ e^{-x}\cos y.$$

EXERCISE 4(C)

Solve the following partial differential equations:

1. $(D^3 + 2D^2D - DD'^2 - 2D'^3)\ z = (y+2)e^x$ **Ans.** $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y-2x) + ye^y$

2. $(D^3 - 3DD'^2 - 2D'^3)z = \cos(x+2y) - e^y(3+2x)$

Ans. $z = \phi_1(y-x) + x\,\phi_2(y-x) + \phi_3(y+2x) + (1/27)\times\sin(y+2x) + xe^x.$

3. $(D^3 - D^2D' - 2DD'^2)z = e^{x+2y}(x^2 + 4y^2)$.

Ans. $z = \phi_1(y) + \phi_2(y+2x) + \phi_3(y-x)\ -(1/81)\times(9x^2 + 36y^2 - 18x - 72y + 76)\ e^{x+2y}$

---

* We shall use the method of separation for solving partial differential equation. For details refer part III ''Boundary value problems'' in author's ''Advanced Differential Equations.''

4.14. SOLUTIONS UNDER GIVEN GEOMETRICAL CONDITIONS:

We have seen that solutions obtained in above methods involve arbitrary functions of x and y. We shall now determine these under the given geometrical conditions. This will lead to required surface satisfying the given differential equation under the given geometrical conditions.

4.15. SOLVED EXAMPLES BASED ON ART. 4.14.

Ex. 1. *Find a surface passing through the two lines* $z = x = 0,\ z - 1 = x - y = 0$ satisfying $r - 4s + 4t = 0$. **[Meerut 97,2000; I.A.S. 1996; Bhopal 2010]**

Sol. The given equation may be written as $\partial^2 z/\partial x^2 - 4(\partial^2 z/\partial x\partial y) + 4(\partial^2 z/\partial y^2) = 0$

or $(D^2 - 4DD' + 4D'^2)z = 0$ or $(D - 2D')^2 z = 0.$

Its solution is $z = \phi_1(y + 2x) + x\phi_2(y + 2x).$ ϕ_1, ϕ_2 being arbitrary functions ...(1)

Since (1) passes through $z = x = 0$, we have $0 = \phi_1(y)$ which gives $\phi_1(y + 2x) = 0.$

$\therefore$ (1) becomes $z = x\,\phi_2(y + 2x).$ ...(2)

Since (2) passes through $z - 1 = x - y = 0$, *i.e.* $z = 1$ and $y = x$, we get

$1 = x\,\phi_2(3x)$ or $\phi_2(3x) = 3/(3x)$ so that $\phi_2(y + 2x) = 3/(y + 2x).$

$\therefore$ from (2), we have $3x = z(y + 2x)$, which is the required surface.

Ex. 2. *Find the surface satisfying the equation* $r + t - 2s = 0$ *and the conditions that* $bz = y^2$ *when* $x = 0$ *and* $az = x^2$ *when* $y = 0$.

Sol. Re-writing the given equation, $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 - 2(\partial^2 z/\partial x\partial y) = 0$

or $(D^2 - 2DD' + D'^2) = 0$ or $(D - D')^2 z = 0.$

Its solution is $z = \phi_1(y + x) + x\,\phi_2(y + x).$ ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Since $z = y^2/b$ when $x = 0$, (1) gives $y^2/b = \phi_1(y),$ $\Rightarrow$ $\phi_1(y + x) = (y + x)^2/b.$ ...(2)

Again since $z = x^2/a$ when $y = 0$, (1) gives $x^2 a = x\phi_2(x) + \phi_1(x).$ ...(3)

Since from (2), $\phi_2(x) = x^2/b$, (3) becomes $\dfrac{x^2}{a} = x\phi_2(x) + \dfrac{x^2}{b}$ *i.e.* $\phi_2(x) = \dfrac{b - a}{ab}x$

which gives $\phi_2(y + x) = \dfrac{b - a}{ab}(y + x).$ ...(4)

Using (2) and (4) in (1), the required surface is

$$z = \frac{b - a}{ab}x(y + x) + \frac{(y + x)^2}{b} = \frac{y + x}{b}\left(\frac{b - a}{a}x + y + x\right) \quad \text{or} \quad z = (y + x)\left(\frac{x}{a} + \frac{y}{b}\right)$$

Ex. 3. *Find a surface satisfying* $r - 2s + t = 6$ *and touching the hyperbolic paraboloid* $z = xy$ *along its section by the plane* $y = x$. **[Meerut 1997]**

Sol. Re-writing the given equation, $\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x\partial y) + \partial^2 z/\partial y^2 = 6$

or $(D^2 - 2DD' + D'^2)z = 6$ or $(D - D')^2 z = 6.$ ... (1)′

Its C.F. $= \phi_1(y + x) + x\,\phi_2(y + x),$ ϕ_1, ϕ_2 being arbitrary functions

$$\text{Now, P.I.} = \frac{1}{(D - D')^2}.6 = \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2} 6 = \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \ldots\right)6 = \frac{1}{D^2}\,6 = 3x^2.$$

$\therefore$ General solution of (1)′ is $z = \text{C.F.} + \text{P.I.} = \phi_1(y + x) + x\phi_2(y + x) + 3x^2.$ ...(1)

Since the required surface (1) touches the given surface $z = xy$ ...(2)

along the section $y = x$, the values of p and q for the two surfaces must be equal for any point on the plane $y = x.$ ...(3)

Now equating the values of p and q from (1) and (2), we have

$$p = \phi_2(y + x) + x\phi_2'(y + x) + \phi_1'(y + x) + 6x = y \quad ...(4)$$

and $$q = x\phi_2'(y + x) + \phi_1'(y + x) = x. \quad ...(5)$$

Subtracting (5) from (4) and using (3), we get

$$\phi_2(2x) = -6x = -3 \times (2x)$$

which gives $$\phi_2(y + x) = -3(y + x). \quad ...(6)$$

From (6), $\phi_2'(y + x) = -3$. Then (5) becomes

$-3x + \phi_1'(y + x) = x$ so that $\phi_1'(2x) = 2 \times (2x)$, as $y = x$

Now, $\phi_1'(2x) = 2(2x) \Rightarrow \phi_1'(x) = 2x.$ ... (7)

Integrating (7), $\phi_1(x) = x^2 + c$ which gives $\phi_1(y + x) = (y + x)^2 + c.$ ...(8)

Putting the values of $\phi_2(y + x)$ and $\phi_1(y + x)$ given by (6) and (8) in (1), we get

$z = x\{-3(y + x)\} + (y + x)^2 + c + 3x^2$ or $z = x^2 - xy + y^2 + c.$ ...(9)

Equating the values of z from (2) and (9), we get

$xy = x^2 - xy + y^2 + c$ or $x^2 = x^2 - x^2 + x^2 + c$, using (3)

giving $c = 0$. Hence the required surface is $z = x^2 - xy + y^2.$

Ex. 4. *A surface is drawn satisfying* $r + t = 0$ *and touching* $x^2 + z^2 = 1$ *along its section by* $y = 0$. *Obtain its equation in the form* $x^2(x^2 + z^2 - 1) = y^2(x^2 + z^2)$. **[Meerut 1998]**

Sol. Given equation $(D^2 + D'^2)z = 0$ *i.e.* $(D + iD)(D - iD')z = 0.$

$\therefore$ Its solution is $z = \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

The given surface is $x^2 + z^2 = 1$ or $z = (1 - x^2)^{1/2}.$ ...(2)

Since (1) and (2) touch along their common section by $y = 0,$ ...(3)

the values of p and q from (1) and (2) must be the same.

$$\therefore \quad p = i\,\phi_1'(y + ix) - i\,\phi_2'(y - ix) = -\frac{x}{(1 - x^2)^{1/2}} \quad \text{and} \quad q = \phi_1'(y + ix) + \phi_2'(y - ix) = 0.$$

Using (3), these reduce to

$$\phi_1'(ix) - \phi_2'(-ix) = \frac{ix}{(1 + x^2 i^2)^{1/2}} \quad \text{and} \quad \phi_1'(ix) + \phi_2'(-ix) = 0, \quad \text{noting that } i^2 = -1$$

Solving these for $\phi_1'(ix)$ and $\phi_2'(-ix)$, $\phi'_1(ix) = \dfrac{ix}{2(1 + x^2 i^2)^{1/2}}$, $\phi'_2(-ix) = \dfrac{-ix}{2(1 + x^2 i^2)^{1/2}}$.

Writing $ix = X$ and $-ix = Y$, these give $\phi_1'(X) = \dfrac{X}{2(1 + X^2)^{1/2}}$ $\phi_2'(Y) = \dfrac{Y}{2(1 + Y^2)^{1/2}}$

Integrating, $\phi_1(X) = (1/2) \times (1 + X^2)^{1/2} + c_1$, $\phi_2(Y) = (1/2) \times (1 + Y^2)^{1/2} + c_2$

These give $\phi_1(y + ix) = (1/2) \times \{1 + (y + ix)^2\}^{1/2} + c_1$, $\phi_2(y - ix) = (1/2) \times \{1 + (y - ix)^2\}^{1/2} + c_2$.

Putting these in (1) and writing $c_1 + c_2 = c$, $z = (1/2) \times [\sqrt{\{1 + (y + ix)^2\}} + \sqrt{\{1 + (y - ix)^2\}}] + c$ ...(4)

Now equating two values of z from (2) and (4) at $y = 0$, we get

$$(1/2) \times [\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)}] + c = \sqrt{(1 - x^2)} \quad \text{so that} \quad c = 0.$$

Then, (4) gives $2z = \sqrt{\{1+(y+ix)^2\}} + \sqrt{\{1+(y-ix)^2\}}$. Squaring its both sides gives

or $$4z^2 = \{1+(y+ix)^2\} + (1+(y-ix)^2\} + 2\sqrt{[\{1+(y+ix)^2\}\ \{1+(y-ix)^2\}]},$$

or $$2z^2 = (1+y^2-x^2) + \sqrt{[\{1+(y+ix)^2\}\{1+(y-ix)^2\}]}. \quad ...(5)$$

Squaring both sides of (5), we get

$$4z^4 = (1+y^2-x^2)^2 + \{1+(y+ix)^2\}\{1+(y-ix)^2\} + 2(1+y^2-x^2)\sqrt{[\{1+(y+ix)^2\}\{1+(y-ix)^2\}]}$$

or $$4z^4 = (1+y^2-x^2)^2 + \{(1+y^2-x^2)+2ixy\}\{(1+y^2-x^2)-2ixy\}$$
$$+ 2(1+y^2-x^2)\{2z^2-(1+y^2-x^2)\}, \text{ using (5)}$$

or $$4z^4 = (1+y^2-x^2)^2 + (1+y^2-x^2)^2 + 4x^2y^2 + 4z^2(1+y^2-x^2) - 2(1+y^2-x^2)^2$$

or $4z^4 = 4x^2y^2 + 4z^2(1+y^2-x^2)$ or $z^2(x^2+z^2-1) = y^2(x^2+z^2)$.

Ex.5. *Find a surface satisfying the equation* $D^2z = 6x+2$ *and touching* $z = x^3+y^3$ *along its section by the plane* $x + y + 1 = 0$. **Ans.** $z = x^3+y^3+(x+y+1)^2$

MISCELLANEOUS PROBLEMS ON CHAPTER 4

1. Solution of differential equation $(D^3-6D^2D'+11DD'^2-6D'^3)z = 0$ is

(*a*) $z = \phi_1(y+x)+\phi_2(y+2x)+\phi_3(y-3x)$ (*b*) $z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$

(*c*) $z = \phi_1(y-x)+\phi_2(y+2x)+\phi_3(y+3x)$ (*d*) $z = \phi_1(y+x)+\phi_2(y+2x)+\phi_3(y+3x)$

Sol. Ans (*a*). Refer solved Ex. 2 (*a*) of Art. 4.5. **[Agra 2005]**

2. P.I. of the equation $r - 2s + t = \cos(2x + 3y)$ is

(*a*) $-\cos(2x + 3y)$ (*b*) $\cos(2x + 3y)$ (*c*) $\sin(2x + 3y)$ (*d*) None of these **[Kanpur 2004]**

Sol. Ans (*a*). Proceed like Ex. 4, Art. 4.9.

3. Auxialary equation of $r - 2s + t = \sin(2x + 3y)$ is

(*a*) $m^2-2m+1 = \sin(2x+3y)$ (*b*) $m^2+2m+1 = \sin(2x+3y)$

(*c*) $(m-1)^2 = 0$ (*d*) $(m+1)^2 = 0$ **[Bhopal 2010]**

Ans. (*c*)

4. Solve $(D^2-DD'-6D'^2)z = \cos(2x+y)$ **[Agra 2009, 10]**

Ans. $z = \phi_1(y+3x)+\phi_2(y-2x)-(1/4)\times\cos(2x+y)$

5

Non–homogeneous Linear Partial Differential Equations with Constant Coefficients

5.1. Non–homogeneous linear partial differential equations with constant coefficients.

Definition. A linear partial differential equation with constant coefficients is known as non–homogeneous linear partial differential equation with coefficients if the orders of all the partial derivatives involved in the equation are not equal.

For example, $(\partial^2 z/\partial x^2)-(\partial^3/\partial y^3)+\partial z/\partial x+z=x+y$ is a non–homogenous partial differential equation with constant coefficients.

5.2. Reducible and irreducible linear differential operators.

A linear differentiable operator $F(D, D')$ is known as *reducible*, if it can be written as the product of linear factors of the form $aD+bD'+c$ with a, b and c as constants.

$F(D, D')$ is known as *irreducible*, if it is not reducible.

For example, the operator $D^2-D'^2$ which can be written in the form $(D+D')(D-D')$ is reducible, whereas the operator $D^2-D'^3$ which cannot be decomposed into linear factors is irreducible.

5.3. Reducible and irreducible linear partial differential equations with constant coefficients. [Delhi Maths (H) 2001, 2004, 09]

A linear partial differential equation with constant coefficients $F(D, D')z=f(x,y)$ is known as reducible, if $F(D,D')$ is reducible.

$F(D,D')z=f(x,y)$ is known as irreducible if $F(D,D')$ is irreducible.

For example, $(D^2-D'^2)z=x^2y^3$ is a reducible partial differential equation, with constant coefficients, since $D^2-D'^2=(D+D')\ (D-D')$ whereas $(D^2-D'^3)\ z=x^2y^3$ is an irreducible partial differential equation with constant coefficients, since $D^2-D'^3$ cannot be decomposed into linear factors.

5.4. Theorem. *If the operator $F(D, D')$ is reducible, the order in which the linear factors occur is unimportant.*

Proof. In order to prove the theorem we must show that

$$(a_rD+b_r\ D'+c_r)\ (a_sD+b_sD'_r+c_s)=(a_sD+b_sD'+c_s)\ (a_rD+b_rD'+c_r) \quad \ldots (1)$$

for any reducible operator can be written in the form

$$F(D,\ D')=\prod_{r=1}^{n}(a_rD+b_rD'+c_r) \quad \ldots (2)$$

The proof of (1) is immediate, since both sides are equal to

$$a_ra_sD^2+(a_sb_r+a_rb_s)\ DD'+b_rb_sD'^2+(c_sa_r+c_ra_s)D+(c_sb_r+c_rb_s)D'+c_sc_r$$

5.5. Determination of complementary function (C.F.) of a reducible non–homogeneous linear partial differential equation with constant coefficients given by

$$F(D, D')z = 0 \quad \ldots (1)$$

Let $F(D, D') = (b_1D - a_1D' - c_1)(b_2D - a_2D' - c_2)..... (b_nD - a_nD' - c_n)$, ...(2)

where a's, b's and c's are constants. Then (1) becomes

$$(b_1D - a_1D' - c_1)(b_2D - a_2D' - c_2)...(b_nD - a_nD' - c_n)\, z = 0. \quad \ldots(3)$$

Equation (3) shows that any solution of the equation

$$(b_rD - a_rD' - c_r)z = 0,\ r = 1, 2, \ldots n \quad \ldots(4)$$

is a solution of (3) *i.e.* $b_rp - a_rq = c_rz$ which is Lagrange's equation.

Its Lagrange's auxiliary equations are $\dfrac{dx}{b_r} = \dfrac{dy}{-a_r} = \dfrac{dz}{c_rz}$. ...(5)

Proceeding as usual two independent integrals of (5) are $b_ry + a_rx = c_1$

and $z = c_2e^{(c_r/b_r)x}$, if $b_r \neq 0$ or $z = c_2'e^{-(c_r/a_r)y}$, if $a_r \neq 0$

$\therefore$ the general solution of (4) is $z = e^{(c_r/b_r)x}\phi_r(b_ry + a_rx)$, if $b_r \neq 0$ ...(6)

or $z = e^{-(c_r/a_r)y}\psi_r(b_ry + a_rx)$, if $a_r \neq 0$, ...(7)

where ϕ_r and ψ_r are arbitrary functions.

The general solution of (3) is the sum of the solutions of the equations of the form (4) corresponding to each factor in (2).

Case of repeated factors. Let two times repeated factor of (2), be $bD - aD' - c$.

Consider the equation $(bD - aD' - c)\,(bD - aD' - c)z = 0$. ...(8)

Let $(bD - aD' - c)z = v$. ...(9)

Then (8) reduces to $(bD - aD' - c)\,v = 0$. ...(10)

As before, the general solution of (10) is $v = e^{(c/b)x}\phi(by + ax)$, if $b \neq 0$. ...(11)

or $v = e^{-(c/a)y}\psi(by + ax)$, $a \neq 0$. ...(12)

where ϕ and ψ are arbitrary functions

Substituting from (11) is (9), we have

$(bD - aD' - c)z = e^{(c/b)x}\,\phi(by + ax)$ or $bp - aq = cz + e^{(c/b)x}\phi(by + ax)$. ...(13)

Lagrange's auxiliary equations of (13) are $\dfrac{dx}{b} = \dfrac{dy}{-a} = \dfrac{dz}{cz + e^{(c/b)x}\phi(by + ax)}$. ...(14)

Taking the first two fractions of (14), $adx + bdy = 0$ so that $by + ax = \lambda$, (say). ...(15)

where λ is an arbitrary constant.

Taking first and third fractions of (14), we get

$\dfrac{dz}{dx} - \dfrac{c}{b}z = \dfrac{1}{b}e^{(c/b)x}\phi(by + ax)$ or $\dfrac{dz}{dx} - \dfrac{c}{b}z = \dfrac{1}{b}e^{(c/b)x}\phi(\lambda)$, using (15)

This is linear equation. Its I.F. $= e^{-\int (c/b)dx} = e^{-(c/b)x}$ and solution is given by

$$ze^{-(c/b)x} = \int \frac{1}{b}e^{(c/b)x}\phi(\lambda)e^{-(c/b)x}\,dx, \text{ that is,}$$

$ze^{-(c/b)x} - (x/b)\,\phi(\lambda) = \mu$ or $ze^{-(c/b)x} - (x/b)\,\phi(by + ax) = \mu$, by (15) ...(16)

where μ is an arbitrary constant.

From (15) and (16), the general solution of (13) or (8) is

$ze^{-(c/b)x} - (x/b)\phi(by + ax) = \phi_1(by + ax)$ or $z = e^{(c/b)x}[\phi_1(by + ax) + x\,\phi_2(by + ax)]$, if $b \neq 0$ ...(17)

where ϕ_1 and ϕ_2 are arbitrary functions.

Taking (12) and (9), we obtain as before.

$$z = e^{-(c/a)y}[\psi_1(by + ax) + y\,\psi_2(by + ax)], \text{ if } a \neq 0. \quad ...(18)$$

where ψ_1 and ψ_2 are arbitrary functions.

In general, if $(bD - aD' - c)$ is repeated r times, then

$$z = c^{(c/b)x}\sum_{i=1}^{r} x^{i-1}\phi_i(by + ax), \text{ if } b = 0 \qquad \text{or} \qquad z = c^{-(c/a)y}\sum_{i=1}^{r} y^{i-1}\psi_i(by + ax), \text{ if } a \neq 0.$$

5.6. Working rule for finding C.F. of reducible non–homogeneous linear partial differential equation with constant coefficients. For proofs refer Art. 5.5.

Let the given differential equation be $F(D, D') = f(x, y)$.

Factorize $F(D, D')$ into linear factors. Then use the following results:

Rule I. Corresponding to each non–repeated factor $(bD - aD' - c)$, the part of C.F. is taken as $e^{(cx/b)}\phi(by + ax)$, if $b \neq 0$

We now have three particular cases of Rule I

Rule IA. Take $c = 0$ in rule I. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is $\phi(by + ax)$, $b \neq 0$.

Rule IB. Take $a = 0$ in rule I. Hence corresponding to each linear factor $(bD - c)$, the part of C.F. is $e^{(cx/b)}\phi(by)$, $b \neq 0$.

Rule IC. Take $a = c = 0$ and $b = 1$ in rule I. Hence corresponding to linear factor D, the part of C.F. is $\phi(y)$.

Rule II. Corresponding to a repeated factor $(bD - aD' - c)^r$, the part of C.F. is taken as

$$e^{(cx/b)}\,[\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + ... + x^{r-1}\phi_r(by + ax)], \text{ if } b \neq 0$$

We now have three particular cases of Rule II.

Rule II A. Take $c = 0$ in rule II. Hence corresponding to each repeated factor $(bD - aD')^r$, the part of C.F. is $\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + ... + x^{r-1}\phi_r(by + ax)$, $b \neq 0$.

Rule II B. Take $a = 0$ in rule II. Hence corresponding to a repeated factor $(bD - c)^r$, the part of C.F. is $e^{(cx/b)}\,[\phi_1(by) + x\phi_2(by) + x^2\phi_3(by) + ... + x^{r-1}\phi_r(by)]$, $b \neq 0$

Rule II C. Take $a = c = 0$ and $b = 1$ in rule II. Hence corresponding to repeated factor D^r, the part of C.F. is $\phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + ... + x^{r-1}\phi_r(y)$.

Rule III. Corresponding to each non–repeated linear factor $(bD - aD' - c)$, the part of C.F. is taken as $e^{-(cy/a)}\phi(by + ax)$, if $a \neq 0$.

We now have three particular cases of rule III.

Rule III A. Take $c = 0$ in rule III. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is $\phi(by + ax)$, $a \neq 0$.

Rule III B. Take $b = 0$ in rule III. Hence corresponding to each linear factor $(aD' + c)$, the part of C.F. is $e^{-(cy/a)}\phi(ax)$, $a \neq 0$.

Rule III C. Take $b = c = 0$ and $a = 1$ in rule III. Hence corresponding to linear factor D', the part of C.F. is $\phi(x)$

Rule IV. Corresponding to a repeated factor $(bD - aD' - c)^r$, the part of C.F. is taken as

$e^{-(cy/a)}$ [$\phi_1(by+ax) + y\,\phi_2(by+ax) + y^2\phi_3(by+ax) + \ldots + y^{r-1}\phi_r(by+ax)$], if $a \neq 0$

We now have three particular cases of rule IV.

Rule IV A. Take $c = 0$ in rule IV. Hence corresponding to repeated factor $(bD - aD')^r$, the part of C.F. is $\phi_1(by+ax) + y\,\phi_2(by+ax) + y^2\phi_3(by+ax) + \ldots + y^{r-1}\phi_r(by+ax)$, $a \neq 0$.

Rule IV B. Take $b = 0$ in rule IV. Hence corresponding to a repeated factor $(aD' + c)^r$, the part of C.F. is $e^{-(cy/a)}[\phi_1(ax) + y\,\phi_2(ax) + y^2\phi_3(ax) + \ldots + y^{r-1}\phi_r(ax)]$, $a \neq 0$

Rule IV C. Take $b = c = 0$ and $a = 1$ in rule IV. Hence corresponding to repeated factor D'^r, the part of C.F. is $\phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \ldots + y^{r-1}\phi_r(x)$.

5.7. SOLVED EXAMPLES BASED ON ART. 5.6.

Ex. 1. *Solve* $(D^2 - D'^2 + D - D')z = 0$.

Sol. The given equation can be re-written as $(D - D')(D + D' + 1)z = 0$.

Here R.H.S. $= 0 \Rightarrow$ P.I. $= 0$. Hence the required solution is $z =$ C.F.

or $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Ex. 2. *Solve* $(D^2 - a^2D'^2 + 2abD + 2a^2bD')z = 0$.

Sol. The given equation can be re–written as

$[(D + aD')(D - aD') + 2ab(D + aD')]z = 0$ or $(D + aD')(D - aD' + 2ab)z = 0$.

Hence general solution is $z =$ C.F. $= \phi_1(y - ax) + e^{-2abx}\phi_2(y + ax)$,

where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 3. *Solve* $r + 2s + t + 2p + 2q + z = 0$.

Sol. The given equation can be re–written as

$(\partial^2 z/\partial x^2) + 2(\partial^2 z/\partial x\partial y) + (\partial^2 z/\partial y^2) + 2(\partial z/\partial x) + 2(\partial z/\partial y) + z = 0$

or $(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0$

or $[(D + D')^2 + 2(D + D') + 1]z = 0$ or $(D + D' + 1)^2 z = 0$.

There are repeated linear factors. So the required general solution is

$z =$ C.F. $= e^{-x}[\phi_1(y - x) + x\phi_2(y - x)]$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 4. *Solve* $DD'(D - 2D' - 3)z = 0$.

Sol. Using rules I, IC, III C of Art. 5.6, the required solution is $z =$ C.F., i.e.,

$z = \phi_1(y) + \phi_2(x) + e^{3x}\phi_3(y + 2x)$, where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Ex. 5. *Solve* $(2D - 3)\,(3D - 5D' - 7)^2 z = 0$

Sol. Using rules IB and II of Art. 5.6, the required solution is $z =$ C.F., i.e.,

$z = e^{3x/2}\phi_1(2y) + e^{7x/3}\{\phi_2(3y + 5x) + x\,\phi_3(3y + 5x)\}$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Ex. 6. *Solve* $(3D - 5)\,(7D' + 2)\,DD'(2D + 3D' + 5)z = 0$

Sol. Using rules IB, IIIB, IC, IIIC and I, the required solution is $z =$ C.F. i.e.,

$$z = e^{5x/3}\phi_1(3y) + e^{-(2y/7)}\phi_2(7x) + \phi_3(y) + \phi_4(x) + e^{-(5x/2)}\phi_5(2y - 3x)$$

Ex. 7. *Solve the partial differential equation* $t + s + q = 0$.

Sol. Re-writing the given equation, $\partial^2 z/\partial x^2 + \partial^2 z/\partial x\partial y + \partial z/\partial y = 0$

or $(D^2 + DD' + D')z = 0$ or $D'(D + D' + 1) = 0,$

whose general solution is $z = \phi_1(x) + e^{-x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

EXERCISE 5(A)

Solve the following partial differential equations :

1. $(D + D' - 1)(D + 2D' - 2)z = 0$ **Ans.** $z = e^x\phi_1(y - x) + e^{2x}\phi_2(y - 2x)$

2. $(D - D' + 1)(D + 2D' - 3)z = 0$ **Ans.** $z = e^{-x}\phi_1(y + x) + e^{2x}\phi_2(y - 3x)$

3. $(DD' + aD + bD' + ab)z = 0$ **Ans.** $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x)$

4. $r + 2s + t + 2p + 2q + z = 0$ **Ans.** $z = e^{-x}\{\phi_1(y - x) + x\,\phi_2(y - x)\}$

5. $(D + 1)(D + D' - 1)z = 0$ **Ans.** $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x)$

6. $(D^2 - D'^2 + D - D')z = 0$ **Ans.** $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$

7. $\partial^2 z/\partial x^2 + \partial^2 z/\partial x\partial y - 6(\partial^2 z/\partial y^2) = 0$ **Ans.** $z = \phi_1(y + 2x) + \phi_2(y - 3x)$

8. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$ **Ans.** $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$

9. $s + p - q - z = 0$ **Ans.** $z = e^x\phi_1(y) + e^{-y}\phi_2(x)$

10. $(D^2 - DD' + D' - 1)z = 0$ **Ans.** $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x)$

11. $(D^2 + DD' + D' - 1)z = 0$ **Ans.** $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x)$

5.8. Method of finding C.F. of irreducible linear partial differential equation with constant coefficients, namely, $F(D, D')z = f(x, y)$ ... (1)

When the operator $F(D, D')$ in (1) is irreducible, it is not always possible to find a solution with the full number of arbitrary functions, but it is possible to construct solutions which contain as many arbitrary constants as we wish. We now state and prove a theorem which will be used to find C.F. of (1).

Theorem. *To show that* $F(D, D')\, e^{ax+by} = F(a, b)\, e^{ax+by}$

Proof. We know that $F(D, D')$ consists of terms of the form $C_{rs}\, D^r\, D'^s$.

Also $D^r(e^{ax+by}) = a^r e^{ax+by}$ and $D'^s(e^{ax+by}) = b^s\, e^{ax+by}$

so that $(C_{rs}\, D^r D'^s)(e^{ax+by}) = C_{rs} a^r b^s\, e^{ax+by}$

The theorem follows by combining the terms of the operator $F(D, D')$.

We now discuss method of finding C.F. of (1). Consider $F(D, D')\, z = 0$ ... (2)

From the above theorem we see that e^{hx+ky} is a solution of (2) provided $F(h, k) = 0$, so that

$$z = \sum_i A_i\, e^{h_i x + k_i y} \quad \text{... (3)}$$

in which A_i, h_i, k_i are all constants, is also a solution provided that h_i, k_i are connected by the relation $F(h_i, k_i) = 0$ ... (4)

Thus we can construct solution of (2) containing as many arbitrary constants as we need. The series (3) may not be finite but if it is infinite, it is necessary that it should be uniformly convergent if it has to be a solution of (2).

Remark. We can also present C.F. of irreducible equation (1) in the following manner

$$C.F. = \Sigma A e^{hx+ky}$$

where A, h, k are orbitrary constants such that $F(h, k) = 0$

Working rule for finding C.F. of irreducible non–homogeneous linear partial differential equation with constant coefficients, namely, $F(D, D')z = 0$

Step 1. If necessary, factorize $F(D, D')$ in the form $F_1(D, D')\, F_2(D, D')$, where $F_1(D, D')$ consist of a product of linear factors in D, D' and $F_2(D, D')$ consists of a product of irreducible factors in D, D'.

Step 2. Using Art. 5.6, write down the part of C.F. corresponding to factors of $F_1(D, D')$.

Step 3. Using Art 5.8, write down the part of C.F. corresponding to factors of $F_2(D, D')$.

Step 4. Adding the C.F. corresponding to $F_1(D, D')$ obtained in step 2 and the C.F. corresponding to $F_2(D, D')$ obtained in step 3, we obtain the C.F. of the given equation

$F(D, D')z = 0$, *i.e.*, $F_1(D, D')F_2(D, D')z = 0$.

5.9 SOLVED EXAMPLES BASED ON ART. 5.8

Ex. 1. *Solve* $(D - D'^2)z = 0$.

Sol. Here $D - D'^2$ is not a linear factor in D and D'. Let $z = Ae^{hx + ky}$ be a trial solution of the given equation. Then $Dz = Ahe^{hx + ky}$ and $D'^2z = Ak^2e^{hx + ky}$.

Putting these values in the given differential equation, we get

$A(h - k^2)e^{hx + ky} = 0$ so that $h - k^2 = 0$ or $h = k^2$.

Replacing h by k^2, the most general solution of the given equation is

$$z = \Sigma A e^{k^2x+ky},$$ where A and k are arbitrary constants.

Ex. 2. *Solve* $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$.

Sol. $(D - 2D' - 1)$ being linear in D and D', the part of C.F. corresponding to it is $e^x\phi(y + 2x)$, where ϕ is an arbitrary function.

To find C.F. corresponding non–linear factor $D - 2D'^2 - 1$, we now proceed as follows :

Let a trial solution of $(D - 2D'^2 - 1)z = 0$ ...(1)

be $z = Ae^{hx + ky}$ ...(2)

$\therefore$ $Dz = Ahe^{hx + ky}$ and $D'^2z = Ak^2e^{hx + ky}$. Hence (1) becomes

$A(h - 2k^2 - 1)e^{hx + ky} = 0$ or $h - 2k^2 - 1 = 0$ or $h = 2k^2 + 1$.

Replacing h by $2k^2 + 1$ in (2), the solution of (1) *i.e.* the part of C.F. corresponding to $(D - 2D'^2 - 1)$ in the given equation is given by

$$\Sigma A e^{(2k^2+1)x+ky},$$ A and k being arbitrary constants.

$\therefore$ The required solution is $z = e^x\phi(y + 2x) + \Sigma A e^{(2k^2+1)x+ky}$.

Ex. 3. *Solve* $(\partial^2z/\partial x^2) + (\partial^2z/\partial y^2) = n^2z$.

Sol. The given equation can be written as $(D^2 + D'^2 - n^2)z = 0$. ...(1)

Let a trial solution of (1) be $z = Ae^{hx + ky}$. ...(2)

$\therefore \quad D^2z = Ah^2e^{hx+ky}$ and $D'^2z = Ak^2e^{hx+ky}$. Hence (1) gives

$A(h^2 + k^2 - n^2)e^{hx+ky} = 0$ or $h^2 + k^2 = n^2$. ...(3)

Taking α as parameter, we see that (2) is satisfied if $h = n\cos\alpha$ and $k = n\sin\alpha$.

Putting these values in (2), the required general solution is

$z = \Sigma Ae^{n(x\cos\alpha + y\sin\alpha)}$, A and α being arbitrary constants.

Ex. 4. *Solve* $(2D^4 - 3D^2D' + D'^2)z = 0$

Sol. Re–writing the given equation, we have $(2D^2 - D')(D^2 - D')z = 0$ ... (1)

Let $z = A\,e^{hx+ky}$ be a solution of $(2D^2 - D')z = 0$

$\therefore (2h^2 - k)Ae^{hx+ky} = 0$ so that $2h^2 - k = 0$ or $k = 2h^2$

Hence C.F. corresponding to $(2D^2 - D')$ is ΣAe^{hx+2h^2y} ... (2)

Again, let $z = A'e^{h'x+k'y}$ be a solution of $(D^2 - D')z = 0$

$\therefore (h'^2 - k')A'\,e^{h'x+k'y} = 0$ so that $h'^2 - k' = 0$ or $k' = h'^2$.

Hence, C.F. corresponding to $(D^2 - D')$ is $\Sigma A'e^{h'x+h'^2y}$ ... (3)

From (2) and (3), the general solution of (1) is given by z = Total C.F., i.e.,

$z = \sum_i A_ie^{h_ix+2h_i^2y} + \sum_i A_i'e^{h_i'x+h_i'^2y}$, where A_i, h_i, A_i' and h_i' are arbitrary constants.

Ex. 5. Solve $(D + 2D' - 3)(D^2 + D')z = 0$

Sol. C.F. corresponding to linear factor $(D + 2D' - 3)$ is $e^{3x}\phi(y - 2x)$.

We now find C.F. corresponding to irreducible factor $(D^2 + D')$.

Let $z = Ae^{hx+ky}$ be a solution of $(D^2 + D')z = 0$.

$\therefore \quad (h^2 + k)Ae^{hx+ky} = 0$ so that $h^2 + k = 0$ or $k = -h^2$.

Hence C.F. corresponding to $(D^2 + D')$ is ΣAe^{hx-h^2y}.

Therefore, the general solution of given equation is $z = e^{3x}\phi(y - 2x) + \sum_i A_ie^{h_ix-h_i^2y}$,

where ϕ is an arbitrary function and A_i, h_i are arbitrary constants.

Ex. 6. *Solve* $(D^2 - D')\,z = 0$. **[Delhi Maths (H) 2009]**

Sol. Given equation is $F(D, D')z = 0$, where $F(D, D') = D^2 - D'$.

Let $z = e^{hx+ky}$ be a trial solution of the given equation. Then, the required solution is

$z = \sum_i A_i\ e^{h_ix+k_iy}$, where $F(h_i, k_i) = h_i^2 - k_i = 0$ so that $k_i = h_i^2$

Hence the required solution is $z = \sum_i A_i\ e^{h_ix+h_i^2y}$, A_i, h_i being arbitrary constants.

Ex. 7. *Show that* $\partial^2 z/\partial x^2 = (1/k) \times (\partial z/\partial t)$ *possesses solutions of the form*

$$\sum_{n=0}^{\infty} C_n \cos(nx + \in_n)\, e^{-kn^2 t}.$$

Sol. Re–writing giving equation, we get $\{D^2 - (1/k) \times D'\}\, z = 0$, ... (1)

where $D = \partial/\partial x$, $D' = \partial/\partial t$. Note that here we have t in place of usual independent variable y. Let $z = e^{ax+bt}$ be a trial solution of (1). Then, $a^2 - (1/k)\, b = 0$ so that $a^2 = b/k$. This relation is satisfied if we take $a = \pm in$ and $b = -kn^2$. Then solution of (1) will be of the form

$z = \sum_{n=0}^{\infty} C_n\, e^{\pm inx - kn^2 t}$, which can be re-written as $z = \sum_{n=0}^{\infty} C_n \cos(nx + \in_n) e^{-kn^2 t}$

Ex. 8. *Write the form of solution possessed by the equation* $\partial^2 y/\partial t^2 + 2k\,(\partial y/\partial t) = c^2\,(\partial^2 y/\partial x^2)$ **[Delhi B.Sc (H) 2002]**

Another form. *Show that the equation* $\partial^2 y/\partial t^2 + 2k\,(\partial y/\partial t) = c^2(\partial^2 y/\partial x^2)$ *possesses solutions of the form* $\sum_{r=0}^{\infty} C_r\, e^{-kt} \cos(w_r t + \delta_r) \cos(\alpha_r x + \in_r)$,

where C_r, $\alpha_r \delta_r$, $\in_r$ are constants and $w_r^2 = \alpha_r^2 c^2 - k^2$.

Sol. Re–writing the given equation, we get $(D'^2 + 2k\, D' - c^2 D^2)\, y = 0$ ... (1)

where $D' \equiv \partial/\partial t$, $D \equiv \partial/\partial x$. Here t is independent variable and y is dependent variable.

Let $z = e^{ax + bt}$ be a solution of (1) Then, $b^2 + 2kb - c^2 a^2 = 0$

so that $b = \{-2k \pm (4k^2 + 4c^2 a^2)\}^{1/2}/2 = -k \pm (k^2 - c^2\alpha_r^2)^{1/2}$, where $a^2 = -\alpha_r^2$

or $b = -k \pm iw_r$, where $w_r^2 = c^2\alpha_r^2 - k^2$ ... (2)

Hence the solution of (1) takes the form

$$y = \sum_{r=0}^{\infty} C_r\, e^{\pm i\alpha_r x + (-k \pm iw_r)t} = \sum_{r=0}^{\infty} C_r\, e^{(-k \pm iw_r)t}\, e^{\pm i\alpha_r x}$$

which can also be re–written as $y = \sum_{r=0}^{\infty} C_r\, e^{-kt} \cos(w_r t + \delta_r) \cos(\alpha_r x + \in_r)$,

where C_r, α_r, δ_r, $\in_r$ are constants. Also, by (2), $\omega_r^2 = \alpha_r^2 c^2 - k^2$.

EXERCISE 5(B)

Solve the following partial differential equations:

1. $(D^2 + D + D')z = 0$ **Ans.** $z = \sum_i A_i e^{h_i - (h_i^2 + h_i)y}$, where A_i and h_i are arbitrary constants.

2. $(2D^2 - D'^2 + D)z = 0$

Ans. $z = \sum_i A_i e^{h_i x + k_i y}$, where $2h_i^2 - k_i^2 + h_i = 0$; A_i, h_i, k_i being arbitrary constants.

3. $(D' + 3D)^2 (D^2 + 5D + D')z = 0$ **Ans.** $z = \phi_1(3y - x) + x\,\phi_2(3y - x)$

$+ \sum_i A_i\, e^{h_i x - (h_i^2 + 5h_i)y}$, where ϕ_1, ϕ_2 are arbitrary functions and A_i, h_i are arbitrary constants.

4. $(2D-3D'+7)^2\ (D^2+3D')z=0$. **Ans.** $z=e^{-(7x/2)}\{\phi_1(2y+3x)+x\,\phi_2(2y+x)\}$ $+\sum_i A_i\ e^{h_i x-(h_i^2 y)/3}$, where ϕ_1, ϕ_2 are arbitrary functions and A_i, h_i are arbitrary constants.

5.10. General solution of non–homogeneous linear partial differential equation with constant coefficients.

Let $$F(D, D')z = f(x, y) \quad \text{... (1)}$$

be a non-homogeneous linear partial differential equation with constant coefficients. Let u be the C.F. of (1). Then, by definition u in a solution of $F(D, D')\,z = 0$ so that

$$F(D, D')u = 0 \quad \text{... (2)}$$

Let z' be a particular integral (P.I.) of (1). Hence

$$F(D, D')z' = f(x, y) \quad \text{... (3)}$$

Now, $F(D, D')\,(u+z') = F(D, D')u + F(D, D')z' = f(x, y)$, by (2) and (3),

Thus $u+z'$ is a solution of (1). Hence, a solution of (1) is z = C.F. + P.I.

5.11. Particular integral of non–homogeneous linear partial differential equation

$$\boldsymbol{F(D, D')y = f(x, y)} \quad \text{... (1)}$$

The inverse operator $1/F(D, D')$ of the operator $F(D, D')$ is defined by the following indentity:

$$F(D, D')\left(\frac{1}{F(D, D')}f(x, y)\right) = f(x, y)$$

$$\Rightarrow \qquad \text{Particular integral (P.I.)} = \frac{1}{F(D, D')}f(x, y)$$

5.12. Determination particular integral of non–homogeneous linear partial differential equations (reducible or irreducible), namely,

$$\mathbf{F(D, D')z = f(x, y).} \quad \text{...(1)}$$

The methods of finding particular integrals of non–homogeneous partial differential equations are very similar to those of ordinary linear differential equation with constant coefficients. We now give a list of some cases of finding P.I. of (1).

Case I. When $f(x, y) = e^{ax+by}$ and $F(a, b) \neq 0$.

Then, $$\text{P.I.} = \frac{1}{F(D, D')}e^{ax+by} = \frac{1}{F(a, b)}e^{ax+by}.$$

Thus in this case we replace D by a and D' by b.

Case II. When f(x, y) = sin (ax + by) or cos (ax + by).

Then, $$\text{P.I.} = \frac{1}{F(D, D')}\sin(ax+by) \qquad \text{or} \qquad \text{P.I.} = \frac{1}{F(D, D')}\cos(ax+by)$$

which is evaluated by putting $D^2 = -a^2, D'^2 = -b^2, DD' = -ab$, provided the denominator is non–zero.

Case III. When $f(x, y) = x^m y^n$

Then, $$\text{P.I.} = \frac{1}{F(D, D')}x^m y^n = [F(D, D')]^{-1}\ x^m y^n,$$

which is evaluated by expanding $[F(D, D')]^{-1}$ in ascending powers of D'/D or D/D' or D or D' as the case may be. In practice, we shall expand in ascending powers of D'/D. However note that if

we expand in ascending powers of D/D', we shall get a P.I. of apparently different form. In this connection remember that both forms of P.I. are correct because the two could be transformed into each other with the help of C.F. of the given equation.

Case IV. When f(x, y) = $Ve^{ax + by}$, when V is a function of x and y.

Then $$\text{P.I.} = \frac{1}{F(D, D')} Ve^{ax + by} = e^{ax + by} \frac{1}{F(D + a, D' + b)} e^{ax + by}$$

Remark. If $F(a, b) = 0$ and $f(x, y) = e^{ax + by}$. Then, we have

$$\text{P.I.} = \frac{1}{F(D, D')} e^{ax + by},$$

in which case I fails. However, by treating $e^{ax + by}$ as product of $e^{ax + by}$ with '1' and applying the result of case IV, we can evaluate P.I. as follows :

$$\text{P.I.} = \frac{1}{F(D, D')} e^{ax+by} \cdot 1 = e^{ax + by} \frac{1}{F(D + a, D' + b)} \cdot 1,$$

which can be evaluated as explained in case III by treating $1 = x^0y^0$.

5.13 SOLVED EXAMPLES BASED ON ARTICLES 5.6, 5.8, AND 5.12.

Type 1 : Examples based on case I of Art. 5.12.

Ex. 1. *Solve* $(DD' + aD + bD' + ab)z = e^{mx + ny}$.

Sol. The given equation can be re–written as $(D + b)(D' + a)z = e^{mx + ny}$

$\therefore$ C.F. $= e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $$\text{P.I.} = \frac{1}{(D + b)(D' + a)} e^{mx + xy} = \frac{1}{(m + b)(n + a)} e^{mx + ny}.$$

Hence the required general solution is $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x) + [(m + b)(n + a)]^{-1}e^{mx + ny}$.

Ex. 2. *Solve* $(D^2 - D'^2 + D - D')z = e^{2x + 3y}$. **[Ravishankar 2005]**

Sol. The given equation can be re–written as

$[(D - D')(D + D') + (D - D')]z = e^{2x + 3y}$ or $(D - D')(D + D' + 1)z = e^{2x + 3y}$.

$\therefore$ C.F. $= \phi_1(y + x) + e^{-x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $$\text{P.I.} = \frac{1}{(D - D')(D + D' + 1)} e^{2x + 3y} = \frac{1}{(2 - 3)(2 + 3 + 1)} e^{2x + 3y} = -\frac{1}{6} e^{2x + 3y}$$

Hence the required general solution is z. $z = \phi_1(y + x) + e^{-x}\phi_2(y - x) - (1/6) \times e^{2x + 3y}$.

Ex. 3. *Solve* $(D - D' - 1)(D - D' - 2)z = e^{2x - y}$.

Sol. Try yourself. **Ans.** $z = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x) + (1/2) \times e^{2x - y}$

Ex. 4. *Solve* (*a*) $(D^2 - 4DD' + D - 1)z = e^{3x - 2y}$.

(*b*) $(D^3 - 3DD' + D + 1)z = e^{2x + 3y}$. **[Kanpur 2006]**

Sol. (*a*) Here $(D^2 - 4DD' + D - 1)$ cannot be resolved into linear factors in D and D'. Hence for finding C.F., consider the equation $(D^2 - 4DD' + D - 1)z = 0$. ...(1)

Let a trial solution of (1) be $z = Ae^{hx + ky}$. ...(2)

$\therefore$ $D^2z = Ah^2e^{hx + ky}$, $DD' = Ahke^{hx + ky}$, $Dz = Ahe^{hx + ky}$ and so (1) gives

$A(h^2 - 4hk + h - 1)e^{hx + ky} = 0$ so that $h^2 - 4hk + h - 1 = 0$

giving $$k = (h^2 + h - 1)/4h. \quad ...(3)$$

$\therefore$ C.F. $= \Sigma Ae^{hx + ky}$, when k is given by (3).

Again, $$\text{P.I.} = \frac{1}{D^2 - 4DD' + D - 1} e^{3x - 2y} = \frac{1}{3^2 - 4 \times 3 \times (-2) + 3 - 1} e^{3x - 2y} = \frac{1}{35} e^{3x - 2y}$$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$ *i.e.* $z = \Sigma A e^{hx+ky} + (1/35) \times e^{3x-2y}$, where A, h and k are constants and k and h are related by (3).

(*b*) Proceed as in part (*a*). **Ans.** $z = \Sigma A e^{hx+ky} - (1/7) \times e^{2x+3y}$, where A, h, k are arbitrary constants and k is given by $k = (h^3 + h + 1)/3h$

Ex. 5. *Solve* $(\partial^2 y/\partial x^2) - (\partial^2 y/\partial z^2) = y + e^{x+z}$.

Sol. Re–writing, $(D^2 - D'^2 - 1)y = e^{x+z}$, where $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial z$.

$$\text{C.F.} = \sum A\, e^{hx+kz}, \qquad \text{where} \qquad h^2 - k^2 - 1 = 0 \qquad \ldots (1)$$

$$\text{P.I.} = \frac{1}{D^2 - D'^2 - 1} e^{x+z} = \frac{1}{1-1-1} e^{x+z} = -e^{x+z}$$

$\therefore$ The required solution is $z = \Sigma A\, e^{hx+ky} - e^{x+z}$, where A, h and k are arbitrary constants and h and k are connected by relation (1).

EXERCISE 5(C)

Solve the following partial differential equations:

1. $(D^2 - DD' - 2D)z = e^{2x+y}$. **Ans.** $z = \phi_1(y) + e^{2x}\phi_2(y+x) - (1/2) \times e^{2x+y}$

2. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y}$.

Ans. $z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) - (1/10) \times e^{2x+3y}$

3. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y}$. **Ans.** $z = e^{-2x}\phi_1(y+x) + e^{x}\phi_2(y-x) - (1/4) \times e^{x-y}$

4. $(D^2 + D' + 4)z = e^{4x-y}$.

Ans. $z = \sum\limits_i A_i\, e^{a_i x - (a_i^2+4)y} + (1/19) \times e^{4x-y}$, where A_i and a_i are arbitrary constants.

5. $(D^2 - D'^2 - 3D')z = e^{x+2y}$ **(Purvanchal 2007)**

Ans. $z = \Sigma A e^{(k^2+3k)^{1/2}x + ky} - (1/9) \times e^{x+2y}$

Type 2 : Examples based on case II of Art. 5.12.

Ex. 1. *Solve* $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$.

[Bilaspur 2003; Bhopal 1998; Jiwaji 1997; Ravishankar 2004]

Sol. The given equation can be re–written as $(D + 1)(D + D' - 1)z = \sin(x + 2y)$.

$\therefore$ C.F. $= e^{-x}\phi_1(y) + e^{x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions

and $$\text{P.I.} = \frac{1}{D^2 + DD' + D' - 1} \sin(x + 2y) = \frac{1}{-1^2 - (1 \cdot 2) + D' - 1} \sin(x + 2y)$$

$$= \frac{1}{D' - 4} \sin(x + 2y) = (D' + 4)\frac{1}{D'^2 - 16} \sin(x + 2y) = (D' + 4)\frac{1}{-2^2 - 16} \sin(x + 2y)$$

$$= -(1/20) \times (D' + 4) \sin(x + 2y) = -(1/20) \times [D' \sin(x + 2y) + 4 \sin(x + 2y)]$$

$$= -(1/20) \times [2 \cos(x + 2y) + 4 \sin(x + 2y)].$$

$\therefore$ Solution is $z = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x) - (1/10) \times [\cos(x + 2y) + 2 \sin(x + 2y)]$.

Ex. 2. *Solve* $(\partial^2 z/\partial x^2) - (\partial z/\partial x \partial y) + (\partial z/\partial y) - z = \cos(x + 2y)$.

[Delhi Maths (H) 2001; M.D.U Rohtak 2004]

Sol. The given equation can be re–written as

$(D^2 - DD' + D' - 1)z = \cos(x + 2y)$ or $(D - 1)(D - D' + 1)z = \cos(x + 2y)$.

$\therefore$ C.F. $= e^x \phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 - DD' + D' - 1}\cos(x + 2y) = \frac{1}{-1^2 + (1\cdot 2) + D' - 1}\cos(x + 2y) = \frac{1}{D'}\cos(x + 2y)$$

$= (1/2) \times \sin(x + 2y)$, as $1/D'$ stands for integration w.r.t. y keeping x as constant

Hence the required solution is $z = e^x \phi_1(y) + e^{-x}\phi_2(y + x) + (1/2) \times \sin(x + 2y)$.

Ex. 3. *Solve* $2(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2) - 3(\partial z/\partial y) = 5\cos(3x - 2y)$.

Ans. $z = \phi_1(x) + e^{3x/2}\phi_2(2y - x) + (1/10) \times [4\cos(3x - 2y) + 3\sin(3x - 2y)]$.

Ex. 4. *Solve* $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$. **[KU Kurukshetra 2005]**

Sol. Here C.F. $= e^x \phi_1(y + x) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $$\text{P.I.} = \frac{1}{(D - D' - 1)(D - D' - 2)}\sin(2x + 3y) = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2}\sin(2x + 3y)$$

$$= \frac{1}{-2^2 + 2 \times (2 \times 3) - 3^2 - 3D + 3D' + 2}\sin(2x + 3y)$$

$$= \frac{1}{-3D + 3D' + 1}\sin(2x + 3y) = D\frac{1}{-3D^2 + 3DD' + D}\sin(2x + 3y)$$

$$= D\frac{1}{-3 \times (-2^2) + 3 \times (2 \times 3) + D}\sin(2x + 3y) = D\frac{1}{D - 6}\sin(2x + 3y)$$

$$= D(D + 6)\frac{1}{D^2 - 36}\sin(2x + 3y) = (D^2 + 6D)\frac{1}{-2^2 - 36}\sin(3x + 2y)$$

$= -(1/40) \times [D^2 \sin(2x + 3y) + 6D \sin(2x + 3y)] = -(1/40) \times [-4\sin(2x + 3y) + 12\cos(2x + 3y)]$

$\therefore$ Solution is $z = e^x \phi_1(y + x) + e^{2x}\phi_2(y + x) + (1/10) \times [\sin(2x + 3y) - 3\cos(2x + 3y)]$.

Ex. 5. *Solve* (*a*) $(D - D'^2)z = \cos(x - 3y)$ **[Delhi Maths (Hons.) 1998, 2007, 2009, 2011]**

(*b*) $(D^2 - D')z = \cos(3x - y)$.

Sol. (*a*) Here $(D - D'^2)$ cannot be resolved into linear factors in D and D'. Hence in order to find C.F. of the given equation, consider the equation

$$(D - D'^2)z = 0. \qquad ...(1)$$

Let a trial solution of (1) be $z = Ae^{hx + ky}$. ...(2)

$\therefore$ $Dz = Ahe^{hx + ky}$ and $D'^2 z = Ak^2 e^{hx + ky}$. Then (1) gives

$A(h - k^2)e^{hx + ky} = 0$ so that $h - k^2 = 0$ or $h = k^2$.

$\therefore$ C.F. $= \Sigma\, Ae^{k(kx + y)}$, where A, k are arbitrary constants.

Now, $$\text{P.I.} = \frac{1}{D - D'^2}\cos(x - 3y) = \frac{1}{D - (-3^2)}\cos(x - 3y)$$

$$= (D - 9)\frac{1}{(D + 9)(D - 9)}\cos(x - 3y) = (D - 9)\frac{1}{D^2 - 81}\cos(x - 3y) = \frac{(D - 9)}{-1^2 - 81}\cos(x - 3y)$$

$$= -(1/82) \times [D\cos(x - 3y) - 9\cos(x - 3y)] = -(1/82) \times [-\sin(x - 3y) - 9\cos(x - 3y)].$$

$\therefore$ General solution is $z = \Sigma\, Ae^{k(kx + y)} + (1/82) \times [\sin(x - 3y) + 9\cos(x - 3y)]$

(*b*) **Ans.** $z = \Sigma Ae^{h(x+hy)} - (1/82) \times [\,9\cos(3x-y) - \sin(3x-y)]$, where A and h are arbitrary constants.

Ex. 5. (*c*) *Solve* $(D^2 - D')z = A\cos(lx + my)$, where A, l, m *are constants*.

Sol. Proceed as in Ex. 5(*a*). **Ans.** $z = \Sigma A' e^{hx+h^2y} - \{A/(m^2 + l^4)\} \times \{m\sin(lx+my) + l^2\cos(lx+my)\}$, where A' and h are arbitrary constants.

Ex. 6. *Solve* $(D^2 - DD' - 2D)z = \sin(3x + 4y)$. **[Delhi Maths (Hons.) 1997]**

Sol. The given equation can be re–written as $D(D - D' - 2)z = \sin(3x + 4y)$.

$\therefore$ C.F. $= \phi_1(y) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and P.I. $= \dfrac{1}{D^2 - DD' - 2D}\sin(3x+4y) = \dfrac{1}{-3^2 + (3\times 4) - 2D}\sin(3x+4y) = \dfrac{1}{3-2D}\sin(3x+4y)$

$$= (3+2D)\frac{1}{9-4D^2} = \frac{3+2D}{9-4D^2}\sin(3x+4y) = \frac{3+2D}{9-4(-3^2)}\sin(3x+4y) = \frac{1}{45}[3\sin(3x+4y) + 2D\sin(3x+4y)]$$

$= (1/45) \times [3\sin(3x+4y) + 6\cos(3x+4y)] = (1/15) \times [\sin(3x+4y) + 2\cos(3x+4y)]$.

$\therefore$ Solution is $z = \phi_1(y) + e^{2x}\phi_2(y + x) + (1/15) \times [\sin(3x+4y) + 2\cos(3x+4y)]$

Ex. 7. *Solve* $(3DD' - 2D'^2 - D')z = \sin(2x+3y)$. **[Delhi Maths (H) 2001]**

Sol. Re–writing, given equation becomes $D'(3D - 2D' - 1)\,z = \sin(2x+3y)$

$\therefore$ C.F. $= \phi_1(x) + e^{x/3}\phi_2(2x+3y)$, ϕ_1, ϕ_2 are arbitrary functions

P.I. $= \dfrac{1}{3DD' - 2D'^2 - D'}\sin(2x+3y) = \dfrac{1}{-3\times(2\times 3) - 2\times(-3^2) - D'}\sin(2x+3y)$

$= -(1/D')\sin(2x+3y) = (1/3)\times\cos(2x+3y)$

$\therefore$ General solution is $z = \phi_1(x) + e^{x/3}\phi_2(2x+3y) + (1/3)\times\cos(2x+3y)$.

Ex. 8. $(D+D')\,(D+D'-2)z = \sin(x+2y)$ **[Delhi Maths (H) 2000]**

Sol. Here C.F. $= \phi_1(y-x) + e^{2x}\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary function

P.I. $= \dfrac{1}{(D+D')\,(D+D'-2)}\sin(x+2y) = \dfrac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'}\sin(x+2y)$

$= \dfrac{1}{-1^2 - 2\times(1\times 2) - 2^2 - 2D - 2D'}\sin(x+2y)$

$= -\dfrac{1}{9 + 2(D+D')}\sin(x+2y) = -\{9 - 2\,(D+D')\}\cdot\dfrac{1}{81 - 4\,(D+D')^2}\sin(x+2y)$

$= \dfrac{-(9-2D-2D')}{81-4D^2-4D'^2-8DD'}\sin(x+2y) = -\dfrac{9-2D-2D'}{81-4(-1^2)-4(-2^2)+8\times(1\times 2)}\sin(x+2y)$

$= (1/117)\times\{-9\sin(x+2y) + 2\cos(x+2y) + 4\cos(x+2y)\} = (3/117)\times\{2\cos(x+2y) - 3\sin(x+2y)\}$

$\therefore$ Solution is $z = \phi_1(y-x) + e^{2x}\,\phi_2(y-x) + (3/117)\times\{2\cos(x+2y) - 3\sin(x+2y)\}$

Ex. 9. *Solve* $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x + y)$

Sol. Re–writing the given equation, we get $(D + D')(D - 2D' + 2)z = \sin(2x + y)$.

$\therefore$ C.F. = $\phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\sin(2x + y) = \frac{1}{-2^2 + (2 \times 1) - 2 \times (-1^2) + 2D + 2D'}\sin(2x + y)$$

$$= \frac{1}{2(D + D')}\sin(2x + y) = \frac{D - D'}{2}\frac{1}{D^2 - D'^2}\sin(2x + y) = \frac{D - D'}{2}\frac{1}{-2^2 - (-1^2)}\sin(2x + y)$$

$$= -(1/6) \times (D - D')\sin(2x + y) = -(1/6) \times \{2\cos(2x + y) - \cos(2x + y)\}$$

$\therefore$ General solution is $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/6) \times \cos(2x + y)$

EXERCISE 5 (D)

Solve the following partial differential equations:

1. $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$. **[MDU Rohtak 2005]**

Ans. $z = \phi_1(x) + e^{3x/2}\phi_2(2y - x) + (3/50) \times \{4\cos(3x - 2y) + 3\sin(3x - 2y)\}$

2. $(D^2 + D')(D - D - D'^2)z = \sin(2x + y)$ **Ans.** $z = \sum_i A_i\, e^{a_i x - a_i^2 y} + \sum_i B_i\, e^{(b_i + b_i^2)x + b_i y}$

$-(1/34) \times \{5\sin(2x + y) - 3\cos(2x + y)\}$, which A_i, a_i, B_i, b_i are arbitrary constants

Type 3. Examples based on case III of Art. 5.12.

Ex. 1. *Solve* $s + p - q = z + xy$. **[Delhi Maths (Hons.) 1995, I.A.S. 1991]**

Sol. The given equation can be rewritten as $(\partial^2 z/\partial x\partial y) + (\partial z/\partial x) - (\partial z/\partial y) - z = xy$

or $(DD' + D - D' - 1)z = xy$ or $(D - 1)(D' + 1)z = xy$. ...(1)

$\therefore$ C.F. = $e^x \phi_1(y) + e^{-y}\phi_2(x)$, ϕ_1 and ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D - 1)(D' + 1)}xy = -\frac{1}{(1 - D)(1 + D')}xy = -(1 - D)^{-1}(1 + D')^{-1}xy$$

$$= -(1 + D + ...)(1 - D' + ...)xy = -(1 + D - D' - DD' + ...)xy = -xy - y + x + 1.$$

$\therefore$ The required solution is $z = e^x \phi_1(y) + e^{-y}\phi_2(x) - xy - y + x + 1$.

Ex. 2. *Solve* (*a*) $r - s + 2q - z = x^2y^2$. **[I.A.S. 1993]**

(*b*) $(D^2 - D' - 1)z = x^2y$.

Sol. (*a*) The given equation can be re–written as

$(\partial^2 z/\partial x^2) - (\partial^2 z/\partial x\partial y) + 2(\partial z/\partial y) - z = x^2y^2$ or $(D^2 - DD' + 2D' - 1)z = x^2y^2$. ...(1)

Since $(D^2 - DD' + 2D' - 1)$ cannot be resolved into linear factors in D and D', hence C.F. of (1) is obtained by considering the equation $(D^2 - DD' + 2D' - 1)z = 0$. ...(2)

Let a trial solution of (2) be $z = Ae^{hx + ky}$ ...(3)

$\therefore$ $D^2z = Ah^2e^{hx + ky}$, $DD'z = Ahke^{hx + ky}$, $D'z = Ake^{hx + ky}$. Then (2) gives

$A(h^2 - hk + 2k - 1)e^{hx + ky} = 0$ or $h^2 - hk + 2k - 1 = 0$

so that $k = (1 - h^2)/(2 - h)$. ...(4)

$\therefore$ C.F. = $\Sigma Ae^{hx + ky}$, where A, h, k are arbitrary constants; h, k being related by (4).

$$\text{P.I.} = \frac{1}{D^2 - DD' + 2D' - 1} x^2y^2 = -\frac{1}{1-(D^2 - DD' + 2D')} x^2y^2 = -[1 - (D^2 - DD' + 2D')]^{-1} x^2y^2$$

$$= -[1 + (D^2 - DD' + 2D') + (D^2 - DD' + 2D')^2 + \{D^2 + D'(2 - D)\}^3 + \ldots]x^2y^2$$

$$= -[1 + (D^2 - DD' + 2D') + (D^2D'^2 + 4D'^2 + 4D^2D' - 4DD'^2 + \ldots) + 3D^2D'^2(2 - D)^2 + \ldots]x^2y^2$$

$$= -(1 + D^2 - DD' + 2D' + D^2D'^2 + 4D'^2 + 4D^2D' - 4DD'^2 + 12D^2D'^2 + \ldots)x^2y^2$$

$$= -x^2y^2 - 2y^2 + 4xy - 4x^2y - 8x^2 - 16x - 16y - 52.$$

$\therefore$ Solution is $z = \Sigma A e^{hx + ky} - x^2y^2 - 2y^2 + 4xy - 4x^2y - 8x^2 - 16x - 16y - 52.$

(*b*) **Ans.** $z = \Sigma A e^{hx + (h^2 - 1)y} + x^2 - x^2y - 2y + 4$, A and h being arbitrary constants.

Ex. 3. *Solve* (*a*) $(D^2 - D')z = 2y - x^2$. **[Delhi Maths (H.) 2004, 10; Agra 2005]**

(*b*) $(2D^2 - D'^2 + D)z = x^2 - y$.

Sol. (*a*) Here $D^2 - D'$ cannot be resolved into linear factors in D and D'. Hence to find C.F., we consider the equation $(D^2 - D')z = 0$. ...(1)

Let a trial solution of (1) be $z = Ae^{hx + ky}$. ...(2)

So $D^2z = Ah^2e^{hx + ky}$ and $D'z = Ake^{hx + ky}$. Then (1) gives

$A(h^2 - k)e^{hx + ky} = 0$ or $h^2 - k = 0$ so that $k = h^2$.

$\therefore$ C.F. $= \Sigma A e^{hx + ky} = \Sigma A e^{hx + h^2y}$, A, h being arbitrary constants.

$$\text{Now, P.I.} = \frac{1}{D^2 - D'}(2y - x^2) = \frac{1}{D^2(1 - D'/D^2)}(2y - x^2) = \frac{1}{D^2}\left(1 - \frac{D'}{D^2}\right)^{-1}(2y - x^2)$$

$$= \frac{1}{D^2}\left(1 + \frac{D'}{D^2} + \ldots\right)(2y - x^2) = \frac{1}{D^2}\left\{(2y - x^2) + \frac{1}{D^2}D'(2y - x^2)\right\} = \frac{1}{D^2}\left(2y - x^2 + \frac{1}{D^2}2\right)$$

$$= \frac{1}{D^2}\left(2y - x^2 + 2 \times \frac{x^2}{2}\right) = \frac{1}{D^2}(2y) = 2y \times \frac{x^2}{2} = x^2y.$$

General solution is $z = \Sigma A e^{hx + h^2y} + x^2y$, A and h being arbitrary constants.

(*b*) **Ans.** $z = \Sigma A e^{hx + ky} - (1/2) \times x^2y^2 + (1/6) \times y^3 - (1/12) \times xy^4 - (1/6) \times y^4 - (1/360) \times y^6$, where h and k are connected by the relation $2h^2 - k^2 + h = 0$

Ex. 4. *Solve* $r - s + p = 1$. **[Meerut 1993; Sagar 2003; Vikram 2004]**

Sol. The given equation can be re–written as $(\partial^2z/\partial x^2) - (\partial^2z/\partial x\partial y) + (\partial z/\partial x) = 1$

or $(D^2 - DD' + D)z = 1$ or $D(D - D' + 1)z = 1$. ...(1)

$\therefore$ C.F. $= \phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $$\text{P.I.} = \frac{1}{D(1 + D - D')}1 = \frac{1}{D}(1 + D - D')^{-1}\,1 = \frac{1}{D}[1 - (D - D') + \ldots]1 = \frac{1}{D}1 = x$$

So the required general solution is $z = \phi_1(y) + e^{-x}\phi_2(y + x) + x$.

Ex. 5. *Solve* (*a*) $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$.

(*b*) $(D + D' - 1)(D + 2D' - 3)z = 2x + 3y$. **[Bhopal 2000, 03, 04]**

Sol. (*a*) Here C.F. $= \phi_1(y) + e^x\phi_2(y - x) + e^{2x}\phi_3(y - 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

$$\text{P.I.} = \frac{1}{D(D + D' - 1)(D + 3D' - 2)}(x^2 - 4xy + 2y^2) = \frac{1}{2D}\{1 - (D + D')\}^{-1}\left\{1 - \frac{D + 3D'}{2}\right\}^{-1}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\{1+(D+D')+(D+D')^2+\ldots\}\left\{1+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\ldots\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2 + \frac{(D+D')(D+3D')}{2}+\ldots\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left(1+\frac{3D}{2}+\frac{5D'}{2}+\frac{7D^2}{4}+\frac{19D'^2}{4}+\frac{11DD'}{2}+\ldots\right)(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{(x^2-4xy+2y^2)+3(x-2y)+5(2y-2x)+\frac{7}{2}+19-22\right\}$$

$$= \frac{1}{2D}\left(x^2-4xy+2y^2-7x+4y+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{x^3}{3}-2x^2y+2y^2x-\frac{7x^2}{2}+4xy+\frac{x}{2}\right)$$

Hence the required general solution is z = C.F. + P.I., *i.e.*,
$z = \phi_1(y) + e^x\phi_2(y-x) + e^{2x}\phi_3(y-3x) - (x^3/6) + x^2y + y^2x - (7x^2/4) + 2xy + x/4$.

(*b*) Try yourself. **Ans.** $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-2x) + 2x/3 + y + 23/9$

Ex. 6. *Solve* $(D-1)(D-D'+1)z = 1+xy$. **[Delhi Maths (H) 2001]**

Sol. Here C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$, where ϕ_1 and ϕ_2 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D-1)(D-D'+1)}(1+xy) = -(1-D)^{-1}\{1+(D-D')\}^{-1}(1+xy)$$

$$= -(1+D+\ldots)\{1-(D-D')+(D-D')^2\ldots\}(1+xy) = -(1+D+\ldots)(1-D+D'-2DD'+\ldots)(1+xy)$$

$$= -(1+D'-DD'+\ldots)(1+xy) = -(1+xy+x-1) = -xy-x$$

$\therefore$ The required solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) - xy - x$

Ex. 7. *Solve* $(D^2-D'^2-3D+3D')z = xy$.

Sol. Re–writing, given equation is $(D-D')(D+D'-3)z = xy$.

Its C.F. = $\phi_1(y+x) + e^{3x}\phi_2(x-y)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-3)} = -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\ldots\right)\left\{1+\frac{D+D'}{3}+\frac{(D+D')^2}{9}+\ldots\right\}xy = -\frac{1}{3D}\left(1+\frac{D'}{D}+\ldots\right)\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+\ldots\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+\frac{D'}{D}+\frac{1}{3}D'+\ldots\right)xy$$

$$= -\frac{1}{3D}\left(xy+\frac{1}{3}y+\frac{2}{3}x+\frac{2}{9}+\frac{x^2}{2}\right) = -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{2x}{9}+\frac{x^3}{6}\right)$$

$\therefore$ solution is $z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - (1/6)\times x^2y - (x^2/9) - (2x/27) - (x^3/18)$

Ex. 8. Solve $(D+D'-1)(D+2D'-3)z = 4+3x+6y.$

Sol. Here C.F. $= e^x\phi_1(y-x)+e^{3x}\phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D+D'-1)(D+2D'-1)}(4+3x+6y) = \frac{1}{3}\{1-(D+D')\}^{-1}\left\{1-\frac{D+2D'}{3}\right\}^{-1}(4+3x+6y)$$

$$= \frac{1}{3}(1+D+D'+\ldots)\left(1+\frac{D}{3}+\frac{2D'}{3}+\ldots\right)(4+3x+6y) = \frac{1}{3}\left(1+\frac{4D}{3}+\frac{5D'}{3}+\ldots\right)(4+3x+6y)$$

$$= (1/3)\times(4+3x+6y+4+10) = 6+x+2y.$$

$\therefore$ General solution is $z = e^x\phi_1(y-x)+e^{3x}\phi_2(y-2x)+6+x+2y.$

EXERCISE 5(E)

Solve the following partial differential equations:

1. $(D-D'-1)(D-D'-2)z = x.$ **Ans.** $z = e^x\phi_1(y+x)+e^{2x}\phi_2(y+x)+(2x+3)/4$

2. $(D^2-DD'-2D'^2+2D+2D')z = xy.$ **Ans.** $z = \phi_1(y-x)+e^{-2x}\phi_2(y+2x)+(1/4)\times(x^2y-xy-2x)$ $+(3x^2)/8-(x^3/12)$, ϕ_1, ϕ_2 being arbitrary functions

3. $(D^2-D'^2+D+3D'-2)z = x^2y$ **Ans.** $z = e^{-2x}\phi_1(y+x)+e^x\phi_2(y-x)-(4x^2y+4xy+6x^2+6y$ $+12x+21)/8$, ϕ_1, ϕ_2 being arbitrary functions.

Type 4. Examples based on case IV of Art. 5.12.

Ex. 1. *Solve* $(D^2-D')z = xe^{ax+a^2y}$.

Sol. Since (D^2-D') cannot be resolved into linear factors in D and D', C.F. is obtained by considering the equation $(D^2-D')z = 0.$ ...(1)

Let a trial solution of (1) be $z = Ae^{hx+ky}.$ ...(2)

$\therefore$ $D^2z = Ah^2e^{hx+ky}$ and $D'z = Ake^{hx+ky}.$ Then (1) becomes

$A(h^2-k)e^{hx+ky} = 0$ so that $h^2-k = 0$ or $k = h^2.$

$\therefore$ From (2), C.F. $= \Sigma A e^{hx+h^2y}$, A, h being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-D'}xe^{ax+a^2y} = e^{ax+a^2y}\frac{1}{(D+a)^2-(D'+a^2)}x = e^{ax+a^2y}\frac{1}{D^2+2aD-D'}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left(1+\frac{D}{2a}-\frac{D'}{2aD}\right)^{-1}x = e^{ax+a^2y}\frac{1}{2aD}\left\{1-\left(\frac{D}{2a}-\frac{D'}{2aD}\right)+\ldots\right\}x = e^{ax+a^2y}\frac{1}{2aD}\left(x-\frac{1}{2a}\right)$$

$$= e^{ax+a^2y}\left\{(x^2/4a)-(x/4a^2)\right\}.$$

$\therefore$ General solution is $z = \sum Ae^{hx+h^2y}+e^{ax+a^2y}\{(x^2/4a)-(x/4a^2)\}.$

Ex. 2. *Solve* $(D-3D'-2)^2z = 2e^{2x}\sin(y+3x)$ **[I.A.S. 2005]**

Sol. Here C.F. $= e^{2x}[\phi_1(y+3x)+x\phi_2(y+3x)]$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2}2e^{2x+0.y}\sin(y+3x) = 2e^{2x+0.y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\sin(y+3x)$$

$= 2e^{2x}\dfrac{1}{(D'-3D')^2}\sin(y+3x) = 2e^{2x}\dfrac{x^2}{1^2\,2!}\sin(y+3x)$, using formula (*ii*)of Art. 4.8

$\therefore$ Required solution is $z = e^{2x}[\phi_1(y+3x) + x\phi_2(y+3x)] + x^2e^{2x}\sin(y+3x)$.

Ex. 3. *Solve* $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x\partial y} + 4\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} = e^{x+y}$.

Sol. Given $(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$ or $(D-2D')(D-2D'+1)z = e^{x+y}$.

$\therefore$ C.F. $= \phi_1(y+2x) + e^{-x}\phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I. $= \dfrac{1}{(D-2D'+1)}\left[\dfrac{1}{D-2D'}e^{x+y}\right] = \dfrac{1}{D-2D'-1}\,\dfrac{1}{1-2}e^{x+y}$, by case I of Art. 5.12

$= -e^{x+y}\dfrac{1}{(D+1)-2(D'+1)+1}.1$, using result of case IV of Art. 5.12

$= -e^{x+y}\dfrac{1}{D-2D'}1 = -e^{x+y}\dfrac{1}{D}\left(1-\dfrac{2D'}{D}\right)^{-1}1 = -e^{x+y}\dfrac{1}{D}1 = -xe^{x+y}$

$\therefore$ The required solution is $z = \phi_1(y+2x) + e^{-x}\phi_2(y+2x) - xe^{x+y}$.

Ex. 4. *Solve* $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x+y)$. **[Delhi Maths (H) 1999, 2008]**

Sol. Since $(3D^2 - 2D'^2 + D - 1)$ cannot be resolved into linear factors in D and D', hence

C.F. $= \Sigma Ae^{hx+ky}$, where A, h are arbitrary constants connected by $3h^2 - 2k^2 + h - 1 = 0$.

P.I. $= \dfrac{1}{3D^2 - 2D'^2 + D - 1}4e^{x+y}\cos(x+y) = 4e^{x+y}\dfrac{1}{3(D+1)^2 - 2(D'+1)^2 + (D+1) - 1}\cos(x+y)$

$= 4e^{x+y}\dfrac{1}{3D^2 + 7D - 2D'^2 - 4D' + 1}\cos(x+y) = 4e^{x+y}\dfrac{1}{3(-1^2) + 7D - 2(-1^2) - 4D' + 1}\cos(x+y)$

$= 4e^{x+y}\dfrac{1}{7D - 4D'}\cos(x+y) = 4e^{x+y}(7D+4D')\dfrac{1}{49D^2 - 16D'^2}\cos(x+y)$

$= 4e^{x+y}\dfrac{7D+4D'}{49(-1^2) - 16(-1^2)}\cos(x+y)$

$= -(4/33)\times e^{x+y}(7D+4D')\cos(x+y) = -(4/33)e^{x+y}\times[7D\cos(x+y) + 4D'\cos(x+y)]$

$= -(4/33)\times e^{x+y}[-7\sin(x+y) - 4\sin(x+y)] = (4/3)\times e^{x+y}\sin(x+y)$.

Hence general solution is $z = \Sigma Ae^{hx+ky} + (4/3)\times e^{x+y}\sin(x+y)$.

Ex. 5. *Solve* $(D-1)(D-D'+1)z = e^y$. **[Delhi Maths (H) 2001]**

Sol. Here C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I. $= \dfrac{1}{(D-1)(D-D'+1)}e^y = \dfrac{1}{D-D'+1}\,\dfrac{1}{D-1}\,e^{0\cdot x+1\cdot y} = \dfrac{1}{D-D'+1}\,\dfrac{1}{0-1}\,e^{0\cdot x+1\cdot y}\,1$

$= -e^{0\cdot x+1\cdot y}\dfrac{1}{(D+0)-(D'+1)+1}1 = -e^y\dfrac{1}{D}\left(1-\dfrac{D'}{D}\right)^{-1}1 = -e^y\dfrac{1}{D}\left(1+\dfrac{D'}{D}+....\right) = -xe^y$

$\therefore$ General solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) - x\,e^y$

Ex. 6. *Solve* $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$. **[Delhi Maths (H) 2004]**

Sol. Here C.F. = $e^{2x}\{\phi_1(y+3x)+x\,\phi_2(y+3x)\}$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2}\,2\,e^{2x}\tan(y+3x) = 2\frac{1}{(D-3D'-2)^2}\,e^{2x+0\cdot y}\tan(y+3x)$$

$$= 2e^{2x+0\cdot y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\tan(y+3x) = 2e^{2x}\frac{1}{(D-3D')^2}\tan(y+3x)$$

$= 2\,e^{2x}\times(x^2/2!)\tan(y+3x)$, refer formula (*ii*), of Art. 4.12 of chapter 4

$\therefore$ General solution is $z = e^{2x}\{\phi_1(y+3x)+x\,\phi_2(y+3x)\}+x^2e^{2x}\tan(y+3x)$

Ex. 7. *Solve* $(D^2-D'^2-3D+3D')z = e^{x+2y}$. **[Delhi Maths (H) 2005]**

Sol. Re–writing the given equation, we get $(D-D')(D+D'-3)z = e^{x+2y}$.

Its C.F. $= \phi_1(y+x)+e^{3x}\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D+D'-3}\left\{\frac{1}{D-D'}\,e^{x+2y}\right\} = \frac{1}{D+D'-3}\,\frac{1}{(1-2)}\,e^{x+2y}\cdot 1$$

$$= -e^{x+2y}\frac{1}{D+1+D'+2-3}1 = -e^{x+2y}\frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1}1$$

$$= -e^{x+2y}(1/D)\,(1-D'/D+.....)\,1 = -e^{x+2y}x$$

$\therefore$ General solution is $z = \phi_1(y+x)+e^{3x}\phi_2(y-x)-x\,e^{x+2y}$

Ex. 8. *Solve* (*i*) $(D^2-D')z = e^{x+y}$

(*ii*) $(D^2-D')z = e^{2x+y}$.

Sol. (*i*) C.F. = $\sum A\,e^{hx+ky}$, where $h^2-k=0$ so that $k=h^2$

$\therefore$ $\text{C.F.} = \sum A\,e^{hx+h^2y} = \sum A\,e^{h(x+hy)}$

$$\text{P.I.} = \frac{1}{D^2-D'}e^{x+y}\cdot 1 = e^{x+y}\frac{1}{(D+1)^2-(D'+1)}1 = e^{x+y}\frac{1}{D^2+2D-D'}1$$

$$= -e^{x+y}\frac{1}{D'}\left(1-\frac{D^2+2D}{D'}\right)^{-1}1 = -e^{x+y}\frac{1}{D'}\left(1+\frac{D^2+2D}{D'}+....\right)1 = -e^{x+y}\frac{1}{D'}1 = -e^{x+y}y$$

$\therefore$ The required solution is $z = \sum A\,e^{h(x+hy)} - y\,e^{x+y}$, where A and h are arbitrary constants

(*ii*) C.F. is same as in part (*i*). Its P.I. is given by

$$\text{P.I.} = \frac{1}{D^2-D'}\,e^{2x+y} = \frac{1}{2^2-1}\,e^{2x+y} = \frac{1}{3}\,e^{2x+y}$$

$\therefore$ General solution is $z = \sum A\,e^{h(x+hy)} - (1/3)\times e^{2x+y}$, A, h being arbitrary constants.

Ex. 9. *Solve* $(D+D'-1)\,(D+D'-3)\,(D+D')z = e^{x+y}\sin(2x+y)$

Sol. C.F. = $e^x\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+D'-1)\,(D+D'-3)\,(D+D')}\ e^{x+y}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+1+D'+1-1)\,(D+1+D'+1-3)\,(D+1+D'+1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+1)\,(D+D'-1)\,(D+D'+2)}\sin(2x+y) = e^{x+y}\frac{1}{(D+D'+2)\,(D^2+2DD'+D'^2-1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{D+D'+2}\,\frac{1}{-2^2-2\times(2\times1)-1^2-1}\sin(2x+y) = -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{(D+D')^2-4}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{D^2+2DD'+D'^2-4}\sin(2x+y) = -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{-2^2-2\times(2\times1)-1^2-4}\sin(2x+y)$$

$$= (1/130)\times e^{x+y}(D+D'-2)\sin(2x+y) = (1/130)\times e^{x+y}\{2\cos(2x+y)+\cos(2x+y)-2\sin(2x+y)\}$$

$\therefore$ Solution is $z = e^x\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x)+(1/130)\times e^{x+y}\{3\cos(2x+y)-2\sin(2x+y)\}$

Ex. 10. *Solve* $r-3s+2t-p+2q=(2+4x)e^{-y}$

Sol. Re–writing the given equation $(D^2-3DD'+2D'^2-D+2D')z=(2+4x)\,e^{-y}$

or $(D-2D')\,(D-D'-1)z=(2+4x)\,e^{-y}$

$\therefore$ C.F. = $\phi_1(y+2x)+e^x\phi_2(y+x)$, where ϕ_1, ϕ_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-2D')\,(D-D'-1)}2\,e^{0\cdot x-y}(1+2x) = 2\,e^{0\cdot x-y}\frac{1}{\{D+0-2(D'-1)\}\,\{D+0-(D'-1)-1\}}(1+2x)$$

$$= 2e^{-y}\frac{1}{(D-2D'+2)\,(D-D')}(1+2x) = 2e^{-y}\,\frac{1}{2D}\left(1+\frac{D-2D'}{2}\right)^{-1}\left(1-\frac{D'}{D}\right)^{-1}(1+2x)$$

$$= e^{-y}\frac{1}{D}\left\{1-\frac{1}{2}(D-2D')+\ldots\right\}\left(1+\frac{D'}{D}+\ldots\right)(1+2x) = e^{-y}\frac{1}{D}(1-\frac{D}{2}+\ldots.)(1+2x)$$

$$= e^{-y}(1/D)\,(1+2x-1) = x^2e^{-y}$$

$\therefore$ The required solution is $z=\phi_1(y+2x)+e^x\phi_2(y+x)+x^2e^{-y}$.

Ex. 11. *Solve* $(D^2-D')z=xe^{x+y}$

Sol. As usual C.F. = $\sum A\,e^{hx+ky}$, when $h^2-k=0$ or $k=h^2$. So C.F. $=\Sigma Ae^{hx+h^2y}$

$$\text{P.I.} = \frac{1}{D^2-D'}\,x\,e^{x+y} = e^{x+y}\frac{1}{(D+1)^2-(D'+1)}x = e^{x+y}\frac{1}{D^2+2D-D'}x$$

$$= e^{x+y}\frac{1}{2D}\left(1+\frac{D^2-D'}{2D}\right)^{-1}x = e^{x+y}\frac{1}{2D}\left(1-\frac{D^2-D'}{2D}+\ldots\right)x = e^{x+y}\frac{1}{2D}\left(1-\frac{D}{2}+\frac{D'}{2D}+\ldots.\right)x$$

$$= e^{x+y}(1/2D)(x-1/2) = e^{x+y}(x^2/4-x/4).$$

$\therefore$ Solution is $z=\sum A\,e^{hx+h^2y}+(x/4)\times(x-1)\,e^{x+y}$, A, h being arbitrary constants.

EXERCISE 5(F)

Solve the following partial differential equations:

1. $D(D-2D')\,(D+D')z = e^{x+2y}(x^2+4y^2)$. **Ans.** $z = \phi_1(y)+\phi_2(y+2x)+\phi_3(y-x)$

$-(1/81)\times(9x^2+36y^2-18x-72y+76)e^{x+2y}$ ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

2. $(D^2+DD'+D+D'-1)z = e^{-2x}(x^2+y^2)$.

Ans. $z = \Sigma A e^{hx+ky}+(1/27)\times e^{-2x}(9x^2+9y^2+18x+6y+14)$, where $h^2 + hk + h + k + 1 = 0$.

3. $(D^2D'+D'^2-2)z = e^{2y}\sin 3x - e^x\cos y$. **Ans.** $z = \Sigma A e^{hx+ky} - (1/16)\times e^{2y}\sin 3x$

$+(1/20)\times e^x(3\cos 2y - \sin 2y)$, where h and k are related by $h^2k + k^2 - 2 = 0$.

4. $(D^2-DD'+D'-1)z = e^y$. **Ans.** $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)-xe^y$

5. $(D^2-DD'+D'-1)z = e^x$ **Ans.** $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)-(1/2)\times xe^x$

MISCELLANEOUS EXAMPLES ON ART. 5.12.

Ex. 1. *Solve* (*a*) $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y$. **[Jabalpur 2004; I.A.S. 1992]**

(*b*) $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^x$.

Sol. (*a*) From given equation $(D-1)(D-D'+1)z = \cos(x+2y)+e^y$. ...(1)

$\therefore$ C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\cos(x + 2y)$

$$= \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) = \frac{1}{-1^2 + (1\times 2) + D' - 1}\cos(x+2y)$$

$$= (1/D')\cos(x+2y) = (1/2)\times\sin(x+2y).$$

P.I. corresponding to e^y

$$= \frac{1}{(D-1)(D-D'+1)}e^y = \frac{1}{D-D'+1}.\frac{1}{D-1}e^{0\cdot x+1\cdot y} = \frac{1}{D-D'+1}.\frac{1}{0-1}e^{0\cdot x+1\cdot y}$$

$$= -e^{0\cdot x+1\cdot y}\frac{1}{(D+0)-(D'+1)+1}1 = -e^y\frac{1}{D(1-D'/D)}1 = -e^y\frac{1}{D}\left(1-\frac{D'}{D}+\ldots\right)^{-1}1$$

$$= -e^y(1/D)(1+\ldots).1 = -e^y x.$$

$\therefore$ The general solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + (1/2)\times\sin(x+2y) - xe^y$.

(*b*) As in part (*a*), C.F. $= e^x\,\phi_1(y) + e^{-x}\,\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary function.

and P.I. corresponding to $\cos(x + 2y) = (1/2)\times\sin(x+2y)$.

Now, P.I. corresponding to e^x

$$= \frac{1}{(D-1)\,(D-D'+1)}e^x = \frac{1}{(D-1)\,(D-D'+1)}e^{1\cdot x+0\cdot y} = \frac{1}{(D-1)}\frac{1}{(1-0+1)}e^{1\cdot x+0\cdot y}$$

$$= \frac{1}{2}\frac{1}{D-1}e^{1\cdot x+0\cdot y}\,1 = \frac{1}{2}e^{1\cdot x+0\cdot y}\frac{1}{(D+1)-1}1 = \frac{1}{2}e^x\frac{1}{D}1 = \frac{xe^x}{2}.$$

$\therefore$ The general solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + (1/2)\times\sin(x+2y) + (x/2)\times e^x$.

Ex. 2. *Solve* (*a*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy + \sin(2x + y)$. **[Delhi 2008]**

(*b*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy$.

(*c*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy + \sin(2x + y)$.

Sol. (*a*) The given equation can be rewritten as

$$(D + D')(D - 2D' + 2)z = e^{2x+3y} + xy + \sin(2x + y). \quad ...(1)$$

$\therefore$ C.F. $= \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to e^{2x+3y}

$$= \frac{1}{(D + D')(D - 2D' + 2)} e^{2x+3y} = \frac{1}{(2 + 3)(2 - 6 + 2)} e^{2x+3y} = -\frac{1}{10} e^{2x+3y}.$$

P.I. corresponding to xy

$$= \frac{1}{(D + D')(D - 2D' + 2)} xy = \frac{1}{D(1 + D'/D) \times 2\{1 + (D/2 - D')\}} xy$$

$$= \frac{1}{2D}\left(1 + \frac{D'}{D}\right)^{-1}\left\{1 + \left(\frac{D}{2} - D'\right)\right\}^{-1} xy = \frac{1}{2D}\left(1 - \frac{D'}{D} + \ldots\right)\left\{1 - \left(\frac{D}{2} - D'\right) + \left(\frac{D}{2} - D'\right)^2 + \ldots\right\} xy$$

$$= \frac{1}{2D}\left(1 - \frac{D'}{D} + \ldots\right)\left(1 - \frac{D}{2} + D' - DD' + \ldots\right) xy = \frac{1}{2D}\left(1 - \frac{D'}{D} + \ldots\right)\left(xy - \frac{y}{2} + x - 1\right)$$

$$= \frac{1}{2D}\left[xy - \frac{y}{2} + x - 1 - \frac{1}{D}\left(x - \frac{1}{2}\right)\right] = \frac{1}{2D}\left[xy - \frac{y}{2} + x - 1 - \frac{x^2}{2} + \frac{x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^2 y}{2} - \frac{xy}{2} + \frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{4}\right] = \frac{x^2 y}{4} + \frac{3x^2}{8} - \frac{xy}{4} - \frac{x}{2} - \frac{x^3}{12}.$$

P.I. corresponding to $\sin(2x + y)$

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x + y) = \frac{1}{-2^2 + (2 \times 1) - 2 \times (-1^2) + 2D + 2D'} \sin(2x + y)$$

$$= \frac{1}{2(D + D')} \sin(2x + y) \quad = \frac{1}{2}(D - D')\frac{1}{(D^2 - D'^2)} \sin(2x + y)$$

$$= \frac{1}{2}\frac{1}{-2^2 - (-1^2)}(D - D') \sin(2x + y)$$

$$= -(1/6) \times (D - D') \sin(2x + y) = -(1/6) \times [D \sin(2x + y) - D' \sin(2x + y)]$$

$$= -(1/6) \times [2 \cos(2x + y) - \cos(2x + y)] = -(1/6) \times \cos(2x + y).$$

The required solution is $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/10) \times e^{2x+3y} + (1/4) \times x^2 y + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12) - (1/6) \times \cos(2x + y)$

(*b*) As in part (*a*), C.F. $= \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$.

P.I. corresponding to $e^{2x+3y} = -(1/10) \times e^{2x+3y}$

P.I. corresponding to $xy = (1/4) \times x^2 y + (3/8) \times x^2 - (1/4) \times xy - (1/2) \times x - (1/12) \times x^3$.

$\therefore$ The required general solution is z = C.F. + P.I., *i.e.*

$z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/10) \times e^{2x+3y} + (1/4) \times x^2 y + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12)$.

(*c*) As in part (*a*), C.F. $= \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$.

P.I. corresponding to $xy = (1/4) \times x^2 + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12)$
and P.I. corresponding to $\sin(2x + y) = -(1/6) \times \cos(2x + y)$.

$\therefore$ The required solution is $z = \text{C.F.} + \text{P.I.}$, *i.e.*, $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) + (1/4) \times x^2 + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12) \times - (1/6) \times \cos(2x + y)$.

Ex. 3. *Find a particular integral of the differential equation* : $(D^2 - D')z = e^{x+y} + 5\cos(x + 2y)$.

Sol. P.I. corresponding to e^{x+y}

$$= \frac{1}{D^2 - D'} e^{x+y} = e^{x+y} \frac{1}{(D+1)^2 - (D'+1)} 1 = e^{x+y} \frac{1}{D^2 + 2D - D'} 1$$

$$= e^{x+y} \frac{1}{2D}\left[1 + \left(\frac{D}{2} - \frac{D'}{2D}\right)\right]^{-1} 1 = e^{x+y} \frac{1}{2D}\{1 + \ldots\}1 = \frac{1}{2} x e^{x+y}$$

P.I. corresponding to $5\cos(x + 2y)$

$$= 5\frac{1}{D^2 - D'} \cos(x + 2y) = 5\frac{1}{-1^2 - D'} \cos(x + 2y) = -\frac{5}{D' + 1} \cos(x + 2y)$$

$$= -5(D' - 1)\frac{1}{D'^2 - 1} \cos(x + 2y) = -5\frac{1}{-2^2 - 1}(D' - 1)\cos(x + 2y)$$

$$= (D' - 1)\cos(x + 2y) = D'\cos(x + 2y) - \cos(x + 2y) = -2\sin(x + 2y) - \cos(x + 2y)$$

$\therefore$ Required P.I. $= (x/2) \times e^{x+y} - 2\sin(x + 2y) - \cos(x + 2y)$.

Ex. 4. Solve $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$. **[Delhi Maths (Prog) 2007; Delhi Maths (H) 2007; Meerut 1998; Bhopal 1995; Indore 1998; KU Kurukshetra 2004]**

Sol. The given equation can be re–written as $(D - D')(D + D' - 3)z = xy + e^{x+2y}$.

$\therefore$ C.F. $= \phi_1(y + x) + e^{3x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $xy = \frac{1}{(D - D')(D + D' - 3)} xy = -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 - \frac{D + D'}{3}\right)^{-1} xy$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\right)\left[1 + \frac{D + D'}{3} + \left(\frac{D + D'}{3}\right)^2 + \ldots\right] xy$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\right)\left(1 + \frac{D + D'}{3} + \frac{2DD'}{9} + \ldots\right) xy = -\frac{1}{3D}\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} + \ldots\right) xy$$

$$= -\frac{1}{3D}\left(xy + \frac{y}{3} + \frac{2x}{3} + \frac{1}{D}x + \frac{2}{9}\right) = -\frac{1}{9}\left(\frac{x^2 y}{2} + \frac{xy}{2} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2x}{9}\right)$$

P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D + D' - 3)} \frac{1}{D - D'} e^{x+2y} = \frac{1}{D + D' - 3} \frac{1}{(1 - 2)} e^{x+2y}$$

$$= -\frac{1}{D + D' - 3} e^{1 \cdot x + 2y} 1 = -e^{1 \cdot x + 2y} \frac{1}{(D + 1) + (D' + 2) - 3} 1 = -e^{x+2y} \frac{1}{D + D'} 1$$

$$= -e^{x+2y} \frac{1}{D}\left(1 + \frac{D'}{D}\right)^{-1} 1 = -e^{x+2y} \frac{1}{D}(1 + \ldots)1 = -x\, e^{x+2y}.$$

Hence the required general solution is $z = \text{C.F} + \text{P.I.}$, *i.e.*

$z = \phi_1(y + x) + e^{3x}\phi_2(y - x) - (x^2 y/6) - (xy/6) - (x^2/9) - (x^3/18) - (2x/27) - xe^{x+2y}$.

Ex. 5. *Solve* $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + x$. **[Meerut 2008]**

Sol. Here C.F. $= e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions

Now, P.I. corresponding to e^{2x-y}

$$= \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y} = \frac{1}{\{2-(-1)-1\}\{2-(-1)-2\}} e^{2x-y} = \frac{1}{2} e^{2x-y}$$

and P.I. corresponding to x

$$= \frac{1}{(D-D'-1)(D-D'-2)} x = \frac{1}{2\{1-(D-D')\}\{1-(D-D')/2\}} x$$

$$= \frac{1}{2}[1-(D-D')]^{-1}\left\{1-\frac{D-D'}{2}\right\}^{-1} x = \frac{1}{2}[1+(D-D')+\ldots]\left\{1+\frac{D-D'}{2}+\ldots\right\}x$$

$$= \frac{1}{2}\left\{1+(D-D')+\frac{D-D'}{2}+\ldots\right\}x = \frac{1}{2}\left\{1+\frac{3}{2}D+\ldots\right\}x = \frac{1}{2}\left(x+\frac{3}{2}\right).$$

$\therefore$ General solution is $z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x) + (1/2) \times e^{2x-y} + x/2 + 3/4.$

Ex. 6. *Solve* (*a*) $(D^2 - DD' - 2D)z = \sin(3x+4y) - e^{2x+y}$. **[Meerut 1995]**

(*b*) $(D^2 - DD' - 2D)z = \sin(3x+4y) + x^2y$ **[Agra 2009, 10]**

Sol. (*a*) The given equation can be re–written as $D(D-D'-2)z = \sin(3x+4y) - e^{2x+y}$.

$\therefore$ C.F. $= \phi_1(y) + e^{2x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\sin(3x+4y)$

$$= \frac{1}{D^2 - DD' - 2D}\sin(3x+4y) = \frac{1}{-3^2 + (3\times 4) - 2D}\sin(3x+4y)$$

$$= \frac{1}{3-2D}\sin(3x+4y) = (3+2D)\frac{1}{9-4D^2}\sin(3x+4y) = \frac{3+2D}{9-4(-3^2)}\sin(3x+4y)$$

$= (1/45) \times [3\sin(3x+4y) + 2D\sin(3x+4y)] = (1/45) \times [3\sin(3x+4y) + 6\cos(3x+4y)]$

and P.I. corresponding to $(-e^{2x+y})$

$$= -\frac{1}{D(D-D'-2)} e^{2x+y} = -\frac{1}{2(2-1-2)} e^{2x+y} = \frac{1}{2} e^{2x+y}$$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$, *i.e.*

$z = \phi_1(y) + e^{2x}\phi_2(y+x) + (1/15) \times [\sin(3x+4y) + 2\cos(3x+4y)] + (1/2) \times e^{2x+y}$

(*b*) As in part (*a*), C.F. $= \phi_1(y) + e^{2x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\sin(3x+4y) = (1/15) \times [\sin(3x+4y) + 2\cos(3x+4y)]$.

P.I. corresponding to $x^2y = \dfrac{1}{D(D-D'-2)} x^2y = -\dfrac{1}{2D}\left\{1-\left(\dfrac{D-D'}{2}\right)\right\}^{-1} x^2y$

$$= -\frac{1}{2D}\left\{1+\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2+\left(\frac{D-D'}{2}\right)^3+\ldots\right\}x^2y = -\frac{1}{2D}\left(1+\frac{D}{2}-\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}-\frac{3D^2D'}{8}+\ldots\right)x^2y$$

$$= -\frac{1}{2D}\left(x^2y+xy-\frac{x^2}{2}+\frac{y}{2}-x-\frac{3}{4}\right) = -\frac{1}{2}\left(\frac{x^3y}{3}+\frac{x^2y}{2}-\frac{x^3}{6}+\frac{xy}{2}-\frac{x^2}{2}-\frac{3x}{4}\right)$$

Hence the solution is $z = \text{C.F.} + \text{P.I.}$, *i.e.* $z = \phi_1(y) + e^{2x}\phi_2(y+x) + (1/15) \times [\sin(3x+4y)$
$+ 2\cos(3x+4y)] - (1/6) \times x^3y - (1/4) \times x^2y + (1/12) \times x^3 - (1/4) \times xy - (x^2/4) + 3x/8$

Ex. 7. Solve $(\partial^2z/\partial x^2) - (\partial^2z/\partial y^2) + (\partial z/\partial x) + 3(\partial z/\partial y) - 2z = e^{x-y} - x^2y$. **[Rewa 1999]**

Sol. The given equation can be re–written as $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2y$

or $\{(D-D')(D+D') + 2(D+D') - (D-D'+2)\}z = e^{x-y} - x^2y$

or $\{(D+D')(D-D'+2) - (D-D'+2)\}z = e^{x-y} - x^2y$ or $(D-D'+2)(D+D'-1)z = e^{x-y} - x^2y$.

$\therefore$ C.F. $= e^{-2x}\phi_1(y+x) + e^x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to e^{x-y}

$$= \frac{1}{(D-D'+2)(D+D'-1)}e^{x-y} = \frac{1}{\{1-(-1)+2\}(1-1-1)}e^{x-y} = -\frac{1}{4}e^{x-y}$$

and P.I. corresponding to $(-x^2y)$

$$= \frac{1}{(D-D'+2)(D+D'-1)}(-x^2y) = \frac{1}{2}\left\{1+\frac{D-D'}{2}\right\}^{-1}\{1-(D+D')\}^{-1}x^2y$$

$$= \frac{1}{2}\left\{1-\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2-\left(\frac{D-D'}{2}\right)^3+\ldots\right\}\times\{1+(D+D')+(D+D')^2+(D+D')^3+\ldots\}x^2y$$

$$= \frac{1}{2}\left(1-\frac{D}{2}+\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}+\frac{3D^2D'}{8}+\ldots\right)\times(1+D+D'+D^2+2DD'+3D^2D'+\ldots)x^2y$$

$= (1/2) \times [1 + (1/2) \times D + (3/2) \times D' + (3/4) \times D^2 + (3/2) \times DD' + (21/8) \times D^2D' + \cdots]x^2y$

$= (1/2) \times [x^2y + xy + (3x^2/2) + (3y/2) + 3x + 21/4]$.

Hence general solution is z = C.F. + P.I., *i.e.* $z = e^{-2x}\phi_1(y+x) + e^x\phi_2(y-x)$

$- (1/4) \times e^{x-y} + (1/2) \times x^2y + (1/2) \times xy + (3/4) \times x^2 + (3/4) \times y + (3/2) \times x + 21/8.$

Ex. 8. *Solve* $(D^2 - D')(D - 2D')z = e^{2x+y} + xy$.

Sol. C.F. corresponding to linear factor $(D - 2D')$ is $\phi(y + 2x)$. Now, $(D^2 - D')$ cannot be resolved into linear factor in D and D'. To find C.F. corresponding to it, we consider the equation

$$(D^2 - D')z = 0. \quad \ldots(1)$$

Let a trial solution of (1) be $z = Ae^{hx+ky}$. ...(2)

$\therefore D^2z = Ah^2e^{hx+ky}$ and $D'z = Ake^{hx+ky}$. Then (1) becomes

$A(h^2 - k)e^{hx+ky} = 0$ so that $h^2 - k = 0$ or $k = h^2$.

So from (2), C.F. corresponding to $(D^2 - D')$ is $\sum Ae^{hx+h^2y}$.

Now, P.I. corresponding to e^{2x+y}

$$= \frac{1}{D-2D'}\cdot\frac{1}{D^2-D'}e^{2x+y} = \frac{1}{D-2D'}\frac{1}{2^2-1}e^{2x+y} = \frac{1}{3}\frac{1}{D-2D'}e^{2x+y}\cdot 1$$

$$= \frac{1}{3}e^{2x+y}\frac{1}{(D+2)-2(D'+1)}1 = \frac{1}{3}e^{2x+y}\frac{1}{D(1-2D'/D)}1 = \frac{1}{3}e^{2x+y}\frac{1}{D}\left(1-\frac{2D'}{D}\right)^{-1}1$$

$= (1/3) \times e^{2x+y} \times (1/D)\ (1 + \ldots)\ 1 = (x/3) \times e^{2x+y}$

and P.I. corresponding to xy

$$= \frac{1}{(D-2D')(D^2-D')}xy = \frac{1}{(-2D')(1-D/2D')(-D')(1-D^2/D')}xy$$

$$= \frac{1}{2D'^2}\left(1-\frac{D}{2D'}\right)^{-1}\left(1-\frac{D^2}{D'}\right)^{-1}xy = \frac{1}{2D'^2}\left(1+\frac{D}{2D'}+\ldots\right)(1+\ldots)xy$$

$$= \frac{1}{2D'^2}\left(1+\frac{D}{2D'}+\ldots\right)xy = \frac{1}{2D'^2}\left(xy+\frac{1}{2D'}y\right) = \frac{1}{2D'^2}\left(xy+\frac{y^2}{4}\right) = \frac{1}{2}\left(\frac{xy^3}{6}+\frac{y^4}{3\times4\times4}\right).$$

$\therefore$ General solution $z = \phi(y + 2x) + \sum Ae^{hx+h^2y} + (x/3) \times e^{2x+y} + (xy^3)/12 + y^4/96,$

where ϕ is an arbitrary function and A and h are arbitrary constants.

Ex. 9. *Solve* $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y + xy + 1$.

Sol. The given equation can be re–written as

$$(D - 1)(D - D' + 1)z = \cos(x + 2y) + e^y + xy + 1. \quad ...(1)$$

Its C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. corresponding to $\cos(x + 2y)$

$$= \frac{1}{D^2 - DD' + D' - 1}\cos(x + y) = \frac{1}{-1^2 + (1\times 2) + D' - 1}\cos(x + 2y)$$

$$= (1/D')\cos(x + 2y) = (1/2)\times\sin(x + 2y),$$

P.I. corresponding to e^y *i.e.* $e^{0\cdot x + 1\cdot y}$

$$= \frac{1}{D^2 - DD' + D' - 1}e^{0\cdot x + 1\cdot y}\cdot 1 = e^{0\cdot x + 1\cdot y}\frac{1}{(D + 0)^2 - (D + 0)(D' + 1) + (D' + 1) - 1}1$$

$$= e^y\frac{1}{D^2 - DD' + D'}1 = e^y\frac{1}{D'}\left\{1 + \left(\frac{D^2}{D'} - D\right)\right\}^{-1}1 = e^y\frac{1}{D'}\{1 + \ldots\}1 = e^y y$$

and P.I. corresponding to $(xy + 1)$

$$= \frac{1}{(D - 1)(D - D' + 1)}(xy + 1) = -(1 - D)^{-1}\{1 + (D - D')\}^{-1}(xy + 1)$$

$$= -\{1 + D + \ldots\}\{1 - (D - D') + (D - D')^2 - \ldots\}(xy + 1) = -(1 + D + \ldots)(1 - D + D' - 2DD' + \ldots)(xy + 1)$$

$$= -(1 + D + \ldots)(xy + 1 - y + x - 2) = -(1 + D + \ldots)(xy - y + x - 1) = -(xy - y + x - 1 + y + 1) = -(xy + x)$$

$\therefore$ Solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (1/2)\times\sin(x + 2y) + ye^y - (xy + x)$.

EXERCISE 5(G)

Solve the following partial differential equations:

1. $(D^2 - DD' - 2D^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x + y)$ **[KU Kurukshetra 2004]**

Ans. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/10)\times e^{2x+3y} - (1/6)\times\cos(2x + y)$.

2. $(D^2 - DD' + D' - 1)\,z = e^y + xy$. **[Delhi Maths (H) 2005]**

Hint : Do like solved Ex. 9, Art. 5.13 **Ans.** $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + ye^y - xy - x + 1$

5.14. General method of finding particular integral for only reducible non–homogeneous linear partial differential equation, namely,

$$\boldsymbol{F(D, D')z = f(x, y)}$$

Let $F(D, D') = (a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)\ldots(a_nD + b_nD' + c_n)$

$\therefore$ P.I. of the given equation $= \dfrac{1}{F(D, D')}f(x, y)$

or $$\text{P.I.} = \frac{1}{(a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)\ldots(a_nD + b_nD' + c_n)}f(x, y) \quad ... (1)$$

In order to evaluate P.I. given by (1), we consider a solution of the following equation (assuming that $a \neq 0$)

$(aD + bD + c)z = f(x, y)$ or $a\,p + b\,q = f(x, y) - c\,z$ ... (2)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations are

$$\frac{dx}{a}=\frac{dy}{b}=\frac{dz}{f(x,y)-cz}$$

Taking the first two fraction of (3), $ady - bdx = 0$ ... (3)

Integrating, $ay - bx = d$, d being an arbitrary constant ... (4)

From (4), we have $y=(d+bx)/a$, if $a\neq 0$ ... (5)

Taking the first and last fractions of (3) and using (5), we get

$$\frac{dz}{dx}=\frac{f(x,y)-cz}{a}=-\frac{cz}{a}+\frac{1}{a}f\left(x,\frac{d+bx}{a}\right) \quad \text{or} \quad \frac{dz}{dx}+\frac{c}{a}z=\frac{1}{a}f\left(x,\frac{d+bx}{a}\right) \quad \text{... (6)}$$

which is linear differential equation whose I.F. $= e^{\int (c/a)dx}=e^{cx/a}$

$\therefore$ Solution of (6) is $z e^{cx/a}=\frac{1}{a}\int f\left(x,\frac{d+bx}{a}\right)dx$

so that $z=\frac{e^{-(cx/a)}}{a}\int f\left(x,\frac{d+bx}{a}\right)dx, \quad a\neq 0$ and $d = ay - bx.$

$\therefore$ From (2), $\frac{1}{(aD+bD'+c)}f(x,y)=\frac{e^{-(cx/a)}}{a}\int f\left(x,\frac{d+bx}{a}\right)dx$ ... (7)

where $ay - bx = d$ and $a\neq 0$ d being arbitrary constant.

Similarly, if $b\neq 0$, we can show that

$$\frac{1}{(aD+bD'+c)}f(x,y)=\frac{e^{-(cy/b)}}{b}\int e^{cy/b}f\left(\frac{d+ay}{b}\right)dy, \quad \text{... (8)}$$

where $bx - ay = d$ and $b\neq 0$, d being an arbitrary constant.

Results (7) and (8) will be used to evaluate P.I. given by (1).

5.15. Working rule for finding P.I. of any reducible linear partial differential equation (homogeneous or non–homogeneous), namely,

$$\boldsymbol{F(D, D')z = f(x, y)} \quad \text{... (1)}$$

Rule I. $\frac{1}{aD+bD'+c}f(x,y)=\frac{e^{-(cx/a)}}{a}\int e^{cx/a}f\left(x,\frac{d+bx}{a}\right)dx,\ a\neq 0$ where $ay-bx=d$

Note that constant d must be replaced by $ay-bx$ after integration is performed.

Rule II. $\frac{1}{aD+bD'+c}f(x,y)=\frac{e^{-(cy/b)}}{b}\int e^{cy/b}f\left(\frac{d+ay}{b}\right)dy$, $b\neq 0$, where $bx - ay = d$

Note that constant d must be replaced by $bx-ay$ after integration is performed.

We now consider some special cases of the above rules.

Rule III. $\frac{1}{aD+c}f(x,y)=\frac{e^{-(cx/a)}}{a}\int e^{cx/a}f(x,d/a)\,dx$, where $ay=d$

Rule IV. $\frac{1}{D-mD'}f(x,y)=\int f\,(x,\,d-mx)\,dx$, where $y+mx=d$

Rule V. $\frac{1}{bD'+c}f\,(x,y)=\frac{e^{-(cy/b)}}{b}\int e^{cy/b}f(d/b,\,y)\,dy$, where $bx = d$

Rule VI. $\frac{1}{D'-mD} f(x,y) = \int f(d-my,\ y)\ dy$, where $x+my=d$

Note 1. Results IV and VI have already been obtained in Art. 4.12 of chapter 4.

Note 2. Suppose $F(D, D')$ can be factored as $\prod_{r=1}^{n}(a_rD+b_rD'+c_r)$, then

P.I. for (1) $= \frac{1}{F(D,D')} f(x,y) = \frac{1}{(a_1D+b_1D'+c_1)(a_2D+b_2D'+C_2)....(a_nD+b_n\ D'+c_n)} f(x,y)$,

which is evaluated by using the above six rules for each factor, in succession, from right to the left.

5.16 SOLVED EXAMPLES BASED ON ART. 5.15.

Ex. 1. *Solve* $(D+D')\ (D+D'-2)z = \sin(x+2y)$ **[Delhi Maths (H) 2000]**

Sol. Here C.F. $= \phi_1(y-x)+e^{2x}\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary function

P.I. $= \frac{1}{D+D'-2}\left\{\frac{1}{D+D'}\sin\ (x+2y)\right\} = \frac{1}{D+D'-2}\int \sin\ (3x+2d)\,dx$, where, $y-x=d$,

[using rule IV of Art. 5.15]

$= \frac{1}{D+D'-2}\left\{-\frac{\cos(3x+2d)}{3}\right\} = -\frac{1}{3}\ \frac{1}{D+D'-2}\cos\ (2x+y)$

$= -\frac{1}{3}\ e^{2x}\int e^{-2x}\cos(3x+2d)dx$, where $y-x=d$ [using rule I of Art. 5.15]

$= -\frac{1}{3}e^{2x}\frac{1}{(-2)^2+3^2}e^{-2x}\{-2\cos(3x+2d)+3\sin(3x+2d)\} = \frac{2}{39}\cos(x+2y)-\frac{1}{13}\sin(x+2y)$

$\therefore$ Solution is $z = \phi_1(y-x)+e^{2x}\phi_2(y-x)+(2/39)\times\cos(x+2y)-(1/13)\times\sin(x+2y)$

Ex. 2. *Solve* $(D^3-DD'^2-D^2+DD')z = (x+2)/x^3$

Sol. Re–writing, the given equation $D(D-D')\ (D+D'-1)z = (x+2)/x^3$

Its C.F. $= \phi_1(y)+\phi_2(y+x)+e^x\phi_3(y-x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

P.I. $= \frac{1}{(D-D')(D+D'-1)}\frac{1}{D}\left(\frac{1}{x^2}+\frac{2}{x^3}\right) = \frac{1}{(D+D'-1)\ (D-D)}\left(-\frac{1}{x}-\frac{1}{x^2}\right)$

$= \frac{1}{D+D'-1}\int\left(-\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{1}{D+D'-1}\left(-\log x+\frac{1}{x}\right) = e^x\int e^{-x}\left(-\log x+\frac{1}{x}\right)dx$

$= -e^x\int e^{-x}\log x\,dx+e^x\int e^{-x}\frac{1}{x}dx = -e^x\left[(-e^{-x})\log x-\int(-e^{-x})\frac{1}{x}dx\right]+e^x\int e^{-x}\frac{1}{x}dx = \log x$

[on integration by parts first integral only]

$\therefore$ General solution is $z = \phi_1(y)+\phi_2(y+x)+e^x\phi_3(y-x)\ +\log x.$

Ex. 3. *Solve* $(D^2+DD'+D'-1)\ z = 4\sinh x$.

Sol. Re–writing, the given equation $(D+1)\ (D+D'-1)\ z = 2(e^x-e^{-x})$.

C.F. $= e^{-x}\phi_1(y)+e^x\phi_2(y-x)$, where ϕ_1, ϕ_2 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D+1)(D+D'-1)} 2(e^x - e^{-x}) = \frac{1}{(D+1)} 2e^x \int e^{-x}(e^x - e^{-x})dx$$

$$= \frac{1}{(D+1)} 2e^x \left(x + \frac{1}{2} e^{-2x} \right) = \frac{1}{D+1}(2xe^x + e^{-x}) = e^{-x} \int e^x (2x\ e^x + e^{-x})dx$$

[using rule III of Art. 5.15]

$$= 2e^{-x} \int x\ e^{2x} dx + e^{-x} x = 2e^{-x} \left[x \times (e^{2x}/2) - \int 1 \cdot (e^{2x}/2) dx \right] + xe^{-x}$$

$$= x\ e^x - e^{-x} \int e^{2x} dx + xe^{-x} = xe^x - e^{-x} \times (1/2) \times e^{2x} + xe^{-x} = (x - 1/2)e^x + xe^{-x}$$

$\therefore$ General solution is $z = e^{-x}\phi_1(y) + e^x \phi_2(y-x) + (x - 1/2)e^x + xe^{-x}$

Ex. 4. *Solve* $(D^2 - DD' + D' - 1)z = 1 + xy + e^y + \cos(x + 2y)$ **[Delhi Maths (H) 2001]**

Sol. Re–writing the given equation $(D-1)\ (D - D' + 1)z = 1 + xy + e^y + \cos(x + 2y)$

C.F. = $e^x \phi_1(y) + e^{-x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-1)\ (D - D' + 1)} \{1 + xy + e^y + \cos\ (x + 2y)\}$$

$$= \frac{1}{D-1} e^{-x} \int e^x \{1 + x\ (d - x) + e^{d-x} + \cos(2d - x)\} dx, \text{ where } d = y + x$$

$$= \frac{1}{D-1}\ e^{-x} \left\{ \int\ (1 + d\,x - x^2) e^x dx + \int e^a dx + \int e^x \cos(x - 2d) dx \right\}$$

$$= \frac{1}{D-1} e^{-x} [(1 + dx - x^2)e^x - (d - 2x)e^x + (-2)e^x + e^d x + \frac{e^x}{1^2 + 1^2} \{\cos\ (x - 2d) + \sin(x - 2d)\}]$$

$$= \frac{1}{D-1} [(1 + d\,x - x^2 - d + 2x - 2) + e^{d-x} x + (1/2)\ \times \{\cos(x - 2d) + \sin(x - 2d)\}]$$

$$= \frac{1}{D-1} [-1 + x(y + x) - x^2 - (y + x) + 2x + e^y x + (1/2)\ \times \{\cos(-2y - x) + \sin(-2y - x)\}]$$

$$= \frac{1}{D-1} \{xy - y + x - 1 + x\,e^y + (1/2) \times \cos(2y + x) - (1/2) \times \sin(2y + x)\}$$

$$= \frac{1}{D-1} \{(x-1)\ (y+1) + xe^y + (1/2) \times \cos(2y + x) - (1/2) \times \sin(2y + x)\}$$

$$= e^x \int e^{-x} \{(x-1)\ (k+1) + xe^k + (1/2) \times \cos(2k + x) - (1/2) \times \sin(2k + x)\}, \text{ where } y = k$$

$$= e^x \left[(k+1) \int e^{-x}(x-1)dx + e^k \int xe^{-x} dx + \frac{1}{2} \int e^{-x} \cos(2k + x)]\ dx - \frac{1}{2} \int e^{-x} \sin(2k + x) dx \right]$$

$$= e^x (k+1)\ \{(-e^{-x})\ (x+1) - (e^{-x})\ (1)\} + e^x e^k \{(-e^{-x})(x) - (e^{-x})(1)\}]$$

$$+ \frac{e^x}{2} \frac{e^{-x}}{(-1)^2 + 1^2} \{-\cos(2k + x) + \sin(2k + x)\}\ - \frac{e^x}{2} \frac{e^{-x}}{(-1)^2 + 1^2} \{-\sin(2k + x) - \cos(2k + x)]$$

$$= -(k+1)\,x - e^k\ (x+1) + (1/2)\ \times \sin\ (2k + x)] = -(y+1)x - e^y\ (x+1) + (1/2) \sin\ (2y + x)$$

$\therefore$ solution is $z = e^x \phi_1(y) + e^{-x}\phi_2(y+x) - x(y+1) - (x+1)e^y + (1/2) \times \sin(2y + x))$

Ex 5. *Solve* $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy + \sin(2x + y)$

Ans. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) + (x/24) \times (6xy - 6y + 9x - 2x^2 - 12) - (1/6) \times \cos(2x + y)$.

5.17. Solutions under given geometrical conditions

We have seen that solutions of non–homogeneous linear partial differential equations involve arbitrary functions of x and y. We shall now determine these functions under the given geometrical conditions. This will lead to the required surface satisfying the given differential equation under the prescribed geometrical conditions.

Example. *Find a surface satisfying* $r + s = 0$, *i.e.*, $(D^2 + DD')z = 0$ *and touching the elliptic paraboloid* $z = 4x^2 + y^2$ *along its section by the plane* $y = 2x + 1$. **[I.A.S. 1994]**

Sol. Given $(D^2 + DD')z = 0$. or $D(D + D') = 0$ ...(1)

$\therefore$ Solution of (1) is $z = \text{C.F.} = \phi_1(y) + \phi_2(y - x)$, ...(2)

where ϕ_1 and ϕ_2 are arbitrary functions.

Since (2) touches the curve given by $z = 4x^2 + y^2$ ...(3)

and $y = 2x + 1$, ...(4)

values of p $(= \partial z/\partial x)$ and q $(= \partial z/\partial y)$ obtained from (2) and (3) must be equal for any point on (4).

$\therefore$ $-\phi_2'(y - x) = 8x$ for $y = 2x + 1$ or $\phi_2'(x + 1) = -8x$. ...(5)

and $\phi_1'(y) + \phi_2'(y - x) = 2y$ for $y = 2x + 1$ or $\phi_1'(2x + 1) + \phi_2'(x + 1) = 4x + 2$ ...(6)

From (5), $\phi_2'(x) = 8 - 8x$

Integrating it, $\phi_2(x) = 8x - 4x^2 + c_1$, c_1 being an arbitrary constant ...(7)

Subtracting (5) from (6), $\phi_1'(2x + 1) = 12x + 2 = 6(2x + 1) - 4$

so that $\phi_1'(x) = 6x - 4$.

Integrating it, $\phi_1(x) = 3x^2 - 4x + c_2$, c_2 being an arbitrary constant ...(8)

From (8), $\phi_1(y) = 3y^2 - 4y + c_2$.

and from (7), $\phi_2(y - x) = 8(y - x) - 4(y - x)^2 + c_1$.

Putting the above values of $\phi_1(y)$ and $\phi_2(y - x)$ in (2), we get

$$z = 3y^2 - 4y + c_2 + 8(y - x) - 4(y - x)^2 + c_1$$

or $z = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3$, where $c_3 = c_1 + c_2$. ...(9)

Equating the values of z from (3) and (9), we get

$4x^2 + y^2 = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3$, where $y = 2x + 1$.

$\therefore$ $c_3 = 8x^2 + 2y^2 - 4y + 8x - 8xy = 8x^2 + 2(2x + 1)^2 - 4(2x + 1) + 8x - 8x(2x + 1) = -2$

Hence, from (9), the required surface is $4x^2 - 8xy + y^2 - 4y + z + 2 = 0$.

MISCELLANEOUS PROBLEM ON CHAPTER 5

1. Find the solution of the equation $\partial^2 z / \partial x^2 + \partial^2 z / \partial y^2 = e^{-x} \cos y$, which tends to zero as $x \to \infty$ and has the value *cos y* when $x = 0$ **Ans.** $z = (1 - x/2)e^{-x} \cos y$

6

Partial Differential Equations Reducible to Equations with Constant Coefficients

6.1. INTRODUCTION

In chapters 4 and 5, we have discussed methods of solving linear partial differential equations with constant coefficients. In this chapter, we propose to discuss the method of solving the so-called *Euler-Cauchy type partial differential equations* of the form

$$a_0x^n\frac{\partial^n z}{\partial x^n}+a_1x^{n-1}y\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2x^{n-2}y^2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+\ldots+a_ny^n\frac{\partial^n z}{\partial y^n}+\ldots=f(x,y), \qquad \ldots(1)$$

having variable coefficients in particular form (namely the term $\partial^n z/\partial x^n$ is multiplied by x^n, $\partial^n z/\partial y^n$ is multiplied by y^n, $\partial^n z/\partial x^r\partial y^{n-r}$, $r = 1, 2, \ldots, n-1$ is multiplied by $x^r y^{n-r}$ and so on. If $D \equiv \partial/\partial x$ and $D' \equiv \partial/\partial y$, then (1) can be re-written as

$$(a_0x^nD^n+a_1x^{n-1}y\,D^{n-1}D'+a_2\,x^{n-2}y^2\,D^{n-2}D'^2+\ldots+a_ny^nD'^n+\ldots)z=f(x,y) \qquad \ldots(2)$$

Examples of such equations are: $(x^2D^2-y^2D'^2)z=xy$; $x^2D^2-y^2D'^2+xD-yD'=\log x$

6.2. METHOD OF REDUCING EULER-CAUCHY TYPE EQUATION TO A LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Consider Euler-Cauchy type equation

$$a_0x^nD^n+a_1x^{n-1}yD^{n-1}D'+a_2x^{n-2}y^2D^{n-2}D'^2+\ldots+a_ny^nD'^n+\ldots)\,z=f(x,y) \qquad \ldots(1)$$

where $D \equiv \partial/\partial x$ and $D' \equiv \partial/\partial y$. Define two new variables u and v by

$$x=e^u \quad \text{and} \quad y=e^v \quad \text{so that} \quad u=\log x \quad \text{and} \quad v=\log y \qquad \ldots(2)$$

Let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/dv$ $\ldots(3)$

Now, $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}=\frac{1}{x}\frac{\partial z}{\partial u}, \text{ using (2)}$$

$$\therefore\ x\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u} \quad \text{so that} \quad x\frac{\partial}{\partial x}=\frac{\partial}{\partial u} \quad \text{or} \quad xD=D_1 \qquad \ldots(4)$$

Again, $$x\frac{\partial}{\partial x}\left(x^{n-1}\frac{\partial^{n-1}z}{\partial x^{n-1}}\right)=x^n\frac{\partial^n z}{\partial x^n}+(n-1)x^{n-1}\frac{\partial^{n-1}z}{\partial x^{n-1}}$$

$$\therefore\quad x^n\frac{\partial^n z}{\partial x^n}=\left(x\frac{\partial}{\partial x}-n+1\right)x^{n-1}\frac{\partial^{n-1}z}{\partial x^{n-1}}$$

so that $$x^nD^n=(D_1-n+1)x^{n-1}D^{n-1}. \qquad \ldots(5)$$

Putting $n = 2, 3, \ldots$ in (5), we have

$x^2D^2=(D_1-1)xD$ or $x^2D^2=D_1(D_1-1)$, using (4) $\ldots(6)$

$x^3D^3 = (D_1 - 2)x^2D^2$ or $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$, using (6) ...(7)

and so on. Similarly, we have

$yD' = D_1'$, $y^2D^2 = D_1'(D_1' - 1)$, $y^3D'^3 = D_1'(D_1' - 1)(D_1' - 2)$ ...(8)

and so on. Also, we have $xyDD' = D_1D_1'$. ...(9)

and $x^my^nD^m D'^n = D_1(D_1 - 1)...(D_1 - m + 1)D_1'(D_1' - 1)...(D_1' - n + 1)$. ...(10)

Using the substitutions (2) and results (4), (5), (6), (7), (8), (9) and (10), the given equation (1) reduces to an equation having constant coefficients and now it can easily be solved by the methods already discussed for homogeneous (refer Chapter 4) and non-homogeneous (refer Chapter 5) linear equations with constant coefficients. Finally, with help of (2), the solution is obtained in terms of old variables x and y.

6.3. WORKING RULE FOR SOLVING EULER-CAUCHY TYPE PARTIAL DIFFERENTIAL EQUATION

$$(a_0x^nD^n + a_1x^{n-1}y\,D^{n-1}D' + a_2x^{n-2}y\,D^{n-2}\,D'^2 + ... + a_ny^nD'^n + ...)z = f(x, y) \quad ...(1)$$

Step 1 Introduce two new variables u and v :

$x = e^u$ and $y = e^v$, *i.e.*, $u = \log x$ and $v = \log y$ ...(2)

Step 2 We have $D \equiv \partial / \partial x$ and $D' \equiv \partial / \partial y$. Also let $D_1 \equiv \partial / \partial u$, $D_1' \equiv \partial / \partial v$

Step 3 Use the following results in (1)

$$\left.\begin{aligned} &xD = D_1, \quad yD' = D_1', \quad x^2D^2 = D_1(D_1 - 1), \quad y^2D'^2 = D_1'(D_1' - 1) \\ &x^3D^3 = D_1(D_1 - 1)(D_1 - 2), \quad y^3D'^3 = D_1'(D_1' - 1)(D_1' - 2) \text{ and so on} \end{aligned}\right\} \quad ...(3)$$

Step 4 Using (2) and (3) in (1), we obtain the following linear partial differential (homogeneous or non-homogeneous)

$$(b_0\,D_1^n + b_1D_1^{n-1}D_1' + b_2D_1^{n-2}D_1'^2 + ... + b_nD_1'^n + ...)z = g(u, v) \quad ...(4)$$

Step 5 If (4) is homogeneous linear partial differential equation, then it is solved with help of methods of chapter 4. Again, if (4) is non-homogeneous linear partial differential equation, then it is solved with help of methods of chapter 5.

Step 6 Using $u = \log x$ and $v = \log y$ in the solution obtained in step 5, we finally obtain the required solution in terms of the original variables x and y.

6.4. SOLVED EXAMPLES BASED ON ART. 6.3.

Ex. 1. *Solve* $x^2(\partial^2z/\partial x^2) - y^2(\partial^2z/\partial y^2) - y(\partial z/\partial y) + x(\partial z/\partial x) = 0$. **(Jabalpur 1996)**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2 - yD' + xD)z = 0$ becomes

$$[D_1(D_1 - 1) - D_1'(D_1' - 1) - D_1' + D_1]z = 0$$

or $(D^2_1 - D'^2_1)z = 0$ or $(D_1 - D_1')(D_1 + D_1')z = 0$.

Hence the required general solution is $z = \text{C.F.} = \phi_1(v + u) + \phi_2(v - u)$

or $z = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$, using (1)

or $z = \phi_1 \log (xy) + \phi_2 \log (y/x)$ or $z = f_1(xy) + f_2(y/x)$, where f_1, f_2 are arbitrary functions.

Ex.2. *Solve* $x^2(\partial^2z/\partial x^2) + 2xy(\partial^2z/\partial x\partial y) + y^2(\partial^2z/\partial y^2) = 0$. **[Delhi Maths (H) 1994, CDLU 2004]**

or Solve $x^2r + 2xys + y^2t = 0$ **(Purvanchal 2007)**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation can be written as $(x^2D^2 + 2xyDD' + y^2D'^2)z = 0$ which reduces to

$$[D_1(D_1 - 1) + 2DD' + D'(D' - 1)]z = 0$$

$[(D_1 + D_1')^2 - (D_1 + D_1')]z = 0$ or $(D_1 + D_1')(D_1 + D_1' - 1)z = 0.$

Hence the required general solution is $z = \text{C.F.} = \phi_1(v - u) + e^u\phi_2(v - u)$

or $z = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

or $z = \phi_1 \log (y/x) + x\phi_2 \log (y/x)$ or $z = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

Ex. 3. *Solve* $x^2(\partial^2z/\partial x^2) - 3xy(\partial^2z/\partial x\partial y) + 2y^2(\partial^2z/\partial y^2) + 5y(\partial z/\partial y) - 2z = 0.$

Sol. Let $x = e^u$, $y = e^v$, so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$, $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - 3xyDD' + 2y^2D'^2 + 5yD' - 2)z = 0$ becomes

$$[D_1(D_1 - 1) - 3D_1D_1' + 2D_1'(D_1' - 1) + 5D_1' - 2]z = 0$$

or $(D_1^2 - 3D_1D_1' + 2D_1'^2 - D_1 + 3D_1' - 2)z = 0$ or $(D_1 - D_1' - 2)(D_1 - 2D_1' + 1)z = 0.$

Hence general solution is $z = \text{C.F.} = e^{2u}\phi_1(v + u) + e^{-u}\phi_2(v + 2u)$

or $z = (e^u)^2\phi_1(\log y + \log x) + (e^u)^{-1}\phi_2(\log y + 2 \log x) = x^2\phi_1 \log (xy) + x^{-1}\phi_2 \log (yx^2)$, using (1)

or $z = x^2f_1(xy) + x^{-1}f_2(yx^2)$, f_1, f_2 being arbitrary functions.

Ex. 4. *Solve* $(x^2D^2 + 2xyDD' + y^2D'^2)z = x^my^n$, where $(m + n) \neq 0, 1.$

[Delhi Maths (H) 1994, KU Kurukshetra 2004]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/dv$.

Then the given equation reduces to $[D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1)]z = e^{mu}.e^{nv}$

or $[(D_1 + D_1')^2 - (D_1 + D_1')]z = e^{mu + nv}$ or $(D_1 + D_1')(D_1 + D_1' - 1) = e^{mu + nv}$

Here C.F. $= \phi_1(v - u) + e^u\phi_2(v - u) = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

$\therefore$ C.F. $= \phi_1 \log (y/x) + x\phi_2 \log (y/x) = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{(D_1 + D_1')(D_1 + D_1' - 1)} e^{mu + nv} = \frac{1}{(m + n)(m + n - 1)} e^{mu + nv}$$

$$= \frac{1}{(m + n)(m + n - 1)} (e^u)^m (e^v)^n = \frac{1}{(m + n)(m + n - 1)} x^m y^n, \text{ using (1).}$$

$\therefore$ Required general solution is $z = f_1(y/x) + xf_2(y/x) + [1/\{(m + n)(m + n - 1)\}]x^my^n.$

Ex. 5. *Solve* $x^2(\partial^2z/\partial x^2) - 4xy(\partial^2z/\partial x\partial y) + 4y^2(\partial^2z/\partial y^2) + 6y(\partial z/\partial y) = x^3y^4.$

[Jabalpur 2004; Vikram 2004; Meerut 1999; Delhi Maths (H) 1995]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')z = x^3y^4$ becomes

$$[D_1(D_1 - 1) - 4D_1D_1' + 4D_1'(D_1' - 1) + 6D_1']z = e^{3u}e^{4v}$$

or $$[(D_1^2 - 4D_1D_1' + 4D_1'^2) - (D_1 - 2D_1')]z = e^{3u + 4v}$$

or $[(D_1 - 2D_1')^2 - (D_1 - 2D_1')]z = e^{3u + 4v}$ or $(D_1 - 2D_1')(D_1 - 2D_1' - 1)z = e^{3u + 4v}$

Here C.F. $= \phi_1(v + 2u) + e^u\phi_2(v + 2u) = \phi_1(\log y + 2 \log x) + x\phi_2(\log y + 2 \log x)$, using (1)

$\therefore$ C.F. $= \phi_1 \log (yx^2) + x\phi_2 \log (yx^2) = f_1(yx^2) + xf_2(yx^2)$, where f_1 and f_2 are arbitrary functions.

Now, P.I. $= \dfrac{1}{(D_1 - 2D_1')(D_1 - 2D_1' - 1)} e^{3u+4v} = \dfrac{1}{\{3-(2\times 4)\}\times\{3-(2\times 4)-1\}} e^{3u+4v}$

$= (1/30)\times (e^u)^3 (e^v)^4 = (1/30)\times x^3 y^4$, using (1)

$\therefore$ The required general solution is $z =$ C.F. + P.I. or $z = f_1(yx^2) + xf_2(yx^2) + (1/30)\times x^3 y^4$.

Ex. 6. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x \partial y) - x(\partial z/\partial x) = x^3/y^2$.

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 + 2xyDD' - xD)z = x^3 y^{-2}$ becomes

$$[D_1(D_1 - 1) + 2D_1D_1' - D_1]z = (e^u)^3 (e^v)^{-2}$$

or $(D_1^2 + 2D_1D_1' - 2D_1)z = e^{3u-2v}$ or $D_1(D_1 + 2D_1' - 2)z = e^{3u-2v}$

$\therefore$ C.F. $= \phi_1(v) + e^{2u}\phi_2(v-2u) = \phi_1(v) + (e^u)^2\phi_2(v-2u) = \phi_1(\log y) + x^2\phi_2(\log y - 2\log x)$, using (1)

$= \phi_1(\log y) + x^2\phi_2(\log(y/x^2)) = f_1(y) + x^2 f_2(y/x^2)$, where f_1 and f_2 are arbitrary functions.

Now, P.I. $= \dfrac{1}{D_1(D_1 + 2D_1' - 2)} e^{3u-2v} = \dfrac{1}{3(3-4-2)} (e^u)^3 (e^v)^{-2} = -\dfrac{x^3 y^{-2}}{9} = -\dfrac{x^3}{9y^2}$

Hence the required general solution is $z =$ C.F. + P.I. or $z = f_1(y) + x^2 f_2(y/x^2) - (x^3/9y^2)$.

Ex. 7. *Solve* $x^2 r - 3xys + 2y^2 t + px + 2qy = x + 2y$.

Sol. The given equation can be re-written as

$$x^2(\partial^2 z/\partial x^2) - 3xy(\partial^2 z/\partial x \partial y) + 2y^2(\partial^2 z/\partial y^2) + x(\partial z/\partial x) + 2y(\partial z/\partial y) = x + 2y$$

or $(x^2D^2 - 3xyDD' + 2y^2D'^2 + xD + 2yD')z = x + 2y$. ...(1)

Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(2)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

$\therefore$ (1) becomes $[D_1(D_1 - 1) - 3D_1D_1' + 2D_1'(D_1' - 1) + D_1 + 2D_1']z = e^u + 2e^v$

or $(D_1^2 - 3D_1D_1' + 2D_1'^2)z = e^u + 2e^v$ or $(D_1 - D_1')(D_1 - 2D_1')z = e^u + 2e^v$.

$\therefore$ C.F. $= \phi_1(v+u) + \phi_2(v+2u) = \phi_1(\log y + \log x) + \phi_2(\log y + 2\log x)$

or C.F. $= \phi_1 \log(xy) + \phi_2 \log(x^2 y) = f_1(xy) + f_2(x^2 y)$, where f_1 and f_2 are arbitrary functions.

Also, P.I. $=$

$$\frac{1}{(D_1 - D_1')(D_1 - 2D_1')}(e^u + 2e^v) = \frac{1}{(D_1 - D_1')(D_1 - 2D_1')} e^{1\cdot u + 0\cdot v} + 2\frac{1}{(D_1 - D_1')(D_1 - 2D_1')} e^{0\cdot u + 1\cdot v}$$

$$= \frac{1}{(1-0)(1-0)} e^u + 2\frac{1}{(0-1)(0-2)} e^v = x + y, \text{ using (2)}$$

Hence the required general solution is $z =$ C.F. + P.I. or $z = f_1(xy) + f_2(x^2 y) + x + y$.

Ex. 8. *Find the general solution of* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x \partial y) + y^2(\partial^2 z/\partial y^2) + nz = n\{x(\partial z/\partial x) + y(\partial z/\partial y)\} + x^2 + y^2 + x^3$.

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then, the given equation reduces to

$$[x^2D^2 + 2xyDD' + y^2D'^2 - n(xD + yD') + n]z = x^2 + y^2 + x^3$$

or $[D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1) - n(D_1 + D_1') + n]z = e^{2u} + e^{2v} + e^{3u}$

or $\{(D_1 + D_1')^2 - (D_1 + D_1') - n(D_1 + D_1' - 1)\}z = e^{2u} + e^{2v} + e^{3u}$

or $\{(D_1 + D_1')(D_1 + D_1' - 1) - n(D_1 + D_1' - 1)\}z = e^{2u} + e^{2v} + e^{3u}$

or $(D_1 + D_1' - 1)(D_1 + D_1' - n)z = e^{2u} + e^{2v} + e^{3u}$.

$\therefore$ C.F. $= e^u\phi_1(v-u) + e^{nu}\phi_2(v-u) = e^u\phi_1(v-u) + (e^u)^n\phi_2(v-u)$

$= x\phi_1(\log y - \log x) + x^n\phi_2(\log y - \log x) = x\phi_1 \log(y/x) + x^n\phi_2 \log(y/x)$, using (1)

$= xf_1(y/x) + x^n f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{Also, P.I.} = \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)}(e^{2u} + e^{2v} + e^{3u}) = \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{2u + 0\cdot v}$$

$$+ \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{0\cdot u + 2v} + \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{3u + 0\cdot v}$$

$$= \frac{(e^u)^2}{(2 + 0 - 1)(2 + 0 - n)} + \frac{(e^v)^2}{(0 + 2 - 1)(0 + 2 - n)} + \frac{(e^u)^3}{(3 + 0 - 1)(3 + 0 - n)} = \frac{x^2 + y^2}{2 - n} + \frac{x^3}{2(3 - n)}$$

Hence general solution is $z = xf_1(y/x) + x^n f_2(y/x) + (x^2 + y^2)/(2 - n) + x^3/2(3 - n)$.

Ex. 9. *Solve* $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) = xy$ or $(x^2D^2 - y^2D'^2)z = xy$.

(Bilaspur 1999, Jabalpur 2003, Jiwaji 2003, 04, Vikram 2004, Ravishankar 2010, I.A.S. 1987, Rohilkhand 1995, Delhi Maths (H) 2004, 06)

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2)z = xy$ becomes

$$[D_1(D_1 - 1) - D_1'(D_1' - 1)]z = e^u e^v \quad \text{or} \quad (D_1^2 - D_1'^2 - D_1 + D_1')z = e^{u+v}$$

or $[(D_1 - D_1')(D_1 + D_1') - (D_1 - D_1')]z = e^{u+v}$ or $(D_1 - D_1')(D_1 + D_1' - 1)z = e^{u+v}$.

$\therefore$ C.F. $= \phi_1(v + u) + e^u\phi_2(v - u) = \phi_1(\log y + \log x) + x\phi_2(\log y - \log x)$, using (1)

or C.F. $= \phi_1 \log(xy) + x\phi_2 \log(y/x) = f_1(xy) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{Also, P.I.} = \frac{1}{(D_1 - D_1')(D_1 + D_1' - 1)} e^{u+v} = \frac{1}{D_1 - D_1'} \frac{1}{(1 + 1 - 1)} e^{u+v} = \frac{u}{1!} e^{u+v} = ue^u e^v = xy \log x$$

Hence the required general solution is z = C.F. + P.I. or $z = f_1(xy) + xf_2(y/x) + xy \log x$.

Ex. 10. *Solve* $yt - q = xy$.

Sol. The given equation can be rewritten as $y(\partial^2 z/\partial y^2) - (\partial z/\partial y) = xy$

or $y^2(\partial^2 z/\partial y^2) - y(\partial z/\partial y) = xy^2$ or $(y^2D'^2 - yD')z = xy^2$. ...(1)

Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ... (2)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then (1) becomes $[D_1'(D_1' - 1) - D_1']z = e^u e^{2v}$ or $D_1'(D_1' - 2)z = e^{u+2v}$

$\therefore$ C.F. $= \phi_1(u) + e^{2v}\phi_2(u) = \phi_1(\log x) + y^2\phi_2(\log x)$, by (2)

$= f_1(x) + y^2 f_2(x)$, f_1 and f_2 being arbitrary functions.

$$\text{Also, P.I.} = \frac{1}{(D_1' - 2)D_1'} e^{u+2v} = \frac{1}{D_1 - 2} \frac{1}{2} e^{u+2v} = \frac{1}{2} e^{u+2v} \frac{1}{D' + 2 + 2} \cdot 1$$

$$= \frac{1}{2} e^{u+2v} \frac{1}{D'} 1 = \frac{1}{2} e^u \times (e^v)^2 \times v = \frac{xy^2}{2} \log y.$$

Hence the required general solution is z = C.F. + P.I. or $z = f_1(x) + y^2 f_2(x) + (1/2) \times xy^2 \log y$.

Ex. 11. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x \partial y) + y^2(\partial^2 z/\partial y^2) = (x^2 + y^2)^{n/2}$. **[Delhi Maths 1999]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then, the given equation can be re-written as $(x^2D^2 + 2xyDD' + y^2D'^2)z = (x^2 + y^2)^{n/2}$

or $[D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1)]z = (e^{2u} + e^{2v})^{n/2}$

or $[(D_1 + D_1')^2 - (D_1 + D_1')]z = (e^{2u} + e^{2v})^{n/2}$ or $(D_1 + D_1')(D_1 + D_1' - 1)z = (e^{2u} + e^{2v})^{n/2}$.

$\therefore$ C.F. $= \phi_1(v-u) + e^u\phi_2(v-u) = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

or C.F. $= \phi_1 \log(y/x) + x\phi_2 \log(y/x) = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{(D_1+D_1')(D_1+D_1'-1)}(e^{2u}+e^{2v})^{n/2} = \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu}\{1+e^{2(v-u)}\}^{n/2}$$

$$= \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu}\left\{1+\frac{n}{2}e^{2(v-u)}+\frac{(n/2)\{(n/2)-1\}}{2!}e^{4(v-u)}+\ldots\right\}$$

$$= \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu+0\cdot v} + \frac{n}{2}\frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{(n-2)u+2v}$$
$$+ \frac{(n/2)\{(n/2)-1\}}{2!}\frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{(n-4)u+4v}+\ldots$$

$$= \frac{1}{(n+0)(n+0-1)}e^{nu} + \frac{n}{2}\frac{1}{\{(n-2)+2\}\{(n-2)+2-1\}}e^{(n-2)u+2v}$$
$$+ \frac{(n/2)[(n/2)-1]}{2!}\frac{1}{\{(n-4)+4\}\{(n-4)+4-1\}}e^{(n-4)u+4v}+\ldots$$

$$= \frac{e^{nu}}{n^2-n}\left[1+\frac{n}{2}e^{2(v-u)}+\frac{(n/2)\{(n/2)-1\}}{2!}e^{4(v-u)}+\ldots\right] = \frac{e^{nu}}{n^2-n}\{1+e^{2(v-u)}\}^{n/2}$$

$$= \frac{1}{n^2-n}\{e^{2u}+e^{2u}e^{2(v-u)}\}^{n/2} = \frac{1}{n^2-n}(e^{2u}+e^{2v})^{n/2} = \frac{1}{n^2-n}(x^2+y^2)^{n/2}, \text{ using (1)}$$

Hence the required general solution is $z = f_1(y/x) + xf_2(y/x) + \{1/(n^2-n)\}(x^2+y^2)^{n/2}$.

Ex.12. *Solve* $x^2r - y^2t + px - qy = \log x$. **[KU Kurukshatra 2004; Meerut 2008]**

Or $(x^2D^2 - y^2D'^2 + xD - yD')z = \log x$. **[Delhi Maths (G) 2004; I.A.S. 1997]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2 + xD - yD')z = \log x$ becomes

$$[D_1(D_1-1) - D_1'(D_1'-1) + D_1 - D_1']z = u \quad \text{or} \quad (D_1^2 - D_1'^2)z = u$$

or $$(D_1 + D_1')(D_1 - D_1')z = u.$$

$\therefore$ C.F. $= \phi_1(v-u) + \phi_2(v+u) = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$

or C.F. $= \phi_1 \log(xy) + \phi_2 \log(y/x) = f_1(xy) + f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

and $$\text{P.I.} = \frac{1}{D_1^2 - D_1'^2}u = \frac{1}{D_1^2(1 - D_1'^2/D_1^2)}u = \frac{1}{D_1^2}\left(1-\frac{D_1'^2}{D_1^2}\right)^{-1}u = \frac{1}{D_1^2}\left(1+\frac{D_1'^2}{D_1^2}+\ldots\right)u$$

$= u^3/6 = (\log x)^3/6$, using (1)

$\therefore$ Required solution is $z = f_1(xy) + f_2(y/x) + (1/6) \times (\log x)^3$, f_1, f_2 being arbitrary functions.

Ex. 13. *Solve* $(x^2D^2 - xyDD' - 2y^2D'^2 + xD - 2yD')z = \log(y/x) - (1/2)$.

[Delhi B.Sc. (Hons) III 2011]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to

$$[D_1(D_1-1) - D_1D_1' - 2D_1'(D_1'-1) + D_1 - 2D_1']z = \log y - \log x - (1/2)$$

or $$(D_1^2 - D_1D_1' - 2D_1'^2)z = v - u - (1/2) \quad \text{or} \quad (D_1 - 2D_1')(D_1 + D_1')z = v - u - (1/2).$$

$\therefore$ C.F. $= \phi_1(v+2u) + \phi_2(v-u) = \phi_1(\log y + 2\log x) + \phi_2(\log y - \log x)$

or C.F. $= \phi_1(\log(yx^2)) + \phi_2(\log(y/x)) = f_1(yx^2) + f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{D_1^2 - D_1D_1' - 2D_1'^2}\left(v - u - \frac{1}{2}\right) = \frac{1}{D_1^2\,(1 - D_1'/D_1 - 2D_1'^2/D_1^2)}\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{D_1^2}\left\{1 - \left(\frac{D_1'}{D_1} + \frac{2D_1'^2}{D_1^2}\right)\right\}^{-1}\left(v - u - \frac{1}{2}\right) = \frac{1}{D_1^2}\left(1 + \frac{D_1'}{D_1} + \ldots\right)\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{D_1^2}\left\{v - u - \frac{1}{2} + \frac{1}{D_1}D_1'\left(v - u - \frac{1}{2}\right)\right\} = \frac{1}{D_1^2}\left(v - u - \frac{1}{2} + \frac{1}{D_1}\cdot 1\right) = \frac{1}{D_1^2}\left(v - u - \frac{1}{2} + u\right)$$

$$= \frac{1}{D_1^2}\left(v - \frac{1}{2}\right) = \left(v - \frac{1}{2}\right)\frac{u^2}{2} = \frac{1}{2}u^2v - \frac{1}{4}u^2 = \frac{(\log x)^2 \log y}{2} - \frac{(\log x)^2}{4}, \text{ by (1)}$$

$\therefore$ Required solution is $z = f_1(yx^2) + f_2(y/x) + (1/2) \times (\log x)^2 \log y - (1/4) \times (\log x)^2$.

Ex. 14. *Solve* $(x^2D^2 - 4y^2D'^2 - 4yD' - 1)z = x^2y^2 \log y$. **[Delhi Maths (H) 2006]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to $[D_1(D_1 - 1) - 4\,D_1'\,(D_1' - 1) - 4\,D_1' - 1]z = e^{2u}e^{2v}v$

or $$(D^2_1 - D_1 - 4\,D_1'^2 - 1)z = e^{2u + 2v}v. \quad \ldots(2)$$

Here $(D^2_1 - D_1 - 4\,D_1'^2 - 1)$ cannot be resolved into linear factors in D_1 and D_1'. To find C.F. corresponding to it, we consider the equation.

$$(D^2_1 - D_1 - 4\,D_1'^2 - 1)z = 0. \quad \ldots(3)$$

Let a trial solution of (3) be $z = Ae^{hu + kv}$ ...(4)

$\therefore$ $D_1^2 z = Ah^2e^{hu + kv}$, $D_1z = Ahe^{hu + kv}$, and $D_1'^2\ z = Ak^2e^{hu + kv}$.

Then, (3) $\Rightarrow$ $A(h^2 - h - 4k^2 - 1)e^{hu + hv} = 0$ $\Rightarrow$ $h^2 - h - 4k^2 - 1 = 0$. ...(5)

$\therefore$ C.F. of (2) $= \Sigma Ae^{hu + kv} = \Sigma A(e^u)^h\,(e^v)^k = \Sigma Ax^hy^k$

$$\text{P.I. of (2)} = \frac{1}{D_1^2 - D_1 - 4D_1'^2 - 1}e^{2u + 2v}v = e^{2u + 2v}\frac{1}{(D_1 + 2)^2 - (D_1 + 2) - 4(D_1' + 2)^2 - 1}v$$

$$= e^{2u + 2v}\frac{1}{D_1^2 + 3D_1 - 4D_1'^2 - 16D_1' - 15}v$$

$$= e^{2u + 2v}\frac{1}{(-15) \times [1 + (16/15) \times D_1' + (4/15) \times D_1'^2 - (1/5) \times D_1 - (1/15) \times D_1^2]}v$$

$$= e^{2u + 2v}\frac{1}{(-15)}\left[1 + \left\{\frac{16}{15}D_1' + \frac{4}{15}D_1'^2 - \frac{1}{5}D_1 - \frac{1}{15}D_1^2\right\}\right]^{-1}v$$

$$= \frac{e^{2u + 2v}}{(-15)}\left(1 - \frac{16}{15}D_1' + \ldots\right)v = \frac{e^{2u + 2v}}{(-15)}\left(v - \frac{16}{15}\right) = \frac{(e^u)^2 \times (e^v)^2(16 - 15v)}{225}$$

$= (1/225) \times x^2y^2\,(16 - 15 \log y)$, using (1).

The required general solution is $z = \Sigma Ax^hy^k + (1/225) \times x^2y^2(16 - 15 \log y)$

where $h^2 - h - 4k^2 - 1 = 0$, and A, h and k are arbitrary constants.

Ex. 15. *Solve* $(x^2D^2 + 2xy\,DD' + y^2D'^2)\,z = x^2y^2$ **[Delhi Maths (H) 2007]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$ ... (1)

Here $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$ let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$

$\therefore$ Given equation reduces to

$$\{D_1(D_1-1)+2D_1D_1'+D_1'(D_1'-1)\}z = e^{2u}\ e^{2v}$$

or $$(D_1+D_1')\ (D_1+D_1'-1)\ z = e^{2u+2v}$$

Its C.F. $= \phi_1(v-u)+e^u\phi_2(v-u) = \phi_1(\log y-\log x)+x\ \phi_2(\log y-\log x)$, by (1)

$= \phi_1\{\log(y/x)\}+x\ \phi_2\{\log(y/x)\} = \psi_1(y/x)+x\psi_2(y/x)]$. ψ_1, ψ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D_1+D_1')\ (D_1+D_1'-1)}e^{2u+2v} = \frac{1}{(2+2)(2+2-1)}e^{2u+2v} = \frac{(e^u)^2(e^v)^2}{12} = \frac{x^2y^2}{12}$$

Hence the required solution is given by $z = \psi_1(y/x)+x\ \psi_2(y/x)+(1/12)\times x^2y^2$.

Ex. 16. *Solve* $(x^2D^2-4xyDD'+4y^2D'^2+4yD'+xD)z = x^2y$

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$ ... (1)

Here $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial x$, . Let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$

Then given equation reduces to $\{D_1\ (D_1-1)-4\ D_1D_1'+4D_1'(D_1'-1)+4D_1'+D_1)\}z = e^{2u}e^v$

or $(D_1^2-4D_1D_1'+4D_1'^2)z = e^{2u+v}$ or $(D_1-2D_1')^2z = e^{2u+v}$.

C.F. $= \phi_1\ (v+2u)+u\ \phi_2(v+2u)\ = \phi_1\ (\log y+2\log x)+\log x\ \phi_2(\log y+2\log x)$, by (1)

$= \phi_1(\log yx^2)+\log x\ \phi_2(\log yx^2) = \psi_1(yx^2)+\log x\ \psi_2(yx^2)$, ψ_1, ψ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D_1-2D_1')^2}\ e^{2u+v} = \frac{u^2}{2!}\ e^{2u+v} = \frac{1}{2}(\log x)^2x^2y$$

$\therefore$ General solution is $z = \psi_1\ (yx^2)+(\log x)\ \psi_2(yx^2)+(1/2)\times x^2y\ (\log x)^2$.

Ex. 17. *Solve* $(x^2D-2xy\ DD'+y^2D'^2-xD+3yD')z = 8y/x$ **[Delhi Maths (H) 2005]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$ ... (1)

Here $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$. Let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to $\{D_1(D_1-1)-2D_1D_1'+D_1'(D_1'-1)-D_1+3D_1'\}z = 8e^v/e^u$

or $\{(D_1^2-2D_1\ D_1'+D_1'^2)-2\ (D_1-D_1')\}z = 8e^{v-u}$ or $(D_1-D_1')\ (D_1-D_1'-2)z = 8\ e^{v-u}$

C.F. $= \phi_1\ (v+u)+e^{2u}\phi_2\ (v+u)\ = \phi_1\ (\log y+\log x)+(e^u)^2\phi_2(\log y+\log x)$, using (1)

$= \phi_1(\log xy)+x^2\phi_2(\log xy) = \psi_1(xy)+x^2\psi_2(xy)$, $\psi_1\ \ \psi_2$ being arbitrary functions

$$\text{P.I.} = \frac{1}{(D_1-D_1')\ (D_1-D_1'-2)}8\ e^{-u+v} = 8\frac{1}{(-1-1)\ (-1-1-2)}e^{-u+v} = \frac{e^v}{e^u} = \frac{y}{x}$$

$\therefore$ The required solution is $z = \psi_1(xy)+x^2\psi_2(xy)+y/x$.

Ex. 18. $(x^2D^2-2xy\ DD'-3y^2D'^2+xD-3yD')z\ =\ x^2y\cos(\log x^2)$

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$ ... (1)

Here $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$. Let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to

$$\{D_1(D_1-1) - 2D_1\ D_1' - 3D_1'\ (D_1'-1)\ +\ D_1 - 3\ D_1'\}z\ =\ e^{2u}\ e^{v}\cos 2u\,.$$

or $(D_1^2 - 2D_1D_1' - 3D_1'^2)z = e^{2u+v}\cos 2u$ or $(D_1 - 3D_1')\,(D + D_1')z = e^{2u+v}\cos 2u$

Its C.F. = $\phi_1\,(u+3u) + \phi_2(v-u)\ =\ \phi_1(\log y + 3\log x) + \phi_2(\log y - \log x)$, using (1)

$= \phi_1(\log yx^3) + \phi_2(\log(y/x)) = \psi_1(x^3y) + \psi_2(y/x)$, $\psi_1\ \psi_2$ being arbitrary functions.

P.I. $= \dfrac{1}{(D_1 - 3D_1')\,(D_1 + D_1')}\,e^{2u+v}\cos 2u = e^{2u+v}\ \dfrac{1}{\{(D_1+2) - 3(D_1'+1)\}\,(D_1 + 2 + D_1' + 1)}\cos 2u$

$= \dfrac{(e^u)^2(e^v)}{(D_1 - 3D_1' - 1)\,(D_1 + D_1' + 3)}\cos 2u\ =\ \dfrac{x^2y}{D_1^2 - 2D_1D_1' - 3D_1'^2 + 2D_1 - 10D_1' - 3}\cos(2u + 0\cdot v)$

$= \dfrac{x^2y}{-2^2 - 2(-2\cdot 0) - 3\cdot 0^2 + 2D_1 - 10D_1' - 3}\cos 2u = \dfrac{x^2y}{2D_1 - 10D_1' - 7}\cos 2u = x^2y\dfrac{2D_1 - 10D_1' + 7}{(2D_1 - 10D_1')^2 - 49}\cos 2u$

$= x^2y\dfrac{2D_1 - 10D_1' + 7}{4D_1^2 - 40D_1D_1' + 100D_1'^2 - 49}\cos(2u + 0.v) = x^2y\dfrac{2D_1 - 10D_1' + 7}{4(-2^2) - 40(-2\cdot 0) + 100(-0^2) - 49}\cos(2u + 0\cdot v)$

$= -\,(1/65)\,x^2y\,(-\,4\sin 2u + 7\cos 2u) = (1/65)\,x^2y\,\{4\sin(2\log x) - 7\cos(2\log x)\}$

$\therefore$ Required solution is $z = \psi_1(x^3y) + \psi_2(y/x) + (1/65)x^2y\ \{4\sin(\log x^2) - 7\cos(\log x^2)\}$.

Ex. 19. *Solve* $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) + x(\partial z/\partial x) - y(\partial z/\partial y) = x^2y^4$ *by reducing it to the equation with constant coefficients.* **[I.A.S. 2001]**

Sol. Re-writing, the given equation $(x^2D^2 - y^2D'^2 + xD - yD')z = x^2y^4$ ... (1)

Let $x = e^u$ and $y = e^v$ so that $u = \log x$ and $v = \log y$ ... (2)

Here $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$. Let $D_1 \equiv \partial/\partial u$, $D_1' \equiv \partial/\partial v$. Then (1) becomes

$\{D_1\ (D_1 - 1) - D_1'(D_1' - 1) + D_1 - D_1'\}z\ =\ e^{2u}e^{4v}$ or $(D_1^2 - D_1'^2)z = e^{2u+4v}$

or $(D_1 - D_1')\,(D_1 + D_1')z = e^{2u+4v}$ ... (3)

C.F. $= \phi_1(v+u) + \phi_2\,(v-u) = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)\ =\ \phi_1(\log xy) + \phi_2(\log(y/x))$

or C.F. $= \psi_1(xy) + \psi_2(y/x)$, ψ_1, ψ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D_1^2 - D_1'^2}\ e^{2u+4v} = \frac{1}{(2^2 - 4^2)}e^{2u+4v} = -\frac{1}{12}\ (e^u)^2\ (e^v)^4 = -\frac{1}{12}x^2y^4$$

$\therefore$ $z = \psi_1\ (xy) + \psi_2(y/x) - (1/12)\times x^2y^4$ is the required solution.

Remark. Sometimes typical substitutions are employed to reduce a given equation into a partial differential equation with constant coefficients as shown in the following Ex. 20.

Ex. 20. *Solve* $\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2} - \frac{1}{x^3}\frac{\partial z}{\partial x} = \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2} - \frac{1}{y^3}\frac{\partial z}{\partial y}$.

Sol. Let $x^2/2 = u, \quad y^2/2 = v$ so that $dx/du = 1/x, \quad dy/dv = 1/y.$ ...(1)

Now, $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} = \frac{1}{x}\frac{\partial z}{\partial x}, \text{ using (1).} \quad ...(2)$$

and $$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)\frac{dx}{du} = \left(\frac{1}{x}\frac{\partial^2 z}{\partial x^2} - \frac{1}{x^2}\frac{\partial z}{\partial x}\right)\frac{1}{x}, \text{ using (1)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial u^2} = \frac{1}{x^2}\frac{\partial^2 z}{\partial x^2} - \frac{1}{x^3}\frac{\partial z}{\partial x}. \quad ...(3)$$

Similarly, $$\frac{\partial^2 z}{\partial v^2} = \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2} - \frac{1}{y^3}\frac{\partial z}{\partial y}. \quad ...(4)$$

Using (3) and (4), the given equation reduces to

$\partial^2 z/\partial u^2 = \partial^2 z/\partial v^2$ or $(D_1^2 - D_1'^2)z = 0$ or $(D_1 - D_1')(D_1 + D_1')z = 0,$ ...(5)

where $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$. Hence solution of (5) is

$$z = \phi_1(v + u) + \phi_2(v - u) = \phi_1\{(1/2) \times (x^2 + y^2)\} + \phi_2\{(1/2) \times (y^2 - x^2)\}$$

or $z = f_1(y^2 + x^2) + f_2(y^2 - x^2), f_1, f_2$ being arbitrary functions.

EXERCISE 6

Solve the following partial differential equations:

1. $x^2(\partial^2 y/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) + x(\partial z/\partial x) + y(\partial z/\partial y) - z = 0$

Ans. $z = xf_1(y/x) + x^{-1}f_2(y/x), f_1, f_2$ being arbitrary functions

2. $(x^2D^2 - y^2D'^2)z = x^2y.$ **Ans.** $z = f_1(xy) + x\,f_2(y/x) + (1/2) \times x^2y$

3. $(x^2D^2 - xy\,DD' - 2y^2D'^2 + xD - 2yD')z = \log(y/x)$ **(Delhi Maths (H) 2005)**

Ans. $z = f_1(yx^2) + f_2(y/x) + (1/2) \times (\log x)^2 \log y, f_1, f_2$ being arbitrary functions

4. $(x^2D^2 - 2xy\,DD' - 3y^2D'^2 + xD - 3yD')z = x^2y\sin(\log x^2)$ **[Nagpur 2010]**

Ans. $z = f_1(x^3y) + f_2(y/x) - (1/65) \times \{4\cos(\log x^2) + 7\sin(\log x^2)\}$

6.5. SOLUTIONS UNDER GIVEN GEOMETRICAL CONDITIONS

We have seen that solution of Euler-Cauchy type partial differential equations involve arbitrary functions of x and y. We shall now determine these functions under the given geometrical conditions. This will lead to the required surface satisfying the given differential equation under the prescribed geometrical conditions.

Illustrative example. *Find a surface satisfying equation* $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$ *and touching the hyperbolic paraboloid* $z = x^2 - y^2$ *along its section by the plane* $y = 1$.

[Meerut 1998]

Sol. Re–writing given equation, $$2x^2\frac{\partial^2 z}{\partial x^2} - 5xy\frac{\partial^2 z}{\partial x\partial y} + 2y^2\frac{\partial^2 z}{\partial y^2} + 2\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = 0.$$

or $$\{2x^2D^2 - 5xyDD' + 2y^2D'^2 + 2(xD + yD')\}z = 0 \quad ...(1)$$

Put $x = e^u, \quad y = e^v$ so that $u = \log x$ and $v = \log y.$

If $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$, then (1) reduces to

$$[2D_1(D_1 - 1) - 5D_1D_1' + 2D_1'(D_1' - 1) + 2(D_1 + D_1')]z = 0$$

or $(2D_1^2 - 5D_1D_1' + 2D_1'^2) = 0$ or $(2D_1 - D_1')(D_1 - 2D_1') = 0.$

$\therefore$ solution is $z = \text{C.F.} = \phi_1(2v + u) + \phi_2(u + 2v)$, ϕ_1, ϕ_2 being arbitrary function

or $z = \phi_1(2 \log y + \log x) + \phi_2(\log y + 2 \log x) = \phi_1(\log y^2x) + \phi_2(\log yx^2)$

or $z = f_1(y^2x) + f_2(yx^2)$, f_1 and f_2 being arbitrary functions ...(2)

The given surface is $z = x^2 - y^2.$ ...(3)

Now (2) and (3) are to touch each other along the section by the plane

$$y = 1. \quad ...(4)$$

Therefore the values of p and q for (2) and (3) must be equal at $y = 1$. Equating values of p and q from (2) and (3), we get

$$y^2f_1'(y^2x) + 2xyf_2'(x^2y) = 2x \quad ...(5)$$

and $$2xyf_1'(y^2x) + x^2f_2'(x^2y) = -2y. \quad ...(6)$$

Putting $y = 1$, (5) and (6) reduce to

$f_1'(x) + 2xf_2'(x^2) = 2x$ and $2xf_1'(x) + x^2f_2'(x^2) = -2.$

Solving these, $f_1'(x) = -(2/3) \times x - (4/3) \times x^{-1}$ ...(7)

and $f_2'(x^2) = (2/3) \times x^{-2} + (4/3)$ ...(8)

Integrating (7), $f_1(x) = -(1/3) \times x^2 - (4/3) \times \log x + c_1$

which gives $f_1(y^2x) = -(1/3) \times y^4x^2 - (4/3) \times \log (y^2x) + c_1.$ ...(9)

Writing X for x^2 in (8), $f_2'(X) = (2/3) \times (1/X) + (4/3)$

Integrating it, $f_2(X) = (2/3) \times \log X + (4/3) \times X + c_2$

which gives $f_2(yx^2) = (2/3) \times \log (yx^2) + (4/3) \times (yx^2) + c_2$ ...(10)

Putting the values of $f_1(y^2x)$ and $f_2(yx^2)$ from (9) and (10) in (2) and writing $c_1 + c_2 = c/3$, the complete solution is

$$z = -(1/3) \times y^4x^2 - (4/3) \times \log (y^2x) + (2/3) \times \log(yx^2) + (4/3) \times (yx^2) + c/3$$

or $$3z = -y^4x^2 - 4(\log x + 2 \log y) + 2(\log y + 2 \log x) + 4yx^2 + c$$

or $$3z = -y^4x^2 - 6 \log y + 4yx^2 + c.$$

Now equating values of z from (3) and (11) and putting $y = 1$, we have

$$x^2 - 1 = (1/3)[-x^2 - 6 \log 1 + 4x^2 + c], \text{ giving } c = -3.$$

So the required surface is $3z = 4yx^2 - y^4x^2 - 6 \log y - 3.$

MISCELLANEOUS PROBLEMS ON CHAPTER 6

1. Show that a linear partial differential equation of the type

$$\sum C_{qs}x^q y^s \frac{\partial^{q+s} z}{\partial x^q \partial y^s} = f(x, y)$$

may be reduced to one with constant coefficients by the substitutions $\log x = \xi$, $\log y = \eta$.

(Meerut 2008)

2. Find the general solution of $x^2(\partial^2 z / \partial x^2) + y^2(\partial^2 z / \partial y^2) = z$ **[Pune 2010]**

Sol. Let $x = e^u$ and $y = e^v$ so that $u = \log x$ and $v = \log y$ ... (1)

Also, let $D \equiv \partial / \partial x$, $D' \equiv \partial / \partial y$, $D_1 \equiv \partial / \partial u$ and $D_1' = \partial / \partial v$ ... (2)

Then, the given equation $(x^2 D^2 + y^2 D'^2 - 1)\, z = 0$ reduce to

$$\{D_1(D_1 - 1) + D_1'(D_1' - 1) - 1\}z = 0 \quad \text{or} \quad (D_1^2 + D_1' - D_1 - D_1' - 1)z = 0 \quad ... (3)$$

Let $z = A\, e^{hu + kv}$ be a trial solution of (3). Then, we have

$D_1 z = Ah\, e^{hu+kv}$, $D_1^2 z = Ah^2 e^{hu+kv}$, $D_1' z = Ak\, e^{hu+kv}$ and $D_1'^2 z = Ak^2\, e^{hu+kv}$

Substituting the above values of $D_1 z, D_1^2 z, D_1' z$ and $D_1'^2 z$ in (3), we have

$A(h^2 + k^2 - h - k - 1)\, e^{hu+kv} = 0$ so that $h^2 + k^2 - h - k - 1 = 0$, taking $A \neq 0$ ... (4)

Hence the required solution is given by

$z = \sum_i A_i e^{h_i u + k_i v}$ or $z = \sum_i A_i e^{h_i \log x + k_i \log y}$, using (1)

or $z = \sum A_i x^{h_i} y^{k_i}$, A_i, h_i and k_i being arbitrary constants.

7

Partial Differential Equations of order Two With Variable Coefficients

7.1 INTRODUCTION

In the present chapter, we propose to discuss partial differential equations of order two with variable coefficients. An equation is said to be of order two, if it involves at least one of the differential coefficients $r\ (= \partial^2 z/\partial x^2)$, $s\ (= \partial^2 z/\partial x \partial y)$, $t\ (= \partial^2 z/\partial y^2)$, but none of higher order ; the quantities p and q may also enter into the equation. Thus, the general form of a second order partial differential equation is

$$f(x, y, z, p, q, r, s, t) = 0. \qquad ...(1)$$

The most general linear partial differential equation of order two in two independent variables x and y with variable coefficients is of the form

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \qquad ...(2)$$

where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

In what follows, we shall show how a large class of second order partial differential equations may be solved by using the methods of solving ordinary differential equations.

Note that x and y, being independent variables, are constant with respect to each other in differentiation and integration. To understand this, note the solution of the following equation.

$$s = 2x + 2y. \qquad ...(3)$$

From (3), $\dfrac{\partial^2 z}{\partial x \partial y} = 2x + 2y$ or $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = 2x + 2y.$ ...(4)

Integrating (4) w.r.t. 'x', $(\partial z/\partial y) = x^2 + 2xy + f(y)$, where $f(y)$ in an arbitrary functions of y. ...(5)

Integrating (5) w.r.t. 'y', $z = x^2 y + xy^2 + F(y) + g(x),$

where F *and* g are arbitrary functions and $F(y)$ is given by $F(y) = \int f(y)\, dy.$

In what follows we shall use the following results.

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y} \quad \text{and} \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

We shall now consider some special types of equations based on (2).

7.2. Type I. Under this type, we consider equations of the form

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{F}{R} = f_1(x, y), \qquad t = \frac{\partial^2 z}{\partial y^2} = \frac{F}{T} = f_2(x, y), \qquad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{F}{S} = f_3(x, y).$$

These are homogeneons linear partial differential equations with constant coefficients and can be solved by methods discussed in chapter 4. However a more direct method of solving such equation will be used in practice.

7.3 SOLVED EXAMPLES BASED ON ART 7.2.

Ex 1. *Solve the following partial differential equations:*

(*i*) $r = 6x$. **[Agra 2009; Bhopal 2010]** (*ii*) $ar = xy$ **[Meerut 2001; Vikram 2003]**

(*iii*) $r = x^2e^y$ **[Indore 2004]** (*iv*) $r = 2y^2$

(*v*) $r = \sin(xy)$

Sol. (*i*) Given equation can be written as $\partial^2 z/\partial x^2 = 6x$. ...(1)

Integrating (1) with respect to 'x', $\partial z/\partial x = 3x^2 + \phi_1(y)$, ...(2)

where $\phi_1(y)$ is an arbitrary function of y.

Integrating (2) with respect to 'x', $z = x^3 + x\phi_1(y) + \phi_2(y)$,

where $\phi_2(y)$ is an arbitrary function of y.

(*ii*) Given equation can be written as $\partial^2 z/\partial x^2 = (1/a) \times xy$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial x = (y/a) \times (x^2/2) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = (y/6a) \times x^3 + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

(*iii*) Try yourself. **Ans.** $z = (e^y/12) \times x^4 + x\phi_1(y) + \phi_2(y)$.

(*iv*) Try yourself. **Ans.** $z = x^2y^2 + x\phi_1(y) + \phi_2(y)$.

(v) Given equation can be written as $\partial^2 z/\partial x^2 = \sin(xy)$. ...(1)

Integrating (1) w.r.t. 'x'. $\partial z/\partial x = -(1/y) \times \cos(xy) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = -(1/y^2) \times \sin(xy) + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 2. *Solve* (*i*) $t = \sin(xy)$ **(Meerut 2008)** (*ii*) $t = x^2 \cos(xy)$.

Sol.(*i*) Given equation can be written as $\partial^2 z/\partial y^2 = \sin(xy)$. ...(1)

Integrating (1) w.r.t. 'y', $\partial z/\partial y = -(1/x) \times \cos(xy) + \phi_1(x)$. ...(2)

Integrating (2) w.r.t., 'y', $z = -(1/x^2) \times \sin(xy) + y\phi_1(x) + \phi_2(x)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) Given equation can be written as $\partial^2 z/\partial y^2 = x^2 \cos(xy)$. ...(1)

Integrating (1) w.r.t. 'y', $\partial z/\partial y = x \sin(xy) + \phi_1(x)$. ...(2)

Integrating (2) w.r.t. 'y', $z = -\cos(xy) + y\,\phi_1(x) + \phi_2(x)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 3. *Solve the following partial differential equations:*

(*i*) $xys = 1$ **[Agra 2007; Rewa 2004, Vikram 2005]** (*ii*) $xy^2s = 1 - 2x^2y$

(*iii*) $\log s = x + y$ (*iv*) $s = x - y$

(*v*) $s = x^2 - y^2$ (*vi*) $x^2s = \sin y$

(*vii*) $s = (x/y) + a$ (*viii*) $s = 0$.

Sol. (*i*) Re-written the given equation, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{1}{xy}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = (1/y) \times \log x + \phi_1(y)$.

Integrating (2) w.r.t. 'y', $z = \log x \log y + \int \phi_1(y)dy + \psi_2(x)$

or $z = \log x \log y + \psi_1(y) + \psi_2(x)$, taking $\psi_1(y) = \int \phi_1(y)dy$.

which is the required general solution, ψ_1, ψ_2 being arbitrary functions.

(*ii*) Given equation is $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{1}{xy^2} - \dfrac{2x}{y}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = (1/y^2) \times \log x - (x^2/y) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = -(1/y) \times \log x - x^2 \log y + \int \phi_1(y)dy + \psi_2(x)$

or $z = -(1/y) \times \log x - x^2 \log y + \psi_1(y) + \psi_2(x)$, taking $\psi_1(y) = \int \phi_1(y)dy$.

which is the required general solution, ψ_1, ψ_2 being arbitrary functions.

(*iii*) The given equation $\log s = x + y$ can be rewritten as

$s = e^{x+y}$ or $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = e^x \cdot e^y$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = e^x e^y + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = e^x e^y + \int \phi_1(y)dy + \psi_2(x)$

or $z = e^{x+y} + \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)dy$, ψ_1, ψ_2 being arbitrary functions

(*iv*) Try yourself. **Ans.** $z = (1/2) \times (x^2y - xy^2) + \psi_1(y) + \psi_2(x)$.

(*v*) Try yourself. **Ans.** $z = (1/3) \times (x^3y - xy^3) + \psi_1(y) + \psi_2(x)$.

(*vi*)Given equation can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\sin y}{x^2}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = -(1/x) \times \sin y + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = (1/x) \cos y + \int \phi_1(y)dy + \psi_2(x)$

or $z = (1/x) \cos y + \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)dy$, ψ_1, ψ_2 being arbitrary functions

(*vii*) Try yourself. **Ans.** $z = (1/2) \times x^2 \log y + axy + \psi_1(y) + \psi_2(x)$.

(*viii*) Try yourself. **Ans.** $z = \psi_1(y) + \psi_2(x)$.

Ex. 4. *Solve* (*i*) $xr = p$ **[Agra 2007]** (*ii*) $rx = (n-1)p$.

Sol.(*i*) Given equation can be rewritten as

$x\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial x}$ or $\dfrac{\partial^2 z/\partial x^2}{\partial z/\partial x} = \dfrac{1}{x}$.

Integrating, $\log(\partial z/\partial x) = \log x + \log \phi_1(y)$ or $\partial z/\partial x = x\phi_1(y)$.

Integrating it w.r.t. x, $z = (x^2/2) \times \phi_1(y) + \phi_2(y)$, where $\phi_1(y)$ and $\phi_2(y)$ are arbitrary functions.

(*ii*) Given $x\dfrac{\partial^2 z}{\partial x^2} = (n-1)\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial^2 z/\partial x^2}{\partial z/\partial x} = \dfrac{n-1}{x}$.

Integrating, $\log(\partial z/\partial x) = (n-1)\log x + \log \phi_1(y)$ or $\partial z/\partial x = x^{n-1}\phi_1(y)$.

Integrating it, $z = (x^n/n) \times \phi_1(y) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 5. *Solve* (*i*) $xr + 2p = 0$ (*ii*) $2yq + y^2t = 1$.

Sol.(*i*) The given equation can be rewritten as

$x\dfrac{\partial p}{\partial x} + 2p = 0$ or $x^2\dfrac{\partial p}{\partial x} + 2xp = 0$ or $\dfrac{\partial}{\partial x}(x^2p) = 0$. ...(1)

Integrating (1) w.r.t. 'x', $x^2p = \phi_1(y)$ or $p = \partial z/\partial x = (1/x^2) \times \phi_1(y)$.

Integrating it w.r.t. 'x', $z = -(1/x) \times \phi_1(y) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) The given equation can be rewritten as

$2yq + y^2\dfrac{\partial q}{\partial y} = 1$ or $\dfrac{\partial}{\partial y}(y^2q) = 0$. ...(1)

Integrating (1) w.r.t. 'y', $y^2q = \phi_1(x)$ or $q = \partial z/\partial y = (1/y^2) \times \phi_1(x)$.

Integrating it, $z = -(1/y) \times \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 6. *Solve* $xs + q = 4x + 2y + 2$.

Sol. The given equation can be re-written as

$$x\frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 4x + 2y + 2 \qquad \text{or} \qquad \frac{\partial}{\partial y}(xp + z) = 4x + 2y + 2.$$

Integrating it w.r.t. 'y', $xp + z = 4xy + y^2 + 2y + \phi_1(x)$

or $x\dfrac{\partial z}{\partial x} + z = 4xy + y^2 + 2y + \phi_1(x)$ or $\dfrac{\partial}{\partial x}(xz) = 4xy + y^2 + 2y + \phi_1(x).$

Integrating it w.r.t. 'x', $xz = 2x^2y + xy^2 + 2xy + \int \phi_1(x)dx + \psi_2(y)$

or $\therefore$ Required solution is $xz = 2x^2y + xy^2 + 2xy + \psi_1(x) + \psi_2(y)$, where $\psi_1(x) = \int \phi_1(x)dx$.

Ex. 7. *Solve* $ys + p = \cos(x + y) - y \sin(x + y)$. **[Meerut 1995]**

Sol. The given equation can be rewritten as

$$y\frac{\partial q}{\partial x} + \frac{\partial z}{\partial x} = \cos(x+y) - y\sin(x+y) \qquad \text{or} \qquad \frac{\partial}{\partial x}(yq + z) = \cos(x+y) - y\sin(x+y).$$

Integrating it w.r.t. 'x', $yq + z = \sin(x + y) + y\cos(x + y) + \phi_1(y)$.

or $y\dfrac{\partial z}{\partial y} + z = \sin(x+y) + y\cos(x+y) + \phi_1(y)$ or $\dfrac{\partial(yz)}{\partial y} = \sin(x+y) + y\cos(x+y) + \phi_1(y).$

Integrating it w.r.t. 'y', $yz = \int \sin(x+y)dy + \int y\cos(x+y)\,dy + \int \phi_1(y)\,dy + \psi_2(y)$

or $yz = \int \sin(x+y)dy + y\sin(x+y) - \int \sin(x+y)\,dy + \psi_1(y) + \psi_2(y)$

[Integrating by parts and taking $\psi_1(y) = \int \phi_1(y)\,dy$]

Required solution is $yz = y\sin(x + y) + \psi_1(y) + \psi_2(y)$, ψ_1, ψ_2 being arbitrary functions.

7.4. Type II. Under this type, we consider equations of the form:

$Rr + Pp = F$, *i.e.*, $R\dfrac{\partial p}{\partial x} + Pp = F$; $Ss + Pp = F$, *i.e.*, $S\dfrac{\partial p}{\partial y} + Pp = F$,

$Ss + Qq = F$, *i.e.*, $S\dfrac{\partial q}{\partial x} + Qq = F$; $Tt + Qq = F$, *i.e.*, $T\dfrac{\partial q}{\partial y} + Qq = F$.

These will be treated as ordinary linear differential equations of order one in which p (or q) is the dependent variable.

7.5 SOLVED EXAMPLES BASED ON ART 7.4

Ex. 1. *Solve* (*i*) $t - xq = x^2$. **[Ravishanker 2010; Nagpur 1996]**

(*ii*) $yt - q = xy$. **[Meerut 1997]**

Sol. (*i*) The given equation can be rewritten as $(\partial q/\partial y) - xq = x^2$, ...(1)

which is linear differential equation in variables q and y, regarding x as constant.

Integrating factor (I.F.) of (1) $= e^{\int(-x)dy} = e^{-xy}$ and solution of (1) is

$q(I.F.) = \int (x^2)(I.F.)dy + \phi_1(x)$ or $qe^{-xy} = \int x^2 e^{-xy}\,dy + \phi_1(x)$

or $qe^{-xy} = x^2 \times (-1/x) \times e^{-xy} + \phi_1(x)$ or $q = \partial z / \partial y = -x + e^{xy}\phi_1(x)$.

Integrating it w.r.t. 'y', $z = -xy + (1/x) \times \phi_1(x)\, e^{xy} + \psi_2(x)$

or $z = -xy + \psi_1(x)e^{xy} + \psi_2(x)$, where $\psi_1(x) = (1/x) \times \phi_1(x)$.

It is the required solution, ψ_1, ψ_2 being arbitrary functions.

(*ii*) The given equation can be rewritten as $y(\partial q/\partial y) - q = xy$ or $(\partial q/\partial y) - (1/y) \times q = x$,

which is differential equation linear in variables q and y, regarding x as constant.

I.F. of (1) $= e^{\int(-1/y)dy} = e^{-\log y} = 1/y$ and solution of (1) is

$$q \times \frac{1}{y} = \int \left(x \times \frac{1}{y} \right) dy + \phi_1(x) \qquad \text{or} \qquad \frac{q}{y} = x \log y + \phi_1(x)$$

or $q = xy \log y + y\phi_1(x)$ or $\partial z/\partial y = xy \log y + y\phi_1(x)$.

Integrating it, $z = x\left[(y^2/2) \times \log y - \int \left\{ (y^2/2) \times (1/y) \right\} dy \right] + (y^2/2) \times \phi_1(x) + \phi_2(x)$

or $z = (1/2) \times xy^2 \log y - (1/4) \times xy^2 + (1/2) \times y^2 \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 2. *Solve* $xs + q = 4x + 2y + 2$.

Sol. Re-writing $x\dfrac{\partial q}{\partial x} + q = 4x + 2y + 2$ or $\dfrac{\partial q}{\partial x} + \dfrac{1}{x} q = 4 + \dfrac{2y}{x} + \dfrac{2}{x}$.

Its I.F. $= e^{\int (1/x) dx} = e^{\log x} = x$ and hence its solution is

$$qx = \int x\left(4 + \frac{2y}{x} + \frac{2}{x} \right) dx + \phi_1(y) = 2x^2 + 2xy + 2x + \phi_1(y)$$

or $q = \partial z/\partial y = 2x + 2y + 2 + (1/x) \times \phi_1(y)$.

Integrating, $z = 2xy + y^2 + 2y + (1/x) \times \int \phi_1(y)\, dy + \psi_2(x)$

or $z = 2xy + y^2 + 2y + (1/x) \times \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)\, dy$.

Ex. 3. *Solve* $xr + p = 9x^2y^3$. **[Ranchi 2010]**

Sol. The given equation can be re-written as $x\dfrac{\partial p}{\partial x} + p = 9x^2y^3$ or $\dfrac{\partial p}{\partial x} + \dfrac{1}{x} p = 9xy^3$.

Its I.F. $= e^{\int (1/x) dx} = e^{\log x} = x$ and hence solution is

$px = \int \{ x \times (9xy^3) \} dx + \phi_1(y)$ or $px = 3x^3y^3 + \phi_1(y)$

or $p = 3x^2y^3 + (1/x) \times \phi_1(y)$ or $(\partial z/\partial x) = 3x^2y^3 + (1/x) \times \phi_1(y)$.

Integrating, $z = x^3y^3 + \phi_1(y) \log x + \phi_2(y)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 4. *Solve* $ys - p = xy^2 \cos(xy)$.

Sol. Re-writing given equation, $y\dfrac{\partial p}{\partial y} - p = xy^2 \cos(xy)$ or $\dfrac{\partial p}{\partial y} - \dfrac{1}{y} p = xy \cos(xy)$.

Its I.F. $= e^{\int (-1/y) dy} = e^{-\log y} = 1/y$ and so its solution is

$$p \times (1/y) = \int (1/y) \times [xy \cos(xy)] dx + \phi_1(x) = \sin(xy) + \phi_1(x)$$

or $p = y \sin(xy) + y\,\phi_1(x)$ or $\partial z/\partial x = y \sin(xy) + y\phi_1(x)$.

Integrating, $z = -\cos(xy) + y \int \phi_1(x)\, dx + \psi_2(y)$

or $z = -\cos(xy) + y\psi_1(x) + \psi_2(y)$, where $\psi_1(x) = \int \phi_1(x)\, dx$.

Ex. 5. *Solve* $t - xq = -\sin y - x \cos y$.

Sol. Re-writting, $(\partial q/\partial y) - xq = -\sin y - x \cos y$, which is linear differential equation in q and y.

Its I.F. $= e^{\int (-x) dy} = e^{-xy}$ and so its soltion is

$$qe^{-xy} = -\int e^{-xy} (\sin y + x \cos y) dy + \phi_1(x) = -\int e^{-xy} \sin y\, dy - x \int e^{-xy} \cos y\, dy + \phi_1(x)$$

$$= -\int e^{-xy} \sin y\, dy - x \left[-\frac{1}{x} e^{-xy} \cos y - \int \left(\frac{1}{x} e^{-xy} \sin y \right) dy \right] + \phi_1(x)$$

$$= -\int e^{-xy} \sin y \, dy + e^{-xy} \cos y + \int e^{-xy} \sin y \, dy + \phi_1(x)$$

or $(\partial z/\partial y)\, e^{-xy} = e^{-xy} \cos y + \phi_1(x)$ or $(\partial z/\partial y) = \cos y + e^{xy}\, \phi_1(x)$.

Integrating, $z = \sin y + (1/x) \times e^{xy}\, \phi_1(x) + \psi_2(x)$

or $z = \sin y + e^{xy}\, \psi_1(x) + \psi_2(x)$, where $\psi_1(x) = (1/x) \times \phi_1(x)$.

Ex. 6. *Solve* $xys - qy = x^2$. **[Delhi Maths Hons. 1992, 93]**

Sol. Re-writing the given equation, we have

$$xy\frac{\partial q}{\partial x} - qy = x^2 \quad \text{or} \quad \frac{\partial q}{\partial x} - \frac{1}{x} q = \frac{x}{y}. \qquad ...(1)$$

which is linear differential equation in variables q and x.

Integrating factor of (1) $= e^{-f(1/x)\,dx} = e^{-\log x} = (1/x)$. Hence solution of (1) is given by

$$q \times \frac{1}{x} = \int \{(x/y) \times (1/x)\}\, dx = \frac{x}{y} + f(y) \quad \text{or} \quad \frac{\partial z}{\partial y} = \frac{x^2}{y} + x\, f(y). \qquad ...(2)$$

Integrating (2), $z = x^2 \log y + x\phi_1(y) + \phi_2(x)$, where $\phi_1(y)$ and $\phi_2(x)$ are arbitrary functions.

Ex. 7. *Solve* $xs + q - xp - z = (1 - y)\,(1 + \log x)$.

Sol. Re-writing the given equations $\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{1}{x}\dfrac{\partial z}{\partial y} - \left(\dfrac{\partial z}{\partial x} + \dfrac{z}{x}\right) = \dfrac{1-y}{x}\,(1 + \log x)$

or $$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x} + \frac{z}{x}\right) - \left(\frac{\partial z}{\partial x} + \frac{z}{x}\right) = \frac{1-y}{x}\,(1 + \log x). \qquad ...(1)$$

Let $$u = (\partial z/\partial x) + (z/x). \qquad ...(2)$$

$\therefore$ (1) $\Rightarrow$ $\dfrac{\partial u}{\partial y} - u = \dfrac{1-y}{x}\,(1 + \log x)$, which is linear differential equation ...(3)

Integrating factor of (3) $= e^{-\int dy} = e^{-y}$ and so solution of (3) is

$$ue^{-y} = \int \frac{1-y}{x}\,(1 + \log x) e^{-y}\, dy = \frac{1 + \log x}{x} \int (1-y)\, e^{-y} dy$$

$$= \frac{1 + \log x}{x}\left[(1-y)(-e^{-y}) - \int (-1)(e^{-y}) dy\right] = \frac{1 + \log x}{x}\,[-e^{-y} + y\, e^{-y} + e^{-y}] + \phi(x)$$

or $$u = (y/x) \times (1 + \log x) + e^y\, \phi(x). \qquad ...(4)$$

Then, using (2), $\dfrac{\partial z}{\partial x} + \dfrac{z}{x} = \dfrac{y}{x}(1 + \log x) + e^y \phi(x)$, which is linear differential equation ...(5)

Integrating factor of (5) $= e^{\int (1/x)\,dx} = e^{\log x} = x$ and solution of (5) is

$$zx = \int x\left[\frac{y}{x}(1 + \log x) + e^y \phi(x)\right] dx + \phi_2(y) = y \int (1 + \log x)\, dx + e^y \int x\, \phi(x) dx + \phi_2(y)$$

or $zx = y\left[(1 + \log x) \times x - \int \{(1/x) \times x\}\, dx\right] + e^y \phi_1(x) + \phi_2(y)$ or $zx = xy \log x + e^y \phi_1(x) + \phi_2(y)$.

Ex. 8. *Solve* $ys + p = \cos(x + y) - y \sin(x + y)$. **[Meerut 1995]**

Sol. Re-writing given equation, $\dfrac{\partial p}{\partial y} + \dfrac{1}{y} p = \dfrac{1}{y}\cos(x + y) - \sin(x + y)$, ...(1)

which is linear differential equation whose I.F. $= e^{\int (1/y) dy} = y$ and so solution of (1) is

$$py = \int y\left[\frac{1}{y}\cos(x+y) - \sin(x+y)\right] dy = \sin(x+y) - \int y \sin(x+y)\, dy$$

$$= \sin(x+y) - \left[-y\cos(z+y) - \int \{-\cos(x+y)\}\, dy\right]$$

$$= \sin(x+y) + y\cos(x+y) - \sin(x+y) + F(x)$$

$\therefore$ $y(\partial z/\partial x) = y \cos(x + y) + F(x)$ or $(\partial z/\partial x) = \cos(x + y) + (1/y) \times F(x)$.

Integrating, $z = \sin(x+y) + (1/y) \times \phi_1(x) + (1/y) \times \phi_2(y)$, where $\phi_1(x) = \int F(x)\,dx$

or $yz = y \sin(x+y) + \phi_1(x) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 9. *Solve* $yt + 2q = (9y+6)e^{2x+3y}$.

Sol. Re-writing, $\dfrac{\partial q}{\partial y} + \dfrac{2}{y}q = \left(9 + \dfrac{6}{y}\right)e^{2x+3y}$, which is linear differential equations ...(1)

whose integrating factor $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and so solution of (1) is

$$qy^2 = \int y^2\left(9 + \frac{6}{y}\right)e^{2x+3y}dy = e^{2x}\int (9y^2 + 6y)e^{3y}\,dy$$

or $$qy^2 = e^{2x}\left[(9y^2+6y)\left(\frac{1}{3}e^{3y}\right) - (18y+6)\left(\frac{1}{9}e^{3y}\right) + 18\left(\frac{1}{27}e^{3y}\right)\right] + \phi_1(x)$$

[using chain rule of integrating by parts]

or $qy^2 = 3y^2e^{2x+3y} + \phi_1(x)$ or $y^2(\partial z/\partial y) = 3y^2\,e^{2x+3y} + \phi_1(x)$.

$\therefore$ $(\partial z/\partial y) = 3e^{2x+3y} + (1/y^2) \times \phi_1(x)$.

Integrating, $z = e^{2x+3y} - (1/y) \times \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 10. *Solve* (*i*) $2yq + y^2t = 1$ (*ii*) $xr + p = 9x^2y^2$.

Sol. (*i*) Re-writing given equation $2yq + y^2(\partial q/\partial y) = 1$ or $\partial q/\partial y + (2/y)q = 1/y^2$, which is linear differential equation in variables q and y, regarding x as constant.

Its I.F. $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and solution is $qy^2 = \int (1/y^2)\,dy + \phi_1(x)$

or $qy^2 = y + \phi_1(x)$ or $\partial z/\partial y = (1/y) + (1/y^2) \times \phi_1(x)$.

Integrating it w.r.t. 'y', $z = \log y - (1/y) \times \phi_1(x) + \phi_2(x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

(*ii*) Do as in Ex. 3 of Art 7.5. **Ans.** $z = x^3y^2 + \log x\ \phi_1(y) + \phi_2(y)$.

7. 6. Type III. Under this type, we consider equations of the form

$Rr + Ss + Pp = F$ or $R(\partial p/\partial x) + S(\partial p/\partial y) = F - Pp$

and $Ss + Tt + Qq = F$ or $S(\partial q/\partial x) + T(\partial q/\partial y) = F - Qq$.

These are linear partial differential equations of order one with p (or q) as dependent variable and x, y, as independent variables. In such situations we shall apply well known Lagrange's method (for more details refer chapter 2).

Recall that $Pp + Qq = R$ is solved by considering its auxiliary equatrions $dx/P = dy/Q = dz/R$. Sometimes the given equation can be reduced to $Pp + Qq = R$ with help of integration of the given equation.

7.7 SOLVED EXAMPLES BASED ON ART 7.6

Ex. 1. *Solve* $t + s + q = 0$. **[Meerut 1994]**

Sol. Re-writing the given equation, $(\partial q/\partial y) + (\partial p/\partial y) + (\partial z/\partial y) = 0$.

Integrating w.r.t. 'y', $q + p + z = f(x)$ or $p + q = f(x) - z$, ...(1)

which is in Lagrange's form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z}. \qquad ...(2)$$

From first and second fractions of (2), $dx - dy = 0$

Integrating, $x - y = c_1$, c_1 being an arbitrary constant. ...(3)

From first and third fraction of (2), $(dz/dx) = f(x) - z$ or $(dz/dx) + z = f(x)$.

Its I.F. $= e^{\int dx} = e^x$ and hence its solution is $ze^x = \int e^x f(x)\,dx + c_2$

or $ze^x - \phi(x) = c_2$, where $\phi(x) = \int e^x(x)\,dx$ and c_2 is an arbitrary constant ...(4)

From (3) and (4), the required general solution is

$ze^x - \phi(x) = \psi(x-y)$ or $ze^x = \phi(x) + \psi(x-y)$, where ϕ and ψ are arbitrary functions.

Ex. 2. *Solve* $p + r + s = 1$. **[Kanpur 2004; Meerut 2005, 10]**

Sol. Re-writing the given equation $(\partial z/\partial x) + (\partial p/\partial x) + (\partial q/\partial x) = 1$

Integrating w.r.t. 'x', $z + p + q = x + f(y)$ or $p + q = x + f(x) - z$, ...(1)

which is in Langrange's form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x + f(y) - z}. \quad \text{... (2)}$$

From first and second fractions of (2), $dx - dy = 0$.

Integrating, $x - y = c_1$, c_1 being an arbitrary constant. ...(3)

From second and third fractions of (2), $(dz/dy) = x + f(y) - z$ or $(dz/dy) + z = x + f(y)$.

Its I.F. $= e^{\int dy} = e^y$ and hence its solution is

$$ze^y = \int \{x + f(y)\}\, e^y dy + c_2 = x \int e^y\, dy + \int e^y f(y)\, dy + c_2$$

or $ze^y - xe^y - \phi(y) = c_2$, where $\phi(y) = \int e^y f(y)\,dy$. ...(4)

From (3) and (4), the required general solution is $ze^y - xe^y - \phi(y) = \psi(x-y)$

or $(z-x)e^y = \phi(y) + \psi(x-y)$, where ϕ and ψ are arbitrary functions.

Ex. 3. *Solve* $s - t = x/y^2$. **[Ravishankar 2005; I.A.S. 1988]**

Sol. The given equation can be re-written as $\frac{\partial p}{\partial y} - \frac{\partial q}{\partial y} = \frac{x}{y^2}$ or $\frac{\partial}{\partial y}(p - q) = xy^{-2}$.

Integrating it w.r.t. 'y', $p - q = -(x/y) + f(x)$, ...(1)

which is in Lagrange's form $Pp + Qq = R$. Its auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{-(x/y) + f(x)}. \quad \text{... (2)}$$

Taking first two fractions of (2), $dx + dy = 0$ so that $x + y = c_1$. ...(3)

Taking first and third fractions of (2), $dz = [-(x/y) + f(x)]\,dx$

or $dz = \left[-\{x/(c_1 - x)\} + f(x)\right]dx$, since from (3), $y = c_1 - x$

or $dz = [1 - \{c_1/(c_1 - x)\} + f(x)]\,dx$.

Integrating, $z = x + c_1 \log(c_1 - x) + \phi(x) + c_2$, where $\phi(x) = \int f(x)\,dx$

or $z - x - (x + y)\log y - \phi(x) = c_2$, using (3). ...(4)

From (3) and (4), the required general solution is $z - x - (x+y)\log y - \phi(x) = \psi(x+y)$

or $z = x + (x+y)\log y + \phi(x) + \psi(x+y)$, where ϕ and ψ are arbitrary functions.

Ex. 4. *Solve* $xyr + x^2 s - yp = x^3 e^y$.

Sol. Re-writing the given equations, $xy(\partial p/\partial x) + x^2(\partial p/\partial y) = yp + x^3 e^y$. ...(1)

Here Lagrange's auxiliary equations for (1) are $\frac{dx}{xy} = \frac{dy}{x^2} = \frac{dp}{yp + x^3 e^y}$. ...(2)

From the first two fractions of (2), $2x\,dx - 2y\,dy = 0$ so that $x^2 - y^2 = c_1$. ... (3)

From second and third fractions of (2), $dp/dy = (yp + x^3 e^y)/x^2$

or $\dfrac{dp}{dy}-\dfrac{yp}{x^2}=xe^y$ or $\dfrac{dp}{dy}-\dfrac{y}{y^2+c_1}p=(y^2+c_1)^{1/2}e^y$,

$[\because$ from (3), $x^2=y^2+c_1$ so that $x=(y^2+c_1)^{1/2}]$

Its I.F. $=e^{-\int\{y/(y^2+c_1)\}dy}=e^{-(1/2)\times\log(y^2+c_1)}=(y^2+c_1)^{-1/2}$ and solution of above equation is

$$p(y^2+c_1)^{-1/2}=\int\{(y^2+c_1)^{-1/2}\cdot e^y(y^2+c_1)^{1/2}\}\,dy+c_2=\int e^y dy+c_2=e^y+c_2$$

or $px^{-1}=e^y+c_2$, as from (3) $y^2+c_1=x^2$. ... (4)

From (3) and (4), the general solution of (1) is $(p/x)-e^y=f(x^2-y^2)$

or $p=x\,e^y+x\,f(x^2-y^2)$ or $\partial z/\partial x=x\,e^y+x\,f(x^2-y^2)$, f being an arbitrary function.

Integrating the above equation w.r.t. 'x', $z=e^y\int x\,dx+\int x\,f(x^2-y^2)dx+\phi(y)$

or $z=(1/2)\times x^2e^y+\psi(x^2-y^2)+\phi(y)$, where $\psi(x^2-y^2)=\int x\,f(x^2-y^2)dx$

which is the required solution, ϕ and ψ being arbitrary functions.

Ex. 5. (*i*) Solve $xr+ys+p=10xy^3$ **[Delhi Maths Hons. 1993]**

(*ii*) $xs+yt+q=10x^3y$.

Sol. Re-writing the given equation $x(\partial p/\partial x)+y(\partial p/\partial y)=10xy^3-p$. ...(1)

Its Lagrange's auxiliary equations are $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dp}{10xy^3-p}$. ...(2)

Taking the first two ratios (2), $(1/x)dx=(1/y)dy$ so that $x/y=c_1$. ...(3)

Taking the second and third ratios of (2), we have

$\dfrac{dp}{dy}=\dfrac{10xy^3-p}{y}$ or $\dfrac{dp}{dy}+\dfrac{1}{y}p=10xy^2$ or $\dfrac{dp}{dy}+\dfrac{1}{y}p=10c_1y^3$, using (3). ...(4)

I.F. of (4) $=e^{\int(1/y)dy}=e^{\log y}=y$ and so solution is

$py=\int y(10c_1y^3)\,dy=2c_1y^5+c_2$ or $py-2xy^4=c_2$, using (3) ...(5)

From (3) and (5), the general solution of (1) is

$py-2xy^4=\phi(x/y)$ or $(\partial z/\partial x)=2xy^3+(1/y)\times\phi(x/y)$.

Integrating w.r.t. 'x', $z=x^2y^3+\phi_1\left(\dfrac{x}{y}\right)+\phi_2(y)$, where $\dfrac{\partial}{\partial x}\phi_1\left(\dfrac{x}{y}\right)=\dfrac{1}{y}\phi\left(\dfrac{x}{y}\right)$.

(*ii*) Do yourself. **Ans.** $zx=x^3y^2+\phi_1(y/x)+\phi_2(x)$.

Ex. 6. *Solve* $sy-2xr-2p=6xy$.

Sol. Re-writing the given equation, $2x(\partial p/\partial x)-y(\partial p/\partial y)=-(6xy+2p)$. ...(1)

Lagrange's auxiliary equations are $\dfrac{dx}{2x}=\dfrac{dy}{-y}=\dfrac{dp}{-6xy-2p}$. ...(2)

Taking the first and second ratios in (2), we get

$(1/x)dx+(2/y)dy=0$ so that $xy^2=c_1$. ...(3)

Now, each ratio of (2) $=\dfrac{2y^3dx-(2yp+2xy^2)\,dy+y^2dp}{0}$

$\Rightarrow (2yp+2xy^2)dy-2y^3dx-y^2dp=0\Rightarrow 2y\,(p+2xy)dy-y^2(dp+2xdy+2ydx)=0.$

$\Rightarrow 2y\,(p+2xy)dy-y^2d\,(p+2xy)=0\Rightarrow -\dfrac{2dy}{y}+\dfrac{d(p+2xy)}{p+2xy}=0$

Integrating it, $-2\log y+\log(p+2xy)=\log c_1$, being an arbitrary constant

or $\log\{(p+2xy)/y^2\}=\log c_1$ or $(p+2xy)/y^2=c_1$. ...(4)

From (3) and (4), the general solution of (1) is

$(p + 2xy)/y^2 = \phi(xy^2)$ or $(\partial z/\partial x) = -2xy + y^2\phi(xy^2)$. ... (5)

Integrating (5) w.r.t. 'x', $z = -x^2y + \phi_1(xy^2) + \phi_2(y)$, where $\frac{\partial}{\partial x}\phi_1(xy^2) = y^2\phi(xy^2)$.

Ex. 7. *Solve* $r + (y/x)s = 15xy^2$.

Sol. Re-writing the given equation, $(\partial p/\partial x) + (y/x)(\partial p/\partial y) = 15xy^2$. ...(1)

So Lagrange's auxiliary equations are $\frac{dx}{1} = \frac{dy}{y/x} = \frac{dp}{15xy^2}$. ...(2)

From (2), $(1/y)\,dy = (1/x)\,dx \Rightarrow \log y - \log x = \log c_1 \Rightarrow y/x = c_1$. ...(3)

Taking the first and third ratios of (2), $dp = 15xy^2dx = 15c_1^2x^3dx$, by (3)

Integrating, $p = (15/4) \times c_1^2x^4 + c_2 \Rightarrow p - (15/4) \times (y/x)^2 \times x^4 = c_2$, by (3)

or $p - (15/4) \times x^2y^2 = c_2$, c_2 being an arbitrary constant ...(4)

Using (3) and (4), the general solution of (1) is

$p - \frac{15}{4}x^2y^2 = \phi\left(\frac{y}{x}\right)$ or $\frac{\partial z}{\partial x} = \frac{15}{4}x^2y^2 + \phi\left(\frac{y}{x}\right)$. ...(5)

Integrating (5) w.r.t. 'x', $z = \frac{5}{4}x^3y^2 + y\int \frac{1}{(-y^2/x^2)}\phi\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) + \phi_2(y)$

or $z = (5/4) \times x^3y^2 + y\phi_1(y/x) + \phi_2(y)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 8. *Solve the following partial differential equations* :

(*i*) $2xr - ys + 2p = xy^2$. (*ii*) $2yt - xs + 2q = 4yx^2$.

Ans. (*i*) $z = \phi_1(xy^2) + \phi_2(y) + (1/4) \times x^2y^2$. (*ii*) $z = \phi_1(x^2y) + \phi_2(x) + x^2y^2$.

Ex. 9. *Solve* $s + r = x\cos(x + y)$

Sol. Re-writing the given equation, $\partial p/\partial x + \partial p/\partial y = x\cos(x + y)$ ... (1)

Its Lagrange's auxiliary equations are $\frac{dx}{1} = \frac{dy}{1} = \frac{dp}{x\cos(x+y)}$ .... (2)

Taking the first two ratios, $dx - dy = 0$ so that $x - y = c_1$ ... (3)

Taking the first and the last fractions of (2) and using (3), we get

$dp = x\cos(x + x - c_1)$ or $dp = x\cos(2x - c_1)$

Integrating, $p = (1/2) \times x\sin(2x - c_1) - (1/2) \times \int \sin(2x - c_1)dx + c_2$

or $p - (1/2) \times x\sin(2x - c_1) - (1/4) \times \cos(2x - c_1) = c_2$

or $p - (1/2) \times x\sin(x + y) - (1/4) \times \cos(x + y) = c_2$, using (2). ... (4)

From (3) and (4), the general solution of (1) is given by

$$p - (1/2) \times x\sin(x + y) - (1/4) \times \cos(x + y) = f(x - y).$$

or $\partial z/\partial x = (1/2) \times x\sin(x + y) + (1/4) \times \cos(x + y) + f(x - y)$ ... (5)

Integrting (5) w.r.t. 'x', $z = (1/2) \times [-x\cos(x + y) + \int \cos(x + y)\,dx]$

$+ (1/4) \times \sin(x + y) + \phi_1(x - y) + \phi_2(y)$, where $\phi_1(x - y) = \int f(x - y)\,dx$

or $z = -(x/2) \times \cos(x + y) + (3/4) \times \sin(x + y) + \phi_1(x - y) + \phi_2(y)$.

Ex. 10. *Solve* $yt + xs + q = 8y\,x^2 + 9y^2$.

Sol. Re-writing the given equation, $x(\partial q/\partial x) + y(\partial q/\partial y) = 8yx^2 + 9y^2 - q$ ... (1)

Its Lagrange's auxiliary equations are $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dq}{8yx^2 + 9y^2 - q}$ ... (2)

Taking the first two ratios of (2), $\log y - \log x = \log c_1$ or $y/x = c_1$ ... (3)

Taking the last two ratios of (2), we get

$$\frac{dq}{dy} = 9y + 8x^2 - \frac{q}{y} \quad \text{or} \quad \frac{dq}{dy} + \frac{1}{y}q = 9y + 8x^2 = 9y + \frac{8y^2}{c_1^2}, \text{ by (3)}$$

which is linear differential equation. Its I.F. $= e^{\int (1/y)\,dy} = e^{\log y} = y$ and solution is

$$qy = \int y(9y + 8y^2/c_1^2)\,dy + c_2 = 3y^3 + (2y^4/c_1^2) + c_2$$

or $qy - 3y^3 - (2y^4/c_1^2) = c_2$ or $qy - 3y^3 - 2y^2x^2 = c_2$, by (3) ... (4)

From (3) and (4), the general solution of (1) is

$qy - 3y^3 - 2y^2x^2 = f(y/x)$ or $\partial z/\partial y = 3y^2 + 2x^2y + (1/y) \times f(y/x)$

Integrating it w.r.t. 'y' while treating x as constant, we get

$$z = y^3 + x^2y^2 + \int \frac{1}{y} f(y/x)\,dy + \phi_1(x) \quad \text{or} \quad z = y^3 + x^2y^2 + \int \frac{f(y/x)}{(y/x)} d\left(\frac{y}{x}\right) + \phi_1(x)$$

or $z = y^3 + x^2y^2 + \phi_2(y/x) + \phi_1(x)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 11. *Solve* $xyr + x^2s - yp = x^3e^y$

Sol. Re-writing the given equation, $xy\,(\partial p/\partial x) + x^2\,(\partial p/\partial y) = yp + x^3e^y$ ... (1)

Its Lagrange's auxiliary equations are $\dfrac{dx}{xy} = \dfrac{dy}{x^2} = \dfrac{dp}{yp + x^3e^y}$ ... (2)

Taking the first two ratios of (2), $2xdx - 2ydy = 0$ so that $x^2 - y^2 = c_1$ ... (3)

Taking the first and the last ratios of (2), we get

$$\frac{dp}{dx} = \frac{yp + x^3e^y}{xy} = \frac{p}{x} + \frac{x^2e^y}{y} \quad \text{or} \quad \frac{dp}{dx} - \frac{1}{x}p = \frac{x^2}{(x^2 - c_1)^{1/2}}\, e^{(x^2 - c_1)^{1/2}}, \text{ by (3)}$$

Its I.F. $= e^{\int (-1/x)\,dx} = e^{-\log x} = 1/x$ and solution is

$$p \times \frac{1}{x} = \int \frac{1}{x} \frac{x^2}{(x^2 - c_1)^{1/2}}\, e^{(x^2 - c_1)^{1/2}} dx = \int e^t\, dt = e^t + c_2$$

[on putting $(x^2 - c_1)^{1/2} = t$ and $\{x/(x^2 - c_1^2)^{1/2}\}dx = dt$]

or $(p/x) - e^{(x^2 - c_1)^{1/2}} = c_2$ or $(p/x) - e^y = c_2$, using (3) ... (4)

From (3) and (4), the general solution of (1) is

$(p/x) - e^y = f(x^2 - y^2)$ or $\partial z/\partial x = x\,e^y + xf(x^2 - y^2)$ ... (5)

Integrating (5) w.r.t. 'x' (while treating y as constant), we get

$z = (1/2) \times x^2e^y + \phi_1(x^2 - y^2) + \phi_2(y)$, where $\phi_1(x^2 - y^2) = \int x\,f(x^2 - y^2)\,dx$.

7.8. Type IV. Under this type, we consider equations of the form

$$Rr + Pp + Zz = F \quad \text{or} \quad R\frac{\partial^2 z}{\partial x^2} + P\frac{\partial z}{\partial x} + Zz = F \quad ...(1)$$

and $Tt + Qq + Zz = F$ or $T\dfrac{\partial^2 z}{\partial y^2} + Q\dfrac{\partial z}{\partial y} + Zz = F,$ ...(2)

which are linear ordinary differential equations of order two with x as independent variable in (1) and y as independent variable in (2).

7.9 SOLVED EXAMPLES BASED ON ART 7.8

Ex. 1. *Solve* $t - 2xq + x^2z = (x-2)e^{3x+2y}$. **[Delhi Math (G) 1999; Poona 1996]**

Sol. Taking $D' \equiv \partial/\partial y$, the given equation becomes

$(D'^2 - 2x\,D' + x^2)z = (x-2)e^{3x+2y}$ or $(D'-x)^2 = (x-2)\,e^{3x+2y}$ ...(1)

$\therefore$ Complementary function of (1) $= e^{xy}\ \{\phi_1(x) + x\phi_2(x)\}$.

and particular integral of (1) $= \dfrac{1}{(D'-x)^2}(x-2)e^{3x+2y} = \dfrac{(x-2)e^{3x+2y}}{(2-x)^2} = \dfrac{e^{3x+2y}}{x-2}$.

$\therefore$ Required solution is $z = e^{xy}\ \{\phi_1(x) + x\phi_2(x)\} + \dfrac{e^{3x+2y}}{x-2}$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 2. *Solve* (*i*) $t - q - (1/x)\ \{(1/x) - 1\}z = xy^2 - x^2y^2 + 2x^3y - 2x^3$. **[Calicut 1999]**

(*ii*) $r - p - (1/y)\ \{(1/y) - 1\}z = x^2y - x^2y^2 + 2xy^3 - 2y^3$.

Sol. (*i*) Let $D' \equiv \partial/\partial y$. Then given equation can be re-written as

$$[D'^2 - D' - (1/x)\ \{(1/x) - 1\}]\ z = xy^2 - x^2y^2 + 2x^3y - 2x^3. \quad ...(1)$$

or $$\left(D' - \frac{1}{x}\right)\left\{D' + \left(\frac{1}{x} - 1\right)\right\} z = xy^2 - x^2y^2 + 2x^3y - 2x^3. \quad ...(1)'$$

So C.F. $= e^{y/x}\,\phi_1(x) + e^{y-(y/x)}\phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions

In order to determine a particular integral of (1), we assume that

$$z = F_1y^2 + F_2y + F_3, \text{ where } F_1, F_2, F_3 \text{ are functions of } x \text{ or constants.} \quad ...(2)$$

$$(2) \Rightarrow \partial z/\partial y = 2F_1y + F_2 \quad \Rightarrow \quad \partial^2 z/\partial y^2 = 2F_1.$$

so that $q = 2F_1y + F_2$ and $t = 2F_1$. ...(3)

Using (2) and (3), given equation reduces to

$$2F_1 - (2F_1y + F_2) - \frac{1}{x}\left(\frac{1}{x} - 1\right)(F_1y^2 + F_2y + F_3) = xy^2 - x^2y^2 + 2x^3y - 2x^3.$$

Equation coefficients of various powers of y in the above identity, we obtain

$$-\{(1/x^2) - (1/x)\}F_1 = x\,(1-x), \quad ...(4)$$

$$-2F_1 - \{(1/x^2) - (1/x)\}F_2 = 2x^3 \quad ...(5)$$

and $$2F_1 - F_2 - \{(1/x^2) - (1/x)\}F_3 = -2x^3. \quad ...(6)$$

From (4), $F_1 = -x^3$. Then, from (5), $F_2 = 0$. So (6) $\Rightarrow$ $F_3 = 0$.

$\therefore$ from (2), P.I. $= -x^3y^2$ and so the required solution is

$$z = e^{y/x}\phi_1(x) + e^{y-(y/x)}\phi_2(x) - x^3y^2.$$

(*ii*) Do your as in part (*i*). **Ans.** $z = e^{(x/y)}\phi_1(y) + e^{x-(x/y)}\phi_2(y) - x^2y^3$.

7.10. SOLUTIONS OF EQUATIONS UNDER GIVEN GEOMETRICAL CONDITIONS.

Working rule. As explained in this chapter, we first find the solution of the given equation containing some arbitrary functions of x and y, which are determined with help of the given geometrical conditions. Substituting the values of arbitrary functions in the general solution, we shall obtain surfaces which satisfy the given geometrical conditions.

7.11 SOLVED EXAMPLES BASED ON ART 7.10

Ex. 1.(a) *Find the surface satisfying* $t = 6x^2y$ *containing two lines* $y = 0 = z$ *and* $y = 2 = z$. **[Kanpur 2001; Sagar 2004]**

Sol. Re-writing the given equation, we get $\partial q/\partial y = 6x^2y$.

Integrating it w.r.t. 'y', $q = 3x^2y^2 + f(x)$ or $\partial z/\partial y = 3x^2y^2 + f(x)$.

Integrating it w.r.t. 'y', $z = x^2y^3 + y\,f(x) + \phi(x)$, ...(1)

which is the general solution, f and ϕ being arbitrary functions.

Since (1) contains the given lines $y = 0 = z$ and $y = 2 = z$, we get $0 = \phi(x)$ ...(2)

and $2 = 8x^2 + 2\,f(x) + \phi(x)$. ...(3)

Using (2), (3) becomes $2 = 8x^2 + 2\,f(x)$ or $f(x) = 1 - 4x^2$.

Putting $\phi(x) = 0$ and $f(x) = 1 - 4x^2$ in (1), the required surface is $z = x^2y^3 + y(1 - 4x^2)$.

Ex. 1.(b) *Find a surface satisfying* $t = 6x^3y$ *and containing the two lines* $y = 0 = z$, $y = 1 = z$.

Sol. Re-writing the given equation, we get $(\partial q/\partial y) = 6x^3y$

Integrating w.r.t. y, $q = 3x^3y^2 + f(x)$ or $\partial z/\partial y = 3x^3y^2 + f(x)$

Integrating it w.r.t. y $z = x^3y^3 + y\,f(x) + \phi(x)$. ...(1)

Since (1) contains the lines $y = 0 = z$ and $y = 1 = z$, we get $0 = \phi(x)$ ...(2)

and $1 = x^3 + f(x) + \phi(x)$. ...(3)

From (2) and (3), $\phi(x) = 0$ and $f(x) = 1 - x^3$.

Putting these values in (1), the required surface is $z = x^3y^3 + y(1 - x^3)$.

Ex. 2. *Find the surface passing through the parabolas* $z = 0,\ y^2 = 4ax$ and $z = 1, y^2 = -4ax$ *and satisfying the equation* $xr + 2p = 0$. **[Kanpur 2000; Agra 1996 ; Meerut 1993 ; I.A.S. 2006]**

Sol. Re-writing the given differential equation, $x(\partial p/\partial x) + 2p = 0$

or $x^2(\partial p/\partial x) + 2px = 0$ or $\partial(x^2p)/\partial x = 0$

Integrating it w.r.t. x, $x^2p = f(y)$ or $p = f(y)/x^2$ or $(\partial z/\partial x) = (1/x^2) \times f(y)$.

Integrating it w.r.t. x, $z = -(1/x) \times f(y) + \phi(y)$. ...(1)

Since (1) passes through $z = 0,\ y^2 = 4ax$, $0 = -(4a/y^2) \times f(y) + \phi(y)$. ...(2)

Again since (1) passes through $z = 1,\ y^2 = -4a\,x$, $1 = (4a/y^2) \times f(y) + \phi(y)$. ...(3)

Adding (2) and (3), $1 = 2\phi(y)$ so that $\phi(y) = 1/2$. ...(4)

Putting $\phi(y) = 1/2$ in (2), we get $f(y) = y^2/8a$. ...(5)

Putting the values of $\phi(y)$ and $f(y)$ given by (4) and (5) in (1), the desired surface is

$z = -y^2/(8ax) + 1/2$ or $8axy = 4ax - y^2$.

Ex. 3. *Show that a surface satisfying* $r = 6x + 2$ *and touching* $z = x^3 + y^3$ *along its section by the plane* $x + y + 1 = 0$ *is* $z = x^3 + y^3 + (x + y + 1)^2$. **[Agra 1994; KU Kurukshetra 2004]**

Sol. Given $r = 6x + 2$ or $\partial p/\partial x = 6x + 2$. ...(1)

Integrating (1) w.r.t. x, $p = 3x^2 + 2x + f(y)$ or $\partial z/\partial x = 3x^2 + 2x + f(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = x^3 + x^2 + xf(y) + F(y)$, ...(3)

where $f(y)$ and $F(y)$ are arbitrary functions.

The given surface is $z = x^3 + y^3$ ...(4)

and the given plane is $x + y + 1 = 0$. ...(5)

Since (3) and (4) touch each other along their section by (5), the values of p and q at any point on (5) must be equal. Thus we must have

$3x^2 + 2x + f(y) = 3x^2$ ...(6)

and $xf'(y) + F'(y) = 3y^2$. ...(7)

From (5) and (6), $f(y) = -2x = 2(y + 1)$ ...(8)

From (8), $f'(y) = 2$. Using this value, (7) gives

$2x + F'(y) = 3y^2$ or $F'(y) = 3y^2 - 2x$ or $F'(y) = 3y^2 + 2(y + 1)$, using (5)

Integrating it, $F(y) = y^3 + y^2 + 2y + c$, ...(9)

where c is an arbitrary constant. Using (8) and (9), (3) gives

$$z = x^3 + x^2 + 2x(y + 1) + y^3 + y^2 + 2y + c \quad ...(10)$$

Now at the point of contact of (4) and (10) values of z must be the same and hence we have

$x^3 + x^2 + 2x(y + 1) + y^3 + y^2 + 2y + c = x^3 + y^3$ or $x^2 + 2x(y + 1) + y^2 + 2y + c = 0$

or $x^2 + 2x(-x) + (x + 1)^2 - 2(x + 1) + c = 0$, as from (5), $y + 1 = -x$ and $y = -(x + 1)$

which gives $c = 1$. Putting $c = 1$ in (10), the required surface is

$z = x^3 + x^2 + 2x(y + 1) + y^3 + y^2 + 2y + 1$ or $z = x^3 + y^3 + (x + y + 1)^2$

Ex. 4(a). *Show that a surface passing through the circle $z = 0$, $x^2 + y^2 = 1$ and satisfying the differential equation $s = 8xy$ is $z = (x^2 + y^2)^2 - 1$.* **[Agra 1993 ; Meerut 1994]**

Sol. Re-writing the given equation, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 8xy$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = 4x^2y + f(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = 2x^2y^2 + \int f(y)\,dy + \phi_1(x)$

or $z = 2x^2y^2 + \phi_2(y) + \phi_1(x)$, ...(3)

where $\phi_2(y) = \int f(y)\,dy$ and ϕ_1 ϕ_2 are arbitrary functions.

Given circle is given by $x^2 + y^2 = 1$ and $z = 0$. ...(4)

Putting $z = 0$ in (3), we have $2x^2y^2 + \phi_2(y) + \phi_1(x) = 0$. ...(5)

Now, $x^2 + y^2 = 1 \Rightarrow (x^2 + y^2)^2 = 1^2 \Rightarrow 2x^2y^2 + x^4 + y^4 = 1$. ...(6)

Comparing (5) and (6), $\phi_2(y) + \phi_1(x) = x^4 + y^4 - 1$.

Substituting the above value of $\phi_2(y) + \phi_1(x)$ in (3), we have

$z = 2x^2y^2 + x^4 + y^4 - 1$ or $z = (x^2 + y^2)^2 - 1$.

Ex. 4(b). *Find the surface passing through the cirlce $x^2 + y^2 = a^2$, $z = 0$ and satisfying the differential equation $s = 8xy$.*

Sol. Proceed as in Ex. 4(a).

Ex. 5. *Show that a surface of revolution satisfying the differential equation $r = 12x^2 + 4y^2$ and touching the plane $z = 0$ is $z = (x^2 + y^2)^2$.* **[Kanpur 1999; Agra 2000, 02 ; Meerut 1993, 97]**

Sol. The given equation can be re-written as

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 12x^2 + y^2$ or $\frac{\partial p}{\partial x} = 12x^2 + 4y^2$. ...(1)

Integrating (1) w.r.t. 'x', $p = \partial z/\partial x = 4x^3 + 4xy^2 + f(y)$. ...(2)

Integrating (2) w.r.t., 'x', $z = x^4 + 2x^2y^2 + xf(y) + g(y)$. ...(3)

Given the required surface (3) touches the plane $z = 0$. Now, for $z = 0$, $\partial z/\partial x = 0$ and so (2) reduces to $4x^3 + 4xy^2 + f(y) = 0$ or $-f(y) = 4x^3 + 4xy^2$. ...(4)

Since L.H.S. of (4) is function of y alone and R.H.S. is not a function of y alone, (4) shows that we must take each side of (4) equal to zero. Thus, we take $f(y) = 0$ ...(5)

and $4x^3 + 4xy^2 = 0$ so that $x^2 = -y^2$.

Putting $z = 0$, $x^2 = -y^2$ and $f(y) = 0$ in (3), we have

$0 = y^4 - 2y^4 + 0 + g(y)$ so that $g(y) = y^4$. ...(6)

Putting the values of $f(y)$ and $g(y)$ given by (5) and (6) in (3), desired surface is

$z = x^4 + 2x^2y^2 + y^4$ or $z = (x^2 + y^2)^2$.

8

Classification of P.D.E. Reduction to Canonical or Normal Forms. Riemann Method

8.1. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER.

Consider a general partial differential equation of second order for a function of two independent variables x and y in the form:

$$R\,r + S\,s + T\,t + f(x, y, z, p, q) = 0, \qquad ...(1)$$

where R, S and T are continuous functions of x and y only possessing partial derivatives defined in some domain D on the xy-plane. Then (1) is said to be

(i) *Hyperbolic* at a point (x, y) in domain D if $S^2 - 4RT > 0$

(ii) *Parabolic* at a point (x, y) in domain D if $S^2 - 4RT = 0$

(iii) *Elliptic* at a point (x, y) in domain D if $S^2 - 4RT < 0$.

Observe that the type of (1) is determined solely by its principal part ($R\,r + S\,s + T\,t$, which involves the highest order derivatives of z) and that the type will generally change with position in the xy-plane unless R, S and T are constants

Remark. Some authors use u in place of z. Then, we have

$r = \partial^2 u / \partial x^2$, $s = \partial^2 u / \partial x \partial y$ and $t = \partial^2 u / \partial t^2$. etc.

Examples: (i) Consider the one-dimensional wave equation $\partial^2 z / \partial x^2 = \partial^2 z / \partial y^2$ *i.e.* $r - t = 0$.
Comparing it with (1), here $R = 1$, $S = 0$ and $T = -1$.
Hence $S^2 - 4RT = 0 - \{4 \times 1 \times (-1)\} = 4 > 0$ and so the given equation is hyperbolic.

(ii) Consider the one-dimensional diffusion equation $\partial^2 z / \partial x^2 = \partial z / \partial y$ *i.e.* $r - q = 0$.
Comparing it with (1), here $R = 1$ and $S = T = 0$.
Hence $S^2 - 4RT = 0 - (4 \times 1 \times 0) = 0$ and so the given equation is parabolic.

(iii) Consider two dimensional Laplace's equation $\partial^2 z / \partial x^2 + \partial^2 z / \partial y^2 = 0$ *i.e.* $r + t = 0$.
Comparing it with (1), here $R = 1$, $S = 0$ and $T = 1$.
Hence $S^2 - 4RT = 0 - (4 \times 1 \times 1) = -4 < 0$ and so the given equation is elliptic.

Ex. 2. *Classify the following partial differential equations:*

(i) $2(\partial^2 u / \partial x^2) + 4(\partial^2 u / \partial x \partial y) + 3(\partial^2 u / \partial y^2) = 2$ **[Meerut 2006]**

(ii) $\partial^2 u / \partial x^2 + 4(\partial^2 u / \partial x \partial y) + 4(\partial^2 u / \partial y^2) = 0$ **[I.F.S. 2005]**

(iii) $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$ **[Delhi Maths (G) 2006]**

(iv) $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$ **[Delhi Maths (Prog) 2007]**

(v) $x(xy-1)r - (x^2y^2 - 1)s + y(xy-1)t + xp + yq = 0$ **[Delhi 2008]**

(vi) $(x-y)(xr - xs - ys + yt) = (x+y)(p-q)$ **[Delhi BA (Prog) II 2011]**

Sol. (*i*) Re-writing the given equation, we get $2r + 4s + 3t - 2 = 0$ ...(1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, u, p, q) = 0,$ we get $R = 2,\ S = 4$ and $T = 3.$ So $S^2 - 4RT = (4)^2 - (4 \times 2 \times 3) = -8 < 0,$ showing that the given equation is elliptic at all points .

(*ii*) Re-writing the given equation, we get $r + 4s + 4t = 0$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, u, p, q) = 0,$ we get $R = 1,\ S = 4$ and $T = 4.$ So $S^2 - 4RT = (4)^2 - (4 \times 1 \times 4) = 0,$ showing that the given equation is parabolic at all points.

(*iii*) Given $xyr - (x^2 - y^2)s - xyt + py - qx - 2(x^2 - y^2) = 0$ ...(1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0,$ we get $R = xy,\ S = -(x^2 - y^2)$ and $T = -xy.$ So, here $S^2 - 4RT = (x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2 > 0,$

showing that the given equation is hyperbolic at all points.

(*iv*) Hyperbolic (*v*) Hyperbolic

(*vi*) Hyperbolic

8.2. CLASSIFICATION OF A PARTIAL DIFFERENTIAL EQUATION IN THREE INDEPENDENT VARIABLES.

A linear partial differential differential equation of the second order in 3 independent variables $x_1,\ x_2,\ x_3$ is given by

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad \text{... (1)}$$

where $a_{ij}\ (= a_{ji})$, b_i and c are constants or some functions of the independent variables $x_1,\ x_2, x_3$ and u is the dependent variable.

Since $a_{ij} = a_{ji}$, $A = [a_{ij}]_{3\times 3}$ is a real symmetric matrix of order 3 × 3. The eigen values of matrix A are roots of the characteristic equation of A, namely, $|A - \lambda I| = 0.$

With help of matrix A, (1) is classified as follows:

I. If all the eigenvalues of A are non-zero and have the same sign, except precisely one of them, then (1) is known as *hyperbolic type of equation*.

II. If $|A| = 0$, *i.e.,* any one of the eigenvalues of A is zero, then (1) is known as *parabolic type of equation*

III. If all the eigenvalues of A are non-zero and of the same sign, then (1) is known as *elliptic type of equation*.

Note. the matrix A can be remembered as indicated below:

$$A = \begin{bmatrix} \text{Coeff. of } u_{xx} & \text{Coeff. of } u_{xy} & \text{Coeff of } u_{xz} \\ \text{Coeff. of } u_{yx} & \text{Coeff. of } u_{yy} & \text{Coeff of } u_{yz} \\ \text{Coeff. of } u_{zx} & \text{Coeff. of } u_{zy} & \text{Coeff of } u_{zz} \end{bmatrix}$$

8.2.A SOLVED EXAMPLES BASED ON ART. 8.2

Ex. 1. *Classify* $u_{xx} + u_{yy} = u_{zz}$ **[Delhi Maths (H) 2007; Kanpur 2011]**

The matrix A of the given equation is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

The eigenvalues of A are given by $|A - \lambda I| = 0$, *i.e.*,

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad -(1+\lambda)\,(1-\lambda)^2 = 0.$$

Hence $\lambda = -1$, 1, 1, showing that all the eigenvalues are non-zero and have the same sign except one. Hence the given equation is of hyperbolic type.

Ex. 2. *Classify* $u_{xx} + u_{yy} + u_{zz} + u_{yz} + u_{zy} = 0$.

Sol. The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + u_{yz} + 0 \cdot u_{zx} + u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$, using properties of determinents

Since $|A| = 0$, the given equation is of parabolic type.

Ex. 3. *Classify* $u_{xx} + u_{yy} + u_{zz} = 0$ **[Meerut 2007, 08; Kanpur 2011]**

Sol. The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + 0 \cdot u_{yz} + 0 \cdot u_{zx} + 0 \cdot u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The eigen values of A are given by $|A - \lambda I| = 0$,

i.e. $$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad (1-\lambda)^3 = 0 \quad \text{giving} \quad \lambda = 1,1,1.$$

Since all eigenvalues are non-zero and of the same sign, the given equation is of parabolic type.

Ex. 4. *Classify the following equations:*

(*i*) $u_{xx} + u_{yy} = u_z$ **[Kanpur 2011]** (*ii*) $u_{xx} + 2\,u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$. **[Delhi 2008]**

Sol. Try yourself **Ans.** (*i*) parabolic (*ii*) parabolic

8.3. Cauchy's problem for second order partial differential equation. Characteristic equation and characteristic curves (or simply characteristics) of the second order partial differential equations. (Delhi Maths (H) 2001)

Cauchy' problem. Consider the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \ldots (1)$$

in which R, S and T are functions of x and y only. The Cauchy's problem consists of the problem of determining the solution of (1) such that on a given space curve C it takes on prescribed values of z and $\partial z / \partial n$, where n is the distance measured along the normal to the curve.

As an example of Cauchy's problem for the second order partial differential equation, consider the following problem :

To determine solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ with the following data prescribed on the x-axis: $z(x, 0) = f(x)$, $z_y(x, 0) = g(x)$. Observe that y-axis is the normal to the given curve (x-axis here)

Characteristic equations and characteristic curves.

Corresponding to (1), consider the λ-quadratic

$$R\lambda^2 + S\lambda + T = 0 \quad \ldots (2)$$

where $S^2 - 4RT \geq 0$, (2) has real roots. Then, the ordinary differential equations

$$(dy/dx) + \lambda(x, y) = 0 \quad \ldots (3)$$

are called the *characteristic equations*.

The solutions of (3) are knwon as *characteristic curves* or simply the *characteristics* of the second order partial differential equation (1).

Now, consider the following three cases:

Case (*i*) If $S^2 - 4RT > 0$ (*i.e.,* if (1) is hyperbolic), then (2) has two distinct real roots λ_1, λ_2 say so that we have two characteristic equations $(dy/dx) + \lambda_1(x, y) = 0$ and $(dy/dx) + \lambda_2(x, y) = 0$.

Solving these we get two distinct families of characteristics.

Case (*ii*). If $S^2 - 4RT = 0$ (*i.e.* (1) is parabolic), then (2) has two equal real roots λ, λ so that we get only one characteristic equation (3). Solving it, we get only one family of characteristics.

Case (*iii*) If $S^2 - 4RT < 0$ (*i.e.* (1) is elliptic), then (2) has complex roots. Hence there are no real characteristics. Thus we get two families of complex characteristics when (1) is elliptic

8.4 ILLUSTRATIVE SOLVED EXAMPLES BASED ON ART. 8.3

Ex. 1. *Find the characteristics of $y^2 r - x^2 t = 0$* **[I.A.S. 2009]**

Sol. Given $y^2 r - x^2 t = 0 \quad \ldots (1)$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = 0$ and $T = -x^2$. Then $S^2 - 4RT = 0 - 4 \times y^2 \times (-x^2) = 4x^2y^2 > 0$ and hence (1) is hyperbolaic everywhere except on the coordinate axes $x = 0$ and $y = 0$.

The λ-quadratic is $R\lambda^2 + S\lambda + T = 0$ or $y^2\lambda^2 - x^2 = 0 \quad \ldots (2)$

Solving (2), $\lambda = x/y, -x/y$ (two distinct real roots). Corresponding characteristic equations are

$(dy/dx) + (x/y) = 0$ and $(dy/dx) - (x/y) = 0$

or $x\,dx + y\,dx = 0$ and $xdx - y\,dy = 0$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$, which are the required families of characteristics. Here these are families of circles and hyperbolas respectively.

Ex. 2. *Find the characteristics of $x^2 r + 2xys + y^2 t = 0$.*

Sol. Given $x^2 r + 2xys + y^2 t = 0 \quad \ldots (1)$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$. Then, $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$ and hence (1) is parabolic everywhere.

The λ-quadratic is $R\lambda^2 + S\lambda + T = 0$ or $x^2\lambda^2 + 2xy\lambda + y^2 = 0 \quad \ldots (2)$

Solving (2), $(x\lambda + y)^2 = 0$ so that $\lambda = -y/x, -y/x$ (equal roots). The characteristic equation is $(dy/dx) - (y/x) = 0$ or $(1/y)\,dy - (1/x)\,dx = 0$ giving $y/x = c_1$ or $y = c_1 x$, which is the required family of characteristics. Here it represents a family of straight lines passing through the origin.

Ex. 3. *Find the characteristics of $4r + 5s + t + p + q - 2 = 0$.*

Sol. Try yourself. **Ans.** $y - x = c_1$ and $y - (x/y) = c_2$.

Ex. 4. *Find the characteristics of $(\sin^2 x)\, r + (2 \cos x)\, s - t = 0$*

Sol. Try yourself **Ans.** $y + \operatorname{cosec} x - \cot x = c_1$, $y + \operatorname{cosec} x + \cot x = c_2$

8.5. Laplace transformation. Reduction to Canonical (or normal) forms.

[Himanchal 2007; Avadh 2001; Delhi Maths (H) 2004, 09]

Consider partial differential equation of the type $Rr + Ss + Tt + f(x, y, z, p, q) = 0, \quad \ldots (1)$

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary. Laplace transformation on (1) consists of changing the independent variables x, y to new set of continuously differentiable independent variables u, v where

$$u = u(x, y) \qquad \text{and} \qquad v = v(x, y) \qquad ...(2)$$

are to be chosen so that the resulting equation in independent variables u, v is transformed into one of three canonical forms, which are easily integrable. From (2), we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \text{and} \qquad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \qquad ...(3)$$

$$(3) \Rightarrow \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}. \qquad ...(4)$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right), \text{ by (3) and (4)}$$

$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right), \text{by (3) and (4)}$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right), \text{ by (3) and (4)}$$

$$= \frac{\partial u}{\partial y}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}.$$

Putting the above values of p, q, r, s, t, in (1) and simplifying, we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial v^2} + F(u, v, z, \partial z/\partial u, \partial z/\partial v) = 0, \qquad ...(5)$$

where

$$A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2, \qquad ...(6)$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}, \qquad ...(7)$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \qquad ...(8)$$

and $F(u, v, z, \partial z/\partial u, \partial z/\partial v)$ is the transformed form of $f(x, y, z, p, q)$.

Now we shall find out u and v so that (5) reduces to simplest possible form. The method of evaluation of desired values of u and v becomes easy when the discriminant $S^2 - 4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad ...(9)$$

is everywhere either positive, negative or zero, and now we shall present these three cases separately.

Case I. Let $S^2 - 4RT > 0$. When this condition is satisfied, then the roots λ_1, λ_2 of the equation (9) are real and distinct. The coefficients of $\partial^2 z/\partial u^2$ and $\partial^2 z/\partial v^2$ in the equation (5) will vanish if we choose u and v such that

$$\partial u/\partial x = \lambda_1 (\partial u/\partial y) \qquad ...(10)$$

and

$$\partial v/\partial x = \lambda_2 (\partial v/\partial y). \qquad ...(11)$$

Since λ_1 is a root of (9), we have $R\lambda_1^2 + S\lambda_1 + T = 0.$ ...(12)

Using (10), (6) gives $A = (R\lambda_1^2 + S\lambda_1 + T)\,(\partial u/\partial y)^2 = 0$, by (12) ...(13)

Again, since λ_2 is a root of (9), we have $R\lambda_2^2 + S\lambda_2 + T = 0$ ...(14)

Using (11), (8) gives $C = (R\lambda_2^2 + S\lambda_2 + T)\,(\partial v/\partial y)^2 = 0$, by (14) ...(15)

Re-writing (10), we have $(\partial u/\partial x) - \lambda_1(\partial u/\partial y) = 0.$ ...(16)

Lagrange's auxiliary equation for (16) are $dx/1 = dy/(-\lambda_1) = du/0$ ...(17)

Taking third fraction of (17), $du = 0$ so that $u = c_1$, c_1 being an arbitrary constant ...(18)

Taking first and second fractions of (17), we get $(dy/dx) + \lambda_1 = 0$ ...(19)

Let the solution of (19) be $f_1(x, y) = c_2$, c_2 being an arbitrary constant ...(20)

From (18) and (20), the general solution of (16) [*i.e.* (10)] is $u = f_1(x, y).$ ...(21)

Similarly, the general solution of (11) can be taken as $v = f_2(x, y).$ ...(22)

Here f_1 and f_2 are arbitrary function

We can easily verify that $$AC - B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$$

or $$B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2, \text{ as } A = C = 0. \quad ...(23)$$

Let the Jacobian J of u and v be non-zero, i.e., let

$$J = \partial(u, v)/\partial(x, y) = (\partial u/\partial x)\,(\partial v/\partial y) - (\partial u/\partial y)(\partial v/\partial x) \neq 0$$

Since $S^2 - 4RT > 0$, (23) shows that $B^2 > 0$. Hence we may divide both sides of (5) by B^2. Then noting that $A = C = 0$, (5) transforms to the form $\partial^2 z/\partial u\,\partial v = \phi\left(u, v, z, \partial z/\partial u, \partial z/\partial v\right)$, ...(24)

which is the canonical form of (1) in this case.

Case II. Let $S^2 - 4RT = 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are real and equal. We now take u exactly as in case I and take v to be any function of x, y which is independent of u. We have, as in case I, $A = 0$. Also, since $S^2 - 4RT = 0$, (23) shows that $B^2 = 0$ so that $B = 0$.

Moreover in this case $C \neq 0$, otherwise v would be a function of u and consequently v would not be independent of u as already assumed.

Putting $A = 0$, $B = 0$ and dividing by C, (5) transforms to the form

$$\partial^2 z/\partial v^2 = \phi(u, v, z, \partial z/\partial u, \partial z/\partial v). \quad ...(25)$$

which is the canonical form of (1) in this case.

Case III. Let $S^2 - 4RT < 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are complex. Hence this case III is formally the same as case I. Therefore, proceeding as in case I, we find that (1) reduces to (24) but that the variables u, v instead of being real are now complex conjugates. To obtain a real canonical form we make further transformation $u = \alpha + i\beta$ and $v = \alpha - i\beta$ so that

$\alpha = (u + v)/2$, and $\beta = i\,(v - u)/2.$ ...(26)

Now, $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right), \text{ by (26)} \quad ...(27)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right), \text{ by (26)} \quad ...(28)$$

$$\therefore \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right) \times \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right), \text{ by (27) and (28)}$$

$$= \frac{1}{4}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) - i\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)\right] = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + i\frac{\partial^2 z}{\partial \alpha \partial \beta} - i\frac{\partial^2 z}{\partial \beta \partial \alpha} + i\frac{\partial^2 z}{\partial \beta^2}\right)$$

or $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right), \quad \text{as} \quad \frac{\partial^2 z}{\partial \alpha \partial \beta} = \frac{\partial^2 z}{\partial \beta \partial \alpha} \qquad ...(29)$$

Putting $u = \alpha + i\beta$, $v = \alpha - i\beta$ and using (27), (28) and (29), (24) reduces to

$$(\partial^2 z/\partial \alpha^2) + (\partial^2 z/\partial \beta^2) = \psi(\alpha, \beta, z, \partial z/\partial \alpha, \partial z/\partial \beta), \qquad ...(30)$$

which is the canonical form of (1) in this case.

8.6 Working rule for reducing a hyperbolic equation to its canonical form

Step 1. Let the given equation $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ ... (1)

be hyperbolic so that $S^2 - 4RT > 0$.

Step 2. Write λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ ... (2)

Let λ_1 and λ_2 be its two distinct roots of (2).

Step 3. Then corresponding characteristic equations are

$(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$

Solving these, we get $f_1(x, y) = c_1$ and $f_2(x, y) = c_2$ ... (3)

Step 4. We select u, v such that $u = f_1(x, y)$ and $v = f_2(x, y)$ ... (4)

Step 5. Using relations (4), find p, q, r, s and t in terms of u and v as shown in Art. 8.5.

Step 6. Substituting the values of p, q, r, s, t obtained in step 4 in (1) and simplifying we shall get the following canonical form of (1):

$$\partial^2 z/\partial u \partial v = \phi\,(u, v, z,\ \partial z/\partial u,\ \partial z/\partial u)\,.$$

8.7. SOLVED EXAMPLES BASED ON ART. 8.6

Ex.1. (*a*) *Write canonical form of* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$. **[Sagar 2004; Delhi Maths (H) 2002]**

(*b*) Reduc $3(\partial^2 z/\partial x^2) + 10\,(\partial^2 z/\partial x \partial y) + 3\,(\partial^2 z/\partial y^2) = 0$ to canonical form and hence solve it **(Himanchal 2008)**

Sol. (*a*) Re-writing the given equation, we get $r - t = 0$ ... (1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$ so that $S^2 - 4RT = 4 > 0$, showing that (1) is hyperbolic

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$

Hence $\lambda = 1, -1$. So $\lambda_1 = 1, \lambda_2 = -1$ (Real and distinct roots).

Then the characteristic equations $dy/dx + \lambda_1 = 0$, $dy/dx + \lambda_2 = 0$ reduces to

$(dy/dx) + 1 = 0$ and $(dy/dx) - 1 = 0$.

Integrating these, $y + x = c_1$ and $y - x = c_2$.

In order to reduce (1) to its canonical form, we choose

$u = y + x$ and $v = y - x$ ... (2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (3)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (4)$$

From (3) and (4), $\frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$. ... (5)

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

or $$r = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad ... (6)$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

or $$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \ldots (7)$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0 \quad \text{or} \quad \frac{\partial^2 z}{\partial u \partial v} = 0.$$

(*b*) $\partial^2 z / \partial u \partial v = 0;\ z = f(y - 3x) + g(3y - x)$

Ex. 2. *Reduce* $\partial^2 z / \partial x^2 = (1 + y)^2 (\partial^2 z / \partial y^2)$ *to canonical form*

Sol. Re-writing the given equation, $r - (1 + y^2)\, t = 0 \quad \ldots (1)$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$, and $T = -(1 + y)^2$ so that $S^2 - 4RT = (1 + y^2) > 0$ for $y \neq -1$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - (1 + y)^2 = 0$ so that $\lambda = 1 + y, -(1 + y)$. Hence the corresponding characteristic equations are given by

$$(dy/dx) + (1 + y) = 0 \quad \text{and} \quad (dy/dx) - (1 + y) = 0$$

Integrating these, $\log(1 + y) + x = C_1$ and $\log(1 + y) - x = C_2$.

In order to reduce (1) to its canonical form, we choose

$$u = \log(1 + y) + x \quad \text{and} \quad v = \log(1 + y) - x \quad \ldots (2)$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)} \quad \ldots (3)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{1 + y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \quad \ldots (4)$$

From (3) $$\partial / \partial x \equiv \partial / \partial u - \partial / \partial v \quad \ldots (5)$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

or $$r = \partial^2 z / \partial u^2 - 2\,(\partial^2 z / \partial u\, \partial v) + \partial^2 z / \partial v^2 \quad \ldots (6)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{\frac{1}{1 + y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\} = -\frac{1}{(1 + y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1 + y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4)}$$

$$= -\frac{1}{(1 + y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1 + y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -\frac{1}{(1 + y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1 + y}\left[\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right)\frac{1}{y + 1} + \left(\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}\right)\frac{1}{1 + y}\right], \text{ by (2)}$$

or $$t = \frac{1}{(1 + y)^2}\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \quad \ldots (7)$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0 \quad \text{or} \quad 4\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}..$$

Ex. 3. *Reduce the differential equation* $t - s + p\ - q(1 + 1/x) + (z/x) = 0$ *to canonical form.*

[Delhi Maths (H) 2004]

Sol. Given $0 \cdot r - s + t + p - q\,(1 + 1/x) + (z/x) = 0$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 0$, $S = -1$ and $T = 1$.

Hence $S^2 - 4RT = 1 > 0$, showing that the given equation is hyperbolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $-\lambda + 1 = 0$ giving $\lambda = 1$. Hence the corresponding characteristic equation $dy/dx + \lambda = 0$ yields $dy/dx + 1 = 0$ or $dx + dy = 0$

Integrating it, $x + y = c$, c being an arbitrary constant

Choose $u = x + y$ and $v = x$, ...(2)

where we have chosen $v = x$ in such a manner that u and v are independent as verified below:

Jacobian of u and $v = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow u$ and v are independent functions.

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$, using (2) ...(3)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}, \text{ using (2)} \qquad ...(4)$$

From (4), we have $\partial/\partial y \equiv \partial/\partial u$ ...(5)

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)} \qquad ...(6)$$

and $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial u}\right)$, using (5)

or $t = \partial^2 z/\partial u^2$. ...(6)

Using (2), (3), (4), (6) and (7), (1) reduces to

$$-\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v}\right) + \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\left(1 + \frac{1}{v}\right) + \frac{z}{v} = 0$$

or $\partial^2 z/\partial u\,\partial v - (\partial z/\partial v) + (1/v) \times (\partial z/\partial u) - (z/v) = 0$, which is the required canonical form.

Ex. 4. *Reduce the equation yr + (x + y) s + xt = 0 to canonical form and hence find its general solution.* **(Delhi Maths (Hons) 2007)**

Sol. Given $yr + (x + y)\,s + xt = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y$, $S = (x + y)$ and $T = x$ so that $S^2 - 4RT = (x + y)^2 - 4xy = (x - y)^2 > 0$ for $x \neq y$ and so (1) is hyperbolic. Its λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $y\lambda^2 + (x + y)\,\lambda + x = 0$ or $(y\lambda + x)\,(\lambda + 1) = 0$

so that $\lambda = -1$, $-x/y$. Then the corresponding characteristic equations are given by

$(dy/dx) - 1 = 0$ and $(dy/dx) - (x/y) = 0$

Integrating these, $y - x = c_1$ and $y^2/2 - x^2/2 = c_2$

In order to reduce (1) to its canonical form, we choose

$u = y - x$ and $v = y^2/2 - x^2/2$ ... (2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad ... (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right), \text{ using (3)}$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v}\right] = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) - \frac{\partial z}{\partial v}$$

$$= -\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] - \frac{\partial z}{\partial v}$$

$$= -\left(-\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial v \partial u}\right) - x\left(-\frac{\partial^2 z}{\partial u \partial v} - x\frac{\partial^2 z}{\partial v^2}\right) - \frac{\partial z}{\partial v}, \text{ using (2)}$$

$$\therefore \qquad r = \frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v} \qquad \ldots (5)$$

Now, $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u} + y\dfrac{\partial z}{\partial v}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}\right) + \dfrac{\partial}{\partial y}\left(y\dfrac{\partial z}{\partial v}\right)$, using (4)

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\} + \frac{\partial z}{\partial v}$$

$$\therefore\ t = \frac{\partial^2 z}{\partial u^2} + y\frac{\partial^2 z}{\partial u \partial v} + y\left(\frac{\partial^2 z}{\partial u \partial v} + y\frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial v} = \frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v} \qquad \ldots(6)$$

Also, $s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u} + y\dfrac{\partial z}{\partial v}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right) + \dfrac{\partial}{\partial x}\left(y\dfrac{\partial z}{\partial v}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right) + y\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial v}\right)$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\} = -\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial v \partial u} - y\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}, \text{ using (2)}$$

$$\therefore \qquad s = -\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2} \qquad \ldots (7)$$

Using (5) (6) and (7) in (1), we get $\quad y\left(\dfrac{\partial^2 z}{\partial u^2} + 2x\dfrac{\partial^2 z}{\partial u \partial v} + x^2\dfrac{\partial^2 z}{\partial v^2} - \dfrac{\partial z}{\partial v}\right)$

$$+(x+y)\left\{-\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}\right\} + x\left(\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\right) = 0$$

or $\quad \{4xy - (x+y)^2\}\dfrac{\partial^2 z}{\partial u \partial v} - y\dfrac{\partial z}{\partial v} + x\dfrac{\partial z}{\partial v} = 0$ $\quad$ or $\quad (y-x)^2\dfrac{\partial^2 z}{\partial u \partial v} + (y-x)\dfrac{\partial z}{\partial v} = 0$

or $\quad u^2\dfrac{\partial^2 z}{\partial u \partial v} + u\dfrac{\partial z}{\partial v} = 0$, by (2) $\quad$ or $\quad u\dfrac{\partial^2 z}{\partial v \partial v} + \dfrac{\partial z}{\partial v} = 0$, as $u \neq 0$ ... (8)

(8) is the required canonical form of (1).

Solution of (8). Multiplying both sides of (8) by v, we get

$$uv\,(\partial^2 z/\partial u \partial v) + v(\partial z/\partial v) = 0 \qquad \text{or} \qquad (uv\,DD' + vD')z = 0 \qquad \ldots (9)$$

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial v$. To reduce (9) into linear equation with constant coefficients, we take new variables X and Y as follows. For details refer Art. 6.3.

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $Y = \log v$ ... (10)

Let $D_1 \equiv \partial / \partial X$ and $D_1' \equiv \partial / \partial Y$. Then (9) reduces to

$(D_1\ D_1' + D_1')\, z = 0$ or $D_1'\ (D_1 + 1)\ z = 0$

Its general solution is $z = e^{-X}\ \phi_1\ (Y) + \phi_2(X) = u^{-1}\ \phi_1(\log v) + \phi_2(\log u)$ [See Art. 5.6]

or $z = u^{-1}\ \psi_1(v) + \psi_2(u) = (y - x)^{-1}\ \psi_1(y^2 - x^2) + \psi_2(y - x)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex.5. *Reduce the equation* $r - (2\sin x)s - (\cos^2 x)t - (\cos x)q = 0$ *to canonical form and hence solve it.* **(Himanchal 2008)**

Sol. Given $r - (2\sin x)\ s - (\cos^2 x)t - (\cos x)q = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = -2\sin x$ and $T = -\cos^2 x$ so that $S^2 - 4RT = 4(\sin^2 x + \cos^2 x) = 4 > 0$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - (2\sin x)\lambda - \cos^2 x = 0$ so that $\lambda = \sin x + 1,\ \sin x - 1$. Hence the corresponding characteristic equations become

$dy/dx + \sin x + 1 = 0$ and $dy/dx + \sin x - 1 = 0$

Integrating these, $y - \cos x + x = c_1$ and $y - \cos x - x = c_2$

Choose $u = y - \cos x + x$ and $v = y - \cos x - x$ ... (2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (1 + \sin x)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}, \text{ by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{... (4)}$$

From (4), we have $\partial / \partial y = \partial / \partial u + \partial / \partial v$ ... (5)

$$\therefore \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (4) and (5)}$$

or $$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{... (6)}$$

Now, $s = \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial y}\left\{(1 + \sin x)\dfrac{\partial z}{\partial u} + (\sin x - 1)\dfrac{\partial z}{\partial v}\right\}$, by (3)

$$= (\sin x + 1)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + (\sin x - 1)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$

$$= (\sin x + 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right\} + (\sin x - 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}$$

$$= (\sin x + 1)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + (\sin x - 1)\left(\frac{\partial^2 z}{\partial u \partial x} + \frac{\partial^2 z}{\partial v^2}\right)$$

or $$s = \sin x\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} \quad \text{... (7)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{(\sin x + 1)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}\right\} = \cos x\frac{\partial z}{\partial u} + (\sin x + 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \cos x\frac{\partial z}{\partial v} + (\sin x - 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(\sin x+1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\}+(\sin x-1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$= \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(\sin x+1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u^2}+(\sin x-1)\frac{\partial^2 z}{\partial v^2}\right\}+(\sin x-1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u\partial v}+(\sin x-1)\frac{\partial^2 z}{\partial v^2}\right\}$$

$$\therefore \quad r=\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(1+\sin x)^2\frac{\partial^2 z}{\partial u^2}+(\sin x-1)^2\frac{\partial^2 z}{\partial v^2}-2\cos^2 x\frac{\partial^2 z}{\partial u\partial v} \quad \text{... (8)}$$

Using (4) (6), (7) and (8) in (1), we get

$$\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(1+2\sin x+\sin^2 x)\frac{\partial^2 x}{\partial u^2}+(\sin^2 x+1-2\sin x)\frac{\partial^2 z}{\partial v^2}-2\cos^2 x\frac{\partial^2 z}{\partial u\,\partial v}$$

$$-2\sin x\left\{\sin x\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}\right\}-\cos^2 x\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=0$$

or $$(1+2\sin x+\sin^2 x-2\sin^2 x-2\sin x-\cos^2 x)\times(\partial^2 z/\partial u^2)+(\sin^2 x+1-2\sin x-2\sin^2 x$$
$$+2\sin x-\cos^2 x)\times(\partial^2 z/\partial v^2)-(2\cos^2 x+4\sin^2 x+2\cos^2 x)\times(\partial^2 z/\partial u\,\partial v)=0$$

or $$\partial^2 z/\partial u\,\partial v=0\text{, on simplification.} \quad \text{... (9)}$$

(9) is the required canonical form of (1).

Solution of (9). Integrating (9) w.r.t. 'u', $\partial z/\partial v=\phi(v),\phi$ being an arbitrary function ... (10)

Integrating (10) w.r.t. 'v', $z=\int \phi(v)\,dv+F(u)=G(v)+F(u)$,

where $G(v)=\int \phi(v)\,dv$, F and G are arbitrary functions.

$\therefore$ $z=G(y-\cos x-x)+F(y-\cos x+x)$ is the required solution.

Ex. 6. *Reduce $\partial^2 z/\partial x^2=x^2(\partial^2 z/\partial y^2)$ to canonical form.*

[Agra 2005; Himanchal 2005; Delhi B.Sc. (Prog) II 2002, 07; Kurukshetra 2004; Ravishankar 2004; Nagpur 2010, Kanpur 2011]

Sol. Re-writing the given equation becomes $r-x^2t=0$. ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have $R=1,\ S=0,\ T=-x^2$.

Now, the λ-quadratic $R\lambda^2+S\lambda+T=0$ gives $\lambda^2-x^2=0$ so that $\lambda=\pm x$.

$\therefore$ Here $\lambda_1=x$ and $\lambda_2=-x$ (Real and disticnt roots)

Hence characterstic equations $dy/dx+\lambda_1=0$ and $dy/dx+\lambda_2=0$

become $dy/dx+x=0$ and $dy/dx-x=0$.

Integrating these, $y+(x^2/2)=c_1$ and $y-(x^2/2)=c_2$.

Hence in order to reduce (1) to canonical form, we change x, y, to u, v by taking

$u=y+(x^2/2)$ and $v=y-(x^2/2)$ ...(2)

Now, $$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=x\frac{\partial z}{\partial u}-x\frac{\partial z}{\partial v}\text{, using (2)} \quad \text{...(3)}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\text{, using (2)} \quad \text{...(4)}$$

$$\therefore\ r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right\}=x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+1.\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\text{, using (3)}$$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\ =x^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial^2 v^2}\right)+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

and $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}\right) = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right) = \dfrac{\partial^2 z}{\partial u^2} + 2\dfrac{\partial^2 z}{\partial u \partial v} + \dfrac{\partial^2 z}{\partial v^2}$, using (4)

Putting the above values of r and t in (1), we get

$$x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $\dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4x^2}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$ or $\dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4(u-v)}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$, by (2)

which is the required canonical form of the given equation.

Ex. 7. *Reduce the equation* $(n-1)^2(\partial^2 z/\partial x^2) - y^{2n}(\partial^2 z/\partial y^2) = n y^{2n-1}(\partial z/\partial y)$ *to canonical form, and find its general solution.* **[Delhi Maths. (H) 2000, 01, 05; Himanchal 2004; Ravishankar 2004]**

Sol. Given $(n-1)^2 r - y^{2n} t - n y^{2n-1} q = 0.$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = (n-1)^2$, $S = 0$, $T = -y^{2n}$.

Now, the λ—quadratic $R\lambda^2 + S\lambda + T = 0$ gives

$(n-1)^2 \lambda^2 - y^{2n} = 0$ so that $\lambda = \pm (n-1)^{-1} y^n$.

$\therefore$ Here $\lambda_1 = (n-1)^{-1} y^n$ and $\lambda_2 = -(n-1)^{-1} y^n$.

Hence, characterstic equations $dy/dx + \lambda_1 = 0$ and $dy/dx + \lambda_2 = 0$

become $dy/dx + (n-1)^{-1} y^n = 0$ and $dy/dx - (n-1)^{-1} y^n = 0.$

Integrating these, $x - y^{-n+1} = c_1$ and $x + y^{-n+1} = c_2$.

Hence in order to reduce (1) to canonical form, we change x, y to u, v by taking

$u = x - y^{-n+1}$ and $v = x + y^{-n+1}$. ...(2)

$\therefore\ p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$ so that $\dfrac{\partial}{\partial x} \equiv \dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}$,

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = (n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right),$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{(n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)^2 y^{-2n}\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

Substituting the above values of r, t, q in (1) and simplifying, we obtain

$$\partial^2 z / \partial u \partial v = 0, \quad \text{...(3)}$$

which is the required canonical form of the given equation

Integrating (3) w.r.t. 'v', $\partial z/\partial u = F(u)$, where $F(u)$ is an arbitrary function of u, ...(4)

Integrating (4) w.r.t. 'u', $z = G(u) + H(v)$,

where $G(u) = \int F(u)\, du$ and $G(u), H(v)$ are arbitrary functions

Using (2), the solution of the given equation is $z = G(x - y^{-n+1}) + H(x + y^{-n+1})$.

Ex. 8. *Reduce the equation* $(y-1)\,r-(y^2-1)s+y(y-1)t+p-q=2ye^{2x}(1-y)^3$ to *canonical form and hence solve it.* **[Delhi B.Sc. (Hons) III 2008; Rohilkhand 1992]**

Sol. Given $(y-1)\,r-(y^2-1)s+y(y-1)t+p-q-2ye^{2x}(1-y)^3=0.$ ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$R=y-1,\quad S=-(y^2-1)$ and $T=y(y-1).$ ...(2)

$\therefore$ The λ–quardratic $R\lambda^2-S\lambda+T=0$ gives

$(y-1)\lambda^2-(y^2-1)\lambda+y(y-1)=0 \quad\Rightarrow\quad \lambda_1=1$ and $\lambda_2=y$ (real and distinct roots)

Hence characterstic equations $(dy/dx)+\lambda_1=0$ and $(dy/dx)+\lambda_2=0$ become

$(dy/dx)+1=0$ and $(dy/dx)+y=0.$

Integrating these, $x+y=c_1$ and $y\,e^x=c_2.$

To reduce (1) to canonical form, we change the independent variables x, y, to new independent variables u, v by taking

$$u=x+y \quad\text{and}\quad v=y\,e^x. \qquad ...(3)$$

$$\therefore\quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v}=\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v},\ \text{by (3)} \qquad ...(4)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},\ \text{by (3)} \qquad ...(5)$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2v\frac{\partial^2 z}{\partial u\partial v}+v^2\frac{\partial^2 z}{\partial v^2}+v\frac{\partial z}{\partial v},\ \text{by (4)}$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)+e^x\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}=\frac{\partial^2 z}{\partial u^2}+(e^x+v)\frac{\partial^2 z}{\partial u\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial z}{\partial y}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Substituting the above values in (1) and simplifying, we have

$$(1-y)^3e^x\frac{\partial^2 z}{\partial u\partial v}=2y\,e^{2x}(1-y)^3 \quad\text{or}\quad \frac{\partial^2 z}{\partial u\partial v}=2v, \qquad ...(6)$$

which is the canonical form of (1).

Integrating (6) w.r.t. 'v', $\partial z/\partial u=v^2+\phi(u)$, $\phi(u)$ being an arbitrary function ...(7)

Integrating (7) w.r.t. 'u', $z=uv^2+\phi_1(u)+\phi_2(v)$, where $\phi_1(u)=\int\phi(u)\,du$

$\therefore$ Using (3) $z=(x+y)y^2e^{2x}+\phi_1(x+y)+\phi_2(ye^x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 9. *Solve* $x^2(y-1)\,r-x\,(y^2-1)\,s+y(y-1)\,t+xyp-q=0.$

Sol. Given $x^2(y-1)\,r-x(y^2-1)\,s+y(y-1)t+xyp-q=0.$ ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0,$ we get

$R=x^2(y-1),\quad S=-x\,(y^2-1)$ and $T=y\,(y-1).$

$\therefore$ λ–quadratic $R\lambda^2+S\lambda+T=0$ reduces to

$x^2(y-1)\,\lambda^2-x(y^2-1)\,\lambda+y(y-1)=0 \quad\Rightarrow\quad \lambda_1=y/x$ and $\lambda_2=1/x$ (real and distinct)

So characterstic equations $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$ become

$(dy/dx) + (y/x) = 0$ and $(dy/dx) + (1/x) = 0$

Integrating these, $xy = c_1$ and $xe^y = c_2$ so for canonical form, we take

$u = xy$ and $v = xe^y$. ...(2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v}, \text{ by (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}, \text{ by (2)} \qquad ...(4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] = y^2\frac{\partial^2 z}{\partial u^2} + 2ye^x\frac{\partial^2 z}{\partial u\partial v} + e^{2y}\frac{\partial^2 z}{\partial y^2},$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^y\frac{\partial z}{\partial v} + xe^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + xe^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v} + xy\frac{\partial^2 z}{\partial u^2} + (yxe^y + e^y x)\frac{\partial^2 z}{\partial u\partial v} + xe^{2y}\frac{\partial^2 z}{\partial v^2}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + xe^y\frac{\partial z}{\partial v} + xe^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + xe^y\frac{\partial z}{\partial v} + xe^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x^2\frac{\partial^2 z}{\partial u^2} + 2x^2e^y\frac{\partial^2 z}{\partial u\partial v} + x^2e^{2y}\frac{\partial^2 z}{\partial y^2} + xe^y\frac{\partial z}{\partial v}.$$

Substituting the above values in (1) and simplifying, we get $\partial^2 z / \partial u\partial v = 0$, ... (5)

which is canonical form of (1).

Integrating (5) w.r.t. 'u', $\partial z/\partial v = \phi(v)$, $\phi(v)$ being an arbitrary function.

Integrating it w.r.t. 'v', $z = \phi_1(v) + \phi_2(u)$, where $\phi_1(v) = \int \phi(v)\, dv$.

$\therefore$ $z = \phi_1(xe^y) + \phi_2(xy)$, by (2). This is the required solution, ϕ_1, ϕ_2 being arbitrary functions

Ex. 10. *Solve* (*i*) $xyr - (x^2 - y^2)\, s - xyt + py - qx = 2(x^2 - y^2)$. **[Delhi Maths (H) 2006]**

(*ii*) $x(y - x)\, r - (y^2 - x^2)\, s + y\,(y - x)\, t + (y + x)(p - x) = 2x + 2y + 2$.

Sol. (*i*) Given $xyr - (x^2 - y^2)s - xyt + py - qx - 2(x^2 - y^2) = 0$ ...(1)

Comparing (*i*) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$R = xy$, $S = -(x^2 - y^2)$ and $T = -xy$.

So λ–quadratic $R\lambda^2 + S\lambda + T = 0$ becomes $xy\lambda^2 - (x^2 - y^2)\lambda - xy = 0$ giving $\lambda = -y/x, x/y$.

$$\therefore \quad \frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} + \frac{y}{x} = 0.$$

Integrating, $y/x = c_1$, and $x^2 + y^2 = c_2$. So, we take

$u = y/x$ and $v = x^2 + y^2$. ...(2)

$\therefore$ Proceeding as usual, we obtain

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\left(-\frac{y}{x^2}\right)\frac{\partial z}{\partial u}+2x\frac{\partial z}{\partial v},\quad q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{x}\frac{\partial z}{\partial u}+2y\frac{\partial z}{\partial v},$$

$$r=\left(-\frac{y}{x^2}\right)^2\frac{\partial^2 z}{\partial u^2}+2\times(2x)\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial v\partial u}+4x^2\frac{\partial^2 z}{\partial v^2}+\frac{2y}{x^3}\frac{\partial z}{\partial u}+2\frac{\partial z}{\partial v}$$

$$s=\left(-\frac{y}{x^2}\right)\left(\frac{1}{x}\right)\frac{\partial^2 z}{\partial u^2}+\left\{2y\left(-\frac{y}{x^2}\right)+2x\times\frac{1}{x}\right\}\frac{\partial^2 z}{\partial u\partial v}+4xy\frac{\partial^2 z}{\partial v^2}-\frac{1}{x^2}\frac{\partial z}{\partial u}$$

and

$$t=\left(\frac{1}{x}\right)^2\frac{\partial^2 z}{\partial u^2}+2\times\frac{1}{x}\times(2y)\frac{\partial^2 z}{\partial u\partial v}+4y^2\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial v}.$$

Substituting these in (1) we get

$$(x^2+y^2)^2\frac{\partial^2 z}{\partial u\partial v}=(y^2-x^2)x^2 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial u\partial v}=\frac{(y^2-x^2)x^2}{(x^2+y^2)^2}=\frac{u^2-1}{(u^2+1)^2},\ \text{by (2)} \quad ...(3)$$

Integrating (3) w.r.t. 'u', we have

$$\frac{\partial z}{\partial v}=\int\frac{u^2-1}{(u^2+1)^2}du+\phi(v)=\int\frac{du}{u^2+1}-2\int\frac{du}{(u^2+1)^2}+\phi(v) \qquad ... (4)$$

We have,

$$\int 1\cdot\frac{1}{u^2+1}du=u\times\frac{1}{u^2+1}-\int u\times\left(\frac{-2u}{(u^2+1)^2}\right)du,\ \text{integrating by parts}$$

or

$$\int\frac{du}{u^2+1}=\frac{u}{u^2+1}+2\int\frac{(u^2+1)-1}{(u^2+1)^2}du=\frac{u}{u^2+1}+2\int\frac{du}{u^2+1}-2\int\frac{du}{(u^2+1)^2}$$

Then,

$$\int\frac{du}{u^2+1}-2\int\frac{du}{(u^2+1)^2}=-\frac{u}{u^2+1} \qquad ... (4)$$

Using (5), (4) gives $\partial z/\partial v=-u/(u^2+1)+\phi(v)$, $\phi(v)$ being an arbitrary function ...(6)

Integrating (6) w.r.t. v, $z=-(uv)/(u^2+v^2)+\phi_1(v)+\phi_2(u)$, where $\phi_1(v)=\int\phi(v)\,dv$

$\therefore$ Using (2), $z=-xy+\phi_1(x^2+y^2)+\phi_2(y/x)$, ϕ_1,ϕ_2 being arbitrary functions.

(*ii*) **Hint.** Since $R=x(y-x)$, $S=-(y^2-x^2)$, $T=y(y-x)$, so here $\lambda_1=y/x$, $\lambda_2=1$.

So we get $(dy/dx)+(y/x)=0$ and $(dy/dx)+1=0$ as characteristic equations

These give $xy=c_1$ and $x+y=c_2$. Hence take

$u=xy$ and $v=x+y$. ...(1)

As usual, $p=y\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$ and $q=x\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$,

$$r=y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2},\ t=x^2\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2},\ s=xy\frac{\partial^2 z}{\partial u^2}+(x+y)\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial u}.$$

$\therefore$ Given equation becomes $-(y-x)^3\dfrac{\partial^2 z}{\partial v\partial u}=2x+2y+2$ ... (2)

or $$\frac{\partial^2 z}{\partial v \partial u} = -\frac{2(x+y+1)}{(y-x)^3} = -\frac{2(x+y+1)}{[(y+x)^2-4xy]^{3/2}} = \frac{2(v+1)}{(v^2-4u)^{3/2}}, \text{ by (1)}$$

Integrating (2) w.r.t. 'u', we get $$\frac{\partial z}{\partial v} = \frac{v+1}{\sqrt{(v^2-4u)}} + \phi(v). \quad \ldots (3)$$

Integrating, (3) w.r.t. v, $$z = \sqrt{(v^2-4u)} + \log\left[v + \sqrt{(v^2-4u)}\right] + \phi_1(v) + \phi_2(u)$$

or $z = x - y + \log(2x) + \phi_1(x+y) + \phi_2(xy)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 11. *Solve* (*i*) $y(x+y)(r-s) - xp - yq - z = 0$ **[Delhi Maths (H) 1998]**

(*ii*) $xys - x^2 r - px - qy + z = -2xy^2 y$.

Sol. (*i*) Given $$y(x+y)\,r - y\,(x+y)s - xp - yq - z = 0. \quad \ldots(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, $R = y\,(x+y)$, $S = -y\,(x+y)$, $T = 0$.

So, the λ–quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$y(x+y)\lambda^2 - y(x+y)\lambda = 0$, giving $\lambda = 0, 1$. Thus $\lambda_1 = 1$ and $\lambda_2 = 0$ and so

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} + 1 = 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Integrating these, $x + y = c_1$, and $y = c_2$.

So we take $u = x + y$ and $v = y$ ...(2)

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}, \text{ by (2)} \quad \ldots(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \quad \ldots.(4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2}, \text{ by (3)} \quad \ldots(5)$$

$$s = \frac{\partial^2 z}{\partial v \partial u} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}, \text{ using (3) and (4)} \quad \ldots(6)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values in (1), we have

$$y(x+y)\left(-\frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial z}{\partial u} - y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0 \quad \text{or} \quad uv\frac{\partial^2 z}{\partial v \partial u} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} + z = 0.$$

or $$\frac{\partial^2 z}{\partial v \partial u} + \frac{1}{v}\frac{\partial z}{\partial u} + \frac{1}{u}\frac{\partial z}{\partial v} + \frac{1}{uv}z = 0 \quad \text{or} \quad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0. \quad \ldots(7)$$

Let $$\partial z / \partial v + (z/v) = w. \quad \ldots(8)$$

Then, the above equation (7) becomes $\partial w / \partial u + w/u = 0$.

Integrating, $wu = \phi(v)$ or $w = (1/u) \times \phi(v)$.

Substituting this value of w in (8), we have $$\frac{\partial z}{\partial v} + \frac{1}{v}z = \frac{1}{u}\phi(v), \quad \ldots(9)$$

I.F. of (9) $= e^{\int (1/v)dv} = v$ and solution of (9) is

$$zv = \frac{1}{u}\int \phi(v)dv + \phi_2(u) \quad \text{or} \quad z = \frac{1}{uv}\phi_1(v) + \frac{1}{v}\phi_2(u), \text{ where } \phi_1(v) = \int \phi(v)\,dv$$

or $z = \dfrac{1}{y(x+y)}\,\phi_1(y) + \dfrac{1}{y}\phi_2(x+y)$, by (2); ϕ_1, ϕ_2 being arbitrary functions

(*ii*) **Hint.** Given $xys - x^2r - px - qy + z = -2x^2y.$ ...(1)

Here, $R = -x^2,\ S = xy,\ T = 0$ and λ–quadratic is $-x^2\lambda^2 + xy\lambda = 0$

so that $\lambda_1 = y/x$ and $\lambda_2 = 0$. Hence, characterstic equations

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Integrating these, $xy = c_1, \quad y = c_2.$ So we take $u = xy$ and $v = y.$ ...(2)

Then, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}y = v\frac{\partial z}{\partial u}, \text{ by (2)} \quad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\,\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \quad ...(4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = v\frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right) = v^2\,\frac{\partial^2 z}{\partial u^2}, \text{ by (3)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = v\frac{\partial}{\partial u}\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = v\left(\frac{1}{v}\frac{\partial z}{\partial u} + \frac{u}{v}\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right), \text{ by (3) and (4)}$$

Substituting these values in (1), we have

$$xy\left(\frac{\partial z}{\partial u} + u\frac{\partial^2 z}{\partial u^2} + v\frac{\partial^2 z}{\partial u \partial v}\right) - x^2v^2\frac{\partial^2 z}{\partial u^2} - v\frac{\partial z}{\partial u}x - y\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + z = -2x^2y$$

or $$u\frac{\partial z}{\partial u} + u^2\frac{\partial^2 z}{\partial u^2} + uv\frac{\partial^2 z}{\partial u \partial v} - u^2\frac{\partial^2 z}{\partial u^2} - u\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} + z = -2(u^2/v^2)v, \text{ by (2)}$$

or $$uv\frac{\partial^2 z}{\partial u \partial v} - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} + z = -\frac{2u^2}{v} \quad \text{or} \quad \frac{\partial^2 z}{\partial u \partial v} - \frac{1}{v}\frac{\partial z}{\partial u} - \frac{1}{u}\frac{\partial z}{\partial v} + \frac{z}{uv} = -\frac{2u}{v^2}.$$

or $$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} - \frac{z}{v}\right) - \frac{1}{u}\left(\frac{\partial z}{\partial v} - \frac{z}{v}\right) = -\frac{2u}{v^2}. \quad ...(5)$$

Let $$\partial z/\partial v - z/v = w. \quad ...(6)$$

Then (5) becomes $\dfrac{\partial w}{\partial u} - \dfrac{1}{u}w = -\dfrac{2u}{v^2}$, which is linear differential equation ...(7)

I.F. of (7) $= e^{-\int (1/u)\,du} = e^{-\log u} = e^{\log u^{-1}} = (1/u)$ and so its solution is

$$\frac{w}{u} = -\int\left(\frac{2u}{v^2} \times \frac{1}{u}\right)du = -\frac{2u}{v^2} + \phi(v) \quad \text{or} \quad w = -\frac{2u^2}{v^2} + u\,\phi(v)$$

Substituting this value of w in (6), we get $$\frac{\partial z}{\partial v} - \frac{1}{v}z = -\frac{2u^2}{v^2} + u\phi(u).$$

Its I.F. $= e^{-\int (1/v)\,dv} = e^{-\log v} = e^{\log v^{-1}} = (1/v)$ and so its solution is

$$\frac{z}{v} = \int \frac{1}{v}\left[-\frac{2u^2}{v^2} + u\,\phi(v)\right]dv = \frac{u^2}{v^2} + u\psi(v) + \phi_2(u)$$

or $z = (u^2/v) + uv\psi(v) + v\phi_2(u) = (u^2/v) + u\phi_1(v) + v\phi_2(u)$ or $z = x^2y + xy\phi_1(y) + y\phi_2(xy)$, by (2).

Ex. 12. *Solve* $x^2r - y^2t + px - qy = x^2$. **[Kurukshetra 2003; Delhi Maths (H) 1998]**

Sol. Given $x^2r - y^2t + (px - qy - x^2) = 0.$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we get

$R = x^2, \quad S = 0$ and $T = -y^2.$ ...(2)

Now, the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ and (2) give

$x^2\lambda^2 - y^2 = 0$ so that $\lambda = \pm y/x$. (real and distinct roots)

Take $\lambda_1 = y/x$ and $\lambda_2 = -y/x$.

Hence characterstic equations $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$

become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

or $(1/x)dx + (1/y)dy = 0$ and $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x + \log y = \log c_1$ and $\log x - \log y = \log c_2$

or $xy = c_1$ and $x/y = c_2$.

To reduce (1) to canonical form, we change the independent variables x, y to new independent variables u, v by taking

$$u=xy \quad \text{and} \quad v = x/y. \qquad ...(3)$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}, \text{ using (3).} \qquad ...(4)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}, \text{ using (3).} \qquad ...(5)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= y\left(\frac{\partial^2 z}{\partial u^2}\times y + \frac{\partial^2 z}{\partial v\partial u}\times\frac{1}{y}\right) + \frac{1}{y}\left(\frac{\partial^2 z}{\partial v\partial u}\times y + \frac{\partial^2 z}{\partial v^2}\times\frac{1}{y}\right), \text{ using (3)}$$

$$\therefore \quad r = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}. \qquad ...(6)$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - \left[-\frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x}{y^2}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2}\times x + \frac{\partial^2 z}{\partial v\partial u}\times\left(-\frac{x}{y^2}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial^2 z}{\partial u\partial v}\times x + \frac{\partial^2 z}{\partial v^2}\times\left(-\frac{x}{y^2}\right)\right]$$

$$\therefore \quad t = x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}. \qquad ...(7)$$

Substituting the values of r, t, p and q given by (6), (7) (3) and (4) in (1), we obtain

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial u^2}\right) - y^2\left(x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right)$$

$$+ x\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) - y\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) - x^2 = 0$$

or $4x^2\dfrac{\partial^2 z}{\partial u\partial v} = x^2$ so that $\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial v}\right) = \dfrac{1}{4}$, ...(8)

which is the canonical form of (1).

Now, integrating (8) w.r.t. 'u', $\partial z/\partial u = (u/4) + f(v)$. ...(9)

Integrating (9) w.r.t. 'v', $z = (uv)/4 + \int f(v)\,dx + \phi(u)$

or $z = (uv)/4 + \psi(v) + \phi(u)$, where $\psi(v) = \int f(v)\,dv$

or $z = x^2/4 + \psi(x/y) + \phi(xy)$, which is the required solution, ϕ, ψ being arbitrary functions.

Ex. 13. *(a) Reduce* $x^2\,(\partial^2 z/\partial x^2) - y^2\,(\partial^2 z/\partial y^2) = 0$ *to canonical form and hence solve it.*

(b) Reduce $y^2(\partial^2 z/\partial x^2) - x^2(\partial^2 z/\partial y^2) = 0$ *to canonical form.*

Sol. (*a*) Re-writing the given equation, $x^2\,r - y^2\,t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2\,y^2 > 0$ for $x \neq 0$, $y \neq 0$ and hence (1) is hyperbolic. The λ-quadrate equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ so that $\lambda = y/x,\ -y/x$ and hence the corresponding characteristic equations become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

Integrating these, $xy = c_1$ and $x/y = c_2$

In order to reduce (1) to its canonical form, we choose $u = xy$ and $v = x/y$ ... (2)

Now, doing exactly as in solved Ex. 12, we get

$$r = y^2\,\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} \quad \text{and} \quad t = x^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$$

Putting these values of r and t in (1), we get

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right) - y^2\left(x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $4x^2\,\dfrac{\partial^2 z}{\partial u \partial v} - \dfrac{2x}{y}\dfrac{\partial z}{\partial v} = 0$ or $2xy\,\dfrac{\partial^2 z}{\partial u \partial v} - \dfrac{\partial z}{\partial v} = 0$

or $2u\,(\partial^2 z/\partial u \partial v) - (\partial z/\partial v) = 0$, using (2). ... (3)

This is the required canonical form of (1).

We now proceed to find solution of (1). Multiplying both sides of (3) by v, we get

$$2uv\,\frac{\partial^2 z}{\partial u\,\partial v} - v\,\frac{\partial z}{\partial v} = 0 \quad \text{or} \quad (2uv\,DD' - vD')z = 0 \qquad \text{... (4)}$$

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial v$. We now reduce (4) to a linear equation with constant coefficients by usual method (refer Art. 6.3 of chapter 6).

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $y = \log v$ ... (5)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$. Then (4) reduces to

$(2D_1D_1' - D_1')z = 0$ or $D_1'\,(2D_1 - 1)\,z = 0$

Its general solution is given by (use Art. 5.6 of chapter 5)

$z = e^{X/2}\,\phi_1(Y) + \phi_2(X) = u^{1/2}\phi_1(\log v) + \phi_2(\log u) = u^{1/2}\psi_1(v) + \psi_2(u)$, using (5)

$= (xy)^{1/2}\psi_1(x/y) + \psi_2(xy) = x(y/x)^{1/2}\,\psi_1(x/y) + \psi_2(xy) = xf(x/y) + \psi_2(xy)$, using (2)

where f and ψ_2 are arbitrary functions

(*b*) Try yourself. Choose $u = (y^2 - x^2)/2$, $v = (y^2 + x^2)/2$.

Ans. $\dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{2(u^2 - v^2)}\left(v\dfrac{\partial z}{\partial u} - u\dfrac{\partial z}{\partial v}\right)$.

Ex. 14. *Reduce the equation* $x(xy-1)r-(x^2y^2-1)s+y(xy-1)t+(x-1)p+(y-1)q=0$ *to canonical form and hence solve it.*

Sol. Comparing the given equation with $Rr+Ss+Tt+f(x,y,z,p,q)=0$,

here, $R=x(xy-1)$, $S=-(x^2y^2-1)$, $T=y(xy-1)$. ...(1)

Now, the λ-quadratic equation $R\lambda^2+S\lambda+T=0$ and (1) give

$x(xy-1)\lambda^2-(x^2y^2-1)\lambda+y(xy-1)=0$ or $x\lambda^2-(xy+1)\lambda+y=0$

or $(x\lambda-1)(\lambda-y)=0$ so that $\lambda=1/x,\ y$. Take $\lambda_1=1/x$ and $\lambda_2=y$.

Hence characterstic equations $(dy/dx)+\lambda_1=0$ and $(dy/dx)+\lambda_2=0$

become $(dy/dx)+(1/x)=0$ and $(dy/dx)+y=0$

or $dy+(1/x)dx=0$ and $(1/y)dy+dx=0$. ...(2)

Intergrating (2), $y+\log x=\log c_1$ and $\log y+x=\log c_2$

or $\log e^y+\log x=\log c_1$ and $\log y+\log e^x=\log c_2$

$x\,e^y=c_1$ and $y\,e^x=c_2$.

To reduce the given equation to canonical form, we change the independent variables x, y to new independent variables u, v, by taking

$u=x\,e^y$ and $v=y\,e^x$. ...(3)

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^y\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v},\ \text{using (3)} \qquad ...(4)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},\ \text{using (3).} \qquad ...(5)$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(e^y\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+ye^x\frac{\partial z}{\partial v}+y\,e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+y\,e^x\frac{\partial z}{\partial v}+ye^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+y\,e^x\frac{\partial z}{\partial v}+y\,e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$\therefore \quad r=e^{2y}\frac{\partial^2 z}{\partial u^2}+2y\,e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y^2e^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{\partial z}{\partial v}.$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\,e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial z}{\partial u}+x\,e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial z}{\partial v}+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\frac{\partial z}{\partial u}+xe^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$=x\,e^{2y}\frac{\partial^2 z}{\partial u^2}+(xy+1)\,e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y\,e^{2x}\frac{\partial^2 z}{\partial v^2}+e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}$$

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\,e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial u}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial^2 z}{\partial u^2}xe^y+\frac{\partial^2 z}{\partial u\partial v}e^x\right]+e^x\left[\frac{\partial^2 z}{\partial u\partial v}xe^y+\frac{\partial^2 z}{\partial v^2}e^x\right],$$

$$\therefore \qquad t = x^2 e^{2y}\frac{\partial^2 z}{\partial u^2} + 2x e^{x+y}\frac{\partial^2 z}{\partial u \partial v} + x e^{y}\frac{\partial z}{\partial u} + e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Putting the above values of r, s, t, p, q in the given equation and simplifying, we obtain the required canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) = 0. \qquad ...(6)$$

Integrating (6) w.r.t. 'v', $\partial z/\partial u = f(u)$, f being an arbitrary function ...(7)

Integrating (7) w.r.t. 'u', $z = \int f(u)\,du + \psi(v)$ or $z = \phi(u) + \psi(v)$, where $\phi(u) = \int f(u)\,du$.

Using (3), the required solution is $z = \phi(xe^{y}) + \psi(ye^{x})$, ϕ and ψ being arbitrary functions.

Ex. 15. (*a*) *Reduce the one-dimensional wave equation* $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *(c > 0) to canonical form and hence find its general solution.*

(b) Find the D'Alembert's solution of the Cauchy's problem: $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *(c > 0) satisfying z (x, 0) = f (x) and* $z_t(x,0) = g(x)$ *where f (x) and g (x) are given functions representing the initial displacement and initial velocity, respectively. Also,* $z_t = \partial z/\partial t$

Sol. (*a*) Given $\partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$, $c > 0$. ... (1)

To re-write (1), put $y = ct$, ... (2)

Then, (1) reduces to $\partial^2 z/\partial x^2 - (\partial^2 z/\partial y^2) = 0$ or $r - t = 0$ ... (3)

Proceed now exactly as in solved Ex. 1 to reduce (3) to its canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = 0 \qquad ... (4)$$

where $u = y + x$, $v = y - x$ or $u = ct + x$ and $v = ct - x$. ... (5)

Integrating (4) w.r.t. 'u', $\partial z/\partial v = f(v)$, where f is an arbitrary function ... (6)

Integrating (6) w.r.t. 'v', $z = \int f\ (v)\ dv + \psi(u) = F(v) + \psi(u)$, where $f(v) = \int f(v)\,dv$

or $z(x,t) = F(ct - x) + \psi\ (ct + x)$, using (5)

or $z(x,t) = \phi(x - ct) + \psi(x + ct)$, ... (7)

where we take $\phi\ (x - ct) = F\ (ct - x)$ and ϕ, ψ as arbitrary functions.

(7) is the required general solution of (1).

(*b*) We are to solve $\partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$ ... (*i*)

subject to the conditions $z(x,0) = f(x)$ ... (*ii*)

and $(\partial z/\partial t)_{t=0} = g(x)$ ... (*iii*)

Proceed exactly as in part (*a*) and get solution of (*i*) as

$$z(x,t) = \phi(x - ct) + \psi(x + ct) \qquad ... (iv)$$

Differentiating (*iv*) partially w.r.t. 't', we get

$$\partial z/\partial t = -c\ \phi'(x - ct) + c\ \psi'(x + ct) \qquad ... (v)$$

where dash denotes the derivative w.r.t. the argument. Putting $t = 0$ in (*iv*) and (*v*) and using (*ii*) and (*iii*) respectively, we get

$$\phi(x) + \psi(x) = f(x) \qquad ... (vi)$$

and $$-c\ \phi'(x) + c\ \psi'(x) = g(x) \qquad ... (vii)$$

Integrating (*vii*), $$-c\ \phi(x) + c\ \psi\ (x) = \int_a^x g(u)\ du, \qquad ... (viii)$$

where a is an arbitrary constant. Solving (*vi*) and (*viii*) for $\phi(x)$ and $\psi(x)$, we have

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_a^x g(x)\, du, \qquad \text{and} \qquad \psi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_a^x g(u)\, du$$

so that
$$\phi\,(x - ct) = \frac{1}{2} f(x - ct) - \frac{1}{2c}\int_a^{x-ct} g(u)\, du \qquad \ldots (ix)$$

and
$$\psi\,(x + ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c}\int_a^{x+ct} g(u)\, du \qquad \ldots (x)$$

Using (*ix*) and (*x*) in (*iv*), we get the required so called D'Alembert's solutiion of the Cauchy problem (which represents the vibrations of an infinite string in the present problem)

$$z(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{a} g(u)\, du + \int_a^{x+ct} g(u)\, dx\right]$$

or
$$z(x,t) = \frac{1}{2}\{f\,(x - ct) + f(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du \qquad \ldots (xi)$$

Particular Case I. If in the above problem, we take $g\,(x) = 0$ so that the initial velocity of the string is zero, then (*xi*) reduces to

$$z(x,t) = \{f(x - ct) + f(x + ct)\}/2\,,$$

where $f\,(x - ct)$ represents a right travelling wave travelling with the speed c (along OX) and $f\,(x + ct)$ represents a left travelling wave travelling with the speed c.

Particular case II. If $f\,(x) = \sin x$ and $g\,(x) = \cos x$ in the above problem, then the corresponding solution (*xi*) reduces to

$$z(x,t) = \frac{1}{2}\{\sin\,(x - ct) + \sin(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos u\, du$$

or $z(x,t) = \sin x \cos ct + (1/2c) \times \{\sin\,(x + ct) - \sin(x - ct)\}$ or $z(x,t) = \sin x \cos ct + (1/c) \times \cos x \sin ct$.

Particular case III. If $f(x) = \sin x$ and $g(x) = x^2$, then (*xi*) gives

$z(x,\ t) = \sin x \cos ct + x^2 t + (c^3 t^3)/3$, on simplification.

8.8 Working rule for reducing a parabolic equation to its canonical form.

Step 1. Let the given equation
$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \ldots (1)$$

be parabolic so that
$$S^2 - 4RT = 0\,.$$

Step 2. Write λ-quadratic equation
$$R\lambda^2 + S\lambda + T = 0 \qquad \ldots (2)$$

Let λ_1, λ_1 be two equal roots of (2)

Step 3. Write the characteristic equation corresponding to $\lambda = \lambda_1$, *i.e.,* $(dy/dx) + \lambda_1 = 0$

Solving it, we get $f_1(x, y) = C_1$, C_1 being an arbitrary constant ... (3)

Step 4. Choose $u = f_1(x, y)$ and $v = f_2(x, y)$ ... (4)

where $f_2\,(x, y)$ is an arbitrary function of x and y and is independent of $f_1\,(x, y)$. For this verify that Jacobian J of u and v given by (4) is non-zero,

i.e.
$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0 \qquad \ldots (5)$$

Step 5. Using relations (4), find p, q, r, s and t in terms of u and v as shown in Art. 8.5.

Step 6. Substituting the values of p, q, r, s and t obtained in step (1) and simplifying we get the following canonical forms of (1)

$$\partial^2 z/\partial u^2 = \phi\left(u, v, z, \partial z/\partial u, \partial z/\partial v\right) \quad \text{or} \quad \partial^2 z/\partial v^2 = \phi\left(u, v, z, \partial z/\partial u, \partial z/\partial v\right)$$

8.9 SOLVED EXAMPLES BASED ON ART. 8.8

Ex. 1. *Reduce the equation* $\partial^2 z/\partial x^2 + 2(\partial^2 z/\partial x \partial y) + \partial^2 z/\partial y^2 = 0$ *to canonical form and hence solve it.* **[Delhi Maths (H) 2000, 06; 08; Jabalpur 2004; Delhi Maths (Prog) II 2008; Delhi B.Sc. (Prog) II 2008, 11; Himanchal 2001; 05 Rajasthan 2003; Lucknow 2010]**

Sol. Re-writing the given equation, we get $r + 2s + t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2$, $T = 1$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadrate equation reduces to $\lambda^2 + 2\lambda + 1 = 0$ so that $\lambda = -1, -1$ (equal roots).

The corresponding characteristic equation is $(dy/dx) - 1 = 0$ or $dx - dy = 0$

Integrating, $x - y = c$, c being an arbitrary constant.

Choose $u = x - y$ and $v = x + y$, ... (2)

where we have chosen $v = x + y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1 \cdot 1 + 1 \cdot 1 = 2 \neq 0 .$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{... (4)}$$

From (3) and (4), $$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{... (5)}$$

$\therefore$ $$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4) and (5)}$$

$$= -\frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 v}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{... (7)}$$

and $$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4) and (5)}$$

$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \quad \text{... (8)}$$

Using (6), (7) and (8) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial v^2} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0 \qquad \text{... (9)}$$

To find the required solution. Integrating (9) partially w.r.t. 'v', we get

$$\partial z / \partial v = \phi(u), \ \phi \text{ being an arbitrary function.} \qquad \text{... (10)}$$

Integrating (10) partially w.r.t 'v', $z = \int \phi(u)\, dv + \psi(u) = v\phi(u) + \psi(u)$

or $z = (x+y)\,\phi(x-y) + \psi(x-y)$, which is the desired solution, ϕ, ψ being arbitrary functions.

Ex. 2. *Reduce the equation* $y^2(\partial^2 z/\partial x^2) - 2xy\,(\partial^2 z/\partial x\,\partial y) + x^2(\partial^2 z/\partial x^2) = (y^2/x)(\partial z/\partial x) + (x^2/y)(\partial z/\partial y)$ *to canonical form and hence solve it.* [**Nagpur 2005; Delhi Maths (H) 2001, 05, 09; Avadh 2001, Himanchal 2009; Delhi B.Sc. (Prog) II 2007; Meerut 2005, 06, 11; G.N.D.U. Amritsar 2005**]

Sol. Re-writing the given equation, $y^2 r - 2xys + x^2 t - (y^2/x)\,p - (x^2/y)q = 0 \qquad \text{... (1)}$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = -2xy$, $T = x^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$y^2\lambda^2 - 2xy\,\lambda + x^2 = 0$ or $(y\lambda - x)^2 = 0$ so that $\lambda = x/y,\ x/y$.

The corresponding characteristic equation is $dy/dx + x/y = 0$

or $x\,dx + y\,dy = 0$ so that $x^2/2 + y^2/2 = C_1$

Choose $u = x^2/2 + y^2/2$ and $v = x^2/2 - y^2/2, \qquad \text{... (2)}$

where we have chosen $v = x^2/2 - y^2/2$ in such a manner that u and v are independent functions as verified below

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -2xy \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right), \text{ using (2)} \qquad \text{... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right] = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ by (4)}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{... (6)}$$

and $$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left\{y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

or $$s = xy\,(\partial^2 z/\partial u^2 - \partial^2 z/\partial v^2) \quad \text{... (7)}$$

Using (3), (4), (5), (6) and (7) in (1) and simplifying, we get

$4x^2y^2\,(\partial^2 z/\partial v^2) = 0$ so that $\partial^2 z/\partial v^2 = 0$, ... (8)

which is the required canonical form.

Integrating (8) partially w.r.t. 'v', $\partial z/\partial v = \phi(u)$, ϕ being arbitrary function. ... (9)

Integrating (9) partially w.r.t. 'v', $z = v\ \phi(u) + \psi(u)$, ψ being arbitrary function.

or $$z = [(x^2 - y^2)/2]\,\phi\{(x^2 + y^2)/2\} + \psi\,\{(x^2 + y^2)/2\}, \text{ using (2)}$$

or $$z = (x^2 - y^2)\,F(x^2 + y^2) + G\,(x^2 + y^2),\ \text{F, G being arbitrary functions}$$

Ex. 3. *(a) Reduce $r + 2xs + x^2 t = 0$ to canonical form*

(b) Reduce $r - 6s + 9t + 2p + 3q - z = 0$ to canonical form

(c) Reduce $r - 2s + t + p - q = 0$ to canonical form and hence solve it.

Sol. (*a*) Given $$r + 2xs + x^2 t = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2x$ and $T = x^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + 2\lambda x + x^2 = 0$ or $(\lambda + x^2) = 0$ so that $\lambda = -x, -x$.

The corresponding characteristic equation is $(dy/dx) - x = 0$ or $dy - x\,dx = 0$

Integrating, $y - x^2/2 = c_1$, c_1 being an arbitrary constant. ... (2)

Choose $u = y - x^2/2$ and $v = x$ ... (2)

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -1 \neq 0$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}, \text{ using (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= -\frac{\partial z}{\partial u} - x\left(-x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = x^2\frac{\partial^2 z}{\partial u^2} - 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} \quad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} = -x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}, \text{ by (4)} \quad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} = \frac{\partial^2 z}{\partial u^2}, \text{ by (4)} \quad \text{... (7)}$$

Using (5), (6) and (7) in (1), we finally obtain $\partial^2 z/\partial v^2 = \partial z/\partial u$, which is required canonical form.

3. (*b*) Hint. Here $\lambda = 3$, $u = y + 3x$. Choose $v = y$. The canonical form will be

$$\partial^2 z/\partial v^2 = z/9 - (\partial z/\partial u) + (1/3)\times(\partial z/\partial v).$$

3. (c) Hints. Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = \partial z/\partial v$.

Solution is $z = \phi(x+y) + e^y \psi(x+y)$, ϕ, ψ being arbitrary functions

Ex. 4. *Reduce the following to canonical form and hence solve*

(*a*) $x^2 r + 2xy\, s + y^2 t = 0$

(*b*) $r - 4s + 4t = 0$

(*c*) $x^2 r + 2xys + y^2 t + xyp + y^2 q = 0$

(*d*) $2r - 4s + 2t + 3z = 0$.

Sol. (a) Given $x^2 r + 2xys + y^2 t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$x^2\lambda^2 + 2xy\lambda + y^2 = 0$ or $(x\lambda + y)^2 = 0$ giving $\lambda = -y/x,\ -y/x$.

The corresponding characteristic equation is $dy/dx - y/x = 0$

or $(1/y)dy - (1/x)dx = 0$ so that $\log y - \log x = c_1$ or $y/x = c_1$

Choose $u = y/x$ and $v = y$, ... (2)

where we have chosen $v = y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -\frac{y}{x^2} \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\frac{y}{x^2}\frac{\partial z}{\partial u}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial z}{\partial u}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial u} - \frac{y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right), \text{ by (3)}$$

$$= \frac{2y}{x^3}\frac{\partial z}{\partial u} - \frac{y}{x^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{2y}{x^3}\frac{\partial z}{\partial u} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial u^2} \quad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}.$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} - \frac{y}{x^3}\frac{\partial^2 z}{\partial u^2} - \frac{y}{x^2}\frac{\partial^2 z}{\partial u \partial v} \quad \text{... (6)}$$

$$t = \frac{\partial^2 y}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \text{ using (4)}$$

$$= \frac{1}{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + \frac{2}{x}\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial v^2} \quad \text{... (7)}$$

Using (5), (6) and (7) in (1), we finally get as the canonical form $\partial^2 z / \partial v^2 = 0$ ... (8)

Integrating (8) partially w.r.t. 'v', $\partial z / \partial v = \phi(u)$ ... (9)

Integrating (9) partially w.r.t 'v', $z = v\phi(u) + \psi(u)$

or $z = y\phi(y/x) + \psi(y/x), \phi, \psi$ being arbitrary functions.

(*b*) Hint. Here $\lambda = 2$, $u = y + 2x$. Choose $v = y$. The canonical form is $\partial^2 z / \partial v^2 = 0$ and solution is $z = y\phi(y + 2x) + \psi(y + 2x)$.

(*c*) Hint. Here $\lambda = -y/x$, $u = y/x$. Choose $v = y$. The canonical form is $\partial^2 z / \partial v^2 = -(\partial z / \partial v)$ and solution is $z = \phi(y/x) + e^{-y}\psi(y/x)$

(*d*) **Hint.** Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z / \partial v^2 = -(3z/2)$ and solution is $z = e^{(i\sqrt{3}/2)y}\phi(y + x) + e^{-(i\sqrt{3}/2)}\psi(y + x)$,

Ex. 5. *Reduce the following in canonical form and solve them*

(*a*) $r - 2s + t + p - q = e^x(2y - 3) - e^y$

(*b*) $r - 2s + t + p - q = e^{x+y}$

Sol. (*a*) Given $r - 2s + t + p - q - e^x(2y - 3) + e^y = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = -2$ and $T = 1$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 - 2\lambda + 1 = 0$ or $(\lambda - 1)^2 = 0$ so that $\lambda = 1, 1$ (equal roots)

So the corresponding characteristic equation is $dy/dx + 1 = 0$ or $dx + dy = 0$

Integrating it, $x + y = c_1$, c_1 being an arbitrary constant.

Choose $u = x + y$ and $v = y$ ... (2)

where we have chosen $v = y$ in such a manner that u and v are independent functions as verified below

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1 \neq 0 .$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}, \text{ using (2)} \quad \ldots (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \ldots (4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2} \text{ by (3)} \quad \ldots (5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (4)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \ldots (6)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}, \text{ using (4)} \quad \ldots (7)$$

Using (2) (3), (4), (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial u^2} - 2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u} - \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = e^{u-v}(2v-3) - e^v$$

or $$\partial^2 z/\partial v^2 - \partial z/\partial v = e^{u-v}(2v-3) - e^v \quad \ldots (8)$$

which is the required canonical form of (1) Let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$.

Then (8) can be re-written as $$D'(D'-1)\, z = e^{u-v}(2v-3) - e^v, \quad \ldots (9)$$

which is non-honmogeneous linear partial differential equation with constant coefficients. To solve it, we shall use results of chapter 5. Accordingly, we have

$$\text{C.F.} = \phi(u) + e^v\, \psi(u) = \phi(x+y) + e^y \psi(x+y), \text{ by (2)}$$

P.I. corresponding to $e^{u-v}(2v-3)$

$$= \frac{1}{D'(D'-1)} e^{u+(-1)v}(2v-3) = e^{u+(-1)v} \frac{1}{(D'-1)\,(D'-1-1)}(2v-3)$$

$$= (1/2) \times e^{u-v}\, (1-D')^{-1}(1-D'/2)^{-1}(2v-3) = (1/2) \times e^{u-v}\, (1+D'+\ldots)(1+D'/2+\ldots)\ (2v-3)$$

$$= (1/2) \times e^{u-v}\,(1+3D'/2+\ldots)\,(2v-3) = (1/2) \times e^{u-v}\,(2v-3+3) = v\, e^{u-v} = y\, e^{x+y-y} = y e^x, \text{ using(2)}$$

P.I. Corresponding to $(-e^v)$

$$= \frac{1}{D'(D'-1)}(-e^v) = -\frac{1}{D'-1}\frac{1}{D'}e^v = -\frac{1}{D'-1}\,(e^v \times 1) = -e^v \frac{1}{D'+1-1} 1 = -e^y \frac{1}{D'} 1$$

$$= -e^v v = -e^y y, \text{ using (2)}$$

Hence the required general solution is given by $y = \phi\,(x+y) + e^y\ \psi\ (x+y) + y\, e^x - y\, e^y$

or $$y = \phi\,(x+y) + e^y\ \psi\ (x+y) + y e^x - (x+y)\, e^y + x\, e^y$$

or $$y = \phi\,(x+y) + e^y\ \{\phi\,(x+y) + (x+y)\} + y\, e^x + x\, e^y$$

or $$y = \phi\ (x + y) + e^y\ F\ (x + y) + y\ e^x + x\ e^y,$$

where ϕ and F are arbitrary functions and $F\ (x + y) = \phi\ (x + y) + x + y$

(*b*) Hint. Here $\lambda = 1$, $u = x + y$, choose $v = y$. The canonical form is $\partial^2 z / \partial v^2 = \partial z / \partial v + e^u$ and solution is $z = \phi\ (x+y) + e^y \psi(x+y) - y\ e^{x+y}$, ϕ, ψ being arbitrary functions.

Ex. 6. *Reduce the equation $x^2r - 2xys + y^2t - xp + 3yq = 8y/x$ to canonical form.*

[Delhi B.Sc. (H) 1999]

Sol. Given $$x^2r - 2xy\ s + y^2t - xp + 3yq - 8y/x = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, y, p, q) = 0$, here $R = x^2$, $S = -2xy$, $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$x^2\lambda^2 - 2xy\lambda + y^2 = 0$ or $(x\lambda - y)^2 = 0$ so that $\lambda = y/x,\ y/x$.

The corresponding characteristic equation is

$dy/dx + y/x = 0$ or $(1/y)\ dy + (1/x)\ dx = 0$ so that $xy = C_1$

Choose $u = xy$ and $v = x$ ... (2)

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -x \neq 0\ .$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v},\ \text{by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u},\ \text{by (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= y\left(y\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right) + y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}, \quad \text{.... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial z}{\partial u} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{\partial z}{\partial u} + xy\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial u \partial v} \quad \text{... (6)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}\right) = x\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right),\ \text{by (4)}$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] = x^2\frac{\partial^2 z}{\partial u^2},\ \text{by (2)} \quad \text{... (7)}$$

Using (2), (3), (4), (5), (6) and (7) in (1), we have

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-2xy\left(\frac{\partial z}{\partial u}+xy\frac{\partial^2 z}{\partial u^2}+x\frac{\partial^2 z}{\partial u\partial v}\right)+y^2x^2\frac{\partial^2 z}{\partial u^2}-x\left(y\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+3y\ x\frac{\partial z}{\partial u}-\frac{8y}{x}=0$$

or $x^2\dfrac{\partial^2 z}{\partial v^2}--x\dfrac{\partial z}{\partial v}=\dfrac{8y}{x}$ or $v^2\dfrac{\partial^2 z}{\partial v^2}-v\dfrac{\partial z}{\partial v}=\dfrac{8u}{v^2}$, by (2)

or $(v^2D'^2-vD')z=8u/v^2$, where $D\equiv\partial/\partial u,\quad D'\equiv\partial/\partial v$ ... (8)

As explained in chapter 6, we shall reduce (8) to linear partial differential equation with constant coefficients and then use methods of chapter 5 to solve the resulting equation.

To solve (8), let $u=e^X$ and $v=e^Y$ so that $X=\log u,\quad Y=\log v$ ... (9)

Then (8) becomes $\{D'(D'-1)-D'\}z=8e^{X-2y}$ or $D'(D'-2)z=8e^{X-2y}$.

C.F. $=\phi(X)+e^{2Y}\psi(X)=\phi(\log u)+v^2\psi(\log u)$, using (9)

$=F(u)+v^2G(u)=F(xy)+x^2G(xy)$, using (2)

P.I. $=\dfrac{1}{D'(D'-1)}8e^{X-2Y}=8e^{X-2Y}\dfrac{1}{(D'-2)(D'-2-2)}\cdot 1$

$$=\frac{8e^X}{(e^Y)^2}\times\frac{1}{8}\left(1-\frac{D'}{2}\right)^{-1}\left(1-\frac{D'}{4}\right)^{-1}\cdot 1=\frac{u}{v^2}\left(1+\frac{D'}{2}+...\right)\left(1+\frac{D'}{4}+...\right)\cdot 1=\frac{u}{v^2}\times 1=\frac{xy}{x^2}=\frac{y}{x},\text{ by (2)}$$

$\therefore$ Required solution is $z=F(xy)+x^2G(xy)+y/x$, F, G being arbitrary functions.

8.10 Working rule for reducing an elliptic equation to its canonical form.

Step 1. Let the given equation $Rr+Ss+Tt+f(x,y,z,p,q)=0$ ... (1)

be ellipitic so that $S^2-4RT<0$. ... (2)

Step 2. Write λ quadratic equation $R\lambda^2+S\lambda+T=0$ ... (2)

Let roots λ_1, λ_2 of (2) be complex conjugates.

Step 3. Then corresponding charactersitic equations are

$(dy/dx)+\lambda_1=0$ and $dy/dx+\lambda_2=0$

Solving these, we shall obtain solutions of the form

$f_1(x,y)+i\ f_2(x,y)=c_1$ and $f_1(x,y)-i\ f_2(x,y)=c_2$ ... (3)

Step 4. Choose $u=f_1(x,y)+i\,f_2(x,y),\quad v=f_1(x,y)-i\ f_2(x,y)$

Let α and β be two new real independent variables such that $u=\alpha+i\beta$ and $v=\alpha-i\beta$,

so that $\alpha=f_1(x,y)$ and $\beta=f_2(x,y)$ ... (4)

Step 5. Using relations (4), find *p*, *q*, *r*, *s* and *t* in terms of α and β (in place of *u* and *v* as we did in Art 8.6 and 8.8 corresponding to the cases of hyperbolic and parabolic equations).

Step 6. Substituting the values of *p*, *q*, *r*, *s* and *t* and relations (4) in (1) and simplifying we shall get the following canonical form of (1)

$$\partial^2 z/\partial\alpha^2+\partial^2 z/\partial\beta^2=\phi(\alpha,\beta,z,\partial z/\partial\alpha,\ \partial z/\partial\beta).$$

8.11 SOLVED EXAMPLES ON ART 8.10

Ex. 1. *Reduce the following partial differential equations to canonical forms:*

$(a)\ \partial^2 z/\partial x^2+x^2(\partial^2 z/\partial y^2)=0$ *or* $r+x^2t=0$

[Delhi B.Sc. (Prog) II 2010; Delhi B.Sc. (Hons) III 2011; Kanpur 2011]

(*b*) $y^2(\partial^2 z/\partial y^2)+\partial^2 z/\partial x^2 = 0$ **[Delhi Math (Hons.) 1995, 98, 2005]**

Sol. (*a*) Re-writing the given equations, we get $r + x^2 t = 0$ ... (1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R = 1,\ S = 0,\ T = x^2$ so that $S^2 - 4RT = -4x^2 < 0,\ x \neq 0,$ showing that (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + x^2 = 0$ giving $\lambda = ix, -ix$.
The corresponding characteristic equations are given by

$dy/dx + ix = 0$ and $dy/dx - ix = 0$

Integrating, $y + i\ (x^2/2) = c_1$ and $y - i\ (x^2/2) = c_2$.

Choose $u = y + i(x^2/2) = \alpha + i\beta$ and $v = y - i(x^2/2) = \alpha - i\beta$,

where $\alpha = y$ and $\beta = x^2/2$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}, \text{ by (2)} \quad \text{.... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}, \text{ by (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right), \text{ by (3)}$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right] = \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} \quad \text{... (5)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}, \text{ by (4)} \quad \text{... (6)}$$

Using (5) and (6) in (1) the required canonical form is

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0 \quad \text{or} \quad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta}, \text{ as } \beta = \frac{x^2}{2}.$$

(*b*) Do as in part (a). **Ans.** $\partial^2 z/\partial\alpha^2 + \partial^2 z/\partial\beta^2 = -(1/2\alpha)\times(\partial z/\partial\alpha)$, where $\alpha = y^2/2,\ \beta = x$.

Ex. 2. *Reduce* $y^2(\partial^2 z/\partial x^2) + x^2(\partial^2 z/\partial y^2) = 0$ *to canonical form*

Sol. Re-writing the given equation, we get $y^2 r + x^2 t = 0$ ... (1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R = y^2,\ S = 0,\ T = x^2$ so that $S^2 - 4RT = -4x^2y^2 < 0$ for $x \neq 0,\ y \neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$y^2\lambda^2 + x^2 = 0$ or $\lambda^2 = -x^2/y^2$ so that $\lambda = ix/y,\quad -ix/y$

The corresponding characteristic equations are

$dy/dx + ix/y = 0$ and $dy/dx - ix/y = 0$

Integrating, $y^2 + ix^2 = C_1$ and $y^2 - ix^2 = C_2$

Choose $u = y^2 + ix^2 = \alpha + i\beta$ and $v = y^2 - ix^2 = \alpha - i\beta$,

where $\alpha = y^2$ and $\beta = x^2$ ... (2)

are now two new independent variables

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = 2x\frac{\partial z}{\partial \beta}, \text{ by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = 2y\frac{\partial z}{\partial \alpha}, \text{ by (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial \beta}\right) = 2\frac{\partial z}{\partial \beta} + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right), \text{ by (3)}$$

$$= 2\frac{\partial z}{\partial \beta} + 2x\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\} = 2\frac{\partial z}{\partial \beta} + 4x^2\frac{\partial^2 z}{\partial \beta^2} \quad \text{... (5)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial \alpha}\right) = 2\frac{\partial z}{\partial \alpha} + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= 2\frac{\partial z}{\partial \alpha} + 2y\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right\} = 2\frac{\partial z}{\partial \alpha} + 4y^2\frac{\partial^2 z}{\partial \alpha^2} \quad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$2y^2\frac{\partial z}{\partial \beta} + 4x^2y^2\frac{\partial^2 z}{\partial \beta^2} + 2x^2\frac{\partial z}{\partial \alpha} + 4x^2y^2\frac{\partial^2 z}{\partial \alpha^2} = 0 \quad \text{or} \quad 2\alpha\beta\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) + \alpha\frac{\partial z}{\partial \beta} + \beta\frac{\partial z}{\partial \alpha} = 0$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}\right) = 0$$

Ex. 3. *Reudce* $\partial^2 z/\partial x^2 + y^2(\partial^2 z/\partial y^2) = y$ *to canonical form.*

Sol. Re-writing the given equation, we get $r + y^2 t - y = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f\ (x, y, z, p, q) = 0$, here R = 1, S = 0 and $T = y^2$ so that $S^2 - 4RT = -4y^2 < 0$ for $y \neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + y^2 = 0 \Rightarrow \lambda = iy,\ -iy$.

The corresponding characteristic equations are given by

$dy/dx + iy = 0$ and $dy/dx - iy = 0$

Integrating these, $\log y + ix = c_1$ and $\log y - ix = c_2$

Choose $u = \log y + ix = \alpha + i\beta$ and $v = \log y - ix = \alpha - i\beta$,

where $\alpha = \log y$ and $\beta = x$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial z}{\partial \beta}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial \alpha}, \text{ using (2)} \quad \text{.... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) = \frac{\partial^2 z}{\partial \beta^2}, \text{ by (3)} \quad \text{... (5)}$$

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right)=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial \beta}{\partial y}\right)\right\}$$

$$=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y}\left(\frac{\partial^2 z}{\partial \alpha^2}\frac{1}{y}\right)=\frac{1}{y^2}\left(\frac{\partial^2 z}{\partial \alpha^2}-\frac{\partial z}{\partial \alpha}\right) \quad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial \beta^2}+\frac{\partial^2 z}{\partial \alpha^2}-\frac{\partial z}{\partial \alpha}-y=0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=\frac{\partial z}{\partial \alpha}+e^{\alpha}, \quad \text{using (2)}$$

Ex. 4. *Reduce* $x\,(\partial^2 z/\partial x^2)+\partial^2 z/\partial y^2=x^2$ *(x > 0) to canonical form.* **[Delhi Maths(H) 2007, 11]**

Sol. Re-writing the given equation, we get $xr+t-x^2=0,\ (x>0)$ ... (1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R = x$, $S = 0$ and $T = 1$ so that $S^2-4RT=-4x<0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to

$x\lambda^2+1=0$ or $\lambda^2=-(1/x^2)$ so that $\lambda=i/x^{1/2},\ -i/x^{1/2}$

The corresponding characteristic equations are given by

$dy/dx+i\,x^{-1/2}=0$ and $dy/dx-i\,x^{-1/2}=0$.

Integrating these, $y+2i\,x^{1/2}=C_1$ and $y-2i\,x^{1/2}=C_2$

Choose $u=y+2i\,x^{1/2}=\alpha+i\beta$ and $v=y-2ix^{1/2}=\alpha-i\beta$,

where $\alpha=y$ and $\beta=2x^{1/2}$ ... (2)

are now two new independent variables.

Now, $$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x}=x^{-1/2}\frac{\partial z}{\partial \beta}, \text{ by (2)} \quad \text{... (3)}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}=\frac{\partial z}{\partial \alpha}, \text{ by (2)} \quad \text{... (4)}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x^{-1/2}\frac{\partial z}{\partial \beta}\right)=-\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta}+x^{-1/2}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

or $$r=-\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta}+x^{-1/2}\left(x^{-1/2}\frac{\partial^2 z}{\partial \beta^2}\right)=-\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta}+\frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \quad \text{... (5)}$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)=\frac{\partial^2 z}{\partial \alpha^2}, \text{ using (4)} \quad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$x\left(-\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta}+\frac{1}{x}\frac{\partial^2 z}{\partial \beta^2}\right)+\frac{\partial^2 z}{\partial \alpha^2}=x^2 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=x^2+\frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta}$$

or $\partial^2 z/\partial\alpha^2 + \partial^2 z/\partial\beta^2 = (\beta^2/4) + (1/\beta)\times(\partial z/\partial\beta)$, as $\beta = 2\,x^{1/2}$.

Ex. 5. *Reduce* $\dfrac{\partial^2 z}{\partial x^2} + 2\,\dfrac{\partial^2 z}{\partial x \partial y} + 5\,\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} - 3z = 0$ *to canonical form.*

Sol. Re-writing the given equation, we get $r + 2s + 5t + p - 2q - 3z = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f\ (x, y, z, p, q) = 0$, here $R = 1$, $S = 2$ and $T = 5$ so that $S^2 - 4RT = -16 < 0$, showing (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + 2\lambda + 5 = 0$ so that $\lambda = \{-2 \pm (4-20)^{1/2}\}/2 = -1 \pm 2i$

The corresponding characterisitc equations are given by

$dy/dx + (-1+2i) = 0$ and $dy/dx + (-1-2i)x = 0.$

Integrating these, $y + (-1+2i)x = C_1$ and $y + (-1-2i)x = C_2$

Let $u = y - x + 2ix = \alpha + i\beta$ and $v = y - x - 2ix = \alpha - i\beta$,

where $\alpha = y - x$ and $\beta = 2x$ ... (2)

are now two new independent variables.

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial \alpha}\dfrac{\partial \alpha}{\partial x} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial \beta}{\partial x} = -\dfrac{\partial z}{\partial \alpha} + 2\dfrac{\partial z}{\partial \beta}$, using (2) ... (3)

$q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial \alpha}\dfrac{\partial \alpha}{\partial y} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial \beta}{\partial y} = \dfrac{\partial z}{\partial \alpha}$, using (2) ... (4)

$t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial \alpha}\left(\dfrac{\partial z}{\partial \alpha}\right) = \dfrac{\partial^2 z}{\partial \alpha^2}$, using (3) ... (5)

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) + 2\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= -\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}\right\} + 2\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

$$= -\left(-\frac{\partial^2 z}{\partial \alpha^2} + 2\frac{\partial^2 z}{\partial \beta \partial \alpha}\right) + 2\left(-\frac{\partial^2 z}{\partial \alpha \partial \beta} + 2\frac{\partial^2 a}{\partial \beta^2}\right), \text{ by (2)}$$

$\therefore$ $r = \partial^2 z/\partial\,\alpha^2 + 4\,(\partial^2 z/\partial\beta^2) - 4\,(\partial^2 z/\partial\alpha\partial\beta)$ ... (6)

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}$$

or $s = -(\partial^2 z/\partial\alpha^2) + 2\,(\partial^2 z/\partial\alpha\ \partial\beta)$, using (2) ... (7)

Using (3), (4), (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial \alpha^2} + 4\,\frac{\partial^2 z}{\partial \beta^2} - 4\frac{\partial^2 z}{\partial \alpha \partial \beta} + 2\left(-\frac{\partial^2 z}{\partial \alpha^2} + 2\frac{\partial^2 z}{\partial \alpha \partial \beta}\right) + 5\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta} - 2\,\frac{\partial z}{\partial \alpha} - 3z = 0$$

or $4\left(\dfrac{\partial^2 z}{\partial \alpha^2} + \dfrac{\partial^2 z}{\partial \beta^2}\right) = 3z + 3\,\dfrac{\partial z}{\partial \alpha} - 2\dfrac{\partial z}{\partial \beta}$ or $\dfrac{\partial^2 z}{\partial \alpha^2} + \dfrac{\partial^2 z}{\partial \beta^2} = \dfrac{3z}{4} + \dfrac{3}{4}\dfrac{\partial z}{\partial \alpha} - \dfrac{1}{2}\dfrac{\partial z}{\partial \beta}$,

which is the required canonical form of given equation (1).

8.12. The solution of linear hyperbolic equations. It what follows we aim at sketching the existence theorems for two types of initial conditions on the linear hyperbolic equation

$$\partial^2 z/\partial x\partial y = f(x, y, z, p, q). \qquad ...(1)$$

For both kinds of initial condition, we assume that the function $f(x, y, z, p, q)$ satisfies the following two conditions :

(*i*) f is continuous at all points of a rectangular region R defined by $\alpha < x < \beta$, $\gamma < y < \delta$ for all values of x, y, z, p, q concerned.

(*ii*) f satisfies the so called Lipschitz condition, namely,

$$|f(x, y, z_2, p_2, q_2) - f(x, y, z_1, p_1, q_1)| \le M\{|z_2 - z_1| + |p_2 - p_1| + |q_2 - q_1|\}$$

in all bounded subrectangles r of R.

We now state (without proof) two existence theorems.

Theorem 1. Initial conditions of the first kind. *If F(x) and G(x) are defined in the open intervals (α, β), (γ, δ), respectively, and have continuous first derivatives, and if (ξ, η) is a point inside R such that $F(\xi) = G(\eta)$, then (1) has at least one integral $z = \phi(x, y)$ in R such that*

$$\phi(x, y) = \begin{cases} F(x), \text{ when } y = \eta \\ G(y), \text{ when } x = \xi. \end{cases}$$

Theorem II. ***Initial conditions of the second kind.*** *Let C_1 be a space curve defined by $x = x(\lambda)$, $y = y(\lambda)$, $z = z(\lambda)$ in terms of a single parameter λ and also let C_0 be the projection of C_1 on the xy-plane. If we are given (x, y, z, p, q) along a strip C_1, then (1) has an integral which takes on the given values of z, p, q along the curve C_0. This intergral exists at every point of the region R, which is defined as the smallest rectangle completely enclosing the curve C_0.*

8.13. Riemann method of solution of general linear hypobolic equation of the second order. [Himanchal 2002; Meerut 2005, 07, 08; Delhi Maths (Hons.) 1995, 1999, 2000]

Assume that the given linear hyperbolic equation is reducible to canonical form

$$L(z) = f(x, y), \qquad ...(1)$$

where L denotes the linear operator given by

$$L \equiv \frac{\partial^2}{\partial x\partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c, \qquad ...(2)$$

where a, b, c are functions of x and y only.

Let w be another function with continuous derivatives of the first order. Again, let M be another operator defined by the relation

$$Mw = \frac{\partial^2 w}{\partial x\partial y} - \frac{\partial(aw)}{\partial x} - \frac{\partial(bw)}{\partial y} + cw. \qquad ...(3)$$

The operator M defined by (4) is called the *adjoint operator* to the operator L.

$$\therefore \quad w\,Lz - z\,Mw = w\left(\frac{\partial^2 z}{\partial x\partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + c\right) - z\left(\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial(aw)}{\partial x} - \frac{\partial(bw)}{\partial y} + cw\right)$$

$$= \left(w\frac{\partial^2 z}{\partial x\partial y} - z\frac{\partial^2 w}{\partial x\partial y}\right) + \left(wa\frac{\partial z}{\partial x} + z\frac{\partial(aw)}{\partial x}\right) + \left(wb\frac{\partial z}{\partial y} + z\frac{\partial(bw)}{\partial y}\right) = \frac{\partial}{\partial y}\left(w\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial y}\right) + \frac{\partial(awz)}{\partial x} + \frac{\partial(bwz)}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(awz - z\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(bwz + w\frac{\partial z}{\partial x}\right) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}, \qquad ...(4)$$

where $U = awz - z(\partial w/\partial y)$ and $V = bwz + w(\partial z/\partial x)$. ...(5)

Now if C' is a closed curve enclosing an area S, then

$$\int\int_S (w\,Lz - z\,Mw)\,dxdy = \int\int_S \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)dxdy = \int_{C'} (Udy - Vdx), \text{ by Green's theorem} \qquad ...(6)$$

Assume that the values of z and $\partial z/\partial x$ (or $\partial z/\partial y$) are prescribed along a curve C in the xy-plane (refer figure 1) and further assume that we are required to determine the solution of (1) at the point $P(\xi, \eta)$ agreeing with these boundary conditions. Draw PA, PB parallel to x-axis and y-axis and cutting the curve C in the points A and B respectively. The closed circuit $PABP$ can be taken as the closed curve C'. Then (6) reduces to

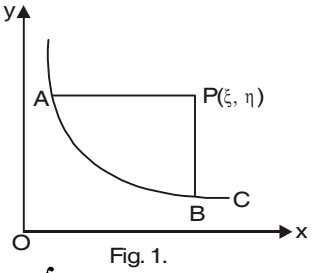


Fig. 1.

$$\iint_S (w\,Lz - z\,Mw)\,dxdy = \int_{AB}(Udy - Vdx) + \int_{BP}(Udy - Vdx) + \int_{PA}(Udy - Vdx)$$
$$= \int_{AB}(Udy - Vdx) + \int_{BP} Udy - \int_{PA} Vdx, \qquad ...(7)$$

where we have used the following facts:

along BP, x = constant so that $dx = 0$ and along PA, y = constant so that $dy = 0$.

Now, $\int_{PA} Vdx = \int_{PA}\left(bwz + w\frac{\partial z}{\partial x}\right)dx$, by (5)

$$= \int_{PA} bwzdz + \int_{PA} w\frac{\partial z}{\partial x}dx = \int_{PA} bwzdz + [wz]_P^A - \int_{PA} z\frac{\partial w}{\partial x}dx, \text{ integrating by parts}$$
$$= [wz]_A - [wz]_P + \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx. \qquad ...(8)$$

Using (5) and (8), (7) becomes

$$\iint_S (w\,Lz - z\,Mz)\,dxdy = \int_{AB}(Udy - Vdx) + \int_{BP}\left(awz - z\frac{\partial w}{\partial y}\right)dy - [wz]_A + [wz]_P - \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx.$$

$$\therefore [wz]_P = [wz]_A + \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx - \int_{BP} z\left(aw - \frac{\partial w}{\partial y}\right)dy$$
$$- \int_{AB}(Udy - Vdx) + \iint_S (w\,Lz - z\,Mw)\,dxdy \qquad ...(9)$$

So far we have treated w as an arbitrary function. Now, we choose a function $w(x, y, \xi, \eta)$ which has the following four properties, namely,

(*i*) $Mw = 0$, (*ii*) $w = 1$, when $x = \xi$, $y = \eta$ *i.e.*, at $P(\xi, \eta)$,

(*iii*) $\partial w/\partial x = bw$ when $y = \eta$, (*iv*) $\partial w/\partial y = aw$ when $x = \xi$.

Such a function $w(x, y, \xi, \eta)$ is known as *Green's function* for the problem or sometimes a *Riemann-Green function*. Using the above four properties of w, (9) may be re-written as

$$[z]_P = [wz]_A - \int_{AB}(Udy - Vdx) + \iint_S w\,Lz\,dxdy$$
$$= [wz]_A - \int_{AB}\left(awz - z\frac{\partial w}{\partial y}\right)dy + \int_{AB}\left(bwz + w\frac{\partial z}{\partial x}\right)dx + \iint_S (wf)dxdy, \text{ using (1) and (5)}$$
$$= [wz]_A - \int_{AB} wz(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\,dxdy. \qquad .(10)$$

Equation (10) may be used to determine the value of z at the point P when $\partial z/\partial x$ is prescribed along the curce C.

Suppose, in place of the prescribed value of $\partial z/\partial x$, we are now given a prescribed value of $\partial z/\partial y$. Then, we make use of the following relation

$$\int_{AB} d(wz) = \int_{AB}\left(\frac{\partial(wz)}{\partial x}dx + \frac{\partial(wz)}{\partial y}dy\right)$$

$$\Rightarrow \quad 0 = [wz]_B - [wz]_A - \int_{AB}\left(\frac{\partial(wz)}{\partial x}dx + \frac{\partial(wz)}{\partial y}dy\right). \qquad ...(11)$$

Adding the corresponding sides of (10) and (11), we get

$$[z]_P = [wz]_B - \int_{AB} wz\,(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) - \int_{AB}\left(\frac{\partial(wz)}{\partial x}dx + \frac{\partial(wz)}{\partial y}dy\right) + \iint_S (wf)\,dxdy$$

$$= [wz]_B - \int_{AB} wz\,(ady - bdx) - \iint_{AB}\left(z\frac{\partial w}{\partial x}\,dx + w\frac{\partial z}{\partial y}\,dy\right) + \iint_S (wf)\,dxdy \qquad ..(12)$$

Equation (12) may be used to determine the z at the point P when $\partial z/\partial y$ is precribed along the curve C.

Finally, by adding (10) and (12), we get the following symmetrical result which can be used to find value of z at the point P when both $\partial z/\partial x$ and $\partial z/\partial y$ are prescribed along the curve C.

$$[z]_P = \frac{1}{2}\{[wz]_A + [wz]_B\} - \int_{AB} wz\,(ady - bdx) + \iint_S (wf)\,dxdy$$

$$-\frac{1}{2}\int_{AB} w\left(\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) - \frac{1}{2}\int_{AB} z\left(\frac{\partial w}{\partial x}dx - \frac{\partial w}{\partial y}dy\right). \qquad ...(13)$$

By means of whichever of the formulas (10), (12) and (13) is suitable, we may determine the solution of (1) at any point in terms of the prescribed values of z, $\partial z/\partial x$ or/and $\partial z/\partial y$ along a given curve C.

We now discuss four particular cases:

Particular Case I *Determine the solution of*

$$\partial^2 z/\partial x\,\partial y + a(\partial z/\partial x) + b(\partial z/\partial y) + cz = f(x, y) \qquad \ldots (i)$$

which satisfies the boundary conditions that z and $\partial z/\partial x$ *are prescribed along curve C in the xy-plane.* **[Delhi Maths (H) 1995, 99, 2000, 06, 08; Meerut 2010]**

Hint. Proceed as in Art. 8.13 upto equation (10), *i.e.,*

$$[z]_P = [wz]_A - \int_{AB} wz\,(a\,dy - b\,dx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf dxdy \qquad \ldots (ii)$$

Relation (*ii*) may be used to determine the value of z at the point P when z and $\partial z/\partial x$ are prescribed along a curve C.

Particular case II *To determine the solution of the equation*

$$\partial^2 z/\partial x\,\partial y = f(x, y) \qquad \ldots (iii)$$

which satisfies the boundary conditions that z and $\partial z/\partial x$ *are prescribed along a curve C in the xy-plane.* **[Meerut 2010; Delhi. Maths (H) 1995, 99, 2000, 06, 08, 09, 10]**

Hint. First state and prove that above particular case I. Note that (*ii*) is solution of (*i*). Comparing (*iii*) with (*i*), we have $a = b = c = 0$ and hence for the present equation (*iii*), (*ii*) gives

$$[z]_P = [wz]_A + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\;dx\,dy \qquad \ldots (iv)$$

where the Green's function w satisfies the following four properties (refer Art. 8.13 and note that $a = b = c = 0$ for the present case).

(*a*) $\partial^2 w/\partial x\,\partial y = 0$ at all points of S (*b*) $w = 1$ at $P(\xi, \eta)$

(*c*) $\partial w/\partial x = 0$ when $y = \eta$ (*d*) $\partial w/\partial y = 0$ when $x = \xi$

Hence Green's function can be taken as $w = 1$ so as to satisfy the above four conditions. Substituting $w = 1$ in (*iv*), the required solution takes the following form

$$[z]_P = [z]_A + \int_{AB}\frac{\partial z}{\partial x}\,dx + \iint_S f(x, y)\,dx\,dy \qquad \ldots (v)$$

The relation (*v*) may be used to determine the value of z at the point P when z and $\partial z/\partial x$ are prescribed along a curve C.

Particular Case III *Determine the solution of*

$$\partial^2 z/\partial x\partial y + a\,(\partial z/\partial x) + b(\partial z/\partial y) + cz = f(x, y) \qquad \ldots (vi)$$

which satisfies the boundary conditions that z and $\partial z/\partial y$ *are prescribed along a curve C in the xy-plane.*

Hint. Proceed as in Art. 8.13 upto equation (12), *i.e*.

$$[z]_P = [wz]_B - \iint_{AB} wz\,(ady - bdx) - \iint_{AB} \left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy \right) + \iint_S wf\ dx\,dy \qquad \ldots (vii)$$

Relation (*vii*) may be used to determine the value of *z* at the point *P* when *z* and $\partial z/\partial y$ are prescribed along a curve C in the *xy*-plane.

Particular Case IV To *determine the solution* of

$$\partial^2 z/\partial x\partial y = f(x, y) \qquad \ldots (viii)$$

which satisfies the boundary conditions that z and $\partial z/\partial y$ *are prescribed along a curve C in the xy-plane*. **[Delhi Maths (H) 1974, 97, 2001]**

Hint. First state and prove the above particular case III. Note that (*vii*) is solution of (*vi*). Comparing (*viii*) with (*vi*), we have a = b = c = 0 and hence for the present equation (*viii*), (*vii*) reduces to

$$[z]_P = [wz]_B - \iint_{AB} \left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy \right) + \iint_S wf\ dx\,dy \qquad \ldots (ix)$$

where the Green's function *w* satisfies the following four properties (refer Art. 8.13 and note that *a* = *b* = *c* = 0 for the present case)

(*a*) $\partial^2 w/\partial x\partial y = 0$ at all points of *S* (*b*) $w = 1$ at $P(\xi, \eta)$

(*c*) $\partial w/\partial x = 0$ when $y = \eta$ (*d*) $\partial w/\partial y = 0$ when $x = \xi$.

Hence Green's function can be taken as $w = 1$ so as to satisfy the above four conditions. Substituting $w = 1$ in (*ix*), the required solution takes the following form.

$$[z]_P = [z]_B - \iint_{AB} \frac{\partial z}{\partial y}\,dy + \iint_S f(x, y)\ dx\ dy\,. \qquad \ldots (x)$$

The relation (*x*) may be used to determine the value of z at the point *P* when *z* and $\partial z/\partial y$ are prescribed along a curve.

Note: Relations (10), (12) (13) (*ii*), (*v*), (*vii*) and (*x*) must be remembered and may be used directly in solving problems based on them.

8.14 SOLVED EXAMPLES BASED ON ART 8.13

Ex 1. *Find the solution, valid when x, y>0, xy > 1 of the equation* $\partial^2 z/\partial x\partial y = 1/(x + y)$ *such that z = 0,* $p = (2y)/(x + y)$ *on the hyperbola xy = 1.* **[Meerut 2007 Delhi Maths (H) 1999, 2006 Himanchal 1997, K.U. Kurukshatra 1999]**

Sol. Comparing the given equation with $L(z) = f(x, y)$, we have

$a = b = c = 0$ and $f(x, y) = 1/(x + y)$. Hence the adjoint operator *M* of the operator *L* is given by $M \equiv \partial^2/\partial x\partial y$.

So Green's function can be taken as $w = 1$. ...(1)

In the present problem, the values of *z* and $\partial z/\partial x$ (=*p*) are given by

$$z = 0, \qquad \partial z/\partial x = (2y)/(x + y), \qquad \ldots(2)$$

along the curve *C*, which is hyperbola.

$$xy = 1. \qquad \ldots(3)$$

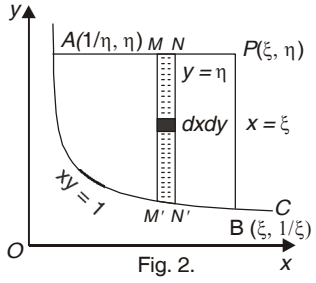


Fig. 2.

Then we wish to find the solution of given equation at the point $P(\xi, \eta)$ agreeing with these boundary conditions. Through P we draw PA parallel to the x-axis and cutting $xy = 1$ in the point A and PB parallel to the y-axis and cutting $xy = 1$ in B. Then region enclosed by $xy = 1, x = \xi, y = \eta$ is denoted by S. Now, we know that (refer equation (10) of Art 8.13.)

$$[z]_P = [wz]_A - \int_{AB} wz(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\, dxdy$$

or
$$[z]_P = \int_{AB}\frac{2y}{x+y}dx + \iint_S \frac{1}{x+y}dxdy, \text{ by (1) and (2).} \qquad ...(4)$$

Now,
$$\int_{AB}\frac{2y}{x+y}dx = 2\int_A^B \frac{xy}{x^2+xy}dx = 2\int_{1/\eta}^{\xi}\frac{1}{1+x^2}dx = 2\{\tan^{-1}\xi - \tan^{-1}(1/\eta)\} \qquad ...(5)$$

and
$$\iint_S \frac{1}{1+x}dxdy = \int_{x=1/\eta}^{\xi}\left\{\int_{y=1/x}^{\eta}\frac{1}{x+y}dy\right\}dx, \qquad ...(6)$$

since to integrate over area bounded by $PABP$, we first integrate along the strip $MNN'M'$ by fixing x and varying y from $y = 1/x$ at M' to $y = \eta$ at M and then integrate from A to P (keeping y fixed) by varying x from $x = 1/\eta$ to $x = \xi$. Evaluating the double integral on R.H.S. of (6) by the usual rule,

$$\iint_S \frac{1}{x+y}dxdy = \int_{1/\eta}^{\xi}\left[\log(x+y)\right]_{1/x}^{\eta}dx = \int_{1/\eta}^{\xi}[\log(x+\eta) - \log(x+1/x)]\,dx$$

$$= \int_{1/\eta}^{\xi}\{\log(x+\eta) - \log(1+x^2) + \log x\}\,dx$$

$$= \left[\{\log(x+\eta) - \log(1+x^2) + \log x\}x\right]_{1/\eta}^{\xi} - \int_{1/\eta}^{\xi} x\left(\frac{1}{x+\eta} - \frac{2x}{1+x^2} + \frac{1}{x}\right)dx$$

$$= \xi\{\log(\xi+\eta) - \log(1+\xi^2) + \log\xi\} - \frac{1}{\eta}\left\{\log\left(\frac{1}{\eta}+\eta\right) - \log\left(1+\frac{1}{\eta^2}\right) + \log\frac{1}{\eta}\right\}$$

$$- \int_{1/\eta}^{\xi}\left(\frac{2}{1+x^2} - \frac{\eta}{x+\eta}\right)dx, \text{ on re-arranging*}$$

$$= \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} - \left[2\tan^{-1}x - \eta\log(x+\eta)\right]_{1/\eta}^{\xi} = \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} - 2\left(\tan^{-1}\xi - \tan^{-1}\frac{1}{\eta}\right) + \eta\log\frac{\eta(\xi+\eta)}{1+\eta^2} \qquad ...(7)$$

Using (5) and (7), (4) reduces to
$$[z]_P = \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} + \eta\log\frac{\eta(\xi+\eta)}{1+\eta^2}. \qquad ...(8)$$

Replacing ξ and η by x and y respectively in (8), the value of z (*i.e.*, solution of the given equation) at any point (x, y) is given by
$$z = x\log\frac{x(x+y)}{1+x^2} + y\log\frac{y(x+y)}{1+y^2}.$$

Ex. 2. *Prove that, for the equation* $(\partial^2 z/\partial x\partial y) + (z/4) = 0$, *the Green's function is* $w(x, y; \xi, \eta) = J_0\sqrt{(x-\xi)(y-\eta)}$, *where* $J_0(z)$ *denotes Bessel's function of the first kind of order zero.*

[Himanchal 1998, 2004, Kerala 2001; Kurukshetra 2000; Nagpur 2000, 03, 05 Delhi Maths Hons. 2004, 07, 09]

Sol. Here
$$L(z) = (\partial^2 z/\partial x\partial y) + (z/4) = 0. \qquad ...(1)$$

$$\therefore \quad L \equiv \frac{\partial^2}{\partial x\partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c = \frac{\partial^2}{\partial x\partial y} + \frac{z}{4}. \qquad \Rightarrow \qquad a = 0, \quad b = 0, \quad c = z/4. \qquad ...(2)$$

---

* $x\left(\frac{1}{x+\eta} - \frac{2x}{1+x^2} + \frac{1}{x}\right) = \frac{x}{x+\eta} - \frac{2x^2}{1+x^2} + 1 = \frac{(x+\eta)-\eta}{x+\eta} - 2\frac{(1+x^2)-1}{1+x^2} + 1 = -\frac{\eta}{x+\eta} + \frac{2}{1+x^2}$

So the adjoint operator M to the operator L is given by $M \equiv (\partial^2 / \partial x \partial y) + (1/4) = 0.$ ...(3)

Given, $w = J_0\sqrt{(x-\xi)(y-\eta)}\,.$ ...(4)

From (4) $\dfrac{\partial w}{\partial x} = \dfrac{\sqrt{(y-\eta)}}{2\sqrt{(x-\xi)}} J_0'$ ...(5)

From (5), $\dfrac{\partial^2 w}{\partial y \partial x} = \dfrac{1}{4}\dfrac{1}{\sqrt{(x-\xi)(y-\eta)}} J_0' + \dfrac{\sqrt{(y-\eta)}}{2\sqrt{(x-\xi)}} \times \dfrac{\sqrt{(x-\xi)}}{2\sqrt{(y-\eta)}} J_0''$

or $\dfrac{\partial^2 w}{\partial y \partial x} = \dfrac{1}{4}\left\{ J_0'' + \dfrac{1}{\sqrt{(x-\xi)(y-\eta)}} J_0' \right\}.$ ...(6)

So (3) and (6) $\Rightarrow$ $Mw = \dfrac{1}{4}\left\{ J_0'' + \dfrac{1}{\sqrt{(x-\xi)(y-\eta)}} J_0' + J_0 \right\}.$ ...(7)

Now, Bessel's eqution of order zero is given by

$x^2y'' + xy' + x^2y = 0$ or $y'' + (1/x) \times y' + y = 0.$ ...(8)

Since $y = J_0\{\sqrt{(x-\xi)(y-\eta)}\}$ is a solution of (8), we get

$J_0'' + \dfrac{1}{\sqrt{(x-\xi)(y-\eta)}} J_0' + J_0 = 0$ or $Mw = 0$, by (7) ...(9)

Again, (5) $\Rightarrow$ $(\partial w/\partial x) = 0 = bw$ when $y = \eta$, as $b = 0$ ...(10)

Similarly, $(\partial w/\partial y) = 0 = aw$ when $x = \xi$, as $a = 0$ ...(11)

Finally, when $x = \xi$, $y = \eta$, $w = J_0(0) = 1.$ ...(12)

Since w satisfies four properties (9), (10), (11) and (12) of a Green's function, it follows that w must be a Green's function of the given equation (1).

Ex. 3. *Prove that for the equation* $\dfrac{\partial^2 z}{\partial y \partial x} + \dfrac{2}{x+y}\left(\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) = 0$, *the Green's function is*

$$w(x, y\,;\, \xi, \eta) = \frac{(x+y)\{2xy + (\xi-\eta)(x+y) + 2\xi\eta\}}{(\xi+\eta)^3}.$$

Hence find the solution of the differential equation which satisfies the conditions $z = 0$, $\partial z/\partial x = 3x^2$ on $y = x$. **[Bangalore 2003, Himanchel 2001; Kurukshetra 2004; Delhi Maths (H) 2001,05,11]**

Sol. Compare the given equation with $L(z) = f(x, y)$ where $L \equiv \dfrac{\partial^2}{\partial y \partial x} + a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y} + c$, we find

$a = 2/(x+y)$, $b = 2/(x+y)$, $c = 0$, $f(x, y) = 0.$ ...(1)

So the adjoint operator M to the operator L is given by

$$Mw \equiv \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial}{\partial x}\left(\frac{2}{x+y} w\right) - \frac{\partial}{\partial y}\left(\frac{2}{x+y} w\right).$$ ...(2)

Given $w(x, y\,;\, \xi, \eta) = \dfrac{(x+y)\{2xy + (\xi-\eta)(x-y) + 2\xi\eta\}}{(\xi+\eta)^3}.$ ...(3)

(3) $\Rightarrow$ $\dfrac{\partial w}{\partial x} = \dfrac{2xy + (\xi-\eta)(x-y) + 2\xi\eta + (x+y)(2y+\xi-\eta)}{(\xi+\eta)^3}.$ ...(4)

(3) $\Rightarrow$ $\dfrac{\partial w}{\partial y} = \dfrac{2xy + (\xi-\eta)(x-y) + 2\xi\eta + (x+y)(2y-\xi+\eta)}{(\xi+\eta)^3}.$ ...(5)

(5) $\Rightarrow$ $\dfrac{\partial^2 w}{\partial x \partial y} = \dfrac{2y+\xi-\eta+2x-\xi+\eta+2(x+y)}{(\xi+\eta)^3} = \dfrac{4(x+y)}{(\xi+\eta)^3}.$ ..(6)

Using (6) and (3), (2) reduces to

$$Mw = \frac{4(x+y)}{(\xi+\eta)^3} - 2\frac{\partial}{\partial x}\left\{\frac{2xy+(\xi-\eta)(x-y)+2\xi\eta}{(\xi+\eta)^3}\right\} - 2\frac{\partial}{\partial y}\left\{\frac{2xy+(\xi-\eta)(x-y)+2\xi\eta}{(\xi+\eta)^3}\right\}$$

$$= \frac{4(x+y)}{(\xi+\eta)^3} - \frac{2(2y+\xi-\eta)+2(2x-\xi+\eta)}{(\xi+\eta)^3} = 0. \qquad ...(7)$$

At $y = \eta$, $\dfrac{\partial w}{\partial x} = \dfrac{2x\eta+(\xi-\eta)(x-\eta)+2\xi\eta+(x+\eta)(2\eta+\xi-\eta)}{(\xi+\eta)^3}$, by (4)

$$= \frac{2\{x(\xi+\eta)+\eta^2+\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(8)$$

From (1) and (3), $$bw = \frac{2\{2xy+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(9)$$

So at $y = \eta$, (9) reduces to

$$bw = \frac{2\{2x\eta+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3} = \frac{2\{x(\xi+\eta)+\eta^2+\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(10)$$

From (8) and (10), $\partial w/\partial x = bw$ when $y = \eta$. ...(11)

Similarly, $\partial w/\partial y = aw$ when $x = \xi$. ...(12)

From (3), when $x = \xi$, $y = \eta$, we get $$w = \frac{(\xi+\eta)\{2\xi\eta+(\xi-\eta)^2+2\eta\}}{(\xi+\eta)^3} = 1. \qquad ...(13)$$

Since w satisfies four properties (7), (11), (12) and (13) of a Green's function, it follows that w must be a Green's function of the given equation.

To find the solution of the equation. In the present problem, the values of z and $\partial z/\partial x$ $(= p)$ are given by

$z = 0$ and $\partial z/\partial x = 3x^2$. ...(14)

along the line AB, $y = x$. ...(15)

Then we wish to find the solution of given equation at the point $P(\xi, \eta)$ agreeing with these boundary conditions. Through P we draw PA parallel to the x-axis and cutting $y = x$ in the point A and PB parallel to the y-axis and cutting $y = x$ in B. Then triangular region enclosed by straight lines $y = x$, $y = \eta$ and $x = \xi$ is denoted by S. Then we know that (refer equation (10) of Art 8.13).

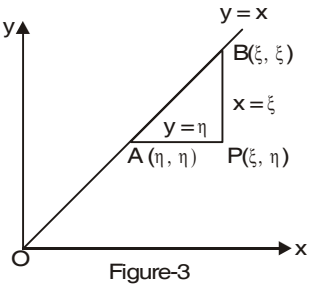


Figure-3

$$[z]_P = [wz]_A - \int_{AB} wz(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\, dxdy. \qquad ..(16)$$

Now on line AB, from (3), $$w = \frac{4x(x^2+\xi\eta)}{(\xi+\eta)^2}, \text{ as } y = x. \qquad ...(17)$$

Using (1), (14) and (17), (16) reduces to

$$[z]_P = \int_A^B \frac{4x(x^2+\xi\eta)}{(\xi+\eta)^3} 3x^2 dx = \frac{12}{(\xi+\eta)^3}\int_\eta^\xi (x^5 + \xi\eta x^3)dx$$

$$= \frac{12}{(\xi+\eta)^3}\left[\frac{x^6}{6} + \xi\eta\frac{x^4}{4}\right]_\eta^\xi = \frac{12}{(\xi+\eta)^3}\left[\frac{\xi^6-\eta^6}{6} + \frac{\xi\eta}{4}(\xi^4-\eta^4)\right]$$

$$= (\xi+\eta)^{-3}\left\{2(\xi^3+\eta^3)(\xi^3-\eta^3) + 3\xi\eta(\xi^2-\eta^2)(\xi^2+\eta^2)\right\}$$

$$= (\xi+\eta)^{-3}\left\{2(\xi^3+\eta^3)\ (\xi-\eta)\ (\xi^2+\eta^2+\xi\eta)+3\xi\eta(\xi-\eta)(\xi+\eta)(\xi^2+\eta^2)\right\}$$

$$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^3+\eta^3)\ (\xi^2+\eta^2)+2\xi\eta(\xi^3+\eta^3)-3\xi\eta(\xi+\eta)\,(\xi^2+\eta^2)\right\}$$

$$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^3+\eta^3)\ (\xi^2+\eta^2)+6\xi\eta(\xi+\eta)\,(\xi^2+\eta^2)\ -3\xi\eta(\xi+\eta)(\xi^2+\eta^2)+2\xi\eta(\xi^3+\eta^3)\right\}$$

$$= (\xi+\eta)^{-3}(\xi-\eta)\left[2(\xi^2+\eta^2)\left\{\xi^3+\eta^3+3\xi\eta\,(\xi+\eta)\right\}\ -\xi\eta\left\{3(\xi+\eta)\,(\xi^2+\eta^2)-2\,(\xi^3+\eta^3)\right\}\right]$$

$$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^2+\eta^2)(\xi+\eta)^3-\xi\eta(\xi+\eta)^3\right\} = (\xi-\eta)(2\xi^2+2\eta^2-\xi\eta)$$

or
$$[z]_P = 2\xi^3+3\xi\eta^2-3\xi^2\eta-2\eta^3. \qquad ....(18)$$

Replacing ξ and η by x and y respectively in (18), the value of z (*i.e.*, solution of the given equation) at any point (x, y) is given by

$$z = 2x^3 + 3xy^2 - 3x^2y - 2y^3.$$

Ex.4. *Obtain the solution of* $\partial^2 z/\partial x\partial y = 1/(x+y)$ *such that* $z = 0$, $p = 2y/(x+y)$ *on* $y = x$.

[Delhi Maths (H) 1998]

Sol. Here we are solve
$$\partial^2 z/\partial x\partial y = 1/(x+y)\,, \qquad ... (1)$$

where $z = 0$ and $p = \partial z/\partial x = 2y/(x+y)$ on $y = x$ ... (2)

Here the given curve C is straight line $y = x$. Then we wish to find the solution of (1) at $P(\xi, \eta)$ agreeing with boundary conditions (2). Through P we draw PA parallel to the x-axis and cutting $y = x$ at the point A and PB parallel to the y-axis and cutting $y = x$ in B. The triangular-region enclosed by stright lines $y = x$, $y = \eta$, $x = \xi$ is denoted by S (draw figure as shown in figure 3 of solved Ex. 3). Then we know that (refer particular case II of Art. 8.13).

$$[z]_P = [z]_A + \int_{AB}\frac{\partial z}{\partial x}dx + \iint_S f(x,y)\ dx\ dy \qquad ... (3)$$

Comparing (1) with $\partial^2 z/\partial x\ \partial y = f(x,y)$, here $f(x,y) = 1/(x+y)$.

Since A lies on given curve AB and it is given that $z = 0$ on AB, hence $[z]_A = 0$.

From (2), $\partial z/\partial x = 2y/(x+y)$ on AB. *i.e.* $y = x$

so that $\partial z/\partial x = 2x/(x+x) = 1$ on $y = x$.

Using the above facts, (3) reduces to

$$[z]_P = \int_\eta^\xi dx + \int_{x=\eta}^{\xi}\left\{\int_{y=\eta}^{x}\frac{1}{x+y}dy\right\}dx = \xi-\eta+\int_\eta^\xi\left[\log(x+y)\right]_{y=\eta}^{x}dx = \xi-\eta+\int_\eta^\xi[\log(2x)-\log(x+\eta)]\,dx$$

$$= \xi-\eta+\left[x\log\frac{2x}{x+\eta}\right]_\eta^\xi-\int_\eta^\xi\left(\frac{1}{x}-\frac{1}{x+\eta}\right)x\ dx = \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}-\eta\log 1-\int_\eta^\xi\frac{\eta}{x+\eta}dx$$

$$= \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}-\eta\left[\log\,(x+\eta)\right]_\eta^\xi = \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}+\eta\log\frac{2\eta}{\xi+\eta}$$

Replacing ξ and η by x and y respectively in the above equation, the value of z (*i.e.,* solution of (1) at any point (x, y) is given by

$$z = x-y+x\log\ \{2x/(x+y)\}+y\log\{2y/(x+y)\}\,.$$

8.15. Riemann-Volterra method for solving the Cauchy problem for the one-dimensional wave equation

The entire procedure of solution will become clear from the following solved examples.

Ex. 1. *Using Riemann-Volterra method, solve* $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$, *when* z, $\partial z/\partial x$, $\partial z/\partial y$ *are prescribed along a curve C in the xy-plane*

Sol. Given $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$ or $r - t = 0$ ... (1)

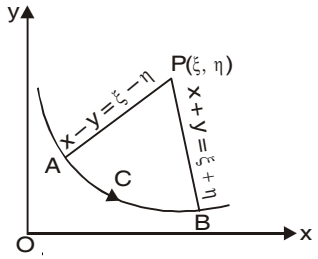
Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, p) = 0$, here R = 1, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by

$dy/dx + 1 = 0$ and $dy/dx - 1 = 0$

Integrating these,

$x + y = C_1$ and $x - y = C_2$, ... (2)

which are characteristics of (1) and these are two families of straight lines. Let $P(\xi, \eta)$ be any point in xy-plane. We now obtain characteristics of (1) passing through P. So putting $x = \xi$ and $y = \eta$ in (2), we have $C_1 = \xi + \eta$ and $C_2 = \xi - \eta$. Hence the characteristics of (1) passing through P are given by



$x + y = \xi + \eta$ and $x - y = \xi - \eta$, ... (3)

which have been shown by straight lines *PB* and *PA* respectively in the figure. Let the characteristics *PA* and *PB* cut the given curve *C* in *A* and *B* respectively. Let C' denote the closed curve *PABP* (which is made up of straight line *PA*, curve *C* (*i.e. AB*) and straight line *BP*). Let *S* be the region enclosed by C'.

Integrating both sides of (1) over S, we have

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dx\, dy = 0 \quad \text{or} \quad \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\, dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0$$, by Green's theorem.*

or $$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0$$

Equation of *BP* is $x + y = \xi + \eta$ and hence $dx = -dy$ on *BP*. Similarly, equation of *PA* is $x - y = \xi - \eta$ and hence $dx = dy$ on *PA*. Using these facts in the above equation, we get

$$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_B^P \left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A \left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) = 0$$

or $$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) - \int_B^P dz + \int_P^A dz = 0$$, as $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

or $$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) - (z_P - z_B) + (z_A - z_P) = 0$$

---

* **Green's theorem.** Let C' be a closed curve bounding the region S on *xy*-plane and $u(x, y)$, $v(x, y)$ be differential functions in *S* and continusou on C', then $\oint_{C'} (udx + vdy) = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$

$$\therefore \qquad z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right), \qquad \ldots (4)$$

which is the required solution of (1) at any point P.

Ex. 2. *Solve one dimensional wave equation by Riemann Volterra method.*

[Kurukshetra 2001; Delhi Maths (H) 1996]

or *Solve homogeneous one-dimensional wave equation* $\partial^2 z/\partial x^2 = (1/c^2) \times (\partial^2 z/\partial t^2)$, *when* $z, \partial z/\partial x, \partial z/\partial t$ *are prescribed along a curve C.*

Sol. Given $\partial^2 z/\partial x^2 = (1/c^2) \times (\partial^2 z/\partial t^2)$ ... (*i*)

Let *y* be a new variable such that $y = ct$ ... (*ii*)

Then (*i*) becomes $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$ or $r - t = 0$ ... (*iii*)

for which z, $\partial z/\partial x$ and $\partial z/\partial y$ are now prescribed along *C*.

Proceed with (*iii*) as we did in solved Ex. 1 upto equation (4).

Ex. 3. *Find z (x, y) such that* $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ *and* $z = f(x)$ *and* $\partial z/\partial y = g(x)$ *on y = 0.*

[Kanpur 2003; Delhi Maths (H) 2000, 08]

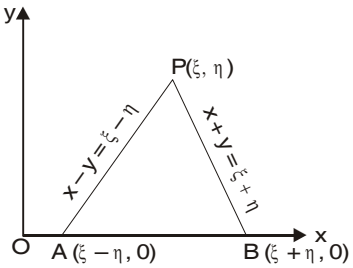
Sol. Given $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$ or $r - t = 0$, ... (1)

where $z(x,0) = f(x)$, *i.e.*, $z = f(x)$ on $y = 0$ *i.e.* $x-$axis ... (2)

and $(\partial z/\partial y)_{y=0} = g(x)$ *i.e.*, $\partial z/\partial y = g(x)$ on $y = 0$ *i.e.* $x-$axis ...(3)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by $dy/dx + 1 = 0$ and $dy/dx - 1 = 0$

Integrating these $x + y = C_1$ and $x - y = C_2$ ... (4) which are the characteristics of (1) and these are two families of straight lines. Let $P(\xi, \eta)$ be any point in *xy*-plane. We now obtain characteristics of (1) passing through *P*. So putting $x = \xi$ and $y = \eta$ in (4), we get $C_1 = \xi + \eta$ and $C_2 = \xi - \eta$. Hence the characteristics of (1) passing through *P* are given by



$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta \qquad \ldots (5)$$

which have been shown by straight lines *PB* and *PA* respectively in the figure. Let the characteristics *PA* and *PB* cut given curve (here $y = 0$ *i.e.* *x*-axis) in $A(\xi - \eta, 0)$ and $B(\xi + \eta, 0)$ respectively. Let C' denote the closed curve *PA BP* (which is made up of straight lines *PA*, *AB* and *BP*). Let *S* be the region enclosed by C'.

Integrating both sides of (1) over *S*, we have

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dx dy = 0 \qquad \text{or} \qquad \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\, dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0$$, by Green's theorem

or $$\int_{AB}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0$$

On AB (*i.e., x*-axis), $y = 0$ so that $dy = 0$. Also, from (3), $\partial z/\partial y = g(x)$ on $y = 0$. On BP (*i.e., x* + *y* = $\xi + \eta$), $dx = -dy$. Similarly, on PA (*i.e.,* $x - y = \xi - \eta$), $dx = dy$. Using these facts, the above equation reduces to

$$\int_A^B g(x)dx + \int_B^P \left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A \left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) = 0$$

or $$\int_A^B g(x)\,dx - \int_B^P dz + \int_P^A dz = 0, \quad \text{as } \frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx = dz$$

or $$\int_A^B g(x) - (z_P - z_B) + z_A - z_P = 0 \qquad \text{or} \qquad z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_A^B g(x)\,dx \qquad \ldots (6)$$

From (2), $z = f(x)$ on $y = 0$ (*i.e., x*-axis). Since *x*-coordinates of A and B are $\xi - \eta$ and $\xi + \eta$ respectively, it follows that $z_A = f(\xi - \eta)$ and $z_B = f(\xi + \eta)$. Hence (6) reduces to

$$z_P = \frac{1}{2}\{f(\xi - \eta) + f(\xi + \eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} g(x)\,dx \qquad \ldots (7)$$

Replacing ξ and η by x and y respectively in (7), the value of z (*i.e.,* solution of (1) at any point P (x, y)) is given by

$$z(x,y) = \frac{1}{2}\{f(x-y) + f(x+y)\} + \frac{1}{2}\int_{x-y}^{x+y} g(u)\,du$$

Ex. 4. *Find the solution of one-dimensional non-homogeneous wave equation* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x,y) = 0$ *by Riemann-Vatterra method.*

[A.M.I.E. 2005; Delhi Maths (H) 1998, 2002; Kanpur 1998]

Sol. Given $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x,y) = 0$ or $r - t + f(x,y) = 0$. ... (1)

Suppose that z, $\partial z/\partial x$ and $\partial z/\partial y$ are prescribed along a given curve C. Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$, here $R = 1$, $S = 0$, $T = -1$ and so λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ giving $\lambda = 1, -1$. The corresponding characteristic equations $dy/dx + 1 = 0$ and $dy/dx - 1 = 0$ give, on integration,

$x + y = C_1$ and $x - y = C_2$ ... (2)

which are characteristics of (1). Draw figure as in solved Ex. 1. Let $P(\xi, \eta)$ be any point in *xy*-plane. Then characteristics of (1) passing through $P(\xi, \eta)$ are given by

$x + y = \xi + \eta$ and $x - y = \xi - \eta$ ... (3)

which have been shown by straight lines PB and PA respectively. Let the characteristics PA and PB cut the given curve C in A and B respectively. Let C' denote the closed curve $PABP$ and let S denote the region enclosed by C'.

Integrating both sides of (1) over S, we have

$$\iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\,dy + \iint_S f(x,y)\,dx\,dy = 0$$

or $$\oint_{C'}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S f(x,y)\,dx\,dy = 0, \text{ by Green's theorem}$$

or $$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S f(x,y)\,dxdy = 0$$

Equation of *BP* is $x + y = \xi + \eta$ and so $dx = -dy$ on *BP*. Similarly, equation of *PA* is $x - y = \xi - \eta$ and so $dx = dy$ on *PA*. Using these facts, the above equation reduces to

$$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_B^P \left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A \left\{\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right\} + \iint_S f(x, y)\, dx\, dy = 0$$

or
$$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) - \int_B^P dz + \int_P^A dz + \iint_S f(x,y)\, dx\, dy = 0$$

or
$$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) - (z_P - z_B) + z_A - z_P + \iint_S f(x,y)\, dx\, dy = 0$$

or
$$z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \frac{1}{2}\iint_S f(x,y)\, dx\, dy = 0,$$

which is the required solution of (1) at any point *P*.

Ex. 5. *Solve* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 1$, *when z (x, 0) = sin x,* z_y *(x, 0) = x.*

Sol. Given $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 - 1 = 0$ or $r - t - 1 = 0$ ... (1)

where $z(x, 0) = \sin x$, *i.e.,* $z = \sin x$ on $y = 0$, *i.e., x*-axis ... (2)

and $z_y\,(x, 0) = x$, *i.e.,* $\partial z/\partial y = x$ on $y = 0$, *i.e., x*-axis ... (3)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are

$dy/dx + 1 = 0$ and $dy/dx - 1 = 0$

Integrating these, $x + y = C_1$ and $x - y = C_2$ ... (4)

which are the characteristics of (1) and these are two families of straight lines Draw a figure as in solved Ex. 3. Let $P(\xi, \eta)$ be any point in *xy*-plane. Putting $x = \xi$, $y = \eta$ in (4), we get $C_1 = \xi + \eta$, $C_2 = \xi - \eta$. Hence the characteristics of (1) passing through *P* are given by

$x + y = \xi + \eta$ and $x - y = \xi - \eta$ ... (5)

which have been shown by straight lines *PB* and *PA* respectively in the figure. Let the characteristics *PA* and *PB* cut the given line $y = 0$ *i.e., x*-axis in $A(\xi - \eta, 0)$ and $B(\xi + \eta, 0)$ respectively. Let *C′* denote the closed curve *PABP* and let *S* denote the region enclosed by C'.

Integrating both sides of (1) over *S*, we have

$$\iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} - \iint_S dx\, dy = 0$$

or
$$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) - \iint_S dx\, dy = 0, \text{ by Green's theorem} \quad \text{... (5)}$$

Now, $\iint_S dx\, dy$ = area of the triangle *SAB* = (1/2) × *AB* × perpendicular distance of *P* from *AB*

$= (1/2) \times \{\xi + \eta - (\xi - \eta)\} \times \eta = \eta^2$

Hence (5) reduces to

$$\int_{AB} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) - \eta^2 = 0 \quad \text{... (6)}$$

On AB (*i.e., x*-axis), $y = 0$ so that $dy = 0$. Also from (3), $\partial z/\partial y = x$ on $y = 0$. On BP (*i.e.,* $x + y = \xi + \eta$), $dx = -dy$. Similarly on PA (*i.e.,* $x - y = \xi - \eta$), $dx = dy$. Using these facts, (6) reduces to

$$\int_A^B x\,dx + \int_B^P \left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A \left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) - \eta^2 = 0$$

or $$\int_A^B x\,dx - \int_B^P dz + \int_P^A dz - \eta^2 = 0, \text{ as } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

or $$\int_A^B x\,dx - (z_P - z_B) + z_A - z_P - \eta^2 = 0 \quad \text{or} \quad z_P = \frac{1}{2}(z_Z + z_B) + \frac{1}{2}\int_A^B x\,dx - \frac{1}{2}\eta^2 \quad \ldots (7)$$

From (2), $z = \sin x$ on $y = 0$ (*i.e., x*-axis). Since x-coordinates of A and B are $\xi - \eta$ and $\xi + \eta$ respectively, it follows that $z_A = \sin(\xi - \eta)$ and $z_B = \sin(\xi + \eta)$. Hence (7) reduces to

$$z_P = \frac{1}{2}\{\sin(\xi - \eta) + \sin(\xi + \eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} x\,dx - \frac{\eta^2}{2} \quad \text{or} \quad z_P = \sin\xi\cos\eta - \frac{1}{4}\left[x^2\right]_{\xi-\eta}^{\xi+\eta} - \frac{\eta^2}{2}$$

or $$z_P = \sin\xi\cos\eta - (1/4) \times \{(\xi + \eta)^2 - (\xi - \eta)^2\} - (1/2) \times \eta^2$$

or $$z_P = \sin\xi\cos\eta - \xi\eta - (\eta^2/2) \quad \ldots (8)$$

Replacing ξ and η by x and y respectively in (8), the value of z (*i.e.,* solution of (1)) at any point (x, y) is given by $z(x, y) = \sin x \cos y - xy - (y^2/2)$.

Ex. 6. *A function z (x, y) satisfies the non-homogeneous equation* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x, y) = 0$ *and the initial conditions* $z = \partial z/\partial y = 0$ *when* $y = 0$. *Show that (using Riemann - Volterra method)*

$$z(x, y) = \frac{1}{2}\iint_\Gamma f(u,v)\,du\,dv,$$

where Γ is the triangle cut off from the upper half of uv-plane by two characteristics through the point (x, y). **[Delhi Maths (Hons) 2002, 07, 11; A.M.I.E. 2005; Amaravati 2003; Kanpur 1999; Rohilkhand 2004]**

Sol. Given $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x, y) = 0$ or $r - t + f(x, y) = 0$, ... (1)

where $z(x, 0) = 0$, *i.e.,* $z = 0$ on $y = 0$ (x-axis) ... (2)

and $(\partial z/\partial y)_{y=0} = 0$, *i.e.,* $(\partial z/\partial y) = 0$ on $y = 0$ (x-axis) ... (3)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$ and so λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ giving $\lambda = 1, -1$. The corresponding characteristic equations are given by

$dy/dx + 1 = 0$ and $dy/dx - 1 = 0$

Integrating these, $x + y = c_1$ and $x - y = c_2$, ... (4)

which are characteristics of (1). Draw figure as in solved Ex. 3. Let $P(\xi, \eta)$ be any point in xy-plane. We now obtain characteristics of (1) passing through P. So putting $x = \xi$ and $y = \eta$ in (4), we get $C_1 = \xi + \eta$ and $C_2 = \xi - \eta$. Hence the characteristics of (1) passing through P are given by

$x + y = \xi + \eta$ and $x - y = \xi - \eta$ ... (5)

which have been shown by straight lines PB and PA respectively in the figure. Let the characteristics PA and PB cut $y = 0$ (*i.e., x*-axis) at A and B respectively. Let C' denote the closed curve $PA\,BP$ and let Γ denote the triangular region enclosed by C'.

Integrating both sides of (1) over Γ, we have

$$\iint_\Gamma \left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right) dx\ dy + \iint_\Gamma f(x,y)\ dx\,dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy \right) + \iint_\Gamma f(x,y)\ dx\,dy = 0$$, using Green's theorem

or $$\int_{AB}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_\Gamma (x,y)\ dx\,dy = 0 \quad \text{... (6)}$$

On AB (*i.e.,* x-axis), $y = 0$ so that $dy = 0$. Also, from (3), $\partial z/\partial y = 0$ on $y = 0$. On BP (*i.e.,* $x + y = \xi + \eta$), $dx = -dy$. Similarly, on PA (*i.e.,* $x - y = \xi - \eta$), $dx = dy$. Using these facts, (6) reduces to

$$\int_B^P \left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx \right) + \int_P^A \left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx \right) + \iint_\Gamma (x,y)\ dxdy = 0$$

or $$-\int_B^P dz + \int_P^A dz + \iint_\Gamma f(x,y)dxdy = 0, \text{ as } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

or $$-(z_P - z_B) + z_A - z_P + \iint_\Gamma f(x,y)\,dx\,dy = 0 \quad \text{... (7)}$$

From (2), $z = 0$ on $y = 0$ (*i.e.,* x-axis) and so $z_A = z_B = 0$, because A and B both lie on $y = 0$. Hence (7) becomes

$$2z_P = \iint_\Gamma f(x,y)\ dxdy \qquad \text{or} \qquad z\,(x,\,y) = \frac{1}{2}\iint_\Gamma f(u,v)\,du\ dv\,,$$

which gives the value of z (*i.e.,* solution of (1)) at any point (x, y)

Ex.7. *Find the solution of the non-homogeneous wave equation* $\partial^2 z/\partial x^2 - (1/c^2)(\partial^2 z/\partial t^2) + f(x,t) = 0$ *with initial conditions* $z\,(x, 0) = f(x)$, $z_t\,(x, t) = g\,(x)$.

Sol. Let y be a new variable such that $y = c\,t$ ... (1)

Then the given problem may be re-written as

$$\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + F(x,y) = 0 \qquad \text{or} \qquad r - t + F(x,y) = 0\,, \quad \text{... (2)}$$

with the modified initial conditions given below

$z\,(x,\,0) = f\,(x)$, *i.e.,* $z = f\,(x)$ on $y = 0$, *i.e.,* x-axis ... (3)

$(\partial z/\partial y)_{y=0} = G\,(x)$, *i.e.,* $\partial z/\partial y = G(x)$ on $y = 0$, *i.e.,* x-axis ... (4)

Here $F\,(x,\,y) = f\,(x,\,t)$ and $G\,(x) = (1/c) \times g\,(x)$ ... (5)

Comparing (2) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$, here $R = 1$, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by

$$dy/dx + 1 = 0 \qquad \text{and} \qquad dy/dx - 1 = 0 \quad \text{... (6)}$$

Integrating these, $x + y = C_1$ and $x - y = C_2$,

which are characteristics of (1). Draw a figure same as in Ex. 3.

Let $P(\xi, \eta)$ be any point in xy-plane. Then characteristics (2) passing through $P(\xi, \eta)$ are

$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta \quad \text{... (7)}$$

which have been shown by straight lines PB and PA respectively in the figure. Here lines given by (7) cut x-axis (*i.e.,* $y = 0$) in $A(\xi - \eta, 0)$ and $B(\xi + \eta, 0)$ respectively. Let C' denote the closed curve $PABP$ and let S be the region enclosed by C'.

Integrating both sides of (2) over S, we have

$$\iint_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \right\} dx\,dy + \iint_S F(x,y)\, dx\,dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy \right) + \iint_S F(x,y)\, dx\,dy = 0$$, using Green's theorem

or $$\int_{AB}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S F(x,y)\, dx\,dy = 0 \quad \text{... (8)}$$

On AB (*i.e.,* x-axis), $y = 0$ so that $dy = 0$. Also, from (4), $\partial z/\partial y = G(x)$ on $y = 0$. On BP (*i.e.,* $x + y = \xi + \eta$), $dx = -$ dy. Similarly, on PA (*i.e.,* $x - y = \xi - \eta$), $dx = dy$. Using these facts, (8) reduces to

$$\int_A^B G(x)\,dx + \int_B^P \left(-\frac{\partial z}{\partial y}\,dy - \frac{\partial z}{\partial x}dx \right) + \int_P^A \left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx \right) + \iint_S F(x,y)\, dx\,dy = 0$$

or $$\int_A^B G(x)dx - \int_B^P dz + \int_P^A dz + \iint_S F(x,y)\,dx\,dy = 0$$

or $$\int_A^B G(x)\, dx - (z_P - z_B) + z_A - z_P + \iint_S F(x,y)\, dx\,dy = 0$$

or $$z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_A^B G(x)\ \ dx + \frac{1}{2}\iint_S F(x,y)\, dx\,dy \quad \text{... (9)}$$

From (3), $z = f(x)$ on $y = 0$ (*i.e.,* x-axis). Since x-coordinates of A and B are $\xi - \eta$ and $\xi + \eta$ respectively, it follows that $z_A = f(\xi - \eta)$ and $z_B = f(\xi + \eta)$. Hence (9) reduces to

$$z_p = \frac{1}{2}\{f(\xi - \eta) + f(\xi + \eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} G(x)\, dx + \frac{1}{2}\iint_s F(x,y)\,dx\,dy \quad \text{... (10)}$$

Replacing ξ and η by x and y (= ct) respectively and using (5), (10) reduces to

$$z(x,y) = \frac{1}{2}\{f(x-y) + f(x+y)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)du + \frac{1}{2}\iint_s f(x,t)dx\,dt$$

Miscellaneous Problems on chapter 8

Ex. 1. $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2)$ is hyperbolic or parabolic. Classify it. **[Agra 2008]**

Hint. See Art 8.1 **Ans.** Hyperbolic

Ex. 2. The equation $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$ is

(a) parabolic (b) hyperbolic (c) elliptic (d) Nonw of these **[Agra 2007]**

Sol. Ans. (**b.**) See Art 8.1.

Ex. 3. Classify and solve the following equation $\partial^2 z/\partial x^2 = x^2(\partial^2 z/\partial y^2)$. **[Bhopal 2010]**

9

Monge's Methods

9.1 INTRODUCTION

The most general form of partial differential equation of order two is

$$f(x, y, z, p, q, r, s, t) = 0. \qquad ...(1)$$

It is only in special cases that (1) can be integrated. Some well known methods of solutions were given by Monge. His methods are applicable to a wide class (but not all) of equations of the form (1). Monge's methods consists in finding one or two first integrals of the form

$$u = \phi(v), \qquad ...(2)$$

where u and v are known functions of x, y, z, p and q and ϕ is an arbitrary function. In other words, Monge's methods consists in obtaining relations of the form (2) such that equation (1) can be derived from (2) by eliminating the arbitrary function. A relation of the form (2) is known as an *intermediate integral* of (1). Every equation of the form (1) need not possess an intermediate integral. However, it has been shown that most general partial differential equations having (2) as an intermediate integral are of the following forms

$$Rr + Ss + Tt = V \quad \text{and} \quad Rr + Ss + Tt + U(rt - s^2) = V, \qquad ...(3)$$

where R, S, T, U and V are functions of x, y, z, p and q. Even equations (3) need not always possess an intermediate integral. In what follows we shall assume that an intermediate integral of (3) exists.

9.2. MONGE'S METHOD OF INTERGRATING Rr + Ss + Tt = V. [Agra 2005; Delhi Maths (Hons) 2000, 02, 08, 09, 11; Garhwal 1994; Patna 2003; Kanpur 1997; Meerut 2000]

Given $$Rr + Ss + Tt = V, \qquad ...(1)$$

where R, S, T and V are functions of x, y, z, p and q.

We know that $p = \partial z/\partial x$, $q = \partial z/\partial y$,

$$\left.\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}, & t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}, \\ s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x} \quad \text{and} & s &= \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y} \end{aligned}\right\} \qquad ...(2)$$

Now, $$dp = (\partial p/\partial x)\,dx + (\partial p/\partial y)dy = r\,dx + s\,dy, \text{ using (2)} \qquad ...(3)$$

and $$dq = (\partial q/\partial x)dx + (\partial q/\partial y)dy = s\,dx + t\,dy, \text{ using (2)} \qquad ...(4)$$

From (3) and (4), $$r = (dp - s\,dy)/dx \quad \text{and} \quad t = (dq - s\,dx)/dy \qquad ...(5)$$

Substituting the values of r and s given by (5) in (1), we get

$$R\left(\frac{dp - s\,dy}{dx}\right) + Ss + T\left(\frac{dq - s\,dx}{dy}\right) = V \quad \text{or} \quad R(dp - s\,dy)dy + Ss\,dx\,dy + T(dq - s\,dx)dx = V\,dx\,dy$$

or $$(R\,dp\,dy + T\,dq\,dx - V\,dx\,dy) - s\{R(dy)^2 - S\,dx\,dy + T(dx)^2\} = 0. \qquad ...(6)$$

Clearly any relation between x, y, z, p and q which satisfies (6) must also satisfy the following two simultaneous equations

$$Rdpdy + Tdq\,dx - Vdxdy = 0. \quad \ldots(7)$$

and $$(dy)^2 - Sdxdy + T(dx)^2 = 0. \quad \ldots(8)$$

The equations (7) and (8) are called *Monge's subsidiary equations* and the relations which satisfy these equations are called *intermediate integrals*.

Equation (8) being a quadratic, in general, it can be resolved into two equations, say

$$dy - m_1\,dx = 0 \quad \ldots(9)$$

and $$dy - m_2\,dx = 0. \quad \ldots(10)$$

Now the following two cases arise :

Case I. When m_1 and m_2 are distinct in (9) and (10).

In this case (7) and (9), if necessary by using well known result $dz = pdx + qdy$, will give two integrals $u_1 = a$ and $v_1 = b$, where a and b are arbitrary constants. These give

$$u_1 = f_1(v_1), \quad \ldots(11).$$

where f_1 is an arbitrary function. It is called an *intermediate integral* of (1).

Next, taking (7) and (10) as before, we get another intermediate integral of (1), say

$$u_2 = f_2(v_2), \text{ where } f_2 \text{ is an arbitrary function.} \quad \ldots(12)$$

Thus we have in this case two distinct intermediate integrals (11) and (12). Solving (11) and (12), we obtain values of p and q in terms of x, y and z. Now substituting these values of p and q in well known relation $dz = pdx + qdy$ …(13)

and then integrating (13), we get the required complete integral of (1).

Case II . When $m_1 = m_2$ i.e., (8) is a perfect square.

As before, in this we get only one intermediate integral which is in Lagrange's form

$$Pp + Qq = R. \quad \ldots(14)$$

Solving (14) with help of Lagrange's method (refer Art. 2.3, chapter 2), we get the required complete integral of (1).

Remark 1. Usually while dealing with case I, we obtain second intermediate integral directly by using symmetry. However sometimes in absence of any symmetry, we find the complete integral with help of only one indetermediate integral. This is done with help of using Lagrange's method.

Remark 2. While obtaining an intermediate integral, remember to use the relation $dx = pdx + qdy$ as explained below :

(*i*) $pdx + qdy + 2xdx = 0$ can be re–written as $dz + 2xdx = 0$ so that $z + x^2 = c$.

(*ii*) $xdp + ydq = dx$ can be re–written as $xdp + ydq + pdx + qdy = dx + pdx + qdy$

or $d(xp) + d(yq) = dx + dz$ so that $xp + yq = x + z + c$, on integration

Remark 3. While integrating, we shall use the following types of calculations. In what follows, f and g are arbitrary functions and k and a are a constants.

(*i*) $\int k\,f(t)\,dt = g(t)$ (*ii*) $\int k\frac{1}{t}f(t)\,dt = g(t).$ (*iii*) $\int k\frac{1}{t^2}f(t^2)\,d(t^2) = g(t^2)$

(*iv*) $\int k\,f(x+y)\,d(x+y) = g(x+y).$ (*v*) $\int k\,t^2\,f\left(\frac{1}{t}\right)d\left(\frac{1}{t}\right) = \int \frac{k}{(1/t)^2}f\left(\frac{1}{t}\right)d\left(\frac{1}{t}\right) = g\left(\frac{1}{t}\right)$

(*vi*) $\int \frac{k}{t^2}f(at^2)\,d(t^2) = \int \frac{k}{(at^2)}f(at^2)\,d(at^2) = g(at^2)$

Proof of (vi). Putting $at^2 = u$, and $d(at^2) = du$ we have

$$\int \frac{k}{t^2}f(at^2)\,d(t^2) = \int \frac{k}{u}f(u)\,d(u) = g(u) = g(at^2), \text{ as } u = at^2.$$

Similarly, other results can be proved. In examination we shall not use substitution as explained above. With good practice, the students will be able to write direct results of integration very easily.

Important Note. For sake of convenience, we have divided all questions based on $Rr + Ss + Tt = V$ in four types. We shall now discuss them one by one.

9.3. Type 1. When the given equation Rr + Ss + Tt = V leads to two distict intermediate intergrals and both of them are used to get the desired solution.

Working rule for solving problems of type 1.

Step 1. Write the given equation in the standard form $Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations:

$Rdpdy + Tdqdx - Vdxdy = 0$ ...(1) $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ... (2)

Step 3. Factorise (1) into two distinct factors.

Step 4. Using one of the factors obtained in (1), (2) will lead to an intermediate integral. In general, the second intermediate integral can be obtained from the first one by inspection, taking advantage of symmetry. In absence of any symmetry, the second factor obtained in step 3 is used in (2) to arrive at second intermediate integral. You should use remark 2 of Art. 9.2 while finding intermediate integrals.

Step 5. Solve the two intermediate integrals obtained in step 4 and get the values of p and q.

Step 6. Substitute the values of p and q in $dz = pdx + q\,dy$ and integrate to arrive at the required general solution. You should use remark 3 of Art. 9.2 while integrating $dz = pdx + qdy$.

9.4. SOLVED EXAMPLES BASED ON ART. 9.3.

Ex. 1. (a) *Solve* $r = a^2t$. **[Agra 2008; Lucknow 2010; Patna 2003; Meerut 2008]**

(b) $r = t$. **[Agra 2006]**

(c) Solve one-dimensions wave equation by Monge's method: $\partial^2 y / dx^2 = a^2(\partial^2 y / \partial t^2)$. **[Meerut 2003]**

Sol. (*a*) Given equation is $r - a^2t = 0$.

Comparing it with $Rr + Ss + Tt = V$, we have $R = 1$, $S = 0$, $T = -a^2$, $V = 0$.

Hence Monge's subsidiary equations

$Rdpdy + Tdq\,dx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T\,(dx)^2 = 0$

become $dpdy - a^2dqdx = 0$ ...(1)

and $(dy)^2 - a^2(dx)^2 = 0$. ...(2)

Equation (2) may be factorised as $(dy - adx)\,(dy + adx) = 0$

Hence two systems of equations to be considered are

$dpdy - a^2\,dqdx = 0$, $dy - adx = 0$. ...(3)

and $dpdy - a^2\,dqdx = 0$, $dy + adx = 0$. ...(4)

Integrating the second equation of (3), we get $y - ax = c_1$. ...(5)

Eliminating dy/dx between the equations of (3), we get

$dp - adq = 0$ so that $p - aq = c_2$. ...(6)

Hence the intermediate integral corresponding to (3) is $p - aq = \phi_1(y - ax)$. ...(7)

Similarly another intgermediate integral corresponding to (4) is $p + aq = \phi_2(y + ax)$. ...(8)

Here ϕ_1 and ϕ_2 are arbitrary functions.

Solving (7) and (8) for p and q, we have

$p = (1/2) \times \{\phi_2(y + ax) + \phi_1(y - ax)\}$ and $q = (1/2a) \times \{\phi_2(y + ax) - \phi_1(y - ax)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2) \times \{\phi_2(y + ax) + \phi_1(y - ax)\}dx + (1/2a) \times \{\phi_2(y + ax) - \phi_1(y - ax)\}dy$$
$$= (1/2a) \times \phi_2(y + ax)\,(dy + adx) - (1/2a) \times \phi_1(y - ax)(dy - adx)$$

Integrating, $z = \psi_2(y + ax) + \psi_1(y - ax)$, ψ_1, ψ_2 being arbitrary functions.

(*b*) This is a particular case of part (*a*). Here $a = 1$. **Ans.** $z = \psi_2(y + x) + \psi_1(y - x)$.

(*c*) Refer part (*a*). Note that $\partial^2 y / \partial x^2 = r$ and $\partial^2 y / \partial t^2 = t$

Ex. 2. *Solve* $r + (a + b)s + abt = xy$. **[Vikram 2003]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = 1$, $S = a + b$, $T = ab$, $V = xy$. The usual Monge's subsidiary equations

$Rdpdy + Tqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$.

become $dp\,dy + a\,b\,dq\,dx - xy\,dx\,dy = 0$ ...(1)

and $(dy)^2 - (a + b)\,dxdy + ab\,(dx)^2 = 0$. ...(2)

Factorizing, (2) gives $(dy - bdx)\,(dy - adx) = 0$.

Hence two systems to be considered are

$dp\,dy + ab\,dq\,dx - xy\,dx\,dy = 0$, $dy - b\,dx = 0$. ...(3)

and $dp\,dy + ab\,dq\,dx - xy\,dx\,dy = 0$, $dy - a\,dx = 0$. ...(4)

Integrating the second equation of (3), $y - bx = c_1$. ...(5)

Eliminating dy/dx between the equations of (3), we get

$dp + a\,dq - xy\,dx = 0$ or $dp + a\,dq - x(c_1 + bx)\,dx = 0$, by (5) ...(6)

Integrating (6), $p + aq - (c_1/2)x^2 - (b/3)x^3 = c_2$ or $p + aq - (x^2/2)\,(y - bx) - (b/3)x^3 = c_2$, using (5)

or $p + aq - (1/2) \times yx^2 + (1/6) \times bx^3 = c_2$. ...(7)

Using (5) and (7), the first intermediate integral corresponding to (3) is

$p + aq - (1/2) \times yx^2 + (1/6) \times bx^3 = \phi_1(y - bx)$, ϕ_1 being an arbitrary function ...(8)

Similarly, another intermediate integral corresponding to (4) is

$p + bq - (1/2) \times yx^2 + (1/6) \times ax^3 = \phi_2(y - ax)$, ϕ_2 being an arbitrary functionJ ...(9)

Solving (8) and (9) for p and q, we have

$$p = (1/2) \times x^2y - (1/6) \times (a + b)x^3 + (a - b)^{-1}\,[a\phi_2(y - ax) - b\phi_1(y - ax)]$$

and $$q = (1/6) \times x^3 + (a - b)^{-1}\,[\phi_1(y - bx) - \phi_2(y - ax)].$$

Substituting these values in $dz = pdx + qdy$, we get

$$dz = (1/2) \times x^2ydx - (1/6) \times (a + b)x^3dx + (a - b)^{-1}\,[\phi_2(y - bx)dx - \phi_1(y - ax)dx] + (1/6) \times x^3dy + (a - b)^{-1}\,[\phi_1(y - bx)dy - \phi_2(y - ax)dy]$$

or $$dz = (1/6) \times (3x^2ydx + x^3dy) - (1/6) \times (a + b)\,x^3dx - (b - a)^{-1}\,[\phi_2(y - bx)dx - \phi_1(y - ax)dx] - (b - a)^{-1}\,[(\phi_1(y - bx)dy - \phi_2(y - ax)dy]$$

or $$dz = (1/6) \times d(x^3y) - (1/6) \times (a + b)\,x^3dx + (b - a)^{-1}\phi_2(y - ax)\,(dy - adx) - (b - a)^{-1}\phi_1(y - bx)\,(dy - bdx)$$

or $$dz = (1/6) \times d(x^3y) - (1/6) \times (a + b)\,x^3dx + (b - a)^{-1}\,\phi_2(y - ax)\,d(y - ax) - (b - a)^{-1}\,\phi_1(y - bx)\,d(y - bx)$$

Integrating, $z = (1/6) \times x^3y - (1/24) \times (a + b)x^4 + \psi_2(y - ax) + \psi_1(y - bx)$,

where ψ_1 and ψ_2 are arbitrary functions.

Ex. 3. *Solve* $r - t\cos^2 x + p \tan x = 0$. **[K.U. Kurukshetra 2005; Meerut 1993]**

Sol. Given $r - t\cos^2 x = -p \tan x$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = 1$, $S = 0$, $T = -\cos^2 x$ and $V = -p \tan x$. ...(2)

Monge's subsidiary equations are $Rdp\,dy + Tdq\,dx - V\,dx\,dy = 0$ ...(3)

and $R\,(dy)^2 - S\,dx\,dy + T\,(dx)^2 = 0$ ...(4)

Putting the values of R, S, T and V, (3) and (4) become

$dp\,dy - \cos^2 x\,dq\,dx + p \tan x\,dx\,dy = 0$ ...(5)

and $(dy)^2 - \cos^2 x\,(dx)^2 = 0$ ...(6)

Equation (6) may be factoriesed as $(dy - \cos x\, dx)(dy + \cos x\, dx) = 0$

$\therefore \quad dy - \cos x\, dx = 0$ ...(7)

or $dy + \cos x\, dx = 0$ ...(8)

Putting the value of dy from (7) in (5), we get

$dp \cos x\, dx - \cos^2 x\, dq\, dx + p \tan x\, dx \cos x\, dx = 0$ or $dp - \cos x\, dq + p \tan x\, dx = 0$

or $\sec x\, dp + p \sec x \tan x\, dx - dq = 0$ or $d(p \sec x) - dq = 0.$

Integrating it, $p \sec - q = c_1$, c_1 being an arbitrary constant ....(9)

Integrating (7), $y - \sin x = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one integral of (1) is $p \sec x - q = f(y - \sin x)$. ...(11)

In a similar manner, (8) and (5) give another integral of (1)

$p \sec x + q = g(y + \sin x)$. ...(12)

Solving (11) and (12) for p and q, we find

$p = (f + g)/2 \sec x = (1/2) \times (f + g) \cos x$ and $q = (g - f)/2$ ...(13)

Now, $dz = p\, dx + q\, dy$ or $dz = (1/2) \times (f + q) \cos x\, dx + (1/2) \times (g - f)\, dy$, by (13)

or $dz = -(1/2) \times f(y - \sin x)(dy - \cos x\, dx) + (1/2) \times g(y + \sin x)(dy + \cos x\, dx)$

Integrating, $z = F(y - \sin x) + G(y + \sin x)$, F and G being arbitrary functions.

Ex. 4. *Solve* $t - r \sec^4 y = 2q \tan y$. **[Delhi Maths Hons 1995; Kanpur 1995; Meerut 1995]**

Sol. Given $t - r \sec^4 y = 2q \tan y$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = -\sec^4 y$, $S = 0$, $T = 1$, $V = 2q \tan y$. ...(2)

Monge's subsidiary equations are $Rdp\, dy + T\, dq\, dx - V\, dx\, dy = 0$ ...(3)

and $R(dy)^2 - S\, dxdy + T(dx)^2 = 0$ ...(4)

Putting the values of R, S, T and V, (3) and (4) become

$-\sec^4 y\, dp\, dy + dq\, dx - 2q \tan y\, dx\, dy = 0$ ...(5)

and $-\sec^4 y\, (dy)^2 + (dx)^2 = 0.$ ...(6)

Equation (6) may be factorised as $(dx - \sec^2 y\, dy)(dx + \sec^2 y\, dy) = 0$ so that

$dx - \sec^2 y\, dy = 0$ ...(7)

or $dx + \sec^2 y\, dy = 0.$ ...(8)

Putting the value of dx from (7) in (5), we get

$-\sec^4 y\, dp\, dy + dq \sec^2 y\, dy - 2q \tan y\, dy \times \sec^2 y\, dy = 0$ or $-dp + \cos^2 y\, dq - 2q \sin y \cos y\, dy = 0$

or $dp - (\cos^2 x\, dq - q \times 2 \sin y \cos y\, dy) = 0$ or $dp - d(q \cos^2 y) = 0.$

Integrating it, $p - q \cos^2 y = c_1$, c_1 being an arbitrary constant ...(9)

Integrating (7), $x - \tan y = c_2$, being an arbitrary constant ...(10)

From (9) and (10), one integral of (1) is $p - q \cos^2 y = f(x - \tan y)$. ...(11)

Similarly, from (8) and (5) the other integral of (1) is $p + q \cos^2 y = g(x + \tan y)$. ...(12)

Solving (11) and (12) for p and q, we find

$p = (f + g)/2$ and $q = (g - f)/(2 \cos^2 y) = (1/2) \times (g - f) \times \sec^2 y$ ...(13)

Now, we have $dz = pdx + qdy$

or $dz = (1/2) \times (f + g)dx + (1/2) \times (g - f) \times \sec^2 y\, dy$, using (13)

or $dz = (1/2) \times f(x - \tan y)(dx - \sec^2 y\, dy) + (1/2) \times g(x + \tan y)(dx + \sec^2 y\, dy)$

or $dz = (1/2) \times f(x - \tan y)\, d(x - \tan y) + (1/2) \times g(x + \tan y)\, d(x + \tan y)$.

Integrating, $z = F(x - \tan y) + G(x + \tan y)$, F, G being arbitrary functions.

Ex. 5. *Solve* $q(yq + z)r - p(2yq + z)s + yp^2 t + p^2 q = 0$. **[Delhi 2008]**

Sol. As usual, here Monge's subsidiary equations are

$q(yq + z)dp\, dy + yp^2 dqdx + p^2 qdxdy = 0$ ... (1)

and $q(yq + z)(dy)^2 + p(2yq + z)dxdy + yp^2(dx)^2 = 0.$ ... (2)

On factorization, (2) gives $(qdy + pdx)\ \{(yq + z)dy + ypdx\} = 0.$

Hence two systems to be considered are

$$q(yq + z)dpdy + yp^2dqdx + p^2qdxdy = 0, \qquad qdy + pdx = 0 \quad \ldots (3)$$

and $$q(yq + z)dpdy + yp^2dqdx + p^2q\ dxdy = 0, \qquad (yq + z)dy + ypdx = 0 \ \ldots (4)$$

Using $dz = pdx + qdy$, the second equation of (3) reduces to

$dz = 0$ so that $z = c_1$. ... (5)

From second equation of (3), $qdy = -pdx$. Hence first equation of (3) reduces to

$(yq + z)dp - ypdq - pqdy = 0$ or $(yq + z)dp - p\ d(yq) = 0$

or $(yq + z)dp - pd(yq + z) = 0,$ as $dz = 0,$ by (5)

or $$\frac{d(yq+z)}{yq+z} - \frac{dp}{p} = 0 \qquad \text{so that} \qquad \log (yq + z) - \log p = \log c_1$$

or $(yq + z)/p = c_2$, c_2 being an arbitrary constant ... (6)

From (5) and (6), the intermediate integral corresponding to (3) is

$(yq + z)/p = \phi_1(z)$ or $yq + z = p\phi_1(z),$ ...(7)

where ϕ_1 is an arbitrary function.

Using $dz = pdx + qdy$, the second equation of (4) becomes

$y(qdy + pdx) + zdy = 0$ or $ydz + zdy = 0$ or $d(yz) = 0.$

Integrating it, $yz = c_3$, c_3 being an arbitrary constant ...(8)

From second equation of (4), $(yq + z)dy = -ypdx.$

Using this fact, first equation of (4) reduces to

$qdp - pdq - (pq/y)dy = 0$ or $-(1/p)dp + (1/q)dq + (1/y)dy = 0.$

Integrating, $-\log p + \log q + \log y = \log c_1$ or $(yq)/p = c_2$ ...(9)

From (8) and (9), another intermediate integral corresponding to (4) is

$(qy)/p = \phi_2(yz)$, where ϕ_2 is an arbitrary function. ...(10)

Solving (7) and (10) for p and q, we have $p = \dfrac{z}{\phi_1(z) - \phi_2(yz)}$, $q = \dfrac{z\phi_2(yz)}{y\{\phi_1(z) - \phi_2(yz)\}}.$

Substituting these in $dz = pdx + qdy$, $dz = \dfrac{z}{\phi_1(z) - \phi_2(yz)}\{dx + (1/y) \times \phi_2(yz)\ dy\}$

or $\phi_1(z)dz = zdx + \phi_2(yz)\dfrac{zdy + ydz}{y}$ or $\dfrac{\phi_1(z)\,dz}{z} = dx + \dfrac{\phi_1(yz)\,d(yz)}{yz}.$

Integrating, $\psi_1(z) = x + \psi_2(yz)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex. 6. *Solve* $(r - t)xy - s(x^2 - y^2) = qx - py.$ **[Delhi Maths 2005, Kurukshetra 2005 (H)]**

Sol. Usual Monge's auxiliary equations are

$$xydpdy - xydqdx - (qx - py)dxdy = 0 \quad \ldots(1)$$

and $$xy(dy)^2 + (x^2 - y^2)\ dxdy - xy\ (dx)^2 = 0. \quad \ldots(2)$$

On factorizing, (2)gives $(xdy - ydx)\ (ydx + xdy) = 0.$

Hence, two systems to be considered are

$$xydpdy - xydqdx - (qx - py)\ dxdy = 0, \qquad xdy - ydx = 0 \quad \ldots(3)$$

and $$xydpdy - xydqdx - (qx - py)\ dxdy = 0, \qquad ydx + xdy = 0. \quad \ldots(4)$$

Second equation of (3) gives $y/z = c_1$, c_1 being an arbitrary constant ...(5)

Using second equation, first equation of (3) reduces to

$ydp - xdq - qdx + pdy = 0$ or $d(yp - xq) = 0$

Integrating, $yp - xq = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), intermediate integral corresponding to (3) is

$yp - xq = \phi_1(y/x)$, where ϕ_1 is an arbitrary function. ...(7)

Second equation of (4) gives $x^2 + y^2 = c_3$, c_3 being arbitrary constant ...(8)

Using second equation, first equation of (4) reduces to

$xdp + ydq + qdy + pdx = 0$ or $d(xp) + d(yq) = 0$

Integrating, $xp + yq = c_4$, c_4 being an arbitrary constant ...(9)

From (8) and (9), another intermediate integral corresponding to (4) is

$xp + yq = \phi_2(x^2 + y^2)$, where ϕ_2 is an arbitrary function. ...(10)

Solving (7) and (10) for p and q, we have

$$p = \frac{1}{x^2+y^2}\left\{y\phi_1\left(\frac{y}{x}\right)+x\phi_2(x^2+y^2)\right\} \quad \text{and} \quad q = \frac{1}{x^2+y^2}\left\{y\phi_2(x^2+y^2)-x\phi_1\left(\frac{y}{x}\right)\right\}.$$

Substituting these values in $dz = pdx + qdy$, we get

$$dz = \frac{1}{x^2+y^2}\left[\left\{y\phi_1\left(\frac{y}{x}\right)+x\phi_2(x^2+y^2)\right\}dx+\left\{y\phi_2(x^2+y^2)-x\phi_1\left(\frac{y}{x}\right)dy\right\}\right]$$

or $$dz = \frac{ydx-xdy}{x^2+y^2}\phi_1\left(\frac{y}{x}\right)+\frac{xdx+ydy}{x^2+y^2}\phi_2(x^2+y^2)$$ or $$dz = -\frac{\phi_1(y/x)}{1+(y/x)^2}d\left(\frac{y}{x}\right)+\frac{1}{2}\frac{\phi_2(x^2+y^2)}{x^2+y^2}d(x^2+y^2).$$

Integrating, $z = \psi_1(y/x) + \psi_2(x^2 + y^2)$, ψ_1, ψ_2 being arbitrary functions.

Ex. 7. *Solve* $(r - s)\, x = (t - s)\, y$. **(M.D.U Rohtak 2005)**

Sol. Usual Monge's subsidiary equations are $xdpdy - ydqdx = 0$ ...(1)

and $x(dy)^2 + (x - y)\, dxdy - y(dx)^2 = 0.$ ...(2)

Factorising, (2) $\Rightarrow$ $(xdy - ydx)\,(dy + dx) = 0.$

Hence two systems to be considered are

$xdpdy - ydqdx = 0$, $xdy - ydx = 0$ ...(3)

and $xdpdy - ydqdx = 0$, $dy + dx = 0.$ ...(4)

Integrating second equation of (3), $y/x = c_1$, c_1 being an arbitrary constant ...(5)

Eliminating dy/dx between equations of (3), we get

$dp - dq = 0$ so that $p - q = c_2$, c_2 being an arbitrary constant ...(6)

Hence the intermediate integral corresponding to (3) is $p - q = \phi_1(y/x)$. ...(7)

Integrating second equation of (4), $x + y = c_3$, c_3 being an arbitrary constant ...(8)

Eliminating dy/dx between equations of (4), we get

$xdp + ydq = 0$ or $xdp + ydq + pdx + qdy = pdx + qdy$

or $d(xp) + d(yq) - dz = 0$, as $dz = pdx + qdy$.

Integrating, $xp + yq - z = c_4$, c_4 being an arbitrary constant ...(9)

Hence the intermediate integral corresponding to (4) is

$xp + yq - z = \phi_2(x + y)$ or $xp + yq = z + \phi_2(x + y)$, ...(10)

Solving (7) and (10) for p and q, we have

$$p = \frac{1}{x+y}\left\{z+\phi_2(x+y)+y\phi_1\left(\frac{y}{x}\right)\right\} \quad \text{and} \quad q = \frac{1}{x+y}\left\{z+\phi_2(x+y)-x\phi_1\left(\frac{y}{x}\right)\right\}.$$

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \frac{1}{x+y}\left[\left\{z+\phi_2(x+y)+y\phi_1\left(\frac{y}{x}\right)\right\}dx+\left\{z+\phi_2(x+y)-x\phi_1\left(\frac{y}{x}\right)\right\}dy\right]$$

$$\Rightarrow \quad \frac{(x+y)\,dx-zdx}{(x+y)^2} = \frac{\phi_2(x+y)\,d(x+y)}{(x+y)^2}+\frac{(ydx-xdy)\,\phi_1(y/x)}{(x+y)^2}$$

$$\Rightarrow \qquad d\left(\frac{z}{x+y}\right) = \frac{\phi_2(x+y)}{(x+y)^2}d(x+y) - \frac{\phi_1(y/x)}{1+(y/x)^2}d\left(\frac{y}{x}\right).$$

Integrating, $z/(x+y) = \psi_2(x+y) + \psi_1(y/x)$, ψ_1, ψ_2 being arbitrary functions.

Ex. 8. *Solve* $r + ka^2t - 2as = 0$.

Sol. Given $r - 2as + ka^2t = 0.$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we have $R = 1$, $S = -2a$, $T = ka^2$, $V = 0$.

Hence the Monge's subsidiary equations

$Rdp\,dy + Tdq\,dx - Vdx\,dy = 0$ and $R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0$

become $dp\,dy + ka^2\,dq\,dx = 0$ ...(2)

and $(dy)^2 + 2a\,dx\,dy + ka^2\,(dx)^2 = 0.$ ...(3)

From (3), $dy = [-2a\,dx \pm \{4a^2(dx)^2 - 4ka^2\,(dx)^2\}^{1/2}]/2 = -a\,dx \pm a\sqrt{(1-k)}\,dx$

or $dy + a\,\{1 \pm \sqrt{(1-k)}\}dx = 0$ or $dy + a\,(1 \pm l)\,dx = 0$, where $l = \sqrt{(1-k)}$.

Hence (3) reduces to the following two equations :

$$dy + a(1 + l)dx = 0 \qquad ...(4)$$

and $$dy + a(1 - l)dx = 0. \qquad ...(5)$$

From (2) and (4) , eliminating dy, we have

$dp\{-a\,(1 + l)\,dx\} + ka^2\,dqdx = 0$ or $(1 + l)dp - ka\,dq = 0.$

Integrating it, $(1 + l)p - kaq = c_1$, c_1 being an arbitrary constant ...(6)

Again, integrating (4), $y + a(1 + l)x = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), first intermediate integral is

$$(1 + l)p - kaq = f_1\{y + a(1 + l)x\}, \text{ where } f_1 \text{ is an arbitrary function.} \qquad ...(8)$$

Similary, from (2) and (5), second intermediate intgegral is given by (replacing l by $-l$ in (8) since (5) differs from (4) in having $-l$ in place of l)

$$(1 - l)p - kaq = f_2\{y + a(1 - l)x\}, \text{ where } f_2 \text{ is an arbitrary function} \qquad ...(9)$$

Solving (8) and (9) for p and q, $p = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]$

and $q = (1/2akl) \times [(1 - l)f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}].$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]dx + (1/2akl) \times [(1 - l)\,f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}]dy$$

or $$dz = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]dx + \frac{1}{2al(1-l^2)}\,[(1 - l)\,f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}]dy,$$ as $l = (1 - k)^{1/2} \Rightarrow k = 1 - l^2$

$$\text{or } dz = (1/2l)\,[dx f_1\{y + a(1 + l)x\} - dx f_2\{y + a(1 - l)x\}] + \frac{1}{2al}\left[\frac{dy}{1+l} f_1\{y + a(1+l)x\} - \frac{dy}{1-l} f_2\{y + (1-l)x\}\right]$$

$$= \frac{1}{2al(l+1)} f_1\{y + a(1 + l)x\}\{dy + a(1 + l)dx\} - \frac{1}{2al(1-l)} f_2\{y + a(1 - l)x\}\ \{dy + a(1 - l)dx\}$$

$$\text{or } dz = \frac{1}{2al(l+1)} f_1\{y + a(1 + l)x\}\ d\{y + a(1 + l)x\} - \frac{1}{2al(1-l)} f_2\{y + a(1 - l)x\}\ d\{y + a\,(1 - l)x\}.$$

Integrating, $z = F_1\{y + a(1 + l)x\} + F_2\{y + a\,(1 - l)x\}$, where F_1 and F_2 are arbitrary functions.

Ex. 9. *Solve* $x^{-2}r - y^{-2}t = x^{-3}p - y^{-3}q$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^{-2}$, $S = 0$, $T = y^{-2}$, $V = x^{-3}p - y^{-3}q$. Then Monge's subsidiary equations

$Rdpdy + Tdqdx + Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $x^{-2}dpdy + y^{-2}dqdx - (x^{-3}p - y^{-3}q)\,dxdy = 0.$ ...(1)

and $x^{-2}(dy)^2 - y^{-2}(dx)^2 = 0.$ ...(2)

Multiplying both sides of (1) by x^3y^3, we get

$xy^3dpdy - x^3ydqdx - py^3dxdy + qx^3dxdy = 0.$ ...(3)

Again, (2) $\Rightarrow$ $x^2y^2(y^2dy^2 - x^2dx^2) = 0$ or $x^2y^2(ydy + xdx)\,(ydy - xdx) = 0$

Hence (2) is equivalent to the equations

$ydy + xdx = 0$ *i.e.*, $ydy = -xdx$ ...(4)

and $ydy - xdx = 0.$ ...(5)

Integrating (4), $y^2/2 + x^2/2 = c_1/2$ or $x^2 + y^2 = c_1$. ...(6)

From (3), $xy^2dp(ydy) - x^2ydq(xdx) - py^2dx(ydy) + qx^2dy(xdx) = 0$

or $xy^2dp(-xdx) - x^2ydq(xdx) - py^2dx\,(-xdx) + qx^2dy(xdx) = 0$, using (4)

or $-xy^2dp - x^2ydq + py^2dx + qx^2dy = 0$ or $y^2(xdp - pdx) + x^2(ydq - qdy) = 0$

or $\dfrac{xdp - pdx}{x^2} + \dfrac{ydq - qdy}{y^2} = 0$ or $d\left(\dfrac{p}{x}\right) + d\left(\dfrac{q}{y}\right) = 0.$

Integrating, $(p/x) + (q/y) = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), an intermediate integral is

$(1/x)p + (1/y)q = f(x^2 + y^2)$, where f is an arbitrary function. ...(8)

Similarly, from (3) and (5), another intermediate integral is

$(1/x)p - (1/y)q = g(x^2 - y^2)$, where g is an arbitrary function ...(9)

Solving (8) and (9) for p and q, we obtain

$p = (x/2) \times \{f(x^2 + y^2) + g(x^2 - y^2)\}$ and $q = (y/2) \times \{f(x^2 + y^2) - g(x^2 - y^2)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$dz = (x/2) \times \{f(x^2 + y^2) + g(x^2 - y^2)\}dx + (y/2) \times \{f(x^2 + y^2) - g(x^2 - y^2)\}dy$

or $dz = (1/4) \times f(x^2 + y^2)\,(2xdx + 2ydy) + (1/4) \times g(x^2 - y^2)\,(2xdx - 2ydy)$ ... (10)

Putting $x^2 + y^2 = u$, $x^2 - y^2 = v$ so that $2xdx + 2ydy = du$ and $2xdx - 2ydy = dv$, (10) gives

$dz = (1/4) \times f(u)\,du + (1/4) \times g(v)dv,$ ... (11)

Integrating (11), $z = F(u) + G(v) = F(x^2 + y^2) + G(x^2 - y^2)$,

where F and G are arbitrary functions.

Ex. 10. *Solve* $rx^2 - 3s\,xy + 2t\,y^2 + px + 2qy = x + 2y$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^2$, $S = -3xy$, $T = 2y^2$, $V = x + 2y - px - 2qy$.

Hence Monge's subsidiary equations are

$Rdpdy + Tdqdy - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $x^2\,dpdy + 2y^2dqdx - (x + 2y - px - 2qy)\,dxdy = 0$ ...(1)

and $x^2(dy)^2 + 3xy\,dxdy + 2y^2(dx)^2 = 0.$ ...(2)

Here (2) $\Rightarrow$ $(xdy + 2ydx)\,(xdy + ydx) = 0.$

Hence (2) resolves into the following two equations

$xdy + 2ydx = 0$ *i.e.*, $2ydx = -xdy$ ...(3)

and $xdy + ydx = 0.$ ...(4)

Re–writing (3), $(1/y)dy + 2(1/x)dx = 0$

Integrating, $\log y + 2 \log x = \log c_1$ or $yx^2 = c_1$. ...(5)

Re–writing (1), $(xdp)\,(xdy) + ydq\,(2ydx) - dx(xdy) - dy\,(2ydx) + pdx\,(xdy) + qdy\,(2ydx) = 0$

or $(xdp)\,(xdy) + ydq(-xdy) - dx(xdy) - dy(-xdy) + pdx(xdy) + qdy(-xdy) = 0$, using (3)

or $xdp - ydq - dx + dy + pdx - qdy = 0$

or $(xdp + pdx) - (ydq + qdy) - dx + dy = 0$ or $d(xp) - d(yq) - dx + dy = 0$.

Integrating, $xp - yq - x + y = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), an intermediate integral is

$xp - yq - x + y = f(x^2y)$, where f is an arbitrary function. ...(7)

Similarly from (1) and (4), another intermediate integral is

$xp - 2yq - x + 2y = g(xy)$, where g is an arbitrary function. ...(8)

Solving (7) and (8) for p and q, we have

$p = (1/x) \times \{x + 2f(x^2y) - g(xy)\}$, and $q = (1/y) \times \{y + f(x^2y) - g(xy)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/x) \times \{x + 2f(x^2y) - g(xy)\}dx + (1/y) \times \{y + f(x^2y) - g(xy)\}]dy$$

or $$dz = dx + dy + f(x^2y)\left(\frac{2}{x}dx + \frac{1}{y}dy\right) - g(xy)\left(\frac{dx}{x} + \frac{dy}{y}\right)$$

or $$dz = dx + dy + f(x^2y)\, d[\log (x^2y)] - g(xy)\, d[\log (xy)].$$

Integrating, $z = x + y + F(x^2y) + G(xy)$, G, and F being arbitrary functions.

Ex. 11. *Find the general solution of the equation $r + 4t = 8xy$, by Monge's method. Find also the particular solution for which $z = y^2$ and $p = 0$, when $x = 0$* **[Delhi Maths (Hons) 2006, 09]**

Sol. Given $r + 4t = 8xy$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 1$, $S = 0$, $T = 4$ and $V = 8xy$. Hence Monge's subsidians equations $Rdp\,dy + Tdq\,dx - Vdxdy = 0$ and $R(dy)^2 - Sdx\,dy + T(dx)^2 = 0$ become

$$dpdy + 4\,dqdx - 8xydxdy = 0 \quad \text{... (2)}$$

and $$(dy)^2 + 4\,(dx)^2 = 0 \quad \text{... (3)}$$

Re–writing (3), $dy^2 - 4i^2dx^2 = 0$ or $(dy - 2idx)\,(dy + 2idx) = 0$

so that $dy - 2idx = 0$ or $dy = 2idx$ ... (4)

and $dy + 2idx = 0$ or $dy = -2idx$ ... (5)

We first consider (4) and (2). Integrating (4), $y - 2ix = C_1$ ... (6)

Using (4) and (6), (2) gives $dp\,(2i\,dx) + 4\,dq\,dx - 8x(C_1 + 2ix)\,(2i\,dx)\,dx = 0$

or $i\,dp + 2dq - 8xi\,(C_1 + 2ix) = 0$, by (6) or $idp + 2dq - 8C_1ix\,dx + 16x^2dx = 0$

Integrating, $ip + 2q - 4C_1ix^2 + (16/3)x^3 = C_2$, C_2 being an arbitrary constant

or $ip + 2q - 4ix^2(y - 2ix) + (16/3) \times x^3 = C_2$, by (6) ... (7)

From (6) and (7) first intermediate integral of (1) is $ip + 2q - 4ix^2(y - 2ix) + (16/3)x^3 = f(y - 2ix)$

or $ip + 2q = (8/3) \times x^3 + 4ix^2y + f(y - 2ix)$, f being an abritrary function ... (8)

Similarly considering the pair (5) and (2), the second intermediate integral of (1) is

$$ip - 2q = -(8/3) \times x^3 + 4ix^2 y + g\ (y + 2ix)\text{, } g \text{ being an arbitrary function} \quad \text{... (9)}$$

Solving (8) and (9) for p and q, $p = \{8ix^2 y + f(y - 2ix) + g(y + 2ix)\}/2i$

and $q = \{(16/3) \times x^3 + f(y - 2ix) - g(y + 2ix)\}/4$

Putting the above values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2i) \times \{8ix^2 y + f(y - 2ix) + g(y + 2ix)\}dx + (1/4) \times \{(16/3) \times x^3 + f(y - 2ix) - g(y - 2ix)\}dy$$

$$= (4/3) \times (3x^2 y dx + x^3 dy) + (1/4) \times f(y - 2ix)d(y - 2ix) - (1/4) \times g(y + 2ix)d(y + 2ix)$$

$$\therefore \quad dz = (4/3) \times d(x^3 y) + (1/4) \times f(y - 2ix)\ d(y - 2ix) - (1/4) \times g(y + 2ix)d(y + 2ix)$$

Integrating, $z = (4/3) \times x^3 y + F(y - 2ix) + G\ (y + 2ix)$, ... (10)

which is the general solution of (1) containing F and G as arbitrary functions

To find particular solution of (1) Given conditions are

$z = y^2$ and $p = \partial z/\partial x = 0$ when $x = 0$ ... (11)

From (11), $\partial z/\partial y = 2y$ when $x = 0$ ... (12)

Differentiating (10) partially w.r.t. 'x' and 'y', we get

$$\partial z/\partial x = 4x^2 y - 2i\ F'(y - 2ix) + 2i\ G'(y + 2ix) \quad \text{... (13)}$$

and $$\partial z/\partial y = (4/3) \times x^3 + F'(y - 2ix) + G'(y + 2ix) \quad \text{... (14)}$$

Using (11) and (12), (10), (13) and (14) reduce to

$$F(y) + G(y) = y^2 \quad \text{... (15)}$$

$$F'(y) - G'(y) = 0 \quad \text{... (16)}$$

and $$F'(y) + G'(y) = 2y \quad \text{... (17)}$$

From (16) and (17), $F'(y) = y$ and $G'(y) = y$

Integrating these, $F(y) = y^2/2$ and $G(y) = y^2/2$ ... (18)

which also satisfy (15).

From (18), $F(y - 2ix) = (y - 2ix)^2/2$ and $G(y + 2ix) = (y + 2ix)^2/2$

Putting these values in (10), the required particular solution is

$z = (4/3) \times x^3 y + (y - 2ix)^2/2 + (y + 2ix)^2/2$ or $z = (4/3) \times x^3 y + y^2 - 4x^2$.

9.5. Type 2. When the given equation *Rr + Ss + Tt = V* leads to two distinct intermediate integrals and only one is employed to get the desired solution.

Working rule for solving problems of type 2.

Step 1. Write the given equation in the standard form $Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ ... (1) $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ... (2)

Step 3. Factorise (1) into two distinct factors.

Step 4. Take one of the factors of step 3 and use (2) to get an intermediate integral. Don't find second intermediate integral as we did in type 1. If required use remark 1 of Art. 9.2.

Step 5. Re–write the intermediate integral of the step 4 in the form of Lagrange equation, namely, $Pp + Qq = R$ (refer chapter 2). Using the well known Lagrange's method we arrive at the desired general solution of the given equation.

9.6 SOLVED EXAMPLES BASED ON ART. 9.5.

Ex. 1. *Solve* $(r - s)y + (s - t)x + q - p = 0$.

Sol. The given can be written as $yr + s(x - y) - tx = p - q.$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = y$, $S = x - y$, $T = -x$ and $V = p - q$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $ydpdy - xdqdx + (q - p)dxdy = 0$ ...(1)

and $y(dy)^2 - (x - y)\, dxdy - x\,(dx)^2 = 0.$ ...(2)

Re–writing (2), $(dy + dx)\,(ydy - xdx) = 0.$

so that $dy + dx = 0$ or $dy = -dx$ ...(3)

and $ydy - xdx = 0.$ ...(4)

Using (3), (1) becomes $-ydpdx - xdqdx + q\,dx(-dx) - p\,dxdy = 0$

or $ydp + xdq + qdx + pdy = 0$ or $(ydp + pdy) + (xdq + qdx) = 0$

or $d(yp) + d(xq) = 0$ so that $yp + xq = c_1.$ ...(5)

Integrating (3), $x + y = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), one intermediate integral is $yp + xq = f(x + y),$ ...(7)

which is of the Lagrange's form and so its subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{f(x+y)}. \qquad ...(8)$$

From first and second fractions of (8), $2xdx - 2ydy = 0.$

Integrating, $x^2 - y^2 = a$, a being an arbitrary constant ...(9)

Taking first and third fractions of (8), we get

$$\frac{dx}{y} = \frac{dz}{f(x+y)} \quad \text{or} \quad \frac{dx}{(x^2-a)^{1/2}} = \frac{dz}{f\left[x + (x^2-a)^{1/2}\right]}, \text{ as (9)} \Rightarrow y = (x^2 - a)^{1/2}$$

or $dz = f[x + (x^2 - a^2)^{1/2}]\,(x^2 - a^2)^{-1/2}dx$ ...(10)

Put $x + (x^2 - a)^{1/2} = v$ so that $\left[1 + x/(x^2 - a)^{1/2}\right]dx = dv$ ...(11)

or $\dfrac{x + (x^2 - a)^{1/2}}{(x^2 - a)^{1/2}}\, dx = dv$ or $\dfrac{dx}{(x^2 - a)^{1/2}} = \dfrac{dv}{v}$, using (11)

Then, (10) reduces to $dz - (1/v)\, f(v)dv = 0.$

Integrating, $z - F(v) = b$ or $z - F[x + (x^2 - a)^{1/2}] = b,$ by (11)

or $z - F(x + y) = b,$ as $y = (x^2 - a)^{1/2},$ by (9) ...(12)

From (9) and (12), the required general solution is $z - F(x + y) = G(x^2 - y^2)$

or $z = F(x + y) + G(x^2 - y^2)$, where F and G are arbitrary functions.

Ex. 2. *Solve* : $q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0.$ **[Meerut 1994; I.A.S. 1974]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we find

$R = q(1 + q),$ $S = -(p + q + 2pq),$ $T = p(1 + p),$ $V = 0$ ...(1)

Monge's subsidiary equations are $Rdpdy + Tdq\,dx - Vdxdy = 0$ ...(2)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ...(3)

Using (1), (2) and (3) become $(q + q^2)dpdy + (p + p^2)dqdx = 0$ ...(4)

and $(q + q^2)\,(dy)^2 + (p + q + 2pq)dxdy + (p + p^2)\,(dx)^2 = 0.$ ...(5)

In order to factorise (5), we re–write it as

$$q(1+q)(dy)^2 + (p+pq)dxdy + (q+pq)dxdy + p(1+p)(dx)^2 = 0$$

or $$q(1+q)(dy)^2 + p(1+q)dxdy + q(1+p)dxdy + p(1+p)(dx)^2 = 0$$

or $$(1+q)dy(qdy+pdx) + (1+p)dx(qdy+pdx) = 0$$

or $$(qdy+pdx)\,[(1+q)dy + (1+p)dx] = 0. \quad \ldots(6)$$

Then, from (6), we get $qdy + pdx = 0$ *i.e.*, $qdy = -pdx$ ...(7)

and $$(1+q)dy + (1+p)dx = 0. \quad \ldots(8)$$

Keeping (7) in view, (4) may be re–written as $(1+q)dp\,(qdy) - (1+p)dq\,(-pdx) = 0$

From (7), qdy and $(-pdx)$ are equivalent. Hence dividing each term of the above equation by qdy, or its equivalent $(-pdx)$, we get

$(1+q)dp - (1+p)dq = 0$ or $dp/(1+p) - dq/(1+q) = 0.$

Integrating it, $\log(1+p) - \log(1+q) = \log c_1$ or $(1+p)/(1+q) = c_1$. ...(9)

Using $dz = pdx + qdy$, (7) becomes $dz = 0$ so that $z = c_2$. ...(10)

From (9) and (10), one intermediate integral of (1) is given by

$(1+p)/(1+q) = f(z)$ or $p - f(z)q = f(z) - 1,$ ...(11)

which is of the form $Pp + Qq = R$. Here Lagrange's auxiliary equations for (11) are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{f(z)-1}. \quad \ldots(12)$$

Choosing 1, 1, 1 as multipliers, each fraction in (12) $= \dfrac{dx+dy+dz}{1-f(z)+f(z)-1} = \dfrac{dx+dy+dz}{0}$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_2$. ...(13)

From first and third fractions in (12), we get $dx - [f(z) - 1]^{-1}\,dz = 0.$

Integrating it, $x + F(z) = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$$x + F(z) = G(x + y + z),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 3. *Solve* $(x-y)\,(xr - xs - ys + yt) = (x+y)(p-q)$.

[Delhi Maths (H) 97, 2000; Meerut 1999; Garhwal 1996]

Sol. Given $(x-y)xr - (x^2 - y^2)s + (x-y)yt = (x+y)(p-q)$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = x(x-y),$ $S = -(x^2 - y^2),$ $T = y(x-y),$ $V = (x+y)(p-q).$ ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ ...(3)

and $$R(dy)^2 - Sdxdy + T(dx)^2 = 0. \quad \ldots(4)$$

Using (2), (3) and (4) become

$$x(x-y)dpdy + y(x-y)dqdx - (x+y)(p-q)dxdy = 0 \quad \ldots(5)$$

and $$x(x-y)(dy)^2 + (x^2 - y^2)dxdy + y(x-y)(dx)^2 = 0. \quad \ldots(6)$$

Since $x^2 - y^2 = (x-y)(x+y)$, dividing (6) by $(x-y)$ gives

$xdy^2 + (x+y)dxdy + ydx^2 = 0$ or $(xdy + ydx)\,(dx + dy) = 0$

Thus we get $xdy + ydx = 0$ or $xdy = -ydx$ ...(7)

and $$dx + dy = 0. \quad \ldots(8)$$

Keeping (7) in view, (5) may be rewritten as

$$(x-y)dp\,(xdy) - (x-y)\,dq\,(-ydx) - (p-q)\,dx\,(xdy) + (p-q)dy(-ydx) = 0.$$

From (7), $x\,dy$ and $(-y\,dx)$ are equal. So dividing each term of the above equation by $x\,dy$, or its equivalent $(-y\,dx)$, we get

$(x-y)\,dp - (x-y)dq - (p-q)\,dx + (p-q)dy = 0$ or $(x-y)\,(dp - dq) - (p-q)\,(dx - dy) = 0$

or $\frac{dp-dq}{p-q}-\frac{dx-dy}{x-y}=0$ so that $\frac{p-q}{x-y}=c_1$ ...(9)

Integrating (7), $xy = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one intermediate integral of (10) is

$(p-q)/(x-y) = f(xy)$ or $p - q = (x - y)f(xy)$ ...(11)

which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{(x-y)f(xy)}. \qquad ...(12)$$

Taking the first two fractions of (12), we get

$dx + dy = 0$ so that $x + y = c_3$, c_3 being an arbitrary constant ...(13)

Taking $y\,f(xy)$, $x\,f(xy)$, 1 as multipliers, each fraction of (12) $= \frac{y\,f(xy)\,dx+x\,f(xy)\,dy+dz}{0}$

so that $f(xy) \times (ydx + x\,dy) + dz = 0$ or $f(xy) \times d(xy) + dz = 0$.

Integrating it, $F(xy) + z = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$F(xy) + z = G(x + y)$, where F and G are arbitrary functions.

Ex. 4. $xy\,(t - r) + (x^2 - y^2)\,(s - 2) = py - qx$. **[Delhi Maths (H) 2001]**

Sol. Given $-xyr + (x^2 - y^2)\,s + xyt = py - qx + 2(x^2 - y^2)$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = -xy$, $S = x^2 - y^2$, $T = xy$, $V = py - qx + 2(x^2 - y^2)$. ...(2)

Monge's subsidiary equations are $Rdp\,dy + Tdq\,dx - V\,dx\,dy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$. ...(4)

Using (2), (3) and (4) become

$$-xy\,dp\,dy + xydqdx - [py - qx + 2(x^2 - y^2)]dxdy = 0 \qquad ...(5)$$

and $-xy\,(dy)^2 - (x^2 - y^2)dxdy + xy(dx)^2 = 0$. ...(6)

From (6), $xy(dy)^2 + x^2dxdy - y^2dxdy - xy(dx)^2 = 0$

or $xdy(ydy + xdx) - ydx\,(ydy + xdx) = 0$ or $(xdy - ydx)\,(ydy + xdx) = 0$.

So, we get $xdx + ydy = 0$, *i.e.*, $xdx = -ydy$ ...(7)

and $xdy - ydx = 0$. ...(8)

Keeping (7) in view, (5) may be re–written as

$xdp(-ydy) + ydq\,(xdx) + pdx\,(-ydy) + qdy(xdx) - 2xdy(xdx) - 2ydx(-ydy) = 0$.

From (7), xdx and $(-ydy)$ are equivalent. So dividing each term of the above equation by xdx, or its equivalent $(-ydy)$, we get

$xdp + ydq + pdx + qdy - 2xdy - 2ydx = 0$ or $(xdp + pdx) + (ydq + qdy) - 2(xdy + ydx) = 0$.

Integrating it, $xp + yq - 2xy = c_1$, being an arbitrary constant ...(9)

Integrating (7), $x^2/2 + y^2/2 = c_2/2$ or $x^2 + y^2 = c_2$. ...(10)

From (9) and (10), one intgegral of (1) is

$xp + qy - 2xy = f(x^2 + y^2)$ or $xp + yq = 2xy + f(x^2 + y^2)$, ...(11)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations for (11)are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{2xy+f(x^2+y^2)}. \qquad ..(12)$$

Taking the first two fractions in (12), we get

$\log y - \log x = \log c_3$ or $y/x = c_3$ or $y = xc_3$ ...(13)

Taking the first and the last fractions in (12) and using $y = xc_3$ in it, we get

$dz = (1/x) \times [2c_3x^2 + f(x^2 + x^2c_3^2)]dx$ or $dx = 2c_3xdx + (1/x) \times f\{(1 + c_3^2)x^2\}dx$

or $dz = 2c_3xdx + (1/2x^2) \times f\{(1 + c_3^2)x^2\}d(x^2)$.

Integrating $z - 2c_3(x^2/2) + F\{(1 + c_3^2)x^2\} = c_4$ or $z - (y/x)x^2 + F\{(1 + y^2/x^2)x^2\} = c_4$, by (13)

or $z - xy + F(x^2 + y^2) = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$z - xy + F(x^2 + y^2) = G(y/x)$, where F and G are arbitrary functions.

Ex. 5. *Solve* $x^2r - y^2t - 2xp + 2z = 0$.

Sol. Given $x^2r - y^2t = 2xp - 2z.$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = x^2$, $S = 0$, $T = -y^2$, $V = 2xp - 2z$.

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $x^2dpdy - y^2dqdx - (2xp - 2z)dxdy = 0$ ...(2)

and $x^2(dy)^2 - y^2(dx)^2 = 0.$ ...(3)

On factorizing, $(3) \Rightarrow (xdy - ydx)\,(xdy + ydx) = 0$

Thus, we have $xdy - ydx = 0$ *i.e.*, $xdy = ydx.$ ...(4)

and $xdy + ydx = 0.$ ...(5)

Re–writing (2), $xdp\,(xdy) - ydq(ydx) - 2(xp - z)\,(xdy)(1/x)dx = 0$

or $xdp(xdy) - ydq(xdy) - 2(xp - z)\,(xdy)\,(1/x)dx = 0$, using (4)

or $xdp - ydq - 2(xp - z)\,(1/x)dx = 0$

or $xdp - dz + pdx + qdy - ydq - 2(xp - z)\,(1/x)dx = 0$ as $dz = pdx + qdy \Rightarrow -dz + pdx + qdy = 0$

or $d(xp - z) - d(yq) + 2qdy - 2(xp - z)\,(1/x)dx = 0$

or $d(xp - yq - z) + 2qy(1/x)dx - 2(xp - z)\,(1/x)dx = 0$, as from (4), $dy = (y/x)dx$

or $d(xp - yq - z) - 2\,(xp - yq - z)\,(1/x)\,dx = 0$ or $\dfrac{d(xp - yq - z)}{xp - yq - z} - \dfrac{2dx}{x} = 0.$

Integrating, $\log(xp - yq - z) - 2\log x = \log c_1$ or $(xp - yq - z)/x^2 = c_1.$ ...(6)

From (4), $(1/y)dy - (1/x)dx = 0$ so that $\log y - \log x = \log c_2$

or $y/x = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), an intermediate integral is

$(xp - yq - z)/x^2 = \phi_1(y/x)$ or $xp - yq = z + x^2\phi_1(y/x).$ ...(8)

Lagrange's auxiliary equations for (8) are $\dfrac{dx}{x} = \dfrac{dy}{-y} = \dfrac{dz}{z + x^2\phi_1(y/x)}.$ ...(9)

From the first two ratios of (9), we get

$(1/x)\,dx + (1/y)\,dy = 0$ so that $xy = c_3.$ ...(10)

Taking the second and third ratios of (9), we get

$$\frac{dz}{dy} + \frac{z}{y} = -\frac{x^2}{y}\phi_1\left(\frac{y}{x}\right) = -\frac{c_3^2}{y^3}\phi_1\left(\frac{y^2}{c_3}\right), \text{ by (10)}$$

Its I.F $= e^{(1/y)dy} = y$ and so solution is $zy = -\int \frac{c_3^2}{y^2}\phi_1\left(\frac{y^2}{c_3^2}\right)dy + c_4$

or $zy + \dfrac{c_3^{3/2}}{2}\int\left(\frac{c_3}{y^2}\right)\phi_1\left(\frac{y^2}{c_3}\right)\left(\frac{\sqrt{c_3}}{y}\right)d\left(\frac{y^2}{c_3}\right) = c_4$ or $zy + c_3^{3/2}\psi_1\left(\frac{y^2}{c_3}\right) = c_4$

or $zy + (xy)^{3/2}\,\psi_1(y/x) = c_4$, using (10). ...(11)

From (10) and (11), the required general solution is

$zy + (xy)^{3/2}\,\psi_1(y/x) = \psi_2(xy)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex. 6. *Solve* $(r - t)xy - s(x^2 - y^2) = qx - py$.

Sol. Given $xyr - (x^2 - y^2)s - xyt = qx - py$. ...(1)

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $xy\,dpdy - xy\,dqdx - (qx - py)\,dxdy = 0$ ...(2)

and $xy\,(dy)^2 + (x^2 - y^2)\,dxdy - xy\,(dx)^2 = 0.$ ...(3)

Now, $(3) \Rightarrow$ $(xdx + ydy)\,(xdy - ydx) = 0$

Hence, $xdx + ydy = 0$ *i.e.*, $xdx = -ydy$ ...(4)

and $xdy - ydx = 0$ ...(5)

Re–writing (2), $(xdp)\,(ydy) - ydq(xdx) - qdy(xdx) + pdx\,(ydy) = 0$

or $(xdp)\,(ydy) - ydq(-ydy) - qdy(-ydy) + pdx(ydy) = 0$, using (4)

or $xdp + ydq + qdy + pdx = 0$ or $d(xp) + d(yq) = 0.$

Integrating, $xp + yq = c_1$, c_1 being an arbitrary constant ...(6)

Integrating (4) $x^2/2 + y^2/2 = c_2/2$ or $x^2 + y^2 = c_2.$ ...(7)

From (6) and (7), are intremediate integral is

$xp + yq = f(x^2 + y^2)$, f being an arbitrary function. ...(8)

Lagrange's subsidiary equations for (8) are $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{f(x^2 + y^2)}.$ ...(9)

Taking the first and second fractions of (9), $(1/y)dy - (1/x)dx = 0.$

Integrating, $\log y - \log x = \log a$ or $y/x = a,$ ...(10)

where a is an arbitrary constant.

Taking the first and third fraction of (9), we get

$\dfrac{dx}{x} = \dfrac{dz}{f(x^2 + y^2)}$ or $\dfrac{dx}{x} = \dfrac{dz}{f(x^2 + a^2x^2)}$, using (10)

or $dz = (1/x) \times f[x^2\,(1 + a^2)]\,dx = (1/x^2) \times f[x^2\,(1 + a^2)]\,xdx.$ ...(11)

Putting $x^2(1 + a^2) = v$ and $2x(1 + a^2)dx = dv,$ (11) gives

$$dz = \frac{1 + a^2}{v}\ f(v) \times \frac{1}{2(1 + a^2)}dv = \left(\frac{1}{2v}\right) f(v)dv.$$

Integrating, $z = F(v) + b$ or $z - F[x^2\,(1 + a^2)] = b$

or $z - F(x^2 + x^2a^2) = b$ or $z - F(x^2 + y^2) = b$, using (10). ...(12)

Here b is an arbitrary constant. From (10) and (12), general solution of (1) is

$z - F(x^2 + y^2) = G(y/x)$ or $z = F(x^2 + y^2) + G(y/x),$

where F and G are arbitrary functions.

Ex. 7. *Solve* $2xr - (x + 2y)s + yt = [(x + 2y)\,(2p - q)]/(x - 2y)$

Sol. Comparing the given equation with $Rr + Ss + Tr = V$, we have

$R = 2x,$ $S = -(x + 2y),$ $T = y,$ $V = [(x - 2y)\,(2p - q)]/(x - 2y).$

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $2xdpdy + ydqdx - \dfrac{x + 2y}{x - 2y}\,(2p - q)\,dxdy = 0$ ...(1)

and $2x(dy)^2 + (x + 2y)dxdy + y(dx)^2 = 0.$ ...(2)

The equation (2) can be resolved into the following two equations

$xdy + ydx = 0$ *i.e.*, $xdy = -ydx$ ...(3)

and $dx + 2ydy = 0.$ ...(4)

Re–writing (1), $2dp(xdy) + dq(ydx) - \frac{2p-q}{x-2y}[(xdy)dx + 2(ydx)dy]$

or $2dp(-ydx) + dq(ydx) - \frac{2p-q}{x-2y}\{(-ydx)dx + 2(ydx)dy\} = 0$ using (3)

or $-2dp + dq - \frac{2p-q}{x-2y}(-dx + 2dy) = 0$ or $\frac{2dp-dq}{2p-q} - \frac{dx-2dy}{x-2y} = 0.$

Intergrating, $\log(2p - q) - \log(x - 2y) = \log c_1$ or $(2p - q)/(x - 2y) = c_1.$ ...(5)

Re–writing (3), $(1/y)dy + (1/x)dx = 0$ so that $\log x + \log y = \log c_2$

$\therefore$ $xy = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), an intermediate integral is

$(2p - q)/(x - 2y) = f(xy)$ or $2p - q = (x - 2y)\, f(xy),$ ...(7)

where f is an arbitrary function. The equation (7) is of Lagrange's form $Pp + Qq = R$. So Lagrange's, subsidiary equation for (7) are

$$\frac{dx}{2} = \frac{dy}{-1} = \frac{dz}{(x-2y)\,f(xy)}. \quad ...(8)$$

Taking the first and second fractions of (8), $dx + 2dy = 0.$

Integrating, $x + 2y = a$, a being an arbitrary constant ...(9)

Taking $y\,f(xy)$, $x\,f(xy)$, 1 as multipliers, each fraction of (8)

$$= \frac{y\,f(xy)\,dx + x\,f(xy)\,dy + dz}{2y\,f(xy) - x\,f(xy) + (x-2y)\,f(xy)} = \frac{f(xy)(ydx + x\,dy) + dz}{0}$$

This $\Rightarrow$ $f(xy)\,d(xy) + dz,$ as $ydx + xdy = d(xy)$

Integrating, $F(xy) + z = b$, b being an arbitrary constant. ...(10)

From (9) and (10), the required complete integral is

$F(xy) + z = G(x + y)$, F and G being arbitrary functions.

Ex. 8. *Solve $xr + (x + y)s + yt + p + q = 0$ by Monge's method.*

Sol. Given $xr + (x + y)s + yt = -(p + q)$ ...(1)

Comparing (1) with $Rr + Sr + Tt = V$, here $R = x$, $S = x + y$, $T = y$ and $V = -(p + q)$. Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$xdpdy + ydqdx + (p + q)dxdy = 0$ ...(2)

and $x(dy)^2 - (x + y)dxdy + y(dx)^2 = 0$ ...(3)

Re–writing (3), $(xdy - ydx)\,(dy - dx) = 0$

so that $xdy - ydx = 0$ ...(4)

and $dy - dx = 0$ *i.e.*, $dy = dx$ ...(5)

For the required solution, we consider relation (5) only.

Integrating (5), $x - y = c_1$, being an arbitrary constant ... (6)

Using (5), (2) becomes $xdpdx + ydqdx + (p + q)(dx)^2 = 0$

or $xdp + ydq + pdx + qdx = 0$, on dividing by dx (as $dx \neq 0$)

or $(xdp + pdx) + (ydq + qdx) = 0$ or $(xdp + pdx) + (ydq + qdy) = 0$ by(5)

or $d(xp) + d(yq) = 0$ so that $xp + yq = c_2.$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$$xp + yq = f(x - y), f \text{ being an arbitrary function} \quad ...(8)$$

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{f(x-y)} \quad ...(9)$$

Taking the first two fractions of (9), $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x - \log y = \log c_3$ or $x/y = c_3$ ...(10)

Now, each fraction of (9) $= \dfrac{dx - dy}{x - y} = \dfrac{d(x-y)}{x-y}$ ...(11)

Combining this fraction with last fraction of (9), we get

$$\frac{dz}{f(x-y)} = \frac{d(x-y)}{x-y} \quad \text{or} \quad dz = \frac{f(x-y)}{x-y}d(x-y) = \frac{f(u)du}{u}, \text{ if } u = x - y$$

Integrating, $z = F(u) + c_4 = F(x-y) + c_4$, where $F(u) = \int \frac{1}{u} f(u)\, du$

or $z - F(x - y) = c_4$, c_4 being an arbitrary constant ...(12)

From (10) and (12), the required solution is

$z - F(x-y) = G(x/y)$ or $z = G(x/y) + F(x-y)$,

where F and G are arbitrary functions.

Ex. 9. *Solve* $rq^2 - 2pqs + p^2t = pt - qs$ *by Monge's method.* **[Delhi Maths (Hons) 2002]**

Sol. Given $q^2r - q(2p-1)s + p(p-1)t = 0$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = q^2$, $S = -q(2p-1)$, $T = p(p-1)$, $V = 0$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\, dxdy + T(dx)^2 = 0$ become

$$q^2dpdy + p(p-1)dqdx = 0 \quad ...(2)$$

and $$q^2(dy)^2 + q(2p-1)dxdy + p(p-1)(dx)^2 = 0 \quad ...(3)$$

Re–writing (3), $(qdy + pdx)\{qdy + (p-1)dx\} = 0$

so that $qdy + pdx = 0$ *i.e.*, $qdy = -pdx$ ...(4)

and $$qdy + (p-1)dx = 0 \quad ...(5)$$

For the required solution, we consider relation (4) only.

Since $dz = pdx + qdy$, (4) reduces to $dz = 0$ and so $z = c_1$ ...(6)

Re–writing (2), $(qdp)(qdy) + (p-1)dq(pdx) = 0$

or $(qdp)(-pdx) + (p-1)dq(pdx) = 0$, since from (4), $qdy = -pdx$

or $-qdp + (p-1)dq = 0$ or $\{1/(p-1)\}dp - \{1/q\}dq = 0$

Integrating, $\log(p-1) - \log q = \log c_2$ or $(p-1)/q = c_2$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$(p-1)/q = f(z)$ or $p - qf(z) = 1$, ...(8)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{1} \quad ...(9)$$

From the first and the last fractions of (9), $dx - dz = 0$ so that $x - z = c_3$ ...(10)

From the last two fractions of (9), $dy - f(z)dz = 0$

Integrating, $y - F(z) = c_4$, where $F(z) = \int f(z)dz$ ...(11)

From (10) and (11), the required solution is $y - F(z) = G(x - z)$

or $y = F(z) + G(x - z)$, where F, G are arbitrary functions.

Ex. 10. *Solve* $e^{2y}(r - p) = e^{2x}(t - q)$ *by Monge's method.*

Sol. Given $e^{2y}r - e^{2x}t = pe^{2y} - qe^{2x}$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = e^{2y}$, $S = 0$, $T = -e^{2x}$ and $V = pe^{2y} - qe^{2x}$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $e^{2y}dxdy - e^{2x}dqdx - (pe^{2y} - qe^{2x})dxdy = 0$ ...(2)

and $e^{2y}(dy)^2 - e^{2x}(dx)^2 = 0$ ...(3)

From (3), $(e^y dy - e^x dx)(e^y dy + e^x dx) = 0$

so that $e^y dy - e^x dx = 0$, that is, $e^x dx = e^y dy$ ...(4)

and $e^y dy + e^x dx = 0$ ...(5)

For the required solution, we consider relation (4) only.

Integrating (4), $e^x - e^y = c_1$, c_1 being arbitrary constant ...(6)

Rewriting (2), $(e^y dp)(e^y dy) - (e^x dq)(e^x dx) - p(e^y dy)(e^y dx) + q(e^x dx)(e^x dy) = 0$

or $(e^y dp)(e^x dx) - (e^x dq)(e^x dx) - p(e^x dx)(e^y dx) + q(x^x dx)(e^y dy) = 0$, by (4)

or $e^y dp - e^x dq - pe^y dx + qe^x dy = 0$ or $\{d(e^y p) - pe^y dy\} - \{d(e^x q) - qe^x dx\} = pe^y dx - qe^x dy$

or $d(e^y p) - d(e^x q) = pe^y(dx + dy) - qe^x(dx + dy)$ or $d(e^y p - e^x q) = (e^y p - e^x q)(dx + dy)$

or $$\frac{d(e^y p - e^x q)}{e^y p - e^x q} = d(x + y)$$

Integrating, $\log(e^y p - e^x q) - \log c_2 = x + y$ or $(e^y p - e^x q)/c_2 = e^{x+y}$

or $(e^y p - e^x q)/e^{x+y} = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), one intermediate integral of (1) is

$(e^y p - e^x q)/e^{x+y} = f(e^x - e^y)$ or $e^y p - e^x q = e^{x+y} f(e^x - e^y)$

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{e^y} = \frac{dy}{-e^x} = \frac{dz}{e^{x+y} f(e^x - e^y)} \quad ...(8)$$

From the first two fractions of (8), $e^x dx + e^y dy = 0$ so that $e^x + e^y = c_3$ ...(9)

Taking the first and third fraction of (8) and noting that $e^y = c_3 - e^x$ from (9), we get

$$\frac{dx}{e^y} = \frac{dz}{e^x e^y f(e^x - c_3 + e^x)}$$ or $dz = e^x f(2e^x - c_3)dx$

or $dz - (1/2) \times f(2e^x - c_3)d(2e^x - c_3) = 0$ or $dz - (1/2) \times f(u)du = 0$, taking $u = 2e^x - c_3$

Integrating, $z - F(u) = c_3$, where $F(u) = \int (1/2) \times f(u)du$

or $z - F(2e^x - c_3) = c_4$ or $z - F(e^x - e^y) = c_4$, by (9) ...(10)

From (9) and (10), the required solution is $z - F(e^x - e^y) = G(e^x + e^y)$

or $z = F(e^x - e^y) + G(e^x + e^y)$, where F, G are arbitrary functions.

Ex. 11. *Solve $x^2r - y^2t = xp - yq$ by Monge's method.*

Sol. Given $x^2r - y^2t = xp - yq$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = x^2$, $S = 0$, $T = -y^2$ and $V = xp - yq$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$x^2dpdy - y^2dqdx - (xp - yq)dxdy = 0 \quad ...(2)$$

and $$x^2(dy)^2 - y^2(dx)^2 = 0 \quad ...(3)$$

Re–writing (3), $(xdy - ydx)(xdy + ydx) = 0$

so that $xdy - ydx = 0$ that is, $xdy = ydx$ ...(4)

and $xdy + ydx = 0$ ...(5)

From (4), $(1/y)dy - (1/x)dx = 0$ so that $y/x = c_1$ ...(6)

For the required solution, we consider relation (4) only.

Re–writing (2), $(xdp)(xdy) - (ydq)(ydx) - (pdx)(xdy) + (qdy)(ydx) = 0$

or $(xdp)(ydx) - (ydq)(ydx) - (pdx)(ydx) + (qdy)(ydx) = 0$, by (4)

or $xdp - ydq - pdx + qdy = 0$ or $\{d(xp) - pdx\} - \{d(yq) - qdy\} - pdx + qdy = 0$

or $d(xp - yq) - 2pdx + 2qdy = 0$ or $d(xp - yq) - 2p\,dx + 2(y/x)dx = 0$, by (4)

or $d(xp - yq) - (2/x)(xp - yq)dx = 0$ or $\dfrac{d(xp - yq)}{xp - yq} - \dfrac{2dx}{x} = 0$

Integrating, $\log(xp - yq) - 2\log x = c_2$ or $(xp - yq)/x^2 = c_2$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$(xp - yq)/x^2 = f(y/x)$ or $xp - yq = x^2 f(x/y)$ ...(8)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{x^2 f(y/x)} \quad ...(9)$$

Taking the first two ratios of (9), $(1/x)\,dx + (1/y)\,dy = 0$ so that $\log x + \log y = c_3$

or $xy = c_3$, c_3 being an arbitrary constant ...(10)

Taking the first and last fractions of (9), we get

$dz = x\,f(y/x)dx$ or $dz = x\,f(c_3/x^2)$, since by (10), $y = c_3/x$

$$\therefore \quad z = \int \left(-\frac{x^4}{2c_3}\right) f\left(\frac{c_3}{x^2}\right)\left(-\frac{2c_3}{x^3}\right)dx = \int \left(-\frac{c_3^2}{2c_3t^2}\right) f(t)dt$$, putting $\dfrac{c_3}{x^2} = t$ and $-\dfrac{2c_3}{x^3}dx = dt$

or $z = -\dfrac{c_3}{2}\int \dfrac{f(t)}{t^2}dt + c_4 = c_3F(t) + c_4$, where $F(t) = -\dfrac{1}{2}\int \dfrac{f(t)}{t^2}dt$

or $z - c_3F(c_3/x^2) = c_4$ or $z - xyF(y/x) = c_4$, by (10) ... (11)

From (10) and (11), the required solution is

$z - xy\,F(y/x) = G(xy)$ or $z = x^2(y/x)\,F\,(y/x) + G(xy)$

or $z = x^2H(y/x) + G(xy)$ where $H(y/x) = (y/x)\;F(y/x)$ and H, G are arbitrary functions.

Ex. 12. *Solve* $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$.

and hence find the surface satisfying the above equation and touching the hyperbolic paraboloid $z = x^2 - y^2$ *along its section by the plane* $y = 1$. **[Meerut 2001, I.A.S. 1978, Ranchi 2010]**

Sol. Given $2x^2r - 5xys + 2y^2t = -2(px + qy)$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = 2x^2$, $S = -5xy$, $T = 2y^2$, $V = -2(px + qy)$

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$.

become $2x^2dpdy + 2y^2dqdx + 2(px + qy)dxdy = 0$. ...(2)

and $2x^2(dy)^2 + 5xydxdy + 2y^2(dx)^2 = 0$. ...(3)

Re–writing (3), $(xdy + 2ydy)\,(2xdy + ydx) = 0$.

so that $xdy + 2ydx = 0$, *i.e.*, $xdy = -2ydx$ ...(4)

and $2xdy + ydx = 0$. ...(5)

Keeping (4) in view, (2) may be re–written as

$2xdp(xdy) - ydq\,(-2ydx) + 2pdx\,(xdy) - qdy\,(-2ydx) = 0$.

or $2xdp(xdy) - ydq\,(xdy) + 2pdx\,(xdy) - qdy\,(xdy) = 0$, using (4)

or $2xdp - ydq + 2pdx - qdy = 0$ or $2(xdp + pdx) - (ydq + qdy) = 0$

or $2d(xp) - d(yq) = 0$ so that $2xp - yq = c_1$. ...(6)

From (4), $(1/y)dy + 2(1/x)dx = 0$ so that $\log y + 2\log x = \log c_2$

or $\log y + \log x^2 = \log c_2$ or $x^2y = c_2$. ...(7)

From (6) and (7), one intermediate integral is

$2xp - yq = f(x^2y)$, f being an arbitrary function. ...(8)

which is of Lagrange's form. Hence Lagrange's subsidiary equations are

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{f(x^2y)}. \qquad ...(9)$$

Taking the first two fractions of (9), $2(1/y)dy + (1/x)dx = 0$.

Integrating, $2\log y + \log x = \log a$ or $y^2x = a$ or $x = a/y^2$. ...(10)

Taking the second and third fractions of (9) and using (10), we get

$$\frac{dy}{-y} = \frac{dz}{f(a^2/y^3)} \quad \text{or} \quad dz + \frac{1}{y} f\left(\frac{a^2}{y^3}\right) dy = 0. \qquad ...(11)$$

Putting $(a^2/y^3) = v$ so that $-(3a^2/y^4)\,dy = dv$, (11) gives

$$dz + \frac{1}{y} f(v) \times \left(-\frac{y^4}{3a^2}\right) dv = 0 \quad \text{or} \quad dz - \frac{f(v)}{3(a^2/y^3)}\, dv = 0$$

or $dz - (1/3v) \times f(v)\,dv = 0$, as $v = a^2/y^3$.

Integrating, $z - F(v) = b$ or $z - F(a^2/y^3) = b$, b being an arbitrary constant.

or $z - F(x^2y) = b$, as $y^2x = a$. ...(12)

From (10) and (12), the required complete solution is

$z - F(x^2y) = G(xy^2)$, F and G being arbitrary fucntions.

or $z = F(x^2y) + G(xy^2)$. ...(13)

Second Part. The given surface is $z = x^2 - y^2$. ...(14)

(13) $\Rightarrow$ $p = \partial z/\partial x = 2xy\,F'(x^2y) + y^2G'(xy^2)$ and $q = \partial z/\partial y = x^2F'(x^2y) + 2xyG'(xy^2)$. ...(15)

From (14), $p = \partial z/\partial x = 2x$ and $q = \partial x/\partial y = -2y$. ...(16)

Since (13) and (14) touch each other along their section by the plane $y = 1$, the values of p and q given by (15) and (16) at any point on $y = 1$ must be equal

Thus, $2xyF'(x^2y) + y^2G'(xy^2) = 2x$, where $y = 1$ ...(17)

and $x^2F'(x^2y) + 2xy\ G'(xy^2) = -2y$, where $y = 1$. ...(18)

From (17), $2xF'(x^2) + G'(x) = 2x$. ...(19)

From (18), $x^2F'(x^2) + 2xG'(x) = -2$. ...(20)

Solving (19) and (20) for $F'(x^2)$ and $G'(x)$, we have

$$F'(x^2) = (4/3) + (2/3) \times (1/x^2). \quad ...(21)$$

and $$G'(x) = -(2/3) \times x - (4/3) \times (1/x). \quad ...(22)$$

(21) $\Rightarrow$ $F'(u) = (4/3) + (2/3) \times (1/u)$, on putting $x^2 = u$

Integrating, $F(u) = (4/3) \times u + (2/3) \times \log u + c_1$, c_1 being an arbitrary constant

This $\Rightarrow$ $F(x^2y) = (4/3) \times x^2y + (2/3) \times \log (x^2y) + c_1$. ...(23)

Integrating (22), $G(x) = -(2/3)(x^2/2) - (4/3) \log x + c_2$, being an arbitrary constant

This $\Rightarrow$ $G(xy^2) = -(1/3) \times x^2y^4 - (4/3) \times \log (xy^2) + c_2$. ...(24)

Putting values of $F(x^2y)$ and $G(xy^2)$ given by (23) and (24) in (13), we get

$$z = (4/3) \times x^2y + (2/3) \times \log (x^2y) + c_1 - (1/3) \times x^2y^4 - (4/3) \times \log (xy^2) + c_2$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log (x^2y) - 2 \log (xy^2)] + c, \text{ taking } c_1 + c_2 = c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log (x^2y) - \log (xy^2)^2]$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log \{(x^2y)/(x^2y^4)\} + c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times \log y^{-3} + c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y + c. \quad ...(25)$$

Now at the point of contact of (14) and (25), the values of z must be the same and hence

$$x^2 - y^2 = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y + c, \text{ where } y = 1$$

$\Rightarrow$ $x^2 - 1 = (4/3) \times x^2 - (1/3) \times x^2 + c$, putting $y = 1$

$\Rightarrow$ $x^2 - 1 = x^2 + c$ $\Rightarrow$ $c = -1$.

Putting $c = -1$ in (25), the required surface is

$z = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y - 1$ or $3z = 4x^2y - x^2y^4 - 6 \log y - 3$.

9.7. Type 3. When the given equation Rr + Ss + Tt = V leads to two identical intermediate intergrals.

Working rule for solving problems of type 3

Step 1. Write the given equation in the standard form $Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rpdy + Tdqdx - Vdx\,dy = 0$ ... (1) $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$ ... (2)

Step 3. R.H.S. of (2) reduces to a perfect square and hence it gives only one distinct factor in place of two as in type 1 and type 2.

Step 4. Start with the only one factor of step 3 and use (2) to get an intermediate integral.

Step 5. Re–write the intermediate integral of the step 4 in the form of $Pp + Qq = R$ and use Lagrange's method to obtain the required general solution of the given equation.

9.8. Solved examples based on Art 9.7

Ex. 1. *Solve* : $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$

[Meerut 2002, Delhi Maths (H) 1999 2007, 10; Rohailkhand 1997; Kanpur 1994]

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, ...(1)

$R = (1 + q)^2$, $S = -2(1 + p + q + pq)$, $T = (1 + p)^2$, $V = 0$. ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$. ...(4)

Using (2), (3) and (4) become

$$(1 + q)^2 dpdy + (1 + p)^2 dqdx = 0 \quad ...(5)$$

and $(1 + q)^2(dy)^2 + 2(1 + p + q + pq)dxdy + (1 + p)^2 (dx)^2 = 0$. ...(6)

Since $1 + p + q + pq = (1 + p)(1 + q)$, (6) becomes $[(1 + q)dy + (1 + p)dx]^2 = 0$

so that $(1 + q)dy + (1 + p)dx = 0$ or $(1 + q)dy = -(1 + p)dx$. ...(7)

Keeping (7) in view, (5) may be re–written as

$$(1 + q)dp \{(1 + q)dy\} - (1 + p)dq \{-(1 + p)dx\} = 0. \quad ...(8)$$

Dividing each term of (8) by $(1 + q)dy$, or its equivalent $-(1 + p)dx$, we get

$(1 + q)dp - (1 + p)dq = 0$ or $dp/(1 + p) - dq/(1 + q) = 0$.

Integrating it, $(1 + p)/(1 + q) = c_1$, c_1 being an arbitrary constant ...(9)

From (7), $dx + dy + pdx + qdy = 0$ or $dx + dy + dz = 0$, as $dz = pdx + qdy$

Integrating it, $x + y + z = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one intermediate integral of (1) is

$(1 + p)/(1 + q) = F(x + y + z)$ or $1 + p = (1 + q) F(x + y + z)$

or $p - q F(x + y + z) = F(x + y + z) - 1$, ...(11)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-F(x+y+z)} = \frac{dz}{F(x+y+z)-1} \quad ...(12)$$

Choosing 1, 1, 1 as multipliers, each fraction of (12) $= (dx + dy + dz)/0$

so that $dx + dy + dz = 0$ giving $x + y + z = c_2$ ... (13)

Using (13) and taking the first two fractions of (12), we have

$dx = -dy/F(c_2)$ or $dy + F(c_2)dx = 0$.

Integrating it, $y + xF(c_2) = c_3$ or $y + x F(x + y + z) = c_3$ ...(14)

From (13) and (14), the required general solution is

$y + x F(x + y + z) = G(x + y + z)$, F, G being arbitrary functions.

Ex. 2. *Solve* $y^2r + 2xys + x^2t + px + qy = 0$. **[Bilaspur 2004]**

Sol. Given $y^2r + 2xys + x^2t = -(px + qy)$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = y^2$, $S = 2xy$, $T = x^2$, $V = -(px + qy)$. ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx + Vdxdy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$. ...(4)

Using (2), (3) and (4) become

$$y^2 dpdy + x^2 dqdx + (px + qy)\, dxdy = 0 \quad ...(5)$$

and $y^2(dy)^2 - 2xydxdy + x^2(dx)^2 = 0$. ...(6)

From (6), $(xdx - ydy)^2 = 0$ so that $xdx - ydy = 0$ or $xdx = ydy$. ...(7)

Keeping (7) in view, (5) may be re–written as

$$ydp\,(ydy) + xdq\,(xdx) + pdy\,(xdx) + qdx\,(ydy) = 0. \quad ...(8)$$

Dividing each term of (8) by xdx, or its equivalent ydy, we get

$ydp + xdq + pdy + qdx = 0$ or $(ydp + pdy) + (xdq + qdx) = 0$

Integrating it, $yp + xq = c_1$, being an arbitrary constant ...(9)

Integrating (7), $x^2/2 - y^2/2 = c_2/2$ or $x^2 - y^2 = c_2$. ...(10)

From (9) and (10), one intermediate integral of (1) is $yp + xq = F(x^2 - y^2)$, ...(11)
which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{F(x^2 - y^2)}. \qquad ...(12)$$

From the first two fractions of (2), $xdx - ydy = 0$ so that $x^2 - y^2 = c_2$. ...(13)
Taking the last two fractions and using (13), we get

$$\frac{dy}{(y^2 + c_2)^{1/2}} = \frac{dz}{F(c_2)} \quad \text{or} \quad dz - F(c_2)\frac{dy}{(y^2 + c_2)^{1/2}} = 0.$$

Integrating, $z - F(c_2) \log [y + (y^2 + c_2)^{1/2}] = c_3$

or $z - F(x^2 - y^2) \log [y + \sqrt{(y^2 + x^2 - y^2)}\,] = c_3$, using (13)

or $z - F(x^2 - y^2) \log (x + y) = c_3$, c_3 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is
$z - F(x^2 - y^2) \log (x + y) = G(x^2 - y^2)$, F, G being arbitrary functions.

Ex. 3(a). *Obtain the integral of $q^2r - 2pqs + p^2t = 0$ in the form $y + xf(z) = F(z)$.*
[Delhi Maths Hons. 1999, 2007; Meerut 1994, 95; Nagpur 2005]

(b) *Show also that this solution represents a surface generated by straight lines that are parallel to a fixed plane.*

Sol. (*a*) Given $q^2r - 2pqs + p^2t = 0$. ...(1)
As ususal Monge's subsidiary equations are $q^2dpdy + p^2dp\,dx = 0$ ...(2)
and $q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0$ or $(qdy + pdx)^2 = 0$. ...(3)
From (3), we have $qdy + pdx = 0$ or $qdy = -pdx$. ...(4)
In view of (4), (2) may be re–written as $qdp\,(qdy) - pdq\,(-pdx) = 0$. ...(5)
Dividing each term of (5) by qdy, or its equivalent $(-pdx)$, we find
$qdp - pdq = 0$ or $(1/p)dp - (1/q)\,dp = 0$.
Integrating it, $p/q = c_1$, c_1 being an arbitrary constant ...(6)
From (4), $dz = 0$, (as $dz = pdx + qdy$) so that $z = c_2$. ...(7)
From (6) and (7), one integral of (1) is $p/q = f(z)$ or $p - f(z)q = 0$, ...(8)
which is of the form $Pp + Qq = R$. Here f is an arbitrary function. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{0}. \qquad ...(9)$$

The last fraction in (9) gives $dz = 0$ so that $z = c_2$ ...(10)
From the first two fractions in (9) and (10), we find

$$\frac{dx}{1} = \frac{dy}{-f(c_2)} \quad \text{or} \quad dy + f(c_2)dx = 0.$$

Integrating, $y + xf(c_2) = c_3$ or $y + xf(z) = c_3$, by (10). ...(11)
From (10) and (11), the required integral is $y + xf(z) = \mathrm{F}(z)$. ...(12)

Part (*b*). Let $z = k$, k being an arbitrary constant. Then (12) is the locus of the straight lines given by the intersection of the planes

$$z = k \quad \text{and} \quad y + xf(k) - F(k) = 0. \qquad ...(13)$$

Clearly the lines are parallel to the plane $z = 0$ (which is a fixed plane) because these lie on the plane $z = k$ for different values of k.

Ex. 4. *Solve $y^2r - 2ys + t = p + 6y$.* **[Agra 1993; Bhopal 2004; Vikram 2004; Meerut 2009; Delhi Maths Hons 1994, 98, 2006, 09, 10]**

Sol. As usual Monge's subsidiary equations are

$$y^2dpdy + dqdx - (p + 6y)dxdy = 0 \qquad ...(1)$$

and $y^2(dy)^2 + 2ydydx + (dx)^2 = 0$ or $(ydy + dx)^2 = 0$. ...(2)

From (2), $ydy + dx = 0$ or $dx = -ydy$. ...(3)

Putting the value of dx from (3) in (1), we find

$$y^2dpdy + dq(-ydy) - (p + 6y)\,dy\,(-ydy) = 0$$

or $ydp - dq + (p + 6y)\,dy = 0$ or $(ydp + pdy) - dq + 6ydy = 0$.

Integrating it, $yp - q + 3y^2 = c_1$, c_1 being an arbitrary constant ...(4)

Integrating (4), $y^2/2 + x = c_2/2$ or $y^2 + 2x = c_2$. ...(6)

From (5) and (6), one integral of (1) is

$yp - q + 3y^2 = F(y^2 + 2x)$ or $yp - q = F(y^2 + 2x) - 3y^2$, ...(7)

which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{F(y^2+2x)-3y^2}. \qquad ...(8)$$

From the first two fractions of (8), $2ydy + 2dx = 0$ so that $y^2 + 2x = c_2$. ...(9)

Taking the last two fractions of (8) and using (9), $dz + [F(c_2) - 3y^2]dy = 0$.

Integrating, $z + yF(c_2) - y^3 = c_2$ or $z + yF(y^2 + 2x) - y^3 = c_3$. ...(10)

From (9) and (10), the required general solution is

$$z + yF(y^2 + 2x) - y^3 = G(y^2 + 2x),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 5. *Solve* $(b + cq)^2r - 2(b + cq)\,(a + cp)s + (a + cp)^2t = 0$

Sol. Usual Monge's subsidiary equations are $(b + cq)^2\,dpdy + (a + cp)^2\,dqdx = 0$. ...(1)

and $(b + cq)^2\,(dy)^2 + 2(b + cq)\,(a + cp)\,dxdy + (a + cp)^2\,(dx)^2 = 0$. ...(2)

(2) $\Rightarrow$ $\{(b + cq)dy + (a + cp)dx\}^2 = 0$ ...(3)

or $(b + cq)dy + (a + cp)dx = 0$ or $adx + bdy + c(pdx + qdy) = 0$

or $adx + bdy + cdz = 0$, as $dz = pdx + qdy$.

Integrating, $ax + by + cz = c_1$, c_3 being an arbitrary constant ...(4)

From (3), $(b + cq)dy = -(a + cp)dx$. So (1) reduces to $(b + cq)dp - (a + cp)dq = 0$

or $\frac{dp}{a+cp} - \frac{dq}{b+cq} = 0$ so that $\frac{a+cp}{b+cq} = c_2$ ...(5)

So the intermediate integral of the given equation is $(a + cp)/(b + cq) = \phi_1(ax + by + cz)$

or $cp - c\phi_1(ax + by + cz)q = -a + b\,\phi_1(ax + by + cz)$. ...(6)

Lagrange's auxiliary equations are

$$\frac{dx}{c} = \frac{dy}{-c\,\phi_1(ax+by+cz)} = \frac{dz}{-a+b\,\phi_1(ax+by+cz)}. \qquad ...(7)$$

Using a, b, c as multipliers, each fraction of (7) $= (adx + bdy + cdz)/0$

$\therefore$ $adx + bdy + cdz = 0$ so that $ax + by + cz = c_3$. ...(8)

Using (8) and taking the first two ratios of (7), we get

$dx = -dy/\phi_1(c_3)$ or $dy + \phi_1(c_3)dx = 0$.

Integrating, $y + x\phi_1(c_3) = c_4$ or $y + x\phi_1(ax + by + cz) = c_4$. ...(9)

From (8) and (9), the required solution is

$$y + x\phi_1(ax + by + cz) = \phi_2(ax + by + cz),\ \phi_1,\ \phi_2 \text{ being arbitrary functions.}$$

Ex. 6. *Solve* $x^2r - 2xs + t + q = 0$. **[K.U. Kurukshetra 2004; Ravishankar 2005]**

Sol. Usual Monge's subsidiary equations are $x^2dpdy + dqdx + qdxdy = 0$ ...(1)

and $x^2(dy)^2 + 2xdxdy + (dx)^2 = 0$. ...(2)

Now, $(2) \Rightarrow (xdy + dx)^2 = 0 \Rightarrow xdy + dx = 0$ ...(3)

$(3) \Rightarrow (dx)/x + dy = 0 \Rightarrow y + \log x = c_1$. ...(4)

Using (3), (1) reduces to $x^2 dpdy + dq\,(-x\,dy) + q(-x\,dy)dy = 0$

or $dp - \left(\frac{dq}{x} - \frac{qdx}{x^2}\right) = 0$ or $d\left(p - \frac{q}{x}\right) = 0$.

Integrating, $p - (q/x) = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), the intermediate integral of the given equation is

$p - (q/x) - \phi_1(y + \log x)$ or $xp - q = x\phi_1(y + \log x)$. ...(6)

Lagrange's auxiliary equations for (6) are $\frac{dx}{x} = \frac{dy}{-1} = \frac{dz}{x\,\phi_1(y + \log x)}$. ...(7)

Taking the first two fractions of (7), $(1/x)dx + dy = 0 \Rightarrow y + \log x = c_3$. ...(8)

Using (8), first and third fractions of (7) give $\frac{dx}{x} = \frac{dz}{x\,\phi_1(c_3)} = \Rightarrow z - x\phi_1(c_3) = c_4$

or $z - x\phi_1(y + \log x) = c_4$, c_4 being an arbitrary constant ...(9)

From (8) and (9) the required solution is

$z - x\phi_1(y + \log x) = \phi_2(y + \log x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 7. *Solve* $(y - x)\,(q^2 r - 2pqs + p^2 t) = (p + q)^2\,(p - q)$.

Sol. The usual Monge's subsidiary equations are

$(y - x)\,(q^2 dpdy + p^2 dqdx) - (p + q)^2\,(p - q)dxdy = 0$ ...(1)

and $q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0$. ...(2)

$(2) \Rightarrow (qdy + pdx)^2 = 0$ or $qdy + pdx = 0$. ...(3)

$dz = pdx + qdy$ and $(3) \Rightarrow dz = 0 \Rightarrow z = c_1$. ...(4)

Using (3), (1) reduces to $(y - x)\,(qdp - pdq) - (p^2 - q^2)\,(dx - dy) = 0$

or $q^2 d\left(\frac{p}{q}\right) - (p^2 - q^2)\,\frac{d(x-y)}{y-x} = 0$ or $\frac{d(x-y)}{x-y} + \frac{d(p/q)}{(p/q)^2 - 1} = 0$

Integrating, $\log (x - y) + \frac{1}{2}\log\frac{(p/q)-1}{(p/q)+1} = \frac{1}{2}\log c_2$ or $(x - y)^2\frac{p-q}{p+q} = c_2$. ...(5)

From (4) and (5), the intermediate integral of the given equation is

$(x - y)^2\frac{p-q}{p+q} = \phi_1(z)$ or $(x - y)^2(p - q) = (p + q)\phi_1(z)$

or $p\{(x - y)^2 - \phi_1(z)\} - q\{(x - y)^2 + \phi_1(z)\} = 0$. ...(6)

Here Lagrange's subsidiary equation for (6) are

$$\frac{dx}{(x-y)^2 - \phi_1(z)} = \frac{dy}{-\{(x-y)^2 + \phi_1(z)\}} = \frac{dz}{0}. \quad ...(7)$$

Now, the third fraction of (7) $\Rightarrow dz = 0$ so that $z = a$, ...(8)

where 'a' is an arbitrary constant.

Now, each fraction of (7) $= \frac{dx + dy}{-2\phi_1(z)} = \frac{dx - dy}{2(x-y)^2} \Rightarrow d(x + y) = -\phi_1(a)\frac{d(x-y)}{(x-y)^2}$, by (8).

Integrating it, $x + y - \phi_1(a)\,(x - y)^{-1} = b$ or $x + y - \phi_1(z)\,(x - y)^{-1} = b$, using (8). ...(9)

From (8) and (9), the required general solution is

$x + y - (x - y)^{-1}\,\phi_1(z) = \phi_2(z)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 8. *Solve* $x^2r + 2xys + y^2t = 0$. **[Meerut 2003, Garhwal 1993; Delhi Maths (H) 2001]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^2$, $S = 2xy$, $T = y^2$. Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$

become $x^2dpdy + y^2dqdx = 0$ ...(1)

and $x^2(dy)^2 - 2xydxdy + y^2(dx)^2 = 0.$ ...(2)

Now, (2) gives $(xdy - ydx)^2 = 0$ so that $xdy - ydx = 0.$ ...(3)

Re–writing (1), $(xdp)\,(xdy) + (ydx)\,(ydq) = 0$

or $(xdp)(xdy) + (xdy)\,(ydq) = 0$ [$\because$ from (3), $ydx = xdy$]

or $xdp + ydq = 0$ or $xdp + ydq + pdx + qdy = pdx + qdy$

or $d(xp) + d(yq) - dz = 0,$ as $dz = pdx + qdy$.

Integrating (1) $xp + yq - z = c_1$, c_1 being an arbitrary constant ...(4)

Now (3) gives $(1/y)dy - (1/x)dx = 0.$

Integrating, $\log y - \log x = \log c_2$ or $y/x = c_2.$ ...(5)

From (4) and (5), the intermediate integral of the given equation is

$xp + yq - z = f(y/x)$ or $xp + yq = z + f(y/x),$ ...(6)

where f is an arbitrary function. Lagrange's subsidiary equation for (6) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + f(y/x)}. \quad \text{...(7)}$$

Taking the first two fractions of (7), $(1/y)dy - (1/x)dx = 0.$

Integrating, $\log y - \log x = \log a$ so that $y/x = a.$ ...(8)

Taking the last two fractions of (7) and using (8), we get $\dfrac{dz}{z + f(a)} - \dfrac{dy}{y} = 0.$

Integrating it, $\log [z + f(a)] - \log y = \log b$, b being an arbitrary constant

so that $[z + f(a)]/y = b$ or $[z + f(y/x)]/y = b$, using (8) ...(9)

From (8) and (9), the required solution is

$[z + f(y/x)]/y = g(y/x)$ or $z = yg(y/x) - f(y/x)$, where f and g are arbitrary functions.

Ex. 9. *Solve* $r - 2s + t = \sin(2x + 3y)$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = 1$, $S = -2$, $T = 1$, $V = \sin(2x + 3y)$. So Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $dpdy + dqdx - \sin(2x + 3y)dxdy = 0.$ ...(1)

and $(dy)^2 + 2\,dxdy + (dx)^2 = 0.$ ...(2)

Now, (2) gives $(dy + dx)^2 = 0$ so that $dy + dx = 0.$ ...(3)

From (3), $dy = -dx$. Then, (1) becomes $-dpdx + dqdx + \sin(2x + 3y)dxdy = 0$

or $dp - dq + \sin(2x + 3y)dy = 0$, as $dx \neq 0.$ ...(4)

Now, integrating (3), $x + y = c_1$, c_1 being an arbitrary constant ...(5)

From (4), $dp - dq + \sin[2(x + y) + y]dy = 0$ or $dp - dq + \sin(2c_1 + y)dy = 0$, using (5).

Integrating, $p - q - \cos(2c_1 + y) = c_2$

or $p - q - \cos(2x + 3y) = c_2,$ as $c_1 = x + y$ ...(6)

From (5) and (6), an intermediate integral is

$p - q - \cos(2x + 3y) = f(x + y)$ or $p - q = \cos(2x + 3y) + f(x + y),$ ...(7)

where f is an arbitrary function. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\cos(2x+3y)+f(x+y)}. \quad ...(8)$$

Taking the first two fractions of (8), $dx + dy = 0$ so that $x + y = a$ ... (9)

Taking the last two fractions of (8) and using (9), we get

$$\frac{dy}{-1} = \frac{dz}{\cos(2a+y)+f(a)} \quad \text{or} \quad dz + [\cos(2a + y) + f(a)]dy = 0.$$

Integrating it, $z + \sin(2a + y) + y f(a) = b$, b being an arbitrary constant

or $z + \sin(2x + 3y) + y f(x + y) = b$, using (9). ...(10)

From (9) and (10) the required complete integral is

$z + \sin(2x + 3y) + y f(x + y) = g(x + y)$, f and g being an arbitrary functions.

Ex. 10. *Solve* $q^2r - 2pqs + p^2t = pq^2$. **[I.A.S. 1986]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = q^2$, $S = -2pq$, $T = p^2$, $V = pq^2$. The Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $q^2dpdy + p^2dqdx - pq^2dxdy = 0$ ...(1)

and $q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0.$ ...(2)

Re–writing (2), $(qdy + pdx)^2 = 0$ so that $pdx + qdy = 0.$ ...(3)

Since $dz = pdx + qdy$, (3) $\Rightarrow$ $dz = 0$ so that $z = c_1$. ...(4)

Re–writing (1), $(qdy)(qdp) + (pdx)(pdq) - (qdy)(pqdx) = 0$

or $(qdy)(qdp) - (qdy)(pdq) - (qdy)(pqdx) = 0$, as from (3), $pdx = -qdy$

or $qdp - pdq - pqdx = 0$ or $(1/p)dp - (1/q)dq = dx.$

Integrating, $\log p - \log q - \log c_2 = x$ or $p/(c_2q) = e^x$

or $(p/q)e^{-x} = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), the intermediate integral of the given equation is

$(p/q)e^{-x} = f(z)$ or $px^{-x} - f(z)q = 0.$ ...(6)

Lagrange's auxiliary equations for (6) are $\frac{dx}{e^{-x}} = \frac{dy}{-f(z)} = \frac{dz}{0}.$ ...(7)

The last fraction of (7) $\Rightarrow$ $dz = 0$ so that $z = a.$ ...(8)

Taking the first fractions of (7) and using (8), we get

$$\frac{dx}{e^{-x}} = \frac{dy}{-f(a)} \quad \text{or} \quad e^x f(a)dx + dy = 0.$$

Integrating, $e^x f(a) + y = b$ or $e^x f(z) + y = b$, as from (8), $a = z$ ...(9)

From (8) and (9), the required complete integral is

$e^x f(z) + y = g(z)$, where f and g are arbitrary functions.

Ex. 11. *Solve* $q^2r - 2q(1+p)s + (1+p)^2t = 0$ *by Monge's method.*

Sol. Given $q^2r - 2q(1+p)s + (1+p)^2t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = q^2$, $S = -2q(1+p)$ and $T = (1+p)^2$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$q^2 dpdy + (1+p)^2\ dqdx = 0 \qquad \ldots (2)$$

and $$q^2 (dy)^2 + 2q(1+p)dxdy + (1+p)^2\ (dx)^2 = 0 \qquad \ldots (3)$$

Rewriting (3), $\{qdy + (1+p)dx\}^2 = 0$ or $qdy + (1+p)\,dx = 0$ ... (4)

From (4), $dx + (pdx + qdy) = 0$ or $dx + dz = 0$, as $dz = pdx + qdy$

Integrating, $x + z = C_1$, C_1 being an arbitrary constant ... (5)

Re–writing (2), $(qdy)\ (qdp) + [(1+p)\ dx] \times \{(1+p)dq\} = 0$

or $(qdy)\ (qdp) + (-qdy)\ [(1+p)\ dq] = 0$, using (4)

or $qdp - (1+p)\,dq = 0$ or $\{1/(1+p)\}dp - (1/q)dq = 0$

Integrating, $\log(1+p) - \log q = \log C_2$ or $(1+p)/q = C_2$ ... (6)

From (5) and (6), the intermediate integral of (1) is

$(1+p)/q = f(x+z)$ or $p - q\,f(x+z) = -1$ ... (7)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(x+z)} = \frac{dz}{-1} \qquad \ldots (8)$$

Taking the first and last ratios, $dx + dz = 0 \Rightarrow x + z = C_3$ ... (9)

Using (9) and taking the first two ratios of (8), we get

$dy + f(C_3)dx = 0$ so that $y + xF(C_3) = C_4$

or $y + x\ f(x+z) = C_4$, using (9) ... (10)

From (9) and (10), the required general solution is

$y + xf(x+z) = g(x+z)$, f, g are arbitrary functions

Ex. 12. *Solve* $(x-y)\ (x^2 - 2xys + y^2 t) = 2xy\ (p-q)$. **[Delhi B.Sc. (Hons) 2011]**

Sol. Given $x^2(x-y)r - 2xy(x-y)s + y^2(x-y)t = 2xy(p-q)$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = x^2(x-y), S = -2xy\ (x-y), T = y^2\ (x-y)$ and $V = 2xy\ (p-q)$. Hence Monge's subsidiars equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dyxy + T(dx)^2 = 0$ become

$$x^2(x-y)\ dpdy - 2xy\ (p-q)\ dxdy + y^2(x-y)dqdx = 0 \qquad \ldots (2)$$

and $$(x-y)\{x^2(dy)^2 + 2xy\ dx\ dy + y^2(dy)^2\} = 0 \qquad \ldots (3)$$

Since $x \neq y$, (3) gives $(xdy + ydx)^2 = 0$ so that $ydx = -xdy$ ... (4)

From (4), $(1/x)dx + (1/y)dy = 0$ so that $xy = C_1$ ... (5)

Re–writing (2), $x(x-y)\ dp\ (xdy) - 2(p-q)(xdy)\ (ydx) + y(x-y)\,dq(ydx) = 0$

or $x(x-y)\,dp\,(xdy) - 2(p-q)\ (xdy)\ (ydx) + y(x-y)dq(-xdy)$, *by* (4)

or $x(x-y)dp - 2\ (p-q)\ (ydx) - y(x-y)\ dq = 0$

or $(x-y)\,(xdp-ydq)=2y\,(p-q)dx$ or $xdp-ydq=\{2y(p-q)dx\}/(x-y)$

or $(xdp+pdx)-(ydq+qdy)=\{2y(p-q)\,dx\}/(x-y)+pdx-qdy$

or $d(xp)-d(yq)=\{2(p-q)\,ydx+(x-y)p\,dx-(x-y)\,qdy\}/(x-y)$

or $(x-y)d(xp-yq)=2pydx-2qydx+xpdx-ypdx-xqdy+yqdy$

$= pydx-2qydx+xpdx+qydx+yqdy\ =\ -pxdy-qydx+xpdx+yqdy,\ by(4)$

$\therefore\quad (x-y)\,d(xp-yq)=xp(dx-dy)-yq(dx-dy)=(xp-yq)(dx-dy)$

or $\dfrac{d(xp-yq)}{xp-yq}=\dfrac{dx-dy}{x-y}$ or $\dfrac{d(xp-yq)}{xp-yq}-\dfrac{d(x-y)}{x-y}=0.$

Integrating, $\log(xp-yq)-\log(x-y)=\log C_2$ or $(xp-yq)/(x-y)=C_2$ ... (6)

From (5) and (6), the intermediate integral of the given equation is

$(xp-yq)/(x-y)=f(xy)$ or $xp-yq=(x-y)f(xy),$ ... (7)

which is of Lagrange's form. Its auxiliary equations are $\dfrac{dx}{x}=\dfrac{dy}{-y}=\dfrac{dz}{(x-y)f(xy)}$ ... (8)

Taking the first two fractions, $(1/x)dx+(1/y)dy=0$ so that $xy=C_3$ ... (9)

Now, each fraction of (8) $=\dfrac{dx+dy}{x-y}=\dfrac{dz}{(x-y)f(xy)}$

or $dz=f(xy)\,d(x+y)$ or $dz=f(C_3)\,d(x+y)$, by (9)

Integrating, $z-(x+y)\,f(C_3)=C_4$ or $z-(x+y)\,f(xy)=C_4$ ... (10)

From (9) and (10), the required solution is $z-(x+y)f(xy)=g(xy)$

or $z=(x+y)f(xy)+g(xy)$, f and g being arbitrary functions.

9.9 Type 4. When the given equation Rr + Ss + Tt = V fails to yield an intermediate integral as in cases 1, 2 and 3.

Working rule for solving problems of type 4.

Suppose the R.H.S. of $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ neither gives two factors nor a perfect square (as in Types 1, 2 and 3 above). In such cases factors dx, dy, p, $1 + p$ etc. are cancelled as the case may be and an integral of given equation is obtained as usual. This integral is then integrated by methods explained in chapter 7.

9.10 SOLVED EXAMPLES BASED ON ART 9.9

Ex. 1. *Solve* $(q + 1)s = (p + 1)t$. **[Agra 2009]**

Sol. Given $(q + 1)s - (p + 1)t = 0.$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find $R = 0$, $S = (q + 1)$. $T = -(p + 1)$, $V = 0$....(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0.$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0.$ ...(4)

Using (2), (3) and (4) become $-(p + 1)dqdx = 0$ ...(5)

and $-(q + 1)dxdy - (p + 1)(dx)^2 = 0.$ ...(6)

Dividing (5) by $-(p + 1)dx$, we obtain $dq = 0.$ ...(7)

and dividing (6) by $-dx$ we get $(q + 1) + (p + 1)dx = 0.$ ...(8)

From (8), $dx + dy + pdx + qdy = 0$ or $dx + dy + dz = 0$, as $dz = pdx + qdy$

Integrating it, $x + y + z = c_1$, being an arbitrary constant ...(9)

Integrating (7), $q = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), an integral of (1) is

$q = f(x + y + z)$ or $\partial z/\partial y = f(x + y + z)$ ...(11)

Integrating (11) partially w.r.t. y (treating x as constant), we find

$z = F(x + y + z) + G(x)$, F, G being arbitrary functions.

Ex. 2. *Solve* $pq = x(ps - qr)$. **[Delhi. Maths (H) 2002, 08]**

Sol. Given $xqr - xps + 0.t = -pq$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = xq$, $S = xp$, $T = o$ and $V = -pq$

Monge's subsidiary equations $Rdp\,dy + T\,dq\,dx - V\,dx\,dy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $xqdpdy + pqdxdy = 0$. ...(2)

and $xq(dy)^2 + xpdxdy = 0$. ...(3)

Dividing (2) by qdy we get $xdp + pdx = 0$ ...(4)

and dividing (3) by xdy, we get $qdy + pdx = 0$. ...(5)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = c_1$ ...(6)

Integrating (4), $xp = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), one integral of (1) is

$$xp = f(z) \quad \text{or} \quad x\frac{\partial z}{\partial x} = f(z) \quad \text{or} \quad \frac{1}{f(z)}\frac{\partial z}{\partial x} = \frac{1}{x}.$$

Integrating it partially w.r.t. x, $F(z) = \log x + G(y)$, F, G being arbitrary functions.

Ex. 3. *Solve* $pt - sqs = q^3$ **[MDU Rohtak 2004; Ravishankar 2004; Delhi Maths (H) 2005; Meerut 2005; 06 ; Rohilkhand 1994]**

Sol. Given $pt - qs = q^3$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 0$, $S = -q$, $T = p$, $V = q^3$.

$\therefore$ Monge's subsidiary equations $Rdpdy + Tdqdx - Vdxdy = 0$, $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $pdqdx - q^3dxdy = 0$ ...(2)

and $qdxdy + p(dx)^2 = 0$. ...(3)

Dividing (2) by dx, we get $pdq - q^3dy = 0$ ...(4)

and dividing (3) by dx, we get $pdx + qdy = 0$. ...(5)

From (5), $dy = -(pdx)/q$. Putting this value of dy into (4) gives

$pdq - q^3(pdx/q) = 0$ or $(1/q^2)dq + dx = 0$.

Integrating it, $-1/q + x = C_1$, C_1 being an arbitrary constant ...(6)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = C_2$. ...(7)

From (6) and (7), one integral of (1) is

$$-\frac{1}{q} + x = f(z) \quad \text{or} \quad \frac{\partial y}{\partial z} = x - f(z), \text{ as } q = \frac{\partial z}{\partial y},$$

Integrating with respect to z partially (treat x as constant), we obtain

$y = xz - F(z) + G(x)$, F, G being arbitrary functions, where $F(z) = \int f(z)\,dz$.

Ex. 4. *Solve* $z(qs - pt) = pq^2$. **[Delhi Maths (H) 1998; 2004, 11]**

Sol. Given $zqs - zpt = pq^2$. ...(1)

The usual Monge's subsidiary equations are $-zpdqdx - pq^2dxdy = 0$ ...(2)

and $-zqdxdy - zp(dx)^2 = 0$. ...(3)

Dividing (2) by $-pdx$, we get $zdq + q^2dy = 0$ ...(4)

and dividing (3) by $-z\,dx$ we get $qdy + pdx = 0$. ...(5)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = C_1$. ...(6)

Using (6) in (4), $C_1 dq + q^2 dy = 0$ or $(1/q^2)dq + (1/C_1)dy = 0$.

Integrating it, $-1/q + y/C_1 = C_2$ or $-1/q + y/z = C_2$, by (6) ...(7)

From (6) and (7), one integral of (1) is

$$-\frac{1}{q} + \frac{y}{z} = f(z) \quad \text{or} \quad \frac{\partial y}{\partial z} - \frac{1}{z}y = -f(z), \quad \text{as } q = \frac{\partial y}{\partial z}$$

which is linear in variables y and z (treating x as constant).

Its integrating factor (I.F.) $= e^{-(1/z)dz} = e^{-\log z} = z^{-1}$ and so its solution is

$$yz^{-1} = -\int z^{-1} f(z)\,dz + G(x) \quad \text{or} \quad yz^{-1} = F(z) + G(x), \quad \text{where } F(z) = \int f(z)\,dz$$

or $y = zF(z) + zG(x)$ or $y = H(z) + zG(x)$,

where $H(z)[= zF(z)]$ and $G(x)$ are arbitrary functions.

Ex. 5. *Solve* $2yq + y^2 t = 1$.

Sol. Given equation is $0.r + 0.s + y^2.t = 1 - 2yq$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 0$, $S = 0$, $T = y^2$, $V = 1 - 2yq$.

Hence the usual subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $y^2 dqdx - (1 - 2yq)dxdy = 0$ ...(2)

and $y^2(dx)^2 = 0$. ...(3)

From (3), $dx = 0$ so that $x = c_1$. ...(4)

From (2), $y^2 dq + 2yq\,dy - dy = 0$ or $d(y^2 q) - dy = 0$.

Integrating it, $y^2 q - y = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), an intermediate integral is

$y^2 q - y = f(x)$ or $y^2(\partial z/\partial y) - y = f(x)$

or $$\partial z / \partial y = 1/y + (1/y^2) \times f(x)$$ ...(6)

Integrating (6) w.r. t. y, treating x as constant, we get

$z = \log y - (1/y) f(x) + g(x)$ or $yz = y \log y - f(x) + y\,g(x)$,

where f and g being arbitrary functions.

Ex. 6. *Solve* $(e^x - 1)(qr - ps) = pqe^x$.

Sol. Given $q(e^x - 1)r - p(e^x - 1)s = pqe^x$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = q(e^x - 1)$, $S = -p(e^x - 1)$, $T = 0$, $V = pqe^x$.

Then the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $q(e^x - 1)dpdy - pqe^x dxdy = 0$ ...(2)

and $q(e^x - 1)(dy)^2 + p(e^x - 1)dxdy = 0$. ...(3)

Now, (3) $\Rightarrow$ $qdy + pdx = 0$ $\Rightarrow$ $dz = 0$, as $dz = pdx + qdy$.

Integrating, $z = c_1$, c_1 being an arbitrary constant ...(4)

Again, from (2), $(e^x - 1)dp - pe^x dx = 0$ or $$\frac{dp}{p} - \frac{e^x}{e^x - 1}dx = 0$$

Integrating, $\log p - \log(e^x - 1) = \log c_2$ or $p/(e^x - 1) = c_2$. ...(5)

From (4) and (5), an intermediate integral is $p/(e^x - 1) = f(z)$, f being an arbitrary function

or $\dfrac{\partial z}{\partial x} = (e^x - 1)f(z)$, or $\dfrac{1}{f(z)}\dfrac{\partial z}{\partial x} = e^x - 1.$

Integrating w.r.t. 'x', treating y as constant, we get

$F(z) = e^x - x + G(y)$ or $x = e^x + G(y) - F(z)$,

F and G being arbitrary functions, where $\int (1/f(z))dz = F(z)$.

Miscellaneous problems based on types 1, 2, 3 and 4

Solve the following partial differential equations by using Monge's method:

1. $x^2r - y^2t = xy$. **Ans.** $z = xy \log x + x\,F(y/x) + G(xy)$

2. $(1 + pq + q^2)r + s(q^2 - p^2) - (1 + pq + p^2)t = 0$ **Ans.** $z\{2 + (x+y)\}^{1/2} = F(x+y) + G(x-y)$

3. $q\,(1 + q)r - (1 + 2q)\,(1 + p)s + (1 + p)^2 t = 0$ **Ans.** $x = F(x + y + z) + G(x + z)$

4. $x^2r - y^2t - xp + yq = xy$. **Ans.** $z = (xy/4) \times \{(\log x)^2 - (\log y)^2\} + xyF(x/y) + G(xy)$

9.11. Monge's Method of integrating the equation Rr + Ss + Tt + U(rt – s²) = V, where r, s, t have their usual meaning and R, S, T, U, V are functions of x, y, z.

Given $Rr + Ss + Tt + U(rt - s^2) = V.$ ...(1)

We have $dp = (\partial p/\partial x)\,dx + (\partial p/\partial y)\,dy = r\,dx + s\,dy$

and $dq = (\partial q/\partial x)\,dx + (\partial q/\partial y)\,dy = s\,dx + t\,dy$

which give $r = (dp - s\,dy)/dx$ and $t = (dq - s\,dx)/dy$.

Putting these values in (1) and simplifying, we get

$(R\,dp\,dy + T\,dq\,dx - U\,dp\,dq - V\,dx\,dy) - s\{R(dy)^2 - S\,dx\,dy + T(dx)^2 + U\,dp\,dx + U\,dq\,dy\} = 0.$

Hence the usual Monge's subsidiary equations are

$L \equiv R\,dp\,dy + T\,dq\,dx + U\,dp\,dq - V\,dx\,dy = 0$ ...(2)

and $M \equiv R(dy)^2 - S\,dx\,dy + T(dx)^2 + U\,dp\,dx + U\,dq\,dy = 0.$ ...(3)

We cannot factorise M as we did before (see Art 9.1), on account of the presence of the additional terms, $U\,dp\,dx + U\,dq\,dy$. Hence let us factorise $M + \lambda L$, where λ is some multiplier to be determined later. Now, we have

$M + \lambda L \equiv R(dy)^2 + T(dx)^2 - (S + \lambda V)\,dx\,dy + U\,dp\,dx + U\,dq\,dy + \lambda R\,dp\,dy + \lambda T\,dq\,dx + \lambda U\,dp\,dq = 0.$ ...(4)

Factorising L.H.S. of (4), let k and m be constants such that

$$M + \lambda L \equiv (R\,dy + mT\,dx + kU\,dp)\left(dy + \frac{1}{m}dx + \frac{\lambda}{k}dq\right) = 0. \quad ...(5)$$

Comparing coefficients in (4) and (5), we get $R/m + mT = -(S + \lambda V)$, ...(6)

$k = m$ and $R\lambda/k = U.$ ...(7)

Now, the two relations of (7) give $m = R\lambda u$

Putting this valus of m in (6) and simplifying, we get $\lambda^2(UV + RT) + \lambda US + U^2 = 0$, ...(8)

which is quadratic in λ. Let λ_1 and λ_2 be its roots.

When $\lambda = \lambda_1$, (7) $\Rightarrow$ $R\lambda_1/k = U$ $\Rightarrow$ $k = R\lambda_1/U$ $\Rightarrow$ $m = R\lambda_1/U$

Hence (5) gives $$\left(R\,dy + \frac{R\lambda_1}{U}T\,dx + R\lambda_1\,dp\right)\left(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq\right) = 0$$

or $(U\,dy + \lambda_1 T\,dx + \lambda_1 U\,dp)\,(U\,dx + \lambda_1 R\,dy + \lambda_1 U\,dq) = 0.$ ...(9)

Similarly for $\lambda = \lambda_2$, (5) gives

$(U\,dy + \lambda_2 T\,dx + \lambda_2\,U\,dp)\,(U\,dx + \lambda_2 R\,dy + \lambda_2 U\,dq) = 0.$ ...(10)

Now one factor of (9) is combined with one factor of (10) to give an intermediate integral. Exactly similarly, the other pair will give rise to another intermediate integral. In this connection remember that we must combine first factor of (9) with the second factor of (10) and similarly the second factor of (9) with the first factor of (10). Thus for the desired solution the proper method is to combine the factors in the following manner :

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0, \qquad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \qquad ...(11)$$

$$Udy + \lambda_2 Tdx + \lambda_2 Udp = 0, \qquad Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \qquad ...(12)$$

Let equations (11) give two integrals $u_1 = c$ and $v_1 = d_1$ so that one intermediate integral is

$$u_1 = f_1(v_1), f_1 \text{ being an arbitrary function} \qquad ...(13)$$

Similarly, (12) gives second intermediate integral $u_2 = f_2(v_2)$, ...(14)

where f_2 is an arbitrary function

We now solve (13) and (14) for p and q and substitute in $dz = pdx + qdy$, which after integration gives the desired general solution.

Remark 1. There are in all four ways of combining factors of (9) and (10). By combining the first factors in these equations, we would get $u\,dy = 0$ on substraction (after dividing equations by λ_1 and λ_2 respectively) and this would not produce any solution. Similarly, combining the second factors in these equations would give $u\,dx = 0$ and hence would produce no solution. Hence for getting integrals of the given equation we must proceed as explained in (11) and (12).

Remark 2. In what follows we shall use the following two results of equation $a\lambda^2 + b\lambda + c = 0$

(*i*) $a = b = 0$, *i.e.*, the coefficients of λ^2 and λ both equal to zero imply that both roots of the equatin are equal to ∞

(*ii*) $a = 0$ but $b \neq 0$, *i.e.*, the coefficient of λ^2 is zero but that of λ is non–zero imply that one root of the equation is ∞ and the other is $-c/b$.

Remark 3. When the two values of λ are equal, we shall have only one intermediate integral $u_1 = f(v_1)$ and proceed as explained in solved examples of type 1 based on $Rr + Ss + Tt + U(rt - s^2) = V$ given below.

An integral of *a* more general form can be obtained by taking the arbitrary function occuring in the intermediate integral to be linear.

Let $u_1 = mv_1 + n$, where m and n are some constants. Then integrating it by Lagrange's method we find the solution of the given equation.

9.12. Type 1: When the roots of λ –quadratic (8) of Art 9.11 are identical. Solved examples of type 1 based on $Rr + Ss + Tt + U(rt - s^2) = V$

Ex. 1. *Solve* $5r + 6s + 3t + 2(rt - s^2) + 3 = 0$. **[I.A.S. 1973 ; Meerut 1998]**

Sol. Given equation $5r + 6s + 3t + 2(rt - s^2) = -3$. ...(1)

Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have $R = 5$, $S = 6$, $T = 3$, $U = 2$ and $V = -3$. Hence the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes $9\lambda^2 + 12\lambda + 4 = 0$ or $(3\lambda + 2)^2 = 0$ so that $\lambda_1 = \lambda_2 = -2/3$.

There is only one intermediate integral given by the equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $2dy + (-2/3) \times 3dx + (-2/3) \times 2dp = 0$ and $2dx + (-2/3) \times 5dy + (-2/3) \times 2dq = 0$

or $3dy - 3dx - 2dp = 0$ and $3dx - 5dy - 2dq = 0$.

Integrating, $3y - 3x - 2p = c_1$ and $3x - 5y - 2q = c_2$. ...(2)

Hence here the only intermediate integral is

$$3y - 3x - 2p = f(3x - 5y - 2q), \text{ where } f \text{ is an arbitrary function.} \qquad ...(3)$$

Solving the two equations of (2) for p and q, we have

$p = (1/2) \times (3y - 3x - c_1)$ and $q = (1/2) \times (3x - 5y - c_2)$.

Putting these values of p and q in $dz = pdx + qdy$, we have

$$dz = (1/2) \times (3y - 3x - c_1)dx + (1/2) \times (3x - 5y - c_2)dy$$

or $$2dz = 3(ydx + xdy) - 3xdx - 5ydy - c_1dx - c_2dy.$$

Integrating, $$2z = 3xy - (3x^2/2) - (5y^2/2) - c_1x - c_2y + c_3,$$

which is the required complete integral, c_1, c_2 and c_3 being arbitrary constants.

Alternative solution. An integral of a more general form can be obtained by supposing the arbitrary function f occuring in the intermediate integral (3) to be linear, giving

$$3y - 3x - 2p = m(3x - 5y - 2q) + n, \text{ where } m \text{ and } n \text{ are arbitrary constants.} \quad ...(4)$$

Re–writing (4), $$2p - 2mq = 3y - 3x + 5my - 3mx - n. \quad ...(5)$$

Lagrange's auxiliary equations for (5) are $$\frac{dx}{2} = \frac{dy}{-2m} = \frac{dz}{3y - 3x + 5my - 3mx - n}. \quad ...(6)$$

Taking the first two fractions of (6), we have

$$dy + mdx = 0 \quad \text{so that} \quad y + mx = a. \quad ...(7)$$

Now, each fraction of (6) $= \dfrac{3xdx + 5ydy + 2dz}{6x - 10my + 6y - 6x + 10my - 6mx - 2n}$ ...(8)

Hence taking first fraction of (6) and fraction (8), we have

$$\frac{dx}{2} = \frac{3xdx + 5ydy + 2dz}{6y - 6mx - 2n} \quad \text{or} \quad dx = \frac{3xdx + 5ydy + 2dz}{3y - 3mx - n}$$

or $$3xdx + 5ydy + 2dz = (3y - 3mx - n)dx$$

or $$2dz + 3xdx + 5ydy = \{3(a - mx) - 3mx - n\}dx, \text{ using (7)}$$

or $$2dz + 3xdx + 5ydy = (3a - 6mx - n)dx .$$

Integrating, $$2z + (3x^2/2) + (5y^2/2) = 3ax - 3mx^2 - nx + b/2$$

or $$4z + 3x^2 + 5y^2 = 6x(y + mx) - 6mx^2 - 2xn + b, \text{ using (7)}$$

or $$4z - 6xy + 3x^2 + 5y^2 + 2nx = b. \quad ...(9)$$

From (7) and (9), the required general solution is $4z - 6xy + 3x^2 + 2nx = \phi(y + mx)$, where ϕ is an arbitrary function and m and n are arbitrary constants.

Ex. 2. *Solve* $3r + 4s + t + (rt - s^2) = 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get $R = 3$, $S = 4$, $T = 1$, $U = 1$, $V = 1$. Then, λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$

becomes $4\lambda^2 + 4\lambda + 1 = 0$ or $(2\lambda + 1)^2 = 0$ so that $\lambda_1 = \lambda_2 = -1/2$.

There is only one intermediate integral given by the equations

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \quad \text{and} \quad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

or $dy + (-1/2) \times dx + (-1/2) \times dp = 0$ and $dx + (-1/2) \times 3dy + (-1/2) \times dq = 0$

or $-2dy + dx + dp = 0$ and $3dy - 2dx + dq = 0$. ... (1)

Integrating, $-2y + x + p = c_1$ and $3y - 2x + q = c_2$. ...(2)

Hence the only intermediate integral is

$$-2y + x + p = f(3y - 2x + q), \text{ where } f \text{ is an arbitrary function.} \quad ...(3)$$

Solving (2) for p and q, $p = 2y - x + c_1$ and $q = -3y + 2x + c_2$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (2y - x + c_1)dx + (-3y + 2x + c_2)dy$$

or $$dz = 2(ydx + xdy) - xdx - 3ydy + c_1dx + c_2dy.$$

Integrating, $$z = 2xy - (x^2/2) - (3y^2/2) + c_1x + c_2y + c_3,$$

which is the required complete integral, c_1, c_2, c_3 being arbitrary constants.

Alternative solution. In order to get the more general solution, we assume the arbitrary function ϕ in (3) to be linear. Thus, we take

$$-2y + x + p = m(3y - 2x + q) + n,\ m, n \text{ being arbitrary constants}$$

or $$p - mq = 2y - x + 3my - 2mx + n. \quad ...(4)$$

Lagrange's auxiliary equations for (4) are $$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{2y - x + 3my - 2mx + n}. \quad ...(5)$$

Taking the first two fractions of (5), $dy + mdx = 0$ so that $y + mx = a$. ...(6)

Now, each fraction of (5) $= \dfrac{xdx + 3ydy + dz}{x - 3my + 2y - x + 3my - 2mx + n}$ ...(7)

Taking the first fraction of (5) and the fraction (7), we have $$\frac{dx}{1} = \frac{xdx + 3ydy + dz}{2y - 2mx + n}$$

or $$xdx + 3ydy + dz = (2y - 2mx + n)dx$$

or $$xdx + 3ydy + dz = 2(a - mx)dx - 2mxdx + ndx, \text{ using (6)}$$

Integrating, $$(x^2/2) + (3y^2/2) + z = 2ax - mx^2 - mx^2 + nx + b/2$$

or $$x^2 + 3y^2 + 2z - 2x(y + mx) + 2mx^2 - nx = b, \text{ using (6)} \quad ...(8)$$

From (6) and (8), the required general solution is $x^2 + 3y^2 + 2z - 2xy - nx = \phi(y + mx)$, where ϕ is an arbitrary function and m and n are arbitrary constants.

Ex. 3. *Solve* $(q^2 - 1)zr - 2pqzs + (p^2 - 1)zt + z^2(rt - s^2) = p^2 + q^2 - 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have $R = z(q^2 - 1)$, $S = -2pqz$, $T = z(p^2 - 1)$, $U = z^2$ and $V = p^2 + q^2 - 1$.

Hence the λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ becomes

$p^2q^2\lambda^2 - 2pqz + z^2 = 0$ or $(pq\lambda - z)^2 = 0$ so that $\lambda_1 = \lambda_2 = z/pq$.

There is only one intermediate integral given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $z^2dy + \dfrac{z^2(p^2 - 1)}{pq}dx + \dfrac{z^3}{pq}dp = 0$ and $z^2dx + \dfrac{z^2(q^2 - 1)}{pq}dy + \dfrac{z^3}{pq}dq = 0$

or $pqdy + (p^2 - 1)dx + zdp = 0$ and $pqdx + (q^2 - 1)dy + zdq = 0$

or $p(qdy + pdx) - dx + zdp = 0$ and $q(pdx + qdy) - dy + zdq = 0$

or $pdz + zdp - dx = 0$ and $qdz + zdq - dy = 0$, as $dz = pdx + qdy$

or $d(pz) - dx = 0$ and $d(qz) - dy = 0$.

Integrating, $pz - x = c_1$ and $qz - y = c_2$. ...(1)

Hence the only intermediate integral is $pz - x = f(qz - y)$, f being an arbitrary function. ...(2)

Solving (1) for p and q, $p = (c_1 + x)/z$ and $q = (c_2 + y)/z$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (1/z) \times (c_1 + x)dx + (1/z) \times (c_2 + y)dy$ or $zdz = (c_1 + x)dx + (c_2 + y)dy$.

Integrating, $(1/2) \times z^2 = (1/2) \times (c_1 + x)^2 + (1/2) \times (c_2 + y)^2 + (1/2) \times c_3'$.

or $z^2 = x^2 + y^2 + 2c_1x + zc_2y + c_3$, where $c_3 = c_1^{\,2} + c_2^{\,2} + c_3'$

which is the complete integral, c_1, c_2, c_3 being arbitrary constants.

Alternative solution. To find the more general solution, we take the arbitrary function f in (2) to be linear. So, let $pz - x = m(qz - y) + n$, m, n being arbitrary constants.

or $$pz - mqz = x - my + n. \quad ...(3)$$

Lagrange's auxiliary equation for (3) are $$\frac{dx}{z} = \frac{dy}{-mz} = \frac{dz}{x - my + n}. \quad ...(4)$$

Taking the first two fractions of (4), $dy + mdx = 0$ so that $y + mx = a$. ...(5)

Now, each fraction of (4) $= \dfrac{(-x/z)dx-(y/z)dy+dz}{z\times(-x/z)-mz\times(-y/z)+x-my+n}$. ...(6)

Taking the first fraction of (4) and fraction (6), $\dfrac{dx}{z} = \dfrac{-(x/z)dx-(y/z)dy+dz}{n}$

or $-xdx-ydy+zdz = ndx$ or $-2zdz+2xdx+2ydy+2ndx = 0$.

Integrating, $-z^2+x^2+y^2+2nx = b$, b being an arbitrary constant ...(7)

From (5) and (7), the required general solution is $-z^2+x^2+y^2+2nx = \phi(y+mx)$, where ϕ is an arbitrary function and m, n are arbitrary constants.

Ex. 4. *Solve* $2s+(rt-s^2) = 1$. **[Garwhal 1995; Meerut 2000]**

Sol. Comparing the given equation with the equation $Rr+Ss+Tt+U(rt-s^2) = V$, we get $R = 0$, $S = 2$, $T = 0$, $U = 1$, $V = 1$, so λ–quardratic $\lambda^2(UV+RT)+\lambda SU+U^2 = 0$ becomes $\lambda^2+2\lambda+1 = 0$ so that $\lambda_1 = \lambda_2 = -1$.

Since we have equal values of l, there would be only one intermediate integral given by

$Udy+\lambda_1 Tdx+\lambda_1 Udp = 0$ and $Udx+\lambda_2 Rdy+\lambda_2 Udq = 0$

or $dy-dp = 0$ and $dx-dq = 0$, using (1)

which give $y-p = c_1$, and $x-q = c_2$.

Solving these for p and q, $p = y-c_1$ and $q = x-c_2$.

$\therefore$ $dz = pdx+qdy = (y-c_1)dx+(x-c_2)dy = (ydx+xdy)-c_1dx-c_2dy$,

or $dz = d(xy)-c_1dx-c_2dy$.

Integrating, $z = xy-c_1x-c_2y+c_3$, which is solution, c_1, c_2, c_3 being arbitrary constants.

Ex. 5. $z(1+q^2)r-2pqzs+z(1+p^2)t+z^2(s^2-rt)+1+p^2+q^2 = 0$.

Sol. Comparing the give equation with $Rr+Ss+Tt+U(rt-s^2) = V$, we get

$R = z(1+q^2)$, $S = -2pqz$, $T = z(1+p^2)$, $U = z^2$ and $V = -(1+p^2+q^2)$. ... (1)

Hence λ–quadratic *i.e.* $\lambda^2(RT+UV)+\lambda US+U^2 = 0$ gives

$\lambda^2(p^2q^2)-2\lambda zpq+z^2 = 0$ or $(\lambda pq-z)^2 = 0$.

Thus here we obtain $\lambda_1 = \lambda_2 = z/pq$. Hence there would be only one intermediate integral which is given by

$$Udy+\lambda_1 Tdx+\lambda_1 Udp = 0. \quad ...(2)$$

and $$Udx+\lambda_2 Rdy+\lambda_2 Udq = 0 \quad ...(3)$$

Using (1), (2) becomes $pq\,dy+(1+p^2)dx+zdp = 0$ ...(4)

Using (1), (3) becomes $pqdx+(1+q^2)dy+zdq = 0$ ...(5)

Now from (4), $p(pdx+qdy)+dx+zdp = 0$ or $pdz+dx+zdp = 0$, as $dz = pdx+qdy$

or $d(zp)+dx = 0$ so that $zp+x = c_1$. ...(6)

Similarly (5) gives $zq+y = c_2$, c_2 being an arbitrary constant ...(7)

Solving (6) and (7), we get $p = (c_1-x)/z$ and $q = (c_2-y)/z$.

$\therefore$ $dz = pdx+qdy = \{(c_1-x)/z\}dx+\{(c_2-y)/z\}dy$ or $zdz = c_1dx+c_2dy-(xdx+ydy)$.

Integrating, $(1/2)\times z^2 = c_1x+c_2y-(x_2+y^2)/2+c_3/2$ or $z^2 = 2c_1x+2c_2y-x^2-y^2+c_3$, which is complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 6. *Solve* $2r+te^x-(rt-s^2) = 2e^x$.

Sol. Comparing the given equation with $Rr+Ss+Tt+U(rt-s^2) = V$, we get

$R = 2$, $S = 0$, $T = e^x$, $U = -1$ and $V = 2e^x$. ... (1)

Hence the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ gives $\lambda^2(2e^x - 2e^x) + (\lambda \times 0) + 1 = 0$.

Since the coefficient of λ^2 and λ in the above quadratic vanish, it follows from the theory of equations that its both the roots must be infinite. Thus $\lambda_1 = \lambda_2 = \infty$. Since the two roots are equal there would be only one intermediate integral which is given by

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$,

i.e., by $(U/\lambda_1)dy + Tdx + Udq = 0$ and $(U/\lambda_2)dx + Rdy + Udq = 0$,

i.e., by $e^x dx - dp = 0$ using (1) and $2dy - dq = 0$, using (1)

Integrating these $e^x - p = c_1$ and $2y - q = c_2$.

Solving these, $p = e^x - c_1$ and $q = 2y - c_2$.

Now, $dz = pdx + qdy = (e^x - c_1)dx + (2y - c_2)dy$.

Integrating, $z = e^x - c_1 x + y^2 - c_2 y + c_3$,

which is complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 7. *Solve* $r + t - (rt - s^2) = 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$,

$R = 1$, $S = 0$, $T = 1$, $U = -1$, $V = 1$. ...(1)

So λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ becomes $(0 \times \lambda^2) + (0 \times \lambda) + 1 = 0$. Since the coeffieicnts of both λ^2 and λ are zero, so both roots of this quadratic are equal to ∞. So $\lambda_1 = \lambda_2 = \infty$

Now, the only one intermediate integral is given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $\lambda_1 Rdy + Udx + \lambda_1 Udq = 0$

On dividing each term by λ_1 as λ_1 is infinite, the above equations become

or $(1/\lambda_1) \times Udy + Tdx + Udp = 0$ and $Rdy + (1/\lambda_1) \times Udx + Udq = 0$

or $Tdx + Udp = 0$, as $\lambda_1 = \infty$ and $Rdy + Udq = 0$, as $\lambda_1 = \infty$

or $dx - dp = 0$ and $dy - dq = 0$, using (1)

Integrating, $p - x = c_1$ and $q - y = c_2$. ...(2)

Solving (2) for p and q, $p = x + c_1$ and $q = y + c_2$.

Putting these values of p and q in $dz = pdx + qdy$, we get $dz = (x + c_1)dx + (y + c_2)dy$

Integrating, $z = x^2/2 + c_1 x + y^2/2 + c_2 y + c_3$,

which is the required integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 8. *Solve* $2pr + 2qt - 4pq\,(rt - s^2) = 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = 2p$, $S = 0$, $T = 2q$, $U = -4pq$, $V = 1$. ...(1)

Then the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes $(0 \times \lambda^2) + (0 \times \lambda) + 4p^2q^2 = 0$. Since the coefficients of both λ^2 and λ are zero, so both roots of the λ– quadratic are equal to ∞. So $\lambda_1 = \lambda_2 = \infty$.

Now the only intermediate integral is given by the equation

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $\lambda_1 Rdy + Udx + \lambda_1 Udq = 0$

On dividing each term by λ_1 as λ_1 is infinite, the above equations become

$(1/\lambda_1) \times Udy + Tdx + Udp = 0$ and $Rdy + (1/\lambda_1) \times Udx + Udq = 0$

or $2qdx - 4pqdp = 0$ and $2pdy - 4pqdq = 0$, using (1)

or $2pdp - dx = 0$ and $2qdq - dy = 0$.

Integrating, $p^2 - x = c_1$ and $q^2 - y = c_2$.

Hence $p = \pm (c_1 + x)^{1/2}$ and $q = \pm (c_2 + y)^{1/2}$

Putting values of p and q in $dz = pdx + qdy$ gives $dz = \pm (c_1 + x)^{1/2}dx \pm (c_2 + y)^{1/2}dy$.

Integrating, $z = \pm (2/3) \times (c_1 + x)^{3/2} \pm (2/3) \times (c_2 + y)^{3/2} + c_3/2$

or $3z = \pm 2(c_1 + x)^{3/2} \pm 2(c_2 + y)^{3/2} + c_3$,

which is the complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 9. *Solve* $(1 + q^2)r - 2pqs + (1 + p^2)t + (1 + p^2 + q^2)^{-1/2}(rt - s^2) = -(1 + p^2 + q^2)^{3/2}$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = 1 + q^2$, $S = -2pq$, $T = 1 + p^2$, $U = (1 + p^2 + q^2)^{-1/2}$, $V = -(1 + p^2 + q^2)^{3/2}$ ...(1)

Now, the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes

$\lambda^2 \{-(1 + p^2 + q^2) + (1 + q^2)(1 + p^2)\} - 2pq(1 + p^2 + q^2)^{-1/2}\lambda + (1 + p^2 + q^2)^{-1} = 0$

or $p^2q^2(1 + p^2 + q^2)\lambda^2 - 2pq(1 + p^2 + q^2)^{1/2}\lambda + 1 = 0$

or $\{pq(1 + p^2 + q^2)^{1/2}\lambda - 1\}^2 = 0$ so that $\lambda_1 = \lambda_2 = 1/pq(1 + p^2 + q^2)^{1/2}$.

Here there is only intermediate integral given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $$\frac{1}{(1+p^2+q^2)^{1/2}}dy + \frac{1+p^2}{pq\,(1+p^2+q^2)^{1/2}}dx + \frac{dp}{pq(1+p^2+q^2)} = 0, \text{ by (1)}$$

and $$\frac{1}{(1+p^2+q^2)^{1/2}}dx + \frac{1+q^2}{pq\,(1+p^2+q^2)^{1/2}}dy + \frac{dq}{pq(1+p^2+q^2)} = 0, \text{ by (1)}$$

or $pqdy + (1 + p^2)dx + [1/(1 + p^2 + q^2)^{1/2}]dp = 0$ ...(2)

and $pqdx + (1 + q^2)dy + \{1/(1 + p^2 + q^2)^{1/2}\}dq = 0$. ...(3)

Eliminating dy between (2) and (3), $\{(1 + p^2)(1 + q^2) - p^2q^2\}dx + \dfrac{(1+q^2)dp - pqdq}{(1+p^2+q^2)^{1/2}} = 0$

or $$(1 + p^2 + q^2)dx + \frac{(1 + p^2 + q^2)dp - (p^2dp + pqdq)}{(1 + p^2 + q^2)^{1/2}} = 0$$

or $$dx + \frac{dp}{(1+p^2+q^2)^{1/2}} - \frac{p}{2}\frac{2pdp + 2qdq}{(1+p^2+q^2)^{3/2}} = 0 \quad \text{or} \quad dx + d\left\{\frac{p}{(1+p^2+q^2)^{1/2}}\right\} = 0$$

Integrating, $x + p(1 + p^2 + q^2)^{-1/2} = a$, where a is an arbitrary constant. ...(4)

Similarly, eliminating dx between (2) and (3), we have

$y + q(1 + p^2 + q^2)^{-1/2} = b$, where b in an arbitrary constant. ...(5)

From (4) and (5), $x - a = -p(1 + p^2 + q^2)^{-1/2}$, $y - b = -q(1 + p^2 + q^2)^{-1/2}$.

$\therefore$ $\dfrac{x-a}{y-b} = \dfrac{p}{q}$ so that $p = \dfrac{x-a}{y-b}q$. ...(6)

Putting the above value of p in (4), we have

$$x + q\frac{x-a}{y-b}\left\{1 + q^2\frac{(x-a)^2}{(y-b)^2} + q^2\right\}^{-1/2} = a \quad \text{or} \quad (x - a) + \frac{x-a}{y-b}q\left[1 + \frac{(x-a)^2 + (y-b)^2}{(y-b)^2}q^2\right]^{-1/2} = 0$$

or $$1 + \frac{(x-a)^2 + (y-b)^2}{(y-b)^2}q^2 = \frac{q^2}{(y-b)^2} \quad \text{or} \quad (y - b)^2 = q^2[1 - \{(x - a)^2 + (y - b)^2\}].$$

Thus, $q = (y-b)/[1-\{(x-a)^2+(y-b)^2\}]^{1/2}$. ... (7)

Now, (6) and (7) $\Rightarrow$ $p = \dfrac{x-a}{y-b}q = \dfrac{x-a}{[1-\{(x-a)^2+(y-b)^2\}]^{1/2}}$. ... (8)

$$\therefore\ dz = pdx + qdy = \frac{(x-a)\,dx+(y-b)\,dy}{[1-\{(x-a)^2+(y-b)^2\}]^{1/2}}, \text{ by (7) and (8)}$$

Integrating, $z = [1-\{(x-a)^2+(y-b)^2\}]^{1/2} + c$ or $(z-c)^2 = 1-\{(x-a)^2+(y-b)^2\}$

$\therefore$ $(x-a)^2+(y-b)^2+(z-c)^2 = 1$ is the complete integral, a, b, c being arbitrary constants.

9.13 Type 2. When the roots of λ –quadratic (8) of Art 9.11 are distinct.

Solved Examples of Type –2 based on Rr + Ss + Tt + U(rt – s^2) = V

Ex. 1. *Solve* $3s + rt - s^2 = 2$.

Sol. Given $3s + (rt - s^2) = 2$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0, S = 3, V = 0, U = 1, V = 2$. ...(2)

λ–quadratic is $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ ...(3)

Using (2), (3) reduces to $2\lambda^2 + 3\lambda + 1 = 0$ so $\lambda_1 = -1, \lambda_2 = -(1/2)$. ... (4)

Two integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0. \end{aligned}\right\} \quad \text{... (5)}$$

and

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0. \end{aligned}\right\} \quad \text{... (6)}$$

Using (2) and (4), (5) and (6) respectively gives

$$\left.\begin{aligned} dy - dp = 0 \quad \text{or} \quad dp - dy = 0 \\ dx - (1/2)dq = 0 \quad \text{or} \quad dq - 2dx = 0 \end{aligned}\right\} \quad \text{... (5A)}$$

and

$$\left.\begin{aligned} dy - (1/2)dp = 0 \quad \text{or} \quad dp - 2dy = 0 \\ dx - dq = 0 \quad \text{or} \quad dq - dx = 0. \end{aligned}\right\} \quad \text{... (6A)}$$

Integration of (5A) and (6A) respectively gives

$$p - y = c_1, \qquad q - 2x = c_2 \quad \text{...(5B)}$$

and

$$p - 2y = c_3, \qquad q - x = c_4, \quad \text{...(6B)}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

From (5B) and (6B), two intermediate integrals of (1) are given by

$p - y = f(q - 2x)$ and $p - 2y = F(q - x)$, ...(7)

where f and F are arbitrary functions.

Let $q - 2x = \alpha$, ...(8)

and $q - x = \beta$. ...(9)

Then from (7) $p - y = f(\alpha)$, ...(10)

and $p - 2y = F(\beta)$. ...(11)

[If we treat α and β as constants, then solution of four simultaneous equation (8), (9), (10) and (11) would show that x, y, p and q are all constants which is absurd. Hence α and β will be regarded as variables (parameters) and we will get the general solution in parametric form involving α and β as parameters].

Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = \beta - \alpha \quad \text{...(12)}$$

and

$$y = f(\alpha) - F(\beta). \quad \text{...(13)}$$

From (10) $p = y + f(\alpha)$. ...(14)

From (9) $q = x + \beta$. ...(15)

From (12) and (13), $dx = d\beta - d\alpha$, and $dy = f'(\alpha)d\alpha - F'(\beta)d\beta$. ...(16)

$\therefore$ $dz = pdx + qdy = [y + f(\alpha)]dx + (x + \beta)dy$, using (14) and (15)

or $dz = ydx + xdy + f(\alpha)dx + \beta dy = d(xy) + f(\alpha)(d\beta - d\alpha) + \beta[f'(\alpha)d\alpha - F'(\beta)d\beta]$, by (16)

Thus, $dz = d(xy) + [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] - f(\alpha)d\alpha - \beta F'(\beta)d\beta$

or $dz = d(xy) + d[\beta f(\alpha)] - f(\alpha)d\alpha - \beta F'(\beta)d\beta$.

Integrating and using integration by parts in the last term on R.H.S. of the above equation,

we get $z = xy + \beta f(\alpha) - \int f(\alpha)d\alpha - [\beta F(\beta) - \int 1 \cdot F(\beta)d\beta]$

or $z = xy + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)d\alpha + \int F(\beta)d\beta$. ...(17)

Let $\int f(\alpha)d\alpha = \phi(\alpha)$ and $\int F(\beta)d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$ ...(19)

Using (18) and (19), (12), (13) and (17) give

$x = \beta - \alpha$, $y = \phi'(\alpha) - \psi'(\beta)$ $z = xy + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$

which is the required solution in parametric form, ϕ and ψ being arbitrary functions and α and β being parameters.

Ex. 2. *Solve* $r + 4s + t + rt - s^2 = 2$. **[I.A.S. 1979]**

Sol. Given $r + 4s + t + (rt - s^2) = 2$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 1$, $S = 4$, $T = 1$, $U = 1$, $V = 2$. ...(2)

λ–quadratic is $\lambda^2(UV + RT) + \lambda US + U^2 = 0$. ...(3)

Using (2), (3) reduces to $3\lambda^2 + 4\lambda + 1 = 0$ so $\lambda_1 = -1$, $\lambda_2 = -(1/3)$.

Two integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad \text{... (5)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \quad \text{... (6)}$$

Using (2) and (4), (5) and (6) respectively gives

$$\left.\begin{aligned} dy - dx - dp = 0 \quad \text{or} \quad dp + dx - dy = 0 \\ dx - (1/3) \times dy - (1/3) \times dq = 0 \quad \text{or} \quad dq + dy - 3dx = 0 \end{aligned}\right\} \quad \text{... (5A)}$$

$$\left.\begin{aligned} dy - (1/3) \times dx - (1/3) \times dp = 0 \quad \text{or} \quad dp + dx - 3dy = 0 \\ dx - dy - dq = 0 \quad \text{or} \quad dq + dy - dx = 0 \end{aligned}\right\} \quad \text{... (6A)}$$

Integration of (5A) and (6A) respectively gives

$p + x - y = c_1$, $q + y - 3x = c_2$ ...(5*B*)

and $p + x - 3y = c_3$, $q + y - x = c_4$, ...(6*B*)

where c_1, c_2, c_3 and c_4 are arbitrary constants.

From (5B) and (6B), two intermediate integrals of (1) are given by

$p + x - y = f(q + y - 3x)$ and $p + x - 3y = F(q + y - x)$. ...(7)

Let $q + y - 3x = \alpha$, ...(8)

and $q + y - x = \beta$. ...(9)

Then from (7), $p + x - y = f(\alpha)$, ...(10)

and $p + x - 3y = F(\beta)$. ...(11)

Here α and β are treated as parameters. Solving (8) and (9) for x and (10) and (11) for y gives

$$x = (\beta - \alpha)/2 \qquad ...(12)$$

and $$y = [f(\alpha) - F(\beta)]/2 \qquad ...(13)$$

From (10), $$p = y - x + f(\alpha) \qquad ...(14)$$

From (9), $$q = x - y + \beta \qquad ...(15)$$

From (12) and (13), $dx = (1/2) \times (d\beta - d\alpha)$, $dy = (1/2) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]$. ...(16)

$\therefore$ $dz = pdx + qdy = [y - x + f(\alpha)]dx + (x - y + \beta)dy$, by (14) and (15)

$= ydx + xdy - xdx - ydy + f(\alpha)dx + \beta dy$

$= d(xy) - xdx - ydy + f(\alpha) \times (1/2) \times (d\beta - d\alpha) + \beta \times (1/2) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]$, by (16)

$= d(xy) - xdx - ydy + (1/2) \times [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] - (1/2) \times f(\alpha)d\alpha - (1/2) \times \beta F'(\beta)d\beta$

or $$2dz = 2d(xy) - 2xdx - 2ydy + d[\beta f(\alpha)] - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating and using integration by parts in the last term on R.H.S. of the above equation,

we get $$2z = 2xy - x^2 - y^2 + \beta f(\alpha) - \int f(\alpha)d\alpha - [\beta F(\beta) - \int 1 \cdot F(\beta)\, d\beta]$$

or $$2z = 2xy - x^2 - y^2 + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)d\alpha + \int F(\beta)\, d\beta. \qquad ...(17)$$

Let $$\int f(\alpha)d\alpha = \phi(\alpha) \quad \text{and} \quad \int F(\beta)\, d\beta = \psi(\beta) \qquad ...(18)$$

so that $$f(\alpha) = \phi'(\alpha) \quad \text{and} \quad F(\beta) = \psi'(\beta). \qquad ...(19)$$

Using (18) and (19), (12), (13) and (17) give

$2x = \beta - \alpha$, $2y = \phi'(\alpha) - \psi'(\beta)$, $2z = 2xy - x^2 - y^2 + \beta[\phi'(a) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$

which is the required solution in parametric form, α and β being parameters and ϕ and ψ being arbitrary functions.

Ex. 3. *Solve* $rt - s^2 + 1 = 0$

Sol. Given that $$0 \cdot r + 0 \cdot s + 0 \cdot t + (rt - s^2) = -1. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0$, $S = 0$, $T = 0$, $U = 1$ and $V = -1$. ...(2)

Here λ–quadratic $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad ...(3)$$

becomes $\lambda^2 - 1 = 0$ so that $\lambda_1 = -1$ and $\lambda_2 = 1$. ...(4)

Since the two values of λ are distinct, we shall get two intermediate integrals which are given by the following sets of equations

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad ... (5A)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad ... (5B)$$

Using (2) and (4), equations (5) and (6) reduces to

$$\left.\begin{aligned} dy - dp = 0 \quad i.e., \quad dp - dy = 0 \\ dx + dq = 0 \quad i.e., \quad dq + dx = 0 \end{aligned}\right\} \qquad ... (5A)$$

$$\left.\begin{aligned} dy + dp = 0 \quad i.e., \quad dp + dy = 0 \\ dx - dq = 0 \quad i.e., \quad dq - dx = 0 \end{aligned}\right\} \qquad ... (6A)$$

Integrating of (5A) and (6A) respectively gives

$$p - y = c_1, \qquad q + x = c_2. \qquad ...(5B)$$

and $$p + y = c_3, \qquad q - x = c_4, \qquad ...(6B)$$

where c_1, c_2, c_3 are c_4 are arbitrary constants

From (5B) and (6B), two intermediate integrals are given by

$$p - y = f(q + x) \quad \text{and} \quad p + y = F(q - x), \qquad ...(7)$$

where f and F are arbitrary functions.

Let $q + x = \alpha$ ...(8)

and $q - x = \beta$. ...(9)

Then, from (7), $p - y = f(\alpha)$ ...(10)

and $p + y = F(\beta)$. ...(11)

In what follows α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (\alpha - \beta)/2 \quad ...(12)$$

and $$y = [F(\beta) - f(\alpha)]/2 \quad ...(13)$$

From (10), $p = y + f(\alpha)$ ...(14)

From (9), $q = x + \beta$. ...(15)

From (12) and (13), $dx = (1/2) \times (d\alpha - d\beta)$, $dy = (1/2) \times [F'(\beta)d\beta - f'(\alpha)d\alpha]$. ...(16)

$\therefore$ $dz = pdx + qdy = [y + f(\alpha)]dx + (x + \beta)dy$, using (14) and (15)

$= (ydx + xdy) + f(\alpha)dx + \beta dy$

$= d(xy) + f(\alpha) \times (1/2) \times (d\alpha - d\beta) + \beta \times (1/2) \times [F'(\beta)d\beta - f'(\alpha)d\alpha]$, by (16)

$= d(xy) + (1/2) \times f(\alpha)d\alpha - (1/2) \times [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] + (1/2) \times \beta F'(\beta)d\beta$

or $$2dz = 2d(xy) + f(\alpha)d\alpha - d[\beta f(\alpha)] + \beta F'(\beta)d\beta.$$

Integrating both sides and using integration by parts in the last term on the R.H.S., we obtain

$$2z = 2xy + \int f(\alpha)d\alpha + \beta f(\alpha) + \beta F(\beta) - \int F(\beta)\,d\beta. \quad ...(17)$$

Let $\int f(\alpha)\,d\alpha = \phi(\alpha)$ and $\int F(\beta)\,d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$. ...(19)

Using (18) and (19), (12), (13) and (17) may be re–written as

$2x = (\alpha - \beta)$, $2y = \psi'(\beta) - \phi'(\alpha)$, $2z = 2xy - \phi(\alpha) + \beta\{\phi'(\alpha) + \psi'(\beta)\} - \psi(\beta)$

which is the required solution in parametric form, α and β being parameters and ϕ and ψ being arbitrary functions.

Ex. 4. *Solve* $r + 3s + t + (rt - s^2) = 1$. **[Rohilkhand 1995]**

Sol. Given $r + 3s + t + (rt + s^2) = 1$ ... (1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 1$, $S = 3$, $T = 1$, $U = 1$, $V = 1$. ...(2)

Now, λ–quadratic is $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ ...(3)

or $2\lambda^2 + 3\lambda + 1 = 0$ so that $\lambda = -1, \quad -1/2$. Here $\lambda_1 = -1, \quad \lambda_2 = -1/2$. ...(4)

Two intermediate integrals of (1) are giving by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad ... (5)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \quad ... (6)$$

Using (2) and (4), equations (5) and (6) reduces to

$$\left.\begin{aligned} dy - dx - dp = 0 \quad i.e., \quad dp + dx - dy = 0 \\ dx - (1/2) \times dy - (1/2) \times dq = 0 \quad i.e., \quad dq - 2dx + dy = 0 \end{aligned}\right\} \quad ... 5(A)$$

and

$$\left.\begin{aligned} dy - (1/2) \times dx - (1/2) \times dp = 0 \quad i.e., \quad dp + dx - 2dy = 0 \\ dx - dy - dq = 0 \quad i.e., \quad dq - dx + dy = 0 \end{aligned}\right\} \quad ...(6A)$$

Integrating of (5*A*) and (6*A*) respectively gives

$$p + x - y = c_1, \qquad q - 2x + y = c_2 \qquad \text{...(5B)}$$

and $$p + x - 2y = c_3, \qquad q - x + y = c_4, \qquad \text{...(6B)}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediate integrals are given by

$$p + x - y = f(q - 2x + y) \quad \text{and} \quad p + x - 2y = F(q - x + y), \qquad \text{...(7)}$$

where f and F are arbitrary functions

Let $$q - 2x + y = \alpha \qquad \text{...(8)}$$

and $$q - x + y = \beta. \qquad \text{...(9)}$$

Then, from (7) $$p + x - y = f(\alpha) \qquad \text{...(10)}$$

and $$p + x - 2y = F(\beta). \qquad \text{...(11)}$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = \beta - \alpha \qquad \text{...(12)}$$

and $$y = f(\alpha) - F(\beta). \qquad \text{...(13)}$$

From (10), $$p = y - x + f(\alpha) \qquad \text{...(14)}$$

From (9), $$q = x - y + \beta. \qquad \text{...(15)}$$

From (12) and (13), $dx = d\beta - d\alpha$, $dy = f'(\alpha)d\alpha - F'(\beta)d\beta$. ...(16)

$\therefore$ $dz = pdx + qdy = [y - x + f(\alpha)]dx + [x - y + \beta]dy$, using (14) and (15)

$= -(x - y)\,(dx - dy) + f(\alpha)dx + \beta dy$

$= -(x - y)\,d(x - y) + f(\alpha)\,(d\beta - d\alpha) + \beta[f'(\alpha)dx - F'(\beta)d\beta]$, by (16)

$= -(x - y)\,d(x - y) - f(\alpha)\,d\alpha + \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - \beta F'(\beta)d\beta$

or $$dz = -(x - y)\,d(x - y) - f(\alpha)\,d\alpha + d[\beta f(\alpha)] - \beta F'(\beta)d\beta.$$

Integrating both sides and using integration by parts in the last term on the R.H.S., we obtain

$$z = -(1/2) \times (x - y)^2 - \int f(\alpha)\,d\alpha + \beta f(\alpha) - \left[\beta F(\beta) - \int F(\beta)\,d\beta\right]. \qquad \text{...(17)}$$

Let $$\int f(\alpha)\,d\alpha = \phi(\alpha) \quad \text{and} \quad \int F(\beta)\,d\beta = \psi(\beta) \qquad \text{...(18)}$$

so that $$f(\alpha) = \phi'(\alpha) \quad \text{and} \quad F(\beta) = \psi'(\beta). \qquad \text{...(19)}$$

Using (18) and (19), (12), (13) and (17) may be written as

$x = \beta - \alpha$, $y = \phi'(\alpha) - \psi'(\beta)$, $z = -(1/2) \times (x - y)^2 - \phi(\alpha) + \psi(\beta) + \beta[\phi'(\alpha) - \psi'(\beta)]$

which is the required solution in parametric form, α and β being parameters, and ϕ and ψ being arbitrary functions.

Ex. 5. *Solve* $rt - s^2 + a^2 = 0$. **[Rohilkhand 1993]**

Sol. Given that $$0 \cdot r + 0 \cdot s + 0 \cdot t + (rt - s^2) = -a^2. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0$, $S = 0$, $T = 0$, $U = 1$, $V = -a^2$. ...(2)

Then, the λ–quadratic $$\lambda^2(UV + RT) + \lambda SU + U^2 = 0 \qquad \text{...(3)}$$

becomes $-\lambda^2 a^2 + 1 = 0$ or $\lambda = \pm 1/a$. So $\lambda_1 = 1/a$, $\lambda_2 = -1/a$. ...(4)

Two intermediate integrals of (1) are given by the following two sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad \text{... (5)}$$

and

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad \text{... (6)}$$

Using (2) and (4), equations (5) and (6) reduce to

$$dy + (1/a) \times dp = 0 \quad i.e., \quad dp + ady = 0$$
$$dx - (1/a) \times dq = 0 \quad i.e., \quad dq - adx = 0 \qquad \ldots (5A)$$

and

$$dy - (1/a) \times dp = 0 \quad i.e., \quad dp - ady = 0$$
$$dx + (1/a) \times dq = 0 \quad i.e., \quad dq + adx = 0 \qquad \ldots (6A)$$

Integration of (5A) and (6A) respectively gives

$$p + ay = c_1, \qquad q - ax = c_2 \qquad \ldots(5B)$$

and

$$p - ay = c_3, \qquad q + ax = c_4. \qquad \ldots(6B)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediate integrals are given by

$$p + ay = f(q - ax) \quad \text{and} \quad p - ay = F(q + ax). \qquad \ldots(7)$$

where f and F are arbitrary functions

Let $q - ax = \alpha$ ...(8)

and $q + ax = \beta$. ...(9)

Then, from (7) $p + ay = f(\alpha)$ ...(10)

and $p - ay = F(\beta)$. ...(11)

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (1/2a) \times (\beta - \alpha) \qquad \ldots(12)$$

and

$$y = (1/2a) \times [f(\alpha) - F(\beta)]. \qquad \ldots(13)$$

From (10), $p = f(\alpha) - ay$. ...(14)

From (9), $q = \beta - ax$. ...(15)

From (12) and (13), $dx = (1/2a) \times (d\beta - d\alpha)$, $dy = (1/2a) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]$ ...(16)

$\therefore\ dz = pdx + qdy = [f(\alpha) - ay]dx + (\beta - ay)dy$, using (14) and (15)

$$= f(\alpha)dx + \beta dy - a(ydx + xdy)$$
$$= f(\alpha) \times (1/2a) \times (d\beta - d\alpha) + \beta \times (1/2a) \times [f'(\alpha)d\alpha - F'(\beta)d\beta] - ad(xy), \text{ by (16)}$$

or

$$2adz = \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - f(\alpha)d\alpha - 2a^2d(xy) - \beta F'(\beta)d\beta.$$

Integrating both sides and using the formula for integration by parts in the last term on R.H.S., we have

$$2az = \beta f(\alpha) - \int f(\alpha)d\alpha - 2a^2xy - \left[\beta F(\beta) - \int F(\beta)\,d\beta\right]. \qquad \ldots(17)$$

Let $\int f(\alpha)d\alpha = \phi(\alpha)$ and $\int F(\beta)\,d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$. ...(19)

Using (18) and (19), (12), (13) and (17) reduces to

$2ax = \beta - \alpha$, $2ay = \phi'(\alpha) - \psi'(\beta)$, $2az = \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) - 2a^2xy + \psi(\beta)$.

which is the required solution in parametric form, α, β, being parameters and $\phi(\alpha)$ and $\psi(\beta)$ being arbitrary functions.

Ex. 6. *Solve* $7r - 8s - 3t + (rt - s^2) = 36$.

Sol. Given that $7r - 8s - 3t + (rt - s^2) = 36$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 7$, $S = -8$, $T = -3$, $U = 1$, $V = 36$. ...(2)

The λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ ...(3)

becomes $15\lambda^2 - 18\lambda + 1 = 0$ or $(5\lambda - 1)(3\lambda - 1) = 0$. So $\lambda_1 = 1/5$, $\lambda_2 = 1/3$. ...(4)

Two intermediate integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad \text{... (5)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \quad \text{... (6)}$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} dy + (1/5) \times (-3)dx + (1/5) \times dp = 0 \quad i.e., \quad dp - 3dx + 5dy = 0 \\ dx + (1/3) \times 7\, dy + (1/3) \times dq = 0 \quad i.e., \quad dq + 7dy + 3dx = 0 \end{aligned}\right\} \quad \text{... (5A)}$$

$$\left.\begin{aligned} dy + (1/3) \times (-3)dx + (1/3) \times dp = 0 \quad i.e., \quad dp - 3dx + 3dy = 0 \\ dx + (1/5) \times 7dy + (1/5) \times dq = 0 \quad i.e., \quad dq + 7dy + 5dx = 0 \end{aligned}\right\} \quad \text{... (6A)}$$

Integrating of (5*A*) and (6*A*) respectively, gives

$$p - 3x + 5y = c_1, \qquad q + 7y + 3x = c_2 \qquad \text{...(5}B\text{)}$$

and $$p - 3x + 3y = c_3, \qquad q + 7y + 5x = c_4, \qquad \text{...(6}B\text{)}$$

where c_1, c_2 c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediatre integrals are given by

$$p - 3x + 5y = f(q + 7y + 3x) \quad \text{and} \quad p - 3x + 3y = F(q + 7y + 5x) \quad \text{...(7)}$$

where f and F are arbitrary functions

Let $q + 7y + 3x = \alpha$ ...(8)

and $q + 7y + 5x = \beta$. ...(9)

Then, from (7) $p - 3x + 5y = f(\alpha)$ ...(10)

and $p - 3x + 3y = F(\beta)$. ...(11)

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (\beta - \alpha)/2 \qquad \text{...(12)}$$

and $$y = [f(\alpha) - F(\beta)]/2 \qquad \text{...(13)}$$

From (10), $p = f(\alpha) + 3x - 5y$. ...(14)

From (9), $q = \beta - 7y - 5x$. ...(15)

From (12) and (13), $dx = (1/2) \times (d\beta - d\alpha)$, $dy = (1/2) \times \{f'(\alpha)d\alpha - F'(\beta)\, d\beta\}$. ...(16)

$\therefore$ $dz = pdx + qdy = \{f(\alpha) + 3x - 5y\}dx + \{\beta - 7y - 5x)dy$, using (14) and (15)

$$= 3xdx - 7ydy - 5(ydx + xdy) + f(\alpha)dx + \beta dy$$

$$= 3xdx - 7ydy - 5d(xy) + f(\alpha) \times (1/2) \times (d\beta - d\alpha) + \beta \times (1/2) \times \{f'(\alpha)d\alpha - F'(\beta)d\beta\}$$

or $$2dz = 6xdx - 14ydy - 10d(xy) + \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - f(\alpha)d\alpha - \beta F'(\beta)d\beta$$

or $$2dz = 6xdx - 14ydy - 10d(xy) + d\{\beta f(\alpha)\} - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating both sides and using the formula for integrating by parts in the last term on R.H.S., we have

$$2z = 3x^2 - 7y^2 - 10xy + \beta f(\alpha) - \int f(\alpha)\, d\alpha - [\beta F(\beta) - \int F(\beta)\, d\beta]$$

or $$2z = 3x^2 - 7y^2 - 10xy + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)\, d\alpha + \int F(\beta)\, d\beta. \qquad \text{...(17)}$$

Let $\int f(\alpha)\, d\alpha = \phi(\alpha)$ and $\int F(\beta)\, d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$ ...(19)

Using (18) and (19), relation (12), (13) and (17) become

$x = (1/2) \times (\beta - \alpha)$, $y = (1/2) \times [\phi'(\alpha) - \psi'(\beta)]$, $2z = 3x^2 - 7y^2 - 10xy + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$.

which is required solution in parametric form, α and β being parameters and $\phi(\alpha)$ and $\psi(\beta)$ being

arbitrary functions.

9.14 Miscellaneous examples on Rr + Ss + Tt + U(rt – s²) = V.

In some problems only one intermediate integral is possible. Sometimes even after getting two intermediate integrals, it may not be possible to get p and q from those intermediate integrals. In such problems, final solution is obtained by integrating only one intermediate integral by the methods of solution of first order equation, for example, Charpit's method. Again, we can avoid Charpit's method by taking $u_1 = \phi_1(v_1)$ and u_2 = constant = λ (say) to obtain final solution. [Here we have assumed that $u_1 = \phi_1(v_1)$ and $u_2 = \phi_2(v_2)$ are two intermediate integrals]. Since an arbitrary constant can be regarded as a particular case of an arbitrary function, the values of p and q derived from $u_1 = \phi_1(v_1)$ and $u_2 = \lambda$ will make $dz = pdx + qdy$ integrable. The complete integral so obtained will involve one arbitrary funtion ϕ_1 and two arbitrary constants, namely, λ and the constant of integration. To obtain the general integral, express one of the arbitrary constant as an arbitrary function of the other and eliminate this remaining constant between the equation so obtained and that deduced from it by differentiation with respect to that constant.

Ex. 1. *Obtain the intermediate integral of* $2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq$. **[Rohilkhand 1992]**

Sol. Given $2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = 2y$, $S = px + qy$, $T = x$, $U = -xy$, $V = 2 - pq$. ...(2)

Now, λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ ...(3)

reduces to $\lambda^2[2yx - xy(2 - pq)] + \lambda[-xy(px + qy)\} + x^2y^2 = 0$

or $\lambda^2 pq - \lambda(px + qy) + xy = 0$ or $(\lambda p - y)(\lambda q - x) = 0$

or $\lambda = y/p, x/p$ so that $\lambda_1 = y/p$ and $\lambda_2 = x/q$. ...(4)

Two intermediate integrals are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad \text{... (5)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \quad \text{... (6)}$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} -xydy + (y/p)xdx + (y/p)(-xy)dp = 0 \quad i.e., \quad (pdy + ydp) - dx = 0 \\ \text{and} \quad -xydx + (x/q)(2y)dy + (x/q)(-xy)dq = 0 \quad i.e., \quad (qdx + xdq) - 2dy = 0. \end{aligned}\right\} \quad \text{... (5A)}$$

$$\left.\begin{aligned} -xydy + (x/q)xdx + (x/q)(-xy)dp = 0 \quad i.e., \quad -qydy + xdx - xydp = 0 \\ \text{and} \quad -xydx + (y/p)(2y)dy + (y/p)(-xy)dq = 0 \quad i.e., \quad -pxdx + 2ydy - xydq = 0 \end{aligned}\right\} \quad \text{...(6A)}$$

Integrating (5A), $py - x = c_1$ and $qx - 2y = c_2$

Hence one intermediate integral is $py - x = \phi(qx - 2y)$. ...(7)

Note, the equation (6A) cannot be integrated. Hence in this problem we can obtain only one intermediate integral, *i.e.*, (7). Here ϕ is an arbitrary function.

Ex. 2. *Solve* $qr + (p + x)s + yt + y(rt - s^2) + q = 0$. **[Rohilkhand 1992]**

Sol. Given $qr + (p + x)s + yt + y(rt - s^2) = -q$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = q$, $S = p + x$, $T = y$, $U = y$ and $T = -q$. ...(2)

Now, the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ ...(3)

reduces to $(0 \times \lambda^2) + \lambda y(p + x) + y^2 = 0$. Since coefficient of λ^2 is zero, it follows that its one root is ∞. The other root is $-y/(p + x)$.

Let $\lambda_1 = -y/(p + x)$ and $\lambda_2 = \infty$. ...(4)

One intermediate integral is given by the following sets

$$\left.\begin{aligned} &Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \quad \text{,i.e.,} \quad Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ &Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \qquad (1/\lambda_2)\times Udx + Rdy + Udq = 0. \end{aligned}\right\} \quad \text{... (5)}$$

Using (2) and (4), equations of (5) reduce to

$$\left.\begin{aligned} &ydy - \frac{y^2}{p+x}dx - \frac{y^2}{p+x}dp = 0 \quad \text{i.e.} \quad \frac{dp+dx}{p+x} - \frac{dy}{y} = 0 \\ \text{and} \quad &qdy + ydq = 0 \quad \text{i.e.,} \quad d(yq) = 0 \end{aligned}\right\} \quad \text{... (5A)}$$

Integrating (5A), $\log (p + x) - \log y = \log c_1$ or $(p + x)/y = c_1$ ...(6)

and $yq = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), an intermediate integral is given by $qy = f\left(\frac{p+x}{y}\right)$. ...(8)

Charpit's auxiliary equations for (8) are $\frac{dx}{-\frac{1}{y}f'\left(\frac{p+x}{y}\right)} = \frac{dy}{y} = \frac{dp}{\frac{1}{y}f'\left(\frac{p+x}{y}\right)}$. ...(9)

Taking the first and third fractions of (9), we have

$dx + dp = 0$ so that $x + p = c$, where c is an arbitrary constant. ...(10)

Solving (8) and (10) for p and q, $p = c - x$ and $q = (1/y) \times f(c/y)$.

Putting these values of p and q in $dz = pdx + qdy$, $dz = (c - x)dx + (1/y) \times f(c/y)dy$.

Integrating, $z = cx - x^2/2 + F(c/y) + G(\lambda)$,

which is the complete integral, F and G being arbitrary functions.

Ex. 3. *Solve* $qxr + (x + y)s + pyt + xy\,(xt - s^2) = 1 - pq$.

Sol. Given $qxr + (x + y)s + pyt + xy(rt - s^2) = 1 - pq$. ...(1)

Comparing (1) with $Rs + Ss + Tt + U(rt - s^2) = V$, we have

$R = qx$, $S = x+y$, $T = py$, $U = xy$ and $V = 1 - pq$ ...(2)

Now, the λ–qaudratic $\lambda^2(UV+RT) + \lambda US + U^2 = 0$ ...(3)

reduces to $\lambda^2 [qxpy+xy\,(1 - pq)] + \lambda xy\,(x + y) + x^2y^2 = 0$

or $\lambda^2 + (x + y)\lambda + xy = 0$ or $(\lambda + x)\,(\lambda + y) = 0$ so that $\lambda = -x\,, -y$.

Let $\lambda_1 = -x$ and $\lambda_2 = -y$. ...(4)

Two intermediate integrals are given by the following two sets :

$$\left.\begin{aligned} &Udy + \lambda_1 T\,dx + \lambda_1 Udp = 0 \\ &Udx + \lambda_2\, Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad \text{...(5)}$$

and

$$\left.\begin{aligned} &Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ &Udx + \lambda_1\, Rdy + \lambda_1 Udq = 0. \end{aligned}\right\} \quad \text{...(6)}$$

Using (2) and (4), equation (5) and (6) reduce to

$$\left.\begin{aligned} &xydy - xpydx - x^2\, ydp = 0 \quad \text{i.e.,} \quad (xdp + pdx) - dy = 0 \\ \text{and} \quad &xydx - yqxdy - yxydq = 0 \quad \text{i.e.,} \quad (ydq + qdy) - dx = 0 \end{aligned}\right\} \quad \text{...(5A)}$$

$$\left.\begin{aligned} &xydy - ypydx - y\,xy\,dp = 0 \quad \text{i.e.,} \quad xdy - pydx - xydp = 0 \\ \text{and} \quad &xydx - xyqdy - x^2\, ydq = 0 \quad \text{i.e.,} \quad ydx - qxdy - xydq = 0 \end{aligned}\right\} \quad \text{...(6A)}$$

We observe that (5*A*) can be integrated whereas (6*A*) cannot the integrated. So we shall obtain only one intermediate integral with help of (5*A*) :

Integrating (*5A*), $px - y = c_1$ and $qy - x = c_2$

Hence the only intermediate integral of (1) is given by

$px - y = f(qy - x)$, where f is an arbitrary function. ...(7)

A general solution of (1) can be obtained by supposing the arbitrary function f occuring in the intermediate integral (7) to be linear, giving

$px - y = m(qy - x) + n$, where m and n are arbitrary constants

or $xp - myq = y - mx + n$, which is in Lagrange's form

Hence here Lagrange axuiliary equations are $\frac{dx}{x} = \frac{dy}{-my} = \frac{dz}{y - mx + n}$ ...(8)

From first and second fractions of (8), $m(1/x)dx + (1/y)dy = 0$

Integrating, $m \log x + \log y = \log a$ or $x^m y = a.$ ...(9)

Chossing m, $1/m$, 1 as multipliers, each fraction of (8)

$$= \frac{mdx + (1/m)dy + dz}{mx + (1/m)(-my) + y - mx + n} = \frac{mdx + (1/m)dy + dz}{n} \quad ...(10)$$

Taking first fraction of (8) and (10), we have

$\frac{dx}{x} = \frac{mdx + (1/m)dy + dz}{n}$ or $dz + \frac{1}{m}dy + mdx - \frac{n}{x}dx = 0.$

Integrating, $z + (1/m)y + mx - n \log x = b$, b being an arbitrary constant ...(11)

From (9) and (11) the required general solution in

$z + (1/m)y + mx - \log x^n = \psi(x^m y)$, ψ being an arbitrary function

Ex. 4. *Solve* $(rt - s^2) - s(\sin x + \sin y) = \sin x \sin y.$ **[Meerut 1999]**

Sol. Given $0 \cdot r - s(\sin x + \sin y) + 0 \cdot t + (rt - s^2) = \sin x \sin y.$ ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = 0$, $S = -(\sin x + \sin y)$, $T = 0$, $U = 1$, $V = \sin x \sin y.$ ...(2)

Now, the λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0.$ ...(3)

reduces to $\sin x \sin y \, \lambda^2 - (\sin x + \sin y)\lambda + 1 = 0$

or $(\lambda \sin x - 1)(\lambda \sin y - 1) = 0$ so that $\lambda = \text{cosec } x$ or $\text{cosec } y.$

Let $\lambda_1 = \text{cosec } x$ and $\lambda_2 = \text{cosec } y.$ ...(4)

Two intermediate integral are given by the following two sets :

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad ...(5)$$

and

$$\left.\begin{aligned} Udy + \lambda_2 \text{Tdx} + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \quad ...(6)$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} dy + \text{cosec } x \, dp = 0 \\ dx + \text{cosec } y \, dq = 0 \end{aligned}\right\} \quad ...(5A)$$

$dy + \text{cosec } y \, dp = 0$ *i.e.*, $dp + \sin y \, dy = 0$

and $dx + \text{cosec } x \, dq = 0$ *i.e.*, $dq + \sin x \, dx = 0$ ...(6A)

We observe that (5*A*) cannot be integrated where as (6*A*) can be integrated. So we shall obtain only one intermediate integral with help of (6A).

Integrating (6*A*), $p - \cos y = c_1$ and $q - \cos x = c_2.$

Hence the only intermediate integral of (1) is given by

$p - \cos y = f(q - \cos x)$, f being an arbitrary function. ...(7)

A general solution of (1) can be obtained by supposing the arbitrary function f occuring in the the intermediate integral (7) to be linear, giving

$p - \cos y = m[q - \cos x] + n$, m, n being arbitrary constants

or $p - mq = \cos y - m \cos x + n$, which is in Lagrange's from.

Its Lagrange's auxiliary equations are $\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\cos y - m \cos x + n}.$ ...(8)

From the first two fractions of (8), $dy + mdx = 0$ so that $y + mx = a$. ...(9)

Again, taking the first and third fractions of (8), we have

$dz = (\cos y - m \cos x + n)dx = [\cos (a - mx) - m \cos x + n]dx$, as from (9), $y = a - mx$

Intergrating, $z = -(1/m) \times \sin (a - mx) - m \sin x + nx + (1/m) \times b$

or $mz + \sin y + m^2 \sin x - mnx = b$, b being an arbitrary constants ...(10)

From (9) and (10), the required general solution of (1) is

$mz + \sin y + m^2 \sin x - mnx = \phi(y + mx)$, ϕ being an arbitrary function.

Ex. 5. *Solve* $xqr + (p + q)s + ypt + (xy - 1)(rt - s^2) + pq = 0$.

Ans. $z - \log (x - m)^n = \phi\{(x - m)^n (1 - my)$

Ex. 6. *Solve* $2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq$.

Ans. $z + (1/m) \times (a^2 - mx^2)^{1/2} + (x/\sqrt{m}) \times \sin^{-1}(x\sqrt{m}/a) + 2mx = \phi(mx^2 + y^2)$

Ex. 7. *Solve* $ar + bs + ct + e(rt - s^2) = h$, *where* a, b, c, e *and* h *are constants.*

Sol. Comparing with $Rr + Ss + Tt + U(rt - s^2) = V$, here $R = a$, $S = b$, $T = c$, $U = e$, $V = h$.

The λ–quadratic $\lambda^2 (UV + RT) + \lambda SU + U^2 = 0$ gives $(ac + eh)\lambda^2 + \lambda be + e^2 = 0$. ...(1)

Let $\lambda = -e/m$. ...(2)

$\therefore$ (1) reduces to $m^2 - bm + (ac + eh) = 0$. ...(3)

Let m_1 and m_2 be the roots of (3). The first intermediate integral is given by

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$, where $\lambda_1 = -e/m_1$

and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$, where $\lambda_2 = -e/m_2$

i.e., $e\,dy - (e/m_1) \times c\,dx - (e/m_1) \times e\,dp$ and $ex - (e/m_2) \times a\,dy - (e/m_2) \times e\,dq = 0$

i.e., $c\,dx + edp - m_1\,dy = 0$ and $ady + edq - m_2 dx = 0$.

Integrating, $cx + ep - m_1 y = c_1$ and $ay + eq - m_2 x = c_2$.

So the first intermediate integral is $cx + ep - m_1 y = \phi_1(ay + eq - m_2 x)$. ...(4)

Proceding as before, the second intermediate integral is

$cx + cp - m_2 y = \phi_2(ay + eq - m_1 x)$. ...(5)

Notice that p and q cannot be determined from (4) and (5). Hence we proced as follows :

We also have, $cx + ep - m_2 y = c_3$ ...(6)

From (4) and (6), $(m_2 - m_1)y = \phi_1(ay + eq - m_2 x) - c_3$

$\therefore$ $ay + eq - m_2 x = \psi_1\{(m_2 - m_1)y + c_3\}]$ ...(7)

where ψ_1 is the inverse function of ϕ_1.

From (7). $q = (1/e) \times [-ay + m_2 x + \phi_1\{(m_2 - m_1)y + c_3\}]$

and from (6) $p = (-cx + m_2 y + c_3)/e$.

Putting these values in $dz = pdx + qdy$, we get

$edz = -(xdx - aydy + m_2(xdy + ydx) + c_3 dx + \phi_1\{(m_2 - m_1)y + c_3\}dy$

Integrating, $ez = -(1/2) \times cx^2 - (1/2) \times ay^2 + m_2 xy + c_3 x + F\{(m_2 - m_1)y + c_3\} + k$.

EXERCISE

Solve the following partial differential equation:

1. $3r + s + t + (rt - s^2) = -9$ **[K.U. Kurukshetra 2004]**

2. $3s - 2(rt - s^2) = 2$ **3.** $2r - 6s + 2t + (rt - s^2) = 4$

Objective problems

1. The equations $R\,dpdy + T\,dqdx - V\,dxdy = 0$ and $Rdy^2 - S\,dxdy + T\,dx^2 = 0$ are called

Sol. Ans. Monge's subsidiary equations. Refer Art. 9.2. **[Meerut 2003].**

2. Monge's method is used to solve a partial differential equation of

(a) nth order (b) first order (c) second order (d) none of these **[Agra 2007]**

Sol. Ans. (**c**). Refer Art 9.2

10

Transport Equation

10.1 INTRODUCTION

The hyperbolic character of a system of first order differential equations exhibits in the fact that it is possible to have solutions whose derivatives are discontinuous and these discontinuities propagate along the characteristic curves. In this chapter we propose to use the above fact and derive a system of linear homogeneous ordinary differential equations known as *transport equation.*

10.2 An IMPORTANT THEOREM

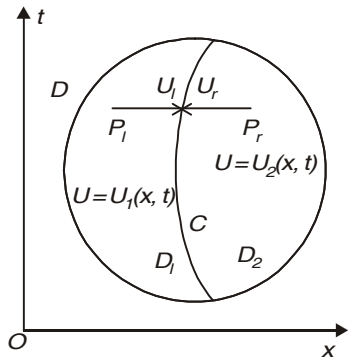
If the first order partial derivatives of a continuous function U (x, t), satisfying a system of quasi-linear equations of first order on both sides of a curve C in xt-plane, are discontinuous across a curve C, then the curve C must be a characteristic curve of the system of equations.

Proof : Suppose that the given first-order system of n quasi-linear partial differential differential equations be given by

$$\left.\begin{aligned} A_{ij}\,(\partial u_j / \partial t) + B_{ij}\,(\partial u_j / \partial x) + C_i = 0, \quad i = 1, 2, ..., n; \quad j = 1, 2, ..., n \\ A(\partial U / \partial t) + B\,(\partial U / \partial t) + C = 0, \end{aligned}\right\} \quad ...(1)$$

or

where the n components $u_1, u_2, ..., u_n$ of the column vector U are dependent variables, A and B are $n \times n$ matrices and C is $a\ n \times 1$ column vector.



Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C such that D_1 is on the left and D_2 on the right of C as shown in the adjoining figure. Let U_1 be the genuine solution of (1) in the domain D_1 and U_2 that in the domain D_2. Suppose that the limiting value U_l *of* U_1 as we approach a point P on C from the domain D_1 and the limiting value U_r as we approach P from the domain D_2 exist and are such that $U_l = U_r$ at every point of the curve C. Let a function U be defined in the domain D such that

$$U = \begin{cases} U_1 & \text{in } D_1 \\ U_2 & \text{in } D_2 \end{cases} \quad ...(2)$$

The function U given by (2) is a genuine solution of (1) in D_1 and D_2 respectively. The function U is continuous in D but its derivatives may be discontinuous across the curve C. Suppose that the limiting values of the derivatives of U as we approach P on the curve C from the two domains D_1 and D_2 exist. Also assume that these derivative, if discontinuous across the curve C, have only a finite jump across the curve C.

Let the equation of the curve C be $\phi(x, t) = 0$ and let $\eta(x,t)$ be any other function independent of ϕ such that ϕ and η are sufficiently smooth and the Jacobian $\partial(\phi, \eta)/\partial(x,t) \neq 0$ in the domain D. Therefore, if we can introduce a new set of independent variables (ϕ, η) in place of (x, t), then U_ϕ represents an exterior derivative and U_η is a tangential derivative along the curve C.

We have
$$U_x = \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = U_\phi\, \phi_x + U_\eta\, \eta_x$$
and
$$U_t = \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = U_\phi \phi_t + U_\eta \eta_t \quad \text{...(3)}$$

Let us now assume that the first order partial derivatives U_x and U_t are discontinuous across the curve C. Since the function U is continuous across C, its tangential derivative U_η is also continuous across C. Hence from the above two relations (3), it follows that the exterior derivative U_ϕ must be discontinuous across C. Let $(U_\phi)_r$ and $(U_\phi)_l$ be the limiting values of U_ϕ across C, as we approach a point P on C from the domains D_1 and D_2 respectively. Then, the jump $[U_\phi]$ in U_ϕ across C is given by

$$[U_\phi] = (U_\phi)_r - (U_\phi)_l \quad \text{....(4)}$$

Since U_η is continuous across the curve C, hence from (3), the jumps in the first order derivatives U_x and U_t are related to $[U_\phi]$ by the relations

$$[U_x] = [U_\phi]\phi_x \quad \text{and} \quad [U_t] = [U_\phi]\phi_t \quad \text{...(5)}$$

The quasi-linear equation (1) is valid everywhere in D except at the points on the curve C. Since all the terms appearing in it other than the first order derivatives are continuous across the curve C, hence taking the limit of (1) as we move from the region D_1 to P and again as we more from the region D_2 to P and then subtracting the equations so obtained, we obtain

$$(A\phi_t + B\phi_x)_{at\,P}\,[U_\phi] = 0 \quad \text{...(6)}$$

Since $[U_\phi]$ is not a zero vector, the matrix $A\phi_t + B\phi_x$ must be singular on the curve C, i.e., at every point of the curve C, we have

or
$$\left.\begin{aligned} &\det\,(A\,\phi_t + B\,\phi_x) = 0 \\ &\det(-\lambda\, A + B) = 0, \text{ where } \lambda = -\phi_t / \phi_x \end{aligned}\right\} \quad \text{...(6)}$$

Hence, it follows that C is a characteristic curve.

Note 1. In (6), det X stands for determinant of the matrix X.

Note 2. If the derivatives of a solution U of system (1) upto order $(r \geq 1)$ are continuous across a curve C and the $(r + 1)$ th derivatives are discontinuous across C, then differentiating (1) r times and then proceeding as discussed above, we can show that C is necessarily a characteristic curve.

10.3 GENERALISED OR WEAK SOLUTION

[Allahabad 2003; G.N.D.U. Amritsar 2004; Kanpur 2003, 05)

Consider a general first order quasi-linear hyperbolic system of first order equations

$$A(x, t)U_t + B(x, t)\,U_x + H(x, t)\,J\,(x, t) = 0, \quad \text{...(1)}$$

where the elements of matrices A, B and H (each of order $n \times n$) and column vector J (of order $n \times 1$) are functions of x and t only. Also note that $U_t = \partial U / \partial t$ and $U_x = \partial U / \partial x$. Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C.

Suppose we have a solution U which satisfies (1) in D_1 and D_2 separately but is itself discontinuous across C. It has been established that discontinuities in the solution cannot be discussed for every function U satisfying (1) in D_1 and D_2. However, such discontinuities can be discussed for a "*generalised"* or '*weak*" solution.

To end, we first reduce the system (1) to the characteristic canonical form and note that $(\partial/\partial t) + \lambda_M(x, t)(\partial/\partial x)$ represents the directional derivatives along the characteristics of the Mth field. Integrating it from a point P_M *to a point* (ξ, τ), *both lying on a* characteristic of the Mth field, we obtain.

$$W_M(\xi, \tau) = -\int_{P_M}^{P} [H_{M_i}\{x_M(t, \xi, \tau), t\}\, W_i\{x_M(t, \xi, \tau), t\}]dt$$

$$+ (W_M)_{at\ P_M} \int_{P_M}^{P} J_M\{x_M(t, \xi, \tau), t\}dt, \text{ for } M = 1, 2, \ldots n \qquad \ldots(2)$$

where $$H_{M_i} = [l^{(M)} A\{\partial r^{(i)}/\partial t + \lambda_M(\partial r^{(i)}/\partial x)\} + (l^{(M)}\ H\, r^{(i)})]/(l^{(M)} A r^{(M)}) \qquad \ldots(3)$$

and $$J_M = (l^{(M)} J)/(l^{(M)}\ A\, r^{(M)}) \qquad \ldots(4)$$

with no sum over M in these expressions and $x = x_M(t,\ \xi, \tau)$ is the characteristic of Mth field through the point P.

We can rewrite in compact form the expression for H_{M_i} in terms of the operator $\tau \equiv A(\partial/\partial t) + B(\partial/\partial x) + H$. Thus, (3) takes the form

$$H_{M_i} = [l^{(M)} \tau\, r^{(i)}]/[l^{(M)}\ A\ r^{(M)}] \qquad \ldots(5)$$

We now define a generalised or weak solution of (1) to be a function $U(x, t)$ obtained from $W_1, W_2, \ldots, W_n$ which satisfy the system of equations (2).

10.4 TRANSPORT EQUATION FOR A LINEAR-HYPERBOLIC SYSTEM

[Calicut 2003 G.N.D.U. (Amritsar 2005; Kanpur 2004; Meerut 2005, 06, 10, 11]

Consider a general first order quasi-linear hyperbolic system of first order equations

$$A(x, t)\, U_t + B(x, t)\, U_x + H(x, t)\, J\,(x, t) = 0, \qquad \ldots(1)$$

where the elements of matrices A, B and H(each of order $n \times n$) and column vector J (of order $n \times 1$) are functions of x and t only. Also note that $U_t = \partial u/\partial t$ and $U_x = \partial U/\partial x$. Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C. Suppose $U(x, t)$ is a weak solution of (1) which is continuous in the domain D except on the curve C and is a genuine solution of (1) in the domains D_1 and D_2. We also suppose that the function U has a jump discontinuity across C.

We now reduce the system (1) to the characteristic canonical form and note that $(\partial/dt) + \lambda_M(x, t)(\partial/\partial x)$ represents the directional derivatives along the characteristics of the Mth field. Integrating it from a point P_M to a point (ξ, τ), both lying on a characteristic of the Mth field, we have

$$W_M(\xi, \tau) = -\int_{P_M}^{P} [H_{M_i}\{x_M(t, \xi, \tau), t\}\, W_i\{x_M(t, \xi, \tau), t]\, dt$$

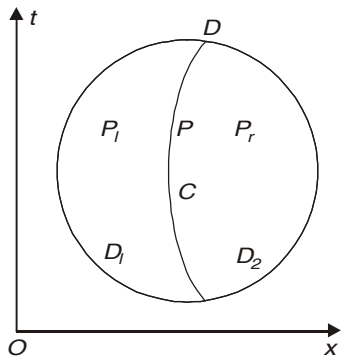
$$+ (W_M)_{at\ P_M} - \int_{P_M}^{P} [J_M\{x_M(t, \xi, \tau), t\}\, dt, \text{ for } M = 1, 2, \ldots n \qquad \ldots(2)$$

where $$H_{M_i} = [l^{(M)} A\{\partial r^{(i)} / \partial t + \lambda_M (\partial r^{(i)} / \partial x)\} + l^{(M)}\ Hr^{(i)}] / [l^{(M)} A\, r^{(M)}] \quad ...(3)$$

and $$J_M = (l^{(M)} J) / [l^{(M)} A\ r^{(M)}] \quad ...(4)$$

with no sum over M in these expressions and $x = x_M(t, \xi, \tau)$ is the characteristic of Mth field through the point P.

In the case when the function U has a jump discontinuity across the curve C, the integrands on the right hand side of (2) are continuous functions of t except for a finite jump across C. On performing the integration in (2) along a characteristic of the Mth family, we find that the characteristic variable W_M is given by a continuous function of this curve. If a curve C is not tangential to a characteristic of the Mth family, W_M must be continuous across the curve C. However, according to our assumption at least one of $W_1, W_2, \ldots, W_n$ must be discontinuous across the curve C. Therefore, it follows that C, the curve of discontinuity, must be a characteristic curve of jth family (say), and the jump in all characteristic variables $W_i,\ i \neq j,$ must be zero across the carve C.



Now, suppose that the curve of discontinuity C is a characteristic curve of the jth family then, the jump $[W_i]$ in W_i satisfies

$$[W_i] = 0, \text{ for } i = j \qquad \text{and} \qquad [W_j] = 0 \quad ...(5)$$

Again, we have $$[U] = r^{(j)}[W_j], \text{ on sum over } j \quad ...(6)$$

Re-writing the equation, we have

$$(\partial W_M / \partial t) + \lambda_M (\partial W_M / \partial x) + C_1\ M_i W_i + J_M = 0, \qquad M = 1, 2, \ldots, n \quad ...(7)$$

Consider two points P_l and P_r on the two sides of C in the regions D_1 and D_2 respectively as shown in the figure. Taking limit as both these points tend to P on the curve C and substracting the results so obtained we obtain

$$\frac{d}{dt}\left[W_j\right] = -H_{jj}[W_j], \text{ no sum over } j \quad ...(8)$$

where $$d / dt \equiv (\partial / \partial t) + (\lambda_j)(\partial / \partial x) \quad ...(9)$$

The above equation (8) is known as the *transport equation*.

Along a given characteristic curve $x = x_j(t)$ of the j the family, the function $H_{jj}(x, t) = H_{jj}(x_j(t), t)$ is a function of t only.

Thus, the transport equation (8) is a linear homogeneous ordinary differential equation of first order and determines the vibration of $[W_j]$, jump in W_j along a characteristic curve of the jth family. From the properties of solutions of linear homogeneous ordinary differential equations, it follows that if there is a discontinuity in U at some point of a characteristic curve C, the discontinuity in U remains non-zero at every point on the curve.

Note. In order to obtain the transport equation for the discontinuities in the derivatives of U of order $n (n \geq 1)$, differentiate (7) $n (n \geq 1)$ times and proceed in exactly same manner as above.

EXERCISE

1. Write short note on the transport equation for a linear hyperbolic system of first order equations. **(Calicut 2003; Meerut 2005; 06)**

2. When all the characteristic velocities λ_i are different from zero, prove that the first order quasi-linear hyperbolic system

$$A(x, t, U)\ (\partial U / \partial t) + B(x, t, U)\ (\partial U / \partial x) + C(x, t, U) = 0$$

can be reduced to a diagonal canonical system of $2n$ equations $(\partial U / \partial t) - RW = 0$ and $(\partial U / \partial t) + A(\partial W / \partial x) + F = 0,$ where the coefficients A, R and F are functions of x, t, U and W

3. Consider the hyperbolic system $u_t + (x+t)v_x = 0,$ $(x+t)v_t + u_x = 0$

Show that the variation in jump $[v]$ along the characteristic curve $(x-t) = C$ (C = constant) is given by

$$[v] = A/(2t+c)^{1/2},$$ A being a constant.

Derive also the transport equation of discontinuities in the first order partial derivatives of u and v.

MISCELLANEOUS PROBLEMS BASED ON THIS PART OF THE BOOK

Ex. 1. *The solution of* $xu_x + yu_y = 0$ *is of the form*

(*a*) $f(y/x)$ (*b*) $f(y + x)$ (*c*) $f(x - y)$ (*d*) $f(xy)$ **[GATE 2008]**

Sol. Ans. (*a*). Given $xu_x + yu_y = 0$ …(1)

which is in the form of Lagrange equation $Pp + Qq = R$, with u in place z. Hence, the Lagrange's auxiliary equations for (1) are given by

$$(dx)/x = (dy)/y = du/0 \quad \ldots(2)$$

Taking the first two fractions of (2), $(1/y)\,dy - (1/x)\,dx = 0$

Integrating, $\log y - \log x = \log c_1$ or $y/x = c_1$ …(3)

Again, the last fraction of (2) yields $du = 0$ so that $u = c_2$ …(4)

From (3) and (4), the required solution is $u = f(y/x)$.

Ex. 2. *Solve* $x(y^2 + z)p + y(z + x^2)q = z(x^2 - y^2)$ **[Madurai Kamraj 2008]**

Sol. Do like Ex. 6, page 2.10. Here Lagrange's auxiliary equations are

or $$\frac{dx}{x(y^2+z)} = \frac{dy}{y(z+x^2)} = \frac{dz}{z(x^2-y^2)} \quad \ldots(1)$$

Choosing $1/x, -(1/y), 1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx - (1/y)dy + (1/z)dz}{y^2 + z - (z + x^2) + x^2 - y^2} = \frac{(1/x)dx - (1/y)dy + (1/z)dz}{0}$$

$\Rightarrow$ $(1/x)\,dx - (1/y)\,dy + (1/z)\,dz = 0$ so that $\log x - \log y + \log z = \log c_1$

or $\log(xz/y) = \log c_1$ or $(xz)/y = c_1$ …(2)

Choosing $x, -y, -1$ as multipliers each fraction of (1)

$$= \frac{xdx - ydy - dz}{x^2(y^2+z) - y^2(z+x^2) - z(x^2-y^2)} = \frac{xdx - ydy - dz}{0}$$

$\Rightarrow$ $xdx - ydy - dz = 0$ or $2xdx - 2ydy - 2dz = 0$

Integrating, $x^2 - y^2 - 2z = c_2$, c_2 being an arbitrary constant …(3)

From (2) and (3), the required solution is given by

$(xz)/y = \phi\,(x^2 - y^2 - 2z)$, ϕ being an arbitrary function

Ex. 3. *If the partial differential* $(x - 1)^2\,u_{xx} - (y - 2)^2\,u_{yy} + 2xu_x + 2yu_y + 2xyu = 0$ *is parabolic in* $S \subseteq R^2$ *but not in* $R^2 \setminus S$, *then S is*

(*a*) $\{(x, y) \in R^2 : x = 0 \text{ or } y = 2\}$ (*b*) $\{(x, y) \in R^2 : x = 1 \text{ and } y = 2\}$

(*c*) $\{(x, y) \in R^2 : x = 1\}$ (*d*) $\{(x, y) \in R^2 : y = 2\}$ **[GATE 2008]**

Sol. Ans. (*a*). Refer Art. 8.1. Here R is the set of all real numbers

Ex. 4. *Find the complete integral of* $xp + 3yq = 2(z - x^2q^2)$ **[Delhi Maths (H) 2008]**

Sol. Here given equation is $f(x, y, z, p, q) = xp + 3yq - 2z + 2x^2q^2 = 0$ … (1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

i.e., $\dfrac{dp}{p+4xq^2-2p}=\dfrac{dq}{3q-2q}=\dfrac{dz}{-px-q(3y+4x^2q)}=\dfrac{dx}{-x}=\dfrac{dy}{-(3y+4x^2q)}$, using (1) …(2)

Taking the second and fourth fractions of (2), we have

$(1/q)\,dq + (1/x)\,dx = 0$ so that $\log q + \log x = \log a$, giving

$qx = a$ so that $q = a/x$, a being an arbitrary constant …(3)

Substituting the value of q given by (3) in (1), we have

$xp + (3ya)/x - 2z + 2a^2 = 0$ or $xp = 2(z - a^2) - (3ya)/x$

Thus, $p = 2(z - a^2)/x - (3ya)/x^2$. …(4)

Substituting values of q and p given by (3) and (4) in $dz = pdx + qdy$, we get

$dz = \{2(z - a^2)/x - (3ya)/x^2\}dx + (a/x)\,dy$ or $x^2dz = 2x\,(z - a^2)\,dx - 3yadx + axdy$

or $x^2dz - 2x(z - a^2)\,dx = axdy - 3yadx$ or $[x^2dz - 2x\,(z - a^2)dx]/x^4 = ax^{-3}dy - 3ayx^{-4}\,dx$

or $d\{(z - a^2)/x^2\} = d\,(ayx^{-3})$

Integrating, $(z - a^2)/x^2 = (ay)/x^3 + b$, b being an arbitrary constant

or $z = a(a + y/x) + bx^2$, which is the required solution.

Ex.5. *Find the general integral of the partial differential equation* $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$. *Also, find the particular integral which passes through the straight line* $x = -1, z = 1$. **(Delhi B.A. (Prog.) 2009)**

Sol. Re-writing the given partial differential equation, we have

$$px\,(z - 2y^2) + qy(z - y^2 - 2x^3) = z\,(z - y^2 - 2x^3) \quad \text{... (1)}$$

Hence the usual Lagrange's, subsidiary equations are given by

$$\frac{dx}{x(z-2y^2)}=\frac{dy}{y(z-y^2-2x^3)}=\frac{dz}{z(z-y^2-2x^3)} \quad \text{…(2)}$$

Taking the last two fractions of (1), we get $(1/y)dy = (1/z)\text{dz}$

Integrating, $\log y = \log z + \log c_1$, c_1 being an arbitrary constant

Thus, $y = c_1z$. ...(3)

Next, taking the first and third fractions of (2) and using (3), we obtain

$$\frac{dx}{x(z-2c_1^2z^2)} = \frac{dz}{z(z-c_1^2z^2-2x^3)} \quad \text{or} \quad \frac{dz}{dx}=\frac{z-c_1^2z^2-2x^3}{x(1-2c_1^2z)}$$

Re-writing it, $(1-2c_1^2z)\,(dz/dx) - (z-c_1^2z^2)\times(1/x) = -2x^2$ …(4)

Putting $z-c_1^2z^2 = v$ so that $(1-2c_1^2z)(dz/dx) = dv/dx$, (4) reduces to

$$(dv/dx) - (1/x)v = -2x^2 \quad \text{…(5)}$$

whose integrating factor is $e^{\int(-1/x)dx} = e^{-\log x} = x^{-1}$ and hence solution of (5) is

$$v \times x^{-1} = \int\{(-2x^2)\times x^{-1}\}\,dx + c_2 = -x^2 + c_2,\ c_2 \text{ being an arbitrary constant}$$

or $(z-c_1^2z^2)/x + x^2 = c_2$ or $(z - y^2)/x + x^2 = c_2$, using (3) …(6)

The required general integral is given by (3) and (6).

We now find the required particular integral. To this end, replacing x by -1 and z by 1 in (3) and (6), we obtain

$y = c_1$ and $-(1 - y^2) + 1 = c_2$ ...(7)

Eliminating y between two relations of (7), we have $c_1^2 = c_2$ …(8)

Substituting the values of c_1 and c_2 given by (3) and (6) in (8), the required particular integral is given by

$$y^2/z^2 = (z - y^2) / x + x^2 \quad \text{or} \quad y^2x = z^2(z - y^2 + x^3)$$

Ex.6. *Solve the partial differential equation $z = px + qy + 3p - 2q$ by Lagrange's method as well as Charpit's method. Hence or otherwise give two different solutions of the above partial differential equation passing through (–3, 2, 0).* **(Delhi B.A. (Prog). II 2009)**

Sol. Solution of the given equation by Lagrange's method:

Re-writing the given equation, $(x + 3)p + (y - 2)q = z$ …(1)

Here the usual Lagrange's subsidiary equations are given by

$$(dx)/(x + 3) = (dy)/(y - 2) = (dz) / z \quad \text{…(2)}$$

Taking the first two fractions of (2), $(dx)/(x + 3) = (dy)/(y - 2)$

Integrating, $\log(x + 3) = \log(y - 2) + \log a$ or $(x + 3) = a\,(y - 2)$ …(3)

Next, taking the last two fractions of (2), $(dy)/(y - 2) = (dz)/z$

Integrating, $\log (y - 2) + \log b = \log z$ or $b\,(y - 2) = z$ ...(4)

From (3) and (4), $(x + 3)/a = (y - 2)/1 = (z - 0)/b$, a and b being arbitrary constants …(5)

which is the required solution of the given equation passing through (–3, 2, 0).

Solution of the given equation by Charpit's method:

Let $f(x, y, z, p, q) = (x + 3)p + (y - 2)q - z = 0$ …(6)

Here Charpit's auxiliary equations $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

yield $\dfrac{dp}{0} = \dfrac{dq}{0} = \dfrac{dz}{-p(x+3) - q(y-2)} = \dfrac{dx}{-(x+3)} = \dfrac{dy}{-(y-2)}$

Hence, $dp = 0$ so that $p = c_1$, c_1 being an arbitrary constant …(7)

From (6) and (7), $(x + 3)c_1 + (y - 2)q - z = 0 \Rightarrow q = \{z - (x + 3)c_1\}/(y - 2)$ …(8)

Substituting the values of p and q given by (7) and (8), we have

$$dz = pdx + qdy = c_1dx + [\{z - (x + 3)c_1\}/(y - 2)]dy$$

or $dz - \dfrac{z}{y-2}dy = c_1dx - \dfrac{(x+3)c_1dy}{y-2}$ or $\dfrac{(y-2)\,dz - z\,dy}{(y-2)^2} = \dfrac{c_1(y-2)\,dx + c_1(x+3)\,dy}{(y-2)^2}$

or $d\left(\dfrac{z}{y-2}\right) = d\left(\dfrac{c_1(x+3)}{y-2}\right)$ giving $\dfrac{z}{y-2} = \dfrac{c_1(x+3)}{y-2} + c_2$

Thus, $z = c_1(x + 3) + c_2\,(y - 2)$,

which is the required second solution of the given equation passing through (–3, 2, 0).

Ex. 7. *Define the singular integral of first order partial differential equation. Is it true that singular integral always exists? Justify your answer.* **[Delhi Math (Hons.) 2009]**

Hint: Refer Art. 3.1.

Ex. 8. *Write the form of the solution of the equation F (D, D') = 0, where F (D, D') is not reducible.* **[Delhi Math (Hons). 2009]**

Sol. Let $z = e^{hx + ky}$ be a trial solution of the given equation. Then, the required solution is

$$z = \sum_i A_i\, e^{h_i x + k_i y},$$ where A_i, h_i and k_i are arbitrary constants,

and h_i, k_i are connected by the relation $F(h_i, k_i) = 0$.

Ex. 9. *Solve* $z = px + qy + p^2 + q^2$. **[Kanpur 2009]**

Sol. Refer Art 3.12. The complete integral is $z = ax + by + a^2 + b^2$ …(1)

Singular integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$0 = x + 2a$ and $0 = y + 2b$ …(2)

From (2), $a = -(x/2)$ and $b = -(y/2)$. Substituting these values of a and b in (1), we get

$z = -(x^2/2) - (y^2/2) + x^2/4 + y^2/4$ or $4z + x^2 + y^2 = 0$.

General integral Take $b = \phi(a)$, where ϕ is an arbitrary function

Then, (1) yields $z = ax + y\,\phi(a) + a^2 + [\phi(a)]^2$ …(3)

Differentiating (3) partially w.r.t. 'a', $0 = x + y\phi'(a) + 2a + 2\phi(a)\phi'(a)$ …(4)

The general integral is obtained by eliminating a between (3) and (4).

Ex. 10. *Classify the following partial differential equation into elliptic, parabolic or hyperbolic and find its degree and order* $x(y-x)\,r - (y^2 - x^2)s + y(y-x)\,t + (y+x)(p-q) = 0$.

Hint. Use Art 1.3, 1.4 and 8.1. **[Delhi BA (Prog.) II 2009]**

Ans. The given equation is hyperbolic, its degree is one and its order is two.

Ex. 11. *Find the characteristic strips of the equation* $xp + yq - pq = 0$ *and then find the equation of the integral surface through the curve* $z = x/2,\ y = 0$ **[Meerut 2011]**

Sol. Given equation is $xp + yq - pq = 0$ ...(1)

We are to find its integral surface passing through the given curve, namely

$z = x/2, \quad y = 0$ …(2)

Re-writing (2) in parametric form, we have

$x = \lambda, \quad y = 0, \quad z = \lambda/2;$ λ being a parameter …(3)

Let the initial values of x_0, y_0, z_0, p_0 and q_0 of x, y, z, p and q be taken as

$x_0 = x_0(\lambda) = \lambda \quad y_0 = y_0(\lambda) = 0, \quad z_0 = z_0(\lambda) = \lambda/2$ …(4 A)

Let p_0 and q_0 be the initial values of p and q corresponding to the initial values of x_0, y_0, z_0. Since the initial values x_0, y_0, z_0, p_0 and q_0 satisfy (1), we have

$x_0 p_0 + y_0 q_0 - p_0 q_0 = 0$ or $\lambda p_0 - p_0 q_0 = 0$ or $q_0 = \lambda$, using (4A) …(5)

Also, we have $z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

so that $1/2 = p_0 + 0$ giving $p_0 = 1/2$, using (4A) …(6)

Thus, from (5) and (6), $p_0 = 1/2$ and $q_0 = \lambda$ …(4 B)

Collecting relations (4 A) and (4 B) together, initial values are given by

$x_0 = \lambda, \quad y_0 = 0, \quad z_0 = \lambda/2, \quad p_0 = 1/2$ and $q_0 = \lambda$, when $t = t_0 = 0$ …(7)

Re-writing (1), let $f(x, y, z, p, q) = xp + yq - pq = 0$ …(8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = x - p \quad \text{…(9)}$$

$$dy/dt = \partial f/\partial q = y - p \quad \text{…(10)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = p(x-q) + q(y-p) = -pq, \text{ using (1)} \quad \text{…(11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = -p - (p \times 0) = -p \quad \ldots(12)$$

$$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = -q - (q \times 0) = -q \quad \ldots(13)$$

From (12), $(1/p)dp = -dt$ so that $\log p - \log c_1 = -t$

Thus, $p = c_1 e^{-t}$, c_1 being an arbitrary constant …(14)

Similarly, (13) yields $q = c_2 e^{-t}$, c_2 being an a arbitrary constant …(15)

Using initial values (7), (14) yields $p_0 = c_1 e^{-t_0}$ giving $c_1 = 1/2$

Hence, (14) reduces to $p = (1/2) \times e^{-t}$ …(16)

Using initial values (7), (15) yields $q_0 = c_2 e^{-t_0}$ giving $c_2 = \lambda$

Hence, (15) reduces to $q = \lambda e^{-t}$ ...(17)

From (9) and (17), $dx/dt = x - \lambda e^{-t}$ or $dx/dt - x = -\lambda e^{-t}$,

whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and hence its solution is given by

$$xe^{-t} = c_3 + \int\{(-\lambda e^{-t}) \times e^{-t}\}\,dt,\ c_3 \text{ being an arbitrary constant}$$

or $$xe^{-t} = c_3 + (\lambda/2) \times e^{-2t} \quad \ldots(18)$$

Using initial values (7), (18) yields $x_0 e^{-t_0} = c_3 + (\lambda/2) \times e^{-2t_0}$

or $\lambda = c_3 + \lambda/2$ so that $c_3 = \lambda/2.$

Hence, (18) yields $xe^{-t} = (\lambda/2) \times (1 + e^{-2t})$

or $$x = (\lambda/2) \times e^{t}(1 + e^{-2t}) \quad \ldots(19)$$

Now, from (10) and (16), $dy/dt = y - (e^{-t})/2$ or $dy/dt - y = -(e^{-t})/2$

whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and hence its solution is given by

$$ye^{-t} = c_4 + \int\{(-e^{-t}/2) \times e^{-t}\}dt,\ c_4 \text{ being an arbitarary constant}$$

or $$ye^{-t} = c_4 + (1/4) \times e^{-2t} \quad \ldots(20)$$

Using initial values (7), (20) yields $y_0 e^{-t_0} = c_4 + (1/4) \times e^{-2t_0}$

or $0 = c_4 + 1/4$ so that $c_4 = -(1/4)$

Hence, (20) reduces to $ye^{-t} = (1/4) \times (e^{-2t} - 1)$

Thus, $$y = (1/4) \times e^{t}\,(e^{-2t} - 1) \quad \ldots(21)$$

From (11), (16) and (17), $dz/dt = -(e^{-t}/2) \times (\lambda e^{-t}) = -(\lambda/2) \times e^{-2t}$

Thus, $(1/z)dz = -(\lambda/2) \times e^{-2t}\,dt$

Integrating, $z = (\lambda/4) \times e^{-2t} + c_5$, c_5 being an arbitrary constant …(22)

Using initial values (7), (22) yields $z_0 = (\lambda/4) \times e^{-2t_0} + c_5$

or $\lambda/2 = (\lambda/4) \times e^{0} + c_5$ so that $c_5 = \lambda/4.$

Hence, (22) reduces to $z = (\lambda/4) \times (e^{-2t} + 1)$ …(23)

The required characteristics of (1) are given by (19), (21) are (23)

In order to obtain the desired integral surface of (1), we now proceed to eliminate two parameters t and λ from (19), (21) and (23).

From (19) and (23), we have $x/z = 2e^t$ giving $e^t = x/2z$. …(24)

From (21), $y = (1/4) \times (1/e^t - e^t) = (1/4) \times (2z/x - x/2z)$, using (24)

or $8xyz = 4z^2 - x^2$, which is the required integral surface of (1).

Ex.12. *Prove that for the equation $z + px + qy - 1 - pq\,x^2y^2 = 0$ the characteristic strips are given by $x = (B + Ce^{-t})^{-1}$, $y = (A + De^{-t})^{-1}$, $z = E - (AC + BD)e^{-t}$, $p = A(B + Ce^{-t})^2$, $q = B(A + De^{-t})^2$, where A, B, C, D and E are arbitrary constants. Hence, find the integral surface which passes through the line $z = 0$, $x = y$.* **[I.A.S 2001]**

Sol. The given equation is $z + px + qy - 1 - pq\,x^2y^2 = 0$ …(1)

Let $f(x, y, z, p, q) = z + px + qy - 1 - pq\,x^2y^2$ …(2)

Then, the characteristic equations of (1) are given by

$$dx/dt = \partial f / \partial p = x - qx^2y^2 \quad \ldots(3)$$

$$dy/dt = \partial f / \partial q = y - px^2y^2 \quad \ldots(4)$$

$$dz/dt = p(\partial f / \partial p) + q(\partial f / \partial q) = p(x - qx^2y^2) + q(y - px^2y^2) = px + qy - 2pqx^2y^2 \quad \ldots(5)$$

$$dp/dt = -(\partial f / \partial x) - p(\partial f / \partial z) = -(p - 2\,pqxy^2) - p = -2p(1 - qxy^2) \quad \ldots(6)$$

$$dq/dt = -(\partial f / \partial y) - q(\partial f / \partial z) = -(q - 2pqx^2y) - q = -2q\,(1 - px^2y) \quad \ldots(7)$$

From (3) and (6), $(1/x)\,(dx/dt) = -(1/2p)\,(dp/dt)$ or $(2/x)\,dx + (1/p)\,dp = 0$

Integrating, $2 \log x + \log p = \log A$ or $x^2p = A$, A being an arbitrary constant …(8)

From (4) and (7), $(1/y)\,(dy/dt) = (-1/2q)\,(dq/dt)$ or $(2/y)dy + (1/q)dq = 0$

Integrating as before, $y^2q = B$, B being an arbitrary constant …(9)

From (3) and (9), $dx/dt = x - Bx^2$ or $x^{-2}(dx/dt) - x^{-1} = -B$ …(10)

Putting $x^{-1} = v$ and $-x^{-2}(dx/dt) = dv/dt$, (10) reduces to

$-(dv/dt) - v = -B$ or $dv/dt + v = B$,

whose integrating factor is $e^{\int dt}$, i.e., e^t and hence its solution is given by

$ve^t = C + \int Be^t\,dt$, C being an arbitrary constant

$e^t / x = c + Be^t$ or $1/x = Ce^{-t} + B$ or $x = (B + Ce^{-t})^{-1}$ …(11)

Similarly, (4) and (8) yield $dy/dt = y - Ay^2$ or $y^{-2}(dy/dt) - y^{-1} = -A$ …(12)

Putting $y^{-1} = u$ and $-y^{-2}(dy/dt) = du/dt$, (12) yields

$-(du/dt) - u = -A$ or $du/dt + u = A$

whose integrating factor is $e^{\int dt}$, i.e., e^t and hence its solution is given by

$ue^t = D + \int Ae^t\,dt$, D being an arbitrary constant

or $e^t / y = D + Ae^t$ or $1/y = De^{-t} + A$ or $y = (A + De^{-t})^{-1}$ …(13)

Using (8) and (9), (5) yields $dz/dt = A/x + B/y - 2AB$

or $dz/dt = A(B + Ce^{-t}) + B(A + De^{-t}) - 2AB$, using (11) and (13)

or $dz/dt = (AC + BD)e^{-t}$ or $dz = (AC + BD)\,e^{-t}\,dt$

Integrating, $z = E - (AC + BD)e^{-t}$, E being an arbitrary constant …(14)

From (8) and (11), $p = Ax^{-2} = A(B + C\,e^{-t})^2$ …(15)

From (9) and (13), $q = By^{-2} = B(A + De^{-t})^2$ …(16)

The required characteristics are given by (11), (13), (14), (15) and (16). We now proceed to find the required integral surface passing through the line given by

$z = 0$ and $x = y$ …(17)

Re-writing (17), $x = \lambda,\quad y = \lambda,\quad z = 0,\quad \lambda$ being a parameter …(18)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = x_0(\lambda) = \lambda,\quad y_0 = y_0(\lambda) = \lambda,\quad z_0 = z_0(\lambda) = 0$ …(19)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since the initial values statisfy (1), we have

$z_0 + p_0x_0 + q_0y_0 - 1 - p_0q_0\,x_0^2y_0^2 = 0$ or $p_0\lambda + q_0\lambda - 1 - p_0q_0\lambda^4 = 0$, using (19)

Thus, $\lambda(p_0 + q_0) = p_0q_0\lambda^4 + 1$ …(20)

Also, we have $z_0'(\lambda) = p_0x_0'(\lambda) + q_0y_0'(\lambda)$

so that, $0 = p_0 + q_0$ giving $q_0 = -p_0$, using (19) …(21)

Using (21), (20) yields $p_0q_0\lambda^4 + 1 = 0$ giving $-p_0^2\lambda^4 + 1 = 0$, using (21)

Thus, $p_0 = 1/\lambda^2$ so that $q_0 = -(1/\lambda^2)$, using (21) …(22)

Using initial values $x = x_0 = \lambda$, $t = t_0 = 0$, (11) reduces to

$\lambda = (B + C)^{-1}$ so that $B + C = 1/\lambda$ …(23)

Using initial values $y = y_0 = \lambda$, $t = t_0 = 0$, (13) reduce to

$\lambda = (A + D)^{-1}$ so that $A + D = 1/\lambda$ …(24)

Using initial values $p = p_0 = 1/\lambda^2$, $t = t_0 = 0$, (15) reduces to

$p_0 = A(B + C)^2$ or $1/\lambda^2 = A \times (1/\lambda)^2$, by (23) so that $A = 1$ …(25)

Using initial values $q = q_0 = -(1/\lambda^2)$, $t = t_0 = 0$, (16) reduces to

$q_0 = B\,(A + D)^2$ or $-(1/\lambda^2) = B \times (1/\lambda^2)$, by (24) so that $B = -1$ …(26)

From (23) and (26), $-1 + C = 1/\lambda$ so that $C = 1 + 1/\lambda$ …(27)

From (24) and (25), $1 + D = 1/\lambda$ so that $D = (1/\lambda) - 1$ …(28)

Using the initial values $z = z_0 = 0$, $t = t_0 = 0$, (14) reduces to

$0 = E - (AC + BD)$ or $E = 1 + 1/\lambda - (1/\lambda - 1)$ or $E = 2$ …(29)

Substituting the values of A, B, C, D and E given by (25), (26), (27), (28) and (29) in (11), (13) and (14), we obtain

$$x = \{-1 + (1/\lambda + 1)e^{-t}\}^{-1} \quad \text{…(30)}$$

$$y = \{1 + (1/\lambda - 1)e^{-t}\}^{-1} \quad \text{…(31)}$$

$$z = 2 - \{1 + 1/\lambda - (1/\lambda - 1)\}e^{-t} = 2(1 - e^{-t}) \quad \text{…(32)}$$

In order to obtain the required surface, we now eliminate λ from (30), (31) and (32).

From (30), $x^{-1} = -1 + (1/\lambda + 1)e^{-t}$ so that $1/x + 1 = (1/\lambda + 1)e^{-t}$ ...(33)

From (31), $y^{-1} = 1 + (1/\lambda - 1)e^{-t}$ so that $1/y - 1 = (1/\lambda - 1)e^{-t}$ ...(34)

Subtracting (34) from (33), $1/x - 1/y + 2 = 2e^{-t}$ or $1/x - 1/y = -2(1 - e^{-t})$ ...(35)

From (32) and (35), we get $1/x - 1/y = -z$ which is the required integral surface.

Ex. 13. The general solution of the partial differential equation $\partial^2 z / \partial x \partial y = x + y$ is of the form (*a*) $(1/2) \times xy(x + y) + F(x) + G(y)$ (*b*) $(1/2) \times xy(x - y) + F(x) + G(y)$

(*c*) $(1/2) \times xy(x - y) + F(x)\ G(y)$ (*d*) $(1/2) \times xy(x + y) + F(x)\ G(y)$ **(GATE 2010)**

Sol. Ans. (*a*). Integrating the give equation w.r.t '*x*', we get

$$\partial z / \partial y = x^2 / 2 + xy + g(y),\ g(y) \text{ being an arbitrary function of } y.$$

Integrating the above equation w.r.t. '*y*', we get

$$z = (x^2 y)/2 + (xy^2)/2 + G(y) + F(x), \quad \text{where} \quad G(y) = \int g(y) dy$$

Ex. 14. Find whether the following is hyperbolic, parabolic or elliptic :

(*i*) $x^2 r - y^2 t - px - qy = x^2$ **[Delhi B.A. (Prog) II 2010, 11]**

(*ii*) $x^2 r + (5/2) \times xys + y^2 t + xp + yq = 0$ **[Delhi B.A. (Prog) II 2010]**

(*iii*) $\partial^2 u / \partial t^2 + \partial^2 u / \partial x \partial t + \partial^2 u / \partial x^2 = 0$ **[Meerut 2010]**

(*iv*) $u_{xx} + u_{yy} + u_{zz} = (1/c^2) \times (\partial u / \partial t)$ **[Meerut 2007, 10]**

(*v*) $(1 - x^2)r - 2xys + (1 - y^2)t + xp + 3x^2 yq = 0$ **[Ravishankar 2010]**

(*vi*) $\partial^2 z / \partial x^2 = x^2 (\partial^2 z / \partial y^2)$ **[Bhopal 2010]**

(*vii*) $\partial^2 u / \partial t^2 + (\partial u / \partial x)\ (\partial u / \partial t) + \partial^2 u / \partial t^2 = 0$ **[Meerut 2011]**

Hint. Use Articles 8.1, 8.2 and 8.2A.

Ans. (*i*) Hyperbolic (*ii*) Hyperbolic (*iii*) Elliptic (*iv*) Parabolic (*v*) Hyperbolic if $x^2 + y^2 > 1$, parabolic if $x^2 + y^2 = 1$, elliptic if $x^2 + y^2 < 1$ (*vi*) Hyperbolic (*vii*) Elliptic

Ex. 15. *The partial differential equation* $x^2 (\partial^2 z / \partial x^2) - (y^2 - 1)x(\partial^2 z / \partial x\ \partial y) + y(y - 1)^2\ (\partial^2 z / \partial y^2) + x(\partial z + \partial x) + y(\partial z / \partial y) = 0$ *is hyperbolic in a region in xy–plane if (a)* $x \neq 0$ *and y = 1 (b) x = 0 and* $y \neq 1$ *(c)* $x \neq 0$, *and* $y \neq 1$ *(d) x = 0 and y = 1.***[GATE 2011]**

Sol. Ans (*c*) The given can be re-written as

$$x^2 r - x(y^2 - 1)s + y(y - 1)^2 t + xp + yq = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = x^2$, $S = -x(y^2 - 1)$ and $T = y(y - 1)^2$. Now, in order that (1) may be hyperbolic we must have $S^2 - rRT > 0$, i.e.,

$$x^2 (y^2 - 1)^2 - 4x^2 y(y - 1)^2 > 0 \quad \text{or} \quad x^2 (y - 1)^2 (y + 1)^2 - 4x^2 y\ (y - 1)^2 > 0$$

or $x^2(y-1)^2\{(y+1)^2-4y\}>0$ or $x^2(y-1)^2(y-1)^2>0,$ ... (2)

Which is true when $x\neq 0$ and $y\neq 1.$

Ex. 16. *The integral surface for the Cauchy problem* $\partial z/\partial x+\partial z/\partial y=1$ *which passes through the circle z = 0,* $x^2+y^2=1$ *is*

(*a*) $x^2+y^2+2z^2+2zx-2yx-2yz-1=0$ (*b*) $x^2+y^2+2z^2+2zx-2yz-1=0$

(*c*) $x^2+y^2+2z^2-2zx-2yz-1=0$ (*d*) $x^2+y^2+2z^2+2zx+2yz-1=0$ **[GATE 2011]**

Sol. Ans. (*c*) In usual symbols, the given equation is $p+q=1$ ... (1)

Lagrange's auxiliary equation of (1) are $(dx)/1=(dy)/1=(dz)/1$ ... (2)

Taking the first two fractions of (2), we get $dx-dy=0$ ... (3)

Integrating (3), $x-y=c_{-1}$, c_1 being an arbitrary constant ... (4)

Next, taking the first and third fractions of (2), we get $dx-dz=0$ ... (5)

Integrating (5), $x-z=c_2$, c_2 being an arbitrary constant ... (6)

The given curve is defined by $x^2+y^2=1,\ z=0$ ... (7)

Putting z = 0 in (6), we have $x=c_2,$ ... (8)

Now, from (4) and (8), we have $c_2-y=c_1$ so that $y=c_1-c_2$ ... (9)

Substituting the values of x and y given by (8) and (9) in (7), we obtain

$c_2^2+(c_2-c_1)^2=1$ or $(x-z)^2+(y-z)^2=1,$ using (4) and (6)

or $x^2+y^2+2z^2-2zx-2yz-1=0.$

Ex. 17. *The integral surfaces satisfying the partial differential equation* $(\partial z/\partial y)+z^2(\partial z/\partial y)=0$ *and spassing through the straight line x = 1, y = z is*

(*a*) $(x-1)z+z^2=y^2$ (*b*) $x^2+y^2-z^2=1$

(c) $(y-z)x+x^2=1$ (*d*) $(x-1)z^2+z=y$ **[GATE 2012]**

Sol. Ans. (d). Given $p+z^2q=0$, where $p=\partial z/\partial x,\ q=\partial z/\partial y$ ... (1)

Lagrange's auxiliary equation (1) are $(dx)/1=(dy)/z^2=(\partial z)/0$ ... (2)

From third fraction of (1), $dz=0$ so that $z=c^1$ ... (3)

Using (3), from first and second fractions of (2), $(dx)/1=(dy)/{c_1}^2$ or $c_1^2dx-dy=0$

Integrating, ${c_1}^2x-y=c_2$ or $z^2x-y=c_2,\ as\ c_1=z$ ... (4)

In order to get the required integral surfaces, we shall use method II given on page 2.28.

The given straight line is represented by $x=1,\quad y=z$ ... (5)

Using (5) in (3) and (4), we get $y=c_1$ and $y^2-y=c_2$ ... (6)

Eliminating *y* from the two equation of (6), we get $c_1^2-c_1=c_2$ ... (7)

Substituting the values of c_1 and c_2 given by (3) and (4) in (7), we get

$z^2-z=z^2x-y$ or $(x-1)z^2+z^2=y$, which is the required integral surface

Ex.18. *The expression* $\frac{1}{D_x^2 - D_y^2}\sin(x-y)$ *is equal to*

(*a*) $-(x/2)\times\cos(x-y)$ (*b*) $-(x/2)\times\sin(x-y)+\cos(x-y)$

(*c*) $-(x/2)\times\cos(x-y)+\sin(x-y)$ (*d*) $(3x/2)\times\sin(x-y)$ **[GATE 2012]**

Sol. Ans. (a). Here note that D_x and D_y stand for D and D' respectively. For solution, processd as in Ex. 7(d). Here, we wish to find only P.I. Thus,

$$\frac{1}{D_x^2 - D_y^2}\sin(x-y) = \frac{1}{D^2 - D'^2}\sin(x-y) = \frac{1}{D+D'}\frac{1}{D-D'}\sin(x-y)$$

$$= \frac{1}{D-D'}\frac{1}{1-(-1)}\int \sin v\, dv, \text{ where } v = x-y$$ [Using formula (i) page 4.9]

$$= \frac{1}{2}\ \frac{1}{D+D'}\ [-\cos(x-y)] = \frac{1}{2}\ \frac{1}{(-1)D-(1)\times D}\ \cos(x-y) = \frac{1}{2}\times\frac{x}{(-1)^1\times 1!}\cos\ (x-y)$$

[Using formula (ii), page 4.9]

$$= -(x/2)\times\cos(x-y)$$

BOUNDARY VALUE PROBLEMS AND THEIR SOLUTIONS BY SEPARATION OF VARIABLES

Where is What

Chapters	*Pages*
1. Heat and wave equations. Method of separation of variables	1.3-1.18
2. Boundary value problems in cartesian co-ordinates	2.1-2.98
Part I : Problems based on one-dimensional heat (or diffusion) equation	2.3-2.27
Part II : Problems based on two-dimensional heat (or diffusion) equation	2.27-2.33
Part III : Problems based on three-dimensional heat (or diffusion) equation	2.33-2.35
Part IV : Problems based on one-dimensional wave equation	2.35-2.64
Part V : Problems based on two-dimensional wave equation	2.64-2.68
Part VI : Problems based on three-dimensions wave equation	2.68-2.70
Part VII : Problems based on two-dimensional Laplace's equation	2.70-2.90
Part VIII : Problems based on three-dimensional Laplace' equation	2.91-2.93
Part IX : Problems based on telegraph (or transmission) line equation	2.93-2.95
3. Boundary value problems in polar co-ordinates	3.1-3.24
4. Boundary value problems in cylindrical co-ordinates	4.1-4.20
5. Boundary value problems in spherical co-ordinates	5.1-5.22

1

Heat and wave equations. Method of separation of variables

1.1. Introduction.

In theoretical physics and engineering, partial differential equations generally arise from the mathematical formulation of real physical problems. In what follows we propose to obtain some important equations governing actual physical problems by making certain simplifying assumptions which would represent reasonable approximations to the problems. The equations so obtained furnish solutions of physical problems in a satisfactory manner.

1.2. Derivation of one dimensional wave equation.

[I.F.S. 2003; Meerut 2000, 10, 11; Nagpur 2002 04, 06; Ravishankar 1996; Rewa 2002; Delhi B.A./B.Sc. (Hons.) III 2012; Kanpur 1997; Pune 2002; Kurukshetra 1997]

Consider an elastic string which is stretched to length l and then fixed at its end points O and A. Suppose that the string is distorted and further suppose that at time $t = 0$ it is released and allowed to vibrate. Let OA be taken as the x-axis and let a perpendicular line through O be taken as the u-axis, We wish to obtain deflection $u(x, t)$ at any point x and at any time $t > 0$. We make the following assumptions:

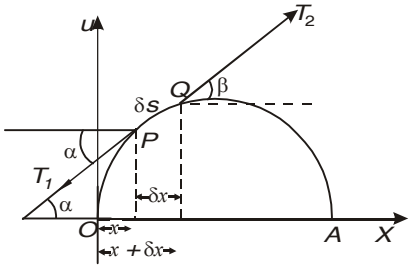
(i) The string is homogeneous, *i.e.*, the mass of the string per unit length is constant ρ (say).

(ii) The entire motion takes place in the xu— plane.

(iii) The string in perfectly elastic and it does not produce resistance to bending.

(iv) The string makes small transverse vibration so that the absolute values of deflection u and the slope $\partial u / \partial x$ at any point of the string are small.

(v) The tension in the string is large so that the force due to weight of the string can be neglected.



Consider the motion of the small portion PQ of length δs of the string. By hypothesis the string produces no resistance to bending so the tensions T_1 and T_2 at P and Q will act tangents at P and Q respectively. Since there is no motion in horizontal direction, we have

$$T_1 \cos\alpha = T_2 \cos\beta = T(\text{constant}), \text{ say.} \quad \ldots (1)$$

Now mass of portion PQ is $\rho\,\delta s$ and acceleration of this element in vertical direction is $\partial^2 u / \partial t^2$. The resultant vertical force acting on PQ is $T_2 \sin\beta - T_1 \sin\alpha$. Hence by $P = mf$, the equation of motion is

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho\,\delta s)\times(\partial^2 u/\partial t^2) \qquad \text{or} \qquad \frac{T_2\sin\beta}{T} - \frac{T_1\sin\alpha}{T} = \frac{\rho\,\delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

[$\because$ to first order of approximation, $\delta s = \delta x$]

or
$$\frac{T_2\sin\beta}{T_2\cos\beta} - \frac{T_1\sin\alpha}{T_1\cos\alpha} = \frac{\rho\delta x}{T}\times\frac{\partial^2 u}{\partial t^2}, \text{ using (1)}$$

or
$$\tan\beta - \tan\alpha = (\rho\delta x/T)\times(\partial^2 u/\partial t^2) \qquad \ldots (2)$$

Since $\tan\alpha$ and $\tan\beta$ are the slopes of the curve of the string at P and Q (*i.e.*, at x and $x+\delta x$ respectively), we have

$$\tan\alpha = (\partial u/\partial x)_x \qquad \ldots (3)$$

and
$$\tan\beta = (\partial u/\partial x)_{x+\delta x} \qquad \ldots (4)$$

Using (3) and (4), (2) reduces to

$$\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x} = \frac{\rho}{T}\frac{\partial^2 u}{dt^2} \qquad \ldots (5)$$

Now as $\delta x \to 0$, (5) reduces to

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \qquad \text{or} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad \ldots (6)$$

where $c^2 = T/\rho$. We write c^2 for T/ρ so as to indicate that the physical constant T/ρ is a positive quantity.

1.3. Derivation of two-diamensional wave equation.

(Garhwal 2005; Delhi B.Sc. (H) Phy. 1995, 2001)

Consider a tightly stretched membrane (*e.g.* the membrane of a drum). Let the membrane be distored and further let at time $t = 0$, it be released and allowed to vibrate. Let $A'B'C'D'$ be the shape of an element of the membrane at any time t. Let $ABCD$ be the projection of $A'B'C'D'$ on the xy-plane. We wish to obtain deflection $u(x, y, t)$ at any point (x, y) and at any time $t > 0$. We make the following assumptions:

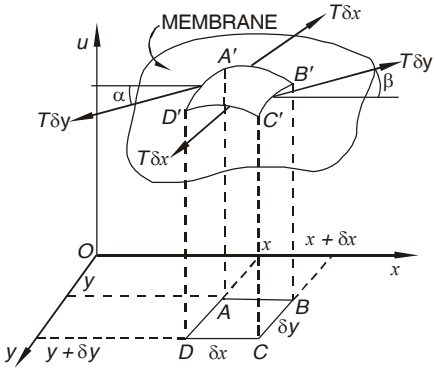
(i) The membrane is homogeneous, *i.e.*, the mass of the membrane per unit area is constant ρ (say).

(ii) The entire motion takes place in a direction perpendicular to xy-plane.

(iii) The string is perfectly elastic and it does not produce resistance to bending.

(iv) The tension T per unit length developed by stretching the membrane is the same at all points and in all directions. We further assume that the absolute value of T does not change during the motion.

(v) The deflection u (x, y, t) is small as compared to the size of the membrane. All angles of inclination are small.



(vi) The slopes $\partial u/\partial x$ and $\partial u/\partial y$ are small so that their higher powers can be neglected.

We now consider motion of the element $A'B'C'D'$. The forces $T\delta y$ on its opposite edges $B'C'$ and $A'D'$ of length δy act at angles α and β to the horizontal. These have a vertical component equal to

$= (T\delta y)\sin\beta - (T\delta y)\sin\alpha = T\delta y(\sin\beta - \sin\alpha)$

$= T\delta y(\tan\beta - \tan\alpha)$, since α and β are small, so $\sin\alpha = \alpha = \tan\alpha$, $\sin\beta = \beta = \tan\beta$.

$= T\delta y[(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x]$

Similarly the forces $T\delta x$ acting on the edges $A'B'$ and $D'C'$ of length δx have a vertical component equal to

$$= T\delta x\{(\partial u/\partial y)_{y+\delta y} - (\partial u/\partial y)_y\}.$$

Now the area of the element $A'B'C'D'$ is $\delta x\,\delta y$ so that its mass is $\rho\,\delta x\,\delta y$. Again the acceleration of the element in vertical direction is $\partial^2 u/\partial t^2$. Hence using $P = mf$, the equation of the motion of the element is

$$T\delta y\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\} + T\delta x\{\partial u/\partial y)_{y+\delta y} - (\partial u/\partial y)_y\} = (\rho\,\delta x\,\delta y)\,(\partial^2 u/\partial t^2)$$

or
$$\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial u)_x}{\delta x} + \frac{(\partial u/\partial y)_{y+\delta y} - (\partial u/\partial y)_y}{\delta y} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Now as $\delta x \to 0$ and $\delta y \to 0$, the above equation reduces to

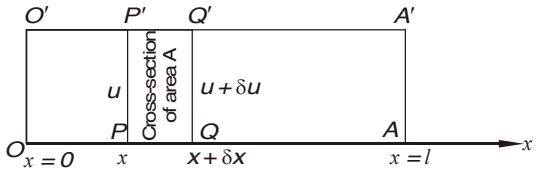
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \qquad \text{or} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2 = T/\rho$. This equation is called *two-dimensional wave equation*.

1.4. Derivation of one-dimensional heat equation.

[Kanpur 2003; Meerut 1998; Nagpur 2005]

Consider the flow of heat by conduction in a bar OA. Let OA be taken as the x-axis. We consider an element $PQQ'P'$ of the bar as shown in the figure. Clearly the temperature $u(x, t)$ of the bar at any point P is function of x and the time t. Suppose that the bar is raised to an assigned temperature distribution at time $t = 0$ and then heat is allowed to flow by conduction. We wish to find $u(x, t)$ at any point x and at any time $t > 0$. We make the following assumptions.



(*i*) The bar is homogenous, *i.e.*, the mass of the bar per unit volume is constant ρ (say).

(*ii*) The sides of the bar are insulated and the loss of heat from the sides by conduction or radiation can be neglected.

(*iii*) The amount of heat crossing any section of the bar is given by $kA\,(\partial u/\partial x)\delta t$, where

A = area of cross section of the bar, $\partial u/\partial x =$ temperature gradient at the section PP'.

$\delta t =$ time of flow of heat, $K =$ themal conductivity of the material of the bar.

Now the quantity Q_1 of heat flowing into the element across the section PP' in time δt is given by $Q_1 = -KA(\partial u/\partial x)_x\,\delta t,$

where the negative sign has been taken because heat flows in the direction of decreasing temperature.

Again the quantity Q_2 of heat flowing out of the element across the section QQ' in time δt is given by $$Q_2 = -KA(\partial u/\partial x)_{x+\delta x}\delta t$$

Hence the quantity of heat retained by the element

$$= Q_1 - Q_2 = KA\,\delta t\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\} \quad \ldots (1)$$

Suppose the above heat raises the temperature of the element by a small quantity δu. Then the same quantity of heat is again given by

$$(\rho A\,\delta x)\sigma\,\delta u, \quad \ldots (2)$$

where σ is specific heat of the bar. Since the expressions (1) and (2) are equal, we have

$$KA\,\delta t\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\} = (\rho A\,\delta x)\sigma\delta u$$

or $$K\frac{\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\}}{\delta x} = \rho\sigma\frac{\delta u}{\delta t} \quad \ldots (3)$$

Now as $\delta x \to 0$ and $\delta t \to 0$, (3) reduces to

$$K\frac{\partial^2 u}{\partial x^2} = \rho\sigma\frac{\partial u}{\partial t} \qquad \text{or} \qquad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad \ldots (4)$$

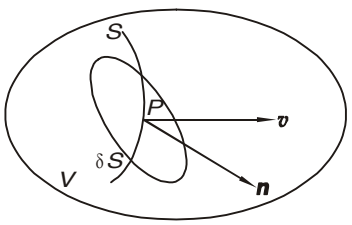
where $k = K/\rho\sigma$ is called the diffusivity of the matrial of the bar.

Note: Equation (4) is also known as *one-dimensional diffusion equation*.

1.5. Derivation of Fourier equation of heat conduction.

[Meerut 2011; Himanchal 2007; Agra 2004; Delhi B.Sc. (H) 1998, Mumbai 2003, 05]

We know that heat flows from points of higher temperature to the points at lower temperature. We consider the flow of heat by conduction across any surface S of a solid body which encloses volume V. Let $P(x, y, z)$ be any point of the solid and let δS be the elementary area of S surrounding P. Let $\boldsymbol{n}$ denote the unit outward drawn vector, normal at P to δS. Let $\boldsymbol{v}$ denote the velocity of the heat (amount of heat flowing per second) at P.



Now, amount of heat leaving δS per unit time $= (\boldsymbol{v}\cdot\boldsymbol{n})\,\delta S$ ... (1)

Physical experiments have shown that $\boldsymbol{v} = -K\,\text{grad}\ u = -K\nabla u$, ... (2)

where K is the termal conductivity of the material of the solid body and $u(x, y, z, t)$ is temperature at $P(x, y, z)$ at any time t. Here grad u or ∇u, is given by

$$\nabla u = \text{grad}\ u = \boldsymbol{i}(\partial u/\partial x) + \boldsymbol{j}(\partial u/\partial y) + k(\partial u/\partial z) \quad \ldots (3)$$

Using (1) and (2), the total amount of heat leaving the entire surface S per second

$$= -K\iint_S (\nabla u\cdot\boldsymbol{n})\,dS \quad \ldots (4)$$

We now consider Gauss's divergence theorem which states, "*If* $\boldsymbol{F}$ *is a vector point function which, together with its derivative in any direction, is uniform, finite and continuous and S is the closed surface enclosing a region V, then*

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iiint_V \nabla\cdot\boldsymbol{F}\,dV. \quad \ldots (5)$$

Using (5), the total amount of heat leaving the entire surface S per second from (4) is given by

$$= -K \iiint_V \nabla \cdot \nabla u \, dV = -K \iiint_V \nabla^2 u \, dV. \qquad \ldots (6)$$

Let the given body be homogeneous, *i.e.*, the mass of the body per unit volume (*i.e.*, density) is constant ρ (say). Let σ be the specific heat of the material of the body. Then the total amount of heat H inside the entire surface S is given by

$$H = \iiint_V \sigma \rho u \, dV.$$

Hence the rate of decrease of H inside the entire surface S *i.e.*, the amount of heat leaving the entire volume V across its surface S per second is

$$-\frac{\partial H}{\partial t} = -\iiint_V \sigma \rho \frac{\partial u}{\partial t} dV \qquad \ldots (7)$$

From (6) and (7), we must have

$$K \iiint_V \nabla^2 u \, dV = \iiint_V \sigma \rho \frac{\partial u}{\partial t} dV \qquad \text{or} \qquad \iiint_V \left[K \nabla^2 u - \sigma \rho \frac{\partial u}{\partial t} \right] dV = 0. \qquad \ldots (8)$$

V being arbitrary and the integrand in (8) being assumed to be continuous, it follows that (8) holds only if its integrand is zero everywhere, *i.e.*,

$$K \nabla^2 u - \sigma \rho \frac{\partial u}{\partial t} = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial t} = \frac{K}{\sigma \rho} \nabla^2 u$$

or

$$\partial u / \partial t = k \nabla^2 u, \qquad \ldots (9)$$

where $k = (K / \sigma \rho)$ is the thermal diffusivity of the material.

Remark. The result (2) is also applicable to simple diffusion representing the flux of a substance which is diffusing within a porous solid. For such a situation u stands for concentration (*i.e.*, the mass of the diffusion substance per unit volume of the solid) and k the coefficient of diffusion. Since the content of the substance in a given region is the integral of u over that region, it follows, as before, that the concentration u satisfies the following equation, known as diffusion equation,

$$\partial u / \partial t = k \nabla^2 u. \qquad \ldots (10)$$

1.6.A. Laplace's equation

We know that the two dimensional heat flow equation in steady state reduces to

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0, \qquad \ldots (1)$$

which is known as *Laplace's equation in two dimensions*.

Likewise the three-dimensional heat flow equation in steady state reduces to

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \partial^2 u / \partial z^2 = 0 \qquad \ldots (2)$$

which is known as *Laplace's equation in three dimensions.*

Laplace's equation has wide applications in physics and engineering. The theory of its solution is known as the *potential theory* and its solutions are known as *potential functions* or *harmonic functions.* A Laplace equation is also known as *potential equation*.

If the problem involves rectangular boundaries, we prefer to take Laplace's equation in cartesian coordinates given by (1) and (2).

1.6.B. Laplace's equation in plane polar coordinates

If the given boundary value problem involves circular boundaries, we would like to use Laplace's equation in polar coordinates (r, θ).

Transformation of Laplace's equation $\partial^2 / \partial x^2 + \partial^2 / \partial y^2 = 0$ *into polar coordinates* (r, θ).

Sol. If (x, y) be the cartesian coordinates of the point P whose polar coordinates are (r, θ), then $x = r\cos\theta$ and $y = r\sin\theta$. ... (1)

From (1), $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$ ... (2)

From (2), $2r\,(\partial r/\partial x) = 2x$ and $2r\,(\partial r/\partial y) = 2y$

so that $\partial r/\partial x = x/r = \cos\theta$ and $\partial r/\partial y = y/r = \sin\theta$. ... (3)

Also, $$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}. \quad \text{... (4)}$$

and $$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y^2/x^2)}\left(\frac{1}{x}\right) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \quad \text{... (5)}$$

Given that $(\partial^2 v/\partial x^2) + (\partial^2 v/\partial y^2) = 0$. ... (6)

Now, $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}, \text{ using (3) and (4)} \quad \text{... (7)}$$

Then, (7) $\Rightarrow$ $$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}. \quad \text{... (8)}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right), \text{ using (7) and (8)}$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 v}{\partial r^2} - \sin\theta\left(-\frac{1}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta}\right)\right] - \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial v}{\partial r} + \cos\theta\frac{\partial^2 v}{\partial\theta\,\partial r} - \frac{1}{r}\left\{\cos\theta\frac{\partial v}{\partial\theta} + \sin\theta\frac{\partial^2 v}{\partial\theta^2}\right\}\right]$$

Thus, $$\frac{\partial^2 v}{\partial x^2} = \cos^2\theta\frac{\partial^2 v}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}. \quad \text{... (9)}$$

Again, $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial y} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}, \text{ using (3) and (5)} \quad \text{... (10)}$$

Then, (10) $\Rightarrow$ $$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}. \quad \text{... (11)}$$

$$\therefore \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \sin\theta \frac{\partial}{\partial r}\left(\sin\theta \frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}\right)$$

$$= \sin\theta\left[\sin\theta \frac{\partial^2 v}{\partial r^2} + \cos\theta\left\{-\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial \theta}\right\}\right]$$

$$+ \frac{\cos\theta}{r}\left[\cos\theta \frac{\partial v}{\partial r} + \sin\theta \frac{\partial^2 v}{\partial \theta\,\partial r} + \frac{1}{r}\left\{-\sin\theta \frac{\partial v}{\partial \theta} + \cos\theta \frac{\partial^2 v}{\partial \theta^2}\right\}\right].$$

Thus, $$\frac{\partial^2 v}{\partial y^2} = \sin^2\theta \frac{\partial^2 v}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial v}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 v}{\partial \theta^2}. \quad \ldots (12)$$

Adding (1) and (2), we obtain $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \quad \ldots (13)$$

Hence, using (6) the Laplace's equation in polar form is given by

$$\partial^2 v/\partial r^2 + (1/r)\times(\partial v/\partial r) + (1/r^2)\times(\partial^2 u/\partial \theta^2) = 0. \quad \ldots (14)$$

1.6.C. Laplace's equation in cylindrical coordinates

If the given boundary value problem involves cylindrical boundaries, we would like to use Laplace's equation in cylindrical coordinates (r, θ, z)

Transformation of Laplace's equation $\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2 = 0$ *in cylindrical coordinates* (r, θ, z).

Sol. If (x, y, z) are the cartesian coordinates of the point P whose cylindrical coordinates are (r, θ, z), then we know that

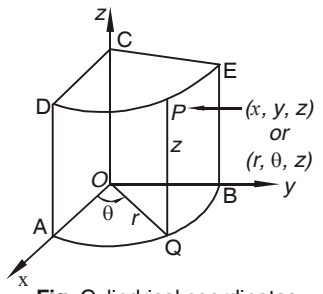


Fig. Cylindrical coordinates of $P(x, y, z)$ are $P(r, \theta, z)$

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z \quad \ldots (1)$$

With $x = r\cos\theta$, $y = r\sin\theta$, proceed as in Art. 1.6B and prove that

$$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = \partial^2 v/\partial r^2 + (1/r)\times(\partial v/\partial r) + (1/r^2)\times(\partial^2 v/\partial \theta^2) \quad \ldots (2)$$

Adding $\partial^2 v/\partial z^2$ on both sides of (2), we have

$$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2 = \partial^2 v/\partial r^2 + (1/r)\times(\partial v/\partial r) + (1/r^2)\times(\partial^2 v/\partial \theta^2) + \partial^2 v/\partial z^2$$

Hence the Laplace's equation $\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2 = 0$ reduces to

$$\partial^2 v/\partial r^2 + (1/r)\times(\partial v/\partial r) + (1/r^2)\times(\partial^2 v/\partial \theta^2) + \partial^2 v/\partial z^2 = 0,$$

which is the required Laplace's equation is cylindrical coordinates.

1.6.D. Laplace's equation in spherical coordinates (*Himanchal 2009*)

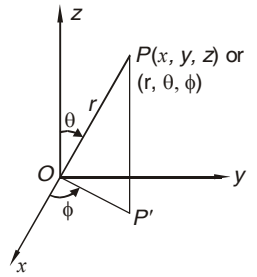


Fig: Shperical coordinates of *P(x, y, z)* are *P(r, θ, ϕ)*

If the given boundary value problem involves spherical boundaries, we would like to use Laplace's equation in spherical coordinates (r, θ, ϕ)

Transformation of Laplaces equation

$$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2 = 0$$

in spherical coordinates (r, θ, ϕ)

Sol. If (x, y, z) are the carterial coordinates of the points P whose spherical polar coordinates are (r, θ, ϕ), then

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta. \quad \ldots (1)$$

$$\therefore \quad r^2 = x^2 + y^2 + z^2, \qquad \tan\theta = (x^2 + y^2)^{1/2}/z, \qquad \tan\phi = y/x$$

or $$r^2 = x^2 + y^2 + z^2, \qquad \theta = \tan^{-1}\left\{(x^2 + y^2)^{1/2}/z\right\}, \qquad \phi = \tan^{-1}(y/x). \quad \ldots (2)$$

From (1) and (2), we have the following relations

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\phi, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi, \qquad \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta, \quad \ldots (3)$$

$$\frac{\partial\theta}{\partial x} = \frac{\cos\theta\cos\phi}{r}, \qquad \frac{\partial\theta}{\partial y} = \frac{\cos\theta\sin\phi}{r}, \qquad \frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}. \quad \ldots (4)$$

$$\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \qquad \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{r\sin\theta} \qquad \frac{\partial\phi}{\partial z} = 0. \quad \ldots (5)$$

Now, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial\theta}\dfrac{\partial\theta}{\partial x} + \dfrac{\partial v}{\partial\phi}\dfrac{\partial\phi}{\partial x} = \sin\theta\cos\phi\dfrac{\partial v}{\partial r} + \dfrac{\cos\theta\cos\phi}{r}\dfrac{\partial v}{\partial\theta} - \dfrac{\sin\phi}{r\sin\theta}\dfrac{\partial v}{\partial\phi}$, by (3), (4), (5)

$$\therefore \quad \frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin\theta\cos\phi\frac{\partial v}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial v}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial v}{\partial\phi}\right)$$

$$= \sin^2\theta\cos^2\phi\frac{\partial^2 v}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 v}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial v}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 v}{\partial r\partial\phi}$$

$$+ \frac{\sin\phi\cos\phi}{r^2}\frac{\partial v}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial v}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 v}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 v}{\partial\theta\,\partial\phi}$$

$$+ \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial v}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial v}{\partial\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}. \quad \ldots (6)$$

Again, $\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial\theta}\dfrac{\partial\theta}{\partial y} + \dfrac{\partial v}{\partial\phi}\dfrac{\partial\phi}{\partial y} = \sin\theta\sin\phi\dfrac{\partial v}{\partial r} + \dfrac{\cos\theta\sin\phi}{r}\dfrac{\partial v}{\partial\theta} + \dfrac{\cos\phi}{r\sin\theta}\dfrac{\partial v}{\partial\phi}$, by (3), (4) and (5)

$$\therefore \qquad \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

$$\therefore \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin\theta\sin\phi\frac{\partial v}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial v}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial v}{\partial\phi}\right)$$

$$= \sin^2\theta\sin^2\phi\frac{\partial^2 v}{\partial r^2} + \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 v}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial v}{\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 v}{\partial r\,\partial\phi}$$

$$-\frac{\sin\phi\cos\phi}{r^2}\frac{\partial v}{\partial\phi} + \frac{\cos^2\sin^2\phi}{r}\frac{\partial v}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 v}{\partial\theta\partial\phi}$$

$$-\frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial v}{\partial r} + \frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} - \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}. \qquad \text{... (7)}$$

Finally, $\dfrac{\partial v}{\partial z} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial z} + \dfrac{\partial v}{\partial\theta}\dfrac{\partial\theta}{\partial z} + \dfrac{\partial v}{\partial\phi}\dfrac{\partial\phi}{\partial z} = \cos\theta\dfrac{\partial v}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial v}{\partial\theta}$, by (3), (4), (5)

Thus, $$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}.$$

$$\therefore \qquad \frac{\partial^2 v}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 v}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}. \qquad \text{... (8)}$$

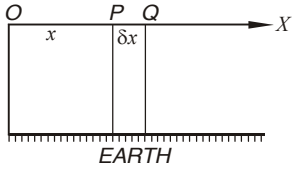
Adding (6), (7) and (8), we see that $(\partial^2 v/\partial x^2) + (\partial^2 v/\partial y^2) + (\partial^2 v/\partial z^2) = 0$ is transformed to

$$\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} = 0,$$

which is the required Laplace's differential equation in spherical polar coordinates.

1.7. Derivation of telegraph (or transmission) line equations.

Consider the flow of electricity in a long insulated cable placed along x-axis. Assuming the flow to be one-dimemional the current I and potential V at any point in the cable is completely determined by one-coordinate x and a time variable t.



Considering the fall of potential in a linear element $PQ(=\delta x)$ situated at point x, we obtain

$$-\delta V = IR\,\delta x + L\delta x(\partial I/\partial t),$$

where R and L are resistance and inductance per unit length respectively.

If there is a capacitance to the earth of C per unit length and conductance G per unit length, then we have $$-\delta I = GV\delta x + C\delta x(\partial V/\partial t)$$

Now as $\delta x \to 0$, the above two equations become $\dfrac{\partial V}{\partial x} + RI + L\dfrac{\partial I}{\partial t} = 0$ ... (1)

and $\dfrac{\partial I}{\partial x} + GV + C\dfrac{\partial V}{\partial t} = 0.$ ... (2)

Differentiating (1) w.r.t. 'x' and (2) w.r.t. 't', we have $\dfrac{\partial^2 V}{\partial x^2} + R\dfrac{\partial I}{\partial x} + L\dfrac{\partial^2 I}{\partial x \partial t} = 0$ ... (3)

and $\dfrac{\partial^2 I}{\partial x \partial t} + G\dfrac{\partial V}{\partial t} + C\dfrac{\partial^2 V}{\partial t^2} = 0$ ... (4)

Eliminating $\partial I / \partial x$ and $\partial^2 I / \partial x \partial t$ from (2), (3) and (4), we get

$$\partial^2 V / \partial x^2 = CL(\partial^2 V / \partial t^2) + (CR + GL)(\partial V / \partial t) + RGV. \quad \text{... (5)}$$

Similarly differentiating (1) w.r.t. 't' and (2) w.r.t. 'x' and eliminating $\partial V / \partial x$ and $\partial^2 V / \partial x \partial t$ as before, we have $\partial^2 I / \partial x^2 = CL(\partial^2 I / \partial t^2) + (CR + GL)(\partial I / \partial t) + RGI.$ ... (6)

Note that (6) is the same as (5) with V replaced by I. The equations (5) and (6) are called *telephone equations*.

Remark 1. If $L = G = O$, the equations (5) and (6) become $\partial^2 V / \partial x^2 = RC(\partial V / \partial t)$ ... (7)

and $\partial^2 I / \partial x^2 = RC(\partial I / \partial t),$ ... (8)

which are known as *telegraph equations*.

Remark 2. If $R = G = 0$, the equations (5) and (6) become $\partial^2 V / \partial x^2 = LC(\partial^2 V / \partial t^2)$ ... (9)

and $\partial^2 I / \partial x^2 = LC(\partial^2 I / \partial t^2),$ ... (10)

which are known as *radio equations*.

Remark 3. If $L = C = 0$, the equations (5) and (6) become $\partial^2 V / \partial x^2 = RGV$ ... (11)

and $\partial^2 I / \partial x^2 = RGI,$ ... (12)

which are equations for *submarine cable*.

1.8. Boundary value problems (B.V.P).

In case of physical and engineering problems, partial differential equations are always accompained by certain initial and boundary conditions and it is required to find the solution satisfying the given conditions. Such problems are known as *boundary value problems.*

Usually we find solutions of partial differential equation first and evaluate the arbitrary functions of the solutions so obtained by using the given conditions. The process of finding the values of arbitrary functions very often becomes laborious. In what follows we discuss an alternative suitable method for such problems.

1.9. Method of separation of variables or product method.

[Bhopal 2001; Poona 2003; Delhi Maths (H) 1997]

While applying the method of separation of variables, we make a certain basic assumption at the outset. Suppose the given partial differential equation involves n independent variable $x_1, x_2, ..., x_n$ and one dependent variables u. Then we first assume that the equation possesses product solution of the form $u(x_1, x_2, ..., x_n) = X_1(x_1)X_2(x_2)...X_n(x_n),$ ... (1)

where X_i is a function of x_i only (i = 1, 2, ..., n). On substitution of (1) into the given equation, we shall obtain n ordinary differential equations one in each of the unknown functions X_i (= 1, 2, ..., n). The entire procedure will be clear from the solved examples of Art. 1.11 of this chapter.

1.10.A. The principle of superposition.

The general linear homogeneous partial differential equation of the second order is

$$A(\partial^2 u/\partial x^2) + B(\partial^2 u/\partial x \partial y) + C(\partial^2 u/\partial y^2) + D(\partial u/\partial x) + E(\partial u/\partial y) + Fu = G. \quad \ldots (1)$$

Suppose that (*i*) u_1, u_2, ... u_n, ... is an infinite set of solutions of (1) in a region R in xy–plane (*ii*) the infinite series $u_1 + u_2 + \ldots + u_n + \ldots$ converges and is differentiable term by term in R. Then by the principle of superposition, the function u, defined by $u = \sum_{n=1}^{\infty} u_n$, is also a solution of (1) in R. Here R denotes the set of all real numbers.

1.10.B. Notations

The conditions which are given for time $t = 0$ are known as *initial conditions* (I.C.). The conditions given at the boundary of the region or interval are called *boundary conditions*. (B.C.).

In what follows we adopt the following standard notations. Suppose $u(x, t)$ and $v(x, y, t)$ are functions of two and three variables respectively. Then we write

$$\partial u/\partial x = u_x = u_x(x,t), \qquad \partial u/\partial t = u_t = u_t(x,t), \qquad (\partial u/\partial x)_{x=\pi} = u_x(\pi,t),$$

$$(\partial u/\partial t)_{t=0} = u_t(x,0), \qquad (\partial v/\partial x) = v_x(x,y,t), \qquad (\partial v/\partial t)_{t=0} = v_t(x,y,0) \text{ etc.}$$

1.11. Solutions of boundary value problems (B.V.P.).

Ex. 1(*a*) *Solve boundary value problem* $\partial u/\partial x = 4(\partial u/\partial y)$, *if* $u\,(0, y) = 8e^{-3y}$.

Sol. Given $$\partial u/\partial x = 4(\partial u/\partial y) \quad \ldots (1)$$

and boundary condition (B.C.) $$u(0, y) = 8e^{-3y}, \quad \ldots (2)$$

Suppose that (1) has solutions of the form $$u(x, y) = X(x)\, Y(y) \quad \ldots (3)$$

where X is a function of x alone and y that of y alone.

Substituting this value of u in (1), we have

$$X'\,Y = 4X\,Y' \qquad \text{or} \qquad X'/4X = Y'/Y, \quad \ldots (4)$$

where the dashes denote derivatives with respect to the relevant variable. Since x and y are independent variables, therefore, (4) can only be true if each side is equal to the same constant, say k. Then (4) gives

$$X' - 4kX = 0 \qquad \text{and} \qquad Y' - kY = 0. \quad \ldots (5)$$

These are ordinary differential equations of first order and first degree with the solutions.

$$X(x) = Ae^{4kx} \qquad \text{and} \qquad Y(y) = Be^{ky}, \quad \ldots (6)$$

where A and B are arbitrary constants.

Substituting these values of X and Y in (3), a solution of (1) is

$$u(x,y) = AB\,e^{4kx+ky} = Ce^{k(4x+y)}, \quad \ldots (7)$$

where $C\,(= AB)$ is another arbitrary constant. Putting $x = 0$ in (7) and using boundary condition (2), we have $8e^{-3y} = C\,e^{ky}$. This is satisfied by talking $C = 8$ and $k = -3$. With these values in (7), the required solution is $u\,(x, y) = 8\,e^{-3(4x+y)}$.

Ex. 1(b) *Solve the boundary value problem* $\partial u/\partial x = 4(\partial u/\partial y)$ *if* $u(0, y) = 8e^{-3y} + 4e^{-5y}$.

Sol. Given $$\partial u/\partial x = 4(\partial u/\partial y) \quad \ldots (1)$$

subject to boundary conditon : $$u(0, y) = 8e^{-3y} + 4e^{-5y} \quad \ldots (2)$$

Proceed as in Ex. 1(a) upto (7) to show that (7) is a solution of (1). Since C and k are arbitrary constants, it follows that $C_1 e^{k_1(4x+y)}$ and $C_2 e^{k_2(4x+y)}$ are solutions of (1). Hence by the

principle of superposition their sum is also a solution of (1). Thus, a solution of (1) may be taken in the form

$$u(x, y) = C_1 e^{k_1(4x+y)} + C_2 e^{k_2(4x+y)} \qquad \ldots (7)'$$

Putting $x = 0$ in $(7)'$ and using boundary condition (2), we have $8e^{-3y} + 4e^{-5y} = C_1 e^{k_1 y} + C_2 e^{k_2 y}$, which is possible if and only if $C_1 = 8$, $k_1 = -3$, $C_2 = 4$ and $k_2 = -5$.

Substituting these values in $(7)'$, the required solution is given by

$$u(x, y) = 8\, e^{-3(4x+y)} + 4e^{-5(4x+y)}.$$

Ex. 2. *Using the method of separation of variables, solve* $\partial u / \partial x = 2(\partial u / \partial t) + u$, *where* $u(x, 0) = 6e^{-3x}$. **[Banaras 1999, Mumbai 2001, Poona 2002]**

Sol. Given $$\partial u / \partial x = 2(\partial u / \partial t) + u \qquad \ldots (1)$$

and boundary condition: $$u(x, 0) = 6e^{-3x} \qquad \ldots (2)$$

Suppose that (1) has solutions of the form $$u(x, y) = X(x)\, T(t) \qquad \ldots (3)$$

where X is a function of x alone and T that of t alone.

Substituting the value of u given by (3) in (1), we get

$$X'T = 2XT' + XT \quad \text{or} \quad X'/X = (2T' + T)/T, \qquad \ldots (4)$$

where the dashes denote derivatives with respect to the relevant variable. Since x and t are independent variables, therefore (4) can only be true if each side is equal to the same constant say k. Then, (4) gives

$$X'/X = k \quad \text{or} \quad dX/dx = k\,X \qquad \ldots (5)$$

and $$(2T' + T)/T = k \quad \text{or} \quad 2(dT/dt) = (k-1)T \qquad \ldots (6)$$

From (5), $(1/X)\, dX = k\, dx$ so that $\log X - \log A = kx$

$\therefore$ $X = Ae^{kx}$, A being an arbitrary constant. $\ldots (7)$

From (6), $(1/T)\, dT = \{(k-1)/2\}dt$ so that $\log T - \log B = \{(k-1)/2\}t$

$\therefore$ $T = B\, e^{\{(k-1)t\}/2}$, B being an arbitrary constant $\ldots (8)$

Substituting the values of X and T given by (7) and (8) in (3), a solution of (1) is given by

$$u(x, t) = AB\, e^{kx} e^{\{(k-1)t\}/2} = C\, e^{kx} e^{\{(k-1)t\}/2}, \qquad \ldots (9)$$

where C (= AB) is another arbitrary constant.

Putting $t = 0$ in (9) and using B.C. (2), we have

$6e^{-3x} = C\, e^{kx}$, which $\Rightarrow$ $C = 6$ and $k = -3$ $\ldots (10)$

Putting $C = 6$ and $k = -3$ in (9), the required solution is

$$u(x, t) = 6\, e^{-3x}\, e^{-2t} \quad \text{or} \quad u(x, t) = 6\, e^{-(3x+2t)}$$

Ex. 3. *Solve by the method of separation of variables.* $\partial^2 z / \partial x^2 - 2(\partial z / \partial x) + \partial z / \partial y = 0$

Sol. Suppose that the given equation has solutions of the form

$$z(x, y) = X(x)\, Y(y), \qquad \ldots (1)$$

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0,$$

where the dashes denote derivatives with respect to the relevant variable. Separating the variables, we get $(X'' - 2X')/X = Y'/Y$ ... (2)

Since x and y are independent variables, so (2) can only be true if each side is equal to the same constant, say k. Then (2) gives

$X'' - 2X' - kX = 0$ or $(D^2 - 2D - k)X = 0$ ... (3)

and $Y' - kY = 0$ or $dY/dy = kY$ ... (4)

Now the auxiliary equation of (3) is $D^2 - 2D - k = 0$ giving $D = \left[2 \pm \sqrt{(4+4k)}\right]/2$

Thus $D = 1 \pm \sqrt{(1+k)}$ and hence general solution of (4) is

$$X(x) = A\, e^{[1+\sqrt{(1+k)}]x} + B\, e^{[1-\sqrt{(1+k)}]x}] \quad \text{... (5)}$$

Again, solution of (4) is given by $Y(y) = C\, e^{ky}$. ... (6)

Substituting these values of X and Y in (1), we get

$$z(x, y) = \{ACe^{[1+\sqrt{(1+k)}]x} + BCe^{[1-\sqrt{(1-k)}]x}\}e^{ky}$$

or $$z(x, y) = \{D\, e^{[1+\sqrt{(1+k)}]x} + E\, e^{[1-\sqrt{(1-k)}]x}\}e^{ky}, \quad \text{... (7)}$$

gives the required solution, where D (= AC) and E (= BC) are new arbitrary constants.

Ex. 4. *Find the most general functions X(x) and Y(y), each of one variable, such that z = XY satisfies the partial differential equation* $\partial^2 z/\partial x^2 = \partial z/\partial y$. *Obtain a solution of this equation which satisfies the boundary conditions, z = 0 when x = 0 and* π; $z = \sin 3x$ *when y = 0 and* $0 < x < \pi$.

Sol. Given $\partial^2 z/\partial x^2 = \partial z/\partial y$ ... (1)

with boundary conditions $z(0, y) = 0$ and $z(\pi, y) = 0$ ... (2)

and initial condition $z(x, 0) = \sin 3x, \quad 0 < x < \pi.$ ... (3)

Let solution of (1) be $z(x, y) = X(x)\ Y(y).$ ... (4)

Substituting the value of z given by (4) into (1), we have

$X''Y = XY'$ or $X''/X = Y'/Y,$ ... (5)

where the dashes denote derivatives with respect to the relevant variable. Since x and y are independent variables, therefore (5) can only be true if each side is equal to the same constant. The given form of the boundary conditions implies that $X(x)$ must be of trigonometric form. To satisfy this requirement, the constant of separation in (5) is taken as $-k^2$. Then (5) gives

$$X'' + k^2 X = 0 \quad \text{... (6)}$$

$$Y' = -k^2 Y \quad \text{... (7)}$$

whose solutions are given by $X(x) = A \cos kx + B \sin kx$ ... (8)

and $Y(y) = Ce^{-k^2 y},$ ... (9)

where A, B and C are arbitrary constants

Putting $x = 0$ in (4) and using (2) [remembering that $Y(y) \neq 0$ for non-trivial solution of (1)],

we have $0 = X(0)\ Y(y)$ so that $X(0) = 0$. ... (10)

Similarly, putting $x = \pi$ in (4) and using (2), we have

$$0 = X(\pi)Y(y) \quad \text{so that} \quad X(\pi) = 0. \quad \text{... (11)}$$

Putting $x = 0$ in (8) and using (10), we get $A = 0$. Next putting $x = \pi$ in (8) and using (11), we get $0 = B \sin k\pi,$ as $A = 0$ ... (12)

From (12), either $B = 0$ or $\sin k\pi = 0.$ But $B = 0$ leads to $X(x) = 0$ so that $z(x, y) = 0$ which does not satisfy (3). Hence $B \neq 0$ and consequently, $\sin k\pi = 0$ where k must be an integer. Thus (8) reduces to

$$X(x) = B \sin kx. \quad \text{... (13)}$$

Using (9) and (13), a solution of (1) is

$$z(x, y) = BC \sin kx\, e^{-k^2 y} = D \sin kx\, e^{-k^2 y}, \quad \text{... (14)}$$

where D (= BC) in an arbitrary constant. Putting y = 0 in (14) and using (3) we have $\sin 3x = D \sin kx$. This is possible if $D = 1$ and $k = 3$. Substituting these values in the (5), the desired solution is $z(x, y) = \sin 3x\, e^{-9y}$.

Ex. 5. *Solve* $\partial u / \partial t = 2(\partial^2 u / \partial x^2)$, $0 < x < 3$, $t > 0$, *given that* $u(0, t) = u(3, t) = 0$, $u(x, 0) = 5 \sin 4\pi x - 3 \sin 8\pi x + 2 \sin 10\pi x$, $|u(x, t)| < M$, M *being a positive real number.*

Sol. Given $2(\partial^2 u / \partial x^2) = \partial u / \partial t$ ... (1)

subject to the boundary conditions: $u(0, t) = u(3, t) = 0$, for all t ... (2)

and the initial condition : $u(x, 0) = 5 \sin 4\pi x - 3 \sin 8\pi x + 2 \sin 10\pi x,\ 0 \le x \le 3$ ... (3)

Also, given that $|u(x, t)| < M$ ... (4)

Suppose that (1) has solutions of the form $u(x, t) = X(x)\ T(t),$ ... (5)

where X is a function of x alone and T that of t alone.

Substituting the value of u given by (5) in (1), we have

$$2X''T = XT' \quad \text{or} \quad X''/X = T'/2T \quad \text{... (6)}$$

Since x and t are independent variables, (6) can only be true if each side is equal to the same constant, say μ. Thus, (6) gives

$$X'' - \mu X = 0 \quad \text{... (7)}$$

and

$$T' = 2\mu T \quad \text{... (8)}$$

Using (2), (5) gives $X(0)\ T(t) = 0$ and $X(3)\ T(t) = 0$ ... (9)

For non-trivial solution of (1), $T(t) \neq 0$ and so (9) gives

$$X(0) = 0 \quad \text{and} \quad X(3) = 0 \quad \text{... (10)}$$

We now solve (7) satisfying (4) and (10). Three cases arise:

Case I. Let $\mu = 0$. Then solutions of (7) and (8) are given by $X(x) = A_1 x + B_1$ and $T(t) = C_1$, where A_1, B_1 and C_1 are arbitrary constants. Hence, using (5), a solution of (1) is given by

$$u(x, t) = X(x)\ T(t) = D_1 x + E_1, \quad \text{... (11)}$$

where D_1 (= $A_1\ C_1$) and E_1 (= $B_1\ C_1$) are new arbitrary constants. The above solution (11) does not satisfy the bounded condition (4). So we reject $\mu = 0$.

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solutions of (7) and (8) are given by $X(x) = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$ and $T = C_1 e^{2\lambda^2 t}$, where A_1, B_1 and C_1 are arbitrary constants. Hence a solution of (1) is given by

$$u(x,t) = X(x)T(t) = (D_1 e^{\lambda x} + E_1 e^{-\lambda x}) e^{2\lambda^2 t},$$

where D_1 (= A_1 C_1) and E_1 (= B_1 C_1) are arbitrary constants. Since above solution does not satisfy (4), so we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0$. Then solution of (7) is given by

$$X(x) = A \cos \lambda x + B \sin \lambda x \quad \text{... (12)}$$

Using boundary condition (10), (12) yields $0 = A$ and $0 = B \sin 3\lambda$ ... (13)

For non-trivial solution we cannot have $A = B = 0$, so we must take $B \neq 0$. Then, (13) reduces to

$\sin 3\lambda = 0$ giving $3\lambda = n\pi$ or $\lambda = n\pi/3,\ n = 1, 2, 3,$ ... (14)

Hence non-zero solutions of (7) are given by $X(x) = B \sin (n \pi x/3)$ ... (15)

Using (14), (8) yields $(1/T)dT = -(2n^2\pi^2/9)dt$

whose solution is $T(t) = C\, e^{-(2n^2\pi^2/9)t}$ ... (16)

From (5), (15) and (16), a solution of (1) is given by

$$u(x,t) = D \sin(n\, \pi\, x/3)\, e^{-(2n^2\pi^2 t/9)}, \quad \text{... (17)}$$

where D (= BC) is another arbitrary constant.

Also, by the principle of superposition, we obtain

$$u(x,t) = D_1 \sin(n_1 \pi x/3) e^{-(2n_1^2\pi^2 t/9)} + D_2 \sin(n_2 \pi x/3) e^{-(2n_2^2\pi^2 t/9)} + D_3 \sin(n_3 \pi x/3) e^{-(2n_3^2\pi^2 t/9)} \quad \text{... (18)}$$

Putting $t = 0$ in (18) and using (3), we have

$$5 \sin 4\pi x - 3 \sin 8\pi x + 2 \sin 10\pi x = D_1 \sin(n_1 \pi x/3) + D_2 \sin(n_2 \pi x/3) + D_3 \sin(n_3 \pi x/3)$$

This is possible if and only if, we have

$D_1 = 5,\ n_1 = 12;\ D_2 = -3,\ n_2 = 24$ and $D_3 = 2;\ n_2 = 30$

Substituting these values in (18), the required solution is

$$u(x,t) = 5\, e^{-32\pi^2 t} \sin 4\pi x - 3\, e^{-128\pi^2 t} \sin 8\pi x + 2 e^{-200\pi^2 t} \sin 10\, \pi x$$

EXERCISE

1. Find the solution of the equation $\partial^2 u/\partial x^2 = 2u + \partial u/\partial y$ in the form $u = f(x)\phi(y)$. Solve the equation subject to the consitions $u = 0$ and $\partial u/\partial x = 1 + e^{-3y}$ when $x = 0$ for all values of y.

[Andhra 2000, Nagpur 1997]

2. Show that $z\ (x,\ y) = 4\ e^{-3x} \cos 3y$ is a solution to the boundary value problem $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0, z(x, \pi/2) = 0, z(x, 0) = 4e^{-3x}$.

3. Using the method of separation of variables, solve

(*i*) $3(\partial u/\partial x) + 2(\partial u/\partial y) = 0, u(x,0) = 4e^{-x}$.

(*ii*) $\partial u/\partial x - u = 2(\partial u/\partial t)$ where $u(x,0) = 6e^{-3x}$.

(*iii*) $4(\partial u/\partial x) + \partial u/\partial y = 3u, u(0,y) = 3e^{-y} - e^{-5y}$. **[Osmania 1999]**

(*iv*) $\partial u/\partial t = \partial u/\partial x - 2u, u(x,0) = 10e^{-x} - 6e^{-4x}$.

(*v*) $\partial u/\partial x = 2(\partial u/\partial y) + u, u(x,0) = 3e^{-5x} + 2e^{-3x}$.

(*vi*) $\partial u/\partial t = \partial^2 u/\partial x^2, u(0,t) = u(4,t) = 0, \quad u(x,0) = 6\sin(\pi x/2) + 3\sin \pi x$.

(*vi*) $u_x + 2u_y = 0, u(0,y) = 4\ e^{-2y}$ [*Delhi B.A./B.Sc. (Hons.) III 2012*]

4. Assuming a solution of $\partial V/\partial t = a^2(\partial^2 V/\partial x^2)$ to be of the form $\phi(\xi)$, where $\xi = x/\sqrt{t}$, obtain the function ϕ and deduce a second solution by differentiating with respect to x.

5. Show that the solution of wave equation $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2)$, where $u = P_0 \cos pt$ (P_0 is a constant) when $x = l$ and $u = 0$ when $x = 0$ is given by

$$u(x, t) = P_0 \text{ cosec } (pl/c) \cos pt \cos (px/c).$$

6. Solve $\partial^2 y/\partial t^2 = Ax + \partial^2 y/\partial x^2, \ 0 < x < 1, \ t = 0,$ given that

$$y(0,t) = y(1, t) = y(x,0) = y_t(x,0) = 0$$

7. For the diffusion equation $u_{xx} = u_t (0 < x < \pi, t > 0)$ subject to $u(0,t) = 0$ and $u(x,0) = 3\sin 2x$, the solution is given by

(*a*) $3e^{-t}\sin 2x$ (*b*) $3e^{-4t}\sin 2x$ (*c*) $4e^{-x-2y}\sin 2x$ (*d*) $3e^{-2t}\sin 2x$ [**GATE 2009**]

8. Laplace equation is (*a*) $\partial^2 z/\partial x^2 = (1/c^2) \times (\partial z/\partial t)$ (*b*) $\partial^2 z/\partial x^2 = \partial^2 z/\partial t^2$

(*c*) $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ (*d*) $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$ **[Bhopal 2010]**

9. Derive the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ by considering the vibrations of a stretched string of length l fixed at the end points. **[Delhi B.A./B.Sc. (Hons.) III 2012]**

Hint. Refer Art. 1.2, page 1.3.

ANSWERS

1. $u = 2^{-1/2}\sinh(x\sqrt{2}) + e^{-3y}\sin x$

3. (*i*) $u = 4e^{(3y - 2x)/2}$ (*ii*) $u = 6e^{-3(x + t)}$

(*iii*) $u = 4e^{x-y} - e^{2x - 5y}$ (*iv*) $u = 10e^{-(x + 3t)} - 6e^{-2(2x + 3t)}$

(*v*) $u = 3e^{-(5x + 3y)} + 2e^{-(3x + 2y)}$ (*vi*) $u = 6e^{-\pi^2 t/4}\sin(\pi x/2) - 3e^{-\pi^2 t}\sin \pi x$

(*vii*) $u = 4e^{4x-2y}$

7. **Ans. (*c*).** Proceed as in Ex. 4, page 1.15 **8.** **Ans. (*c*).** Refer Art. 1.6 A.

2

Boundary Value Problems in Cartesian Co-ordinates

2.1. Introduction

In the present chapter we propose to discuss heat (or diffusion), wave and Laplace (or harmonic) equations in cartesian coordinates.

We present some useful results for ready reference.

1. Heat (Diffusion) Equation : $\nabla^2 u = (1/k)\,(\partial u/\partial t)$.

One-dimensional : $\partial^2 u/\partial x^2 = (1/k)\,(\partial u/\partial i)$

Two-dimensional : $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/k)\,(\partial u/\partial t)$

Three-dimensional : $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/k)\,(\partial u/\partial t)$

Note. Some authors write c^2 instead of k in heat equation.

Remark. 1. ∇^2 is called the *Laplacian operator* and is defined as $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

A function u which satisfies Laplace's equation, is called the *harmonic function*.

2. Wave Equation. $\nabla^2 u = (1/c^2)(\partial^2 u/\partial t^2)$.

One-demensional : $\partial^2 u/\partial x^2 = (1/c^2)\,(\partial^2 u/\partial t^2)$.

Two-dimensional : $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2)\,(\partial^2 u/\partial t^2)$

Three-dimensional : $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/c^2)\,(\partial^2 u/\partial t^2)$

3. Laplace's (or harmonic) Equation : $\nabla^2 u = 0$.

Two dimensional : $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

Three dimensional : $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$.

4. Fourier sine series. If it be required to expand $f(x)$ as a sine series in $0 < x < l$, then its expansion will give the Fourier sine series :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad \text{where} \qquad b_n = \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}\,dx,\ n = 1, 2, 3, \ldots$$

Particular Case : If $l = \pi$, then the above result (4) reduces to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n x, \qquad \text{where} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx,\ n = 1, 2, 3, \ldots$$

5. Fourier Cosine series. If it be required to expand $f(x)$ as a cosine series in $0 < x < l$, then its expansion will give the Fourier cosine series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \quad \text{where} \quad a_0 = \frac{2}{l}\int_0^l f(x)dx\ , \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx, \ n = 1, 2, 3,....$$

Particular case. If $l = \pi$, the above result 5 reduces to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{where} \quad a_0 = \frac{2}{l}\int_0^l f(x)\,dx, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos nx dx, \ n = 1, 2, 3 \ ...$$

Remarks 2. Results (4) and (5) are also known as *half range sine series* and *half range cosine series* respectively.

6. Double Fourier Sine Series. If it be required to expand $f(x, y)$ as a sine series in rectangle $0 \le x \le a, 0 \le y \le b;$ then its expansion will give double Fourier sine series:

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

where $$A_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x, y)\sin\frac{m\pi y}{a}\sin\frac{n\pi y}{b}dxdy, \ m = 1, 2, 3, ...; n = 1, 2, 3, ...$$

Particular Case. If $a = b = \pi$, the above result 6 reduces to

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny,$$

where $$A_{mn} = \frac{4}{\pi^2}\int_{x=0}^{\pi}\int_{y=0}^{\pi} f(x, y)\sin mx \sin\ ny\ dx\ dy, \ m = 1, 2, 3, ... \ n = 1, 2, 3, ...$$

7. Triple Fourier sine series. If it be required to expand $f(x, y, z)$ as a sine series in parallelopiped $0 \le x \le a, \ 0 \le y \le b, \ 0 \le z \le c,$ then its expansion will give triple Fourier sine series:

$$f(x, y, z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \cdot A_{lmn} \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}, \ \text{where}$$

$$A_{lmn} = \frac{8}{abc}\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} f(x, y, z)\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\,dx\,dy\,dz, l = 1, 2, 3,..., m = 1, 2, 3.., n = 1, 2, 3....$$

Particular Case. If $a = b = c = \pi$, the above result 7 reduces to

$$f(x, y, z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn} \sin lx \sin my\ siz\ nz, \ \text{where}$$

$$A_{lmn} = \frac{8}{\pi^3}\int_{x=0}^{\pi}\int_{y=0}^{\pi}\int_{z=0}^{\pi} f(x, y, z)\sin lx \sin my \sin z\ dx\,dydz, l = 1, 2, 3\ ...., m = 1, 2, 3, ..., n = 1,2,3,...$$

8. Method of indicating boundary conditions.

We begin with the case in which the end $x = l$ of the bar is kept at temperature u_0. Then, we write

$$u\ (l,\ t) = u_0, \qquad \text{where } u_0 \text{ is a constant.} \qquad ...\ (i)$$

Condition (i) simply states that the boundary $x = l$ is held by some means at a constant temperature u_0 for all $t > 0$.

Next, we consider the case in which the end $x = l$ of the bar is insulated. Then, we write

$$(\partial u / \partial x)_{x=l} = 0 \qquad ...\ (i)$$

Condition (*ii*) indicates that the boundary $x = l$ is insulated.

From the emprical law of heat transfer, the flux of heat across a boundary (that is, the amount of heat per unit area per unit time conducted across the boundary) is proportional to the normal derivative of the temperatur u. Thus, when the boundary $x = l$ in thermally insulated, no heat flows into or out of the rod and so $(\partial u / \partial x)_{x=l} = 0.$

We shall adopt a similar method of indicating boundary condition in two or three dimensional heat transfer problems.

An improtant note. In what follows, we have explained methods of solving boundary valve problems in some articles and indicated working rules for doing problems. You can solve a problem in one of the two following ways :

Method I. Prove the result of the relevant article completely. Then, compare the given problem with standard boundary value problem and use the relevant working rule as the case may be.

Method II. Proceed exactly as explained in the relevant article without using any result.

We have solved boundary value problem by both of the above two methods to make students familiar with the technique of solving boundary value problem. In examination, you can use any one of the above two methods.

PART I: PROBLEMS BASED ON ONE DIMENSIONAL HEAT (OR DIFFUSION) EQUATION

Situation I. ***When ends of the rod are kept at zero temperature.***

2.2 A. *General solution of heat flow equation* $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ *by the method of separation of variables.* **[Delhi Maths (H) 1997; Meerut 1995; Ravishankar 2002]**

Sol. Given $$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \text{... (1)}$$

Let solution of (1) be of the form $$u(x, t) = X(x)\, T(t) \quad \text{... (2)}$$

We then find, on substituting in (1), that

$$k X'' T = X T' \quad \text{or} \quad X''/X = T'/kT, \quad \text{... (3)}$$

where the dashes denote derivatives with respect to the relevant variable. Clearly the L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone. Since x and t are independent variables, (3) can hold good if each side is equal to a constant, say μ. Then (3) leads to

$$X'' - \mu X = 0 \quad \text{... (4)}$$

and $$T' = \mu k T \quad \text{... (5)}$$

Three cases arise according as μ is zero, positive or negative.

Case I. Let μ = 0. Then solutions of (4) and (5) are

$$X = a_1 x + a_2 \quad \text{and} \quad T = a_3 \quad \text{... (6)}$$

Case II. Let μ *be* + *ve*, say λ^2, where $\lambda \neq 0$. Then (4) and (5) become respectively $X'' - \lambda^2 X = 0$ and $T' = \lambda^2 k T$. Solving these differential equations, we obtain

$$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x} \quad \text{and} \quad T = b_3 e^{\lambda^2 k t} \quad \text{... (7)}$$

Case III. Let μ be – ve, say $-\lambda^2$, where $\lambda \neq 0$. Then (4) and (5) become respectively $X'' + \lambda^2 X = 0$ and $T' = -\lambda^2 k T$. Solving these, we get

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x \quad \text{and} \quad T = c_3 e^{-\lambda^2 k t} \quad \text{... (8)}$$

Thus the various possible solutions are $$u(x, t) = A_1 x + A_2, \quad \text{... (9)}$$

$$u(x,t) = (B_1 e^{\lambda x} + B_2 e^{-\lambda x}) e^{\lambda^2 k t} \quad \text{... (10)}$$

and $$u(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\lambda^2 k t}, \quad \text{... (11)}$$

where $A_1 = a_1 a_3$, $A_2 = a_2 a_3$, $B_1 = b_1 b_3$, $B_2 = b_2 b_3$, $C_1 = c_1 c_3$ and $C_2 = c_2 c_3$ are new arbitrary constants.

Now we have to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problem of heat conduction, temperature $u(x, t)$ must decrease with the increase of time. Accordingly the solution given by (11) is the only suitable solution.

2.2 B Solved examples based on Art. 2.2 A

Example 1. *Explain the method of separation of variables for finding solutions of second order linear partial differential equations. Hence, find the solution of one-dimensional diffusion equation* $\partial^2 z/\partial x^2 = (1/k)(\partial z/\partial t)$ *which tends to zero as* $t \to \infty$. **[Kanpur 2011]**

Sol. ***Method of separation of variables.*** A powerful method of solving the following second-order linear partial differential equation can be used in certain circumstances

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \qquad \ldots (i)$$

where $r = \partial^2 z/\partial x^2, s = \partial^2 z/\partial x \partial y, t = \partial^2 z/\partial y^2$, $p = \partial z/\partial x, q = \partial z/\partial y$ and R, S, T, P, Q, Z and F are functions of x and y.

Let (*i*) possess a solution of the form $z = X(x)\ Y(y)$, ... (*ii*)

where X is a function of x alone and Y is a function of y alone.

Substituting the above value of z in (*i*), (*i*) reduces to

$$(1/X) f(D)\ X = (1/Y)\ g(D')\ \ Y, \qquad \ldots (iii)$$

where $f(D)$, $g(D')$ are quadratic functions of $D(= \partial/\partial x)$ and $D'(= \partial/\partial y)$, respectively. When (*i*) reduces to (*iii*), we say that the equation (*i*) is separable in the variables x, y.

Now, the L.H.S. of (*iii*) is a function of x alone whereas the R.H.S. is a function of y alone and the two can be equal only if each is equal to a constant, λ say. Then (*iii*) gives the following pair of second-order linear ordinary differential equations.

$$f(D)X = \lambda X \qquad \text{and} \qquad g(D')Y = \lambda Y. \qquad \ldots (iv)$$

The problem of finding solutions of the form (ii) of (i) reduces to solving two equations given by (iv).

To find the required solution of the given diffusion equation.

Proceed exactly as in Art. 2.2 A upto equations (11). Then proceed as follows.

We have three possible solutions, namely, (9), (10) and (11). The given condition demands that $u \to 0$ as $t \to \infty$ we therefore reject the solutions given by (8) and (10). Hence, the desired solution is given by (11).

Ex. 2. *Use the method of separation of variables to solve the equation* $\partial^2 v/\partial x^2 = \partial v/\partial t$. *Given that* $v = 0$ *when* $t \to \infty$ *as well as* $v = 0$ *at* $x = 0$ *and* $x = l$.

[Delhi Maths (Hons) 2002, Delhi Maths (Hons) Physics 2001, Lucknow UP (Tech) 2005; Meerut 2000, 02]

Sol. Proceed as in Art. 2.2 A by noting that here $u = v$, $k = 1$ upto equation (11). Thus, we obtain the following three possible types of solutions of $\partial^2 v/\partial x^2 = \partial v/\partial t$.

$$v(x, t) = A_1 x + A_2 \qquad \ldots (9)'$$

$$v(x, t) = (B_1 e^{\lambda x} + B_2 e^{-\lambda x}) e^{\lambda^2 t}, \lambda \neq 0 \qquad \ldots (10)'$$

$$v(x, t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\lambda^2 t}, \lambda \neq 0 \qquad \ldots (11)'$$

We are given that $v \to 0$ as $t \to \infty$. We, therefore, reject the solutions given by (9)′ and (10)′. Hence, the desired solution is given by (11)′, namely,

$$v(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\lambda^2 t}, \lambda \neq 0 \qquad \ldots (12)$$

The given boundary conditions are $v(0, t) = 0$ ... (13)

and $v(l, t) = 0$ ... (14)

Putting $x = 0$ in (12) and using B.C. (13), we obtain $C_1 = 0$ Then, (12) reduces to

$$v(x,t) = C_2 \sin \lambda x\, e^{-\lambda^2 t} \quad \text{... (15)}$$

Putting $x = l$ in (15) and using B.C. (14), we obtain

$$0 = C_2 \sin \lambda l\, e^{-\lambda^2 t} \qquad \text{giving} \qquad C_2 \sin \lambda l = 0 \quad \text{... (16)}$$

Since we are looking for a non-trivial solution, we take $C_2 \neq 0$. Hence (16) reduces to

$$\sin \lambda l = 0 \quad \text{giving} \quad \lambda l = n\pi \quad \text{so that} \quad \lambda = n\pi / l,\ n = 1, 2, 3, ... \quad \text{... (17)}$$

Hence, from (11)′ a solution $v_n(x, t)$ of the given equation for some value of n is given by

$$v_n(x,t) = B_n \sin(n\pi x / l)\, e^{-(n^2\pi^2 t)/l^2}, \text{ by setting } C_2 = B_n. \quad \text{... (18)}$$

Noting that the given equation $\partial^2 v / \partial x^2 = \partial v / \partial t$ is linear, its most general solution is obtained by applying the principle of superposition. Thus, we have

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-(n^2\pi^2 t)l^2}$$

2.3 A. General solution of heat equation when both the ends of a bar are kept at temperature zero and the initial temperature is prescribed.

If both the ends of a bar of length a are at temperature zero and the initial temperature is to be prescribed function f(x) in the bar, then find the temperature at a subsequent time t.

or *The faces x = 0 and x = a of an infinite slab are maintained at zero temperature. Given that the temperature u (x, t) = f(x) at t = 0. Determine the temperature at a subsequent time t.*

[Meerut 2010; Delhi B.A./B.Sc. (Hons) III 2012; Agra 2004; Aligarg 2003; Andhra 2003]

Sol. Here the temperature $u(x, t)$ in the given solid is governed by the one dimensional heat equation

$$k(\partial^2 u / \partial x^2) = \partial u / \partial t \quad \text{... (1)}$$

Since the ends $x = 0$ and $x = a$ are kept at zero temperature, the boundary conditions are

$$u(0, t) = 0 \quad \text{and} \quad u(a, t) = 0, \quad \text{for all } t \quad \text{... (2)}$$

The initial condition is given by $u(x, 0) = f(x),\ 0 < x < a$ ... (3)

Suppose that (1) has solutions of the form $u(x, t) = X(x)\ T(t)$ ... (4)

where X is a function of x alone and T that of t alone.

Substituting this value of u in (1), we get

$$k X'' T = X T' \qquad \text{or} \qquad X'' / X = T' / kT \quad \text{... (5)}$$

Since x and t are independents variables, (5) can only be true if each side is equal to the same constant, say μ. Hence (5) leads to the following equations:

$$X'' - \mu X = 0 \quad \text{... (6)}$$

and

$$T' = \mu k T. \quad \text{...(7)}$$

Using (2), (4) gives $X(0)\ T(t) = 0$ and $X(a)\ T(t) = 0$ ... (8)

Since $T(t) = 0$ leads to $u = 0$, so suppose that $T(t) \neq 0$.

$\therefore$ From (8), $X(0) = 0$ and $X(a) = 0$ ... (9)

We now solve (6) under B.C. (9). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (10)

Using B.C. (9), (10) gives $0 = B$, $0 = Aa + B$ so that $A = B = 0$.

Hence $X(x) \equiv 0$ so that $u \equiv 0$, which does not satisfy (3). So we reject $\mu = 0$

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ... (11)

Using B.C. (9), (11) gives $0 = A + B$ and $0 = Ae^{a\lambda} + Be^{-a\lambda}$ ... (12)

Solving (12), $A = B = 0$ so that $X(x) \equiv 0$ and hence $u = 0$,

which does not satisfy (3). So we also reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0$, Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ... (13)

Using B.C. (9), (13) gives $0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$

So $\sin\lambda a = 0$. We have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $u \equiv 0$ which does not satisfy (3). Solving the trignometric equation $\sin\lambda a = 0$, we have

$\lambda a = n\pi$ so that $\lambda = n\pi / a$, where $n = 1, 2, ...$ ... (14)

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x / a)$ ... (15)

Using (14), (7) reduces to $\dfrac{dT}{T} = -\dfrac{n^2\pi^2 k}{a^2}dt,$ as $\mu = -\lambda^2 = -\dfrac{n^2\pi^2}{a^2}$

or $(1/T)\,dT = -C_n^2 dt,$ where $C_n^2 = (n^2\pi^2 k)/a^2$

whose general solution is $T_n(t) = D_n e^{-C_n^2 t}$ ... (16)

$\therefore$ $u_n(x,t) = X_n(x)T_n(t) = E_n \sin(n\pi x / a)e^{-C_n^2 t}$ ... (17)

are solutions of (1), satisfying (2). Here E_n (= $B_n D_n$) is another arbitrary constant. In order to obtain a solution also satisfying (3), we consider more general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x / a)e^{-C_n^2 t}. \quad \text{... (18)}$$

Substituting $t = 0$ in (18) and using (3), we get $f(x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x / a)$

which is Fourier sine series. So the constants E_n are given by

$$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x / a)dx, \qquad n = 1, 2, 3 \quad \text{... (19)}$$

Hence (18) is the required solution where E_n is given by (19).

2.3 B. Working rule for solving heat equation when both the ends of a bar of length *a* are kept at temperature zero and the initial temperature *f(x)* is prescribed.

Step I. Proceed as in Art. 2.3 A and prove that the solution of the heat equation

$$k(\partial^2 u / \partial x^2) = \partial u / \partial t \quad \text{... (1)}$$

subject to the boundary conditions $u(a, t) = u(a, t) = 0$, for all t ... (2)

and the initial condition $u(x, 0) = f(x)$, $0 < x < a$ ... (3)

is given by $u(x,t) = \sum_{n=0}^{\infty} E_n \sin(n\pi x / a)e^{-C_n^2 t}$ ... (4)

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx,\ n = 1, 2, 3, \ldots \quad \ldots (5)$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \ldots (6)$$

Step II. Compare the given problem with (1), (2) and (3) and find particular values of k, a and $f(x)$.

Step III. Substitute the partiular values of k, a and $f(x)$ in (5) and (6) to get E_n and C_n^2

Step IV. Substitute the values of coefficients E_n and C_n^2 obtained is step III in (4) to arrive at the desired solution of the given boundary value problem.

2.3 C. Solved example based on Art. 2.3 A and Art. 2.3 B

Ex. 1 (*a*) *A rod of length 1 with insulated sides, is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperation u (x, t).* **[Andhra 1997, 2003]**

(b) *A thin rod of length π is first immersed in boiling water so that its temperature is 100°C thoughout. Then the rod is removed from water at t = 0 which is immediately put in ice so that the ends are kept at 0°C. Find w(x, t) if heat equation is $a^2(\partial^2 w/\partial x^2) = \partial w/\partial t$.* **[Nagpur 2002]**

Sol. (*a*) We can prove (actually prove in examination) that the solution of the heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \ldots(1)$$

subject to the boundary conditions $u\ (0,\ t) = u\ (a,\ t) = 0$, for all t. ... (2)

and the initial condition $u(x, 0) = f(x),\ 0 < x < a$ ... (3)

is given by $$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)e^{-C_n^2 t},\ n = 1, 2, 3, \ldots \quad \ldots (4)$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx,\ n = 1, 2, 3 \quad \ldots (5)$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \ldots (6)$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = u_0$. Hence, (5) reduces to

$$E_n = \frac{2}{l}\int_0^1 u_0 \sin\frac{n\pi x}{l}dx = \frac{2u_0}{l}\left[-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^1 = \frac{2u_0}{n\pi}[1-(-1)^n], \quad \text{as} \quad \cos n\pi = (-1)^n$$

$$= \begin{cases} 0, & \text{if } n = 2m, \quad m = 1, 2, 3, \ldots. \\ 4u_0/n\pi, & \text{if } n = 2m-1, \ m = 1, 2, 3, \ldots \end{cases}$$

Hence solution (4) reduces to

$$u(x,t) = \sum_{m=1}^{\infty} E_{2m-1}\sin\frac{(2m-1)\pi x}{l}e^{-C_{2m-1}^2 t} \quad \text{or} \quad u(x,t) = \frac{4u_0}{\pi}\sum_{m=1}^{\infty}\frac{l}{(2m-1)}\sin\frac{(2m-1)\pi x}{l}e^{-C_{2nt-1}^2 t}$$

where $$C_{2m-1}^2 = \{(2m-1)^2\pi^2 k\}/l^2.$$

(*b*) Proceed as in part (*a*). Here $w = u$, $k = a^2$, $a = \pi$ and $u_0 = 100$.

Then, by (6), $C_n^2 = (2m-1)^2 a^2$ and so from (4) the required solution is

$$\therefore \quad w(x,t) = \frac{400}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin(2m-1)x\ e^{-(2m-1)^2 a^2 t}$$

Ex. 2(*a*) *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the conditions* $u(0, t) = u(l, t) = 0$ and $u(x, 0) = lx - x^2$. **[Himanchal 2007; Meerut 2010; Delhi 2007]**

(*b*) *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the conditions* $u(0, t) = u(l, t) = 0$ and $u(x, 0) = x$ when $0 \le x \le l/2$; $u(x, 0) = l - x$ when $l/2 \le x \le l$.

[Meerut 1998, 2000]

Sol. We can prove that the solution of heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \ldots (1)$$

subject to the boundary conditions $u(0, t) = u(a, t) = 0$ for all t ... (2)

and the initial condition $u(x, 0) = f(x),\ 0 < x < a$ ... (3)

is given by $$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a) e^{-C_n^2 t} \quad \ldots (4)$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx, n = 1, 2, 3, \ldots \quad \ldots (5)$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \ldots (6)$$

Part (*a*) : Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = lx - x^2$. Hence, (5) reduces to

$$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left\{\frac{-\cos(n\pi x)/l}{(n\pi)/l}\right\} - (l - 2x)\left\{\frac{-\sin(n\pi x)/l}{(n\pi)^2/l^2}\right\} + (-2)\left\{\frac{\cos(n\pi x)/l}{(n\pi)^3/l^3}\right\}\right]_0^l$$

[Using the chain rule of integration by parts]

$$= (2/l)\left\{(-2l^3/n^3\pi^3)\cos n\pi + (2l^3/n^3\pi^3)\right\} = (4l^2/n^3\pi^3)\left\{1 - (-1)^n\right\}$$

$$\therefore \quad E_n = \begin{cases} (8l^2)/(2m-1)^3\pi^3, \text{ if } n = 2m-1 \text{ (odd) and } m = 1, 2, 3, \ldots \\ 0, \text{ if } n = 2m \text{ (even) where } m = 1, 2, 3, \ldots \end{cases}$$

Then, by (6), $C_n^2 = \{(2m-1)^2\pi^2 k\}/l^2$ and so from (4) the required solution is given by

$$u(x,t) = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 kt\}/l^2}.$$

Part (b) : Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and

$$f(x) = \begin{cases} x, & \text{when } 0 \le x \le l/2 \\ l - x, & \text{when } l/2 \le x \le l \end{cases} \quad \ldots (6)$$

$$\therefore (5) \quad \Rightarrow \quad E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}dx + \int_{l/2}^l f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \int_0^{l/2} \frac{2x}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2}{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \left[\left(\frac{2x}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right) - \left(\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)}{(n\pi)^2/l^2}\right)\right]_0^{l/2} + \left[\left(\frac{2(l-x)}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right) - \left(-\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)/l}{(n\pi)^2/l^2}\right)\right]_{l/2}^{l}$$

[Using chain rule of integration by parts]

$$= -(l/n\pi)\cos(n\pi/2) + (2l/n^2\pi^2)\sin(n\pi/2) + (l/n\pi)\cos(n\pi/2) + (2l/n^2\pi^2)\sin(n\pi/2)$$

$$\therefore \qquad E_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1,2,3,... \\ 4l/(2m-1)^2\pi^2, \text{ if } n = 2m-1 \text{ and } m = 1,2,3,... \end{cases}$$

Then, $\qquad (6) \Rightarrow C_n^2 = \{(2m-1)^2\pi^2 k\}/l^2$

$\therefore$ From (4), $\qquad u(x,t) = \dfrac{4l}{\pi^2}\sum_{m=1}^{\infty}\dfrac{1}{(2m-1)^2}\sin\dfrac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 kt\}/l^2}$

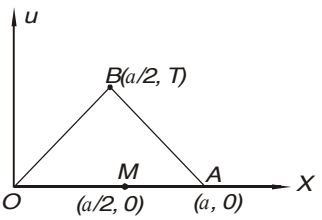
Ex. 2 (c) *Solve* $\partial u/\partial t = \partial^2 u/\partial x^2$, $0 < x < l$, $t > 0$ *given that* $u(0, t) = u(l, t) = 0$ *and* $u(x,0) = x(l-x)$, $0 \le x \le l$. **[I.A.S. 2002]**

Sol. Refer solved Ex. 2(*a*). Here $k = 1$ and hence the solution reduces to

$$u(x,t) = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 t/l^2}$$

Ex. 3. *A homogeneous rod of conducting material of length a has its ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature function u (x, t).*

Sol. We know that $u(x, t)$ is the solution of heat equation $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$. Here the boundary conditions are $u(0, t) = u(a, t) = 0$ for all $t \ge 0$. Let OA be the given rod and M be its middle point. Given that the temperature at the centre M is T and falls uniformly to zero at the two ends O and A of the rod. Hence, the temperature distribution at t = 0 is as given in the adjoining figure. The equations of straight lines OB and BA respectively are given by



$$u - 0 = \frac{T-0}{(a/2)-0}(x-0) \qquad \text{and} \qquad u - 0 = \frac{T-0}{(a/2)-a}(x-a) \qquad \ldots (i)$$

We can prove that the solution of heat equation $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ ... (1)

subject to the boundary conditions $u(a, t) = u(a, t) = 0$, for all t ... (2)

and the initial condition $u(x, 0) = f(x)$, $0 < x < a$ ... (3)

is given by $u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)^{-C_n^2 t}$ ... (4)

where $E_n = \dfrac{2}{a}\int_0^a f(x)\sin(n\pi x/a)\,dx$, $n = 1, 2, 3,$ ... (5)

and $C_n^2 = (n^2\pi^2 k)/a^2$ ... (6)

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = a$. Also, from (*i*), we have

$$u(x,0) = f(x) = \begin{cases} (2Tx)/a, \text{ where } 0 \le x \le a/2 \\ \{2T(a-x)\}/a, \text{ where } a/2 \le x \le a \end{cases} \quad \ldots (7)$$

$$\therefore \quad (5) \Rightarrow \quad E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx = \frac{2}{a}\left[\int_0^{a/2} f(x)\sin\frac{n\pi x}{a}dx + \int_{a/2}^a f(x)\sin\frac{n\pi x}{a}dx\right]$$

$$= \frac{2}{a}\int_0^{a/2}\frac{2Tx}{a}\sin\frac{n\pi x}{a}dx + \frac{2}{a}\int_{a/2}^a \frac{2T(a-x)}{a}\sin\frac{n\pi x}{a}dx, \text{ using (7)}$$

$$= \int_0^{a/2}\frac{4Tx}{a^2}\sin\frac{n\pi x}{a}dx + \int_{a/2}^a \frac{4T(a-x)}{a^2}\sin\frac{n\pi x}{a}dx$$

$$= \left[\left(\frac{4Tx}{a^2}\right)\left(-\frac{\cos(n\pi x)/a}{(n\pi)/a}\right) - \left(\frac{4T}{a^2}\right)\left(-\frac{\sin(n\pi x)/a}{(n\pi)^2/a^2}\right)\right]_0^{a/2}$$

$$+\left[\left\{\frac{4T(a-x)}{a^2}\right\}\left(-\frac{\cos(n\pi x)/a}{(n\pi)/a}\right) - \left(-\frac{4T}{a^2}\right)\left(-\frac{\sin(n\pi x)/a}{(n\pi)^2/a^2}\right)\right]_{a/2}^a$$

$$= -(2T/n\pi)\cos(n\pi/2l) + (4T/n^2\pi^2)\sin(n\pi/2) + (2T/n\pi)\cos(n\pi/2) + (4T/n^2\pi^2)\sin(n\pi/2)$$

$$\therefore \quad E_n = \frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1,2,3,... \\ \{8(-1)^{m+1}T\}/(2m-1)^2\pi^2, \text{ if } n = 2m-1 \text{ and } m = 1,2,3,... \end{cases}$$

Then, by (6), $C_n^2 = \{(2m-1)^2\pi^2 k\}/a^2$ and so from (4), the required solution is given by

$$u(x,t) = \frac{8T}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{a}e^{-\{(2m-1)^2\pi^2 kt\}/a^2}$$

Ex. 4. *Solve the one-dimensional diffusion equation* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *in the range* $0 \le x \le 2\pi, t \ge 0$ *subject to the boundary conditions :* u $(x, 0) = \sin^3 x$ *for* $0 \le x \le 2\pi$ *and* $u(0,t) = u(2\pi,t) = 0$ *for* $t \ge 0$ **[Meerut 2008; Delhi Maths (H), 2004, 06, 09]**

Sol. Proceed upto equation (18) as in Art. 2.3 A by taking $a = 2\pi$ and $f(x) = \sin^3 x$. Then, equation (18) for the present problem reduces to

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{nx}{2}\right)e^{-(n^2kt)/4}, \quad n = 1, 2, 3, \ldots \quad \ldots (i)$$

Putting $t = 0$ in (i) and using given condition u $(x, 0) = \sin^3 x$, we get

$$\sum_{n=1}^{\infty} E_n \sin(nx/2) = \sin^3 x = (3/4)\times\sin x - (1/4)\times\sin 3x$$

$$[\because \sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = (1/4)\times(3\sin x - \sin 3x)]$$

or $$E_1 \sin(x/2) + E_2 \sin x + E_3 \sin(3x/2) + E_4 \sin 2x + E_5 \sin(5x/2) + E_6 \sin 3x + E_7 \sin(7x/2) + \ldots = (3/4)\times\sin x - (1/4)\times\sin 3x. \quad \ldots (ii)$$

Equating the coefficients of the like terms on both sides of (ii), we get

$E_2 = 3/4$, $E_6 = -1/4$ and $E_n = 0$ when $n \neq 2$ or $n \neq 6$. Substituting these values in (*i*), the required solution is

$$u(x, t) = E_2 \sin x\, e^{-kt} + E_6 \sin 3x\, e^{-9kt} = (1/4) \times [3 \sin x\, e^{-kt} - \sin 3x\, e^{-9kt}].$$

Ex. 5. *Determine u such that* $\partial^2 u / \partial x^2 = (l/k)(\partial u / \partial t)$ *and satisfy the conditions (i)* $u \to 0$ *as* $t \to \infty$ *(ii)* $u = \sum_n c_n \cos nx$ *for* $t = 0$. **[Delhi Maths (H) 2004]**

Sol. Given $$\partial^2 u / \partial x^2 = (l/k)(\partial u / \partial t). \quad \text{... (1)}$$

Also given that $u \to 0$ as $t \to \infty$. ... (2)

and $$u(x, 0) = \sum_n c_n \cos nx. \quad \text{... (3)}$$

Let a solution of (1) be $$u(x, t) = X(x)\, T(t), \quad \text{...(4)}$$

where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Substituting (4) in (1), we get $$(1/X)X'' = (1/kT)T' \quad \text{... (5)}$$

Since the L.H.S of (5) is a function of x alone and the R.H.S. of (6) is a function of t alone, hence the two sides of (6) can be equal only if each side is equal to a constant, λ say. In view of condition (2), we choose $\lambda = -n^2$, where n is a non-zero constant. Then (6) gives

$(1/X)X'' = -n^2$ so that $(D^2 + n^2)\, X = 0$, where $D \equiv d/dx$.

and $(1/kT)T' = -n^2$ so that $(1/t)dT = -n^2 k dt$.

Solving these $X_n(x) = a_n \cos nx + b_n \sin nx$ and $T_n(t) = e_n e^{-n^2 kt}$

Keeping (3) and (4) in view, the most general solution of (1) may be written as

$$u(x,t) = \sum_n u_n(x,t) = \sum_n X_n(x) T_n(t) \quad \text{or} \quad u(x,t) = \sum_n (c_n \cos nx + d_n \sin nx) e^{-n^2 kt}, \quad \text{... (1)}$$

where c_n (= $a_n\, e_n$) and d_n (= $b_n\, e_n$) are new arbitrary constants.

Putting $t = 0$ in (1) and using (3), we have $$\sum_n c_n \cos nx = \sum_n (c_n \cos nx + d_n \sin nx),$$

showing that for the present problem $d_n = 0$. Then, from (1) the required solution is

$$u(x,t) = \sum_n c_n \cos nx\, e^{-n^2 kt}$$

Ex. 6. *A uniform rod 20 cm in length is insulated over its sides. Its ends are kept at 0°C. Its initial temperature is* $\sin(\pi x/20)$ *at a distance x from an end. Find temperature u(x, t) at time t. Given that* $\partial u / \partial t = a^2 (\partial^2 u / \partial x^2)$. **[Nagpur 1996]**

Sol. Given $$\partial u / \partial t = a^2 (\partial^2 u / \partial x^2). \quad \text{... (1)}$$

Boundary Conditions (B.C.) : $u(0, t) = u(20, t) = 0$ for all t ... (2)

Initial Conditions (I.C.) : $u(x,0) = \sin(\pi x/20),\ 0 \le x \le 20.$ ... (3)

Let a solution of (1) be $$u(x,t) = X(x)\, T(t). \quad \text{... (4)}$$

Substituting this value of u in (1), we have

$XT' = a^2 X'' T$ or $X''/X = T'/a^2 T.$ ... (5)

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant say μ. Then, (5) gives $X'' - \mu X = 0$ ... (6)

and $T' = \mu a^2 T$ ... (7)

Using (2), (4) gives $X(0)\ T(t) = 0$ and $X(20)\ T(t) = 0$ ... (8)

Since $T(t) \neq 0$, so (8) $\Rightarrow$ $X(0) = 0$ and $X(20) = 0$. ... (9)

We now solve (6) under B.C. (9). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$. ... (10)

Using B.C (9), (10) gives $B = 0$ and $20A + B = 0$ so that $A = B = 0$.

Hence $X(x) \equiv 0$ so that $u = 0$ which does not satisfy (3). Hence reject $\mu = 0$.

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{\lambda x} + B\,e^{-\lambda x}$ ... (11)

Using B.C. (9), (11) gives $A + B = 0$ and $Ae^{20\lambda} + Be^{-20\lambda} = 0$ ... (12)

Solving (12), $A = B = 0$ so that $X(x) \equiv 0$ and hence $u \equiv 0$,

which does not satisfy (3), So we also reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x. \quad ...(13)$$

when A and B are arbitrary constants.

Using B.C. (9), (13) gives A = 0 and $A\cos 20\lambda + B\sin 20\,\lambda = 0$ giving $\sin 20\lambda = 0$. We have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $u \equiv 0$ which does not satisfy (3).

Now $\sin 20\lambda = 0 \Rightarrow 20\lambda = n\pi \Rightarrow \lambda = n\pi/20$, where $n = 1, 2, 3,$ ... (14)

Hence non-zero solution $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x/20)$. ... (15)

Using (14), (7) reduces to $\dfrac{dT}{T} = -\dfrac{n^2\pi^2a^2}{400}dt$, as $\mu = -\lambda^2 = -\dfrac{n^2\pi^2}{400}$

whose general solution is $T_n(t) = D_n e^{-(n^2\pi^2a^2/400)t}$. ... (16)

$$\therefore \quad u_n(x,t) = X_n(x)T_n(t) = E_n \sin(n\pi x/20)e^{-(n^2\pi^2a^2/400)t},$$

are solutions of (1), satisfying (2). Here E_n (= $B_n\ D_n$) is another arbitrary constant. In order to obtain a solution also satisfying (3), we consider the most general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{20} e^{-(n^2\pi^2a^2/400)t} \quad ... (17)$$

Putting time t = 0 in (17) and using (3), we get $\sin\dfrac{\pi x}{20} = \sum_{n=1}^{\infty} E_n \sin\dfrac{n\pi x}{20}$, ... (18)

which is Fourier sine series. So the constants E_n are given by

$$E_n = \frac{2}{20}\int_0^{20} \sin\frac{\pi x}{20}\sin\frac{n\pi x}{20}dx,\ n = 1,2,3,... \quad ... (19)$$

Case I. If $n \neq 1$, then (19) gives

$$E_n = \frac{1}{20}\int_0^{20}\left[\cos\frac{(n-1)\pi x}{20} - \cos\frac{(n+1)\pi x}{20}\right]dx = \frac{1}{20}\left[\frac{20}{(n-1)\pi}\sin\frac{(n-1)\pi x}{20} - \frac{20}{(n+1)\pi}\sin\frac{(n+1)\pi x}{20}\right]_0^{20} = 0$$

Thus, $E_n = 0$ for $n = 2, 3, 4, \ldots$ ... (20)

Case II. If $n = 1$, then (19) gives

$$E_1 = \frac{1}{20}\int_0^{20}\left(2\sin^2\frac{\pi x}{20}\right) = \frac{1}{20}\int_0^{20}\left(1-\cos\frac{2\pi x}{20}\right)dx = \frac{1}{20}\left[x - \frac{10}{x}\sin\frac{\pi x}{10}\right]_0^{20} = 1. \quad \ldots (21)$$

Using (20) and (21) (17) reduces to

$$u(x,t) = E_1 \sin\frac{\pi x}{20}e^{-(\pi^2a^2/400)t} = \sin\frac{\pi x}{20}e^{-(\pi^2a^2/400)t}$$

Ex. 7. *Solve the one-dimensional diffusion equation* $\partial^2u/\partial x^2 = (1/k)(\partial u/\partial t)$ *in the region* $0 \le x \le \pi, t \ge 0$ *when (i) u remains finite as* $t \to \infty$, *(ii) u = 0 if x = 0 or* π, *for all values of t; (iii) At t = 0, u = x for* $0 \le x \le \pi/2$, *and* $u = \pi - x$ *for* $\pi/2 < x \le \pi$.

Sol. Given that $\partial^2u/\partial x^2 = (1/k)(\partial u/\partial t)$ ... (1)

Also, $u(x, t)$ = finite quantity as $t \to \infty$. ... (2)

Boundary conditions are $u(0,t) = (\pi, t) = 0$ for each t. ... (3)

Initial condition is $$u(x,0) = \begin{cases} x, \text{ when } 0 \le x \le \pi/2 \\ \pi - x, \text{ when } \pi/2 \le x \le \pi. \end{cases}$$ ... (4)

Let a solution of (1) be $u(x, t) = X(x)\,T(t)$ ... (5)

Substituting (5) in (1), we get $(1/X)X'' = (1/kT)T'$ ... (6)

Since the L.H.S of (6) is a function of x alone whereas the R.H.S is a function of t alone, hence the two sides can be equal only if each side is equal to a constant, say μ. Hence (6) yields

$$X'' - \mu X = 0 \quad \ldots (7)$$

and $$dT/dt - \mu kT = 0 \quad \ldots (8)$$

In solving (7) and (8), three distinct cases arise :

Case I. When $\lambda = 0$, then the solutions of (7) and (8) will be of the forms

$X = a_1x + a_2$ and $T = a_3$ ... (9)

Case II. When $\mu = \lambda^2, \lambda \ne 0$, then the solutions of (7) and (8) will be of the forms

$X = b_1 e^{\lambda x} + b_2e^{-\lambda x}$ and $T = b_3e^{\lambda^2kt}$ ... (10)

Case III. When $\mu = -\lambda^2, \lambda \ne 0$, then the solutions of (7) and (8) will be of the forms

$X = c_1\cos\lambda x + c_2\sin\lambda x$ and $T = c_3e^{-\lambda^2kt}$ ... (11)

Hence, in view of (5), the various solutions of (1) could be the following

$$u(x, t) = Ax + B \quad \ldots (12)$$

and $$u(x,t) = (Ce^{\lambda x} + De^{-\lambda x})e^{\lambda^2kt} \quad \ldots (13)$$

$$u(x,t) = (E\cos\lambda x + F\sin\lambda x)e^{-\lambda^2 kt}, \qquad \ldots (14)$$

where $A = a_1 a_3$, $B = a_2\, a_3$, $C = b_1\, b_3$, $D = b_2\, b_3$, $E = c_1\, c_3$ and $F = c_2\, c_3$

Now, the condition (2) demands that u should remain finite as $t \to \infty$. We therefore reject solution (13).

Next, in view of B.C. (3), solution (12) gives $0 = A\cdot 0 + B$ and $0 = A\cdot\pi + B$. These give $A = B = 0$ and hence from (12), $u = 0$ for all t. This is a trivial solution. Since we are looking for a non-trivial solution, we reject the solution (12) also. Thus, the only possible solution satisfying the condition (2) is given by (14).

Putting $x = 0$ in (4) and using B.C. $u(0, t) = 0$ given by (3), we obtain $E = 0$. Then, (14) is simplified as

$$u(x,t) = F\sin\lambda x\, e^{-\lambda^2 kt} \qquad \ldots (15)$$

Putting $x = \pi$ in (15) and using the BC $u(\pi,t) = 0$ given by (3), we obtain

$$0 = F\sin\lambda\pi\, e^{-\lambda^2 kt} \quad \text{giving} \quad F\sin\lambda\pi = 0 \qquad \ldots (16)$$

Now, in view of $E = 0$ we must take $F \neq 0$ in order to obtain a non-trivial solution of (1). Accordingly (16) yields

$\sin\lambda\pi = 0$ or $\lambda\pi = n\pi$ so that $\lambda = n$, $n = 1, 2, 3, \ldots$

Thus, from (15), we arrive at a solution of the form

$$u_n(x,t) = F_n \sin nx\, e^{-n^2 kt},\ n = 1,2,3,\ldots$$

Noting that the diffusion equation (1) is linear, its most general solution is obtained by applying the principle of superposition. Thus, the general solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} F_n \sin nx\, e^{-n^2 kt} \qquad \ldots (17)$$

Putting t = 0 in (17), we have

$$u(x,0) = \sum_{n=1}^{\infty} F_n \sin nx, \qquad \ldots (18)$$

which is a half-range Fourier-sine series and, hence

$$F_n = \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} u(x,0)\sin nx\,dx + \int_{\pi/2}^{\pi} u(x,0)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx, \text{ using (5)}$$

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n} -\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

[Using the chain rule of Integrating by parts]

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left(\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right)$$

$$\therefore \qquad F_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3,\ldots \\ 4(-1)^{m+1}/\pi(2m-1)^2, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3,\ldots \end{cases}$$

Substituting the above value of F_n in (17), the required solution is

$$u(x,t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin(2m-1)x\, e^{-(2m-1)^2 kt}$$

Ex. 9. *Make use of the method of separating variables to solve*

$$\partial u/\partial t = c^2(\partial^2 u/\partial x^2),\ t>0,\ 0\le x\le 1 \quad \ldots(1)$$

$$u(0,t) = 2, \qquad u(1,t) = 3 \quad \ldots(2)$$

and

$$u(x,0) = x(1-x) \quad \ldots(3)$$

Sol. Let

$$g(x) = c_1 x + c_2 \quad \ldots(4)$$

such that $g(0) = 2$ and $g(1) = 3$. ...(5)

Using (5), (4) reduces to $2 = c_2$ and $3 = c_1 + c_2$, so that $c_1 = 1$ and $c_2 = 2$, Then (4) becomes

$$g(x) = x + 2 \quad \ldots(6)$$

Let

$$v(x,t) = v(x,t) - g(x) = u(x,t) - x - 2. \quad \ldots(7)$$

Using (7), (1), (2) and (3) respectively may be re-written as

$$\partial v/\partial t = c^2(\partial^2 v/\partial x^2) \quad \ldots(8)$$

$$v(0,t) = 0, \qquad v(1,t) = 0 \quad \ldots(9)$$

and

$$v(x,0) = x(1-x) - x - 2 = -(x^2+2). \quad \ldots(10)$$

Proceed as Art. 2.3B taking $u = v$, $a = 1$, $k = c^2$, $C_n^2 = n^2\pi^2c^2$ and $f(x) = -(x^2+2)$

$$\therefore \qquad v(x,t) = \sum_{n=1}^{\infty} E_n \sin n\pi x\, e^{-n^2\pi^2c^2t} \quad \ldots(11)$$

and $$E_n = -2\int_0^1 (x^2+2)\sin n\pi x\, dx = 2\left[(x^2+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

[Using the chain rule of integration by parts].

$$= -2\left[-\frac{3(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} + \frac{2}{n\pi} - \frac{2}{n^3\pi^3}\right] \quad \ldots(12)$$

Using (11), (7) gives the required solution $u(x,t) = v(x,t) + x + 2$

or $u(x,t) = x + 2 + \sum_{n=1}^{\infty} E_n \sin n\pi x\, e^{-n^2\pi^2c^2t}$, where E_n is given by (12)

2.4 A. General solution of heat equation when both the ends of a bar are insulated and the initial temperature is prescribed. (Meerut 2002)

If both the ends of a bar of length a are insulated and the initial temperature f(x) is prescribed, then to find the temperature at a subsequent time t.

Sol. Here the temperature $u(x, t)$ in the given bar is governed by one dimensional heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \ldots(1)$$

Physical experiments show that the rate of heat flow is proportional to the gradient $\partial u/\partial x$ of the temperature $u(x, t)$. Hence if the ends $x = 0$ and and $x = a$ of the bar are insulated, so that no heat can flow through the ends, we have

the boundary conditions: $u_x(0, t) = u_x(a, t) = 0$ for all t ... (2)

Also, the prescribed initial condition $\Rightarrow$ $u(x, 0) = f(x)$ for all x ... (3)

Suppose that (1) has solutions of the form $u(x, t) = X(x)\ T(t)$, ... (4)

where X is a function of x alone and T that of t alone.

Substituting this value of u in (1), we get

$k X'' T = X T'$ or $X''/X = T'/kT$ ... (5)

Since x and t are independent variables, (5) can be true if each side is equal to the same constant, say μ. Thus, (5) yields. $X'' - \mu X = 0$ ... (6)

and $T' = \mu k T$ ... (7)

Differentiating (4) partially w.r.t. 'x', we get $u_x(x,t) = X'(x)T(t)$ ... (8)

Using (2), (8) gives $X'(0)T(t) = 0$ and $X'(a)T(t) = 0$ ... (9)

Since $T(t) = 0$ leads to $u = 0$, so assume that $T(t) \neq 0$. Hence, from (9), we get

$X'(0) = 0$ and $X'(a) = 0$ ... (10)

Three cases arise :

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$, ... (11)

which yields $X'(x) = A$ ... (12)

Using B.C. (10), (12) gives $A = 0$. Then (11) reduces to $X(x) = B$.

Again, corresponding to $\mu = 0$, (7) yields

$dT/dt = 0$ so that T = constant = $E_0/2B$, say

$\therefore$ Corresponding to $\mu = 0$ a solution of the given boundary value problem from (4) is given by

$u(x, t) = B \times (E_0/2B) = E_0/2$ ... (13)

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = A e^{\lambda x} + B e^{-\lambda x}$ ... (14)

which yields $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ ... (15)

Using B.C (10), (15) gives $0 = \lambda(A - B)$ and $0 = \lambda(Ae^{a\lambda} - Be^{-a\lambda})$.

These equations yield $A = B = 0$ so that $X(x) \equiv 0$ and hence $u = 0$,

which does not satisfy (3). So we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0$. Then solution of (6) is

$X(x) = A\cos\lambda x + B\sin\lambda x$ ... (16)

which yields $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$ ... (16)′

Using B.C. (10), (16)′ gives $0 = B\lambda$ and $0 = -A\lambda\sin\lambda a + B\lambda\cos\lambda a$.

Thus, $B = 0$ and $A\lambda\sin\lambda = 0$. Since $\lambda \neq 0$ and $A \neq 0$ for a non-trivial solution, we must have

$\sin\lambda a = 0$ so that $\lambda a = n\pi$

Thus, $\lambda = n\pi/a$, $n = 1, 2, 3, ...$ ... (17)

Hence non-zero solutions $X_n(x)$ of (6) are given by

$X_n(x) = A_n \cos(n\pi x/a)$, by (16) and (17) ... (18)

Using (17), (7) reduces to

$$\frac{dT}{T} = -\frac{n^2\pi^2 k}{a^2}dt, \qquad \text{as} \qquad \mu = -\lambda^2 = -\frac{n^2\pi^2}{a^2}$$

or $$(1/T)dT = -C_n^2 dt \qquad \text{... (19)}$$

where $$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \text{... (20)}$$

Solving (19), $$T_n(t) = D_n e^{-C_n^2 t} \qquad \text{... (21)}$$

$$\therefore \qquad u_n(x,t) = X_n(X)T_n(t) = E_n \cos(n\pi x/a)e^{-C_n^2 t}, \text{ by (18) and (21)} \qquad \text{... (23)}$$

are solutions of (1). For n = 1, 2, 3, ..., each one these satisfy the boundary conditions (2). Here E_n (= A_n D_n) is another arbitrary constant.

Thus, (13) and (23) constitute a set of infinite solutions of (1). To obtain a solution also satisfying the initial condition (3), we consider a linear combination of these solutions. Hence the complete solution of (1) may be taken in the following form.

$$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} u_n(x,t) \qquad \text{or} \qquad u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a} e^{-C_n^2 t} \qquad \text{... (24)}$$

Substituting t = 0 in (24) and using (3), we have $$f(x) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a}, \qquad \text{... (25)}$$

which is Fourier cosine series. So the constants E_0 and E_n are given by

$$E_0 = \frac{2}{a}\int_0^a f(x)dx \qquad \text{and} \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx, \; n = 1, 2, 3. \ldots \qquad \text{... (26)}$$

Hence (24) is the required solution where E_0 and E_n are given by (26)

2.4 B. Working rule for solving heat equation when both the ends of a bar of length a are insulated and the initial temperature *f*(*x*) is prescribed.

Step I. Proceed as in Art. 2.4 A and prove that the solution of the heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \text{... (1)}$$

subject to the boundary conditions $u_x(0, t) = u_x(a, t) = 0$ for all t ... (2)
and the initial condition $u(x, 0) = f(x)$, for all x ... (3)

is given by $$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a} e^{-C_n^2 t} \qquad \text{... (4)}$$

where $$E_0 = \frac{2}{a}\int_0^a f(x)dx, \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx, \; n = 1, 2, 3\ldots \qquad \text{... (5)}$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \text{... (6)}$$

Step II. Compare the given problem with (1), (2) and (3) and find particular values of k, a and $f(x)$.

Step III. Substitute the particular values of k, a and $f(x)$ in (5) and (6) and calculate E_0, E_n and C_n^2.

Step IV. Substitute the values of coefficients E_0, E_n and C_n^2 obtained is step III in (4) to arrive at the desired solution of the given boundary value problem.

2.4 C. Solved examples based on Art. 2.4 A and Art 2.4 B

Ex. 1 (*a*) *Solve* $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ *for* $0 < x < \pi, t > a$, if $u_x(0,t) = u_x(\pi,t) = 0$ and $u(x, 0) = \sin x$. **[I.A.S. 2002]**

(*b*) *Find the temperature in a laterally insulated bar of length a whose ends are insulated assuming that the initial temperature is* $f(x) = \begin{cases} x, \text{ if } a < x < a/2 \\ a - x, \text{ if } a/2 < x < a \end{cases}$

Sol. We can prove that (prove is examination for complete solution) that the solution of heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \ldots (1)$$

subject to the boundary conditions $u_x(0, t) = u_x(a, t) = 0$ for all t ... (2)

and the initial condition $u(x, t) = f(x),\ 0 < x < a$ ... (3)

is given by $$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a} e^{-C_n^2 t} \quad \ldots (4)$$

where $$E_0 = \frac{2}{a}\int_0^a f(x)dx, \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx,\ n = 1, 2, 3, \ldots \quad \ldots (5)$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \ldots (6)$$

(*a*) Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k,\ a = \pi$ and $f(x) = \sin x$. Hence, from (5), we get

$$E_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\, dx = \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}\times 2 = \frac{4}{\pi} \quad \ldots (7)$$

and $$E_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} \{\sin(n+1) - \sin(n-1)x\}\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

Thus, $$E_n = \frac{1}{\pi}\left[\frac{1-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right] \quad \ldots (8)$$

If $n = 2m - 1$ (odd) with $m = 1, 2, 3, \ldots$, then $E_n = 0$, by (8)

If $n = 2m$ (even) with $m = 1, 2, 3, \ldots$, then from (8), we get

$$E_{2m} = \frac{2}{m}\left(\frac{1}{2m+1} - \frac{1}{2m-1}\right) = -\frac{4}{\pi(4m^2-1)},\ m = 1, 2, 3 \quad \ldots (9)$$

Also, from (6), $$C_{2m}^2 = \{(2m)^2 \times \pi^2 \times k\}/\pi^2 = 4m^2\pi^2$$

Substituting the values of E_0 and E_{2m} given by (7) and (9) in (4), the required solution is

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum \frac{\cos 2mx}{4m^2-1} e^{-4m^2 k t}$$

(*b*) Comparing the given boundary value problem with boundary value problem given by (1), (2) and (3), we have $k = k$, $a = a$ and

$$u(x,0) = f(x) = \begin{cases} x, & \text{if } 0 < x < a/2 \\ a - x, & \text{if } a/2 < x < a \end{cases} \quad \ldots (7)$$

From (5) and (7), we have

$$E_0 = \frac{2}{a}\int_0^a f(x)dx = \frac{2}{a}\left[\int_0^{a/2} f(x)dx + \int_{a/2}^a f(x)\,dx\right]$$

$$= \frac{2}{a}\left[\int_0^{a/2} x\,dx + \int_{a/2}^a (a-x)dx\right] = \frac{2}{a}\left\{\left[\frac{x^2}{2}\right]_0^{a/2} + \left[ax - \frac{x^2}{2}\right]_{a/2}^a\right\}$$

$$= (2/a) \times \left\{a^2/8 + (a^2/2 - 3a^2/8\right\} = (2/a) \times (a^2/4) = a/2$$

$$E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx = \frac{2}{a}\left[\int_0^{a/2} f(x)\cos\frac{n\pi x}{a}dx + \int_{a/2}^a f(x)\cos\frac{n\pi x}{a}dx\right]$$

$$= \frac{2}{a}\int_0^{a/2} x\cos\frac{n\pi x}{a}dx + \frac{2}{a}\int_{a/2}^a (a-x)\cos\frac{n\pi x}{a}dx, \text{ using (7)}$$

$$= \frac{2}{a}\left[(x)\frac{\sin(n\pi x/a)}{(n\pi/a)} - (1)\left(-\frac{\cos(n\pi x/a)}{(n^2\pi^2/a^2)}\right)\right]_0^{a/2} + \frac{2}{a}\left[(a-x)\frac{\sin(n\pi x/a)}{(n\pi/a)} - (-1)\left(-\frac{\cos(n\pi x/a)}{(n^2\pi^2/a^2)}\right)\right]_{a/2}^a$$

[Using chain rule of integrating by parts]

$$= \frac{2}{a}\left[\frac{a}{2} \times \frac{a}{n\pi}\sin\frac{n\pi}{2} + \frac{a^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{a^2}{n^2\pi^2}\right] + \frac{2}{a}\left[-\frac{a^2}{n^2\pi^2}\cos n\pi - \frac{a}{2} \times \frac{a}{n\pi}\sin\frac{n\pi}{2} + \frac{a^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{a}\left[\frac{2a^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{a^2}{n^2\pi^2} - \frac{a^2}{n^2\pi^2}\cos n\pi\right] = \frac{2a}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\,\pi - 1\right)$$

Also, from (6), we have $C_n^2 = (n^2\pi^2 k)/a^2$

Substituting the above values of E_0, E_n and C_n^2 in (4), the required solution is given by

$$u(x,t) = \frac{a}{4} + \frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)\cos\frac{n\pi x}{a}e^{-(n^2\pi^2 kt)/a^2}$$

or $$u(x,t) = \frac{a}{4} - \frac{8a}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{a}e^{-(4\pi^2 kt)/a^2} + \frac{1}{6^2}\cos\frac{6\pi x}{a}e^{-(36\pi kt/a^2} + \ldots\right)$$

Ex. 2. *Find the solution of the one-dimensional diffusion equation* $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ *satisfying the following boundary conditions : (1) u is bounded as* $t \to \infty$ *(ii)* u_x (0, *t*) = 0, u_x (*a*, *t*) = 0 for all *t* (*iii*) *u*(*x*, 0) = *x* (*a* – *x*), 0 < *x* < *a*. **(Himanchal 2007)**

Sol. We know that the bounded solution the diffusion equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \ldots (1)$$

subject to the boundary conditions u_x (0, *t*) = u_x (*a*, *t*) = 0 for all *t* ... (2)

and the initial condition $u(x, 0) = f(x), 0 < x < a$ ... (3)

is given by $$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{a} e^{-C_n^2 t} \quad \text{... (4)}$$

where $$E_0 = \frac{2}{a}\int_0^a f(x)dx, \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx, \ n = 1, 2, 3, \ldots \quad \text{... (5)}$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \text{... (6)}$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = a$ and $f(x) = ax - x^2$. So from (5), we have

$$E_0 = \frac{2}{a}\int_0^a (ax - x^2)dx = \frac{2}{a}\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{a^2}{3}$$

$$E_n = \frac{2}{a}\int_0^a (ax - x^2)\cos\frac{n\pi x}{a}dx$$

$$= \frac{2}{a}\left[(ax - x^2)\frac{\sin(n\pi x/a)}{(n\pi/a)} - (a - 2x)\left(-\frac{\cos(n\pi x/a)}{n^2\pi^2/a^2}\right) + (-2)\left(-\frac{\sin(n\pi x/a)}{n^3\pi^3/a^3}\right)\right]_0^a$$

[Using chain rule of integrating by parts]

$$= \frac{2}{a}\left[-a \times \frac{a^2}{n^2\pi^2}(-1)^n - a \times \frac{a^2}{n^2\pi^2}\right] = -\frac{2a^2}{n^2\pi^2}\left\{1 + (-1)^n\right\}$$

Hence, if $n = 2m$ (even), then $E_n = E_{2m} = -(a^2/m^2\pi^2)$

and if $n = 2m–1$ (odd), then $E_n = E_{2m-1} = 0$.

Also, from (6), $C_n^2 = (n^2\pi^2 k)/a^2 = (4m^2\pi^2 k)/a^2$, if $n = 2m$

Substituting the above values of E_0, E_n and C_n^2 in (4), the required solution is given by

$$u(x,t) = \frac{a^2}{6} + \sum_{m=1}^{\infty}\left(-\frac{a^2}{m^2\pi^2}\right)\cos\frac{2m\pi x}{a} e^{-(4m^2\pi^2 kt)/a^2}$$

or $$u(x,t) = \frac{a^2}{6} - \frac{a^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\frac{2m\pi x}{a} e^{-(4m^2\pi^2 kt)/a^2}$$

2.5. Solution of heat equation when one end is insulated

While finding the solution of heat equation when one end is insulated while the other end is kept at a constant temperature, we proceed as explained in the following example.

Example: *Obtain temperature distribution* $y(x, t)$ *in a uniform bar of unit length whose one end is kept at* $10°$ *C and the other end is insulated. Further it is given that* $y(x, 0) = 1 – x, 0 < x < 1$.

Sol. Suppose the bar be placed along the x-axis with its one end (which is at 10°C) at origin and the other end at $x = 1$ (which is insulated so that flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity). Then we are to solve heat equation

$$\partial y/\partial t = k(\partial^2 y/\partial x^2) \quad \text{... (1)}$$

with boundary conditions $y_x(1, t) = 0, \quad y(0, t) = 10$ ... (2)

and initial conditions $y(x, 0) = 1 - x, \ 0 < x < 1.$ ... (3)

Let $y(x, t) = u(x, t) + 10$ ... (4)

i.e., $u(x, t) = y(x, t) - 10.$ ... (5)

Using (4) or (5), (1), (2) and (3) reduce to $\partial u / \partial t = k(\partial^2 u / \partial x^2)$ ... (6)

$u_x(1, t) = 0, \quad u(0, t) = 0$ ... (7)

and $u(x, 0) = y(x, 0) - 10 = -(x + 9).$ ... (8)

Suppose that (6) has solutions of the form $u(x, t) = X(x)\, T(t)$ ... (9)

Substituting this value of u in (6), we get

$X T' = k X'' T$ or $X'' / X = T' / kT.$ ... (10)

Since x and t are independent variables, (10) can only be true if each side is equal to the same constant, say μ. Hence (10) gives $X'' - \mu X = 0$ ... (11)

and $T' = \mu k T.$ ... (12)

Using (7), (9) gives $X'(1)T(t) = 0$ and $X(0)\, T(t) = 0$ ... (13)

Since T(t) = 0 leads to $u \equiv 0,$ so we suppose that $T(t) \neq 0.$ Then, from (13), we get

$X'(1) = 0$ and $X(0) = 0$ ... (14)

We now solve (11) under B.C. (14). Three cases arise.

Case I. Let μ = 0. Then solution of (11) is $X(x) = Ax + B$ ... (15)

From (15), $X'(x) = A.$ ... (15)′

Using B.C. (14), (15) and (15)′ give $0 = A$ and $0 = B.$

So from (15), $X(x) \equiv 0,$ which leads to $u \equiv 0.$ So reject μ = 0.

Case II. Let $\mu = \lambda^2, \lambda \neq 0.$ Then solution of (11) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ... (16)

so that $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ ... (16)′

Using B.C. (14), (16) and (16)′ give $0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda}$ and $0 = A + B.$

These give $A = B = 0$ so that $X(x) \equiv 0$ and hence $u(x) = 0$ and hence $u(x) \equiv 0.$

So we reject $\mu = \lambda^2.$

Case III. Let $\mu = \lambda^2, \lambda \neq 0.$ Then solution of (12) is

$X(x) = A\cos\lambda x + B\sin\lambda x$ ...(17)

so that $X'(x) = -A\lambda \sin \lambda x + B\lambda \cos\lambda x$ ...(17)′

Using B.C. (14), (17) and (17)′ give $0 = -A\lambda\sin\lambda + B\lambda\cos\lambda$ and $0 = A$

These give $A = 0$ and $\cos\lambda = 0,$ ... (18)

where we have taken $B \neq 0,$ since otherwise $X(x) \equiv 0$ and hence $u \equiv 0.$

Now, $\cos\lambda = 0 \quad \Rightarrow \quad \lambda = (2n-1) \times (\pi/2) = (1/2) \times (2n-1)\pi \quad n = 1, 2, 3, \ldots$

So, $\mu = -\lambda^2 = -(1/4) \times (2n-1)^2 \pi^2.$ ... (19)

Hence non-zero solutions $X_n(x)$ of (17) are given by $X_n(x) = B_n \sin\{(2n-1)\pi x/2\}$

Again using (19), (12) reduces to

$$\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2 k}{4} T \quad \text{or} \quad \frac{dT}{T} = -C_n^2 dt \qquad \ldots (20)$$

where $$C_n^2 = (1/4)\times(2n-1)^2\pi^2 k \qquad \ldots (21)$$

Solving (20), $$T_n(t) = D_n e^{-C_n^2 t}. \qquad \ldots (22)$$

Thus, $$u_n(x,t) = X_n T_n = E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$$

are solutions of (6), satisfying (7). Here E_n (= $B_n D_n$) is another arbitrary constant. In order to obtain a solution also satisfying (8), we consider more general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t} \qquad \ldots (23)$$

Putting $t = 0$ in (23) and using (8), we have $$-(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} \qquad \ldots (24)$$

Multiplying both sides of (24) by $\sin\{(2m-1)\pi x/2\}$ and then integrating with respect to x from 0 to 1, we get

$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = \sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx \qquad \ldots (25)$$

But $$\int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n. \end{cases} \qquad \ldots (26)$$

[Students are advised to prove result (26) in examination for complete solution]

Using (26), (25) gives $$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = E_m$$

$\therefore$ $$E_n = -\int_0^1 (x+9)\sin\frac{(2n-1)\pi x}{2}\,dx$$

or $$E_n = -2\left[(x+9)\left\{\frac{-\cos(2n-1)\pi x}{(2n-1)\pi/2}\right\} - (1)\left\{\frac{-\sin\frac{(2n-1)\pi x}{2}}{(2n-1)^2\pi^2/4}\right\}\right]_0^1$$

[on using chain rule of integration by parts]

$$= \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi} \qquad \left\{\because \cos\{(2n-1)\pi/2\} = 0 \text{ and } \sin\{(2n-1)\pi/2\} = (-1)^{n-1}\right\} \qquad \ldots (27)$$

Using (23) and (4), the required solution is given by

$$y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}.$$

where C_n and E_n are given by (21) and (27) respectively.

2.6. Miscellaneous examples on solution of heat or diffusion equation.

Ex. 1. (*a*) *An insulated rod of length l has its ends A and B kept at a° calsius and b° celsius respectively until steady state conditions prevail. The temperature at each end is suddenly reduced to zero degree celsius and kept so. Find the resulting temperature at any point of the rod taking the end A as origin.*

(*b*) *A rod 30 cm long has its ends A and B kept at 20° and 80° respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0° and kept so. Find the resulting function u(x, t) taking x = 0 at A.*

(*c*) *The temperature of a bar 50 cm long with insulated sides is kept at 0° at one end and 100° at the other end until steady conditions prevail. The two ends are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.*

Sol. (*a*) The temperature function $u\,(x, t)$ is a solution of heat equation

$$\partial^2 u / \partial x^2 = (1/k)\,(\partial u / \partial t). \qquad \ldots (i)$$

When the steady state condition prevails, $\partial u / \partial t = 0$ and therefore (*i*) reduces to ordinary differential equation $d^2u/dx^2 = 0$ ... (*ii*)

Integrating (*ii*), $du/dx = c_1$, ... (*iii*)

Integrating (iii), $u(x) = c_1 x + c_2$, ... (*iv*)

where c_1 and c_2 and arbitrary constants. Putting $x = 0$ and $x = l$ in (*iv*) and using the fact that $u = a$ when $x = 0$ and $u = b$ when $x = l$, we have

$a = (c_1 \times 0) + c_2$ and $b = c_1 l + c_2$ so that $c_1 = (b - a)/l$ and $c_2 = a$.

So, by (*iv*), here $u\,(x, 0) = f(x) = \{(b - a)\,x\}/l + a$. ... (*v*)

Also, given boundary conditions are: $u\,(0, t) = u\,(l, t) = 0$ for all $t \geq 0$.

We can prove that (actually prove in examination as in Art. 2.4 B) that the solution of the

heat equation $k(\partial^2 u / \partial x^2) = \partial u / \partial t$ ... (1)

subject to the boundary conditons $u(0, t) = u(a, t) = 0$ for all $t > 0$ ... (2)

and the initial condition $u(x, 0) = f(x),\ 0 < x < a$ ... (3)

is given by $$u(x,t) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{a} e^{-C_n^2 t} \qquad \ldots (4)$$

where $$E_n = \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a}\,dx, \quad n = 1, 2, 3, \qquad \ldots (5)$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \ldots (6)$$

Comparing the given boundry value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = a + \{(b - a)x\}/l$. Hence, from (5), we have

$$E_n = \frac{2}{l}\int_0^l \left\{\frac{(b-a)x}{l} + a\right\} \sin \frac{n\pi x}{l}\,dx = \left[\frac{2}{l}\left\{\frac{(b-a)x}{l} + a\right\}\left(\frac{-\cos(n\pi x)/l}{(n\pi)/l}\right) - \frac{2(b-a)}{l^2}\left(\frac{-\sin(n\pi x)/l}{(n\pi/l)^2}\right)\right]_0^l$$

[Using chain rule of integration by parts]

$$= -(2b/n\pi)\cos n\pi + (2a/n\pi) = (2/n\pi)\{a - b(-1)^n\}$$

Also, from (6), $C_n^2 = (n^2\pi^2 k)/l^2$

So, by (4), $$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{a-b(-1)^n}{n}\sin\frac{n\pi x}{l}e^{-(n^2\pi^2kt)/l^2}.$$

(*b*) Proceed as in part (*a*) taking $l = 30$, $a = 20$ and $b = 80$.

Ans. $$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1-4(-1)^n}{n}\sin\frac{n\pi x}{30}e^{-(n^2\pi^2kt)/900}$$

(*c*) Proceed as in part (*a*) taking $l = 50$, $a = 0$ and $b = 100$.

Ans. $$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{50}e^{-(n^2\pi^2kt)/2500}.$$

Ex. 2. (*a*) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively untill steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.*

(*b*) *Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* **[Nagpur 1995]**

Sol. Part (*a*). The equation for conduction of heat is

$$\partial u/\partial t = k(\partial^2 u/\partial x^2) \quad \ldots (1)$$

Prior to temperature change at the end *B* when $t = 0$, the heat-flow was independent of time (steady state condition, for which $\partial u/\partial t = 0$). When u depends only on x, (1) reduces to

$d^2u/dx^2 = 0$ so that $u = c_1x + c_2$ ... (2)

Given $u = 0$ for $x = 0$ and $u = 100$ for $x = l$,

$\therefore$ (2) gives, $0 = c_2$ and $100 = lc_1 + c_2$, giving $c_1 = 100/l$, $c_2 = 0$

$\therefore$ (2) becomes $u = (100/l)x$

Hence the boundary conditions for the subsequent flow are

$u(0, t) = 0$, for all t ... (3)

$u(l, t) = 0$, for all t ... (4)

and the initial condition is $u(x, 0) = (100/l)x$ ... (5)

We can prove that the solution of heat equation (refer Art. 2.4 B)

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \ldots (i)$$

subject to the boundary conditions $u(0, t) = u(a, t) = 0$ for all t ... (*ii*)

and the initial condition $u(x, 0) = f(x)$, $0 < x < a$ ... (*iii*)

is given by $$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}e^{-C_n^2 t} \quad \ldots (iv)$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \quad \ldots (v)$$

and $$C_n^2 = (n^2\pi^2k)/a^2 \quad \ldots (vi)$$

Comparing the boundary value problem given by (1), (3), (4) and (5) with the boundary value problem given by (*i*), (*ii*) and (*iii*), we have $a = l$, $k = k$ and $f(x) = (100/l)x$. So, from (*v*) we get

$$E_n = \frac{2}{l}\int_0^1 \frac{100x}{l}\sin\frac{n\pi x}{l}dx = \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - \left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^1 = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Also, from (*vi*) we have $$C_n^2 = (n^2x^2k)/l^2$$

$\therefore$ From (*iv*), the required solution is

$$u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-(kn^2\pi^2 t)/l^2}$$

Part (b). Here the initial condition is (5) as before. The new boundary conditions are

$$u(0, t) = 20, \quad \text{for all } t \qquad \ldots (7)$$

and

$$u(l, t) = 80, \quad \text{for all } t. \qquad \ldots (8)$$

Note that in part (*a*), the two boundary values were both zero and the required solution was easily obtained. In the present part (*b*), the two boundary values are non-zero, so we modify the procedure as follows :

We split the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_1(x) + u_2(x, t) \qquad \ldots (9)$$

where $u_1(x)$ is a solution of (1) involving x only and satisfying (7) and (8); $u_2(x, t)$ is then a function defined by (9). Hence $u_1(x)$ is a steady state solution of the form (2) and $u_2(x, t)$ may be treated as a transient part of the solution, which decreases with increase of t.

Since $u_1(x) = 20$ for $x = 0$ and $u_1(x) = 80$ for $x = l$, (2) gives

$$u_1(x) = 20 + (60x/l) \qquad \ldots (10)$$

Putting $x = 0$ in (9) and using (7), we get

$$u_2(0, t) = u(0, t) - u_1(0) = 20 - 20 = 0 \qquad \ldots (11)$$

$\therefore$ Next, putting $x = l$ in (9) and using (8), we get

$$u_2(l, t) = u(l, t) - u_1(l) = 80 - 80 = 0 \qquad \ldots (12)$$

Again, $u_2(x, 0) = u(x, 0) - u_1(x) = (100x/l) - [(60x/l) + 20]$

Thus,

$$u_2(x, 0) = (40x/l) - 20 \qquad \ldots (13)$$

Hence the boundary conditions and initial condition to the transient solution $u_2(x, t)$ are given by (11), (12) and (13).

So we now solve $\partial u_2 / \partial t = k(\partial^2 u_2 / \partial x^2)$

subject to boundary condition (11) and (12) and initial condition (13) as in Art. 2.3 B and obtain

$$u_2(x,t) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{l} e^{-(kn^2\pi^2 t/l^2)} \qquad \ldots (14)$$

where

$$E_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right) \sin \frac{n\pi x}{l} dx = -\frac{40}{n\pi}(1 + \cos n\pi) = -\frac{40}{n\pi}[1 + (-1)^n]$$

Thus,

$$E_n = \begin{cases} 0, \text{ when } n \text{ is odd} \\ -80/n\pi, \text{ when } n \text{ is even} \end{cases}$$

Taking $n = 2m$, we have $E_n = -(40/m\pi)$

$\therefore$ From (14), we have

$$u_2(x,t) = -\frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} e^{-(4km^2\pi^2 t/l^2)} \qquad \ldots (15)$$

Combining (10) and (15), the required solution is

$$u(x,t) = 20 + \frac{40x}{l} - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} e^{-(4km^2\pi^2 t/l^2)}$$

EXERCISE 2(A)

1. The heat conduction problem of a homogeneous rod of length l where the surface is insulated to prevent heat loss through the boundary is given by the equations : $u_t = ku_{xx}$ for $0 < x < l$, $t > 0$, $u(0,t) = 0$ *for* $t \geq 0$, $u(l,t) = 0$ for $t \geq 0$ and $u(x,0) = f(x)$, for

$0 \le x \le l$. Show that the formal series solution of this problem is given as. $u(x, y) = \sum_{n-1}^{\infty} a_n e^{-(n\pi/l)^2 \times kt} \sin(n\pi x / l)$, where $a_n = \frac{2}{l}\int_0^1 f(x)\sin(n\pi x / l)\, dx$.

[Delhi B.A./B.Sc. (Hons) II 2012]

Hint : Refer Art. 2.3A, page 2.5 by taking $a = l$ and $F_n = a_n$.

2. Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = l$ are kept at temperature zero and whose initial temperature is given by $f(x) = \begin{cases} A, \text{ when } 0 < x < l/2 \\ 0, \text{ when } l/2 < x < l \end{cases}$

Ans. $u(x,t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{n\pi}{4} \sin \frac{n\pi x}{l} e^{-(n^2\pi^2 kt)/l^2}$

3. Determine the solution of one-dimensional heat equation $\partial\theta / \partial t = \partial^2\theta / \partial x^2$ under the boundary conditions $\theta(0,t) = 0, \theta(l,t) = 0$ when $t > 0$ and the initial condition $\theta(x, 0) = x$ when $0 < x < l$, l being the length of the bar. **[Himanchal 2007; Garhwal 2005, Meerut 1997]**

Ans. $\theta(x,t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin \frac{n\pi x}{l}\ e^{-(n^2\pi^2 t/l^2)}$

4. Solve $\partial^2 u / \partial x^2 = h^2(\partial u / \partial t)$ when $u(0, t) = u(l, t) = 0$ and $u(x, 0) = \sin(\pi x/l)$.

Ans. $u(x,t) = \sin(\pi x / l) e^{-\pi^2 t / h^2 l^2}$

5. Show that the solution of $\partial u / \partial t = k(\partial^2 u / \partial x^2)$ subject to the conditions (*i*) u is not infinite for $t \to \infty$ (*ii*) $\partial u / dx = 0$ for $x = 0$ and $x = l$ (*iii*) $u = lx - x^2$ for $t = 0$, between $x = 0$ and $x = l$ is

$$u = \frac{1}{6}l^2 - \frac{l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{l}{n^2} \cos \frac{n\pi x}{l} e^{-(4\pi^2 n^2 kt)/l^2}$$

6. Solve $\partial u / dt = \alpha^2(\partial^2 u / \partial x^2)$, given that (*i*) $u = 0$, when $x = 0$ and $x = l$ for all t,

(*ii*) $u = 3\sin(\pi x / l)$ when $t = 0$ for all x, $0 < x < l$. **Ans.** $u(x,l) = 3\sin(\pi x / l) e^{-(\alpha^2\pi^2 t/l^2)}$

7. Find by the method of separation of variables the solution $u(x, t)$ of the boundary value problem $\partial U / \partial t = 3(\partial^2 U / \partial x^2)$, $t > 0, 0 < x < 2$, $U(0, t) = 0$, $U(2, t) = 0$, $t > 0$; U $(x, 0) = x$, $0 < x < 2$

Ans. $U(x,t) = \sum_{n=1}^{\infty} \frac{\sin(n\pi x/2)}{\sin(n\pi/2)} e^{-(3\pi^2 n^2 t)/4}$ **[Rohilkhand 1997]**

8. A bar of length unity, has its end at $x = 0$ insulated and its end at $x = 1$ is kept at temperature zero. Find an expression for the temperature $u(x, t)$ if

$$u(x,0) = \begin{cases} 1, \text{ where } 0 \le x \le 1/2 \\ 2(1-x) \quad , \text{when } 1/2 \le x \le 1 \end{cases}$$

Ans. $u = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos\{(2n-1)\pi x/2\} \cos\{(2n-1)\pi/4\}}{(2n-1)^2} e^{-\{(2n-1)^2\pi^2 k^2 t\}/4}$

9. The ends A and B of a rod 20 cm long have the temperatures at 30° and 80° until steady state prevails. The temperatures of the ends are changed to 40° and 60° respectively. Find the temperature distribution in the rod at time *t*. **[I.F.S 2003. Rajasthan 2006; I.A.S. 2005]**

Ans. $u = 40 + x - \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{2(-1)^n + 1}{n} \sin \frac{n\pi x}{20} e^{-(c^2 n^2 \pi^2 t)/400}$

10. The temperature at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state condition prevails. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

Ans. $u = 50 - \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{50} e^{-(2n-1)^2 \pi^2 c^2 t/2500}$

11. Show that the solution of the equation $\partial^2\theta/\partial x^2 = \partial\theta/\partial t$ satisfying the conditions $(i)\ \theta \to 0$ as $t \to \infty$ $(ii)\ \theta = 0$ when $x = \pm a$ for all values of $t > 0$ $(iii)\ \theta = x$ when $t = 0$ and $-a < x < a$ is $\theta = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin \frac{n\pi x}{a} \exp\left(-\frac{n^2\pi^2 t}{a^2}\right)$, where $\exp a = e^a$. **[Meerut 2005, 2011]**

12. A rod of length *l* with insulated sides is initially at a uniform temperature U_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Prove that the temperature function $u(x, t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/l) e^{-(c^2\pi^2 n^2 t)/l^2}, \text{ where } b_n \text{ is given by } U_0 = \sum_{n=1}^{\infty} b_n \sin(n\pi x/l)$$

[Andhra 2003; Agra 2004; Kanpur 1997; Aligarh 2003]

Hint. For the required solution, take $\partial u/\partial t = c^2(\partial^2 u/\partial x^2)$ as heat equation

PART II: PROBLEMS BASED ON TWO-DIMENSIONAL HEAT (OR DIFFUSION) EQUATION

2.7 A. General solution of the two dimensional heat (or diffusion) equation

(Delhi Maths (H) 1995; Nagpur 1995, 97; Pune 1998; Rajasthan 2001, 03)

Sol. Two-dimensional heat (or diffusion) equation is

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/k) \times (\partial u/\partial t) \quad \ldots (1)$$

Suppose that (1) has solutions of the form $u(x, y, t) = X(x)\ Y(y)\ T(t)$. ... (2)

Substituting this value of *u* in (1), we have

$$X''\,YT + XY''T = \frac{1}{k} XYT' \qquad \text{or} \qquad \frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{k}\frac{T'}{T} \quad \ldots (3)$$

Since *x*, *y* and *t* are independent variables, (3) is true if each term on each side is a constant, such that

$$\frac{X''}{X} = -n^2, \qquad \frac{Y''}{Y} = -m^2 \qquad \text{and} \qquad \frac{T'}{kT} = -p^2 \quad \ldots (4)$$

with $n^2 + m^2 = p^2$ ... (4)′

We have chosen constants in (4) in such a way so that the solution *u* has the property that $u \to 0$ as $t \to \infty$. This is generally satisfied due to physical conditions of a real problem.

Solving differential equation given by (4), we have

$$X_n(x) = A_n \cos nx + B_n \sin nx, \qquad Y_m(y) = C_m \cos my + D_m \sin my$$

and $$T_p(t) = E_p e^{-p^2 kt} = F_{nm} e^{-(n^2+m^2)kt}, \text{ using } (4)'$$

Hence a suitable solution of (1) is of the form

$$\therefore \quad u_{nm}(x, y, t) = F_{nm} (A_n \cos nx + B_n \sin nx)\ (C_m \cos my + D_m \sin my) e^{-(n^2+m^2)kt}$$

2.7 B. General solution of two-dimensional heat (or diffusion) equation satisfying the given boundary and initial conditions

A thin rectangular plate whose surface is impervious to heat flow has at t = 0 an arbitrary distribution of temperature f(x, y). Its four edges x = 0, x = a, y = 0, y = b are kept at zero temperature. Determine the temperature at a point of the plate as t increases.

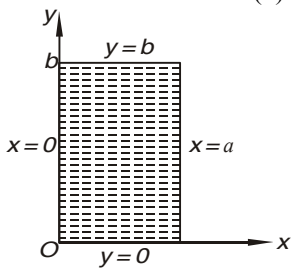
(Delhi Maths Hons. 2000, 01, 05; Meerut 2001, 03, 06)

Sol. Here the temperature $u(x, y, t)$ in the plate is governed by the two-dimensional heat equation

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = (1/k) \times (\partial u / \partial t) \qquad \ldots (1)$$

Since the edges $x = 0$, $x = a$, $y = 0$, $y = b$ are kept at zero temperature, so the boundary conditions are

$$u(0, y, t) = 0 \qquad \ldots 2(a)$$
$$u(a, y, t) = 0 \qquad \ldots 2(b)$$
$$u(x, 0, t) = 0 \qquad \ldots 2(c)$$
$$u(x, b, t) = 0 \qquad \ldots 2(d)$$



The initial condition is $u(x, y, 0) = f(x, y)$ ... (3)

Suppose that (1) has solutions of the form

$$u(x, y, t) = X(x)\, Y(y)\, T(t) \qquad \ldots (4)$$

Substituting this in (1), we have

$$X''YT + XY''T = \frac{1}{k} XYT \qquad \text{or} \qquad \frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{kT} \qquad \ldots (5)$$

where the dashes denote derivatives with respect to relevant variable. Since x, y and t are independent variables, (5) can only be true if each term on each side is equal to a constant.

Let $X''/X = \mu_1$ so that $X'' - \mu_1 X = 0$ ... (6)

Using 2(a) and 2(b), (4) gives

$X(0)\, Y(y)\, T(t) = 0$ and $X(a)\, Y(y)\, T(t) = 0$ ... (7)

Since $Y \equiv 0$ or $T \equiv 0$ leads to $u \equiv 0$, so we suppose that $Y \neq 0$ and $T \neq 0$. Then (7) gives

$X(0) = 0$ and $X(a) = 0$ ... (8)

We now solve (6) satisfiying boundary conditions (8). Three cases *arise*:

Case I. Let $\mu_1 = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (9)

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ and hence $X(x) \equiv 0$. This lead to $u \equiv 0$, which does not satisfy (3). So we reject $\mu_1 = 0$.

Case II. Let $\mu_1 = \lambda_1^2, \lambda_1 \neq 0$. Then solution (6) is $X(x) = Ae^{\lambda_1 x} + Be^{-\lambda_1 x}$ ... (10)

Using B.C. (8), (10) gives

$0 = A + B$ and $0 = Ae^{a\lambda_1} + Be^{-a\lambda_1}$ ... (11)

Solving (11), $A = B = 0$ so that $X(x) \equiv 0$. This lead to $u \equiv 0$, which does not satisfy (3). So we reject $\mu_1 = \lambda_1^2$.

Case III. Let $\mu_1 = -\lambda_1^2, \lambda_1 \neq 0.$ Then solution (6) is $X(x) = A\cos\lambda_1 x + B\sin\lambda_1 x.$ ... (12)

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda_1 a + B\sin\lambda_1 a.$

Hence $A = 0$ and $\sin\lambda_1 a = 0,$

where we have taken $B \neq 0,$ since otherwise $X \equiv 0$ so that $u \equiv 0$ which does not satisfy (3).

Now, $\sin\lambda_1 a = 0$ gives $\lambda_1 a = m\pi,$ $m = 1, 2, 3, ...$

$\therefore$ $\lambda_1 = m\pi/a,$ $m = 1, 2, 3. ...$ ... (13)

Hence non-zero solutions $X_m(x)$ of (6) are given by

$$X_m(x) = B_m \sin(m\pi x/a), \quad m = 1, 2, 3. \quad ... (14)$$

Next, let $Y''/Y = \mu_2$ so that $Y'' - \mu_2 Y = 0.$ ... (15)

Using 2(c) and 2(d), (4) gives as before $Y(0) = 0$ and $Y(b) = 0.$ ... (16)

Solving (15) under B.C. (16) as before, we get

$$Y_n(y) = D_n \sin(n\pi y/b), \quad n = 1, 2, 3, ... \quad ... (17)$$

where $\mu_2 = -\lambda_2^2$ and $\lambda_2 = n\pi/b,$ $n = 1, 2, 3, ...$ ... (18)

In view of (6) and (15), (5) reduces to

$$T'/kT = \mu_1 + \mu_2 = -\lambda_1^2 - \lambda_2^2 = -\pi^2(m^2/a^2 + n^2/b^2)$$

or $T' = -\lambda_{mn}^2 T$ or $(1/T)\,dT = -\lambda_{mn}^2\,dt$ ... (19)

where $\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/b^2),$ $m = 1, 2, 3, ...; n = 1, 2, 3, ...$ ... (20)

Solving (19), $T_{mn}(t) = E_{mn} e^{-\lambda_{mn}^2 t},$ $m = 1, 2, 3, ...,$ $n = 1, 2, 3, ...,$

$\therefore$ $$u_{mn}(x, y, t) = X_m(x) Y_n(y) T_{mn}(t) = F_{mn} \sin\frac{m\pi\pi}{a} \sin\frac{n\pi y}{b} e^{-\lambda_{mn}^2 t}$$

are solutions, of (1), satisfying 2(a), 2(b), 2(c) and 2 (d). Here F_{mn} (= $B_m D_n$) is another arbitrary constant. In order to obtain a soluton also satisfying (3), we consider more general solution

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(x, y, t) \quad i.e., \quad u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-\lambda_{mn}^2 t}. \quad ... (21)$$

Putting $t = 0$ in (21) and using (3), we get

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad ... (22)$$

which is double Fourier sine series. Accordingly, we get

$$F_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,dx\,dy \quad ... (23)$$

Hence (21) is the required solution where F_{mn} is given by (23).

2.7 C. Working rule for solving two-dimesnional heat or diffusion equation

Step I : Proceed as in Art. 2.7 B and show that solution of

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/k) \times (\partial u/\partial t) \quad ... (i)$$

subject to the boundary conditions

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \quad ... (ii)$$

and the initial condition $u(x, y, 0) = f(x, y)$ ... (*iii*)

is given by $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} e^{-\lambda_{mn}^2 t} \quad \text{... (iv)}$$

where $$F_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \quad \text{... (v)}$$

and $$\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/b^2), \qquad m = 1, 2, 3, \ldots, \qquad n = 1, 2, 3, \quad \text{... (vi)}$$

Step II. Compare the given problem with (*i*) (*ii*) and (*iii*) and find particular values of k, a, b and $f(x, y)$.

Step III. Substitute the particular values of a, b and $f(x,y)$ in (*v*) and (*vi*) to compute F_{mn} and λ_{mn}^2.

Step IV. Substitute the values of F_{mn} and λ_{mn}^2 in (*iv*) to arrive at the desired solution of the given boundary value problem.

2.7 D Solved example based on Art. 2.7 A, Art. 2.7B and Art 2.7C

Ex. 1. *The four edges of a thin square plate of area* π^2 *are kept at temperature zero and the faces are perfectly insulated. The initial temperature is assumed to be* $u\,(x, y, 0) = xy(\pi - x)\,(\pi - y)$. *By applying the method of separating variables to the two diminsional heat equation* $u_t = c^2\nabla^2 u$, *determine the temperature* $u(x, y, t)$ *in the plate, where* $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$

Sol. As is Art. 2.7 B, first prove that the solution of two dimensional heat equation

$$k\nabla^2 u = u_t, \qquad i.e., \qquad k(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) = \partial u/\partial t \quad \text{... (i)}$$

subject to the boundary conditions $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$ ... (*ii*)

and the initial condition $u(x, y, 0) = f(x, y)$ ... (*iii*)

is given by $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} e^{-\lambda_{mn}^2 t}, \quad \text{... (iv)}$$

where $$F_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \quad \text{... (v)}$$

and $$\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/b^2), \qquad m = 1, 2, 3, \ldots; \qquad n = 1, 2, 3 \ldots \quad \text{... (vi)}$$

Comparing the given boundary value problem with the boundary value problem given by (*i*), (*ii*) and (*iii*), we have $k = c^2$, $a = b = \pi$ and $f(x, y) = xy\,(\pi - x)\,(\pi - y)$.

$\therefore$ From (*vi*), $$\lambda_{mn}^2 = \pi^2 c^2(m^2/\pi^2 + n^2/\pi^2) = c^2(m^2 + n^2) \quad \text{... (vii)}$$

$\therefore$ (*iv*) gives $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin mx \sin xy\, e^{-c^2(m^2+n^2)t} \quad \text{... (viii)}$$

From (*v*), $$F_{mn} = \frac{4}{\pi^2}\int_{x=0}^{\pi}\int_{y=0}^{\pi} xy(\pi - x)\,(\pi - y)\sin mx \sin ny\, dx\, dy$$

$$= \frac{4}{\pi^2}\left[\int_0^{\pi}(\pi x - x^2)\sin mx\, dx\right]\times\left[\int_0^{\pi}(\pi y - y^2)\sin ny\, dy\right] \quad \text{... (ix)}$$

Using the chain rule of integrating by parts, we get

$$\int_0^{\pi}(\pi x - x^2)\sin mx\,dx = \left[(\pi x - x^2)\left(-\frac{\cos mx}{m}\right) - (\pi - 2x)\left(-\frac{\sin mx}{m^2}\right) + (-2)\left(\frac{\cos mx}{m^3}\right)\right]_0^{\pi} = \frac{2}{m^3}[1-(-1)^m].$$

Similarly, $$\int_0^{\pi}(\pi y - y^2)\sin ny\,dy = \frac{2}{n^3}[1-(-1)^n].$$

$\therefore$ $$F_{mn} = \frac{16}{\pi^2 m^3 n^3}[1-(-1)^m][-1-(-1)^n], \text{ using } (ix)$$

or $$F_{mn} = \begin{cases} 0, \text{ when } \quad m = 2p \quad \text{or} \quad n = 2q \\ \dfrac{64}{\pi^2(2p-1)^3(2q-1)^3}, \quad \text{when } m = 2p-1,\ n = 2q-1 \end{cases} \qquad \ldots (x)$$

Using (*viii*) and (*x*), the desired solution is given by

$$u(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} F_{pq}\sin[(2p-1)x]\sin[(2q-1)y]e^{-c^2\{(2p-1)^2+(2q-1)^2\}t}$$

where $$F_{pq} = 64/\pi^2(2p-1)^3(2q-1)^3.$$

Ex. 2. *Find temperature distribution inside a square plate of side a having boundary condition u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, a, t) = 0 and initial condition*

$$u(x,y,0) = \cos\{\pi(x-y)/a\} - \cos\{\pi(x+y)/a\}$$ **[Delhi B.Sc. (Hons.) Physics 1998]**

Sol. As in Art. 2.7 B, first prove that the solution of two-dimensional heat equation

$$k(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) = \partial u/\partial t \qquad \ldots (i)$$

subject to the boundary conditions $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$ ... (*ii*)

and the initial condition $$u(x, y, a) = f(x, y) \qquad \ldots (iii)$$

is given by $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\lambda_{mn}^2 t} \qquad \ldots (iv)$$

where $$F_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \qquad \ldots (v)$$

and $$\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/b^2), \quad m = 1, 2, 3, ..., \quad n = 1, 2, 3, ... \qquad \ldots (vi)$$

Comparing the given boundary value problem with the boundary value problm given by (*i*), (*ii*) and (*iii*), we have $b = a$ and $f(x,y) = \cos\{\pi(x-y)a\} - \cos\{\pi(x+y)/a\}$

$\therefore$ From (*vi*), $$\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/a^2) = \{\pi^2 k(m^2+n^2)\}/a^2 \qquad \ldots (vii)$$

and from (*iv*), $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}e^{-\lambda_{mn}^2 t} \qquad \ldots (viii)$$

From (*v*), $$F_{mn} = \frac{4}{a^2}\int_{x=a}^{a}\int_{y=0}^{a}\left\{\cos\frac{\pi(x-y)}{a} - \cos\frac{\pi(x+y)}{a}\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,dx dy$$

$$= \frac{4}{a^2}\int_{x=0}^{a}\int_{y=0}^{a}\left(2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,dx\,dy \qquad \ldots (ix)$$

or $$F_{mn} = \frac{2}{a^2}\left(\int_{x=0}^{a} 2\sin\frac{\pi x}{a}\sin\frac{m\pi x}{a}\,dx\right)\times\left(\int_{y=0}^{a} 2\sin\frac{\pi y}{a}\sin\frac{n\pi y}{a}\,dy\right)$$

$$= \frac{2}{a^2}\left(\int_{x=0}^{a}\left\{\cos\frac{(m-1)\pi x}{a} - \cos\frac{(m+1)\pi x}{a}\right\}dx\right)\times\left(\int_{y=0}^{a}\left\{\cos\frac{(n-1)\pi y}{a} - \frac{\cos(n+1)\pi y}{a}\right\}dy\right)$$

$$F_{mn} = \frac{2}{a^2}\times\left[\frac{\sin\{(m-1)\pi x/a\}}{(m-1)\pi/a} - \frac{\sin\{(m+1)\pi x/a\}}{(m+1)\pi/a}\right]_0^a \times\left[\frac{\sin\{(n-1)y\pi/a\}}{(n-1)\pi/a} - \frac{\sin\{(n+1)y\pi/a\}}{(n+1)\pi/a}\right]_0^a \ldots(x)$$

When $m \neq 1$ and $n \neq 1$, then $(x) \Rightarrow F_{mn} = 0$

When $m = 1$ and $n = 1$, (*ix*) reduces to

$$F_{11} = \frac{8}{a^2}\int_{x=0}^{a}\int_{y=0}^{a}\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a} = \frac{2}{a^2}\left(\int_{x=0}^{a} 2\sin^2\frac{\pi x}{a}\,dx\right)\times\left(\int_{y=0}^{a} 2\sin^2\frac{\pi y}{a}\,dy\right)$$

$$= \frac{2}{a^2}\left(\int_{x=0}^{a}\left(1-\cos\frac{2\pi x}{a}\right)dx\right)\times\left(\int_{y=0}^{a}\left(1-\cos\frac{2\pi y}{a}\right)dy\right) = \frac{2}{a^2}\left[x - \frac{a}{2\pi}\sin\frac{2\pi x}{a}\right]_0^a \times\left[y - \frac{a}{2\pi}\sin\frac{2\pi y}{a}\right]_0^a$$

Hence, $F_{11} = (2/a^2)\times a \times a = 2$

Thus, $$F_{mn} = \begin{cases} 2, \text{ if } m = 1, n = 1 \\ 0, \text{ otherwise} \end{cases} \quad \ldots (xi)$$

$\therefore$ from (*vii*), $$\lambda_{11}^2 = \pi^2 k(1/a^2 + 1/a^2) = (2\pi^2 k)/a^2 \quad \ldots (xii)$$

Using (*xi*) and (*xii*), (*viii*) yields the required solution

$$u(x,y,t) = F_{11}\sin(\pi x/a)\sin(\pi y/a)\,e^{-\lambda_{11}^2 t} \quad \textit{i.e.,} \quad u(x,y,t) = 2\sin(\pi x/a)\sin(\pi y/a)e^{-(2\pi^2 kt)/a^2}$$

Ex. 3. *Show that the two dimensional diffusion equation* $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = (1/k)\,(\partial\theta/\partial t)$

can be put in the form, $$\theta(x,y,t) = \sum_{\lambda}\sum_{\mu} A_{\lambda\mu}\cos(\lambda x + \in_{\lambda})\cos(\mu y + \in_{\mu})e^{-(\lambda^2+u^2)kt}$$

[Delhi Maths (Hons) 1994]

Sol. Suppose the given equation has solutions of the form

$$\theta(x,y,t) = X(x)Y(y)T(t) \quad \ldots (1)$$

Substitution this value of θ in the given equation, we get

$$X''YT + XY''T = (1/k)\times XYT' \quad \text{or} \quad X''/X + Y''/Y = (1/k)\times(T'/T) \quad \ldots (2)$$

Since x, y and t are independent variables, (2) is true if each term on each side is a constant, such that

$$X''/X = -\lambda^2, \qquad Y''/Y = -\mu^2 \qquad \text{and} \qquad T'/kT = -\nu^2, \quad \ldots (3)$$

where $$\nu^2 = \lambda^2 + \mu^2 \quad \ldots (4)$$

Solutions of equations in (3) can be put in the form

$$X(x) = \cos(\lambda x + \in_{\lambda}), \qquad Y(y) = \cos(\mu y + \in_{\mu}), \qquad T(t) = A_{\lambda\mu}e^{-(\lambda^2+\mu^2)kt}$$

$\therefore$ By the principle of superposition, the required general solution is

$$\theta(x,y,t) = \sum_{\lambda}\sum_{\mu} A_{\lambda\mu}\cos(\lambda x + \in_{\lambda})\cos(\mu y + \in_{\mu})e^{-(\lambda^2+\mu^2)kt}$$

EXERCISE 2(B)

Ex. 1. Solve the two dimensional diffusion equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/k)\,(\partial u/\partial t)$, in the region $0 < x < a$, $0 < y < b$, $t > 0$ with the following boundary and initial conditions: $u(x, y, t) = 0$ and where C is the boundary of the rectangle defined by $0 \le x \le a$ and $0 \le y \le b$ and $u(x, y, 0) = f(x, y)$, $0 < x < a$, $0 < y < b$.

Sol. This is exactly the same as Art. 2.7B

Ex. 2. A rectangular plate bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$ has an initial distribution given by $V = A\sin(\pi x/a)\sin(\pi y/b)$. The edges are kept at zero temperature and the plane faces are impervious to heat. Find the temperature at any point.

[Meerut 2000, 08, 11, 12; Kurukshetra 2004]

Ans. $u(x, y, t) = A\sin(\pi x/a)\sin(\pi y/a)e^{-\pi^2 kt(1/a^2 + 1/b^2)}$

Ex. 3. Solve $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = \partial V/\partial t$, if $V = 0$ when $t = \infty$, $x = 0$ or l and $y = 0$ or l.

Ex. 4. A square plate with sides of unit length has its faces insulated and its sides kept at 0°C. If the initial temperature is specified, determine the subsequent temperature at any point of the plate.

PART III : PROBLEMS BASED ON THREE-DIMENSIONAL HEAT (OR DIFFUSION) EQUATION

2.8 A. Solution of three-dimensional heat (diffusion) equation. [Kanpur 2000]

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/k)\,(\partial u/\partial t) \quad \ldots (1)$$

Suppose that (1) has solutions of the form

$$u(x, y, z, t) = X(x)\, y(y)\, Z(z)\, T(t) \quad \ldots (2)$$

where X, Y, Z and T are respectively the functions of x, y, z and t alone. Substituting this value of u in (1), we have $X''YZT + XY''ZT + XYZ''T = (1/k)\,XYZT'$

or $$X''/X + Y''/Y + Z''/Z = (1/k)\,(T'/T) \quad \ldots (3)$$

Since x, y, z and t are independent variables, (3) is true only if each term on each side is a constant such that

$$X''/X = -n^2, \quad Y''/Y = -m^2, \quad Z''/Z \equiv -l^2 \quad \text{and} \quad T'/kT = -p^2 \quad \ldots (4)$$

where $$n^2 + m^2 + l^2 = p^2 \quad \ldots (5)$$

We have chosen constants in (4) in such a manner so that the solution $u(x, y, z, t)$ has the property that $u \to 0$ and $t \to \infty$. This is generally satisfied due to physical conditions of actual physical problem. Solving equations in (4), we have

$$X_n(x) = A_n \cos nx + B_n \sin nx, \qquad Y_m(y) = C_m \cos my + D_m \sin my.$$

$$Z_l(z) = E_l \cos lz + F_l \sin lz, \quad \text{and} \quad T_p(t) = G_p e^{-p^2 kt} = H_{nml} e^{-(n^2+m^2+l^2)kt}$$

$\therefore\ u_{nml}(x, y, z, t) = H_{nml}(A_n \cos nx + B_n \sin nx)(C_m \cos my + D_m \sin my)(E_l \cos lz + F_l \sin lz)e^{-(n^2+m^2+l^2)kt}$ are solutions of (1). Hence the general solution of (1) is given by

$$u(x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} u_{nml}(x, y, z, t).$$

2.8 B. Solution of three-dimensional heat (or diffusion) equation satisfying the given boundary and initial conditions.

Find the solution of the three dimensional diffusion equation in the region $0 < x < a$, $0 < y < b$, $0 < z < c$, $t > 0$, $\partial^2 u/\partial u^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/k)(\partial u/\partial t)$ *with the boundary and initial conditions:* $u(0, y, z, t) = 0 = u(a, y, z, t)$; $u(x, 0, z, t) = 0 = u(x, b, z, t)$, $u(x, y, 0, t) = 0 = u(x, y, c, t)$ and $u(x, y, z, 0) = f(x, y, z)$.

Sol. Given
$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/k)(\partial u/\partial t) \quad \text{... (1)}$$
$$u(0, y, z, t) = 0 = u(a, y, z, t), \quad \text{for all } t > 0 \quad \text{... (2)}$$
$$u(x, 0, z, t) = 0 = u(x, b, z, t), \quad \text{for all } t > 0 \quad \text{... (3)}$$
$$u(x, y, 0, t) = 0 = u(x, y, c, t), \quad \text{for all } t > 0 \quad \text{... (4)}$$
and
$$u(x, y, z, 0) = f(x, y, z), \quad 0 < x < a, \quad 0 < y < b, \quad 0 < z < c \quad \text{... (5)}$$

Let a solution of (1) be of the form $u(x, y, z, t) = X(x)\, Y(y)\, Z(z)\, T(t)$ ... (6)

Substituting this in (1), we have $X''/X + Y''/Y + Z''/Z = (1/k)(T'/T)$ ... (7)

Since x, y, z and t are independent variables, (7) is true if each term on each side is a negative constant. [We have chosen negative constant because non-negative constant will give rise to trivial solution of (1)]. Thus, we have

$$X''/X = -\lambda_1^2, \quad Y''/Y = -\lambda_2^2, \quad Z''/Z = -\lambda_3^2 \quad \text{and} \quad T'/kT = -\lambda^2, \quad \text{... (8)}$$

where
$$\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \text{... (9)}$$

Solutions of (8) are given by
$$X(x) = A\cos\lambda_1 x + B\sin\lambda_1 x \quad \text{... (10)}$$
$$Y(y) = C\cos\lambda_2 y + D\sin\lambda_2 y \quad \text{... (11)}$$
$$Z(z) = E\cos\lambda_3 z + \sin\lambda_3 z \quad \text{... (12)}$$
and
$$T(t) = G\, e^{-\lambda^2 kt} \quad \text{... (13)}$$

Put $x = 0$ and $x = a$ by turn in (6). Using (2) and the fact that $Y(y) \neq 0$, $X(z) \neq 0$, $T(t) \neq 0$ for non-trivial solution of (1), we get

$X(0) = 0$ and $X(a) = 0$. ... (14)

Putting $x = 0$ and $x = a$ by turn in (10) and using (14), we get

$0 = A$ and $0 = A\cos\lambda_1 a + B\sin\lambda_1 a$

$\Rightarrow$ $A = 0$ and $\sin\lambda_1 a = 0$,

where we have taken $B \neq 0$, since otherwise $X(x) = 0$ and hence we have trivial solution $u = 0$ for (1)

Now $\sin\lambda_1 a = 0 \Rightarrow \lambda_1 a = l\pi \Rightarrow \lambda_1 = l\pi/a$, $l = 1, 2, 3, \ldots$ ... (15)

and then from (10), $X_l(x) = B_l \sin(l\pi x/a)$, $n = 1, 2, 3, \ldots$ ... (16)

Similarly, boundary condition (3) and (11) give

$$\lambda_2 = m\pi/b, \quad Y_m(y) = D_m \sin(m\pi y/b), \quad m = 1, 2, 3, \ldots \quad \text{... (17)}$$

and finally, boundary condition (4) and (12) give

$$\lambda_3 = n\pi/c, \quad Z_n(z) = F_n \sin(n\pi z/c), \quad n = 1, 2, 3, \ldots. \quad \text{... (18)}$$

Using (15), (17) and (18), (9) reduces to

$$\lambda^2 = \pi^2 (l^2/a^2 + m^2/b^2 + n^2/c^2) \quad \text{... (19)}$$

Using (6), (13), (16), (17) and (18), a solution of (1) is given by

$$H_{lmn} \sin(l\pi x/a) \sin(m\pi y/b) \sin(n\pi z/c) e^{-\lambda^2 kt},$$

where H_{lmn} (= $B_l D_m F_n G$) is another arbitrary constant Now, using the principle of superposition, the general solution of (1) is

$$u(x,y,z,t) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} H_{lmn} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} e^{-\lambda^2 kt} \quad \text{... (20)}$$

where λ is given by (19),

Putting $t = 0$ in (20) and using (5), we have

$$f(x,y,z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} H_{lmn} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} \quad \text{... (21)}$$

which is Fourier sine series for three variables x, y, z and so

$$H_{lmn} = \frac{8}{abc} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} f(x,y,z) \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} \quad \text{... (22)}$$

The required solution of (1) is given by (21) and (22).

Exercise. *The faces of a solid parallelopiped* $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$ *are kept at zero temperature. If, initially the temperature of the solid is given by* $u(x,y,z,0) = f(x,y,z)$

Show that $u(x,y,z,t) = \frac{8}{abc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} F(m,n,q)\ e^{-\lambda^2 kt} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{q\pi z}{c}$, *where*

$$F(m,n,q) = \int_0^a \int_0^b \int_0^c f(x,y,z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{q\pi z}{c}, \quad \text{where} \quad \lambda^2 = \pi^2 (m^2/a^2 + n^2/b^2 + q^2/c^2),$$

[Meerut 2005, 10]

Hint. This is same as Art. 2.8B. Here $l = m$, $m = n$, $n = q$ and $H_{lmn} = F(l,m,n)$. With these changes, we get the desired solution.

PART IV: PROBLEMS BASED ON ONE-DIMENSIONAL WAVE EQUATION

2.9 A. General solution of one-dimensional wave equation $\partial^2 y/\partial x^2 = (1/c^2) \times (\partial^2 y/\partial t^2)$

[Delhi Maths (H) 1997: Meerut 1997, 98, 2006; Rajasthan 1996, 2001; Ravishankar 1995; U.P. (Tech). Lucknow 2003]

Sol. Given $\partial^2 y/\partial x^2 = (1/c^2)\ (\partial^2 y/\partial t^2)$ ... (1)

Let solution of (1) be of the from $y(x, t) = X(x)\ T(t)$ ... (2)

From (1) and (2), $X'' T = (1/c^2) \times X T''$ or $X''/X = T''/c^2 T$

Since x and t are independent variables, hence the above equation can only be true if each side is equal to the same constant, say μ. Thus, we obtain $X'' - \mu X = 0$ ... (3)

and $T'' - c^2 \mu T = 0.$ ... (4)

Now, we solve (3) and (4). Three cases arise:

(*i*) When $\mu = 0$. Then $X = a_1 x + a_2$, $T = a_3 t + a_4$

(*ii*) When μ is positive. Let $\mu = \lambda^2$ (say). Then $X = b_1 e^{\lambda x} + b_2 e^{-\lambda}$, $T = b_3\, e^{c\lambda t} + b_4 e^{-c\lambda t}$

(*iii*) When μ is negative Let$\mu = -\lambda^2$ (say). Then $X = c_1 \cos\lambda x + c_2 \sin\lambda x$, $T = c_3 \cos c\lambda t + c_4 \sin cpt$.

Thus the various possible solutions are :

$$y(x, t) = (a_1 x + a_2)\,(a_3 t + a_4) \quad \ldots (5)$$

$$y(x,t) = (b_1 e^{\lambda x} + b_2 e^{-\lambda x})\,(b_3 c^{c\lambda t} + b_4 e^{-c\lambda t}) \quad \ldots (6)$$

and $$y(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)\,(c_3 \cos c\lambda t + c_4 \sin c\lambda t). \quad \ldots (7)$$

Now we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, $y(x, t)$ must be a periodic function of t. Hence $y(x, t)$ must involve trigonometric terms. Accordingly, the solution given by (7) is the only suitable solution.

2.9 B. Examples based on Art. 2.9A

Ex. 1. *Solve one dimensional wave equation* $\partial^2 y / \partial x^2 = (1/c^2)(\partial^2 y / \partial t^2)$. *Deduce the expression for y satisfying the boundary conditions* $y(0, t) = 0 = y(a, t)$.

[Delhi Maths (Hons) 2000, 07]

Sol. For the first part refer Art. 2.9A.

Deduction. First of all proceed as in Art. 2.9A to get three possible solution (5), (6) and (7), Now, proceed as follows.

Given boundary conditions are : $y(0, t) = y(a, t) = 0$ for all t ... (8)

In view of B.C (8), the first solution (5) gives

$0 = a_2(a_3 t + a_4)$ and $0 = (a_1\, a + a_2)\,(a_3 t + a_4)$ for all t

$\Rightarrow$ $a_2 = 0$ and $a_1 a + a_2 = 0$ $\Rightarrow$ $a_1 = a_2 = 0$

Hence $y(x, t) = 0$ for all t. This is a trivial solution. Since we are looking for a non-trivial solution, we reject solution (5).

Again, in view of B.C. (8), the second solution (6) gives

$0 = (b_1 + b_2)(b_3 e^{c\lambda t} + b_4 e^{-c\lambda t})$ and $0 = (b_1 e^{\lambda a} + b_2 e^{-\lambda a})\,(b_3 e^{c\lambda t} + b_4 e^{-c\lambda t})$

$\Rightarrow$ $b_1 + b_2 = 0$ and $b_1 e^{\lambda a} + b_2 e^{-\lambda a} = 0$, since above relations held for all t.

Thus, we get $b_1 = b_2 = 0$ and hence $y(x, t) = 0$ for all t. This is a trivial solution and so we reject solution (6) also.

Finally, in view of B.C. (8), solution (7) gives

$0 = c_1(c_3 \cos c\lambda t + c_4 \sin c\lambda t)$ and $0 = (c_1 \cos\lambda a + c_2 \sin\lambda a)\,(c_3 \cos c\lambda t + c_4 \sin c\lambda t)$

$\Rightarrow$ $c_1 = 0$ and $c_2 \sin\lambda a = 0$, since the above relations hold for all t.

Now for a non-trivial solution of given wave equation, we cannot take $c_2 = 0$. (because c_1 is already zero and so $c_1 = c_2 = 0$ will lead to a trivial solution).

$\therefore$ $c_2 \sin\lambda a = 0$ $\Rightarrow$ $\sin\lambda a = 0$ $\Rightarrow$ $\lambda a = n\pi,\ n = 1, 2, 3, \ldots$

Thus, $\lambda = n\pi / a,\ n = 1, 2, 3, \ldots.$

Hence from (7), for each n, a solution $y_n(x, t)$ is given by

$$y_n(x,t) = c_2 \sin\frac{n\pi x}{a}\left(c_3 \cos\frac{n\pi ct}{a} + c_4 \sin\frac{n\pi ct}{a}\right)$$

or $$y_n(x,t) = \sin\frac{n\pi x}{a}\left(E_n \cos\frac{n\pi ct}{a} + F_n \sin\frac{n\pi ct}{a}\right),\ n = 1, 2, 3, ...$$

where E_n (= $c_2\ c_3$) and F_n (= $c_2\ c_4$) are new arbitrary constants.

Since the given wave equation is linear, its most general solution is obtained by applying the principle of superposition. Thus, the required solution is

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\left(E_n \cos\frac{n\pi ct}{a} + F_n \sin\frac{n\pi ct}{a}\right)$$

2.10 A. General solution of one-dimensional wave equation satisfying the given boundary and initial conditions. [Himanchal 2007, 208; Meerut 2006, 10]

Consider one-dimensional wave equation

$$\partial^2 y / \partial^2 x^2 = (1/c^2) \times (\partial^2 y / \partial t^2), \quad \ldots (1)$$

where $y(x, t)$ is the deflection of the string. To find how the string moves, we solve (1); more precisely, we obtain a solution of (1) that also satisfies the conditions imposed by the physical system. If the ends of the string are fixed at $x = 0$ and $x = a$, we have the two boundary conditions.

$$y(0, t) = 0 \quad \text{and} \quad y(a, t) = 0 \quad \text{for all } t \quad \ldots (2)$$

The form of the motion of the string will depend on the initial deflection (deflection at $t = 0$) and on the initial velocity (velocity at $t = 0$). Denoting the initial deflection by $f(x)$ and the initial velocity by $g(x)$, we arrive at two initial conditions

$$y(x, 0) = f(x),\ 0 \le x \le a \quad \ldots 3\ (a)$$

and $$(\partial y / \partial t)_{t=0} = g(x) \quad i.e., \quad y_t(x, 0) = g(x), \quad 0 \le x \le a \quad \ldots 3\ (b)$$

Our problem is now to find a solution of (1) satisfying the conditions (2), 3(a) and 3(b)

Suppose that (1) has solution of the form $y(x, t) = X(x)\ T(t)$ ... (4)

Substituting this value of $y\ (x, t)$ in (1), we have

$$X''T = (1/c^2) \times X T'' \quad \text{or} \quad X''/X = T''/c^2 T \quad \ldots (5)$$

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ. Then (5) gives $X'' - \mu X = 0$ ... (6)

and $$T'' - \mu c^2 T = 0 \quad \ldots (7)$$

Using (2), (4) gives $X(0)\ T(t) = 0$ and $X(a)\ T(t) = 0$ ... (8)

Since $T(t) \equiv 0$ *leads to* $y \equiv 0$, so suppose that $T(t) \neq 0$.

Then (8) gives $X(0) = 0$ and $X(a) = 0$. ... (9)

We now solve (6) under B.C. (9). Three cases arise

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$. ... (10)

Using B.C. (9), (10) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ so that $X(x) \equiv 0$. This leads to $y \equiv 0$, which does not satisfy 3(a) and 3(b). So we reject $\mu = 0$.

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ... (11)

Using B.C. (9), (11) gives $0 = A + B$ and $0 = Ae^{a\lambda} + Be^{-a\lambda}$ ... (12)

Solving (12), $A = B = 0$ so that $X(x) \equiv 0$. This leads to $y \equiv 0$ which does not satisfy 3 (a) and 3 (b). So we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x.$ ... (13)

Using B.C. (9), (13) gives $0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$

so that $A = 0$ and $\sin\lambda a = 0,$

where we have taken $B \neq 0,$ since otherwise $X \equiv 0$ so that $y \equiv 0$ which does not satisfy 3 (*a*) and 3 (*b*).

Now $\sin\lambda a = 0$ gives $\lambda a = n\pi$ and so $\lambda = n\pi/a,\ n = 1, 2, 3, ...$ ... (14)

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x/a)$ ... (15)

Using (14), (7) reduces to

$$T'' + (n^2\pi^2c^2/a^2)T = 0, \quad \text{since} \quad \mu = -\lambda^2 = -(n^2\pi^2)/a^2$$

whose general solution is $T_n(t) = C_n\cos\frac{n\pi ct}{a} + D_n\sin\frac{n\pi ct}{a}$ ... (16)

$$\therefore \quad y_n(x,t) = X_n(x)T_n(x) = \left(E_n\cos\frac{n\pi ct}{a} + F_n\sin\frac{n\pi ct}{a}\right)\sin\frac{n\pi x}{a} \quad \text{... (17)}$$

are solutions of (1) satisfying (2). Here E_n ($= C_nB_n$) and F_n ($= B_nD_n$) are new artitrary constants. In order to obtain a solution also satisfying 3 (*a*) and 3 (*b*), we consider more general solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t), \quad i.e., \quad y(x,t) = \sum_{n=1}^{\infty}\left(E_n\cos\frac{n\pi ct}{a} + F_n\sin\frac{n\pi ct}{a}\right)\sin\frac{n\pi x}{a} \quad \text{... (18)}$$

Differentiating (18) partially w.r.t '*t*', we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left(-\frac{n\pi c\,E_n}{a}\sin\frac{n\pi ct}{a} + \frac{n\pi c\,F_n}{a}\cos\frac{n\pi ct}{a}\right)\sin\frac{n\pi x}{a} \quad \text{... (19)}$$

Putting $t = 0$ in (18) and (19) and using 3 (*a*) and 3 (*b*), we get

$$f(x) = \sum_{n=1}^{\infty} E_n\sin\frac{n\pi x}{a} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty}\frac{n\pi cF_n}{a}\sin\frac{n\pi x}{a} \quad \text{... (20)}$$

which are Fourier sine series expansion for $f(x)$ and $g(x)$ respectively. Accordingly, we get

$$E_n = \frac{2}{a}\int_0^a f(x)\frac{n\pi x}{a}dx \quad \text{... (21)}$$

and $\frac{n\pi cF_n}{a} = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi x}{a}dx$ so that $F_n = \frac{2}{n\pi c}\int_0^a g(x)\sin\frac{n\pi x}{a}dx.$ ... (22)

Hence the required solution is given by (18) wherein E_n and F_n are given by (21) and (22) respectively.

Particular Case I. If initial velocity = $y_t(x, 0) = g(x) = 0$, then $F_n = 0$ from (22). Thus, in this case the solution (18) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} E_n\cos\frac{n\pi ct}{a}\sin\frac{n\pi x}{a}. \quad \text{... (23)}$$

Particular Case II. If initial displacement = $y(x, 0) = f(x) = 0$, then $E_n = 0$ by (21). Thus, in this case the solution (18) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} F_n\sin\frac{n\pi ct}{a}\sin\frac{n\pi x}{a} \quad \text{... (24)}$$

2.10 B. Working rule for solving one-dimensional wave equation when both the ends of the string of length *a* are fixed and initial deflection (or shape) and velocity are prescribed.

Step I. Proceed as in Art. 2.10 A and prove the solution of wave equation

$$\partial^2 y/\partial x^2 = (1/c^2) \times (\partial^2 y/\partial t^2) \quad \ldots (1)$$

subject to the boundary conditions $u(0, t) = u(a, t) = 0$ for all t ... (2)

and the initial conditions $y(x, 0) = f(x), \quad y_t(x, 0) = g(x), \quad 0 \le x \le a$ ... (3)

is given by $$y(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos\frac{n\pi ct}{a} + F_n \sin\frac{n\pi ct}{a} \right) \sin\frac{n\pi x}{a} \quad \ldots (4)$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$ and $$F_n = \frac{2}{n\pi c}\int_0^a g(x)\sin\frac{n\pi x}{a}dx \quad \ldots (5)$$

Step II. Compare the given problem with (1), (2) and (3) and find particular values of c, a, $f(x)$ and $g(x)$.

Step III. Substitute the particular values of a, $f(x)$ and $g(x)$ in (5) and compute E_n and F_n.

Step IV. Substitute the values of E_n and F_n in (4) to arrive at the desired solution of the given boundary value problem.

2.10 C. Working rule for solving one-dimensional wave equation when both the ends of the string of length *a* are fixed and the initial velocity of the string is zero *i.e.*, the string starts from the position of rest.

Step I. Proceeds as in Art. 2.10 upto particular case I and prove that the solution of wave equation $$\partial^2 y/\partial x^2 = (1/c^2) \times (\partial^2 y/\partial t^2) \quad \ldots (1)$$

subject to the boundary conditions $y(0, t) = y(a, t) = 0$ for all t ... (2)

the initial deflection $= y(x, 0) = f(x), \quad 0 \le x \le a$ ... 3 (*a*)

and initial velocity $= y_t(x, 0) = 0$ ... 3 (*b*)

is given by $$y(x,t) = \sum_{n=1}^{\infty} E_n \cos\frac{n\pi ct}{a} \sin\frac{n\pi x}{a}, \quad \ldots (4)$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \quad \ldots (5)$$

Step II. Compare the given boundary value problem with (1), (2), 3 (*a*) and 3 (*b*) and get particular values of c, a and $f(x)$. Use (5) to get E_n and then use (4) to get the required solution.

2.10 D. Working rule for solving one-dimensional wave equation when both the ends of the string of length *a* are fixed and the initial deflection of the string is zero.

Step I. Proceed as in Art. 2.10 A upto particular case II. Thus, prove that the solution of wave equation $$\partial^2 y/\partial x^2 = (1/c^2) \times (\partial^2 u/\partial t^2) \quad \ldots (1)$$

subject to the boundary conditions $y(0, t) = y(a, t) = 0$ for all t ... (2)

the initial deflection $= y(x, 0) = 0, \quad 0 \le x \le a$ ... 3(*a*)

and the initial velocity $= y_t(x, 0) = g(x), \quad 0 \le x \le a$ ... 3(*b*)

is given by $$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi ct}{a} \sin\frac{n\pi x}{a}, \quad \ldots (4)$$

where $$F_n = \frac{2}{x\pi c}\int_0^a g(x)\sin\frac{x\pi x}{a}dx \quad \ldots (5)$$

Step II. Compare the given boundary value problem with (1), (2), 3(*a*) and 3(*b*) and compute particular values of c, a and $g(x)$.

Step III. Substitute the values of a and $g(x)$ obtained in step II in (5) to compute F_n.

Step IV. Substitititute the value of F_n obtained in step III in (4) to get the required solution of the given boundary value problem.

An important note. Solution of one dimensional wave equation can be obtained by use any one of the following two methods:

Method I. Use working rule 2.10 B, 2.10 C or 2.10D as the case may be. But before using these working rules, all the results of that particular working rule must be proved in examination for complete solution.

Method II Proceed as explained in Art. 2.10A without using any of the working rules 2.10B, 2.10C or 2.10D.

2.10 E. Solved examples based on articles 2.10A, 2.10B, 2.10C and 2.10D.

Ex. 1. (*a*) *A string of length l has its ends x = 0 and x = l fixed. It is released from rest in the position* $y = \{4\lambda x(l-x)\}/l^2$. *Find an expression for the displacement of the string at any subsequent time.* **[A.M.I.E. 2002, Meerut 2005]**

(*b*) *A taut string of length l has its ends x* = 0 *and x* = l *fixed. The mid-point is taken to a small height h and released from rest at time t* = 0. *Find the displacement function y*(*x, t*).

(*c*) *A string is stretched between two fixed points at a distance l apart. Motion is started by displacing the string in the form* $y = y_0 \sin(\pi x/l)$ *from which it is released at time t* = 0. *Find the displacement at any point at a distance x from one end at time t.*

[Himachal 2007, 08; Kurukshetra 2004]

(*d*) *For the vibrating string of length* π *if initial shape is given by* $y(x, 0) = C \sin x$, *C being constant and string is released from rest, then find the displacement y* (*x*, *t*). **[Nagpur 2002, 06]**

(*e*) *A tightly stretched elastic string of length l, with fixed end points x* = 0 *and x = l is initially in the position of rest. Find the displacement y*(*x, t*).

[Kanpur 1995, Roorkee 1999, 2001]

(*f*) *A tightly stretched elastic string of length* π, *with fixed end points x* = 0 and $x = \pi$ *is initially in the position given by* $y = y_0 \sin^3 x$, y_0 *being constant. Find the displacement y*(*x*, *t*).

[Nagpur, 1996]

(*g*) *Solve the one dimensional wave equation* $\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$, $0 \le x \le 2\pi$, $t \ge 0$ *subject to the following initial and boundary conditions.* (*i*) $y(x, 0) = \sin^3 x$, $0 \le x \le 2\pi$ (*iii*) $(\partial y/\partial t)_{t=0} = 0$, $0 \le x \le 2\pi$ (*iii*) $y(0, t) = y(2\pi, t) = 0$, *for* $t \ge 0$.

(*h*) *Find the deflection y*(*x*, *t*) *of the vibrating string* (*length* = π, *and* $c^2 = 1$) *corresponding to zero initial velocity and initial deflection* $f(x) = k(\sin x - \sin 2x)$ **[Garhwal 2004]**

Sol. First of all, as explained in working rule of Art. 2.10 C, prove that the solution of one-dimensional wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2) \quad \ldots (i)$$

subject to boundary conditions

$$y(0, t) = y(l, 0) = 0 \text{ for all } t. \quad \ldots (ii)$$

and

$$\left.\begin{aligned}\text{initial velocity} &= (\partial y/\partial t)_{t=0} = 0,\ 0 < x < l\\ \text{initial deflection} &= y(x, 0) = f(x),\ 0 < x < l\end{aligned}\right\} \quad \ldots (iii)$$

is $$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \quad \text{... } (iv)$$

where $$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \quad \text{... } (v)$$

[**Note.** Before proceeding with solution of any part of Ex. 1, give the above matter in each part for complete solution]

(*a*) The displacement function $y(x, t)$ is the solution of the wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2) \quad \text{... (1)}$$

subject to boundary conditions : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ... (2)

and initial conditions, namely,

$$\text{Initial velocity} = (\partial y/\partial t)_{t=0} = 0 \quad \text{for} \quad 0 \leq x \leq l. \quad \text{... 3 } (a)$$

and $$\text{initial displacement} = y(x, 0) = f(x) = \{4\pi x\,(l - x)\}/l. \quad \text{... 3 } (b)$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*) and (*iii*), the solution of (1) satisfying the above boundary and initial conditions is given by

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \text{ by } (iv) \quad \text{... (4)}$$

where $$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \text{ by } (v) \quad \text{... (5)}$$

Substituting the value of $f(x)$ given by 3(*a*) in (5), we get

$$E_n = \left(\frac{2}{l}\right)\times\left(\frac{4\lambda}{l^2}\right)\int_0^l \left(lx - x^2\right)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{8\lambda}{l^3}\left[(lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

[using the chain rule of integrating by parts]

$$= \begin{cases} 0, \text{ if } n = 2m \text{ (even) with } m = 1, 2, 3, \ldots \\ (32\lambda)/(2m-1)^3\pi^3, \text{ if } n = 2m - 1 \text{ (odd) with } m = 1, 2, 3, \ldots \end{cases}$$

Substituting the value of E_n in (4), the required expression is

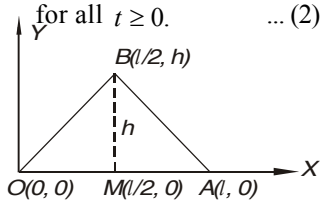
$$y(x, t) = \frac{32\lambda}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi ct}{l}$$

(*b*) The displacement function $y(x, t)$ is the solution of the wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)\times(\partial^2 y/\partial t^2) \quad \text{... (1)}$$

subject to the boundary conditions : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ... (2)

Initial position of the string at $t = 0$ is made up of two straight line segments *OB* and *BA* as shown in the figure and the string is released from rest.



The equation of *OB* is given by

$$y-0=\frac{h-0}{(l/2)-0}(x-0) \qquad \text{or} \qquad y=(2hx)/l, \qquad \text{for } 0\le x\le l/2.$$

The equation of BA is given by

$$y-0=\frac{h-0}{(l/2)-l}(x-l) \qquad \text{or} \qquad y=\{2h(l-x)\}/l, \quad \text{for } l/2\le x\le l.$$

Hence, the initial displacement is given by

$$y(x,0)=f(x)=\begin{cases}(2hx)/l, & 0\le x\le l/2\\ \{2h(l-x)\}/l, & l/2\le x\le l\end{cases} \qquad \ldots 3(a)$$

and
$$\text{the initial velocity}=(\partial y/\partial t)_{t=0}=0 \qquad \ldots 3(b)$$

Comparing the given problem given by (1), (2), 3(a) and 3(b) with the boundary value problem given by (i), (ii) and (iii), the solution of (1) satisfying the above boundary and initial conditions is given by

$$y(x,t)=\sum_{n=1}^{\infty}E_n\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \qquad \ldots (4)$$

where $E_n=\dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}dx=\frac{2}{l}\left[\int_0^{l/2}f(x)\sin\frac{n\pi x}{l}dx+\int_{l/2}^{l}f(x)\sin\frac{n\pi x}{l}dx\right]$

$$=\frac{2}{l}\left[\int_0^{l/2}\frac{2hx}{l}\sin\frac{n\pi x}{l}dx+\int_{l/2}^{l}\frac{2h(l-x)}{l}\sin\frac{n\pi x}{l}dx\right],\ \text{using } 3(a)$$

$$=\frac{4h}{l^2}\int_0^{l/2}x\sin\frac{n\pi x}{l}dx+\frac{4h}{l^2}\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{4h}{l^2}\left[(x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+\frac{4h}{l^2}\left[(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$=\frac{4h}{l^2}\left(-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)+\frac{4h}{l^2}\left(\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$=(8h/n^2\pi^2)\times\sin(n\pi/2)$$

$$=\begin{cases}\{8(-1)^{m+1}h\}/(2m-1)^2\pi^2, & \text{if } n=2m-1 \text{ (odd) and } m=1,2,3,\ldots\\ 0, & \text{if } n=2m \text{ (even) and } m=1,2,3,\ldots\end{cases}$$

[Note that for $n=2m-1$, $\sin(n\pi/2)=\sin\{(2m-1)\times(\pi/2)\}=\sin(m\pi-\pi/2)$
$=\sin m\pi\cos(\pi/2)-\cos m\pi\sin(\pi/2)=0-(-1)^m=(-1)^{m+1}$]

Substituting the above value of E_n in (4), the required displacement function is given by

$$y(x,t) = \frac{8h}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin\frac{(2m-1)\pi x}{l} \cos\frac{(2m-1)\pi ct}{l}$$

(*c*) The required displacement $y(x, t)$ of the string is the solution of the wave equation

$$\partial^2 y / \partial x^2 = (1/c^2)(\partial^2 y / \partial t^2) \quad \text{... (1)}$$

subject to the boundary conditions :

$$y(0, t) = y(l, t) = 0 \quad \text{for all} \quad t \geq 0 \quad \text{... (2)}$$

and the given initial conditions, namely :

$$\text{Initial velocity } = (\partial y / \partial t)_{t=0} = 0 \quad \text{for} \quad 0 \leq x \leq l. \quad \text{... 3}(a)$$

and $$\text{initial displacement } = y(x,0) = y_0 \sin(\pi x / l), \quad 0 \leq x \leq l. \quad \text{... 3}(b)$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*) and (*iii*), the solution of (1) satisfying the above boundary and initial conditons is given by

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}. \quad \text{... (4)}$$

Putting $t = 0$ in (4) and using 3(*b*), we have $$y_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

$\therefore$ $$E_1 \sin(\pi x/l) + E_2 \sin(2\pi x/l) + E_3 \sin(3\pi x/l) + \ldots = y_0 \sin(\pi x/l)$$

Comparing the coefficients of the like terms on both sides, we have

$E_1 = y_0$ and $E_n = 0$ for $n \neq 1$. With these values, (4) reduces to

$$y(x, t) = y_0 \sin(\pi x/l) \cos(\pi ct/l).$$

(*d*) Proceed as in part (*c*) taking $l = \pi$ and $y_0 = C$. **Ans.** $y(x, t) = C \sin x \cos(ct)$.

(*e*) The required displaceent $y(x, t)$ of the string is the solution of the wave equation

$$\partial^2 y / \partial x^2 = (1/c^2)(\partial^2 y / \partial t^2) \quad \text{... (1)}$$

subject to the boundary conditions :

$$y(0, t) = y(l, t) = 0, \quad \text{for all} \quad t \geq 0 \quad \text{... (2)}$$

and the given initial conditions, namely :

$$\text{Initial velocity } = (\partial y / \partial t)_{t=0} = 0, \quad \text{for} \quad 0 \leq x \leq l. \quad \text{... 3}(a)$$

and $$\text{initial displacement } = y(x,0) = y_0 \sin^3(\pi x / l), \; 0 \leq x \leq l. \quad \text{... 3}(b)$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*) and (*iii*), the solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}. \quad \text{... (4)}$$

Putting $t = 0$ in (4) and using 3(*b*), we have $$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

or $$E_1 \sin(\pi x/l) + E_2 \sin(2\pi x/l) + E_3 \sin(3\pi x/l) + E_4 \sin(4\pi x/l) + \ldots$$

$$= (y_0/4) \times \{3\sin(\pi x/l) - \sin(3\pi x/l)\}$$

$$\left[\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta \;\Rightarrow\; \sin^3\theta = (1/4) \times (3\sin\theta - \sin 3\theta)\right]$$

Comparing the coefficients of the like terms on both sides, we have

$E_1 = (3y_0)/4$, $E_2 = 0$, $E_3 = -(y_0/4)$, $E_n = 0$ for all $n \geq 4$.

Substituting these values in (4), the required displacement is given by

$y(x, t) = (3y_0/4) \times \sin(\pi x/l) \cos(\pi ct/l) - (y_1/4) \times \sin(3\pi x/l) \cos(3\pi ct/l)$.

(*f*) The required displacement $y(x, t)$ of the string is the solution of the wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2) \quad \text{... (1)}$$

subject to the boundary conditions :

$$y(0, t) = y(\pi, t) = 0 \quad , \quad \text{for all } t \geq 0 \quad \text{... (2)}$$

and the given initial conditions, namely :

$$\text{Initial velocity} = (\partial y/\partial t)_{t=0} = 0 \ , \qquad \text{for} \quad 0 \leq x \leq \pi. \quad \text{... 3}(a)$$

and $$\text{initial displacement} = y(x, 0) = y_0 \sin^3 x, \ 0 \leq x \leq \pi. \quad \text{... 3}(b)$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*) and (*iii*) the solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin(nx)\cos(nct). \quad \text{... (4)}$$

Putting $t = 0$ in (4) and using 3(*b*), we have $y_0 \sin^3 x = \sum_{n=1}^{\infty} E_n \sin nx$

or $E_1 \sin x + E_2 \sin 2x + E_3 \sin 3x + E_4 \sin 4x + ... = (y_0/4) \times (3 \sin x - \sin 3x)$

so that $E_1 = 3y_0/4$, $E_2 = 0$, $E_3 = -y_0/4$, $E_n = 0$ for all $n \geq 4$. So, (4) gives

$y(x, t) = (3y_0/4) \times \sin x \cos(ct) - (y_0/4) \times \sin(3x) \cos(3ct)$.

(*g*) We are to solve $\partial^2 y/\partial x^2 = (1/c^2) \times (\partial^2 y/\partial t^2)$ ... (1)

subject to the boundary condition $y(0, t) = y(2\pi, t) = 0$, for all t ... (2)

the initial conditions $y(x, 0) = f(x) = \sin^3 x, \ 0 \leq x \leq 2\pi$ ... 3(*a*)

and $y_t(x, 0) = 0, \ 0 \leq x \leq 2\pi$ ... 3(*b*)

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*) and (*iii*), the required solution is given by

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin(n x/2)\cos(nct/2) \quad \text{... (4)}$$

Putting $t = 0$ in (4) and using initial condition $y(x, 0) = \sin^3 x$, we have

$$\sin^3 x = \sum_{n=1}^{\infty} E_n \sin(nx/2)$$

or $E_1 \sin(x/2) + E_2 \sin x + E_3 \sin(3x/2) + E_4 \sin 4x + E_5 \sin(5x/2) + E_6 \sin 3x + ...$

$= (1/4) \times (3 \sin x - \sin 3x)$

Comparing the coefficients of the like terms on both sides, we get $E_2 = 3/4$, $E_6 = -1/4$ and $E_n = 0$ for $n \neq 2$ or $n \neq 6$. With these values of E_2, E_6 etc. (4) reduces to

$y(x, t) = (3/4) \times \sin x \cos ct - (1/4) \times \sin 3x \cos 3ct$.

(*h*) we are to solve $\partial^2 y/\partial x^2 = \partial^2 y/\partial t^2$ ... (1)

subject to the boundary conditions $y(0,t) = y(\pi,t) = 0$ for all t ... (2)

the initial conditions $y(x, 0) = f(x) = k(\sin x - \sin 2x), \ 0 \leq x \leq \pi$ ... 3(*a*)

and $$y_t(x, 0) = 0, \ 0 \le x \le \pi \qquad \ldots 3(b)$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*) and (*iii*), the required solution is given by

$$y(x,t) = \sum_{n=0}^{\infty} E_n \sin nx \cos nx, \text{ as } c^2 = 1 \Rightarrow c = 1 \qquad \ldots (4)$$

where $$E_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2k}{\pi}\int_0^{\pi} (\sin x - \sin 2x) \sin nx \, dx$$

or $$E_n = \frac{2k}{\pi}\left[\int_0^{\pi} \sin x \sin nx \, dx - \int_0^{\pi} \sin 2x \sin nx \, dx\right] \qquad \ldots (5)$$

We can show that $$\int_0^{\pi} \sin px \sin qx \, dx = \begin{cases} 0, \text{ when } p \ne q \\ \pi/2, \text{ when } p = q \end{cases} \qquad \ldots (6)$$

Using (6), (5) $\Rightarrow$ $E_1 = k$, $E_2 = -k$ and $E_n = 0$ for all $n \ge 3$

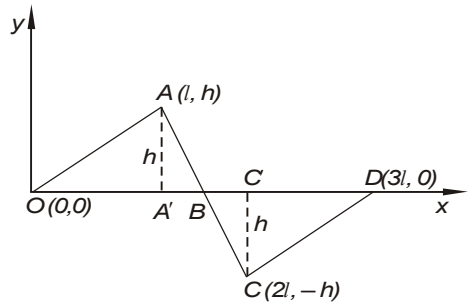
Putting these values in (4), the required deflection is given by

$$y(x, t) = k (\cos t \sin x - \cos 2t \sin 2x)$$

Exercise 2. *The points of trisection of a string are pulled aside through a distance h on opposite sides of the position of equilibrium, and the string is released from rest. Derive an expresion for the string at any subsequent time and show that the middel point of the sting always remains at rest.* [*Meerut 2008*]

Sol. The displacement function $y(x, t)$ is the solution of the one-dimensional wave equation of string (of length $3l$, say) $$\partial^2 y / \partial x^2 = (1/c^2)(\partial^2 y / \partial t^2) \qquad \ldots (1)$$

subject to the boundary conditions $y(0, t) = y(3l, 0) = 0$, for all t $\qquad \ldots (2)$



Let $OA' = A'C' = C'D = l$ so that the A' and C' are the points of trisection of the string of length AD (= $3l$). According to given problems, A' and C' are pulled in the direction of y through a distance h as shown in figure and hence the coordinates of A, C and D are (l, h), $(2l, -h)$ and $(3l, 0)$ respectively.

Initial deflection is given by $OABCD$. Equation of OA is

$$y - 0 = (h/l)(x - 0) \qquad i.e., \qquad y = hx/l.$$

Equations of AC and CD respectively are given by

$$y - h = \frac{-h - h}{2l - l}(x - 1) \qquad \text{or} \qquad y = \frac{h(3l - 2x)}{l}$$

and $$y - (-h) = \frac{0 - (-h)}{3l - 2l}(x - 2l) \qquad \text{or} \qquad y = \frac{h(x - 3l)}{l}$$

Hence, the initial displacement is given by

$$y(x,0) = f(x) = \begin{cases} hx/l, \text{ where } 0 \le x \le l \\ \{h(3l-2x)\}/l, \text{ where } l \le x \le 2l \\ \{h(x-3l)\}/l, \text{ where } 2l \le x \le 3l \end{cases} \quad \ldots 3(a)$$

Since initially the string was in the position of equilibrium, Therefore

$$\text{the initial velocity } = y_t(x, 0) = 0, \ 0 \le x \le 3l \quad \ldots 3(b)$$

As explained in working rule of Art 2.10C, prove that the solution of one dimensional wave equation (1) subject to the

boundary condition $y(0, t) = y(a, t) = 0$, for all t ... (4)

and initial conditions $y(x, 0) = f(x), \quad y_t(x, 0) = 0, \ 0 \le x \le a$ ... (5)

is $$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}\cos\frac{n\pi ct}{a} \quad \ldots (6)$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \quad \ldots (7)$$

Comparing the given problem given by (1), (2), 3(a) and 3(b) with the boundary value problem gievn by (1), (4) and (5) we have $c = c$, $a = 3l$ and $f(x)$ is given by 3(a).

From (7), $E_n = \frac{2}{3l}\int_0^{3l} f(x)\sin\frac{n\pi x}{3l}dx$

$$= \frac{2}{3l}\left[\int_0^l f(x)\sin\frac{n\pi x}{3l}dx + \int_l^{2l} f(x)\sin\frac{n\pi x}{3l}dx + \int_{2l}^{3l} f(x)\sin\frac{n\pi x}{3l}dx\right]$$

$$= \frac{2}{3l}\left[\int_0^l \frac{hx}{l}\sin\frac{n\pi x}{3l}dx + \int_l^{2l} \frac{h(3l-2x)}{l}\sin\frac{n\pi x}{3l}dx + \int_{2l}^{3l} \frac{h(x-3l)}{l}\sin\frac{n\pi x}{3l}dx\right], \text{ using } 3(a)$$

$$= \frac{2h}{3l^2}\left[x\left\{-\frac{\cos(n\pi x/3l)}{n\pi/3l}\right\} - (1)\left\{-\frac{\sin(n\pi x/3l)}{n^2\pi^2/9l^2}\right\}\right]_0^1 + \frac{2h}{3l^2}\left[(3l-2x)\left\{-\frac{\cos(n\pi x/3l)}{n\pi/3l}\right\} - (-2)\left\{-\frac{\sin(n\pi x/3l)}{n^2\pi^2/9l^2}\right\}\right]_l^{2l}$$

$$+ \frac{2h}{3l^2}\left[(x-3l)\left\{-\frac{\cos(n\pi x/3l)}{n\pi/3l}\right\} - (1)\left\{-\frac{\sin(n\pi x/3l)}{n^2\pi^2/9l^2}\right\}\right]_{2l}^{3l}, \quad \text{by chain rule of integration by parts}$$

$$= \frac{2h}{3l^2}\left[-\frac{3l^2}{n\pi}\cos\frac{n\pi}{3} + \frac{9l^2}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{3l^2}{n\pi}\cos\frac{2n\pi}{3} - \frac{18l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{3l^2}{n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{18l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{3l^2}{n\pi}\cos\frac{2n\pi}{3} - \frac{9l}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$= \frac{2h^2}{3l^2}\times\frac{27l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right) = \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \sin\left(n\pi - \frac{n\pi}{3}\right)\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left\{\sin n\pi\cos\frac{n\pi}{3} - \cos n\pi\sin\frac{n\pi}{3}\right\}\right] = [1+(-1)^n]\left(18\times\frac{h}{n^2\pi^2}\right)\sin\frac{n\pi}{3}$$

$\therefore$ If n is odd and equal to $2m - 1$, then $E_n = 0$

and if n is even and equal to $2m$, then

$$E_n = E_{2m} = 2(18h/4m^2\pi^2)\sin(2m\pi/3) = (9h/m^2\pi^2)\sin(2m\pi/3)$$

Putting these value in (6), the required deflection is given by

$$\therefore \quad y(x,t) = \frac{9h}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\frac{2mkct}{3l}\sin\frac{2m\pi x}{3l} \quad \ldots (8)$$

In order to get the displacement of the mid-point of the string, we put $x = 3l/2$ in (8) and obtain $y(3l/2, t) = 0$. This shows that the middle point of the string is always at rest.

Exercise 3. (*a*) *A tightly stretched elastic string of length l, with fixed end points* $x = 0$ *and* $x = l$ *is initially in the position given by* $y = C\sin^3(\pi x/l)$, *C being constant. It is released from the position of rest. Find the displacement* $y(x, t)$. **[Meerut 2012]**

(*b*) *A tightly strecthed elastic string of length* π, *with fixed end points* $x = 0$ *and* $x = 1$ *is initially in the position given by* $y = \sin^3 x$, *C being constant. It is released from the position of rest. Find the displacement* $y(x, t)$. **[Nagpur, 1996]**

Sol. (*a*) The partial differential equation of the transverse vibrations of the given elastic string in given by $\partial^2 y/\partial x^2 = (1/a^2)(\partial^2 y/\partial t^2)$ ... (1)

where $y(x, t)$ is the deflection of the string and a in a constant. Given boundary and initial conditions are:

Boundary condition (B.C.): $y(0, t) = y(l, t) = 0$ for all t. ... (2)

Initial conditions (I.C.): $y(x, 0) = C\sin^3(\pi x/l)$ (Initial deflection) ... 3(*a*)

$(\partial y/\partial t)_{t=0} = y_t(x, 0) = 0$ (Initial velocity). ... 3(*b*)

Let solution of (1) be of the form $y(x, t) = X(x)\,T(t)$ ... (4)

Substituting this value of y in (1), we have

$$X''T = (1/a^2)\,XT'' \quad \text{or} \quad X''/X = T''/a^2T \quad \ldots (5)$$

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ. Then (5) gives $X'' - \mu X = 0$ ... (6)

and $T'' - \mu a^2 T = 0$ ... (7)

Using (2), (4) gives $X(0)\,T(t) = 0$ and $X(l)\,T(t) = 0$. ... (8)

Since $T(t) = 0$ leads to $y = 0$, hence we assume that $T(t) \neq 0$. Then (8) gives

$X(0) = 0$ and $X(l) = 0$. ... (9)

We now solve (6) under B.C. (9). Three cases arise:

Case I. Let $\mu = 0$. Then solution of (6) is given by $X(x) = Ax + B$ ... (10)

Using B.C. (9), (10) gives $0 = B$ and $0 = Al + B$. These give $A = B = 0$ so that $X(x) = 0$. With the help of (4), this leads to $y \equiv 0$, which does not satisfy 3(*a*) and 3(*b*). So we reject $\mu = 0$.

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$. ... (11)

Using B.C. (9), (11) gives $A + B = 0$ and $Ae^{l\lambda} + Be^{-l\lambda} = 0$ ... (12)

Solving (12), $A = B = 0$ and so $X(x) = 0$. Hence as in case I, we reject the possibility $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ... (13)

Using B.C. (9), (13) gives $A = 0$ and $A\cos\lambda l + B\sin\lambda l = 0$ so that $A = 0$ and $\sin\lambda l = 0$, where we have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $y = 0$ which does not satisfy 3(a) and 3(b).

Now, $\sin\lambda l = 0 \quad \Rightarrow \quad \lambda l = n\pi \quad \Rightarrow \quad \lambda = n\pi l, \quad n = 1, 2, 3, \ldots$ ... (14)

Hence, non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x/l)$. ... (15)

Using (14), (7) reduces to

$$(d^2T/dt^2) + (n^2\pi^2a^2/l^2)T = 0, \qquad \text{as} \qquad \mu = -\lambda^2 = -(n^2\pi^2)/l^2$$

whose general solution is $$T_n(t) = C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l}.$$ ... (16)

$$\therefore \qquad y_n(x,t) = X_n(t)T_n(t) = \left(E_n\cos\frac{n\pi at}{l} + F_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

are solutions of (1) satisfying (2) for n = 1, 2, 3, ... Here E_n (= C_nB_n) and F_n (= $B_n\,D_n$) are new arbitrary constants. In order to obtain a solution also satisfying 3(a) and 3(b), we consider most general solution of the form

$$y(x,t) = \sum_{n=1}^{\infty}\left(E_n\cos\frac{n\pi at}{l} + F_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}.$$ ... (17)

Differentiating (17) partially w.r.t. 't', we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left(-\frac{n\pi aE_n}{l}\sin\frac{n\pi at}{l} + \frac{n\pi aF_n}{l}\cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}.$$ ... (18)

Putting t = 0 in (17) and (18) and using 3(a) and 3(b), we get

$$C\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty}E_n\sin\frac{n\pi x}{l} \qquad \text{and} \qquad 0 = \sum_{n=1}^{\infty}\left(\frac{n\pi aF_n}{l}\right)\sin\frac{n\pi x}{l}$$

which are Fourier since series. Accordingly, we have

$$E_n = \frac{2}{l}\int_0^l C\sin^3\left(\frac{\pi x}{l}\right)\sin\frac{n\pi x}{l}\,dx$$ ... (19)

and $$\frac{n\pi aF_n}{l} = \frac{2}{l}\int_0^l 0\,.\sin\left(\frac{n\pi x}{l}\right)dx \;\Rightarrow\; F_n = 0$$ ... (20)

Now, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \;\Rightarrow\; \sin^3\theta = (3\sin\theta - \sin 3\theta)/4$

Hence, $$\sin^3\left(\frac{\pi x}{l}\right) = \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$ ... (21)

$$\therefore (19) \Rightarrow \qquad E_n = \frac{2C}{l}\int_0^l\frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{3C}{2l}\int_0^l\sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\,dx - \frac{C}{2l}\int_0^l\sin\frac{3\pi x}{l}\sin\frac{n\pi x}{l}\,dx.$$ ... (22)

We now show that $$I = \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \begin{cases} l/2, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \quad \ldots (23)$$

If $m = n$, then, we have

$$I = \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \int_0^l \frac{1}{2}\left(1-\cos\frac{2n\pi x}{l}\right)dx = \frac{1}{2}\left[x - \frac{l}{2n\pi}\sin\frac{2n\pi x}{l}\right]_0^l = \frac{l}{2}$$

If $m \neq n$, then we have

$$I = \frac{1}{2}\int_0^l 2\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \int_0^l\left(\cos\frac{(n-m)\pi x}{l} - \cos\frac{(n+m)\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[\frac{1}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l} - \frac{l}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\right]_0^l = 0$$

Using (23), from (22), we have

$$E_1 = \left(\frac{3C}{2l}\right)\times\left(\frac{l}{2}\right) = \frac{3C}{4} \quad \text{and} \quad E_3 = -\left(\frac{C}{2l}\right)\times\left(\frac{l}{2}\right) = -\frac{C}{4} \quad \ldots (24)$$

Also, $E_n = 0$ for $n \neq 1$ and $n \neq 3$.

Using (20) and (24), (17) reduces to

$$y(x,t) = E_1\cos\frac{\pi at}{l}\sin\frac{\pi x}{l} + E_3\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l} = \frac{C}{4}\left[3\cos\frac{\pi at}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}\right] \quad \ldots (25)$$

(*b*) *This is a particular case of part* (*a*) *with* $l = \pi$. *So reproduce the entire solution of part* (*a*) *by replacing* l *by* π. **Ans.** $y(x, t) = (C/4) \times (3 \cos a\,t \sin x - \cos 3\,a\,t \sin 3x)$

Ex. 4(a). *A uniform string of length* l *held tightly between* $x = 0$ *and* $x = l$ *with no initial displacement, is struct at* $x = a$, $0 < a < l$ *with velocity* v_0. *Find the displacement of the string at any time* $t > 0$. **[I.A.S. 2004]**

(*b*). *A tightly stretched string with fixed end points* $x = 0$ *and* $x = l$ *is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity* $kx(l - x)$, *find its displacement.* **[Meerut 2010; I.A.S. 2000; Delhi Maths (H) 2000]**

(c). *A string of length* l *is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity* v *given by* $v = kx$ *if* $0 \le x \le l/2$ *and* $v = k(l - x)$ *if* $l/2 \le x \le l$. *Find the displacement function* $y(x, t)$.

(*d*) *The deflection of a vibrating string of length* l, *is governed by the partial differential equation* $y_{tt} = c^2 y_{xx}$. *The initial velocity is zero. The initial displacement is given by* $y(x,0) = \begin{cases} x/l, & 0 < x < l/2 \\ (l-x)/l, & l/2 < x < l \end{cases}$. *Here* $y_{tt} = \partial^2 y/\partial t^2$ *and* $y_{xx} = \partial y^2/\partial x^2$.

Find the deflection of the string at any instant of time. **[I.A.S. 2006]**

(*e*) *A string of length* l *is initially at rest in its equilibrium position and each of its points is given the velocity* $(\partial y/\partial t)_{t=0} = v_0 \sin^3(\pi x/l)$ *where* $0 < x < l$. *Find the displacement function.* **[Nagpur 1995]**

(*f*) *If the string of length l is initially at rest in equilibrium position and each of its points is given the velocity $v_0 \sin(3\pi x/l)\cos(2\pi x/l)$ where $0 < x < l$ at $t = 0$. Find the displacement function.*

Sol. *First of all, as explained in working rule of Art. 2.10 D, prove that the solution of one-dimensional wave equation* $\partial^2 y/\partial x^2 = (1/c^2)\times(\partial^2 y/\partial t^2)$ ... (*i*)

subject to the boundary conditions $y(0, t) = y(l, t) = 0$, for all t ... (*ii*)

with initial conditions, $y(x, 0) = 0$ and $(\partial y/\partial t)_{t=0} = g(x)$, $0 \le x \le l$ ... (*iii*)

is given by $$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}, \quad \text{... (iv)}$$

where $$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \quad \text{... (v)}$$

(*a*) The required displacement $y(x, t)$ of the string of length l is the solution of the one-dimensional wave-equation (*i*)

subject to the boundary conditions : $y(0, t) = y(l, t) = 0$ for all t. ... (1)

and the given initial conditions, namely,

$$\text{initial displacement } = y(x, 0) = 0,\ 0 \le x \le l \quad \text{... (2)}$$

and $$\text{initial velocity } = y_t(x,0) = g(x) = v_0,\ 0 \le x \le l \quad \text{... (3)}$$

Comparing the given problem given by (*i*), (1) (2) and (3) with the boundary value problem given by (*i*), (*ii*) and (*iii*), required solution is given by (*iv*) and (*v*).

Substituting the value of $g(x)$ given by (3) in (*v*), we get

$$F_n = \frac{2}{n\pi c}\int_0^l v_0 \sin\frac{n\pi x}{l}\,dx = \frac{2v_0}{n\pi c}\left[-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l = \frac{2v_0 l}{cn^2\pi^2}(-\cos n\pi + 1) = \frac{2v_0 l}{cn^2\pi^2}\left\{1-(-1)^n\right\}$$

or $$F_n = \begin{cases} 0, \text{ if } n = 2m \text{ (even) and } m = 1, 2, 3, ... \\ (4v_0 l)/c\pi^2(2m-1)^2, \text{ if } n = 2m-1 \text{ (odd) and } m = 1, 2, 3.... \end{cases}$$

Substituting these values in (*iv*), the required displacement is given by

$$y(x,t) = \frac{4v_0 l}{c\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$$

(*b*) The required displacement $y(x, t)$ of the string is the solution of the wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2) \quad \text{... (1)}$$

subject to the boundary conditions : $y(0, t) = y(l, t) = 0$ for all $t \ge 0$. ... (2)

and the given initial conditions namely:

$$\text{initial displacement} = y(x, 0) = f(x) = 0 \quad \text{... 3(a)}$$

and $$\text{initial velocity} = (\partial y/\partial t)_{t=0} = g(x) = k(lx - x^2). \quad \text{... 3(b)}$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*), (*iii*), the solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}, \quad \text{... (4)}$$

where $$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx = \frac{2k}{n\pi c}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx. \qquad \text{... (5)}$$

$$\therefore\ F_n = \frac{2k}{n\pi c}\left[\left(lx - x^2\right)\left(-\frac{1}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi l}{l}\right)\right]_0^l$$

$$= \frac{2k}{n\pi c}\left[-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right] = \frac{4kl^3}{cn^4\pi^4}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ (even) and } m = 1, 2, 3, ... \\ (8kl^3)/c\pi^4(2m-1)^4, \text{ if } n = 2m-1 \text{ (odd) and } m = 1, 2, 3, ... \end{cases}$$

Substituting these values in (4), the required displacement is given by

$$y(x,t) = \frac{8kl^3}{c\pi^4}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$$

(*c*) The required displacement $y(x, t)$ is the solution of the wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)\,(\partial^2 y/\partial t^2) \qquad \text{... (1)}$$

subject to the boundary conditions : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ... (2)

and the given initial conditions, namely :

$$\text{initial displacement} = y(x, 0) = f(x) = 0 \qquad \text{... (3}a\text{)}$$

and $$\text{initial velocity } = (\partial y/\partial t)_{t=0} = g(x) = \begin{cases} kx, 0 \leq x \leq l/2 \\ k(l-x), l/2 \leq x \leq l \end{cases} \qquad \text{... (3}b\text{)}$$

Comparing the given problem given by (1), (2), (3*a*) and (3*b*) with the boundary value probelm given by (*i*), (*ii*) and (*iii*), the solution of (1) satisfying the above boundary and initial conditons is given by

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}, \qquad \text{... (4)}$$

where $$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{n\pi c}\left[\int_0^{l/2} g(x)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^l g(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2k}{n\pi c}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \frac{2k}{n\pi c}\int_{l/2}^l (l-x)\sin\frac{n\pi x}{l}\,dx, \text{ by 3}(b)$$

$$= \frac{2k}{n\pi c}\left[(x)\left(-\frac{1}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+ \frac{2k}{n\pi c}\left[(l-x)\left(-\frac{1}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^l$$

(using chain rule of integration by parts)

$$= \frac{2k}{n\pi c}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{4kl^2}{cn^2\pi^3}\ \sin\ \frac{n\pi}{2}$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1,2,3,... \\ (-1)^{m+1}\times\{(4kl^2)/c\pi^3(2m-1)^3\}, \text{ if } n = 2m-1 \text{ and } m = 1,2,3,... \end{cases}$$

$$[\because \ \textit{when } n = 2m-1, \sin\frac{n\pi}{2} = \sin\frac{\pi}{2}(2m-1) = \sin\left(m\pi - \frac{\pi}{2}\right)$$

$$= \sin m\pi\cos(\pi/2) - \cos m\pi\sin(\pi/2) = 0-(-1)^m = (-1)^{m+1}]$$

With these values of E_n in (4), the required displacement function is given by

$$y(x,t) = \frac{4kl^2}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\ \pi ct}{l}$$

(*d*) The required deflection $y(x, t)$ of the string is the solution of the one-dimensional wave equation $y_{tt} = c^2 y_{xx}$, *i.e.*, $\partial^2 y/\partial x^2 = (1/c^2)\times(\partial^2 y/\partial t^2)$ ... (1)

subject to the boundary conditions : $y(0, t) = y(l, t) = 0$, for all t ... (2)

and the given initial conditions, namely

$$\text{initial displacement} = y(x, 0) = f(x) = 0,\ 0 \le x \le l \qquad \ldots 3(a)$$

and

$$\text{initial velocity} = y_t(x,t) = g(x) = \begin{cases} x/l, 0 < x < l/2 \\ (l-x)/l, l/2 < x < l \end{cases} \qquad \ldots 3(b)$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*) and (*iii*), the required displacement is given by (*iv*) and (*v*).

$$\therefore \qquad F_n = \frac{2}{n\pi c}\left[\int_0^{l/2} g(x)\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} g(x)\sin\frac{n\pi x}{l}dx\right], \text{ by } (v)$$

$$= \frac{2}{n\pi c}\int_0^{l/2}\frac{x}{l}\sin\frac{n\pi x}{l}dx + \frac{2}{n\pi c}\int_{lh}^{l}\frac{l-x}{l}\sin\frac{n\pi x}{l}dx, \text{ by } 3(b)$$

$$= \frac{2}{n\pi cl}\left[(x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\right] - (1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)_0^{l/2}$$

$$+\frac{2}{n\pi cl}\left[(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

[using chain rule of integration by parts]

$$= \frac{2}{n\pi cl}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \frac{2}{n\pi cl}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4l}{c\pi^3 n^3} \times \sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ \{4l(-1)^{m+1}\} / c\pi^3 (2m-1)^3, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

$$[\because\ n = 2m-1 \ \Rightarrow\ \sin\frac{n\pi}{2} = \sin\frac{\pi}{2}(2m-1) = \sin\left(m\pi - \frac{\pi}{2}\right)$$

$$= \sin m\pi \cos(\pi/2) - \cos m\pi \sin(\pi/2) = 0 - (-1)^m = (-1)^{m+1}]$$

Substituting the above value of F_n in (*iv*), the required deflection is given by

$$y(x,t) = \frac{4l}{c\pi^3} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^3} \sin\frac{(2m-1)\pi x}{l} \sin\frac{(2m-1)\pi ct}{l}$$

(*e*) The required displacement function $y(x, t)$ is the solution of the wave equation

$$\partial^2 y / \partial x^2 = (1/c^2)\,(\partial^2 y / \partial t^2) \qquad \ldots (1)$$

subject to the boundary conditions : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ... (2)

and the given initial conditions, namely :

$$\text{initial displacement} = y(x, 0) = f(x) = 0 \qquad \ldots 3(a)$$

and $$\text{initial velocity} = (\partial y / \partial t)_{t=0} = g(x) = v_0 \sin^3(\pi x / l),\ 0 \leq x \leq l. \qquad \ldots 3(b)$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (*i*), (*ii*), (*iii*), solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}, \qquad \ldots (4)$$

Differentiating both sides of (4) partially w.r.t. '*t*', we have

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} F_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \qquad \ldots (5)$$

Putting $t = 0$ in (5) and using initial condition (5), we have

$$\sum_{n=1}^{\infty} F_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\frac{\pi x}{l} = \frac{v_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$$\left[\because \sin^3\theta = (1/4) \times (3\sin\theta - \sin 3\theta) \ \Rightarrow\ \sin^3(\pi x / l) = (1/4) \times \{3\sin(\pi x / l) - \sin(3\pi x / l)\}\right]$$

or $$(\pi c F_1 / l) \sin(\pi x / l) + (2\pi c F_2 / l) \sin(2\pi x / l) + (3\pi c F_3 / l) \sin(3\pi x / l) + \ldots$$

$$= (3v_0 / 4)\sin(\pi x / l) - (v_0 / 4)\sin(3\pi x / l) \qquad \ldots (6)$$

Equating the coefficients of like terms on both sides of (6), we get

$$(\pi c F_1)/l = 3v_0/4, \qquad (2\pi c F_2)/l = 0, \qquad (3\pi c F_3)/l = -(v_0/4), \ldots$$

so that $B_1 = (3v_0 l)/4\pi c$, $B_2 = 0$, $B_3 = -(v_0 l)/12\pi c$, and $E_n = 0$ for all $n \geq 4$.

Substituting these values in (4), the reqired displacement is given by

$$y(x,t) = \frac{3v_0 l}{4\pi c} \sin\frac{\pi x}{l} \sin\frac{\pi ct}{l} - \frac{v_0 l}{12\pi c} \sin\frac{3\pi x}{l} \sin\frac{3\pi ct}{l}$$

(*f*) The required displacement function $y(x, t)$ is the solution of the wave equation

$$\partial^2 y / \partial x^2 = (1/c^2)\,(\partial^2 y / \partial t^2) \qquad \ldots (1)$$

subject to the boundary conditions : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ... (2)

and the given initial conditions, namely :

$$\text{initial displacement} = y(x, 0) = f(x) = 0 \quad \text{... 3}(a)$$

and $$\text{initial velocity} = (\partial y / \partial t)_{t=0} = g(x) = v_0 \sin(3\pi x / l) \cos(2\pi x / l) \quad \text{... 3}(b)$$

Comparing the given problem given by (1), (2), 3(a) and 3(b) with boundary value problem given by (i), (ii) and (iii) the solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}. \quad \text{... (4)}$$

Differentiating both sides of (4) partially w.r.t 't', we have

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} F_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}. \quad \text{... (5)}$$

Putting $t = 0$ in (5) and using initial condition (5), we have

$$\sum_{n=1}^{\infty} F_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin\frac{3\pi x}{l} \cos\frac{2\pi x}{l} = \frac{v_0}{2}\left[\sin\frac{5\pi x}{l} + \sin\frac{\pi x}{l}\right]$$

$[\because\ 2 \sin A \cos B = \sin (A + B) + \sin (A - B)]$

or $$(\pi c F_1 / l)\sin(\pi x / l) + (2\pi c F_2 / l)\sin(2\pi x / l) + \ldots = (v_0 / 2)\sin(\pi x / l) + (v_0 / 2)\sin(5\pi x / l).$$

Equating the coefficients of like terms, we have

$$(\pi c F_1) / l = v_0 / 2, \qquad (5\pi c F_5) / l = v_0 / 2, \qquad F_2 = F_3 = F_4 = F_6 = F_7 = \ldots\ldots = 0.$$

$$\Rightarrow\ F_1 = (l v_0) / 2\pi c, \qquad F_5 = (l v_0) / 5\pi c, \qquad F_2 = F_3 = F_4 = F_6 = F_7 = \ldots\ldots = 0.$$

With these values, (4) gives the required displacement

$$y(x,t) = (l v_0 / 2\pi c)\sin(\pi x / l)\sin(\pi c t / l) + (l v_0 / 5\pi c)\sin(5\pi x / l)\sin(5\pi c t / l).$$

2.10 F. Miscellaneous solved examples based on one dimensional wave equation

Ex. 1. *Solve the following wave equation*

$$\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2),\ 0 \leq x \leq 1,\ t \geq 0 \quad \text{... (1)}$$

satisfying boundary condition (B.C.) $u(0, t) = 0,\ t > 0$ ... 2(a)

boundary condition (B.C.) $u_x(1, t) = 0,\ t > 0$ ... 2(b)

Initial conditions (I.C.)
$$u(x,0) = \begin{cases} x, & 0 < x < 1/4 \\ (1/2) - x, & 1/4 < x < 1/2 \\ 0, & 1/2 < x < 1 \end{cases} \quad \text{... 3}(a)$$

and $$u_t(x, 0) = 0, \quad 0 < x < 1 \quad \text{... 3}(b)$$

Sol. Suppose (1) has solution of the form $u(x, t) = X(x)\ T(t)$, ... (4)

Substituting this value of u in (1), we have

$$X T'' = c^2 X'' T \qquad \text{or} \qquad X'' / X = T'' / c^2 T \quad \text{... (5)}$$

where the dashes denote derivatives with respect to the relevant variable. Clearly the L.H.S. of (5) is a function of x alone whereas the R.H.S. is a function of t alone. Since x and t are independent variables, (5) can hold good if each side is equal to a constant, say μ. Then (5) gives

$$X'' - \mu X = 0 \quad \text{... (6)}$$

$$T'' - \mu c^2 T = 0 \qquad \ldots (7)$$

Using 2(*a*), (4) gives $X(0)\,T(t) = 0,$ so that $X(0) = 0,$ as T(t) $\neq 0$

Differentiating (4) partially w.r.t. '*x*', we get $u_x(x,t) = X'(x)\,T(t)$ ... (8)

Using 2(*b*), (8) *gives* $X'(1)\,T(t) = 0$ *so that* $X'(1) = 0.$ *Here we have taken* $T(t) \neq 0,$ *since otherwise* $X(x) \equiv 0,$ *so that* $u \equiv 0$ *which does not satisfy* 3(*a*).

Thus, $X(0) = 0$ and $X'(1) = 0$ ... (9)

We now solve (6) under B.C. (9). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is

$$X(x) = Ax + B \quad \text{with} \quad X'(x) = B. \qquad \ldots (10)$$

Using B.C. (9), (10) gives 0 = *B* and 0 = *A*, so that $X(x) \equiv 0$ which leads to $u \equiv 0.$ So we reject $\mu = 0$.

Case II. $\mu = \lambda^2, \lambda \neq 0.$ Then solution of (6) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{with} \quad X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x} \qquad \ldots (11)$$

Using B.C. (9), (11) gives $0 = A + B$ and $0 = \lambda(Ae^{\lambda} - Be^{-\lambda})$ ... (12)

Solving (12), $A = B = 0$, so that $X(x) \equiv 0,$ which leads to $u \equiv 0.$ So we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0.$ Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{with} \quad X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x \qquad \ldots (13)$$

Using B.C. (9), (13) gives $0 = A$ and $0 = B\lambda\cos\lambda$

so that $A = 0$ and $\cos\lambda = 0,$ as $\lambda \neq 0$

where we have taken $B \neq 0,$ since otherwise $X \equiv 0$ so that $u \equiv 0.$

Also, $\cos\lambda = 0$ gives $\lambda = (2n-1)\pi/2,$ $n = 1, 2, 3, \ldots$

Hence non-zero solution $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin\{(2n-1)\pi/2\}$

Also, $\mu = -\lambda^2 = -\{(2n-1)^2\pi^2/4\}$ so that (7) becomes $T'' - \{(2n-1)^2\pi^2/4\}c^2T = 0$

whose general solution is $T_n(t) = C_n\cos\frac{1}{2}(2n-1)\pi ct + D_n\sin\frac{1}{2}(2n-1)\pi ct$ ... (16)

$$\therefore \quad u_n(x,t) = \left[E_n\cos\frac{1}{2}(2n-1)\pi ct + F_n\sin\frac{1}{2}(2n-1)\pi ct\right]\sin\frac{1}{2}(2n-1)\pi x$$

are solutions of (1), satisfying (2). Here E_n (= B_n C_n) and F_n(= B_n D_n) are new arbitrary constants. In order to obtain a solution also satisfying 3(*a*) and 3(*b*), we consider more general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left[E_n\cos\frac{1}{2}(2n-1)\pi ct + F_n\sin\frac{1}{2}(2n-1)\pi ct\right]\sin\frac{1}{2}(2n-1)\pi x \qquad \ldots (17)$$

Differentiating (17) partially w.r.t. '*t*' we get

$$\frac{\partial u}{\partial t} = u_t(x,t) = \sum_{n=1}^{\infty}\left[-\frac{1}{2}(2n-1)\pi c\, E_n \sin\frac{1}{2}(2n-1)\pi ct + \frac{1}{2}(2n-1)\pi c\, F_n \cos\frac{1}{2}(2n-1)\pi ct\right]\sin\frac{1}{2}(2n-1)\pi x \quad \ldots (18)$$

Putting $t = 0$ in (17) and (18), we get

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin\{(2n-1)\pi x/2\} \quad \ldots (19)$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} F_n \{(2n-1)\pi c/2\}\sin\{(2n-1)\pi x/2\} \quad \ldots (20)$$

Multiplying both sides of (19) by $2\ \sin\frac{1}{2}(2m-1)\pi x$ and then integrating w.r.t. x from 0 to 1,

$$\sum_{n=1}^{\infty} E_n \int_0^1 2\sin\frac{1}{2}(n-1)\pi x \sin\frac{1}{2}(2m-1)\pi x\, dx = 2\int_0^1 u(x,0)\sin\frac{1}{2}(2m-1)\pi x\, dx$$

or

$$E_n = 2\int_0^1 u(x,0)\sin\frac{1}{2}(2n-1)\pi x\, dx \quad \ldots (21)$$

where we have used the following results to be proved in examinating for complete solution

$$2\int_0^1 \sin\frac{1}{2}(2n-1)\pi x\ \sin\frac{1}{2}(2m-1)\pi x\, dx = \begin{cases} 0, \text{ if } n \neq m \\ 1, \text{ if } n = m \end{cases}$$

In a similar manner, (20) leads to

$$\frac{1}{2}(2n-1)\pi c\ F_n = 2\int_0^1 u_t(x,0)\sin\frac{1}{2}(2n-1)\pi\, x\, dx \quad \ldots (22)$$

From (21),

$$E_n = 2\int_0^{1/4} u(x,0)\sin\frac{1}{2}(2n-1)\pi x\, dx + 2\int_{1/4}^{1/2} u(x,0)\sin\frac{1}{2}(2n-1)\pi x\, dx$$

$$= 2\int_0^{1/4} x\sin\frac{1}{2}(2n-1)\pi x dx + 2\int_{1/4}^{1/2}\left(\frac{1}{2}-x\right)\sin\frac{1}{2}(2n-1)\pi x\, dx \text{ , using } 3(a)$$

$$= 2\left[x\left\{\frac{-\cos\frac{1}{2}(2n-1)\pi x}{\frac{1}{2}(2n-1)\pi}\right\} - (1)\left\{\frac{-\sin\frac{1}{2}(2n-1)\pi x}{\frac{1}{4}(2n-1)^2\pi^2}\right\}\right]_0^{1/4}$$

$$+2\left[\left(\frac{1}{2}-x\right)\left\{\frac{-\cos\frac{1}{2}(2n-1)\pi x}{\frac{1}{2}(2n-1)\pi}\right\} - (-1)\left\{\frac{-\sin\frac{1}{2}(2n-1)\pi x}{\frac{1}{4}(2n-1)^2\pi^2}\right\}\right]_{1/4}^{1/2} = \frac{8\sin\{(2n-1)\pi/4\}}{(2n-1)^2\pi^2}$$

Using 3(*b*), (22) gives $F_n = 0$

Putting these values of E_n and F_n in (17), the desired solution is given by

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{\sin\{(2n-1)\pi/4\}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{2}\sin\frac{(2n-1)\pi x}{2}$$

Ex. 2(*a*) *Discuss D'Alembert's solution of one dimensional wave equation*

or Show that the general solution of wave equation $c^2(\partial^2 u/\partial x^2) = \partial^2 u/\partial t^2$ *is* $u(x, t) = \phi(x + ct) + \psi(x - ct)$, *where* ϕ *and* ψ *are arbitrary functions.* **[Kanpur 2011]**

[Delhi Maths (H) 2002; Kurukshetra 2001, 05; Meerut 2002, 03; Nagpur 2005]

or *Obtain the solution of one-dimensional wave equation by canonical reduction*

[Himanchal 2007]

(*b*) *Obtain the D'Alembert's solution of the following Cauchy problem of the following infinite string:* $u_{tt} - c^2 u_{xx} = 0,\ x \in \mathbf{R},\ t > 0,\ u(x,0) = f(x),\ x \in \mathbf{R},\ u_t(x,0) = g(x),\ x \in \mathbf{R}$.

[Delhi B.A./B.Sc. (Hons.) III 2012]

(*c*) *One end of a string* ($x = 0$) *is fixed and the point* $x = a$ *is made to oscillate, so that at time t the displacement is g*(*t*). *Show that the displacement u*(*x*, *t*) *of the point x at time t is given by*

$$u(x, t) = f(ct - x) - f(ct + x),$$

where f is a function satisfying the relation $f(t + 2a) = f(t) - g[(t + a)/c]$ for all t. **[I.A.S. 1999]**

Sol. (*a*) Given $\partial^2 u/\partial x^2 = (1/c^2) \times (\partial^2 u/\partial t^2)$ ... (1)

Let v and w be two new independent variables such that

$w = x + ct$ and $v = x - ct$ ... (2)

Now, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial w} + \dfrac{\partial u}{\partial v}$, using (2)

Thus $\partial u/\partial x = \partial u/\partial w + \partial u/\partial v$ so that $\partial/\partial x \equiv \partial/\partial w + \partial/\partial v$ ... (3)

$\therefore$ $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \left(\dfrac{\partial}{\partial w} + \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial u}{\partial w} + \dfrac{\partial u}{\partial v}\right)$, using (3)

Thus, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial w^2} + 2\dfrac{\partial^2 u}{\partial w \partial v} + \dfrac{\partial^2 u}{\partial v^2}$ ... (2)

Again, $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial t} + \dfrac{\partial u}{\partial v}\dfrac{\partial v}{\partial t} = c\dfrac{\partial u}{\partial w} - c\dfrac{\partial u}{\partial v}$, using (2)

$\therefore$ $\dfrac{\partial u}{\partial t} = c\left(\dfrac{\partial u}{\partial w} - \dfrac{\partial u}{\partial v}\right)$ so that $\dfrac{\partial}{\partial t} \equiv c\left(\dfrac{\partial}{\partial w} - \dfrac{\partial}{\partial v}\right)$ ... (5)

$\therefore$ $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial t}\right) = c^2\left(\dfrac{\partial}{\partial w} - \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial u}{\partial w} - \dfrac{\partial u}{\partial v}\right)$, using (5)

Thus $\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial w^2} - 2\dfrac{\partial^2 u}{\partial u \partial v} + \dfrac{\partial^2 u}{\partial v^2}$ ... (6)

Using (4) and (6), (1) reduces to

$$\frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} = \frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} \quad \text{or} \quad \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial w}\right) = 0 \quad \text{... (7)}$$

Integrating (7) w.r.t. 'v', $\partial u/\partial w = F(w)$, where F is an arbitrary function of w ... (8)

Integrating (8) w.r.t, 'w', $u = \int F(w)dw + \psi(v)$, where ψ is an arbitrary function of v.

Then, $u = \phi(w) + \psi(v)$, where $\phi(w) = \int F(w)\,dw$

or $u(x,t) = \phi(x+ct) + \psi(x-ct)$, where ϕ and ψ are arbitrary functions, using (2) ... (9)

(*b*) Consider the wave equation $\partial^2 u/\partial x^2 = (1/c^2) \times (\partial^2 u/\partial t^2)$ ... (1)

Proceed as in part (*a*) and prove that a general solution of (1) is given by

$u(x,t) = \phi(x+ct) + \psi(x-ct)$. where ϕ and ψ are arbitrary functions. ... (2)

Solutions (2) is known as the *D'Alembert's solution of the wave equation (1).*

The functions ϕ and ψ can be obtained by using the given initial conditions:

$u(x, 0) = f(x)$, for all x. ... (3)

and $(\partial u/\partial t)_{t=0} = g(x)$ for all x. ... (4)

From (2), $\partial u/\partial t = c\,\phi'(x+ct) - c\,\psi'(x-ct)$ ... (5)

Putting $t = 0$ in (2) and using (3), we get $\phi(x) + \psi(x) = f(x)$ ... (6)

Putting $t = 0$ in (4) and using (4), we get

$c\,\phi'(x) - c\,\psi'(x) = g(x)$ or $\phi'(x) - \psi'(x) = (1/c) \times g(x)$. ... (7)

Integrating (7), $\phi(x) - \psi(x) = \dfrac{1}{c}\int_a^x g(\theta)d\theta$, ... (8)

where a is arbitrary. Solving (6) and (8), we have

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_a^x g(\theta)d\theta \quad \text{and} \quad \psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_a^x g(\theta)d\theta.$$

$$\therefore\ \phi(x+ct) = \frac{f(x+ct)}{2} + \frac{1}{2c}\int_a^{x+ct} g(\theta)\,d\theta, \qquad \psi(x-ct) = \frac{f(x-ct)}{2} - \frac{1}{2c}\int_a^{x-ct} g(\theta)\,d\theta$$

Substituting these values in (2), we have

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\left[\int_a^{x+ct} g(\theta)d\theta - \int_a^{x-ct} g(\theta)\,d\theta\right]$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\left[\int_a^{x+ct} g(\theta)d\theta + \int_{x-ct}^{a} g(\theta)\,d\theta\right]$$

$$\therefore \qquad u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\theta)\,d\theta. \qquad \text{... (9)}$$

Particular case. If the string is at rest when $t = 0$ so that $g = 0$, then the required solution (9) reduces to $u(x, t) = (1/2) \times \{f(x + ct) + f(x - ct)\}$. ... (10)

(*c*) The displacement $u(x, t)$ of the point x at time t is governed by one-dimensional wave equation $\partial^2 u/\partial x^2 = (1/c^2)\,(\partial^2 u/\partial t^2)$ ... (1)

We are to solve (1) subject to boundary conditions:

$u(0, t) = 0$ and $u(a, t) = g(t)$ ... (2)

Let v and w be two new independent variables such that

$w = ct + x$ and $v = ct - x$ ... (3)

Now, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial w} - \dfrac{\partial u}{\partial v}$, using (3) ... (4)

Then, (4) $\Rightarrow$ $\partial / \partial x \equiv \partial / \partial w - \partial / \partial v$ ... (5)

$\therefore$ $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \left(\dfrac{\partial}{\partial w} - \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial u}{\partial w} - \dfrac{\partial v}{\partial v}\right)$, using (4) and (5)

Thus, $\partial^2 u / \partial x^2 = (\partial^2 u / \partial w^2) - 2(\partial^2 u / \partial w \partial v) + (\partial^2 u / \partial v^2)$ ... (6)

Again, $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial t} + \dfrac{\partial u}{\partial v}\dfrac{\partial v}{\partial t} = c\left(\dfrac{\partial u}{\partial w} + \dfrac{\partial u}{\partial v}\right)$, using (3) ... (7)

Then, (7) $\Rightarrow$ $\partial / \partial t = c(\partial / \partial w + \partial / \partial v)$ ... (8)

$\therefore$ $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial t}\right) = c^2\left(\dfrac{\partial}{\partial w} + \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial u}{\partial w} + \dfrac{\partial}{\partial v}\right)$, using (7) and (8)

Thus, $(1/c^2) \times (\partial^2 u / \partial t^2) = \partial^2 u / \partial w^2 + 2(\partial^2 u / \partial w \partial v) + (\partial^2 u / \partial v^2)$ ... (9)

Using (6) and (9), (1) reduces to

$$\partial^2 u/\partial w^2 - 2(\partial^2 u/\partial w \partial v) + \partial^2 u / \partial v^2 = \partial^2 u / \partial w^2 + 2(\partial^2 u / \partial w \partial v) + d^2 u / \partial v^2$$

$\dfrac{\partial^2 u}{\partial w \partial u} = 0$ or $\dfrac{\partial}{\partial v}\left(\dfrac{\partial u}{\partial w}\right) = 0$ ... (10)

Integrating (10), w.r.t. 'v', $\partial u / \partial w = F(w)$, where F is an arbitrary function ... (11)

Integrating (11), w.r.t. 'w', $u = \int F(w)\, dw + f(v)$, f being an arbitrary function

Thus, $u = G(w) + f(v)$, where $G(w) = \int F(w)\, dw$

or $u\,(x, t) = G\,(ct + x) + f\,(ct - x)$, *using* (3)*; f* and *G* being arbitrary functions. ... (12)

Putting $x = 0$ in (12) and using B.C. u(0, t) = 0, we have

$0 = G\,(ct) + f\,(ct)$ so that $G\,(ct) = -f\,(ct)$ ... (13)

Replacing ct by $ct + x$ in (13), we get $G\,(ct + x) = -f\,(ct + x)$ ... (14)

Using (14), (12) reduces to $u\,(x, t) = f\,(ct - x) - f\,(ct + x)$ ... (15)

Putting $x = a$ in (15) and using B.C. $u\,(a, t) = g(t)$, we have

$$g\,(t) = f\,(ct - a) - f\,(ct + a) \quad \text{...(16)}$$

Replacing t by $(t + a)/c$ in the above identity (16), we get

$g\left(\dfrac{t+a}{c}\right) = f(t) - f(t + 2a)$ so that $f(t + 2a) = f(t) - g\left(\dfrac{t+a}{c}\right)$ ...(17)

Thus, the required displacement $u(x, t)$ is given by (15) where f is a function satisfying the relation (17). This proves the required result.

EXERCISE 2(C)

1. *A string is stretched between the fixed points* (0, 0) *and* (*l*, 0) *and released at rest from the initial deflection given by*

$$f(x) = \begin{cases} (2kx)/l, & \text{when } 0 < x < 1/2 \\ [2k(l-x)]/l, & \text{when } 1/2 < x < 1 \end{cases}$$

Find the deflection of the string at any time t. **[Bhopal 2004; Meerut 1993, 97]**

Ans. $u(x,t) = \dfrac{8k}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2} \cos\dfrac{(2n-1)\pi ct}{l} \sin\dfrac{(2n-1)\pi x}{l}$

2. *Solve* $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2)$ *where* $u(0, t) = u(1, t) = 0$, $u(x,0) = A\sin \pi x$ *and* $(\partial u/\partial t)_{t=0} = 0$. **[Nagpur 2002; Meerut 1997, 99] Ans.** $u(x, t) = A \cos(c\pi t) \sin \pi x$

3. *Find a solution of* $\partial^2 y/\partial t^2 = c^2(\partial^2 y/\partial x^2)$ *such that* (*i*) *y involves x trigonometrically* (*ii*) $y = 0$ *when* $x = 0$ *or* π, *for all values of t* (*iii*) $\partial y/\partial t = 0$ *when* $t = 0$, *for all values of x* (*iv*) *when* $t = 0$, $y = \sin x$ *from* $x = 0$ *to* $x = \pi/2$ *and* $y = 0$ *from* $x = \pi/2$ *to* $x = \pi$.

4. *A tightly stretched violin string of length l and fixed at both ends is plucked at* $x = l/3$ *and assumes initially the shape of a triangle of height a. Find the displacement* $u(x, t)$ *at any distance x and any time t after the string is released from rest.*

5. *A tightly stretched flexible string has its ends fixed at* $x = 0$ *and* $x = l$. *At time* $t = 0$ *the string is given a shape defined by* $f(x) = \mu x(l - x)$, *where* μ *is a constant, and then released. Find the displacement of any point x of the string an any time* $t > 0$.

[Aligarh 2003; I.A.S. 1993, 95, 2009; I.F.S. 2005]

Ans. $u(x,t) = \dfrac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \sin\dfrac{(2n-1)\pi x}{l} \cos\dfrac{(2n-1)\pi ct}{l}$

6. *Solve* $\partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2)$ *under the boundary conditions* $y(0, t) = 0$, $t > 0$; $y(l, t) = 0$. $t > 0$ *and the initial conditions* $y(x, 0) = m(lx - x^2)$, $0 < x < l$ *and* $(\partial y/\partial t)_{t=0} = 0$, $0 < x < l$, *m being being a suitable constant.*

OR

A string is stretched and fastened to two points l apart, Motion is started by displacing the string into the form $y = m(lx - x^2)$ *from which it is released at time* $t = 0$. *Find the displacement of any point on the string at a distance x from one end at time t.*

Ans. $y = \dfrac{8ml^2}{\pi^3} \sum_{m=1}^{\infty} \left[\dfrac{1}{(2m-1)^2} \sin\dfrac{(2m-1)\pi x}{l} \cos\dfrac{(2m-1)\pi ct}{l} \right]$

7. *A string fixed at ends distant* π *is initially in the shape of the string* $y(x, 0) = x(\pi - x)/\pi$ *and is released from rest. Find* $y(x, t)$ **[Nagpur 2005]**

8. *Solve the equation* $\partial^2 y/\partial t^2 = c^2(\partial^2 y/\partial x^2)$ *under the conditions* (*i*) $y(0, t) = y(\pi, 0) = 0$ *and* (*ii*) $y(x, 0) = 0$, $y_t(x, 0) = (\pi x - x^2)/100$. **[Mysore 2004]**

9. *The vibrations of an elastic string is governed by the partial differential equation* $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$. *The length of the string is* π *and the ends are fixed. The initial velocity is zero and the initial deflection is* $u(x, 0) = 2(\sin x + \sin 3x)$. *Show that the deflection* $u(x, t)$ *of the vibrating string for* $t > 0$ *is given by* $u(x, t) = 4 \cos x \cos 2t \sin 2x$.

[A.M.I.E. 2005; Banglaore (B.E.) 2003, 04]

10. *Solve the initial value problem* $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$ *for* $0 \le x \le \pi$ and $t = 0$, $u(x, 0) = 1$, $u_t(x, 0) = 0$ for $0 \le x \le \pi$, $u(0, t) = u(\pi, t) = 0$ *for* $t \ge 0$. **(Meerut 2011)**

Ans. $u(x,t) = (4/\pi) \times \sum_{n=1}^{\infty} (2n-1)^{-1} \sin\{(2n-1)x\} \cos\{(2n-1)t\}$

2.10 G. Derivation of one-dimensional wave eqution in another form and its solution for a hanging chain suspended vertically from one end [Delhi B.Sc (H) Physics 2002]

Sol. Derivation of one-dimensional wave equation

We propose to study the transverse vibration of a tightly stretched string. Consider a perfectly flexible string that is stretched between two points by a constant tension T so great that gravity may be neglected in comparison with it. Let the string be uniform and have a mass per unit length equal to m. Let us take the undisturbed position of the string to be the x-axis and suppose that the motion is confined to the xy-plane. Consider the motion of an element PQ of length ds as shown in figure (*i*). Let the directions of tension at P and Q be tangents making angles θ_1 and θ_2 with lines parallel to x-axis.

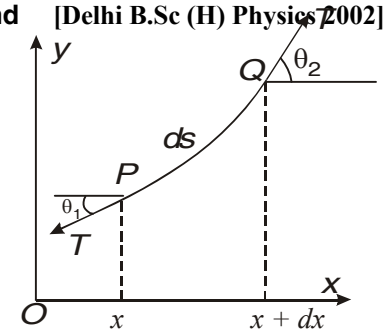


Figure (i)

The net force (F_y, say) in the y-direction is given by

$$F_y = T \sin\theta_2 - \sin\theta_1 \quad \ldots (1)$$

Now, for small oscillations, we may write

$$\sin\theta_2 = \tan\theta_2 = (\partial y/\partial x)_{x+dx} \quad \ldots (2)$$

and

$$\sin\theta_1 = \tan\theta_1 = (\partial y/\partial x)_x \quad \ldots (3)$$

$\therefore$ From (1)

$$F_y = \left(T\frac{\partial y}{\partial x}\right)_{x+dx} - \left(T\frac{\partial y}{\partial x}\right)_x, \text{ using (2) and (3)} \quad \ldots (4)$$

By Taylor's expansion, we have

$$\left(T\frac{\partial y}{\partial x}\right)_{x+dx} = \left(T\frac{\partial y}{\partial x}\right)_x + \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)_x dx, \quad \ldots (5)$$

where we have neglected terms of order dx^2 and higher. Substituting this in (4), we obtain

$$F_y = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)dx \quad \ldots (6)$$

By Newton's law of motion, we have

$$F_y = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)dx = m\,dx\left(\frac{\partial^2 y}{\partial t^2}\right), \quad \ldots (7)$$

where $m\,dx$ represents the mass of the section PQ of the string under consideration and where we have written dx for ds since the displacement is small. $\partial^2 y/\partial t^2$ is acceleration of the section PQ of the string in the y-direction. Thus, we have

$$\frac{\partial}{\partial x}\left(T\frac{\partial T}{\partial x}\right) = m\frac{\partial^2 y}{\partial t^2} \quad \ldots (8)$$

as the equation governing the small oscillations of a flexible string. If the stretching force is constant throughout the string, we may write

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = T\frac{\partial^2 y}{\partial x^2} = m\frac{\partial^2 y}{\partial t^2}$$

or $\partial^2 y/\partial x^2 = (1/c^2)\times(\partial^2 y/\partial t^2)$, where $c = (T/m)^{1/2}$ ... (9)

Remark. Equation (9) is same as already obtained in Art 1.2 of chapter 1. The constant c has the dimensions of length/time and hence the same dimensions as a velocity.

Solution of wave equation for a hanging chain suspended vertically from one end.

Let us consider the small coplanar oscillations of a uniform flexible string or chain hanging from a support under the action of gravity as shown in figure (*ii*). We consider only small deviations y from the equilibrium position; x is measured from the free end of the chain. Then, we propose to determine the position of the chain

$$y = y(x, t), \qquad \text{... (10)}$$

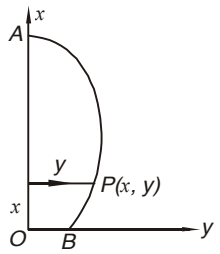


Figure (ii)

When at t = 0, we give the chain an arbitrary displacement

$$y(x, 0) = y_0(x) \qquad \text{... (11)}$$

In this case the tension of the chain is variable and hence the displacement of the chain at any instant is governed by the equation (refer equation (8)

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = m\frac{\partial^2 y}{\partial t^2}, \qquad \text{... (12)}$$

where m is the mass per unit length of the chain. In this case the tension is given by

$$T = mgx. \qquad \text{... (13)}$$

From (12) and (13), $\dfrac{\partial}{\partial x}\left(mgx\dfrac{\partial y}{\partial x}\right) = m\dfrac{\partial^2 y}{\partial t^2}$

or $mg\left(\dfrac{\partial y}{\partial x} + x\dfrac{\partial^2 y}{\partial x^2}\right) = m\dfrac{\partial^2 y}{\partial t^2}$ or $x\dfrac{\partial^2 y}{\partial x^2} + \dfrac{\partial y}{\partial x} = \dfrac{l}{g}\dfrac{\partial^2 y}{\partial t^2}$ ... (14)

Let us assume a solution of (14) of the form

$$y(x, t) = e^{iwt}\, v(x), \quad \text{where } i = \sqrt{-1} \qquad \text{... (15)}$$

where $v(x)$ is a function of x alone and w is to be determined.

From (15), $\dfrac{\partial^2 y}{\partial t^2} = -w^2 e^{iwt} v$, $\dfrac{\partial y}{\partial x} = e^{iwt}\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial^2 y}{\partial x^2} = e^{iwt}\dfrac{\partial^2 v}{\partial x^2}$ ... (16)

Substituting the values of $\partial^2 y/\partial t^2$ and $\partial y/\partial x$ and $\partial^2 y/\partial x^2$ given by (16) in (14), we have

$$xe^{iwt}\frac{\partial^2 v}{\partial x^2} + e^{iwt}\frac{\partial v}{\partial x} = -\frac{w^2}{g}e^{iwt} v \quad \text{or} \quad x\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + \frac{w^2}{g}v = 0 \qquad \text{... (17)}$$

Let us choose a new independent variable Z such that

$$x = (gZ^2)/4w^2 \quad \text{or} \quad Z^2 = (4w^2 x)/g \qquad \text{... (18)}$$

From (18), $2Z(\partial Z/\partial x) = (4w^2)/g$ or $\partial Z/\partial x = 2w^2/gZ$ ... (19)

Now, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial Z}\dfrac{\partial Z}{\partial x}$ or $\dfrac{\partial v}{\partial x} = \dfrac{2w^2}{gZ}\dfrac{\partial v}{\partial Z}$, by (19) ... (20)

Again, $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial Z}\left(\dfrac{\partial v}{\partial x}\right)\dfrac{\partial Z}{\partial x} = \dfrac{\partial}{\partial Z}\left(\dfrac{2w^2}{gZ}\dfrac{\partial v}{\partial Z}\right)\dfrac{\partial Z}{\partial x}$, by (20)

or $\dfrac{\partial^2 v}{\partial x^2} = \left(-\dfrac{2w^2}{gZ^2}\dfrac{\partial v}{\partial Z} + \dfrac{2w^2}{gZ}\dfrac{\partial^2 v}{\partial Z^2}\right)\times\dfrac{2w^2}{gZ}$, by (19) ... (21)

Substituting the values of x, $\partial v/\partial x$ and $\partial^2 v/\partial x^2$ given by (18), (20) and (21) respectively in (17), we have

$$\frac{gZ^2}{4w^2}\left(-\frac{2w^2}{gZ^2}\frac{\partial v}{\partial Z} + \frac{2w^2}{gZ}\frac{\partial^2 v}{\partial Z^2}\right)\times\frac{2w^2}{gZ} + \frac{2w^2}{gZ}\frac{\partial v}{\partial Z} + \frac{w^2}{g}v = 0$$

or $\dfrac{\partial^2 v}{\partial Z^2} + \dfrac{1}{Z}\dfrac{\partial v}{\partial Z} + v = 0$ or $Z^2\dfrac{\partial^2 v}{\partial Z^2} + Z\dfrac{\partial v}{\partial Z} + Z^2 v = 0,$ ... (22)

which is Bessel's equation of zeroth order and its general solution is given

$$v = A\,J_0(Z) + B\,Y_0(Z) \quad \text{... (23)}$$

Also, recall that $Y_0(Z) \to \infty$ as $Z \to 0$ (*i.e.*, $x \to 0$ by (18)). In order to satisfy the condition that the displacement of the string y remains finite when $x = 0$, we must take $B = 0$ in (23).

Substituting $B = 0$ and the value of Z given by (18) in (23), we have

$$v(x) = AJ_0(2w(x/g)^{1/2}) \quad \text{... (24)}$$

So far, the value of w is undetermined. In order to determine it, we make use of the boundary condition $v = 0$ at $x = l$, ... (25)

since the string is fastened at $x = l$, l being the length of the string. Using (25) (24) reduces to

$$AJ_0(2w(l/g)^{1/2}) = 0 \quad \text{... (26)}$$

Since we are looking for a non-trivial solution of (22), A cannot be zero and hence (26) yields

$$J_0(2w(l/g)^{1/2}) = 0 \quad \text{... (27)}$$

Assume that $u = 2w(l/g)^{1/2}$ ... (28)

Hence, (28) takes the form $J_0(u) = 0$ ... (29)

From the table of Bessel functions, the roots of (29) are given by 2.405, 5.52, 8.654, etc. Accordingly the various values of w are given by (note that from (28). $w = (u/2) \times (g/l)^{1/2}$)

$w_1 = (2.405/2) \times (g/l)^{1/2}$, $w_2 = (5.52/2) \times (g/l)^{1/2}$, $w_3 = (8.654/2) \times (g/l)^{1/2}$ etc.

To each value of w we associate a characteristic function v_n of the form

$$v_n = A_n\,J_0\,(2\,w_n\,(x/g)^{1/2}) \quad \text{... (30)}$$

Since the real and imaginary parts of the assumed solution (15) are solutions of the original differential equation (14), we can construct a general solution of (14) satisfying the boundary condition by summing the particular solutions corresponding to the various values of n in the manner.

$$y(x,t) = \sum_{n=1}^{\infty} J_0(2w_n(x/g)^{1/2})\,(A_n \cos w_n t + B_n \sin w_n t), \qquad \ldots (31)$$

where the quantities A_n and B_n are arbitrary constants to be determined from given boundary conditions of the problem. In the case under consideration there is no initial velocity imparted to the chain, so that,

$$(\partial y/\partial t)_{t=0} = 0 \qquad \ldots (32)$$

From (1), $$\partial y/\partial t = \sum_{n=1}^{\infty} J_0(2w_n(x/g)^{1/2})\,(-A_n w_n \sin w_n t + B_n w_n \cos w_n t) \qquad \ldots (33)$$

Substituting $t = 0$ in (33) and using (32), we get $B_n = 0$

$\therefore$ (31) $\Rightarrow$ $$y(x,t) = \sum_{n=1}^{\infty} A_n J_0(2w_n(x/g)^{1/2}) \cos w_n t \qquad \ldots (34)$$

Substituting $t = 0$ in (34) and using (11), we have

$$y_0(x) = \sum_{n=1}^{\infty} A_n J_0(2w_n(x/g)^{1/2}) \qquad \ldots (35)$$

Let $$(x/l)^{1/2} = z \qquad \ldots (36)$$

From (36), when $x = l \Rightarrow z = 1$ and $x = 0 \Rightarrow z = 0$. Also $0 \le x \le l \Rightarrow 0 \le z \le 1$

From (35) and (36), $$y_0(x) = y_0(lz^2) = F(z) = \sum_{n=1}^{\infty} A_n J_0(2w_n z(l/g^{1/2})$$

or $$F(z) = \sum_{n=1}^{\infty} A_n J_0(u_n z), \qquad 0 \le z \le 1 \qquad \ldots (37)$$

[Assuming that $y_0(lz^2) = F(z)$ and using relation (28)]

where the quantities u_n are successive positive roots of the equation (29).

Since (37) is Fourier – Bessel series, we obtain

$$A_n = \frac{2}{J_1^2(u_n)} \int_0^1 z\, J_0(u_n z) F(z)\, dz \qquad \ldots (38)$$

Example. *Solve* $\dfrac{\partial}{\partial x}\left(x\dfrac{\partial y}{\partial x}\right) = \dfrac{\partial^2 y}{\partial t^2}$ *where* $y = y(x, t)$. *The boundary conditions are* $(\partial y/\partial t)_{t=0} = 0$, $u(1, t) = 0$ *and* $y(x, 0) = f(x)$. *Here* $y(x, t)$ *represents displacement.*

[Delhi B.Sc (H) Physics 1998]

Hint. Compare the given equation with (8), we have $g = 1$, $l = 1$, $y_0(x) = f(x)$. With these changes, proceed as in above Art. 2.10 F.

PART V: PROBLEMS BASED ON TWO-DIMENSIONAL WAVE EQUATION

2.11A. General soution of two-dimensional wave equation

To solve $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)$ *subject to the boundary conditions* $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$ *and initial conditions* $u(x, y, 0) = f(x, y)$ *and* $(\partial u/\partial t)_{t=0} = g(x, y)$ *(Madurai Kamraj University 2008)*

Sol. We are to solve $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2) \times (\partial^2 u/\partial t^2)$ ... (1)

subject the boundary conditions $u(0, y, t) = 0$ ... 2(*a*)

$u(a, y, t) = 0$ ... 2(*b*)

$u(x, 0, t) = 0$ ... 2(*c*)

and $u(x, b, t) = 0$ ... 2(*d*)

and the two initial conditions

$u(x, y, 0) = f(x, y)$ [$\because$ given initial displacement = $f(x, y)$] ... 3(*a*)

and $u_t(x, y, 0) = g(x, y)$ [$\because$ given initial velocity = $g(x, y)$] ... 3(*b*)

where $u(x, y, t)$ gives the displacement of the point (x, y) of the vibrating membrane from rest ($u = 0$) at time t.

Suppose that (1) has solutions of the form $u(x, y, t) = X(x)\, Y(y)\, T(t)$ ... (4)

Substituting this in (1), we have $X''YT + XY''T = (1/c^2) XYT''$

or $X''/X + Y''/Y = T''/(c^2 T)$ ... (5)

Since x, y and t are independent variables, (5) can only be true if each term on each side is equal to a constant.

Let $X''/X = \mu_1$ so that $X'' - \mu_1 X = 0$ ... (6)

Using 2(*a*) and 2(*b*), (4) gives

$X(0)\, Y(y)\, T(t) = 0$ and $X(a)\, Y(y)\, T(t) = 0$ ... (7)

We assume that $Y(y) \neq 0$ and $T(t) \neq 0$. So by (7), we have

$X(0) = 0$ and $X(a) = 0$ ... (8)

We now solve (6) under B.C. (8). Three cases arise :

Case I. Let $\mu_1 = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (9)

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ and hence $X(x) \neq 0$. This leads to $u = 0$, which does not satisfy 3(*a*) and 3(*b*). So we reject $\mu_1 = 0$.

Case II. Let $\mu_1 = \lambda_1^2, \lambda_1 \neq 0$. Then solution of (6) is $X(x) = A e^{x\lambda_1} + B e^{-x\lambda_1}$ ... (10)

Using B.C. (8), (10) gives $0 = A + B$ and $0 = Ae^{a\lambda_1} + B e^{-a\lambda_1}$ ... (11)

Solving (11), $A = B = 0$ so that $X(x) \equiv 0$. This leads to $u \equiv 0$. So we reject $\mu_1 = \lambda_1^2$.

Case III. Let $\mu_1 = -\lambda_1^2, \lambda_1 \neq 0$. Then solution of (6) is $X(x) = A\cos\lambda_1 x + B\sin\lambda_1 x$ ... (12)

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda_1 a + B\sin\lambda_1 a$

so that $A = 0$ and $\sin\lambda_1 a = 0,$

where we have taken $B \neq 0,$ since otherwise $X(x) \equiv 0$ so that $u \equiv 0$ which does not satisfy 3(*a*) and 3(*b*).

Now, $\sin\lambda_1 a = 0$ gives $\lambda_1 a = m\pi, \quad m = 1, 2\ 3, \ldots$

Hence $\lambda_1 = m\pi/a, \qquad m = 1, 2, 3, \ldots$ ... (13)

Hence non-zero solutions $X_m(x)$ of (6) are given by

$$X_m(x) = B_m \sin(m\pi x/a), \qquad m = 1, 2, 3, \ldots \quad \ldots (14)$$

Next, let $Y''/Y = \mu_2$ so that $Y'' - \mu_2 y = 0$ ... (15)

Using 2(*c*) and 2(*d*), (4) gives as before

$Y(0) = 0$ and $Y(b) = 0$ ... (16)

Solving (15) under B.C. (16) as before, we get

$$Y_n(y) = D_n \sin(n\pi y / b), \qquad n = 1, 2, 3, \ldots \quad \ldots (17)$$

where $\mu_2 = -\lambda_2^2$ and $\lambda_2 = n\pi / b, \quad n = 1, 2, 3, \ldots$ ... (18)

In view of (6) and (15), (5) reduces to

$T'' / c^2 T = \mu_1 + \mu_2 = -\lambda_1^2 - \lambda_2^2 = -\pi^2 (m^2 / a^2 + n^2 / b^2)$, using (13) and (18)

or $$T'' + \lambda_{mn}^2 T = 0 \quad \ldots (19)$$

where $$\lambda_{mn}^2 = c^2 \pi^2 (m^2 / a^2 + n^2 / b^2) \quad \ldots (20)$$

Solving (19), $$T_{mn}(t) = E_{mn} \cos \lambda_{mn} t + F_{mn} \sin \lambda_{mn} t \quad \ldots (21)$$

$\therefore$ Functions $u_{mn}(x, y, t) = X_m(x)\, Y_n(y)\, T_{mn}(t)$

i.e., $$u_{mn}(x, y, t) = (A_{mn} \cos \lambda_{mn} t + B_{mn} \sin \lambda_{mn} t)\ \sin(m\pi x / a) \sin(n\pi y / b) \quad \ldots (22)$$

are solutions of (11) satisfying 2(*a*), 2(*b*), 2(*c*) and 2(*d*). Here $A_{mn} = (B_m D_n E_{mn})$ and B_{mn} (= $B_m D_n F_{mn}$) are new arbitrary constants. In order to obtain a solution also satisfying 3(*a*) and 3(*b*), weconsider more general solution, that is,

$$u(x, y, t) = \sum_{m=1} \sum_{n=1} (A_{mn} \cos \lambda_{mn} t + B_{mn} \sin \lambda_{mn} t)\ \sin(m\pi x / a) \sin\ (n\pi y / b) \quad \ldots (23)$$

Differentiating (23) pertially w.r.t. '*t*' we get

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-A_{mn} \lambda_{mn} \sin \lambda_{mn} t + B_{mn} \lambda_{mn} \cos \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \ldots (24)$$

Putting $t = 0$ in (23) and (24) and using 3 (*a*) and 3 (*b*), we get

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \ldots (25)$$

and $$g(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (B_{mn} \lambda_{mn}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \ldots (26)$$

which are double Fourier sine series. Accordingly, we get

$$A_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy \quad \ldots (27)$$

and $$B_{mn} \lambda_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy$$

so that $$B_{mn} = \frac{4}{ab \lambda_{mn}} \int_{x=0}^{a} \int_{y=0}^{b} g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy \text{ , } m = 1,2,3 \ldots, n = 1, 2, 3 \ldots \quad \ldots (28)$$

Thus the desired deflection $u(x, y, t)$ of the given membrane is given by (23) wherein A_{mn} and B_{mn} are given by (27) and (28) respectively.

Particular Case I. If the initial velocity $u_t(x, y, 0) = g(x, y) = 0$, then $B_{nm} = 0$ by (28). Thus, in this case the solution (23) reduces to

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos(\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \quad \text{... (29)}$$

Particular Case II. If the initial displacement = $u(x, y, 0) = f(x, y) = 0$, then $A_{mn} = 0$ by (27). Thus, in this case the solution (23) reduces to

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn}\sin(\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad \text{... (30)}$$

Eigenfunctions (or characteristic functions) and eigenvalues (or characteristic values) of the vibrating membrane. Definitions

The numbers λ_{mn} ($m = 1, 2, 3, ..., n = 1, 2, 3, ...$) given by (20) are called the *eigenvalues* and the functions $u_{mn}(x, y, t)$ ($m = 1, 2, 3, ..., n = 1, 2, 3, ...$) are called the corresponding *eigenfunctions* of the vibrating membrane. The *frequency* of u_{mn} is given by $\lambda_{mn}/2\pi$.

2.11 B. Solved examples based on Art. 2.11 A

Ex. 1. *Solve the two-dimensional wave equation* $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = (1/c^2)(\partial^2u/\partial t^2)$ *by the method of separation of variables under the following initial and boundary conditions:*

$u(x, y, 0) = f(x, y)$, $(\partial u/\partial t)_{t=0} = 0, 0 \le x \le a, 0 \le y \le b$ $u(0, y, t) = 0 = u(a, y, t)$ for $t \ge 0$

$u(x, 0, t) = 0 = u(x, b, t)$, for $t \ge 0$ **[Delhi Maths (Hons) 1996]**

Sol. Proceed as in Art. 2.11A upto equation (29).

Ex. 2. *Find the deflection u (x, y, t) of the square membrane of each side unity and c = 1, if the initial velocity is zero and the initial deflection is* $f(x, y) = A\sin\pi x\sin 2\pi y$.

[Kurukshetra 2004]

Sol. The deflection $u(x, y, t)$ of the given membrane is governed by the two dimensional wave equation

$$\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = \partial^2u/\partial t^2 \quad \text{... (1)}$$

subject to the boundary conditions $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ... (2)

and the initial deflection = $u(x, y, 0) = f(x, y) = A\sin\pi x\sin 2\pi y$ ... 3(*a*)

and the initial velocity = $u_t(x, y, 0) = g(x, y) = 0$ ... 3(*b*)

Proceed as in Art. 2.11A upto equation (29) and prove that the solution of

$$\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = (1/c^2)\times(\partial^2u/\partial t^2) \quad \text{... (4)}$$

subject to the boundary conditions $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$ ... (5)

and initial condition: $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = g(x, y) = 0$ ... (6)

is given by $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos(\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad \text{... (7)}$$

where $$A_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \quad \text{... (8)}$$

and $$\lambda_{mn}^2 = c^2\pi^2(m^2/a^2 + x^2/b^2) \quad \text{... (9)}$$

Comparing the given boundary value problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (4), (5) and (6), we have $c = 1$, $a = b = 1$ and $f(x, y) = A\sin\pi x\sin 2\pi y$.

$\therefore$ From (8), $$A_{mn} = 4\int_{x=0}^{1}\int_{y=0}^{1} A \sin \pi x \sin 2\pi y \ \sin m\pi x \ \sin n\pi y \, dx \, dy$$

or $$A_{mn} = A\left(\int_0^1 2\sin \pi x \sin m\pi x \, dx\right) \times \left(\int_0^1 2\sin 2\pi y \sin n\pi y \, dy\right) \quad \ldots (10)$$

But $$\left.\begin{aligned}\int_0^1 2\sin \pi x \sin m\pi x \, dx = 0 \text{ if } m \neq 1\\ = 1 \text{ if } m = 1\end{aligned}\right\} \quad \ldots (11)$$

and $$\left.\begin{aligned}\int_0^1 2\sin 2\pi y \sin n\pi y \, dy = 0 \text{ if } n \neq 2\\ = 1 \text{ if } n = 2\end{aligned}\right\} \quad \ldots (12)$$

For complete solution results (11) and (12) should be proved in examination.

Using (11) and (12), (10) gives $A_{12} = 1$ and $A_{mn} = 0$, otherwise.

Also, (9) $\Rightarrow$ $\lambda_{12} = \pi(1^2 + 2^2)^{1/2} = \pi\sqrt{5}$

Now, from (7) the required solution is given by

$$u(x, y, t) = A_{12} \cos \lambda_{12} t \ \sin \pi x \ \sin 2\pi y = \cos(\pi t\sqrt{5}) \sin \pi x \sin 2\pi y.$$

EXERCISE 2(D)

1. Discuss a solution of the wave equation $\nabla^2 z = (1/c^2)(\partial^2 z / \partial t^2)$ satisfied by a thin membrane bounded by a rectangle $x = 0$, $x = a$, $y = 0$, $y = b$ and $z = f(x, y)$, $\partial z / \partial t = 0$ at $t = 0$.

2. Obtain the solution of the two-dimensional wave equation $\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2)$ subject to the boundary condition that $u = 0$ on the curve Γ (which is a rectangle bounded by $x = 0$, $x = a, y = 0, y = b$) for all t, and the initial onditions $u = f(x, y)$ and $\partial u / \partial t = 0$ at $t = 0$ for all (x, y) in the region S bounded by Γ. **[Delhi Maths (H) 1996, 2006]**

Hint: For complete solution proceed as in Art. 2.11A upto equation (29).

3. Solve the partial differential equation for the vibrations of a square elastic membrane fixed at edges, $\partial^1 u / \partial x^2 + \partial^2 u / \partial y^2 = (1/c^2)(\partial^2 u / \partial t^2)$. The length of each side of the square is L, c is the velocity of elastic waves and $u(x, y, t)$ is the displacement of the membrane. Obtain the lowest frequency of vibrations, if $c = 10{,}000$ metres/sec. and $L = \sqrt{2}$ metres.

PART VI: PROBLEMS BASED ON THREE-DIMENSIONAL WAVE EQUATION

2.12 A. General solution of three-dimensional wave equation

Show that the solution of three dimensional wave equation $\nabla^2 u = (1/c^2)(\partial^2 u / \partial t^2)$ *can be put in the form* $e^{\pm i(lx + my + nz + kct)}$, *provided* $k^2 = l^2 + m^2 + n^2$.

Sol. Given $$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \partial^2 u / \partial z^2 = (1/c^2) \times (\partial^2 u / \partial t^2) \quad \ldots (1)$$

Suppose (1) has solutions of the form $u(x, y, z, t) = X(x)\, Y(y)\, Z(z)\, T(t)$ ... (2)

Substituting this in (1) and simplifying, we have

$$X'' / X + Y'' / Y + Z'' / Z = (1/c^2) \times (T'' / T) \quad \ldots (3)$$

Since x, y, z and t are independent variables, (3) is true if each term on both sides is equal to a constant, that is,

$$X''/X = -l^2, \qquad Y''/Y = -m^2, \qquad Z''/Z = -n^2, \qquad T''/(c^2T) = -k^2, \quad \text{... (4)}$$

provided $$k^2 = l^2 + m^2 + n^2 \quad \text{... (5)}$$

Solutions of differential equations of (4) are of the forms

$$X(x) = e^{\pm ilx}, \qquad Y(y) = e^{\pm imy}, \qquad Z(z) = e^{\pm inz}, \qquad T(t) = e^{\pm ickt} \quad \text{... (6)}$$

Using (2) and (6), a solution of (1) can be put in the form

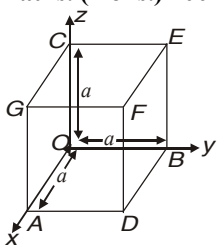
$$u(x, y, z, t) = e^{\pm i(lx+my+nz+ckt)}, \qquad \text{where} \qquad k^2 = l^2 + m^2 + n^2.$$

2.12 B. Solved example based on Art. 2.12A

Ex. 1. *A gas is contained in a cubical box of side a. Show that if c is the velocity of sound in the gas, the period of free oscillations are* $2a/c(n_1^2 + n_2^2 + n_3^2)^{1/2}$, where n_1, n_2, n_3 are integers.

[Delhi Maths. (Hons.) 2001; 2003]

Sol. Let one corner O of the given cubical box (*OBDA*, *CEFG*) be taken as origin as shown in figure. Since side of the box is a, we can take its six faces as $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$ and $z = a$. Thus, the gas is contained in the region defined by $0 \le x < a$, $0 \le y \le a$, $0 \le z \le a$.



We now solve three-dimensional wave equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/c^2)(\partial^2 u/\partial t^2) \quad \text{... (1)}$$

which is to be solved under the boundary condition that at all the six faces of the cube $\partial u/\partial n = 0$, where n is the unit outward drawn normal to surface. Thus, we are to solve (1) subject to the following boundary conditions :

$$(\partial u/\partial x)_{x=0} = (\partial u/\partial x)_{x=a} = 0, \text{ for all } t \ge 0. \quad \text{... (2)}$$

$$(\partial u/\partial y)_{y=0} = (\partial u/\partial y)_{y=a} = 0, \text{ for all } t \ge 0, \quad \text{... (3)}$$

and $$(\partial u/\partial z)_{z=0} = (\partial u/\partial z)_{z=a} = 0, \text{ for all } t \ge 0 \quad \text{... (4)}$$

Since the initial velocity of the gas is zero, we have

$$(\partial u/\partial t)_{t=0} = 0, \text{ for } 0 \le x \le 0, \ 0 \le y \le a, 0 \le z \le a \quad \text{... (5)}$$

Let a solution of (1) be of the form $u(x, y, z, t) = X(x)\, Y(y)\, Z(z)\, T(t)$. ... (6)

Substituting (6) in (1), we have $X''/X + Y''/Y + Z''/Z = (1/c^2)(T''/T)$. ... (7)

Since x, y, z and t are independent variables, (7) is true if each term of both sides is equal to a constant. Since non-negative constants lead to a trivial solution of (1), hence we take

$$X''/X = -\mu_1^2, \qquad Y''/Y = -\mu_2^2, \qquad Z''/Z = -\mu_3^2, \qquad T''/c^2T = -\mu^2 \quad \text{... (8)}$$

where $$\mu^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 \quad \text{... (9)}$$

Solving differential equations of (8), we have $X(x) = A\cos\mu_1 x + B\sin\mu_1 x$, ... (10)

$$Y(y) = C\cos\mu_2 y + D\sin\mu_2 y, \quad \text{... (11)}$$

$$Z(z) = E\cos\mu_3 z + F\sin\mu_3 z \quad \text{... (12)}$$

$$T(t) = G\cos(\mu ct) + H\sin(\mu ct) \quad \text{... (13)}$$

Differentiating (6) partially w.r.t. 'x', we have $\partial u/\partial x = X'(x)\,Y(y)\,Z(z)\,T(t)$ ... (14)

Putting $x = 0$ and $x = a$ by turn in (14) and using (2), we have

$X'(0) = 0$ and $X'(a) = 0$ ... (15)

Differentiating (10), $X'(x) = -A\mu_1 \sin \mu_1 x + B\mu_1 \cos \mu_1 x$ ... (16)

Putting $x = 0$ and $x = a$ by turn in (16) and using (15), we get

$0 = B\mu_1$ and $0 = -A\mu_1 \sin \mu_1 a + B\mu_1 \cos \mu_1 a$

These $\Rightarrow$ $B = 0$ and $\sin \mu_1 a = 0$, as $\mu_1 \neq 0$ and $A \neq 0$

Now $\sin \mu_1 a = 0 \Rightarrow \mu_1 a = n_1 \pi \Rightarrow \mu_1 = n_1 \pi / a$, ... (17)

where n_1 is an integer. With $B = 0$ and $\mu_1 = n_1 \pi / a$, (10) gives

$$X_{n_1}(x) = A_{n_1} \cos(n_1 \pi x / a). \quad \text{... (18)}$$

Similarly, (3) and (11) $\Rightarrow$ $\mu_2 = n_2 \pi / a$ and $Y_{n_2}(y) = D_{n_2} \cos(n_2 \pi y / a)$ ... (19)

and (4) and (12) $\Rightarrow$ $\mu_3 = n_3 \pi / a$, and $Z_{n_3}(z) = F_{n_3} \cos(n_3 \pi z / a)$ ... (20)

(9), (17), (18) and (20) $\Rightarrow$ $\mu^2 = (\pi^2 / a^2) \times (n_1^2 + n_2^2 + n_3^2)$ ... (21)

From (13), (18), (19) and (20), general solution of (1) is given by

$$u(x,y,z,t) = \sum_{n_1,n_2,n_3} I_{n_1,n_2,n_3} \cos(n_1 \pi x/a)\cos(n_2 \pi y/a)\cos(n_3 \pi z/a)[G_\mu \cos(\mu ct) + H_\mu \sin(\mu ct)] \quad \text{...(22)}$$

where $I_{n_1,n_2,n_3} (= A_{n_1} D_{n_2} F_{n_3})$ is another arbitrary constant.

Differentiating (22) partially with respect to 't', we get

$$\frac{\partial u}{\partial t} = \sum_{n_1,n_2,n_3} I_{n_1 n_2 n_3} \cos(n_1 \pi x/a)\cos(n_2 \pi y/a)\cos(n_3 \pi z/a)\,[-G_\mu \mu c \sin(\mu ct) + H_\mu \mu c \cos(\mu ct)] \quad \text{... (23)}$$

Putting $t = 0$ is (23) and using (5), we have $H_\mu = 0$. Hence (22) gives

$$u(x,y,z,t) = \sum_{n_1,n_2,n_3} J_{n_1 n_2 n_3} \cos\left(\frac{n_1 \pi x}{a}\right)\cos\left(\frac{n_2 \pi y}{a}\right)\cos\left(\frac{n_3 \pi z}{c}\right)\cos(\mu ct), \quad \text{... (24)}$$

where $J_{n_1,n_2,n_3} (= I_{n_1 n_2 n_3} G_\mu)$ are new arbitrary constants. From (24), we see that

the periods of the free oscillations of the gas $= \dfrac{2\pi}{\mu c} = \dfrac{2a}{c(n_1^2 + n_2^2 + n_3^2)^{1/2}}$, using (21)

PART VII: PROBLEMS BASED ON TWO-DIMENSIONAL LAPLACE'S EQUATION

The two dimensional heat equation is given by

$$\partial u/\partial t = c^2 \nabla^2 u = c^2(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) \quad \text{... (1)}$$

If the heat flow is steady (that is, time independent) then $\partial u/\partial t = 0$, and (1) reduces to Laplace's equation. $\nabla^2 u = 0$, *i.e.*, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ ... (2)

A heat problem then consists of the above equation (2) to be considered in some region R of the xy-pale and a given boundary conditon on the boundary curve of R. This is called a *boundary value problem*. We shall call it

(*i*) *Dirichlet problem* if u is prescribed on C

(*ii*) *Neumann problem* if the normal derivative $\partial u / \partial n$ is prescribed on C.

(*iii*) *Mixed problem* if u is prescribed on a portion of C and $\partial u / \partial n$ on the remaining part of C.

The solutions of (2) are called *harmonic functions.*

2.13 A. General solution of two-dimensional Laplace's equation

Consider $$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0. \quad \text{... (1)}$$

Suppose that (1) has solutions of the form $u(x, y) = X(x)\, Y(y)$ ... (2)

where X and Y are functions of x and y respectively.

From (2), $\partial^2 u / \partial x^2 = X'' Y$ and $\partial^2 u / \partial y^2 = X Y''$. Hence (2) reduces to

$$X'' Y + X Y'' = 0 \quad \text{or} \quad (1/X) X'' = -(1/Y) Y''. \quad \text{... (3)}$$

Since the L.H.S. of (3) depends only on x and the R.H.S depends only on y, both sides of (3) must be equal to same constant, say μ. This leads to two ordinary differential equations

$$X'' - \mu X = 0 \quad \text{and} \quad Y'' + \mu Y = 0, \quad \text{... (4)}$$

whose solutions depends only on the value of μ. Three cases arise :

Case I. When μ = 0. Then (4) reduces to $X'' = 0$ and $Y'' = 0$.

Solving these, $X = A_1 x + B_1$ and $Y = C_1 y + D_1$.

Then a solution of (1) is $u(x, y) = (A_1 x + B_1)\,(C_1 y + D_1)$. ... (5)

Case II. When $\mu = \lambda^2$, *i.e.*, positive. Here $\lambda \neq 0$. Then (4) reduces to

$$X'' - \lambda^2 X = 0 \quad \text{and} \quad Y'' + \lambda^2 Y = 0.$$

Solving these, $X = A_2 e^{\lambda x} + B_2 e^{-\lambda x}$ and $Y = C_2 \cos \lambda y + D_2 \sin \lambda y$.

or, alternatively, $X = A_2 \cosh \lambda x + B_2 \sinh \lambda x$, $Y = C_2 \cos \lambda y + D_2 \sin \lambda y$

Then a solution of (1) is $u(x, y) = (A_2 e^{\lambda x} + B_2 e^{-\lambda x})(C_2 \cos \lambda y + D_2 \sin \lambda y)$ ... (6)

or alternatively, $u(x, y) = (A_2 \cosh \lambda x + B_2 \sinh \lambda x)\,(C_2 \cos \lambda y + D_2 \sin Dy)$ ... (6)′

Case III. When $\mu = -\lambda^2$, *i.e.*, negative. Here $\lambda \neq 0$. Then (4) reduces to

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad Y'' - \lambda^2 Y = 0.$$

Solving these, $X = A_3 \cos \lambda x + B_3 \sin \lambda x$ and $Y = C_3 e^{\lambda y} + D_3 e^{-\lambda y}$.

or alternatively, $X = A_3 \cos \lambda x + B_3 \sin \lambda x$ and $Y = C_3 \cosh \lambda y + D_3 \sinh \lambda y$

Then a solution of (1) is $u(x, y) = (A_3 \cos \lambda x + B_3 \sin \lambda x)(C_3 e^{\lambda y} + D_3 e^{-\lambda y})$. ... (7)

or alternatively, $u(x, y) = (A_3 \cos \lambda x + B_3 \sin \lambda x)\,(C_3 \cosh \lambda y + D_3 \sinh \lambda y)$ ... (7)′

Out of the above mentioned three types of solutions (5), (6) and (7), we must select an appropriate solution which suits the physical nature of the problem and given boundary conditions.

2.13 B. Dirichlet problem in a rectangle

(Delhi Maths (H) 2001; Rajasthan 2002, Ravishankar 1998)

The Dirichlet problem in a rectangle is defined as follows:

Laplace's equation : $$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0, \quad 0 \le x \le a, \quad 0 \le y \le b \quad \text{... (1)}$$

Boundary conditions : $u(0, y) = 0, \quad u(a, y) = 0, \quad 0 \le y \le b$ ... (2)

$u(x, b) = 0, \quad 0 \le x \le a$ ... 3(*a*)

and $u(x, 0) = f(x), \quad 0 \le x \le a$ ... 3(*b*)

We shall solve (1) subject to the boundary conditions (2), 3(*a*) and 3(*b*).

Suppose (1) has a solution of the form

$$u(x, y) = X(x)\, Y(y) \quad \text{... (4)}$$

Substituting this value of u in (1), we get

$$X''Y + XY'' = 0 \quad \text{or} \quad X''/X = -Y''/Y. \quad \text{... (5)}$$

Since x and y are independent, variables each side of (5) must be equal to the same constant, say μ.

Then (5) gives $X'' - \mu X = 0$ ... (6)

and $Y'' + \mu Y = 0$ ... (7)

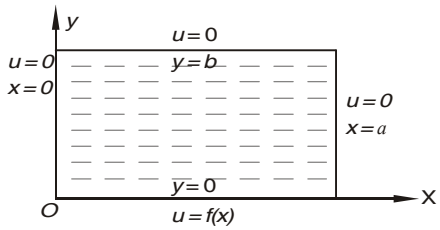


Fig. Dirichlet boundary conditions

Using (2), (4) gives $X(0)\, Y(y) = 0$ and $X(a)\, Y(y) = 0$

giving $X(0) = 0$ and $X(a) = 0$ ... (8)

where we have taken $Y(y) \neq 0$, since otherwise $u = 0$ which does not satisfy 3(*b*).

We now solve (6) under B.C. (8). Three cases arise:

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (9)

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ so that $X(x) \equiv 0$. This leads to $u \equiv 0$ which does not satisfy 3(*b*). So we reject $\mu = 0$.

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{x\lambda} + Be^{-x\lambda}$ ... (10)

Using B.C. (8), (10) gives $0 = A + B$ and $0 = Ae^{a\lambda} + Be^{-x\lambda}$ ... (11)

Solving (11), $A = B = 0$ so that $X(x) = 0$ and hence $u = 0$. So we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0$. Solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ... (12)

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$ so that

$A = 0$ and $\sin\lambda a = 0,$

where we have taken $B \neq 0$, since otherwsie $X(x) = 0$ and hence $u = 0$ which does not satisfy 3(*b*).

Now, $\sin\lambda a = 0 \quad \Rightarrow \quad \lambda a = n\pi \quad \Rightarrow \quad \lambda = n\pi/a, \quad n = 1, 2, 3, \ldots$ ... (13)

Hence non-zero solution $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x/a).$ ... (14)

Using $\mu = -\lambda^2 = -n^2\pi^2/a^2$, (7) reduces to $Y'' - (n^2\pi^2/a^2)Y = 0$ ... (15)

whose solution is $Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}.$ ... (16)

Using 3(*a*), (4) gives $0 = X(x)\, Y(b)$ so that $Y(b) = 0$, where we have taken $X(x) \neq 0$, since otherwise $u = 0$ which does not satisfy 3(*b*).

Again, $Y(b) = 0 \quad \Rightarrow \quad Y_n(b) = 0.$ ... (17)

Putting $y = b$ in (16) and using (17), we have

$$0 = C_n e^{n\pi b/a} + D_n e^{-n\pi b/a} \qquad \text{so that} \qquad C_n = -\{D_n e^{-n\pi b/a}\}/e^{n\pi b/a}. \quad \ldots (18)$$

Putting this value of C_n given by (18) in (16), we get

$$Y_n(y) = D_n(e^{-n\pi y/a}e^{n\pi b/a} - e^{n\pi y/a}e^{-n\pi b/a})/e^{n\pi b/a} \quad \text{or} \quad Y_n(y) = D_n\{e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a}\}/e^{n\pi b/a}$$

or $$Y_n(y) = 2D_n e^{-n\pi b/a}\sinh\{n\pi(b-y)/a\}, \qquad \text{as } e^{\theta} - e^{-\theta} = 2\sinh\theta$$

$$\therefore \qquad u_n(x,y) = X_n(x)Y_n = F_n \sin(n\pi x/a)\sinh\{n\pi(b-y)/a\}$$

are solutions of (1), satisfying (2) and 3(a). Here $F_n(=2B_nD_ne^{-n\pi b/a})$ are new arbitrary constants. In order to obtain a solution also satisfying 3(b), we consider more general solution, namely,

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} F_n \sin(n\pi x/a)\sinh\{n\pi(b-y)/a\} \qquad \ldots (20)$$

Putting $y = 0$ in (20) and using 3(b), we have

$$f(x) = \sum_{n=1}^{\infty}\{F_n \sinh\ (n\pi b/a)\}\sin(n\pi x/a),$$

which is the half range Fourier sine series of $f(x)$ in (0, a). Hence, we get

$$F_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \qquad \text{or} \qquad F_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \quad \ldots (21)$$

Hence (20) is the required solution wherein F_n is given by (21).

2.13 C. Solved examples based on Art. 2.13B or problems similar to the problem based Art 2.13 A and Art 2.13B.

Ex. 1. *Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions*

$$u\ (0, y) = u\ (a, y) = 0 \text{ for } 0 \le y \le b$$

and $$u(x, b) = 0 \qquad \text{and} \qquad u(x, 0) = x\ (a - x) \qquad \text{for } 0 \le x \le a.$$

Sol. First proceed as in Art. 2.13B upto equation (21).

For the present problem, $$u(x, 0) = f(x) = x\ (a - x),\ 0 \le x \le a$$

$$\therefore (21) \quad \Rightarrow \quad F_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx = \frac{2}{a\sinh(n\pi b/a)}\int_0^a (ax - x^2)\sin\frac{n\pi x}{a}dx$$

$$= \frac{2}{a\sinh(n\pi b/a)}\left[(ax - x^2)\left(-\frac{a}{n\pi}\right)\cos\frac{n\pi x}{a} - (a - 2x)\left(-\frac{a^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{a} + (-2)\left(\frac{a^3}{n^3\pi^3}\right)\cos\frac{n\pi x}{a}\right]_0^a$$

[using chain rule of integration by parts]

$$= \frac{2}{a\sinh(n\pi b/a)}\left[-\frac{2a^3(-1)^n}{n^3\pi^3} + \frac{2a^3}{n^3\pi^3}\right] = \frac{4a^2}{n^3\pi^3}[1 - (-1)^n]\operatorname{cosech}\frac{n\pi b}{a}$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ (even) and } m = 1, 2, 3, \ldots \\ \left\{8a^2/\pi^3(2m-1)^3\right\} \operatorname{cosech}\{(2m-1)\pi b/a\}, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting the above value of F_n in (21), the required steady temperature $u(x, y)$ is given by

$$u(x,y) = \frac{8a^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{a}\sinh\frac{(2m-1)(b-y)\pi}{a}\operatorname{cosech}\frac{(2m-1)\pi b}{a}$$

Ex. 2. *Use the method of separation of variables to determine the solution u(x, y) of the problem which consists of Laplace equation* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ *and the boundary conditions*: $u(x,0) = f(x), 0 \le x \le \pi;\ u(x,\pi) = 0, 0 \le x \le \pi, u(0,y) = u(\pi,y) = 0,\ 0 \le y \le \pi.$

Here f(x) is a given function of x.

Sol. Proceed exactly as in Art. 2.13*B* by replacing *a* by π and *b* by π in the entire solution. Finally, we get

$$u(x,y) = \sum_{n=1}^{\infty} F_n \sin nx \sinh n(\pi - y), \qquad \text{where} \qquad F_n = (2/\pi \sinh n\pi)\int_0^{\pi} f(x)\sin nx\, dx.$$

Ex. 3(*a*) *Find the steady temperature distribution in a thin plate bounded by the lines x = 0, x = a, y = 0 and* $y = \infty$ *assuming that heat cannot escape from either surface; the sides x = 0, x = a being kept at temperature zero. The lower edge y = 0 is kept at f(x) and the edge* $y = \infty$ *at temperature zero.* **[Meerut 2000, 06, 10]**

Sol. We know that heat equation in two-dimensions is

$$\partial^2 u/\partial x^2 + \partial u/\partial y^2 = (1/k)\times(\partial u/\partial t)$$

For steady state, $\partial u/\partial t = 0$ so the above equation reduces to

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \qquad \ldots (1)$$

We are to solve (1) under the following conditions.

$$u(0, y) = 0, \qquad u(a, y) = 0 \quad \ldots (2)$$

$$u(x, y) \to 0 \quad \text{as} \quad y \to \infty \quad \ldots (3a)$$

and $$u(x, 0) = f(x) \quad \ldots (3b)$$

Suppose (1) has a solution of the form $u(x, y) = X(x)\, Y(y) \quad \ldots (4)$

Substituting this value of *u* in (1), we get

$$X''Y + XY'' = 0 \qquad \text{or} \qquad X''/X = -Y''/Y \quad \ldots (5)$$

Since *x* and *t* are independent variables, each side of (5) must be equal to the same constant, say μ. Then (5) gives $X'' - \mu X = 0 \quad \ldots (6)$

and $Y'' + \mu Y = 0 \quad \ldots (7)$

Using (2), (4) gives $X(0)\, Y(y) = 0$ and $X(a)\, Y(y) = 0$

giving $X(0) = 0$ and $X(a) = 0, \quad \ldots (8)$

where we have taken $Y(y) \ne 0,$ since otherwise $u \equiv 0$ which does not satsify 3(*b*).

We now solve (6) under B.C. (8). Three cases arise :

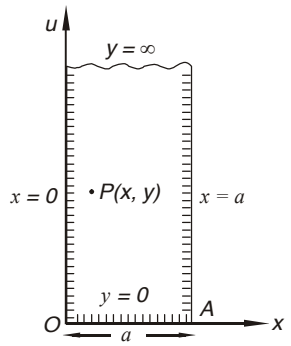
Case I. Let μ = 0. Then solution of (6) is $X(x) = Ax + B \quad \ldots (9)$

Using B.C. (8), (9) gives 0 = *B* and 0 = *Aa* + *B*. These give *A* = *B* = 0 so that $X(x) \equiv 0.$ This leads to $u \equiv 0,$ which does not satisfy 3(*b*). So we reject μ = 0.

Case II. Let $\mu = \lambda^2, \lambda \ne 0.$ These solution of (6) is $X(x) = Ae^{x\lambda} + Be^{-x\lambda} \quad \ldots (10)$

Using B.C. (8), (10) gives $0 = A + B$ and $0 = Ae^{a\lambda} + Be^{-a\lambda} \quad \ldots (11)$

Solving (11), *A* = *B* = 0 so that $X(x) \equiv 0$ and hence $u \equiv 0.$ So we reject $\mu = \lambda^2.$



Case III. Let $\mu = -\lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ... (12)

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$, Then, $A = 0$ and $\sin\lambda a = 0$.

where we have taken $B \neq 0$, since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$ which does not satisfy 3(*b*).

Now, $\sin\lambda a = 0$ gives $\lambda a = n\pi$ so that $\lambda = n\pi/a$, $n = 1, 2, 3, \ldots$ ... (13)

Hence non-zero solutions $X_n(x)$ of (6) are given by

$$X_n(X) = B_n \sin(n\pi x/a) \quad \text{... (14)}$$

Using $\mu = -\lambda^2 = (n^2\pi^2/a^2)$. Hence (7) becomes

$$Y'' - (n^2\pi^2/a^2)Y = 0 \quad \text{... (15)}$$

Solving (15), we have $Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}$ ... (16)

Now, 3(*a*) and (4) $\Rightarrow$ $Y_n(y) \to 0$ as $y \to \infty$.

Hence we must take $C_n = 0$ in (16). Then (16) reduces to

$$Y_n(y) = D_n e^{-n\pi y/a} \quad \text{... (17)}$$

$$\therefore \quad u_n(x, y) = X_n(x)Y_n(y) = E_n \sin\frac{n\pi x}{a} e^{-n\pi y/a}$$

are solutions of (1), satisfying (2) and 3 (*a*). Here E_n (= B_nD_n) are new arbitrary constants. In order to obtain a solution also satisfying 3(b), we consider more general solution, namely,

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a} e^{-n\pi y/a} \quad \text{... (18)}$$

Putting $y = 0$ in (18) and using 3(*b*), we get $f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}$

which is Fourier sine series and hence

$$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx. \quad \text{... (19)}$$

Hence, the required temperature $u(x, y)$ is given by (18) wherein E_n is given by (19).

Ex. 3(*b*) *A rectangular plate with insulated surfaces 8 cm wide and so long compared to its width that it can be considered infinite in the length without introducing an appreciable error. If the temperature along the short edge y = 0 is given by u(x, 0) = 100 sin* ($\pi x/8$) *while the two long edges x = 0 and x = 8 as well as the other short edges are kept at 0°C. Find the steady state temperature function u(x, y).*

Sol. The steady state temperature $u(x, y)$ is the solution of

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0. \quad \text{... (1)}$$

subject to the boundary conditions :

$$u(0, y) = u(8, y) = 0 \quad \text{for } 0 < y < \infty \quad \text{... (2)}$$

$$u(x, y) \to 0 \text{ as } y \to \infty \quad \text{for } 0 < x < 8 \quad \text{... 3}(a)$$

and $$u(x, 0) = 100\sin(\pi x/8) \quad \text{for } 0 < x < 8 \quad \text{... 3}(b)$$

Take $a = 8$ and proceed as in Ex. 3(*a*) upto equation (18). Thus, we have

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/8) e^{-n\pi y/8}. \qquad \ldots (18)$$

Putting $y = 0$ in (18) and using B.C. 3(b), we have

$$100 \sin(\pi x/8) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/8) = E_1 \sin(\pi x/8) + E_2 \sin(2\pi x/8) + \ldots .$$

Comparing the coefficients of like terms, $E_1 = 100$ and $E_n = 0$ for $n \geq 2$.

So, from (18), the required temperature is

$$u(x, y) = 100 \sin(\pi x/8) e^{-\pi y/8}.$$

Ex. 3(*c*) *An infinitely long uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is* π. *The end is maintained at 100°C at all points and the other edges are at 0°C. Find the steady state temperature function u*(*x, y*).

Sol. The steady state temperature $u(x, y)$ is the solution of Laplace equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0. \qquad \ldots (1)$$

subject to the B.C : $u(0, y) = u(\pi, y) = 0$ for all $y \geq 0$ ... (2)

$u(x, y) \to 0$ as $y \to \infty$ for $0 \leq x \leq \pi$ ... 3(a)

and $u(x, 0) = f(x) = 100$ for $0 \leq x \leq \pi$ ... 3(b)

Take $a = \pi$ and proceed as in 3(a) upto equation (19). Thus, we obtain

$$\therefore \quad u(x, y) = \sum_{n=1}^{\infty} E_n \sin nx\, e^{-ny}, \qquad \ldots (18)$$

where $$E_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx. \qquad \ldots (19)$$

Here, $$E_n = \frac{2}{\pi}\int_0^{\pi} 100 \sin nx\, dx = \frac{200}{\pi}\left[-\cos nx\right]_0^{\pi}, \text{ using 3}(b).$$

or $$E_n = \frac{200}{\pi}\left[1 - (-1)^n\right] = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 400/(2m-1)\pi, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting this value of E_n in (18), the required temperature function is given by

$$u(x, y) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2m-1)x}{2m-1} e^{-(2m-1)y}.$$

Ex. 4 (*a*). *Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions.*

$u(0, y) = u(a, y) = 0$ *for* $0 \leq y \leq b$ *and* $u(x, 0) = 0$ *and* $u(x, b) = f(x)$ *for* $0 \leq y \leq a$.

Sol. The heat flow in a body for a two-dimensional case is governed by the equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2)(\partial u/\partial t)$. When the steady state condition prevails, temperature u is independent of t so that $\partial u/\partial t = 0$. Then, the above equation reduces to Laplace's equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \qquad \ldots (1)$$

Given that $u(0, y) = u(a, y) = 0$ for $0 \le y \le b$ ... (2)

Also, given that $u(x, 0) = 0$ for $0 \le x \le a$ ... 3(*a*)

and $u(x, b) = f(x)$ for $0 \le x \le a$ ... 3(*b*)

Suppose (1) has a solution of the form $u(x, y) = X(x)\ Y(y)$ ... (4)

Substituting this value of u in (1), we get

$X'' y + X Y'' = 0$ or $X''/X = -Y''/Y$ ... (5)

Since x and t are independent variables, each side of (5) must be equal to the constant, say μ. Then (5) gives $X'' - \mu X = 0$ ... (6)

and $Y'' + \mu Y = 0$ ... (7)

Using (2), (4) gives $X(0)\ Y(y) = 0$ and $X(a)\ Y(y) = 0$

giving $X(0) = 0$ and $X(a) = 0,$ ... (8)

where we have taken $Y(y) \neq 0,$ since otherwise $u \equiv 0$ which does not satisfy 3 (*b*).

We now solve (6) under B.C. (8). Three cases arise:

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (9)

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ so that $X(x) = 0$. This leads to $u \equiv 0$ which does not satisfy 3(*b*). So we reject $\mu = 0$.

Case II. Let $\mu = \lambda^2, \lambda \neq 0.$ Then solution of (6) is $X(x) = A\ e^{x\lambda} + B\ e^{-x\lambda}$ ... (10)

Using B.C. (8), (10) gives $A + B = 0$ and $0 = e^{a\lambda} + B\ e^{-a\lambda}$ ... (11)

Solving (11), $A = B = 0$ so that $X(x) = 0$ and hence $u \equiv 0.$ So we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0.$ Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ... (12)

Using B.C. (8), (12) gives $0 = A$ and $0 = A \cos \lambda a + B \sin \lambda a$

so that $A = 0$ and $\sin \lambda a = 0$

where we have taken $B \neq 0,$ since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$ which does not satisfy 3(*b*).

Now, $\sin \lambda a = 0$ gives $\lambda a = n\pi, \quad n = 1, 2, 3$

$\therefore \quad \lambda = n\pi/a, \ n = 1, 2, 3, \ldots$ ... (13)

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin (n\pi x/a)$ ... (14)

Using $\mu = -\lambda^2 = -n^2\pi^2/a^2$, (7) becomes $Y'' - (n^2\pi^2/a^2)\ Y = 0$ ... (15)

whose general solution is $Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}$ ... (16)

Using 3(*a*), (4) gives $0 = X(x)\ Y(0)$ so that $Y(0) = 0$, where we have taken $X(x) \neq 0,$ for otherwise we shall get $u = 0$ which does not satisfy 3(*b*).

But $Y(0) = 0 \quad \Rightarrow \quad Y_n(y) = 0$ ... (17)

Putting $y = 0$ in (16) and using (17), we have

$0 = C_n + D_n$ so that $D_n = -C_n$. Then (6) reduces to

$$Y_n(y) = C_n(e^{n\pi y/a} - e^{-n\pi y/a}) = 2\sinh(n\pi y/a), \quad \text{as} \quad e^{\theta} - e^{-\theta} = 2\sinh\theta. \qquad \ldots (18)$$

$\therefore \quad u_n(x, y) = X_n(x)\ Y_n(y) = E_n \sin (n\pi x/a) \sinh (x\pi y/a)$ ... (19)

are solutions of (1), satisfying (2) and 3(*a*). Here E_n (= 2 B_n C_n) are new arbitrary constants. In order to satisfy conditions 3(*b*), we now consider more general solution given by

i.e.,
$$u(x,y)=\sum_{n=1}^{\infty}u_n(x,y)=\sum_{n=1}^{\infty}E_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \quad \ldots (20)$$

Putting $y = b$ in (20) and using 3(b), we get

$$f(x)=\sum_{n=1}^{\infty}\left(E_n\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a},$$

which is the half range Fourier sine series of $f(x)$ in $(0, a)$. Hence, we have

$$E_n\sinh\frac{n\pi b}{a}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi b}{a}dx \qquad \Rightarrow \qquad E_n=\frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx. \quad \ldots (21)$$

Hence, (20) is the required solution wherein E_n is given by (21).

Ex. 4(*b*) *Find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$. The edges $x = 0$, $x = a$, $y = 0$ are kept at temperature zero while the edge $y = b$ is kept at $100°C$.* **[I.A.S. 2008; Nagpur (B.E.) 2003]**

Sol. Temperature $u(x, y)$ is steady state in two-dimensional plate is governed by the Laplace's equation
$$\partial^2u/\partial x^2+\partial^2u/\partial y^2 \quad \ldots (1)$$

Here
$$u(0, y) = 0,\ u(a, y) = 0, \quad \ldots (2)$$
$$u(x, 0) = 0 \quad \ldots 3(a)$$
and
$$u(x, b) = 100 \quad \ldots 3(b)$$

Now proceed as in Ex. 4(a) upto (21). Here $u(x, b) = f(x) = 100$. Hence, from (21), we get

$$E_n=\frac{2}{a\sinh(n\pi b/a)}\int_0^a 100\sin\frac{n\pi x}{a}dx=\frac{200}{a\sinh(n\pi b/a)}\left[-\frac{\cos(n\pi x/a)}{(n\pi/a)}\right]_0^a$$

$$\therefore \quad E_n=\frac{200}{n\pi}[1-(-1)^n]\operatorname{cosech}\frac{n\pi b}{a}=\begin{cases}0, \text{ if } n=2m,\ m=1,2,3,\ldots\\ \dfrac{400\operatorname{cosech}\{(2m-1)\pi b/a\}}{(2m-1)\pi}, \text{ if } n=2m-1,\ m=1,2,3,\ldots\end{cases}$$

With these values of E_n, (20) reduces to

$$u(x,y)=\sum_{m=1}^{\infty}E_{2m-1}\sin\frac{(2m-1)\pi x}{a}\sinh\frac{(2m-1)\pi y}{a}$$

or,
$$u(x,y)=\frac{400}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin\frac{(2m-1)\pi x}{a}\sinh\frac{(2m-1)\pi y}{a}\operatorname{cosech}\frac{(2m-1)\pi b}{a}$$

Ex. 4(*c*) *A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 10$ and $y = 10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 10) = x(10 - x)$ while the other three faces are kept at $0°C$. Find the steady state temperature in the plate.*

Sol. The steady state tempeature $u(x, y)$ is the solution of Laplace equation

$$\partial^2u/\partial x^2+\partial^2u/\partial y^2=0 \quad \ldots (1)$$

subject to boundary conditions
$$u(0, y) = u(10, y) = 0,\ 0 \le y \le 10 \quad \ldots (2)$$
$$u(x, 0) = 0,\ 0 \le x \le 10 \quad \ldots 3(a)$$
and
$$u(x,10) = 10x - x^2,\ 0 \le x \le 10. \quad \ldots 3(b)$$

Comparing the problem given by (1), (2), 3(*a*) and 3(*b*) with problem given in Ex. 4(*a*), here $a = b = 10$ and $u(x, b) = u(x, 10) = f(x) = x(10 - x)$. Proceed as in Ex. 4(*a*) upto (21). Then,

$$(21) \Rightarrow \qquad E_n = \frac{2}{10 \sinh n\pi} \int_0^{10} (10x - x^2) \sinh \frac{n\pi x}{10} dx$$

$$= \frac{1}{5 \sinh n\pi} \left[(10x - x^2) \left(-\frac{10}{n\pi} \right) \cos \frac{n\pi x}{10} - (10 - 2x) \left(-\frac{100}{n^2 \pi^2} \right) \sin \frac{n\pi x}{10} + (-2) \left(\frac{1000}{n^3 \pi^3} \right) \cos \frac{n\pi x}{10} \right]_0^{10}$$

[Using chain rule of integrating by parts]

$$= \frac{1}{5 \sinh n\pi} \left[-\frac{2000(-1)^n}{n^3 \pi^3} + \frac{2000}{n^3 \pi^3} \right] = \frac{400\{1 - (-1)^n\}}{n^3 \pi^3 \sinh n\pi}$$

$$\therefore \qquad E_n = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, ... \\ \{800 \text{ cosech } (2m-1)\pi\} / (2m-1)^3 \pi^3, \text{ if } n = 2m-1, \ m = 1, 2, 3 \end{cases} \qquad ... (i)$$

For the present problem equation (20) of Ex. 4(*a*) takes the form

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sinh \frac{n\pi x}{10} \sinh \frac{n\pi y}{10} \qquad ... (ii)$$

Putting the value of E_n given by (*i*) in (*ii*), the required temperature is given by

$$u(x, y) = \frac{800}{\pi^3} \sum \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi x}{10} \sinh \frac{(2m-1)\pi y}{10} \text{ cosech } (2m-1)\pi.$$

Ex. 5 (*a*) *Solve the Laplace's equation* $\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0$ *subject to the following boundary conditions :* $u(x, 0) = u(x, b) = 0$ *for* $0 \le x \le a$, $u(0, y) = 0$ *and* $u(a, y) = f(y)$ for $0 \le y \le b$.

[This example is similar to the problem of Art. 2.13B. So we shall follow similar method]

Sol. Given Laplace's equation is $\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0.$ ... (1)

Also given $u(x, 0) = u(x, b) = 0$ for $0 \le x \le a,$ ... (2)

$u(0, y) = 0$ for $0 \le y \le b.$ ... 3(*a*)

and $u(a, y) = f(y)$ for $0 \le y \le b.$ ... 3(*b*)

Let a solution of (1) be of the form $u(x, y) = X(x)\ Y(y).$ ... (4)

Using (4), (1) reduces to

$X''Y + XY'' = 0$ or $(1/X)X'' = -(1/Y)Y''.$ ... (5)

Since the L.H.S. of depends only on x and the R.H.S. depends only on y, each side of (5) must be equal to the same constant, say μ. Then, (5) leads to

$$X'' - \mu X = 0 \qquad ... (6)$$

and $$Y'' - \mu Y = 0 \qquad ... (7)$$

Using (2), (4) gives $X(x)\ Y(0) = 0$ and $X(x)\ Y(b) = 0$

so that $Y(0) = 0$ and $Y(b) = 0$ ... (8)

where we have taken $X(x) \neq 0,$ since otherwise $u = 0$ does not satisfy (3*b*).

We now solve (7) under boundary conditions (8). Three cases arise :

Case I. Let $\mu = 0$. Then, solution of (7) is $Y(y) = Ay + B.$ ... (9)

Using B.C. (8), (9) gives $0 = B$ and $0 = Ab + B$. These give $A = B = 0$ so that $Y(y) \equiv 0$. This leads to $u = 0$ which does not satisfy 3(b). So we reject $\mu = 0$.

Case II. Let $\mu = -\lambda^2$ (negative). Here $\lambda \neq 0$. Then solution of (7) is

$$Y(y) = Ae^{y\lambda} + Be^{-y\lambda}. \quad \text{... (10)}$$

Using B.C. (8), (10) gives $0 = A + B$ and $0 = Ae^{b\lambda} + Be^{-b\lambda}$. ... (11)

(11) $\Rightarrow A = B = 0$ so that $X(x) = 0$. This leads to $u = 0$ which does not satisfy 3(b). So we reject $\mu = -\lambda^2$.

Case III. Let $\mu = \lambda^2$ (positive). Here $\lambda \neq 0$. Solution of (7) is

$$Y(y) = A\cos\lambda y + B\sin\lambda y. \quad \text{...(12)}$$

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda b + B\sin\lambda b$ so that $A = 0$ and $\sin\lambda b = 0$, where we have taken $B \neq 0$, since otherwise we shall get $Y(y) \equiv 0$. This leads to $u = 0$ which does not satisfy 3(b).

Now, $\sin\lambda b = 0 \quad \Rightarrow \quad \lambda b = n\pi \quad \Rightarrow \quad \lambda = n\pi / b, \qquad n = 1, 2, 3, ...$

Hence, non-zer0 solutons $Y_n(y)$ of (7) are given by

$$Y_n(y) = B_n \sin(n\pi y / b), \quad n = 1, 2, 3, ... \quad \text{... (13)}$$

Also, then $\mu = -\lambda^2 = -n^2\pi^2 / b^2$. Hence, (6) reduces to

$$X'' - (n^2\pi^2 / b^2)X = 0, \qquad n = 1, 2, 3, ... \quad \text{... (14)}$$

whose solutions are $X_n(x) = C_n e^{n\pi x/b} + D_n e^{-n\pi x/b}.$ ... (15)

Using 3(a), (4) gives $0 = X(0)\ Y(y)$ so that $X(0) = 0$, where we have taken $Y(y) \neq 0$, since otherwise $u = 0$ which does not satisfy 3(b).

Now $X(0) = 0 \Rightarrow X_n(0) = 0$. Putting $n = 0$ in (15) and using $X_n(0) = 0$, we get

$0 = C_n + D_n$ so that $D_n = -C_n$. Then (15) reduces to

$X_n(x) = C_n(e^{n\pi x/b} - e^{-n\pi x/b}) = 2C_n \sinh(n\pi x/b)$, as $e^{\theta} - e^{-\theta} = 2\sinh\theta$ ... (16)

From (4), (13) and (16), we see that a solution $u_n(x, y)$ of (1) is given by

$$u_n(x, y) = X_n(x)\ Y_n(y) = E_n \sinh(n\pi x/b) \sin(n\pi y/b),\ n = 1, 2, 3, ... \quad \text{... (17)}$$

where E_n (= $B_n\ C_n$) are new arbitrary constants. In order to obtain a solution also satisfying B.C. (3b), we consider more general solution

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} E_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b} \quad \text{... (18)}$$

Putting $x = a$ in (18) and using B.C. 3(b), we have $f(y) = \sum_{n=1}^{\infty}\left(E_n \sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b},$

which is the half range Fourier sine series of $f(y)$ in $(0, b)$. Hence

$$E_n \sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy \qquad \text{or} \qquad E_n = \frac{2}{b\sinh(n\pi a/b)}\int_0^b f(y)\sin\frac{n\pi y}{b}. \quad \text{... (19)}$$

Hence, (18) is the required solution wherein E_n is given by (19).

Ex. 5 (*b*) By separating the variables show that $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$ *has solutions of the form* $A\ exp\ (\pm nx \pm iny)$, *where A and n are constants. Deduce that the functions of the form* $V(x, y) = \sum_r A_r \sin(r\pi y/a)e^{-(r\pi x/a)}$ $x \geq 0, 0 \leq y \leq \infty$ *where* A_r *are constants, are plane harmonic functions satisfying the conditions* $V(x, 0) = V(x, a) = 0$ *and* $V(x, y) \to 0$ as $x \to \infty$.

[Himanchal 2000; Rajasthan 2000; Meerut 2005]

OR

Find $V(x, y)$ *such that* $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$ *satisfies the conditions* (*i*) $V \to 0$ as $x \to \infty$ (*ii*) $V(x, 0) = V(x, a) = 0$.

Sol. Given that $$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0 \quad \text{... (1)}$$

First part. Suppose that (1) has a solution of the form $V(x, y) = X(x)\ Y(y)$ ... (2)

Substituting this value of V in (1), we have

$$X''Y + XY'' = 0 \qquad \text{or} \qquad (1/X)X'' = -(1/Y)Y''. \quad \text{... (3)}$$

Since the L.H.S. of (3) depends only on x whereas the R.H.S. depends only on y, the two sides of (3) must be equal to the same constant, say n^2. Then (3) leads to the following two equations. $X'' - n^2X = 0$ and $Y'' + n^2Y = 0$

whose solutions are $X = Ae^{nx} + Be^{-nx}$ and $Y = C\cos ny + D\sin ny$.

Using (2), we see that (1) has solutions of the form

$$V(x, y) = (Ae^{nx} + Be^{-nx})\ (C\cos ny + D\sin ny), \quad \text{... (4)}$$

which may be put in compact form $V(x, y) = A\exp(\pm nx \pm iny)$. ... (5)

Second part. We shall now solve (1) under the given boundary conditions:

$$V(x, y) \to 0 \text{ as } x \to \infty \quad \text{... (6)}$$

and $$V(x, 0) = V(x, a) = 0, \quad \text{... (7)}$$

B.C. (6) is satisfied by (4) by taking $A = 0$. Then (4) may be re-written as (writing $BC = E$ and $BD = F$). $V(x, y) = (E\cos ny + F\sin ny)\ e^{-nx}$. ... (8)

Putting $y = 0$ in (8) and using B.C. (7), $0 = Ee^{-nx}$ for all x so that $E = 0$.

Next, putting $y = a$ in (8) and using B.C. (7), we have

$$0 = F\sin na\ e^{-nx} \text{ for all } x \quad \Rightarrow \quad \sin na = 0,$$

where we taken $F \neq 0$, for otherwise $E = F = 0$ would lead to trivial solution for $V(x, y)$.

Now, $\sin na = 0 \Rightarrow na = r\pi \Rightarrow n = r\pi/a$, $r = 1, 2, 3, ...$

Hence, non-zero solutions $V_r(x, y)$ of (1) are given by (8), in the form

$V_r(x, y) = A_r \sin(r\pi y/a)e^{-(r\pi x/a)}$, $r = 1, 2, 3, ...$; where A_r are constants.

Hence, the most general solution of (1) is of the form

$$V(x, y) = \Sigma V_r(x, y) = \Sigma A_r(r\pi y/a)e^{-(r\pi x/a)},$$

2.13 D. The Neumann problem in a rectangle

The Neumann problem in a rectangle is defined as follows:

Laplace's equation: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0,\ 0 \leq x \leq a,\ 0 \leq y \leq b$ ... (1)

Boundary conditions: $u_x(0, y) = u_x(a, y) = 0,\ 0 \leq y \leq b$ ... (2)

$u_y(x, 0) = 0,\ 0 \leq x \leq a$ ... 3(*a*)

and $$u_y(x, b) = f(x), \qquad 0 \le x \le a \qquad \text{... } 3(b)$$

We shall solve (1) subject to the boundary conditions (2), 3(a) and 3(b).

Suppose (1) has a solution of the form $u(x, y) = X(x)\, Y(y)$ ... (4)

Substituting this value of u in (1), we have

$$X''Y + XY'' = 0 \quad \text{or} \quad X''/X = -Y''/Y \qquad \text{... (5)}$$

Since x and y are independent, each side of (5) must be equal to the same constant, say μ.

Then (5) gives $X'' - \mu X = 0$ ... (6)

and $Y'' + \mu Y = 0$ ... (7)

From (4), $\partial u / \partial x = X'(x)Y(y)$ ... (8)

Putting $x = 0$ and $x = a$ by turn in (8) and using (2), we have

$0 = X'(0)Y(y)$ and $X'(a)Y(y) = 0$

so that $X'(0) = 0$ and $X'(a) = 0,$ ... (9)

where we have taken $Y(y) \neq 0,$ since otherwise $u \equiv 0$ which does not satisfy 3(b).

We now solve (6) satisfying B.C. (9). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (10)

Differentiative (10) w.r.t. 'x', we get $X'(x) = A$ ... (11)

Putting $x = 0$ and $x = a$ in (11) and using (9), we get $A = 0.$

Hence (10) reduces to $X(x) = B$ ... (12)

Case II. Let $\mu = \lambda^2, \lambda \neq 0.$ Then solution of (6) is

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ so that $X'(x) = \lambda(Ae^{\lambda x} - Be^{-\lambda x})$ ... (13)

Using B.C. (9), (13) gives $0 = \lambda(A - B)$ and $0 = \lambda(Ae^{\lambda a} - Be^{-\lambda a})$

These give $A = B = 0$ so that $X(x) = 0$. This leads to $u \equiv 0,$ which does not satisfy 3(b).

Case III. Let $\mu = -\lambda^2, \lambda \neq 0.$ Solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ... (14)

Fron (14), $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$ ... (15)

Using B.C. (9), (14) $\Rightarrow$ $0 = B\lambda$ and $0 = -A\lambda\sin\lambda a + B\lambda\cos\lambda a$ ... (16)

Since $\lambda \neq 0$ and $A \neq 0$ for otherwise $u(x, y)$ would be a trivial solution, therefore

(16) $\Rightarrow B = 0$ and $\sin\lambda a = 0 \Rightarrow B = 0$ and $\lambda a = n\pi, \quad n = 1, 2, 3, \ldots.$

Thus, $B = 0$ and $\lambda = n\pi / a,$ where $n = 1, 2, 3$ ... (17)

Hence non-zero solution of (14) are given by

$$X(x) = A\cos(n\pi x / a), \qquad n = 1, 2, 3 \qquad \text{... (18)}$$

With $n = 0$, (18) reduces to $X(x) = A$ = constant. Also, $n = 0 \Rightarrow \lambda = 0.$ Hence we can include non-zero solution (12) in (18) by taking $n = 0, 1, 2, 3, \ldots$. Thus, all non-zero solutions of (6) are

$$X_n(x) = A_n \cos(n\pi x / a), \qquad n = 0, 1, 2, 3, \ldots \qquad \text{... (19)}$$

Now, $\mu = -\lambda^2 = -(n^2\pi^2 / a^2),$ $n = 0, 1, 2, 3, \ldots$ Hence (7) becomes

$$Y'' - (n^2\pi^2/a^2)Y = 0, \text{ whose general solutio\n is}$$

$Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}$, C_n and D_n being arbitrary constants ... (20)

From (4), $\partial u/\partial y = u_y(x, y) = X(x)Y'(y)$ ... (21)

Putting $y = 0$ in (21) and using B.C. 3(a), we get $0 = X(x)Y'(0)$. Hence $Y'(0) = 0$

($\because$ $X(x) \neq 0$ for otherwise $u(x, y)$ would be a trivial solution.)

Now, $Y'(0) = 0 \quad \Rightarrow \quad Y_n'(0) = 0$ ... (22)

From (20), $Y_n'(y) = C_n(n\pi/a)e^{n\pi y/a} - D_n(n\pi/a)e^{-n\pi y/a}$ ... (23)

Putting $y = 0$ in (23) and using (22), we get $0 = (n\pi/a)\times(C_n - D_n)$ so that $D_n = C_n$

With this value of D_n, (20) reduces to $Y_n(y) = C_n(e^{n\pi y/a} + e^{-n\pi y/a})$

or $Y_n(y) = 2C_n \cosh(n\pi y/a), \quad n = 0, 1, 2, 3, \ldots$ ... (24)

From (19) and (24), we see that

$$u_n(x, y) = X_n(x)Y_n(y) = E_n \cos(n\pi x/a)\cosh(n\pi y/a), \ n = 0, 1, 2, \ldots$$

are solution of (1) satisfying (2) and 3(a). Here E_n (= A_nC_n) are new arbitrary constants. In order to obtain a solution also satisfying 3(b), we obtain the following non-trivial solution after using the superposition principle.

$$U(x, y) = \sum_{n=0}^{\infty} U_n(x, y) = \sum_{n=0}^{\infty} E_n \cos(n\pi x/a)\cosh(n\pi y/a) \quad \ldots (25)$$

From (25), $\partial u/\partial y = u_y(x, y) = \sum_{n=0}^{\infty} (n\pi E_n/a)\cos(n\pi x/a)\sinh(n\pi y/a)$ ... (26)

Putting $y = b$ in (26) and using B.C. 3(b), we have

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi E_n}{a}\sinh\frac{n\pi b}{a}\right)\cos\frac{n\pi x}{a}, \quad 0 \le x \le a,$$

which is Fourier cosine series and hence we have

$$\therefore \quad \frac{n\pi E_n}{a}\sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

Thus, $E_n = \dfrac{2}{n\pi\sinh(n\pi b/a)}\displaystyle\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx, \quad n = 0, 1, 2, 3\ldots$ ... (27)

Hence the required solution is given by (25) wherein E_n are given by (27)

2.13 E. 'Mixed problems' in a rectangle

Ex. 1. *Evaluate the steady temperature in a rectangular plate of length a and width b, the sides of which are kept at temperature zero, the lower end is kept at temperature f(x) and the upper edge is kept insulated.* **[Meerut 1999, 2000, 06]**

Sol. Temperature $u(x, y)$ in steady state in two-dimensional plate is governed by the Laplace's equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ ... (1)

Here boundary conditions are: $u(0, y) = 0, u(a, y) = 0,$ ... (2)

$$u_y(x, b) = 0, \quad i.e., \quad (\partial u / \partial y)_{y=b} = 0 \quad \ldots 3(a)$$

and $$u(x, 0) = f(x). \quad \ldots 3(b)$$

Note the form of condition 3(a) on the upper edge $y = b$ which is being kept insulated.

Suppose (1) has a solution of the form $u(x, y) = X(x)\, Y(y)$ ... (4)

Substituting this values of u in (1), we get $X''Y + XY'' = 0$ or $X''/X = -(Y''/Y)$ ... (5)

Since x and t are independent, each side of (5) must be equal to the same constant, say μ. Then (5) gives

$$X'' - \mu X = 0 \quad \ldots (6)$$

$$Y'' + \mu Y = 0 \quad \ldots (7)$$

Using (2), (4) gives $X(0)\, Y(y) = 0$ and $X(a)\, Y(y) = 0$ giving $X(0) = 0$ and $X(a) = 0$ ... (8)

where we have taken $Y(y) \not\equiv 0,$ since otherwise $u \equiv 0$ which does not satisfy 3(b).

We now solve (6) under B.C. (8). Three cases arise:

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (9)

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ so that $X(x) \equiv 0.$ This leads to $u \equiv 0$ which does not satisfy 3(b). So we reject $\mu = 0$.

Case II. Let $\mu = \lambda^2, \lambda \neq 0.$ Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ... (10)

Using B.C. (8), (10) gives. $0 = A + B$ and $0 = Ae^{a\lambda} + Be^{-a\lambda}$ ... (11)

Solving (11), $A = B = 0$ so that $X(x) \equiv 0$ and hence $u \equiv 0$ which does not satisfy 3(b). So we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0.$ Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ... (12)

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$ so that $A = 0$ and $\sin\lambda a = 0,$

where we have taken $B \neq 0,$ since otherwise $X(x) \not\equiv 0$ and $u \equiv 0$ which does not satisfy 3(b).

Now, $\sin\lambda a = 0$ gives $\lambda a = n\pi$ so that $\lambda = n\pi/a, \quad n = 1, 2, 3, \ldots$ ... (13)

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x/a)$ ... (14)

Using $\mu = -\lambda^2 = -n^2\pi^2/a^2,$ (7) becomes $Y'' - (n^2\pi^2/a^2)Y = 0$ ... (15)

whose general solution is $Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}.$ ... (16)

From (4), $(\partial u/\partial y) = X(x)Y'(y)$ so that $(\partial u/\partial y)_{y=b} = u_y(x, b) = X(x)Y'(b)$ ... (17)

Using 3(a), (17) gives $X(x)Y'(b) = 0$ so that $Y'(b) = 0$ and hence $Y_n'(b) = 0$ ... (18)

Differentiating w.r.t, 'y', (16) gives $Y_n'(y) = (n\pi/a)\, C_n e^{n\pi y/a} - (n\pi/a) D_n e^{-n\pi y/a}.$ ... (19)

Putting $y = b$ in (19) and using (18), we have

$$0 = (n\pi/a)\,(C_n e^{n\pi b/a} - D_n e^{-n\pi b/a}) \quad \text{so that} \quad D_n = (C_n e^{n\pi b/a})/e^{-n\pi b/a}.$$

With this value of D_n, (16) reduces to

$$Y_n(y) = C_n(e^{n\pi y/a}e^{-n\pi b/a} + e^{-n\pi y/a}e^{n\pi b/a})/e^{-n\pi b/a} \quad \text{or} \quad Y_n(y) = C_n e^{n\pi b/a}\{e^{n\pi(b-y)/a} + e^{-n\pi(b-y)/a}\}$$

or $$Y_n(y) = 2C_n e^{n\pi b/a}\cosh\{n\pi(b-y)/a\}, \quad \text{as} \quad e^{\theta} + e^{-\theta} = 2\cosh\theta. \qquad \ldots (20)$$

$$\therefore \quad u_n(x,y) = X_n(x)Y_n(y) = E_n \sin(n\pi x/a)\cosh\{n\pi(b-y)/a\}$$

are solutions of (1) satisfying 3(*a*). Here E_n (= $2B_n\, C_n$) are new arbitrary constants. In order to obtain a solution also satisfying 3(*b*), we consider more general solution, namely,

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)\cosh\{n\pi(b-y)/a\}. \qquad \ldots (21)$$

Putting $y = 0$ in (21) and using 3(*b*), we have

$$f(x) = \sum_{n=1}^{\infty}\{E_n \cosh(n\pi b/a\}\sin(n\pi x/a),$$

which is the half range Fourier since series of $f(x)$ is (0, a). So we get

$$E_n \cosh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \qquad \text{or} \qquad E_n = \frac{2}{a\cosh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \quad \ldots (22)$$

The required temperature is given by (21) wherein E_n is given by (22).

Ex. 2. *Find the steady state temperature distribution in a rectangular plate bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$ if the edge $y = 0$ is insulated, the edges $x = 0$ and $x = a$ are kept at 0°C and edge $y = b$ is kept at temperature $f(x)$.*

Sol. The required temperature $u(x, y)$ in steady state in two dimensional plate is governed by the Laplace equation. $$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0. \qquad \ldots (1)$$

Given that $$u(0, y) = u(a, y) = 0 \text{ for } 0 \le y \le b. \qquad \ldots (2)$$

$$(\partial u/\partial y)_{y=0} = 0\text{, as the edge } y = 0 \text{ is insulated for } 0 \le x \le a. \qquad \ldots 3(a)$$

and $$u(x, b) = f(x) \text{ for } 0 \le x \le a. \qquad \ldots 3(b)$$

Let (1) has a solution of the form $$u(x, y) = X(x)\, Y(y). \qquad \ldots (4)$$

Using (4), (1) gives $X''Y + XY'' = 0$ or $(1/X)X'' = -(1/Y)Y''$. $\ldots (5)$

Since the L.H.S. of (5) depends only on x and the R.H.S. depends only on y, both sides of (5) must be equal to the same constant, say μ. Then (5) gives $X'' - \mu X = 0$ $\ldots (6)$

$$Y'' + \mu Y = 0. \qquad \ldots (7)$$

Using (2), (4) gives $X(0)\, Y(y) = 0$ and $X(a)\, Y(y) = 0$ so that

$X(0) = 0$ and $X(a) = 0$ $\ldots (8)$

where we have taken $Y(y) \neq 0$, since otherwise $u \equiv 0$ which does not satisfy 3(*b*).

We now solve (6) under B.C. (8). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is $$X(x) = Ax + B. \qquad \ldots (9)$$

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ so that $X(x) = 0$. This leads to $u \equiv 0$ which does not satisfy 3(*b*). So reject $\mu = 0$.

Case II. Let $\mu = \lambda^2$ where $\lambda \neq 0$. Solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$. $\ldots (10)$

Using B.C. (8), (10) gives $0 = A + B$ and $0 = Ae^{\lambda a} + Be^{-\lambda a}$. $\ldots (11)$

Solving (11), we get $A = B = 0$ so that $X(x) \equiv 0$. This leads to $u \equiv 0$ which does not satisfy (3*b*). so we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x, \ A \text{ and } B \text{ being arbitrary constants} \quad \ldots (12)$$

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$ so that A = 0 and $\sin\lambda a = 0$, where we have taken $B \neq 0$, since otherwsie we shall get $X(x) \equiv 0$ and hence $u \equiv 0$ which does not satisfy (3*b*).

Now, $\sin\lambda a = 0 \quad \Rightarrow \quad \lambda a = n\pi \quad \Rightarrow \quad \lambda = n\pi/a, \quad n = 1, 2, 3, \ldots$ ... (13)

Hence, non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x/a)$. ... (14)

Then $\mu = -\lambda^2 = -(n^2\pi^2/a^2)$ and hence (7) reduces to $Y'' - (n^2\pi^2/a^2)Y = 0$. ... (15)

whose general solution is $Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}$. ... (16)

From (4), $\partial u/\partial y = X(x)Y'(y)$ so that $(\partial u/\partial y)_{y=0} = X(x)Y'(0)$ ... (17)

Using 3(*a*), (17) gives $X(x)Y'(0) = 0$ so that $Y'(0) = 0$,

where we have taken $X(x) \neq 0$, since otherwise $u \equiv 0$ which does not satisfy 3(*b*).

Now, $Y'(0) = 0 \quad \Rightarrow \quad Y_n'(0) = 0.$ ... (18)

Differentiating both sides of (16) partially, w.r.t, '*y*', we get

$$Y_n'(y) = (n\pi/a)\, C_n e^{n\pi y/a} - (n\pi/a)\, D_n e^{-n\pi y/a}. \quad \ldots (19)$$

Putting $y = 0$ in (19) and using (18), we have

$0 = (n\pi/a)\times(C_n - D_n)$ so that $D_n = C_n$. Then (16) reduces to

$$Y_n(y) = C_n(e^{n\pi y/a} + e^{-n\pi y/a}) = 2C_n\cosh(n\pi y/a), \quad \text{as} \quad e^{\theta} + e^{-\theta} = 2\cosh\theta. \quad \ldots (20)$$

$$\therefore \quad u_n(x, y) = X_n(x)Y_n(n) = E_n \sin(n\pi x/a)\cosh(n\pi y/a)$$

are solutions of (1) satisfying (2) and 3(*a*). Here $E_n = (2B_nC_n)$ are new arbitrary constants. In order to obtain a solution also satisfying 3(*b*), we consider more general solution, namely,

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}. \quad \ldots (21)$$

Putting $y = b$ in (21) and using (3*b*), we have

$$f(x) = \sum_{n=1}^{\infty}\left(E_n\cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a},$$

which is the half range Fourier since series of $f(x)$ in (0, *a*). Hence, we get

$$E_n\cosh\frac{n\pi b}{a} = \frac{2}{a}\int_b^a f(x)\sin\frac{n\pi x}{a}dx \quad \text{or} \quad E_n = \frac{2}{a\cosh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx. \quad \ldots (22)$$

The required temperature is given by (21) wherein E_n is given by (22).

Ex. 3. *A rectangular metal plate is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. The three sides $x = 0$, $x = a$ and $y = b$ are insulated and the side $y = 0$ is kept at temperature $u_0 \cos(\pi x / a)$. Find the steady state temperature at any point of the plate.*

Sol. The steady state temperature $u(x, y)$ is the solution of the Laplace equation

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0, \qquad \ldots (1)$$

Since the sides $x = 0$, $x = a$ and $y = b$ are insulated, we have

$$(\partial u / \partial x)_{x=0} = 0, \qquad (\partial u / \partial x)_{x=a} = 0. \qquad \ldots (2)$$

$$(\partial u / \partial y)_{y=b} = 0 \qquad \ldots 3(a)$$

Also, given that $u(x, a) = u_0 \cos(\pi x / a).$ $\ldots 3(b)$

Suppose (1) has a solution of the form $u(x, y) = X(x)\, Y(y).$ $\ldots (4)$

Substituting (4) in (1), we get

$$X'' Y + X Y'' = 0 \quad \text{or} \quad (1/X) X'' = -(1/Y) Y'' \qquad \ldots (5)$$

Since the L.H.S. of (5) depends only on x and the R.H.S. depends only on y, both sides of (5) must be equal to the same constant, say μ.

Then, (5) yields $X'' - \mu X = 0$ $\ldots (6)$

and $Y'' + \mu Y = 0.$ $\ldots (7)$

From (4), $\partial u / \partial x = X'(x) Y(y).$ $\ldots (8)$

Putting $x = 0$ and $x = a$ by turn in (8) and using (2), we have

$$0 = X'(0) Y(y) \quad \text{and} \quad X'(a) Y(y)$$

giving $X'(0) = 0$ and $X'(a) = 0,$ $\ldots (9)$

where we have taken $Y(y) \neq 0,$ since otherwise $u \equiv 0$ which does not satisfy 3(b). We now solve (6) and B.C. (9). Three cases arise:

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B.$ $\ldots (10)$

Differentiating (10) w.r.t. 'x', we get $X'(x) = A.$ $\ldots (11)$

Putting $x = 0$ and $x = a$ in (11) by turn and using (9), we get $A = 0,$

With $A = 0$, (10) reduces to $X(x) = B.$ $\ldots (12)$

Case II. Let $\mu = \lambda^2,$ where $\lambda \neq 0.$ Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

so that $X'(x) = \lambda(Ae^{\lambda x} - Be^{-\lambda x}).$ $\ldots (13)$

Using B.C. (9), (13) gives $0 = \lambda(A - B)$ and $0 = \lambda(Ae^{\lambda a} - Be^{-\lambda a})$

These give $A = B = 0$ so that $X(x) = 0$. This leads to $u \equiv 0$, which does not satisfy 3(b).

Case III. Let $\mu = -\lambda^2,$ $\lambda \neq 0.$ Solution of (6) is $X(x) = A \cos \lambda x + B \sin \lambda x$ $\ldots (14)$

From (14), we have $X'(x) = -A\lambda \sin \lambda x + B\lambda \cos \lambda x.$ $\ldots (15)$

Using B.C. (9), (14) gives $0 = B\lambda$ and $0 = -A\lambda \sin \lambda a + B\lambda \cos \lambda a.$ $\ldots (16)$

Since $\lambda \neq 0$ and $A \neq 0$ for otherwise $u(x, y)$ would be a trivial solution, therefore

(16) $\Rightarrow B = 0$ and $\sin \lambda a = 0 \Rightarrow B = 0$ and $\lambda a = n\pi$, $n = 1, 2, 3. \ldots$

Thus, $B = 0$ and $\lambda = n\pi / a$ for $n = 1, 2, 3, \ldots$ ... (17)

Hence, non-zero solutions of (14) are $X(x) = A\cos(n\pi x / a),\ n = 1, 2, 3, \ldots$ ... (18)

With $n = 0$, (18) reduces to $X(x) = A$ = constant. Also $n = 0 \Rightarrow \lambda = 0.$

So we can include non-zero solution (12) in (18) by taking $n = 0, 1, 2, 3, \ldots$ Hence, all non-zero solution of (6) are given by

$$X_n(x) = A_n \cos(n\pi x/a),\ n = 0, 1, 2, 3, \ldots \quad \ldots (19)$$

Now, $\mu = -\lambda^2 = -(n^2\pi^2 / a^2),\ n = 0, 1, 2, 3, \ldots.$ Hence (7) becomes

$$Y'' - (n^2\pi^2 / a^2)Y = 0, \text{ whose general solution is}$$

$$Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a} \quad \ldots (20)$$

From (4), $\partial u / \partial y = X(x)Y'(y)$ ... (21)

Putting $y = b$ in (21) and using B.C. 3(a) we get $0 = X(x)Y'(b)$. Hence $Y'(b) = 0.$

($\because$ $X(x) \neq 0$ for otherwise $u(x, y)$ would be a trivial solution)

Now, $Y'(b) = 0 \Rightarrow Y_n'(b) = 0$ ... (22)

From (20), $Y_n'(y) = C_n(n\pi / a)e^{n\pi y/a} - D_n(n\pi / a)e^{-n\pi y/a}.$ ... (23)

Putting $y = b$ in (23) and using (22), we have

$$0 = (n\pi / a) \times (C_n e^{n\pi b/a} - D_n e^{-n\pi b/a}) \Rightarrow D_n = (C_n e^{n\pi b/a}) / e^{-n\pi b/a}.$$

With this value of D_n, (20) reduces to

$$Y_n(y) = C_n(e^{n\pi y/a} e^{-n\pi b/a} + e^{-n\pi y/a} e^{n\pi b/a}) / e^{-n\pi b/a} = C_n e^{n\pi b/a} \left\{ e^{n\pi(b-y)/a} + e^{-n\pi(b-y)/a} \right\}$$

$$\therefore \quad Y_n(y) = 2C_n e^{n\pi b/a} \cosh\left\{n\pi(b-y)/a\right\}, \quad \text{as } e^{\theta} + e^{-\theta} = 2\cosh\theta \quad \ldots (24)$$

where $n = 0, 1, 2, 3, \ldots$ From (19) and (24), we see that

$$u_n(x, y) = X_n(x)Y_n(y) = E_n \cos(n\pi x / a)\cosh\left\{n\pi(b-y)/a\right\}$$

are solutions of (1) satisfying (2) and 3(a). Here $E_n (= 2A_nC_n e^{n\pi b/a})$ are new arbitrary constants. In order to obtain a solution also satisfying 3(b), we consider the most general solution, namely,

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) = \sum_{n=0}^{\infty} E_n \cos(n\pi x / a)\cosh\left\{n\pi(b-y)/a\right\} \quad \ldots (25)$$

Putting $y = a$ in (25) and using B.C. 3(b), we get

$$u_0 \cos(\pi x / a) = \sum_{n=0}^{\infty} E_n \cos(n\pi x / a)\cosh\left\{n\pi(b-a)/a\right\}$$

or $u_0 \cos(\pi x / a) = E_0 + E_1 \cos(\pi x / a)\cosh\{\pi(b-a)/a\} + E_2 \cos(2\pi x / a)\cosh\{2\pi(b-a)/a\}$

$$+ E_3 \cos(3\pi x / a)\cosh\{3\pi(b-a)/a\} + \ldots \quad \ldots (26)$$

Equating the coefficients of like terms on both sides of (26), we get

$E_0 = 0$, $E_1 \cosh \{\pi (b - a)/a\} = u_0$ and $E_n = 0$, for $n \geq 2$

Putting the above values in (25), the required temperature is

$$u(x, y) = u_0 \operatorname{sech}\{(b-a)\pi/a\} \cos(\pi x/a) \cosh\{(b-y)\pi/a\}$$

EXERCISE 2(D)

1. An infinitly long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state. **[Meerut 2007]**

Ans. $u(x, y) = \dfrac{4u_0}{\pi} \sum_{m=1}^{\infty} \dfrac{1}{2m-1} e^{-(2m-1)y} \sin(2m-1)x$

2. Find the steady temperature distribution $u(x, y)$ in the uniform unit square $0 \leq x \leq 1$; $0 \leq y \leq 1$ when the edge $y = 1$ is kept at temperature u_0 and the remaining three sides are kept at temperature zero.

Ans. $u(x, y) = \dfrac{4u_0}{\pi} \sum_{m=1}^{\infty} \dfrac{1}{2m-1} \cdot \dfrac{\sin(2m-1)\pi x \sinh(2m-1)\pi y}{\sinh(2m-1)\pi}$

3. Find the steady temperature distribution $u(x, y)$ in the uniform unit square $0 \leq x \leq 1; 0 \leq y \leq 1$ when the edge $y = 0$, $0 \leq x \leq 1$, is maintained at the temperature $x(1 - x)$, the other three edges being thermally insulated, so that $(\partial u/\partial x) = 0$ along them.

Ans. $u(x, y) = \dfrac{1}{6} - \dfrac{1}{\pi^2} \sum_{n=1}^{\infty} \dfrac{\cos 2n\pi x \cosh 2n\pi y}{n^2 \cosh 2ny}$

4. Solve Laplace's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$ subject to the following boundary conditions : $V = 0$ when y is very large for all x; $V = 0$ when $x = 0$ and 1, for all y; and $V = x(1 - x)$ when $y = 0$, for $0 < x < 1$.

Ans. $V(x, y) = \dfrac{8}{\pi^3} \sum_{m=1}^{\infty} \dfrac{1}{(2m-1)^3} e^{-(2m-1)\pi y} \sin(2m-1)\pi x$

5. A thin uniformly long metal plate is in the form of area enclosed by the lines $y = 0$, $y = b$ and $x = 0$ and extending to infinity on the positive side of the *x-axis.* The temperature is zero along the edges $y = 0$, $y = b$ and at infinity. In the steady state, the temperature u at a point (x, y) of the plate satisfies $\partial^2 u/\partial x^2 + \partial^2 u/\partial x^2 = 0$.

If the edge $x = 0$ is kept at a constant temperature u_0, determine the temperature distribution.

Ans. $u(x, y) = \dfrac{4u_0}{\pi} \sum_{m=1}^{\infty} \dfrac{1}{2m-1} e^{-(2m-1)\pi/l} \sin \dfrac{(2m-1)\pi y}{l}$

6. If $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ and $z = \sin x$ at $y = 0$, for all values of x and $z = 0$ at $y = \infty$, for all values of x; show that $z = e^{-y} \sin x$.

7. Heat flows in a semi-infinite rectangular plate, the end $x = 0$ being kept at temperature θ_0 and the long edges $y = 0$ and $y = a$ at zero temperature. Prove that temperature at any point (x, y) is

$$\frac{4\theta_0}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin \frac{(2m+1)\pi y}{a} e^{-[(2m+1)\pi x/a]}$$

8. Find the steady temperature distribution in a thin rectangular plate bounded by the lines $x = 0$, $x = l$, $y = 0$, $y = h$, assuming that heat cannot escape from either surface; the edges $x = 0$, $x = l$, $y = 0$ are kept at temperature zero while the edge $y = h$ is kept at temperature $f(x)$.

Ans. $u(x, y) = \sum_{n=1}^{\infty} E_n \sinh(n\pi y/l)\sin(n\pi/l)$, where $E_n = (2/l)\,\text{cosech}\,(n\pi h/l)\int_0^h f(x)\sin(n\pi x/l)\,dx$

9. Solve the differential equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ which satisfies the conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, l) = \sin(n\pi x/l)$. **(Kanpur 2011)**

Ans. $u(x, y) = \sin(n\pi x/l)\sinh(n\pi y/l)\,\text{cosech}\,(n\pi a/l)$

10. A long rectangular plate of width a with insulated surfaces has its temperature u equal to zero on both the long sides and one of the short sides so that $u(0, y) = u(a, y) = u(x, \infty) = 0$, $u(x, 0) = kx$. Show that the steady-state temperature within the plate is given by

$$u(x, y) = \frac{2ak}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n\pi y/a}\sin\frac{n\pi x}{a}.$$

11. Solve $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ for $0 < x < \pi, 0 < y < \pi$, with conditions $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

Ans. $u = -\frac{8}{\pi}\sum_{n=1,3,5..}^{\infty}\frac{\sin nx\,\sinh n(\pi - y)}{n(n^2-4)\sinh n\pi}$

12. The edge of breadth b of an infinitely long rectangular conducting strip is maintained at constant temperature T_0, the remaining edges being maintained at zero temperature. Show that the steady temperature distribution in the strip is given by

$$T = \frac{4T_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left\{(2n+1)\frac{\pi y}{b}\right\}e^{-[(2n+1)\,\pi x/b]}$$

13. Discuss briefly the boundary value problem of Laplace's equation associated with the names of Dirichiet's and Neumann. **[I.A.S. 1985]**

14. Solve $\partial^2 u/\partial x^2 + \partial^2 + \partial^2 u/\partial y^2 = 0$ for $0 < x < \pi$, $0 < y < A$, $u(0, y) = u(\pi, y) = u(x, A) = 0$ and $u(x, 0) = f(x)$ where $f(x)$ is continuous, $f(0) = f(\pi)$ and $\int_0^{\pi}[f(x)]^2\,dx$ is finite. **[I.A.S. 1982]**

15. Find the function $f(x, y)$ which satisfies the Laplace's equation inside the rectangle $0 \le x \le \pi$, $0 \le y \le a$ subject to the boundary conditions $f(0, y) = f(\pi, y) = 0$, $0 < y < a$; $f(x, a) = 0$ for $0 \le x \le \pi$ and $f(x, 0) = x(\pi - x)$ for $0 \le x \le \pi$. **[I.A.S. 1980]**

16. Solve $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$ given that (*i*) $V = 0$ when $x = 0$ and $x = c$ (*ii*) $V \to \infty$ as $y \to \infty$ (*iii*) $V = V_0$, when $y = 1$. **Ans.** $V = \frac{4V_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}e^{\{(2n-1)\,(1-y)\}/c}\sin\frac{(2n-1)\pi x}{c}$

17. A square plate has its faces and its edges $x = 0$ and $x = \pi$ $(0 < y < \pi)$ insulated. Its edges $y = 0$ and $y = \pi$ are kept at temperatures zero and $f(x)$ respectively. Derive formula for its steady temperature

$$u(x, y) = \frac{1}{2\pi}a_0 y + \sum_{n=1}^{\infty}a_n\frac{\sinh my}{\sinh m\pi}\cos nx, \quad \text{where} \quad a_n = \frac{2}{a}\int_0^{\pi}f(x)\cos nx\,dx,\ n = 0, 1, 2...$$

PART VIII: SOLUTION OF THREE DIMENSIONAL LAPLACE'S EQUATION

2.14 A. *General solution of three-dimensional Laplace's equation in rectangular cartesian coordinates (x, y, z) by the method of separation of variables.*

Sol. We are to solve three-dimensional Laplace's equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0 \qquad ...(1)$$

Suppose (1) has solutions of the form $u(x, y, z) = X(x)\, Y(y)\, Z(z),$ ... (2)

where X, Y and Z are functions of x, y and z respectively,

Substituting this value of u in (1), we have $X''YZ + XY''Z + XYZ'' = 0$

or $$X''/X + Y''/Y = -(Z''/Z) \qquad ... (3)$$

Since x, y and z are independent variables, (3) can be true if each term on each side is equal to a constant. The following three cases arise:

Case I. When each term in (3) is zero. Then we get

$$X'' = 0, \qquad Y'' = 0, \qquad Z'' = 0$$

giving $X = Ax + B, \qquad Y = Cy + D, \qquad Z = Ez + F.$

Hence a solution of (8) is of the form

$$u(x, y, z) = (Ax + B)\,(Cy + D)\,(Ez + F). \qquad ... (4)$$

Case II. Let $X''/X = \lambda_1^2$, $Y''/Y = \lambda_2^2$ and $\lambda_1^2 + \lambda_2^2 = \lambda^2$. Then (3) gives

$$X'' - \lambda_1^2 X = 0, \qquad Y'' - \lambda_2^2 Y = 0, \qquad \text{and} \qquad Z'' + \lambda^2 Z = 0.$$

$$\therefore\ X = Ae^{x\lambda_1} + Be^{-x\lambda_1}; \qquad Y = Ce^{y\lambda_2} + De^{-y\lambda_2} \qquad \text{and} \qquad Z = E\cos\lambda z + F\sin\lambda z.$$

Hence a solution of (1) is of the form

$$u(x, y, z) = (Ae^{x\lambda_1} + Be^{-x\lambda_1})\,(Ce^{y\lambda_2} + De^{-y\lambda_2})\,(E\cos\lambda z + F\sin\lambda z)$$

A more general solution is given by

$$u(x, y, z) = \sum_{\lambda_1}\sum_{\lambda_2}(Ae^{x\lambda_1} + Be^{-x\lambda_1})\,(Ce^{y\lambda_2} + De^{-y\lambda_2})\,(E\cos\lambda z + F\sin\lambda z)$$

Case III. Let $X''/X = -\lambda_1^2$, $Y''/Y = -\lambda_2^2$ and $(\lambda_1^2 + \lambda_2^2) = \lambda^2$. Then (5) gives

$$X'' + \lambda_1^2 X = 0, \qquad Y'' + \lambda_2^2 Y = 0, \qquad Z'' - \lambda^2 Z = 0$$

$$\therefore \qquad X = A\cos\lambda_1 x + B\sin\lambda_1 x, \qquad y = C\cos\lambda_2 y + D\sin\lambda_2 y, \qquad Z = Ee^{z\lambda} + Fe^{-z\lambda}$$

Hence a general solution is given by

$$u(x, y, z) = \sum_{\lambda_1}\sum_{\lambda_2}(A\cos\lambda_1 x + B\sin\lambda_1 x)\,(C\cos\lambda_2 y + D\sin\lambda_2 y)\,(Ee^{z\lambda} + Fe^{-z\lambda})$$

In fact a number of alternative solutions could be written in a similar manner.

2.14 B. Solved example based on Art 2.14 A

Ex. 1. *Find the potential function* $\psi(x, y, z)$ *in the region* $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$ *satisfying the conditions:* (*i*) $\psi = 0$, on $x = 0, x = a, y = 0, y = b, z = 0$

(*ii*) $\psi = f(x, y)$ on $z = c$, $0 \le x \le a, 0 \le y \le b.$ **[Delhi Maths (H) 2002, 04, 05, 11]**

Sol. We know that potential function $y(x, y, z)$ satisfies Laplace's equation.

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 = 0 \quad \ldots (1)$$

We wish to solve (1) subject to the conditions : $\psi(0, y, z) = 0, \psi(a, y, z) = 0$ ... (2)

$$\psi(x, 0, z) = 0, \psi(x, b, z) = 0 \quad \ldots (3)$$

$$\psi(x, y, 0) = 0 \quad \ldots 4(a)$$

and

$$\psi(x, y, c) = f(x, y) \quad \ldots 4(b)$$

Suppose (1) has solutions of the form $\psi(x, y, z) = X(x)Y(y)Z(z)$ ... (5)

Substituting this in (1) and dividing by XYZ, we get

$$X''/X + Y''/Y = -(Z''/Z) \quad \ldots (6)$$

Since x, y and z are independent variables, each term on each side in (6) must be a constant.

$\therefore$ $X''/X = \mu_1$, $Y''/Y = \mu_2$ so that $\mu_1 + \mu_2 = -(Z''/Z)$

$\therefore$

$$X'' - \mu_1 X = 0. \quad \ldots (7)$$

$$Y'' - \mu_2 Y = 0 \quad \ldots (8)$$

and

$$Z'' + (\mu_1 + \mu_2)Z = 0 \quad \ldots (9)$$

Using (2), (5) gives $X(0)\, Y(y)\, Z(z) = 0$ and $X(a)Y(y)Z(z) = 0$

so that $X(0) = 0$ and $X(a) = 0$, ... (10)

where we have assumed $Y(y) \neq 0$ and $Z(z) \neq 0$ for getting non-zero solution of (1).

We now solve (7) under B.C. (10). Three cases arise:

Case I. Let $\mu_1 = 0$. Then solution of (7) is $X(x) = Ax + B$. ... (11)

Using B.C. (10), (11) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ so that $X \equiv 0$. This leads to $u \equiv 0$. So we reject $\mu_1 = 0$.

Case II. Let $\mu_1 = \lambda_1^2, \lambda_1 \neq 0$. Then solution of (7) is $X(x) = Ae^{\lambda_1 x} + Be^{-\lambda_1 x}$ ... (12)

Using B.C. (10), (12) gives $0 = A + B$ and $0 = Ae^{a\lambda_1} + Be^{-a\lambda_1}$ ... (13)

Solving (13), $A = B = 0$, so that $X \equiv 0$ and hence $u \equiv 0$. So we reject $\mu_1 = \lambda_1^2$.

Case III. Let $\mu_1 = -\lambda_1^2, \lambda_1 \neq 0$. Then solution of (7) is

$X(x) = A \cos \lambda_1 x + B \sin \lambda_1 x$, A and B being arbitrary constants ... (14)

Using B.C. (10), (14) gives

$0 = A$ and $0 = A\cos\lambda_1 a + B \sin\lambda_1 a$

so that $A = 0$, and $\sin\lambda_1 a = 0$,

where we have taken $B \neq 0$, since otherwise $X \equiv 0$, so that $u \equiv 0$.

Now, $\sin\lambda_1 a = 0$ gives $\lambda_1 a = m\pi$ so that $\lambda_1 = m\pi/a$, $m = 1, 2, 3, \ldots$ ... (15)

Hence non-zero solutions $X_m(x)$ of (7) are given by $X_m(x) = B_m \sin(m\pi x/a)$ ... (16)

Using (3) and (5), we get as before $Y(0) = 0$ and $Y(b) = 0$ ... (17)

Solving (8) under B.C. (17) as before, we get non-zero solutions $Y_n(y)$ of (8) given by

$$Y_n(y) = D_n \sin(n\pi y/b) \quad \ldots (18)$$

where $\mu_2 = -\lambda_2^2$ and $\lambda_2 = n\pi/b, \quad n = 1, 2, 3, \ldots$ ... (19)

Now, $\mu_1 + \mu_2 = -(\lambda_1^2 + \lambda_2^2) = -\pi^2(m^2/a^2 + n^2/b^2)$, by (15) and (19)

Let $\mu_1 + \mu_2 = -\lambda_{mn}^2$ so that $\lambda_{mn} = \pi\sqrt{(m^2/a^2 + n^2/b^2)}$ ... (20)

With the above value of $\mu_1 + \mu_2$, (9) becomes $Z'' - \lambda_{mn}^2 Z = 0$, whose solution is

$$Z_{mn}(z) = E_{mn}\cosh(\lambda_{mn}z) + F_{mn}\sinh(\lambda_{mn}z) \quad \ldots (21)$$

Using 4(a), (5) reduces to $X(x)\,Y(y)\,Z(0) = 0$ so that $Z(0) = 0$.

Now, $Z(0) = 0 \Rightarrow Z_{mn}(0) = 0$ ... (22)

Putting $z = 0$ in (21) and using (22), we get $E_{mn} = 0$.

Hence (21) reduces to $Z_{mn}(z) = F_{mn}\sinh(\lambda_{mn}z)$. ... (23)

$$\therefore \quad \psi_{mn}(x, y, z) = X_m Y_n Z_{mn} = G_{mn}\sin(m\pi x/a)\sin(n\pi y/b)\sinh(\lambda_{mn}z)$$

are solutions of (1) satisfying (2), (3) and 4(a). Here G_{mn} ($= B_m D_n F_{mn}$) are new arbitrary constants. In order to obtain a solution also satisfying 4(b), we consider more general solution given by

$$\psi(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\psi_{mn}(x, y, z) \text{ } i.e., \text{ } \psi(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}G_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sinh(\lambda_{mn}z), \quad \ldots (24)$$

Putting $z = c$ in (24) and using 4(b), $f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}G_{mn}\sinh(\lambda_{mn}c)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

which is double Fourier series and so $G_{mn}\sinh(\lambda_{mn}c) = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b}f(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$

so that $G_{mn} = \frac{4\,\text{cosech}\,(\lambda_{mn}c)}{ab}\int_{x=0}^{a}\int_{y=0}^{b}f(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$ ... (25)

Hence the required potential function is given is given by (24) wherein G_{mn} is given by (25) for m, n = 1, 2, 3, ...

Ex. 2. *Find the potential V in the parallelepiped* $0 \le x \le a,\ a \le y \le b, 0 \le z \le c$, satisfying (*i*) $V = 0$ on $x = a, x = 0,\ y = 0, y = b, z = 0$ (*ii*) V = $A\sin(\pi x/a)\sin(\pi y/b), 0 \le x \le a, 0 \le y \le b$. **(Meerut 2011)**

Sol. Take $\psi = V$ and $f(x, y) = A\sin(\pi x/a)\sin(\pi y/b)$ in Ex. 1. Then, (25) yields

$$G_{mn} = \frac{A\,\text{cosch}\,(\lambda_{mn}c)}{ab} \times \left(\int_{x=0}^{a} 2\sin\frac{\pi x}{a}\sin\frac{m\pi x}{a}dx\right) \times \left(\int_{y=0}^{b} 2\sin\frac{\pi y}{b}\sin\frac{n\pi y}{b}dy\right) \quad \ldots (26)$$

But $\int_{x=0}^{a} 2\sin\frac{\pi x}{a}\sin\frac{m\pi x}{a}dx = \begin{cases} 0, \text{ if } m \ne 1 \\ a, \text{if } m = 1 \end{cases}$ and $\int_{y=0}^{b} 2\sin\frac{\pi y}{b}\sin\frac{n\pi y}{b}dy \begin{cases} 0, \text{ if } n \ne 1 \\ b, \text{ if } n = 1 \end{cases}$ ... (27)

Using(27),(26) $\Rightarrow G_{11} = A\,\text{cosech}\,(\lambda_{11}c), G_{mm} = 0$, otherwise. Also(20) $\Rightarrow \lambda_{11} = \pi(1/a^2 + 1/b^2)^{1/2}$

Hence, from(24), V = A cosech $\{\pi c(1/a^2 + 1/b^2)^{1/2}\}$ sin $(\pi x/a)\sin(\pi y/b)$ sinh $\{\pi z(1/a^2 + 1/b^2)^{1/2}\}$

PART IX: PROBLEMS BASED ON TELEGRAPH OR TRANSMISSION LINE

Ex. 1. *Find the current I and voltage V in a transmission line of length l, t seconds after the ends are suddenly grounded given that $I(x, 0) = I_0$ and $V(x, 0) = V_0\sin(\pi x/l)$ and R and G are negligible.*

Sol. Since R and G are neglected, the transmission line equations become

$$\partial V/\partial x = -L(\partial I/\partial t) \quad \ldots (1)$$

$$\partial I / \partial x = -C(\partial V / \partial t) \quad \ldots (2)$$

Differentiating (1) w.r.t 'x' and (2) w.r.t. 't', we get

$$\partial^2 V / \partial x^2 = -L(\partial^2 I / \partial x \partial t) \quad \text{and} \quad \partial^2 I / \partial t \partial x = -C(\partial^2 V / \partial t^2)$$

Eliminating $\partial^2 I / \partial x \partial t$ from these, we obtain $\partial^2 V / \partial x^2 = LC(\partial^2 V / \partial t^2)$ ... (3)

Since the ends are suddenly grounded, boundary conditions are :

$$V(0, t) = 0 \quad \text{and} \quad V(l, t) = 0 \quad \ldots (4)$$

Now, $I(x, 0) = I_0 \Rightarrow (\partial I / \partial x)_{t=0} = 0 \Rightarrow (\partial V / \partial t)_{t=0} = 0$, by (2)

$\therefore$ Initial conditions are given by

$$V(x, 0) = V_0 \sin(\pi x / l) \quad \ldots (5)$$

and

$$(\partial V / \partial t)_{t=0} = 0 \quad \ldots (6)$$

Let $V(x, t) = X(x)\, T(t)$ be a solution of (3). Substituting this in (1), we obtain

$$X'' / X = CLT'' / T = -k^2, \text{ say} \quad \ldots (7)$$

where we have taken $-k^2$ because the initial conditions suggest that $V(x, t)$ must be periodic and hence it must contain trigonometric functions.

From (7), $X'' + k^2 X = 0$ and $CLT'' + k^2 T = 0$

Solving these and using $V = XT$, we get

$$V(x,t) = (c_1 \cos kx + c_2 \sin kx)\left(c_3 \cos \frac{kt}{\sqrt{(LC)}} + c_2 \sin \frac{kt}{\sqrt{(LC)}}\right)$$

Using the boundary conditions (4), we get $c_1 = 0$ and $k = n\pi / l$, n being any integer

$$\therefore \quad V(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi t}{l\sqrt{(LC)}} + c_4 \sin \frac{n\pi t}{t\sqrt{(LC)}}\right) \quad \ldots (9)$$

From (9), $$\frac{\partial V}{\partial t} = c_2 \sin \frac{n\pi x}{l} \times \frac{n\pi}{l\sqrt{(LC)}}\left(-c_3 \sin \frac{n\pi t}{l\sqrt{(LC)}} + c_4 \cos \frac{n\pi t}{l\sqrt{(LC)}}\right)$$

Putting $t = 0$ and using (6), we get $c_2\, c_4 = 0$

$\therefore$ From (9), $V(x,t) = A_n \sin(n\pi x / l)\cos[n\pi t / l\sqrt{LC}]$,

where $A_n = c_2 c_3$. By superposition, the general solution is

$$V(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x / l)\cos[n\pi t / l\sqrt{(LC)}] \quad \ldots (10)$$

Putting $t = 0$ and using (5), we get

$$V_0 \sin(\pi x / l) = \sum_{n=1}^{\infty} A_n \sin(n\pi x / l)$$

which suggest that $A_1 = V_0$ and $A_n = 0$ for remaining values of n. Hence, from (10), the required solution is

$$V(x,t) = V_0 \sin(\pi x / l)\cos[\pi t / l\sqrt{(LC)}] \quad \ldots (11)$$

From (1), (2) and (11), we have

$$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial V}{\partial x} = -\frac{V_0 \pi}{l L}\cos \frac{\pi x}{l}\cos \frac{\pi t}{l\sqrt{(LC)}} \quad \ldots (12)$$

and $$\frac{\partial I}{\partial x} = -c\frac{\partial V}{\partial t} = \frac{cV_0\pi}{l\sqrt{(LC)}}\sin\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{(LC)}} \quad \ldots (13)$$

Integrating (12) and (13), we obtain

$$I(x,t) = -V_0\left(\frac{C}{L}\right)^{1/2}\cos\frac{n\pi}{l}\sin\frac{\pi t}{l\sqrt{(LC)}} + f(x) \text{ and } I(x,t) = -V_0\left(\frac{C}{L}\right)^{1/2}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{(LC)}} + g(t)$$

Since $I(x, 0) = I_0$, so we get $f(x) = g(t) = I_0$. Thus, we have

$$I(x,t) = I_0 - V_0\left(\frac{C}{L}\right)^{1/2}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{(LC)}}. \quad \ldots (14)$$

Hence the required solutions are given by (11) and (14).

EXERCISE 2 (E)

1. Neglecting R and G, find the *e.m.f.* $V(x, t)$ in a line a km long, t second after the ends were suddenly grounded, if initially $I(x, 0) = I_0$ and

(a) $V(x,0) = E_1\sin(\pi x/a) + E_7\sin(7\pi x/a)$ (b) $V(x, 0) = Ex/a$

Ans. (a) $V = E_1\sin\frac{\pi x}{a}\cos\frac{\pi t}{a\sqrt{(LC)}} + E_7\sin\frac{7\pi x}{a}\cos\frac{7\pi t}{a\sqrt{(LC)}}$

(b) $V = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{a}\cos\frac{n\pi t}{a\sqrt{(LC)}}$, where $A_n = \frac{2E}{a^2}\int_0^a x\sin\frac{n\pi x}{a}dx$

2. A transmission line 1000 km long, is initially under steady state conditions with potential 1300 volts at the sending end ($x = 0$) and 1200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential $E(x, t)$.

Ans. $E(x,t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-(n^2\pi^2t)/l^2RC)}$

MISCELLANEOUS EXAMPLES ON CHAPTER 2

Ex. 1. *Solve the equation* $\partial u/\partial t = c^2(\partial^2 u/\partial x^2)$ *by separation of variables method to the conditions:* $u(0, t) = 0 = u(l, t)$, *for all t, and* $u(x, 0) = f(x)$, *for all x in* $[0, l]$. **[I.A.S. 2007]**

Sol. Proceed exactly as in Art 2.3 A by taking $a = l$, $k = c^2$. Then, required solution will be of the form $$u(x,t) = \sum_{n=1}^{\infty} E_n\sin(n\pi x/l)e^{-(n^2\pi^2c^2/l)t}, \text{ where } E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{a}dx.$$

Ex. 2. *Solve* $u_{xx} + u_{yy} = 0$ *in D where* $D = \{(x, y); 0 < x < a, 0, y < b\}$ *is a rectangle in a plane with the boundary conditions :* $u(x, 0) = 0$, $u(x, b) = 0$, $0 \le x \le a$; $u(0, y) = g(y)$, $u_x(a, y) = h(y)$, $0 \le y \le b$. **[I.A.S. 2007]**

***Sol.** Given $$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \qquad 0 < x < a, \qquad 0 < y < b \quad \ldots (1)$$

with boundary conditons $$u(x, 0) = u(x, b) = 0, \qquad 0 \le x \le a \quad \ldots (2)$$

---

* This example is based on method of solving 'Mixed problems' in rectangle given in Art. 2.13E

$$u(0, y) = g(y), \qquad 0 \le y \le b \qquad \ldots 3(a)$$

and $$u_x(a, y) = h(y), \qquad 0 \le y \le b \qquad \ldots 3(b)$$

Suppose (1) has a solution of the form $u(x, y) = X(x)Y(y)$ ... (4)

where $X(x)$ and $Y(y)$ are functions of x and y respectively.

Susbstituting the value of $u(x, y)$ given by (4) in (1), we have

$$X'Y + YY'' = 0 \quad \text{or} \quad X''/X = -(Y''/Y), \qquad \ldots (5)$$

where the dashes denote derivatives with respect to the relevant variables. Since x and y are independent variables, therefore (5) can only be true of each side is equal to the same constant, μ (say). Then (5) leads to the following pair of differential equations

$$X'' - \mu X = 0 \qquad \ldots (6)$$

and $$Y'' + \mu Y = 0 \qquad \ldots (7)$$

Putting $y = 0$ and $y = b$ by turn in (4) and using (2), we get

$$X(x)\,Y(0) = 0 \quad \text{and} \quad X(x)\,Y(b) = 0$$

so that $$Y(0) = 0 \quad \text{and} \quad Y(b) = 0, \qquad \ldots(8)$$

where we have taken $X(x) \not\equiv 0$, since otherwise $u \equiv 0$, which does not satisfy the boundary conditions 3(a) and 3(b).

We now solve (7) satisfying (8). Three cases arise:

Case I. Let $\mu = 0$. Then solution of (7) is $Y(y) = Ay + B$ ... (9)

Using boundary conditons (8), (9) reduces to $0 = B$ and $0 = Ab + B$. Solving these, $A = B = 0$ so that $Y(y) \equiv 0$. This leads to $u \equiv 0$, which does not satisfy the boundary conditions 3(a) and 3(b).

Case II. Let $\mu = -\lambda^2$, whee $\lambda \neq 0$. Then, solution of (7) is

$$Y(y) = Ae^{\lambda y} + Be^{-\lambda y}, \text{ where } A, B \text{ are arbitrary constants} \qquad \ldots (10)$$

Using boundary conditions (8), (10) gives $0 = A + B$ and $0 = Ae^{b\lambda} + Be^{-b\lambda}$. Solving these, we get $A = B = 0$. Hence $Y(y) \equiv 0$. This leads to $u \equiv 0$, which does not satisfy the boundary conditions 3(a) and 3(b).

Case III. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then, solution of (7) is

$$Y(y) = A\cos\lambda y + B\sin\lambda y, \; A, B \text{ being arbitrary constants} \qquad \ldots (11)$$

Using boundary conditons (8), (11) gives $0 = A$ and $0 = A\cos\lambda b + B\sin\lambda b$

so that $A = 0$ and $\sin\lambda b = 0$, where we have taken $B \neq 0$, since otherwise. $Y(y) \equiv 0$ which leads to $u \equiv 0$, which does not satisfy the boundary condition 3(a) and 3(b).

Now, $$\sin\lambda b = 0 \;\Rightarrow\; \lambda b = n\lambda \;\Rightarrow\; \lambda = (n\pi/b), \quad n = 1, 2, 3, \ldots \qquad \ldots (12)$$

Hence, from (11), all non-zero solutions of (7) are given by

$$Y_n(y) = B_n \sin(n\pi y/b), \; n = 1, 2, 3, \ldots \qquad \ldots (13)$$

Again, $$\mu = \lambda^2 \;\Rightarrow\; \mu = n^2\pi^2/b^2, \; n = 1, 2, 3, \ldots \qquad \ldots (14)$$

Using (14), (16) reduces to $X'' - (n^2\pi^2/b^2)X = 0$, whose general solution is

$$X_n(x) = C_n \cosh(n\pi x/b) + D_n \sinh(n\pi x/b), \; n = 1, 2, 3, \ldots \qquad \ldots (15)$$

From (4), (13), and (15), solutions of (1) satisfying (2) are given by

$$u_n(x, y) = X_n(x)Y_n(y) = \sin(n\pi y/b)\{E_n \cosh(n\pi x/b) + F_n \sinh(n\pi x/b)\},\ n = 1, 2, 3, ...$$

where E_n (= B_n C_n) and F_n (= B_n D_n) are new arbitrary constants.

In order to obtain a solution satisfying the boundary conditions 3(*a*) and 3(*b*), we consider a more general solution given by

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} \sin(n\pi y/b)\{E_n \cosh(n\pi x/b) + F_n \sinh(n\pi x/b)\} \quad \text{... (16)}$$

Substituting $x = 0$ in (16) and using the boundary condition 3(a), we get (noting that sinh 0 = 0and cosh 0 = 1)

$$g(y) = \sum_{n=1}^{\infty} E_n \sin(n\pi y/b),$$

which is Fourier sine series. Hence the constants E_n are given by

$$E_n = \frac{2}{b}\int_0^b g(y)\sin(n\pi y/b)dy, \qquad n = 1, 2, 3, ... \quad \text{... (17)}$$

Differentiating (16) partially w.r.t. '*x*', we have

$$u_x(x, y) = \sum_{n=1}^{\infty} \sin(n\pi y/b)\{(n\pi E_n/b)\sinh(n\pi x/b) + (n\pi F_n/b)\cosh(n\pi x/b)\} \quad \text{... (18)}$$

Substituting $x = a$ in (18) and using the boundary condition 3(*b*), we have

$$h(y) = \sum_{n=1}^{\infty} (n\pi/b)\times\{E_n \sinh(n\pi a/b) + F_n \cosh(n\pi a/b)\}\sin(n\pi y/b),$$

which is Fourier sine series. Hence, as before, we have

$$(n\pi/b)\times\{E_n \sinh(n\pi a/b) + F_n \cosh(n\pi a/b)\} = \frac{2}{b}\int_0^b h(y)\sin(n\pi y/b)dy$$

or
$$F_n \cosh(n\pi a/b) = \frac{2}{n\pi}\int_0^b h(y)\sin(n\pi y/b)dy - E_n \sinh(n\pi a/b)$$

or
$$F_n \cosh(n\pi a/b) = \frac{2}{n\pi}\int_0^b h(y)\sin(n\pi y/b)dy - \frac{2\sinh(n\pi a/b)}{b}\int_0^b g(y)\sin(n\pi y/b)dy$$

[using (17)]

or
$$F_n = \frac{2}{\cosh(n\pi a/b)}\int_0^b \left\{\frac{h(y)}{n\pi} - \frac{g(y)\sinh(n\pi a/b)}{b}\right\}\sin\frac{n\pi y}{b}dy,\ n = 1, 2, 3, ... \quad \text{... (19)}$$

The required solution is given by (16) wherein E_n and F_n are given by (17) and (19) respectively.

Ex.3. Statement for linked answer questions (*i*) and (*ii*). Consider the boundary value problem $u_{xx} + u_{yy} = 0,\ x \in (0, \pi),\ y \in (0, \pi),\ u(x, 0) = u(x, \pi) = u(0, y) = 0$.

(*i*) Any solution of this boundary value problem is of the form

(*a*) $\sum_{n=1}^{\infty} a_n \sinh nx \sin ny$ (*b*) $\sum_{n=1}^{\infty} a_n \cosh nx \sin ny$

(c) $\sum_{n=1}^{\infty} a_n \sinh nx \cos ny$ (*d*) $\sum_{n=1}^{\infty} a_n \cosh nx \cos ny$

(*ii*) If an additional boundary condition $u_x(x, y) = \sin y$ is satisfied, then $u(x, \pi/2)$ equals

(*a*) $(\pi/2)\times(e^{\pi} - e^{-\pi})(e^{x} - e^{-x})$ (*b*) $\pi(e^{x} + e^{-x})/(e^{\pi} - e^{-\pi})$

(*c*) $(e^{x} - e^{-x})/(e^{\pi} + e^{-\pi})$ (*d*) $(\pi/2)\times(e^{\pi} + e^{-\pi})\,(e^{x} + e^{-x})$ (GATE 2008)

Sol. Part (i). Proceed as in Ex. 5 (a), page 2.79 upto equation (18) by replacing *a* by π, *b* by π and E_n by a_n and let

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh nx \sin ny \qquad \ldots (1)'$$

Thus, choice (*a*) is correct

Part (*ii*). Differentiating both sides of $(1)'$ partially with respect to '*x*', we get

$$u_x(x, y) = \sum_{n=1}^{\infty} n\, a_n \cosh nx \sin ny \qquad \ldots (2)'$$

Given that $u_x(x, \pi) = \sin y$ ... $(3)'$

Putting $x = \pi$ in $(2)'$ and using $(3)'$, we have $\sin y = \sum_{n=1}^{\infty} n\, a_n \cosh n\pi \sin ny$

which is the half range Fourier sine series of sin *y* in (0, π). Hence

$$n\, a_n \cosh n\pi = \frac{2}{\pi}\int_0^{\pi} \sin y \sin ny\, dy \quad \text{or} \quad a_n = \frac{2}{\pi n \cosh n\pi}\int_0^{\pi} \sin y \sin ny\, dy \qquad \ldots (4)'$$

When $n \neq 1$, $(4)'$ *gives* $$a_n = \frac{1}{\pi n \cosh n\pi}\int_0^{\pi} [\cos(n-1)y - \cos(n+1)y]\, dy$$

or $$a_n = \frac{1}{\pi n \cosh n\pi}\left[\frac{\sin(n-1)y}{n-1} - \frac{\sin(n+1)y}{n+1}\right]_0^{\pi} = 0$$

Next, when $n = 1$, $(4)'$ reduces to

$$a_1 = \frac{2}{\pi \cosh \pi}\int_0^{\pi} \sin^2 y\, dy = \frac{1}{\pi \cosh \pi}\int_0^{\pi} (1 - \cos 2y)\, dy = \frac{1}{\pi \cosh \pi}\left[y - \frac{\sin 2y}{2y}\right]_0^{\pi} = \frac{1}{\cosh \pi}$$

Also $a_n = 0, \forall\, n \geq 2$ Hence, $(1)'$ yields $u(x, y) = a_1 \sinh x \sin y = (\sinh x \sin y)/\cosh \pi$

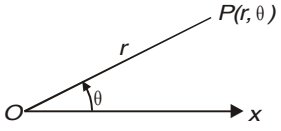
Thus, $u(x, \pi/2) = \sinh x / \cosh \pi = (e^{x} - e^{-x})/(e^{\pi} + e^{-\pi})$ and hence choice (c) is correct.

3

Boundary value problems in polar co-ordinates

3.1. Introduction

While solving boundary value problems, appropriate choice of coordinate system is very useful. In physical problems that involve a circular goemetry (for example, the problem of evaluating the temperature in a circular disc or a sector of a circular disc), it is clear from the nature of the situation that the problem will be simplified if we can use polar coordinates (r, θ) instead of rectangular coordinates (x, y).



Polar coordinates are defined by means of equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad \text{where} \qquad r \geq 0, \qquad 0 \leq \theta \leq 2\pi \quad \ldots (1)$$

The Laplce's equation in polar coordinates is given by

$$\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial r^2) = 0 \quad \ldots (2)$$

The requirement of single-valuedness of $u(r, \theta)$ in a circular region of radius 'a' implies periodicity condition, *i.e.*,

$$u(r, \theta + 2\pi) = u(r, \theta), \qquad 0 \leq r \leq a \quad \ldots (3)$$

In what follows, we shall use the fact that steady state heat flow problems involving circular geometry satisfy the Laplace's equation (2).

We now present some useful results:

I. Half range sine series

(*a*) If $f(\theta)$ is defined in interval $(0, \alpha)$, then half range sine series is given by

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi\theta}{\alpha}, \qquad \text{where} \qquad b_n = \frac{2}{\alpha}\int_0^{\alpha} f(\theta)\sin\frac{n\pi\theta}{\alpha}\,d\theta$$

(*b*) If $f(\theta)$ is defined in interval $(0, \pi)$, then half range sine series is given by

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta, \qquad \text{where} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta$$

Note: Result (*b*) is a particular case of result (a) for $\alpha = \pi$.

II. General Fourier series in $(0, 2\pi)$.

Let $f(\theta)$ be a periodic function with period 2π.

Suppose $f(\theta)$ can be represented as a trigonometric series in the interval $0 < x < 2\pi$ as follows:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n \sin n\theta),$$

then, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta, \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta$

III. Equation $r^2(d^2R/dr^2) + r(dR/dr) + \lambda^2 r^2 R = 0$ is called Bessel's equation of zeroth order. Its solution is $R(r) = A_\lambda J_0(\lambda r) + B_\lambda Y_0(\lambda r),$

where J_0 and Y_0 are known as Bessel's functions of first and second kind respectively.

Also, it is known that $Y_0 \to \infty$ as $r \to 0$.

IV. Fourier–Bessel series.

If $f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r),\ 0 \le r \le a$, then $A_n = \frac{2}{a^2\{J_1(\lambda_n a)\}^2}\int_0^a rJ_0(\lambda_n r) f(r)dr$

3.2 Solution of Laplace's equation in polar coordinates.

Ex. 1. *Solve two dimensional Laplace's equation in plane polar coordinates* (r, θ).

[Delhi Maths (H) 1994]

Sol. *In plane polar coordinates* (r, θ) *Laplace's equation*

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \qquad \ldots (1)$$

reduces to $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \qquad \ldots (2)$$

Suppose (1) has solutions of the form $u(r, \theta) = R(r)\Theta(\theta)$ ... (3)

where R and Θ are functions of r and θ respectively.

Substituting this values of u in (2), we get

$$R''\Theta + (1/r) \times R'\Theta + (1/r^2) \times R\Theta'' = 0 \qquad \ldots (4)$$

or $$(r^2R'' + rR')/R = -\Theta''/\Theta, \qquad \ldots (5)$$

where the dashes denotes derivatives with respect to the relevant variables. Since r and θ are independent, (5) is true only if each side is equal to a constant. Careful inspection of physical conditions of the actual problems guides us towards the correct choice of the proposed constant. On physical grounds we require $u(r, \theta) = u(r, \theta + 2\pi)$. Hence solution must involve trigonometric functions. For this purpose we choose each side of (5) equal to n^2, where n is an integer. Hence (5) gives $$r^2R'' + rR' - n^2R = 0 \qquad \ldots (6)$$

and $$\Theta'' + n^2\Theta = 0 \qquad \ldots (7)$$

From (6), $$r^2(d^2R/dr^2) + r(dR/dr) - n^2R = 0 \qquad \ldots (8)$$

which is linear homogeneous differential equation.

Let $r = e^z$ and $D_1 \equiv r(d/dr) \equiv d/dz$. Then as usual (8) reduces to

$[D_1(D_1 - 1) + D_1 + n^2]R = 0$ or $(D_1^2 - n^2)R = 0$ ... (9)

Its auxiliary equation is $D_1^2 - n^2 = 0$ so that $D_1 = n, -n$.

Hence solution of (9) is $R(r) = A_n e^{nz} + B_n e^{-nz}$

or $R(r) = A_n r^n + B_n r^{-n}$, as $e^z = r$ ... (10)

Solution of (7) is $\Theta(\theta) = C_n \cos n\theta + D_n \sin n\theta$ ... (11)

From (10) and (11), a solution of (2) is

$$u_n(r,\theta) = (A_n r^n + B_n r^{-n})\,(C_n \cos n\theta + D_n \sin n\theta) \quad \text{... (12)}$$

or, by superposition, a more general solution of (2) is

$$u(r,\theta) = \sum_n u_n(r,\theta) = \sum_n (A_n r^n + B_n r^{-n})\,(C_n \cos n\theta + D_n \sin n\theta) \quad \text{... (13)}$$

where the summation over n may be finite of infinite.

In the special case when $n = 0$, (6) and (7) becomes $rR'' + r = 0$ ... (14)

amd $\Theta'' = 0$, *i.e.*, $d^2\Theta/d\theta^2 = 0$ ... (15)

Re-writing (14), $\dfrac{d}{dr}(r\,R') = 0$ ... (16)

Integrating (16), $rR' = E$ or $dR = (E/r)\,dr$ ... (17)

Integrating (17), $R(r) = E \log r + F$, E, F being arbitrary constants ... (18)

From (15), $\Theta(\theta) = G\theta + H$, G and H being arbitrary constants ... (19)

Thus for $n = 0$, a solution of (2) is $u(r,\theta) = (E\log r + F)\,(G\theta + H)$ ... (20)

Hence a more general solution can be obtained by combining the R.H.S. of (13) with a term in the form of the R.H.S. of (20). When the region under consideration includes of the origin, at which $r = 0$, terms involving log r have to be excluded by setting $B_n = 0$ and $E = 0$. This will lead to non-singular solution of (2).

Note 1. Solution (12) is also expressed as $(Ar^n + Br^{-n})e^{\pm in\theta}$.

Note 2. The solution (12) is known as *circular harmonics of degree n*. (20) is called *circular harmonics of degree zero*.

Ex. 2 *Show that if the two dimensional harmonic equation* $\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0$ *is transformed to plane polar coordinates r and* θ *defined by* $x = r\cos\theta$, $y = r\sin\theta$, *it takes the form* $\partial^2 v/\partial r^2 + (1/r)\,(\partial v/\partial r) + (1/r^2)\,(\partial^2 v/\partial\theta^2) = 0$ *and deduce that it has solutions of the form* $(Ar^n + Br^{-n})e^{\pm in\theta}$, *where A and B are contants.* **[Delhi Maths (Hons) 1994]**

Sol. For first part refer chapter 1

Defuction. Proceed as in solved Ex. 1 upto equation (10). Now complete the solution as given below. Writing $A_n = A$ and $B_n = B$, equation (10) may be re-written as

$$R(r) = A\,r^n + Br^{-n}$$

Solution of equation (7) can be put in the form $\Theta(\theta) = e^{\pm in\theta}$.

Hence a solution of (2) can be expressed in the form

$$(Ar^n + Br^{-n})e^{\pm in\theta},\ A \text{ and } B \text{ being constants.}$$

Ex. 3. *Solve the differential equation* $(\partial^2 u/\partial r^2) + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial\theta) = 0$ *subject to the boundary conditions:* (*i*) *u is finite when* $r \to 0$ (*ii*) $u = \Sigma C_n \cos n\theta$ *when* $r = a$. **[Meerut 2011, Delhi Maths (H) 1996, 2005, 09, 11]**

Sol. Given $(\partial^2 u/\partial r^2) + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial\theta) = 0$ ... (1)

Suppose (1) has a solution of the form $u(r,\theta) = R(r)\Theta(\theta)$ ... (2)

where R and Θ are functions of r and θ respectively. Substituting this value of u in (1), we get

$$R''\Theta + (1/r)\times R'\Theta + (1/r^2)\times R\Theta'' = 0 \quad \text{or} \quad (r^2R'' + rR')/R = -\Theta''/\Theta, \quad \text{... (3)}$$

where the dashes denote derivatives with respective to the relevant variables. Since r and θ are independent, (3) is true only if each side in equal to the same constant. Since the required solution involves trigonometric function $\cos n\theta$, we take each side of (5) equal to n^2, where n is an integer.

Hence (3) gives $r^2R'' + rR' - n^2R = 0$ ... (4)

and $\Theta'' + n^2\Theta = 0$ ... (5)

From (4), $r^2(d^2R/dr^2) + r(dR/dr) - n^2R = 0,$ ... (6)

which is linear homogeneous differential equation.

Let $r = e^z$ so that $D_1 \equiv r(d/dr) \equiv d/dz$

Then (6) becomes $[D_1(D_1 - 1) + D_1 - n^2]\,R = 0$ or $(D_1^2 - n^2)R = 0$ ... (7)

Its anxiliary equation is $D_1^2 - n^2 = 0$ so that $D_1 = n, -n.$

Hence solution is (7) is $R = A_ne^{nz} + B_ne^{-nz}$, A_n and B_n being arbitrary as constants.

or $R(r) = A_nr^n + B_nr^{-n}$, as $e^z = r$ ... (8)

Solution of (5) is $\Theta(\theta) = C_n\cos n\theta + D_n\sin n\theta$ ... (9)

From (8) and (9), a solution of (1) is

$$u_n(r,\theta) = (A_nr^n + B_nr^{-n})\,(C_n\cos n\theta + D_n\sin n\theta) \quad \text{... (10)}$$

So by the principal of superposition, a more general solution of (1) is

$$u(r,\theta) = \Sigma_n u_n(r,\theta) = \Sigma_n(A_nr^n + B_nr^{-n})\,(C_n\cos n\theta + D_n\sin n\theta), \quad \text{... (11)}$$

where the summation over n may be finite or infinite.

From given boundary condition (*i*), u is finite when $r \to 0$. Hence we must take $B_n = 0$ in (11). Then (11) reduces to

$$u(r,\theta) = \Sigma A_nr^n(C_n\cos n\theta + D_n\sin n\theta) \quad i.e., \quad u(r,\theta) = \Sigma r^n(E_n\cos n\theta + F_n\sin n\theta) \quad \text{... (12)}$$

where E_n (= A_n C_n) and F_n(= A_nD_n) are new arbitary constants.

Re-writing given boundary conditions (*ii*), we get $u(a,\theta) = \Sigma C_n\cos n\theta$ ... (13)

Putting $r = a$ in (12) and using (13), we get

$$\Sigma C_n\cos n\theta = \Sigma a^n(E_n\cos n\theta + F_n\sin n\theta)$$

so that $F_n = 0$ and $a^n\,E_n = C_n$ *i.e.*, $E_n = C_n/a^n$. So the required solution is

$$u(r,\theta) = \Sigma C_n(r/a)^n\cos n\theta.$$

Ex. 4. *Obtain steady temperature distribution in a semi circular plate of radius a, insulated on both faces, with its curved boundary kept at a constant temperature u_0 and its boundary diameter kept at zero temperature.*

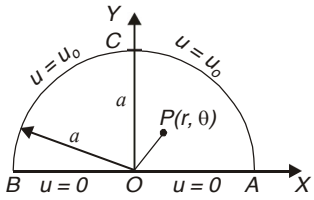
Sol. The partial differential equation governing the steady temperature distribution is the Laplace's equation $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0$

For the present problem for a semi-circular plate the cartesian form of Laplace's equation is not suitable. So we use polar form of it, namely,

$$\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial\theta^2) = 0 \quad \ldots (1)$$

Here we wish to solve (2) under the following boundary conditions

(*i*) $u(r, 0) = 0$, $u(r, \pi) = 0$ (*ii*) $u(a, \theta) = u_0$.



Here conditions (*i*) express the fact that temperature $u(r, \theta)$ is zero along OA ($\theta = 0$) and OB ($\theta = \pi$) so that temperature is zero along the bounding diameter AB. Again condition (*ii*) expresses the fact that temperature on the boundary ACB (where $r = a$) is u_0.

Suppose (1) has a solution of the form $u(r,\theta) = R(r)\Theta(\theta)$, ... (2)

where R and Θ are functions of r and θ respectively.

Substituting this value of u in (1), we get $R''\Theta + (1/r)\times R'\Theta + (1/r^2)\times R\Theta'' = 0$

or $(r^2R'' + rR')/R = -\Theta''/\Theta.$ ... (3)

where the deshes denote derivatives with respect to the relevant variables. Since r and θ are independent, (3) is true only if each side is equal to the same constant. On physical grounds we require $u(r, \theta) = u(r, \theta + 2\pi)$. Hence the solution must involve trigonometric functions. For this we choose each side of (3) equal to n^2 and obtain

$$\Theta'' + n^2\Theta = 0 \quad \ldots (4)$$

and $$r^2R'' + rR' - n^2R = 0 \quad \ldots (5)$$

Now, putting $\theta = 0$ and $\theta = \pi$ in (1) and using boundary condition (*i*), we have

$0 = R(r)\,\Theta(0)$ and $0 = R(r)\,\Theta(\pi)$

so that $\Theta(0) = 0$ and $\Theta(\pi) = 0.$ ... (6)

where we have taken $R(r) \neq 0,$ since otherwise $u \equiv 0$ which does not satisfy given boundary condition (*ii*). The general solution of (4) is

$\Theta(\theta) = C_n\cos n\theta + D_n\sin n\theta$, C_n and D_n being arbitrary constants ... (7)

Using boundary conditions (6), (7) reduces to $0 = C_n$ and $0 = C_n\cos n\pi + D_n\sin n\pi$

so that $C_n = 0$ and $\sin n\pi = 0,$

where we have taken $D_n \neq 0,$ since otherwise $u \neq 0$ which does not satisfy boundary condition (*ii*).

Now, $\sin n\pi = 0 \Rightarrow$ n is an integer.

With the above values of C_n and n, we get $\Theta(\theta) = D_n\sin n\theta$ ... (8)

Next, from (5), $r^2(d^2R/dr^2) + r(dR/dr) - n^2R = 0,$ ... (9)

which is linear homogeneous differential equation.

Let $r = e^z$ so that $r(d/dr) \equiv d/dz \equiv D_1$

Then (9) becomes $[D_1(D_1 - 1) + D_1 - n^2]\,R = 0$ or $(D_1^2 - n^2)R = 0$ ... (10)

Its anxiliary equation is $D_1^2 - n^2 = 0$ giving $D_1 = n, -n.$

Hence solution of (10) is $R = A_n e^{nz} + B_n e^{-nz}$, A_n, B_n being arbitarary constants.

or $$R(r) = A_n r^n + B_n r^{-n}, \text{ as } e^z = r \quad \ldots (11)$$

As $r \to 0$, $r^{-n} \to \infty$ so that $R(r) \to \infty$ and hence $u(r,\theta) \to \infty$. But this contradicts the physical problem in hand. So we take $B_n = 0$ in (11) and obtain

$$R(r) = A_n r^n \quad \ldots (12)$$

From (8) and (12), a solution of (1) is

$$u_n(r, \theta) = A_n D_n r^n \sin n\theta = E_n r^n \sin n\theta \quad \ldots (13)$$

where E_n ($=A_n D_n$) is another arbitrary constant. These give solution satisfying the given boundary condition (*i*)

In order to satisfy boundary condition (ii) also, we consider a more general solution

$$u(r,\theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta \quad \ldots (14)$$

Putting $r = a$ in (14) and using boundary condition (*ii*), we have $u_0 = \sum_{n=1}^{\infty} E_n a^n \sin n\theta$

which is Fourier sine series and hence (refer result I(*b*) of Art. 3.1)

Hence, $$E_n a^n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin n\theta \, d\theta = \frac{2u_0}{\pi}\left[-\frac{\cos n\theta}{n}\right]_0^{\pi} = \frac{2u_0}{n\pi}[1-(-1)^n]$$

$\therefore$ $E_n = 4u_0/(2m-1)\pi a^{2m-1}$, if $n = 2m-1$ while $E_n = 0$, if $n = 2m$

With these values of E_n, (14) reduce to

$$u(r,\theta) = \frac{4u_0}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}\left(\frac{r}{a}\right)^{2m-1}\sin(2m-1)\theta$$

which gives the required temperature distribution.

Ex. 5. *u is a function of r and* θ *satisfying the equation.*

$$\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial\theta^2) = 0$$

within the region of the plane bounded by r = a, r = b, θ = 0, θ = π/2. *Its value along the boundary r = a is* θ(π/2 – θ)*, and it value along the other boundary is zero. Prove that*

$$u = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(r/b)^{4m-2} - (b/r)^{4m-2}}{(a/b)^{4m-2} - (b/a)^{4m-2}}\frac{\sin(4m-2)\theta}{(2m-1)^3}$$ **[Delhi Maths (H) 1998, 99, 2007]**

Sol. Given $$\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial\theta^2) = 0 \quad \ldots (1)$$

We are to solve (1) under the following boundary conditions:

[**Note—**Here shaded area of the figure given on the next page is being considered].

(*i*) $u(r, 0) = 0$, $u(r, \pi/2) = 0$ (*ii*) $u(b, \theta) = 0$ (*iii*) $u(a,\theta) = \theta(\pi/2 - \theta)$.

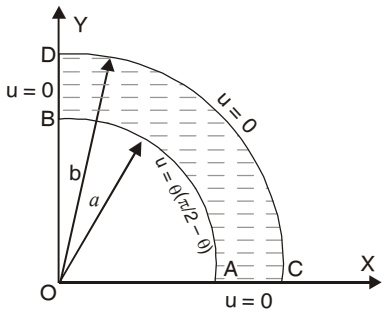
Suppose (1) has a solution of the form $$u(r,\theta) = R(r)\,\Theta(\theta) \quad \ldots (2)$$

where R and Θ are functions of r and θ respectively.

Substituting this value of u in (1), we get

$$R''\Theta + (1/r)R'\Theta = -\Theta''/\Theta, \qquad \text{... (3)}$$

where the dashes denote derivatives with respect to the relevant variables. Since r and θ are independent, (3) is true only if each side in equal to the same constant. On physical grounds we require $u(r,\theta) = u(r,\theta+2\pi)$.



Hence the solution must involve trigonometric functions. For this we choose each side of (3) equal to n^2 and obtain

$$\Theta'' + n^2\Theta = 0 \qquad \text{... (4)}$$

and

$$r^2R'' + rR' - n^2R = 0, \qquad \text{... (5)}$$

Now, putting $\theta = 0$ and $\theta = \pi/2$ in (2) and using boundary conditions (*i*), we have

$0 = R(r)\,\Theta(0)$ and $0 = R(r)\Theta(\pi/2)$

so that $\Theta(0) = 0$ and $\Theta(\pi/2) = 0,$ ... (6)

where we have taken $R(r) \neq 0,$ since otherwise $u = 0$ which does not satisfy given boundary conditons (*iii*). The general solution of (4) is

$$\Theta(\theta) = C_n \cos n\theta + D_n \sin n\theta, \text{ where } C_n \text{ and } D_n \text{ are arbitrary constants} \qquad \text{... (7)}$$

Using boundary conditions (6), (7) reduces to $0 = C_n$ and $0 = C_n \cos(n\pi/2) + D_n \sin(n\pi/2)$

so that $C_n = 0$ and $\sin(n\pi/2) = 0$

where we have taken $D_n \neq 0,$ since otherwise $u = 0$ which does not satisfy boundary conditon (*iii*).

Now, $\sin(n\pi/2) = 0 \quad \Rightarrow \quad n\pi/2 = (2m-1)\pi \quad \Rightarrow \quad n = 4m - 2$ ... (8)

With the above values of C_n and n, (7) becomes $\Theta(\theta) = D_m \sin(4m-2)\theta$ ... (9)

Next, from (5) $r^2(d^2R/dr^2) + r(dR/dr) - r^2R = 0,$ ... (10)

which is linear homogeneous differential equation.

Let $r = e^z$ so that $r(d/dr) \equiv d/dz \equiv D_1$, say

Then (10) becomes

$[D_1(D_1 - 1) + D_1 - n^2]R = 0$ or $(D^2 - n^2)\,R = 0$ ... (11)

Its auxiliary equation is $D_1^2 - n^2 = 0$ giving $D_1 = n, -n$

Hence solution of (11) is $R = A_n\,e^{nz} + B_n\,e^{-nz}$, A_n and B_n being arbitrary constants.

or $R(r) = A_n\,r^n + B_n\,r^{-n}, \quad$ as $\quad e^z = r$ ... (12)

Now, putting $r = b$ in (2) and using boundary condition (*ii*), we get

$0 = R(b)\,\Theta(\theta)$ so that $R(b) = 0,$ ... (13)

where we have taken $\Theta(\theta) \neq 0,$ otherwise $u = 0$ which does not satisfy boundary conditions (*iii*).

Putting $r = b$ in (12) and using (13), we get

$0 = A_n\,b^n + B_n\,b^{-n}$ so that $B_n = -A_n\,b^{2n}$

Using this value of B_n is (12), we get

$$R(r) = A_n\,(r^n - b^{2n}\,r^{-n}) = A_n\,(r^n - b^{2n}/r^n)$$

or $R(r) = A_m\,(r^{4m-2} - b^{8m-4}/r^{4m-2})$, using (8) ... (14)

From (9) and (14), a solution of (1) satisfying boundary conditions (*i*) and (*ii*) is

$$u_m(r,\theta) = A_m D_m \left(r^{4m-2} - \frac{b^{8m-4}}{r^{4m-2}} \right) \sin(rm-2)\theta \text{ or } u_m(r,\theta) = E_m \left(r^{4m-2} - \frac{b^{8m-4}}{r^{4m-2}} \right) \sin(4m-2)\theta \quad \text{... (15)}$$

where E_m ($= A_m D_m$) is new arbitrary constant.

In order to satisfy boundary condition (*iii*) also, we consider a more general solution

$$u(r,\theta) = \sum_{m=1}^{\infty} u_m(r,\theta) = \sum_{m=1}^{\infty} E_m \left(r^{4m-2} - \frac{b^{8m-4}}{r^{4m-2}} \right) \sin(4m-2)\theta \quad \text{... (16)}$$

Putting $r = a$ in (16) and using boundary condition (*iii*), we have

$$\theta\left(\frac{\pi}{2} - \theta\right) = \sum_{m=1}^{\infty} E_m \left(a^{4m-2} - \frac{b^{8m-4}}{a^{4m-2}} \right) \sin(4m-2)\theta$$

$$\therefore \quad E_m \, (a^{4m-2} - b^{8m-4}/a^{4m-2}) = \frac{4}{\pi} \int_0^{\pi/2} \theta\left(\frac{\pi}{2} - \theta\right) \sin(4m-2)\theta \, d\theta$$

$$= \frac{4}{\pi} \left[\theta\left(\frac{\pi}{2} - \theta\right) \left\{ -\frac{\cos(4m-2)}{4m-2} \right\} - \left(\frac{\pi}{2} - 2\theta\right) \left\{ -\frac{\sin(4m-2)\theta}{(4m-2)^2} \right\} + (-2) \left\{ \frac{\cos(4m-2)\theta}{(4m-2)^3} \right\} \right]_0^{\pi/2}$$

[using chain rule of integration by parts]

$$\therefore \quad E_m \left(a^{4m-2} - \frac{b^{8m-4}}{a^{4m-2}} \right) = \frac{4}{\pi} \left[-\frac{2\cos(2m-1)\pi}{8(3m-1)^3} + \frac{2}{8(2m-1)^3} \right] = \frac{1-(-1)^{2m-1}}{\pi(2m-1)^3} = \frac{2}{\pi(2m-1)^3}.$$

Thus, $$E_m = \frac{2}{\pi(2m-1)^3 (a^{4m-2} - b^{8m-4}/a^{4m-2})}$$

With this value, (16) reduce to

$$u(r,\theta) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{r^{4m-2} - (b^{8m-4}/r^{4m-2})}{a^{4m-2} - (b^{8m-4}/a^{4m-2})} \frac{\sin(4m-2)\theta}{(2m-1)^3} = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(r/b)^{4m-2} - (b/r)^{4m-2}}{(a/b)^{4m-2} - (b/a)^{4m-2}} \frac{\sin(4m-2)\theta}{(2m-1)^3}$$

Ex. 6. *Find solution of two-dimensional Laplace's equation* $r^2(\partial^2 u/\partial r^2) + r(\partial u/\partial r) + (\partial^2 u/\partial \theta^2) = 0$ *in polar co-ordinates.*

or Solve heat equation in steady state in two-dimensional polar co-ordinates.

[Delhi B.Sc. Physics (H) 2001]

Sol. Given $$r^2(\partial^2 u/\partial r^2) + r(\partial u/\partial r) + (\partial^2 u/\partial \theta^2) = 0. \quad \text{... (1)}$$

Suppose (1) has a solutions of the form $$u(r,\theta) = R(r)\,\Theta(\theta), \quad \text{... (2)}$$

where R is a function of r only and Θ is a function of θ only.

From (2), $$\partial u/\partial r = R'\Theta, \qquad \partial^2 u/\partial r^2 = R''\Theta, \qquad \partial^2 u/\partial \theta^2 = R\Theta''. \quad \text{... (3)}$$

Using (3), (1) reduces to $$r^2 R''\Theta + r\,R'\Theta + R\Theta'' = 0.$$

or $$(r^2 R'' + r\,R')\,\Theta = -R\Theta'' \qquad \text{or} \qquad (r^2 R'' + r\,R')/R = -\Theta''/\Theta. \quad \text{... (4)}$$

Since the L.H.S. of (4) is a function of r only and the R.H.S. is a function of θ only, both sides of (4) must be equal to the same constant, say μ. Then, (5) leads to the following ordinary differential equations:

$$r^2(d^2R/dr^2) + r(dR/dr) - \mu R = 0 \quad \ldots (5)$$

and

$$d^2\Theta/d\theta^2 + \mu\Theta = 0. \quad \ldots (6)$$

As usual, we first reduce linear homogeneous differential equation (5) into a linear differential equation with constant coefficients.

Re-writing (5), $(r^2 D^2 + rD - \mu)R = 0$, where $D \equiv d/dr$. ... (7)

Let $r = e^z$ (or $z = \log r$) and $D_1 \equiv d/dz$

Then, we know that $rD = D_1$ and $r^2 D^2 = D_1 (D_1 - 1)$

Substituting these in (7), we get

$\{D_1 (D_1 - 1) + D_1 - \mu\} R = 0$ or $(D_1^2 - \mu) R = 0$. ... (8)

Again, let $D_2 \equiv d/d\theta$. Then (6) may be re-written as

$$(D_2^2 + \mu)\Theta = 0. \quad \ldots (9)$$

The solutions of (8) and (9) depend on μ. Three cases arise :

Case I. Let $\mu = 0$. Then (8) and (9) reduce to

$d^2R/dz^2 = 0$ and $d^2\Theta/d\theta^2 = 0$.

Solving these, $R = A_1 z + B_1 = A_1 \log r + B_1$ and $\Theta = C_1\theta + D_1$.

Hence, from (2), the required solution is of the form

$$u(r, \theta) = (A_1 \log r + B_1)(C_1\theta + D_1). \quad \ldots (10)$$

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$, *i.e.*, μ is positive. Then (8) and (9) become

$(D_1^2 - \lambda^2)R = 0$ and $(D_2^2 + \lambda^2)\Theta = 0$.

Solving these, $R = A_2 e^{\lambda z} + B_2 e^{-\lambda z} = A_2 (e^z)^{\lambda} + B_2 (e^z)^{-\lambda} = A_2 r^{\lambda} + B_2 r^{-\lambda}$

and

$$\Theta = C_2 \cos\lambda\theta + D_2 \sin\lambda\theta.$$

Hence, from (2), the required solution is of the form

$$u(r,\theta) = (A_2 r^{\lambda} + B_2 r^{-\lambda})(C_2 \cos\lambda\theta + D_2 \sin\lambda\theta). \quad \ldots (11)$$

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$, *i.e.*, μ is negative. Then (8) and (9) become

$(D_1^2 + \lambda^2)R = 0$ and $(D_2^2 - \lambda^2)\Theta = 0$.

Solving these, $R = A_3 \cos z + B_3 \sin z = A_3 \cos(\log r) + B_3 \sin(\log r)$

and

$$\Theta = C_3 e^{\lambda\theta} + D_3 e^{-\lambda\theta}.$$

Hence, from (2), the required solution is of the form

$$u(r,\theta) = \{A_3 \cos(\log r) + B_3 \sin(\log r)\}(C_3 e^{\lambda\theta} + D_3 e^{-\lambda\theta}). \quad \ldots (12)$$

The correct choice of most suitable solution among (10), (11) and (12) depends upon the physical nature of the problem under consideration and the given boundary conditions.

Ex. 7. (*a*) *A thin semi-circular plate of radius a has its boundary diameter kept at* 0°*C and its circumferene at f*(θ)*. Find the temperature distribution in the steady state.*

(*b*) *A thin semi-circular plate of radius a has its boundary diameter kept at* 0°*C and its circumference at* 100°*C. If u* (*r*, θ) *is the steady state temperature, find u* (*a*/4, π/2).

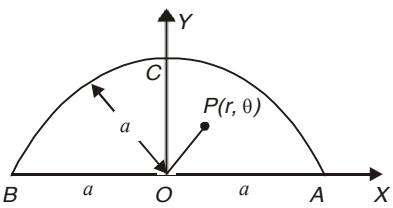
(*c*) *The boundary diameter of a semi-circular plate of radius* 10 *cm is kept at* 0°*C and its temperature along the semi-circular boundary is given by*

$$u(10,\theta)=\begin{cases}50\,\theta, \text{ for } 0\le\theta\le\pi/2\\ 50\,(\pi-\theta), \text{ for } \pi/2\le\theta\le\pi.\end{cases}$$

Find the staady state temperature u (*r,* θ) *at point in the plate.*

(*d*) *A semi-circular plate of radius a is kept at temperature* u_0 *along the bounding diameter and* u_1 *along the circumference. Find the steady state temperature at any point of the plate.*

Sol. Consider a thin semi-circular plate of radius *a* whose surfaces are insulated. Its bounding diameter *AB* is kept at temperature 0°C and its circumference *ACB* at temperature *f*(θ).



The steady state temperature *u* (*r*, θ) is the solution of the Laplace's equation in polar co-ordinates (*r*, θ), namely,

$$r^2(\partial^2u/\partial r^2)+r(\partial u/\partial r)+\partial^2u/\partial\theta^2=0. \quad \text{... (1)}$$

We are to solve (1) under the boundary conditions (B.C.): $u(r, 0) = u(r, \pi) = 0$ ...(2)

and $u(a, \theta) = f(\theta)$. ... (3)

Note that B.C. (2) expresses the fact that the temperature $u(r, \theta)$ is zero along *OA* ($\theta = 0$) and *OB* ($\theta = \pi$) so that temperature is zero along the bounding diameter *AB*. Again, the B.C. (3) expresses the fact that the temperature on the circumference *ACB* (where $r = a$) is $f(\theta)$.

Suppose (1) has a solution of the form $u(r,\theta)=R(r)\,\Theta(\theta)$, ... (4)

where *R* and Θ are functions of *r* and θ respectively.

Using (4), (1) becomes $R''\Theta+(1/r)R'\Theta+(1/r^2)R\Theta''=0$

or $(r^2R''+rR')/R=-\Theta''/\Theta$, ... (5)

where the dashes denote derivatives with respect to the relevant variables. Since the L.H.S. of (5) is function of *r* alone whereas the R.H.S. is a function of θ alone, hence (5) is true only if each of its side is equal to the same constant. For the solution of (1) satisfying the given boundary conditions, we choose this constant as λ^2 Then (5) gives

$r^2R''+rR'-\lambda^2R=0$ or $r^2(d^2R/dr^2) + r(dR/dr) - \lambda^2R = 0$ ... (6)

and $\Theta''+\lambda^2\Theta=0$ ... (7)

Putting $\theta = 0$ and $\theta = \pi$ by turn in (4) and using B.C. (2), we get

$0=R(r)\Theta(0)$ and $0=R(r)(\Theta)(\pi) \Rightarrow \Theta(0)=0$ and $(\Theta)(\pi)=0$, ... (8)

where we have taken $R(r)\neq 0$, since otherwise $u\equiv 0$ which does not satisfy B.C. (3). We now solve (7) under B.C. (8).

Solving (7), $\Theta(\theta)=A\cos\lambda\theta+B\sin\lambda\theta$, *A* and *B* being arbitrary, constants. ... (9)

Putting $\theta = 0$ and $\theta = \pi$ by turn in (9) and using (8), we get

$0 = A$ and $0 = B \sin \lambda\pi$ so that $A = 0$ and $\sin \lambda\pi = 0$, ... (10)

where we have taken $B \neq 0$ for getting non-trivial solution $u(r, \theta)$.

Now, $\sin \lambda\pi = 0 \Rightarrow \lambda\pi = n\pi \Rightarrow \lambda = n$, an integer.

Hence, (9) reduces to $\Theta(\theta) = B \sin n\theta$ ... (11)

We now solve linear homogeneous equation (6). Let $D \equiv d/dr$. With $\lambda = n$, (6) may be re-written as $(r^2 D^2 + rD - n^2)\, R = 0$ ... (12)

Let $r = e^z$ (or $z = \cos r$) and $D_1 \equiv d/dz$. so that $rD = D_1$ and $r^2D^2 = D_1 (D_1 - 1)$.

Then (12) reduces to

$[D_1 (D_1 + 1) + D_1 - n^2]R = 0$ or $(D_1^2 - n^2)R = 0$, whose solution is

$$R(r) = Ce^{nz} + De^{-nz} = C(e^z)^n + D(e^z)^{-n} = Cr^n + Dr^{-n}, \text{ as } r = e^z \quad \text{... (13)}$$

Due to physical nature of the given problem, $u(r, \theta)$ must be finite when $r \to 0$. Hence, $R(r)$ must be finite when $r \to 0$. For this purpose, we take $D = 0$ in (13), for otherwise $r^{-n} \to \infty$ as $r \to 0$. Then (13) reduces to

$$R(r) = C r^n. \quad \text{... (14)}$$

From (4), (11) and (14), non-zero solutions of (1) are given by

$u_n(r, \theta) = B C r^n \sin n\theta = E_n r^n \sin \theta$, taking $E_n = BC$.

Hence, the most general solution of (1) satisfying B.C. (2) is

$$u(r,\theta) = \sum_{n=1}^{\infty} u_n(r,\theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta. \quad \text{... (15)}$$

Putting $r = a$ in (15) and using B.C. (3), we have

$$f(\theta) = \sum_{n=1}^{\infty} \left(E_n a^n\right) \sin n\theta,$$

which is the Fourier half range sine series of $f(\theta)$ in $(0, \pi)$. So, we get (use result I (*b*) of Art 3.1).

$$E_n a^n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta \quad \text{or} \quad E_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta. \quad \text{... (16)}$$

The required solution is given by (15) wherein E_n is given by (16).

(*b*) This is a particular case of part (*a*). Here $f(a, \theta) = f(\theta) = 100$.

Here, as in part (*a*), $u(r,\theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta,$ ... (15)

where $$E_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta d\theta = \frac{2}{\pi a^n}\int_0^{\pi} 100 \sin n\theta\, d\theta = \frac{200}{\pi a^n}[-\cos n\theta]_0^{\pi}$$

$$= \frac{200}{n\pi a^n}[1-(-1)^n] = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 400/(2m-1)\pi a^{2m-1}, \text{ if } n = 2m-1,\ m = 1, 2, 3, \ldots \end{cases}$$

Substituting the above value of E_n in (15), we get

$$u(r,\theta) = \frac{400}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1} \frac{\sin(2m-1)\theta}{2m-1}$$

Putting $r = a/4$ and $\theta = \pi/2$ in above relation, we get

$$u(a/4, \pi/2) = \frac{400}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}\left(\frac{1}{4}\right)^{2m-1} = \frac{400}{\pi}\left[\frac{1}{4} - \frac{1}{3}\left(\frac{1}{4}\right)^3 + \left(\frac{1}{5}\right)\left(\frac{1}{4}\right)^5, \ldots\right]$$

$[\because \sin(2m-1) \times (\pi/2) = \sin(m\pi - \pi/2) = (-1)^{m+1}]$

(*c*) This is a particular case of part (*a*). Here $a = 10$ and

$$f(\theta) = u(10, \theta) = \begin{cases} 50\,\theta, \text{ for } 0 \le \theta \le \pi/2 \\ 50(\pi - \theta), \text{ for } \pi/2 \le \theta \le \pi. \end{cases} \quad \ldots (i)$$

As in part (*a*), $$u(r, \theta) = \sum_{r=1}^{\infty} E_n r^n \sin n\theta \quad \ldots (15)$$

where $$E_n = \frac{2}{\pi(10)^n}\int_0^{\pi} f(\theta)\sin n\theta$$

$$= \frac{2}{\pi(10)^2}\left[\int_0^{\pi/2} f(\theta)\sin n\theta d\theta + \int_{\pi/2}^{\pi} f(\theta)\sin n\theta\, d\theta\right]$$

$$= \frac{2}{\pi(10)^2}\left[\int_0^{\pi/2} 50\,\theta\sin n\theta\, d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta)\sin n\theta\, d\theta\right], \text{ using } (i)$$

$$= \frac{100}{\pi(10)^n}\left[\theta\left(-\frac{\cos n\theta}{n}\right) - (1)\left(-\frac{\sin n\theta}{n^2}\right)\right]_0^{\pi/2} + \frac{100}{\pi(10)^n}\left[(\pi - \theta)\left(-\frac{\cos n\theta}{n}\right) - (-1)\left(-\frac{\sin n\theta}{n^2}\right)\right]_{\pi/2}^{\pi}$$

[Using Chain rule of integration by parts]

$$= \frac{100}{\pi(10)^n}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{200}{\pi n^2}\sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ (-1)^{m+1} \times \left\{200/\pi(2m-1)^2\right\}, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, 4, \ldots \end{cases}$$

[Note that $\sin\{(2m-1) \times (\pi/2)\} = \sin(m\pi - \pi/2) = (-1)^{m+1}$]

Substituting the above value of E_n is (15), we get

$$u(r, \theta) = \frac{200}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{10}\right)^{2m-1}(-1)^{m-1}\frac{\sin(2m-1)\theta}{(2m-1)^2}$$

(*d*) Steady state temperature $u(r, \theta)$ is the solution of

$$r^2(\partial^2 u/\partial r^2) + r(\partial u/\partial r) + \partial^2 u/\partial\theta^2 = 0 \quad \ldots (i)$$

Given boundary conditions are

$$u(r, 0) = u(r, \pi) = u_0,\ 0 < r < a \quad \ldots (ii)$$

and $$u(a, \theta) = u_1,\ 0 \le \theta \le \pi. \quad \ldots (iii)$$

Let $$v(r, \theta) = u(r, \theta) - u_0 \quad \ldots (iv)$$

so that $$u(r, \theta) = v(r, \theta) + u_0 \quad \ldots (v)$$

Substituting the value of u from (v) in (i), we get

$$r^2(\partial^2 v/\partial r^2) + r(\partial v/\partial r) + \partial^2 v/\partial\theta^2 = 0 \quad \ldots (1)$$

Again, using (*ii*) and (*iii*), (*iv*) gives

$$v(r, 0) = v(r, \pi) = 0, \ 0 < r < a \qquad \ldots (2)$$

and
$$v(a, 0) = u_1 - u_0 = f(\theta), \text{ say} \qquad \ldots (3)$$

Note that (1), (2) and (3) are exactly the same as the respective equations (1), (2) and (3) of part (*a*) of the present exercise. Here, we have $u = v$ and $f(\theta) = u_1 = u_0$. So, as in part (*a*), we have

$$v(r, \theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta, \qquad \ldots (15)$$

where
$$E_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta = \frac{2}{\pi a^n}\int_0^{\pi}(u_1 - u_0)\sin n\theta\, d\theta$$

$$= \frac{2(u_1 - u_0)}{\pi a^n}\left[-\frac{\cos n\theta}{n}\right]_0^{\pi} = \frac{2(u_1 - u_0)}{\pi n a^n}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, ... \\ 4(u_1 - u_0)/\pi(2m-1)a^{2m-1}, \text{ if } n = 2m-1,\ m = 1, 2, 3, ... \end{cases}$$

Then, (15) gives
$$v(r, \theta) = \frac{4(u_1 - u_0)}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1}$$

Substituting this value in (v), the required temperature $u(r, \theta)$ is given by

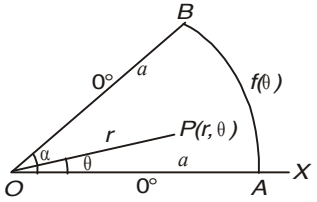
$$u(r, \theta) = u_0 + \frac{4(u_1 - u_0)}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1}$$

Ex. 8. (*a*) *A circular sector is determined by* $0 \le r \le a, 0 \le \theta \le \alpha$. *The temperature is kept* $0°C$ *along the straight edges and at* $f(\theta)$ *along the curved edge. Find the steady state temperature at any point of the sector with its surface insulated.*

(*b*) *Find the steady state temperature at the points in the sector given by* $0 \le \theta \le \pi/4$, $0 \le r \le a$ *of a circular plate if the temperature is maintained at* $0°C$ *along the side edges and at a constant temperature* $u_0\, C$ *along the curved edge.*

(*c*) *A circular sector is determined by* $0 \le r \le a, \ 0 \le \theta \le \pi/2$. *The temperature is kept at* $0°C$ *along the straight edges and at temperature* $50(\pi\theta - 2\theta^2)$ *along the curved edge. Find the steady state temperature at* $(a/2, \pi/4)$.

Sol. (*a*) Consider a thin circular sector OAB of radius a such that $\angle AOB = \alpha$. The steady state temperature $u(r, \theta)$ is the solution of the Laplace's equation in polar co-ordinates (r, θ) given by



$$r^2(\partial^2 u/\partial r^2) + r(\partial u/\partial r) + \partial^2 u/\partial\theta^2 = 0 \qquad \ldots (1)$$

We are to solve (1) under the following boundary conditions (B.C.):

$$u(r, 0) = u(r, \alpha) = 0, \ 0 < r < a \qquad \ldots (2)$$

and
$$u(a, \theta) = f(\theta), \ 0 < \theta < \alpha. \qquad \ldots (3)$$

Note that B.C. (2) expresses the fact that the temperature $u(r, \theta)$ is zero along the straight edges OA and OB of the given sector OAB and the B.C. (3) expresses the fact that the temperature on curved edge AB of the sector is $f(\theta)$.

Suppose (1) has a solution of the form $u(r,\theta) = R(r)\,\Theta(\theta)$, ... (4)

where R and Θ are functions of r and θ respectively.

Using (4), (1) becomes, $R''\Theta + (1/r)R'\Theta + (1/r^2)R\Theta'' = 0$

or $\left(r^2R'' + rR'\right)/R = -\Theta''/\Theta,$ ... (5)

where the dashes denote derivatives with respect to the relevant variables. Since the L.H.S. of (5) is a function of r alone whereas the R.H.S. is a function of θ alone, hence (5) is true only if each of its side is equal to the same constant. For the solution of (1) satisfying the given boundary conditions, we choose this constant as λ^2. Then, (5) gives

$r^2R'' + rR' - \lambda^2 R = 0$ or $r^2(d^2R/dr^2) + r(dR/dr) - \lambda^2 R = 0$ ... (6)

and $\Theta'' + \lambda^2\Theta = 0.$ ... (7)

Putting $\theta = 0$ and $\theta = \alpha$ by turn in (4) and using B.C. (2), we get

$0 = R(r)\,\Theta(0)$ and $0 = R(r)\Theta(\alpha) \Rightarrow \Theta(\theta) = 0$ and $\Theta(\alpha) = 0,$ ... (8)

where we have taken $R(r) \neq 0,$ since otherwise $u \equiv 0$ which does not satisfy B.C. (3).

We now solve (7) under boundary conditons (8).

Solving (7), $\Theta(\theta) = A\cos\lambda\theta + B\sin\lambda\theta,$ A and B being arbitrary constants ... (9)

Putting $\theta = 0$ and $\theta = \alpha$ by turn in (9) and using (8), we get

$0 = A$ and $0 = B\sin\lambda\alpha$ so that $A = 0$ and $\sin\lambda\alpha = 0,$

where we have taken $B \neq 0$ for getting non-trivial solution $u(r, \theta)$.

Now, $\sin\lambda\alpha = 0 \Rightarrow \lambda\alpha = n\pi \Rightarrow \lambda = n\pi/\alpha$, n being an integer. ... (10)

Hence, (9) reduces to $\Theta(\theta) = B\sin(n\pi\theta/\alpha)$ ... (11)

We now solve linear homogeneous equation (6). Let $D \equiv d/dr.$ Then (6) may be re-written

$(r^2D^2 + rD - \lambda^2)\,R = 0$ ... (12)

Let $r = e^z$ (or $z = \log r$) and $D_1 \equiv d/dz.$ Then $rD = D_1$ and $r^2D^2 = D_1(D_1 - 1)$ and hence (12) reduces to $[D_1(D_1 - 1) + D_1 - \lambda^2]\,R = 0$

or $(D_1^2 - \lambda^2)R = 0$ whose solution is

$R(r) = Ce^{\lambda z} + De^{-\lambda z} = C(e^z)^{\lambda} + D(e^z)^{-\lambda} = Cr^{\lambda} + Dr^{-\lambda}$, as $r = e^z$ ... (13)

or $R(r) = Cr^{n\pi/\alpha} + Dr^{-n\pi/\alpha}$, using (10).

Dur to physical nature of the given problem, $u(r, \theta)$ must be finite when $r \to 0.$ Hence, $R(r)$ must be finite when $r \to 0.$ For this problem, take $D = 0$ in (13), for otherwise $r^{-n\pi/a} \to 0$ as $r \to 0.$

Then (13) reduces to $R(r) = Cr^{n\pi/\alpha}.$ ... (14)

From (4), (11) and (14), non-zero solutions of (1) are given by

$$u_n(r,\theta) = BCr^{n\pi/\alpha}\sin(n\pi\theta/\alpha) = E_n r^{n\pi/\alpha}\sin(n\pi\theta/\alpha), \quad \text{taking } E_n = BC$$

Hence, the most general solution of (1) satisfying (2) is

$$u(r,\theta) = \sum_{n=1}^{\infty} u_n(r,\theta) = \sum_{n=1}^{\infty} E_n r^{n\pi/\alpha}\sin(n\pi\theta/\alpha) \quad \text{... (15)}$$

Putting $r = a$ in (15) and using B.C. (3), we get

$$f(\theta) = \sum_{n=1}^{\infty} \left(E_n a^{n\pi/\alpha} \right) \sin(n\pi\theta/\alpha),$$

which is the Fourier half range sine series of $f(\theta)$ in $(0, \alpha)$. Hence, we get (use result I(*b*) of Art. 3.1)

$$E_n a^{n\pi/\alpha} = \frac{2}{\alpha} \int_0^{\alpha} f(\theta) \sin \frac{n\pi\theta}{\alpha} d\theta \qquad \text{or} \qquad E_n = \frac{2}{\alpha a^{n\pi/\alpha}} \int_0^{\alpha} f(\theta) \sin \frac{n\pi\theta}{\alpha} d\theta. \qquad \ldots (16)$$

The required temperature $u\,(r, \theta)$ is given by (15) wherein the constants E_n are given by (16).

Particular case. If $\alpha = \pi$, the thin plate reduces to a semi-circular plate and we get results (15) and (16) of Ex. 7 (*a*).

(*b*) This is a particular case of part (*a*). Here $f(a, \theta) = f(\theta) = u_0$ and $\alpha = \pi/4$. Proceed as in part (*a*). Then, from (15), we have

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n r^{4n} \sin 4n\,\theta \qquad \ldots (i)$$

From (16), $E_n = \dfrac{2}{(\pi/4) \times a^{4n}} \displaystyle\int_0^{\pi/4} u_0 \sin 4n\theta\, d\theta = \frac{8u_0}{\pi a^{4n}} \left[-\frac{\cos 4n\theta}{4n} \right]_0^{\pi/4} = \frac{2u_0}{n\pi a^{4n}} \left[1 - (-1)^n \right]$

$$\therefore \qquad E_n = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ (4u_0)/\pi(2m-1)a^{4(2m-1)}, \text{ if } n = 2m-1,\ m = 1, 2, 3, \ldots \end{cases}$$

Substituting this in (*i*), the required temperature distribution is

$$u(r, \theta) = \frac{4u_0}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a} \right)^{4(2m-1)} \frac{\sin 4(2m-1)\theta}{2m-1}$$

(*c*) This is a particular case of part (*a*). Here $f(a, \theta) = 50(\pi\theta - 2\theta^2)$ and $\alpha = \pi/2$.

Proceed as in part (*a*). Then, from (15), we have

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n r^{2n} \sin 2n\theta. \qquad \ldots (i)$$

From (16), we have

$$E_n = \frac{2}{(\pi/2) \times a^{2n}} \int_0^{\pi/2} 50(\pi\theta - 2\theta^2) \sin 2n\theta\, d\theta = \frac{200}{\pi a^{2n}} \int_0^{\pi/2} (\pi\theta - 2\theta^2) \sin 2n\theta\, d\theta$$

$$= \frac{200}{\pi a^{2n}} \left[(\pi\theta - 2\theta^2) \left(-\frac{\cos 2n\theta}{2n} \right) - (\pi - 4\theta) \left(-\frac{\sin 2n\theta}{4n^2} \right) + (-4) \left(\frac{\cos 2n\theta}{8n^3} \right) \right]_0^{\pi/2}$$

[Using chain rule of integration by parts]

$$= \frac{200}{\pi a^{2n}} \left[-\frac{4\cos n\pi}{8n^3} + \frac{4}{8n^3} \right] = \frac{100}{\pi n^3 a^{2n}} \left[1 - (-1)^n \right]$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 200/\pi(2m-1)^3 a^{2(2m-1)}, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

$\therefore$ (*i*) gives $$u(r,\theta)=\frac{200}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2(2m-1)}\frac{\sin 2(2m-1)\theta}{(2m-1)^3} \quad \ldots (ii)$$

Putting $r = a/2$ and $\theta = \pi/4$ in (*ii*), we get

$$u\left(\frac{a}{2},\frac{\pi}{4}\right)=\frac{200}{\pi}\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^{4m-2}\frac{\sin(m\pi-\pi/2)}{(2m-1)^3}=\frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2^{4m-2}}\frac{1}{(2m-1)^3}$$

$$=(200/\pi)\times\left[(1/2^2)-(1/2^6)\times(1/3^3)+(1/2^{10})\times(1/5^3)-.....\right].$$

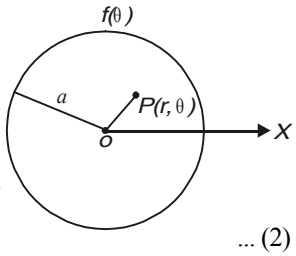
Ex. 9 (*a*) **Interior Dirichlet problem for a circle :** Find the steady state temperature in a circular plate of radius *a* whose circumference is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface.

(*b*) *Find the steady state temperature in a circular plate of radius a which has one half of its circumference at* $0°C$ *and the other half at constant temperature* $u_0 C$.

(*c*) *Find the steady state temperature in a circular plate of radius a which as one half of its circumference at* $0°C$ *and the other half at* $100°C$.

(*d*) *A circular plate of radius a has half of its boundary kept at constant temperature* u_1 *and the other half at constant temperature* u_2, *where* $u_1 \neq u_2$. *Find the steady state temperature of the plate.* **[Delhi B.Sc. Physics (H) 1998]**

Sol. (*a*) Consider a thin circular plate of radius *a* with its surface insulated. The steady state temperature $u(r, \theta)$ is the solution of the Laplace's equation in polar co-ordinates (r, θ), namely,



$$r^2(\partial^2u/\partial r^2)+r(\partial u/\partial r)+\partial^2u/\partial\theta^2=0. \quad \ldots (1)$$

We are to solve (1) under the given boundary condition, namely

$$u(a,\theta)=f(\theta),\ 0\le\theta\le 2\pi. \quad \ldots (2)$$

The required solution $u(r, \theta)$ must be periodic in θ and is finite when $r \to 0$. Keeping these properties of $u(r, \theta)$, we now solve (1).

Suppose (1) has a solution of the form

$$u(r,\theta)=R(r)\Theta(\theta), \quad \ldots (3)$$

where R and Θ are function of r and θ respectively.

Using (3), (1) reduces to $$r^2R''\Theta+rR'\Theta+R\Theta''=0$$

or $$(r^2R''+rR')\Theta=-R\Theta'' \quad \text{or} \quad (r^2R''+rR')/R=-\Theta''/\Theta. \quad \ldots (4)$$

Since the L.H.S. of (4) is a function of r only and the R.H.S. is a function of θ only, the two sides of (4) must be equal to the same constant, say μ. Then, (5) gives

$$r^2(d^2R/dr^2)+r(dR/dr)-\mu R=0 \quad \ldots (5)$$

or $$d^2\Theta/d\theta^2+\mu\Theta=0. \quad \ldots (6)$$

As usual, we first reduce linear homogeneous differential equation (5) into a linear differential equation with constant coefficients.

Rewritten (5), $(r^2D^2+rD-\mu)R=0$, where $D\equiv d/dr$. $\ldots (7)$

Let $r = e^z$ (or $z = \log r$) and $D_1 = d/dz$. Then, we know that $rD = D_1$ and $r^2D^2 = D_1(D_1 - 1)$.

Substituting these in (7), we get

$$\{D_1(D_1 - 1) + D_1 - \mu\}\ R = 0 \qquad \text{or} \qquad (D_1^2 - \mu)R = 0. \qquad \ldots (8)$$

Again, let $D_2 \equiv d/d\theta$. Then, (6) may be re-written as $(D_2^2 + \mu)\Theta = 0.$ ... (9)

The solutions of (8) and (9) depend on μ. Consider the following cases:

Case I. Let $\mu = 0$. Then (8) and (9) reduces to

$$d^2R/dz^2 = 0 \qquad \text{and} \qquad d^2\Theta/d\theta^2 = 0.$$

Solving these, $R(r) = A_1 z + B_1 = A_1 \log r + B_1$ and $\Theta = C_1\theta + D_1$.

Hence, from (3), a solution of (1) is of the form

$$u(r, \theta) = (A_1 \log r + B_1)\,(C_1\theta + D_1). \qquad \ldots (10)$$

Since $u(r, \theta)$ is periodic in θ and is finite when $r \to 0$, so we must taken $A_1 = 0$ and $C_1 = 0$.

Then, (10) reduces to $u(r, \theta) = B_1D_1 = E_0/2$, say ... (11)

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then (8) and (9) become

$$(D_1^2 - \lambda^2)R = 0 \qquad \text{and} \qquad (D_2^2 + \lambda^2)\Theta = 0 \qquad \ldots (12)$$

[Note that we cannot choose $\mu = -\lambda^2$ because it will lead to $(D_2^2 - \lambda^2)\Theta = 0$ whose solution will not contain trigonometric functions and hence periodic nature of $u(r, \theta)$ will not be attained.]

Solving (12), $R(r) = A_2e^{\lambda z} + B_2e^{-\lambda z} = A_2r^{\lambda} + B_2r^{-\lambda}$ and $\Theta(\theta) = C_2 \cos\lambda\theta + D_2 \sin\lambda\theta$ ... (13)

Since $u(r, \theta)$ is periodic in θ with period 2π, we must take $\lambda = n$, where $n = 1, 2, 3, \ldots$

Using (13), (3) reduces to

$$u(r,\theta) = (\mathrm{A}_2r^n + B_2r^{-n})(C_2 \cos n\theta + D_2 \sin n\theta),\ n = 1, 2, 3, \ldots \qquad \ldots (14)$$

Furthermore, since $u(r, \theta)$ is finite when $r \to 0$, we must take $B_2 = 0$, So, (14) becomes

$$u(r,\theta) = A_2r^n(C_2 \cos n\theta + D_2 \sin n\theta) \qquad \text{or} \qquad u(r,\theta) = r^n(E_n \cos n\theta + F_n \sin n\theta), \qquad \ldots (15)$$

where $E_n = (A_2C_2)$ and $F_n\ (= A_2D_2)$ are new arbitrary constants.

Keeping (11) and (15) in view, the most general solution of (1) is

$$u(r,\theta) = \frac{E_0}{2} + \sum_{n=1}^{\infty} r^n(E_n \cos n\theta + F_n \sin n\theta). \qquad \ldots (16)$$

Putting $r = a$ in (16) and using B.C. (2), we get

$$f(\theta) = \frac{E_0}{2} + \sum_{n=1}^{\infty}(E_n a^n \cos n\theta + F_n a^n \sin n\theta), \qquad \ldots (17)$$

which is usual expansion of $f(\theta)$ as Fourier series in $(0, 2\pi)$. So we get (use result 2 of Art 3.1)

$$E_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta \qquad E_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta) \cos n\theta\,d\theta \qquad \text{and} \qquad F_n = \frac{1}{a^n\pi}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta \qquad \ldots (18)$$

Hence, the required temperature is given by (16) wherein the constants E_0, E_n and F_n are given by (18).

(*b*) This a particular case of part (*a*). Here, boundary condition (2) is modified as

$$u(a,\theta) = f(\theta) = \begin{cases} u_0, \text{ for } 0 < \theta < \pi \\ 0, \text{ for } \pi < \theta < 2\pi \end{cases} \qquad \ldots (2)'$$

Now proceed exactly as in part (*a*) by taking (2)′ in plane of (2). Solution is given by (16). To find constants E_0, E_n and F_n, we use (18) and (2)′. Thus, we get

$$E_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{\pi}\left[\int_0^{\pi} f(\theta)\,d\theta + \int_{\pi}^{2\pi} f(\theta)\,d\theta\right] = \frac{1}{\pi}\int_0^{\pi} u_0 d\theta, \text{ using (2)}$$

$$= (1/\pi)\times\left[u_0\theta\right]_0^{\pi} = (1/\pi)\times\left[u_0\pi\right] = u_0$$

$$E_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{a^n\pi}\left[\int_0^{\pi} f(\theta)\cos n\,\theta\,d\theta + \int_{\pi}^{2\pi} f(\theta)\cos n\,\theta\,d\theta\right]$$

$$= \frac{1}{a^n\pi}\int_0^{\pi} u_0\cos n\,\theta\,d\,\theta = \frac{u_0}{a^n\pi}\left[\frac{\sin n\,\theta}{n}\right]_0^{\pi} = 0$$

$$F_n = \frac{1}{a^n\pi}\int_0^{\pi} u_0\sin n\,\theta\,d\,\theta = \frac{u_0}{a^n\pi}\left[-\frac{\cos n\,\theta}{n}\right]_0^{\pi} = \frac{u_0}{a^n n\pi}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ (2u_0)/\pi(2m-1)a^{2m-1}, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting these values in (16), the required temperature is

$$u(r,\theta) = \frac{u_0}{2} + \frac{2u_0}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1}. \qquad \ldots (i)$$

(*c*) This is a particular case of part (*b*). Hence $u_0 = 100$. Hence, from (*i*)

$$u(r,\theta) = 50 + \frac{200}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1}.$$

(*d*) This is a particular case of part (*a*). According to the given problem, the boundary conditions may be taken in the following form

$$u(a,\theta) = f(\theta) = \begin{cases} u_1, \text{ for } 0 < \theta < \pi \\ u_2, \text{ for } \pi < \theta < 2\pi \end{cases} \qquad \ldots (2)'$$

Now proceed exactly as in part (*a*) by taking (2)′ in place of (2). Solution is given to (16). To find E_0, E_n and F_n, we use (18) and (2)′. Thus, we get

$$E_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{1}{\pi}\left[\int_0^{\pi} f(\theta)d\theta + \int_{\pi}^{2\pi} f(\theta)\,d\theta\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} u_1 d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} u_2 d\theta = \frac{u_1}{\pi}\left[\theta\right]_0^{\pi} + \frac{u_2}{\pi}\left[\theta\right]_{\pi}^{2\pi} = u_1 + u_2$$

$$E_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos n\,\theta\,d\,\theta = \frac{1}{a^n\pi}\left[\int_0^{\pi} f(\theta)\cos n\,\theta\,d\,\theta + \int_{\pi}^{2\pi} f(\theta)\cos n\,\theta\,d\,\theta\right]$$

$$= \frac{1}{a^n\pi}\int_0^{\pi} u_1\cos n\,\theta\,d\,\theta + \frac{1}{a^n\pi}\int_{\pi}^{2\pi} u_2\cos n\,\theta d\,\theta = \frac{u_1}{a^n\pi}\left[\frac{\sin n\pi}{n}\right]_0^{\pi} + \frac{u_2}{a^n\pi}\left[\frac{\sin n\,\theta}{n}\right]_{\pi}^{2\pi} = 0$$

Thus, $E_n = 0$, for $n = 1, 2, 3, \ldots$

$$F_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta = \frac{1}{a^n\pi}\left[\int_0^{\pi} f(\theta)\sin n\theta\, d\theta + \int_{\pi}^{2\pi} f(\theta)\sin n\theta\, d\theta\right]$$

$$= \frac{1}{a^n\pi}\int_0^{\pi} u_1 \sin n\theta\, d\theta + \frac{1}{a^n\pi}\int_{\pi}^{2\pi} u_2 \sin n\theta\, d\theta = \frac{u_1}{a^n\pi}\left[-\frac{\cos n\theta}{n}\right]_0^{\pi} + \frac{u_2}{a^n\pi}\left[-\frac{\cos n\theta}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{u_1(1-\cos n\pi)}{n\pi a^n} - \frac{u_2(1-\cos n\pi)}{n\pi a^n} = \frac{(u_1-u_2)\{1-(-1)^n\}}{n\pi\, a^n}$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ (even)}, m = 1,2,3,... \\ 2(u_1-u_2)/(2m-1)\pi a^{2m-1}, \text{ if } n = 2m-1 \text{(odd)}, m = 1,2,3... \end{cases}$$

Substituting the above values of E_0, E_n and F_n in (16), the required solution is given by

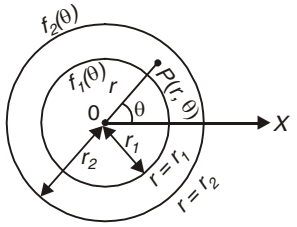
$$u(r,\theta) = \frac{u_1+u_2}{2} + \sum_{m=1}^{\infty} \frac{2(u_1-u_2)r^{2m-1}}{(2m-1)\,\pi\, a^{2m-1}} \sin(2m-1)\theta = \frac{u_1+u_2}{2} + \frac{2(u_1-u_2)}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{(2m-1)}$$

Ex. 10. (*a*) *Consider a circular annulus of inner radius* r_1 *and outer radius* r_2. *Let the surface of the annulus be insulated. Find the steady state temperature at any point* (r, θ) *in the annulus, given that the temperature distribution along the inner circle* $r = r_1$ *and the outer circle* $r = r_2$ *are maintained as* $u(r_1, \theta) = f_1(\theta)$ *and* $u(r_2, \theta) = f_2(\theta)$.

(*b*) *A plate in the form of a ring is bounded by the circles* $r = 2$ *and* $r = 4$. *Its surfaces are insulated and the temperature* $u(r, \theta)$ *along the boundary are* $u(2, \theta) = 6\cos\theta + 10\sin\theta$ *and* $u(4, \theta) = 15\cos\theta + 17\sin\theta$. *Find the steady state temperature* $u(r, \theta)$ *in the ring.*

(*c*) *Along the inner boundary of a circular annulus of radii* 10 *cm and* 20 *cm the temperature is maintained as* $u(10, \theta) = 15\cos\theta$ *and along the outer boundary the temperature* $u(20, \theta) = 30\sin\theta$ *is maintained. Find the steady state temperature at an arbitrary point* (r, θ) *in the annulus.*

Sol. We are given a circular annulus whose inner and outer radii are r_1 and r_2 respectively. The steady state temperature $u(r, \theta)$ at any point $P(r, \theta)$ of the annulus is the solution of the Laplace's equation in polar co-ordinates (r, θ), namely



$$r^2(\partial^2 u/\partial r^2) + r(\partial u/\partial r) + \partial^2 u/\partial\theta^2 = 0. \qquad \ldots (1)$$

Since temperatures along the inner boundary $(r = r_1)$ and outer boundary $(r = r_2)$ are maintained at $f_1(\theta)$ and $f_2(\theta)$ respectively, we have

$$u(r_1, \theta) = f_1(\theta) \quad \text{and} \quad u(r_2, \theta) = f_2(\theta), \quad 0 \le \theta \le 2\pi. \quad \ldots (2)$$

Clearly the temperature function $u(r, \theta)$ must be periodic in θ of period 2π. Accordingly, we now proceed to solve (1).

Suppose (1) has a solution of the form $u(r,\theta) = R(r)\Theta(\theta), \quad \ldots (3)$

where R and Θ are functions of r and θ respectively.

Using (3), (1) reduces to $r^2R''\Theta + rR'\Theta + R\Theta'' = 0$

or $(r^2R'' + rR')\Theta = -R\Theta''$ or $(r^2R'' + rR')/R = -\Theta''/\Theta. \quad \ldots (4)$

Since the L.H.S. of (4) is the function of r only and the R.H.S. is a function of θ only, the two sides of (4) must be equal to the same constant, say μ. Then, (5) gives

$$r^2 (d^2R / dr^2) + r (dR / dr) - \mu R = 0 \qquad \ldots (5)$$

and
$$d^2\Theta / d\theta^2 + \mu\Theta = 0. \qquad \ldots (6)$$

As usual, we first reduce linear homogeneous differnetial equation (5) into a linear differential equation with constant coefficients.

Rewriting (5), $(r^2 D^2 + rD - \mu)\, R = 0,$ where $D \equiv d / dr.$ ... (7)

Let $r = e^z$ (or $z = \log r$) and $D_1 \equiv d / dz.$ Then, we know that $rD_1 = D_1$ and $r^2D^2 = D_1(D_1 - 1)$. Substituting these in (7), we get

$$\{D_1 (D_1 - 1) + D_1 - \mu\}R = 0 \quad \text{or} \quad (D_1^2 - \mu)R = 0 \qquad \ldots (8)$$

Again, let $D_2 \equiv d / d\theta.$ Then, (6) may be re-written as $(D_2^2 + \mu)\Theta = 0.$ ... (9)

The solutions of (8) and (9) depend on μ. Consider the following three cases:

Case I. Let $\mu = 0$. Then (8) and (9) reduce to

$$d^2R / dz^2 = 0 \quad \text{and} \quad d^2\Theta / d\theta^2 = 0$$

Solving these, $R(r) = A_1 z + B_1 = A_1 \log r + B_1$ and $\Theta = C_1\theta + D_1$

Hence, from (3), a solution of (1) is of the form

$$u(r, \theta) = (A_1 \log r + B_1)\,(C_1\theta + D_1) \qquad \ldots (10)$$

Since $u(r, \theta)$ is periodic in θ, we must take $C_1 = 0$. Then the above solution (10) becomes

$$u(r, \theta) = (A_1 \log r + B_1)\, D_1 = (1/2) \times (a_0 \log r + b_0), \qquad \ldots (11)$$

where a_0 (= 2 A_1D_1) and b_0 (= $2B_1D_1$) are new arbitrary constants.

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0.$ Then (8) and (9) become

$$(D_1^2 - \lambda^2)R = 0 \quad \text{and} \quad (D_2^2 + \lambda^2)\Theta = 0 \qquad \ldots (12)$$

[Note that we cannot choose $\mu = -\lambda^2$ because it will lead to $(D_2^2 - \lambda^2)\Theta = 0$ whose solution will not contain trigonometric functions and hence periodic nature of $u(r, \theta)$ will not be attained.]

Solving (12), $R(r) = A_2 e^{\lambda x} + B_2 e^{-\lambda z} = A_2 (e^z)^{\lambda} + B_2 (e^z)^{-\lambda} = A_2 r^{\lambda} + B_2 r^{-\lambda}$

and
$$\Theta(\theta) = C_2 \cos\lambda\theta + D_2 \sin\lambda\theta.$$

Hence, from (3), a solution of (1) is of the form

$$u(r, \theta) = (A_2 r^{\lambda} + B_2 r^{-\lambda})(C_2 \cos\lambda\theta + D_2 \sin\lambda\theta). \qquad \ldots (13)$$

Since $u(r, \theta)$ is periodic in θ with period 2π, we must take $\lambda = n$, where $n = 1, 2, 3, \ldots$. Hence, (13) takes the form

$$u(r, \theta) = (A_2 r^n + B_2 r^{-n})(C_2 \cos n\theta + D_2 \sin n\theta), \quad n = 1, 2, 3, \ldots \qquad \ldots (14)$$

With help of (11) and (14), the most general solution of (1) is

$$u(r, \theta) = \frac{a_0 \log r + b_0}{2} + \sum_{n=1}^{\infty} \left[\left(a_n r^n + b_n r^{-n} \right) \cos n\theta + \left(c_n r^n + d_n r^{-n} \right) \sin n\theta \right], \qquad \ldots (15)$$

which holds for $r_1 \le r \le r_2$. Here a_n (= A_2C_2), b_n (= B_2C_2), c_n (= A_2D_2) and d_n (= B_2D_2) are new arbitrary constants.

Putting $r = r_1$ and $r = r_2$ by turn in (15) and using boundary conditon (2), we have

$$f_1(\theta) = \frac{a_0 \log r_1 + b_0}{2} + \sum_{n=1}^{\infty}\left[\left(a_n r_1^n + b_n r_1^{-n}\right)\cos n\,\theta + \left(c_n r_1^n + d_n r_1^{-n}\right)\sin n\,\theta\right]. \quad \ldots (16)$$

and $$f_2(\theta) = \frac{a_0 \log r_2 + b_0}{2} + \sum_{n=1}^{\infty}\left[\left(a_n r_2^n + b_n r_2^{-n}\right)\cos n\theta + \left(c_n r_2^n + d_n r_2^{-n}\right)\sin n\,\theta\right]. \quad \ldots (17)$$

(16) and (17) are usual expansions of $f_1(\theta)$ and $f_2(\theta)$ as Fourier series in $(0, 2\pi)$. Hence, we have (use result 2 of Art. 3.1)

$$a_0 \log r_1 + b_0 = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)d\theta, \qquad a_0 \log r_2 + b_0 = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\,d\theta \quad \ldots (18)$$

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\cos n\theta\,d\theta, \qquad a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\cos n\theta\,d\theta \quad \ldots (19)$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\sin n\theta\,d\theta, \qquad c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\sin n\theta\,d\theta \quad \ldots (20)$$

Solving pair of equations in (18), (19) and (20) we can get a_0, b_0; a_n, b_n and c_n, d_n respectively. Substituting these value in (15), we obtain the required temperature distribution.

(*b*) This is a particular case of part (*a*). Here $r_1 = 2$, $r_2 = 4$, $f_1(\theta) = 6\cos\theta + 10\sin\theta$ and $f_2(\theta) = 15\cos\theta + 17\sin\theta$. Now, proceed as in part (a) upto equation (17).

Putting the given values of r_1, r_2, $f_1(\theta)$ and $f_2(\theta)$, (16) and (17) reduce to

$$6\cos\theta + 10\sin\theta = (1/2)\times(a_0 \log 2 + b_0) + \sum_{n=1}^{\infty}\left(a_n 2^n + b_n 2^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(c_n 2^n + d_n 2^{-n}\right)\sin n\theta \ldots (i)$$

$$15\cos\theta + 17\sin\theta = (1/2)\times(a_0 \log 4 + b_0) + \sum_{n=1}^{\infty}\left(a_n 4^n + b_n 4^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(c_n 4^n + d_n 4^{-n}\right)\sin n\theta. \ldots (ii)$$

Equating the constants in (*i*) and (*ii*), we get

$(1/2)\times(a_0 \log 2 + b_0) = 0$ and $(1/2)\times(a_0 \log 4 + b_0) = 0 \Rightarrow a_0 = b_0 = 0.$ ... (*iii*)

Equating the coefficients of $\cos\theta$ in (*i*) and (*ii*), we get

$2a_1 + 2^{-1} b_1 = 6$ and $4a_1 + 4^{-1}b_1 = 15$ so that $a_1 = 4$ and $b_1 = -4.$ ... (*iv*)

Equating the coefficients of $\sin\theta$ in (*i*) and (*ii*), we get

$2c_1 + 2^{-1}d_1 = 10$ and $4c_1 + 4^{-1}d_1 = 17$ so that $c_1 = 4$ and $d_1 = 4$ ... (*v*)

Also, we note that $a_n = b_n = c_n = d_n = 0$ for all $n \geq 2.$ ... (*vi*)

Using (*ii*), (*iii*), (*iv*), (*v*) and (*vi*) in (15) of part (*a*), we get

$$u(r, \theta) = (4r - 4r^{-1})\cos\theta + (4r + 4r^{-1})\sin\theta$$

(*c*) This is a particular case of part (*a*). Here $r_1 = 10$, $r_2 = 20$, $f_1(\theta) = 15\cos\theta$ and $f_2(\theta) = 30\sin\theta$. Now, proceed as in a part (*a*) upto equation (17). Putting the given values of r_1, r_2, $f_1(\theta)$ and $f_2(\theta)$, (16) and (17) reduce to

$$15\cos\theta = (1/2)\times(a_0 \log 10 + b_0) + \sum_{n=1}^{\infty}\left(a_n 10^n + b_n 10^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(c_n 10^n + d_n 10^{-n}\right)\sin n\theta \quad \ldots (i)$$

$$30\sin\theta = (1/2)\times(a_0 \log 20 + b_0) + \sum_{n=1}^{\infty}\left(a_n 20^n + b_n 20^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(c_n 20^n + d_n 20^{-n}\right)\sin n\theta \quad \ldots (ii)$$

Equating the constants in (*i*) and (*ii*), we get

$(a_0 \log 10 + b_0)/2 = 0$ and $(a_0 \log 20 + b_0)/2 = 0 \Rightarrow a_0 = b_0 = 0$ ... (*iii*)

Equating the coefficients of cos θ on both sides of (*i*) and (*ii*), we get

$10a_1 + 10^{-1}b_1 = 15$ and $20a_1 + 20^{-1}b_1 = 0 \Rightarrow a_1 = -(1/2)$ and $b_1 = 200$ ... (*iv*)

Equating the coefficients of sin θ on both sides of (*i*) and (*ii*), we get

$10c_1 + 10^{-1}d_1 = 10$ and $20c_1 + 20^{-1}\, d_1 = 30 \Rightarrow c_1 = 2$ and $d_1 = -200$ ... (*v*)

Also, we have $a_n = b_n = c_n = d_n = 0$ for all $n \geq 2$. ... (*vi*)

Using (*iii*), (*iv*), (*v*) and (*vi*), equation (15) of part (*a*) reduces to

$$u(r, \theta) = \left\{-(r/2) + 200r^{-1}\right\}\cos\theta + \left(2r - 200r^{-1}\right)\sin\theta.$$

EXERCISE

1. Write down the Laplace's equation in two-dimensional polar coordinates. Obtain a solution by separating variables. What conditions the arbitrary constants must satisfy if (*i*) the solution is regular at $r = 0$ (*ii*) the solution exists for all values of θ ?

2. Show that the two dimensional harmonic equation $(\partial^2 V/\partial r^2) + (1/r)(\partial V/\partial r) + (1/r^2)(\partial^2 V/\partial\theta^2) = 0$ has solutions of the form $(Ar^n + Br^{-n})e^{\pm in\theta}$, where A, B and n are constants. Determine V satisfying the above equation in the region $0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi$ and satisfying the boundary conditions.

(*i*) V remains finite as $r \to 0$. (*ii*) $V = \sum_n C_n \cos n\theta$ on $r = a$.

[Delhi Maths (H) 1996, 2009; Meerut 1996]

3. The edge $r = a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss from either surface. Find the temperature distribution is steady state.

Ans. $u(r, \theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin\theta)$, where

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta, \qquad A_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \qquad B_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta,$$

4. Solve Dirichlet problem for the unit circle if the boundary function is $f(\theta) = \theta_0$, $-\pi < \theta < \pi$. **[Nagpur 2005]**

3.3 Temperature inside a circular plate

Consider a thin circular plate whose faces are impervious to heat flow and whose circular edge is kept at zero temperature. At $t = 0$, the initial temperature of the plate is a function $f(r)$ of the distance r from the centre of the plate. We propose to find temperature of the plate at any time t.

Due to symmetry the temperature u must be function of r only because $\partial u/\partial\theta = 0$. Hence equation of heat in polar coordinates reduces to

$$\partial u/\partial t = k\{\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r)\}, \qquad 0 \leq r \leq a \quad \text{... (1)}$$

The boundary condition is $u(a, t) = 0$, for all t ... (2)

and the initial condition is $u(r, 0) = f(r),\ 0 \leq r \leq a$ ... (3)

To solve (1), assume the solution of the form $u(r, t) = e^{-mt} R(r)$, ... (4)

where $R(r)$ is a function of r only and m is a constant. From (4), we get

$$\partial u/\partial t = -me^{-mt}R(r), \qquad \partial u/\partial r = e^{-mt}R'(r), \qquad \partial^2 u/\partial r^2 = e^{-mt}R''(r), \quad \text{... (5)}$$

where the dashes denote the derivatives with respect to r. Using (5), (1) reduces to

$$-me^{-mt}R = ke^{-mt}\{R'' + (1/2)\times R'\} \qquad \text{or} \qquad -mR = k\{R'' + (1/r)\times R'\}$$

or $$r^2R'' + rR' - (m/k)\times r^2R = 0 \quad \text{... (6)}$$

Let $$m/k = \lambda^2 \quad \text{... (7)}$$

Then (6) reduces to $$r^2R'' + rR' + \lambda^2 r^2 R = 0,$$

which is Besse's equation of zeroth order and hence its solution is

$$R(r) = A\,J_0(\lambda r) + B\,Y_0(\lambda r), \quad \text{... (8)}$$

where A and B are arbitrary constants and J_0, Y_0 are Bessel's functions of the zeroth order and of the first and second kind respectively.

Since the temperature must remain finite at $r = 0$, we choose $B = 0$. Here we have used the fact that $Y_0(\lambda r) \to \infty$ as $r \to 0$ and hence $u(r,t) \to \infty$ as $r \to 0$ by (4). Thus, (8) reduces to

$$R(r) = A\,J_0(\lambda r) \quad \text{... (9)}$$

Using (9), (4) yields $$u(r, t) = A\,e^{-mt}J_0(\lambda r) \quad \text{... (10)}$$

Putting $r = a$ in (10) and using the boundary condition (2), we have

$0 = A\,e^{-mt}J_0(\lambda a)$ or $$J_0(\lambda a) = 0, \quad \text{... (11)}$$

because for non-trivial solution, A cannot be equal to zero.

Hence only those values of λ are permissible that satisfy equation (11). Let these values be λ_i, $i = 1, 2, 3, \ldots$. Equation (7) gives the following values of m.

$$m_i = k\lambda_i^2 \quad \text{... (12)}$$

A particular solution of (1) that satisfies the boundary condition (2) is given with help of (10) and (12) in the following form: $$u_i(r,t) = A_i e^{-k\lambda_i^2 t}J_0(k_i r) \quad \text{... (13)}$$

If we sum over all possible values of i, we construct the general solution

$$u(r,t) = \sum_{i=1}^{\infty} A_i e^{-k\lambda_i^2 t}J_0(k_i r), \quad \text{... (14)}$$

where the arbitrary constants A_i must be determined from the initial condition (3) Putting $t = 0$ in (14), we have

$$f(r) = \sum_{i=1}^{\infty} A_i J_0(k_i r), \quad 0 \le r \le a \quad \text{... (15)}$$

which is Fourier-Bessel series and hence

$$A_i = \frac{2}{a^2\{J_1(\lambda_i a)\}^2}\int_0^a r\,f(r)J_0(\lambda_i r)\,dr, \quad i = 1, 2, 3, \ldots \quad \text{... (16)}$$

Hence the required temperature distribution $u(r, t)$ is given by (14) wherein the arbitrary constant A_i are given by (16).

3.4 The vibrations of a circular membrane

Consider vibrations of a circular membrane of radius a which is fixed along the boundary $r = a$. We propose to study the symmetrical oscillations of the membrane that is given an initial displacement $f(r)$ at $t = 0$. Also assume that at $t = 0$, the membrane has no initial velocity. Let us consider the symmetrical case where the motion is started in a symmetrical manner about the origin so that the displacement u is a function of r and t and is independent of the angle θ. This is the case of symmetrical oscillations about the origin. Hence wave equation in polar coordinates reduce to

$$\partial^2 u/\partial t^2 = c^2\{\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r)\}, \quad 0 \le r \le a \quad \text{... (1)}$$

The boundary condition is $u(a, t) = 0$, for all t ... (2)

and the inital condition are $u(r, 0) = f(r), \quad 0 \le r \le a$ ... (3*a*)

and $u_t(r, 0) = 0, \quad 0 \le r \le a$ ... (3*b*)

To solve (1), we assume the solution in the form

$$u(r, t) = e^{iwt} R(r), \qquad (i = \sqrt{-1}) \qquad \text{... (4)}$$

where $R(r)$ is a function of r only and w is a constant. From (4), we have

$$\partial^2 u / \partial t^2 = i^2 w^2 e^{iwt} R = -w^2 e^{iwt} R, \qquad \partial u / \partial r = e^{iwt} R', \qquad \partial^2 u / \partial r^2 = e^{iwt} R'' \quad \text{... (5)}$$

Using (5), (1) reduce to $-w^2 e^{iwt} R = c^2 e^{iwt} \{R'' + (1/r) \times R'\}$

or $-w^2 R = c^2 \{R'' + (1/r) \times R'\}$ or $rR'' + rR' + (w/c)^2 r^2 R = 0$ ... (6)

Let $k = w/c$ ... (7)

Then (6) reduces to $r^2 R'' + rR' + k^2 r^2 R = 0,$

which is Bessel's equation of zeroth order and hence its solution is

$$R(r) = A\, J_0 (kr) + B\, Y_0 (kr), \qquad \text{... (8)}$$

where A and B are arbitrary constants and J_0 and Y_0 are the Bessel's functions of zeroth order and of the first and second kind, respectively. Since the amplitude of the oscillations must remain finite at the origin (*i.e.*, at $r = 0$), we must choose $B = 0$. Here we have used the fact that $Y_0(kr) \to \infty$ as $r \to 0$ and hence $u(r,t) \to \infty$ as $r \to 0$ by (4). Thus, (8) reduces to

$$u(r, t) = A\, J_0 (kr) \qquad \text{... (9)}$$

Using (9), (4) yields $u(r, t) = A\, e^{iwt} J_0 (kr)$ ... (10)

Putting $r = a$ in (10) and using the boundary condition (3), we have

$0 = A\, e^{iwt} J_0 (ka)$ or $J_0(ka) = 0$ ... (11)

because for non-trivial solution A cannot be equal to zero.

From a table of Bessel functions, we see that (11) is satisfied if we have $k_1 a = 2.404$, $k_2 a = 5.520$, $k_3 a = 8.653$, $k_4 a = 11.791$ .... Accordingly, from (7), the fundamental angular frequency is given by

$k_1 a = 2.404$ or $(w_1/c) \times a = 2.404$ or $w_1 = (2.404c)/a$ ... (12)

The other possible angular frequencies are given by the above numbers multiplied by c/a. Thus we observe that there exist as infinite number of natural frequencies possible, but these frequencies are not multiplier of each other.

A particular solution of (1) that satisfies the boundary condition (2) is given with help of (7) and (10) in the following form $u(r,t) = A_n J_0 (w_n r / c)\, e^{iw_n t}$

By summing over all the possible values of w we can construct a general solution of (1) that satisfies the initial conditions (3*a*) and (3*b*). We, thus arrive at

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0 (w_n r / c) \cos w_n t \qquad \text{... (13)}$$

Putting $t = 0$ in (12) and using (3*a*), we have $f(r) = \sum_{n=0}^{\infty} A_n J_0 (w_n r / c), \quad 0 \le r \le a$ ... (14)

which is fourier-Bessel series and hence

$$A_n = \frac{2}{a^2 J_1^2 \{(w_n/c) \times a\}} \int_0^a r\, J_0 (w_n r / c)\, f(r)\, dr \qquad \text{... (15)}$$

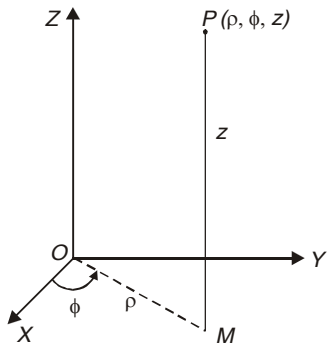
By substituting the above values of A_n into (13), we get the desired subsequent displacement of the membrane.

4

Boundary value problems in cylindrical coordinates

4.1. Introduction

While solving boundary value problems, appropriate choice of coordinate system is very useful. In physical problems that involve a cylindrical surface (for example, the problem of evaluating the temperature in a cylindrical rod), it will be convenient to make use of cylinderical coordinates.



Cylindrical coordinates (ρ, ϕ, z) are defined by means of equations

$$x = \rho \cos \phi, \qquad y = \rho \sin \phi, \qquad z = z \quad \ldots (1)$$

where $\rho \geq 0; \qquad 0 \leq \phi \leq 2\pi; \qquad -\infty < z < \infty.$

The Laplacian operator ∇^2 in cylindrical coordinates, reduces to

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \qquad \ldots (2)$$

We present some useful results for ready reference:

1. Laplace's equation $\nabla^2 \mathbf{u} = 0$ or $$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

2. Heat (Diffusion) equation $\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t}$ or $$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

3. Wave equation $\nabla^2 \mathbf{u} = \frac{1}{c^2}\frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2}$ or $$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Note. Some authors use r and θ for ρ and ϕ respectively while writing the above equations in cylindrical coordinates. The corresponding changes must be made if r and θ are used in any problem.

4. Table of some basic equations and their solutions.

Equations	Solutions
4A $\rho^2 R'' + \rho R' + (m^2\rho^2 - n^2)R = 0.$ Bessel's equation, where n is an interger.	$R(\rho) = A_{mn} J_n(m\rho) + B_{mn} Y_n(m\rho)$, where J_n and Y_n are Bessel's functions of first and second kind respectively. Also note that $Y_n(m\rho) \to \infty$ as $\rho \to 0$.

	Equation	Solution
4B	$\rho^2 R'' + \rho R' + m^2\rho^2 R = 0.$ Bessel's equation of zeroth order.	$R(\rho) = A_m J_0(m\rho) + B_m Y_0(m\rho),$ where $Y_0(m\rho) \to \infty$ as $\rho \to 0$
4C	$\rho^2 R'' + \rho R' + (m^2\rho^2 - n^2)R = 0,$ where n is not an integer	$R(\rho) = A_{mn} J_n(m\rho) + B_{mn} J_{-n}(m\rho),$ where J_n and J_{-n} are Bessel's functions of first kind and $J_{-n}(m\rho) \to 0$ as $\rho \to 0$
4D	$\rho^2 R'' + \rho R' - n^2 R = 0$ (Homogeneous linear equation)	$R(\rho) = A_n r^n + B_n r^{-n},$ when A_n and B_n are arbitrary constants

5 Fourier-Bessel Series

If $f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r),\ 0 \le r \le a,$ then $A_n = \dfrac{2}{a^2[J_1(\lambda_n a)]^2} \int_0^a r J_0(\lambda_n r) f(r) dr$

4.2 Solution of Laplace's equation.

Ex. 1. (*i*) *Obtain solution of Laplace's equation in cylindrical polar coordinates.*
[Gujrat 2001, Delhi Maths (H) 1997, 2000, Ravishanker 1995, 97, Meerut 2001, 03]

(*ii*) *Show that in a cylinderical coordinates Laplace's equation has solution of the form* $R(\rho)e^{\pm mz \pm in\phi}$ *where* $R(\rho)$ *is a solution of Bessel's equation.* **[Delhi Maths (Hons) 2000]**

(*iii*) *Obtain solution of the three dimensional Laplace equation which is finite on the axis.*
[Delhi Maths (H) 1995]

Sol. (*i*) Laplace's equation in cylindrical coordinates (ρ, φ, z) is

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \ldots (1)$$

Let a solution of (1) be of the form $u(\rho, \phi, z) = R(\rho)\,\Phi(\phi)\,Z(z),$ ... (2)

where R, Φ and Z are functions of ρ, ϕ, and z respectively.

Substituting this value of u in (1), we get

$$R''\Phi Z + (1/\rho) \times R'\Phi Z + (1/\rho^2) \times R\Phi''Z + R\Phi Z'' = 0,$$

where the dashes denote derivatives with respect to the relevant variables. Dividing throughout by $R\Phi Z,$ we obtain

$$\frac{R''}{R} + \frac{R'}{\rho R} + \frac{\Phi''}{\rho^2 \Phi} = -\frac{Z''}{Z} \qquad \ldots (3)$$

Since the right hand side depends only on z while the left hand side depends only on ρ and φ, (3) can only true if each side is equal to the same constant, * say $-m^2$. Then (3) gives

$$Z'' - m^2 Z = 0 \qquad \ldots (4)$$

and

$$\frac{R''}{R} + \frac{R'}{\rho R} + \frac{\Phi''}{\rho^2 \Phi} = -m^2 \qquad \ldots (5)$$

---

*We select the constant in the form $-m^2$ because the resulting equation (4) gives solutions which are generally useful for practical problems to be discussed later.

Re-writing (5) by seperating variables, we have

$$\rho^2\left(\frac{R''}{R}+\frac{R'}{\rho R}+m^2\right)=-\frac{\Phi''}{\Phi} \quad \ldots (6)$$

Since ρ and φ are independent, (6) is true only when each side is equal to the same constant. On physical grounds of actual problems, we must involve trigonometric functions in solutions. So we choose the proposed constant to be n^2. Furthermore, in order to satisfy the physical condition $u(\rho, \phi) = u(\rho, \phi + 2\pi)$, we suppose that n is an integer. Then (6) reduces to

$$\Phi'' + n^2\Phi = 0 \quad \ldots (7)$$

and

$$\rho^2 R'' + \rho R' + (m^2\rho^2 - n^2)R = 0 \quad \ldots (8)$$

Solutions of (4), (7) and (8) are

$$Z(z) = A_m e^{mz} + B_m e^{-mz} \quad \ldots (9)$$

$$\Phi(\phi) = C_n \cos n\phi + D_n \sin n\phi \quad \ldots (10)$$

$$R(\rho) = E_{mn} J_n(m\rho) + F_{mn} Y_n(m\rho) \quad \ldots (11)$$

Hence a general solution of (1) is

$$u(\rho, \phi, z) = \sum_n \sum_m (A_m e^{mz} + B_m e^{-mz})\,(C_n \cos n\phi + D_n \sin n\phi)[E_{mn} J_n(m\rho) + F_{mn} Y_n(m\rho)] \quad \ldots (12)$$

Note. Multiplying (9), (10) and (11), a solution of (1) is given by

$$(A_m e^{mz} + B_m e^{-mz})\,(C_n \cos n\phi + D_n \sin n\phi)\,[E_{mn} J_n(m\rho) + F_{mn} Y_n(m\rho)].$$

This is called a *cylindrical harmonic*.

Part (*ii*) Proceed like part (*i*) upto (11). Now solutions (9) and (10) can also be re-written as $e^{\pm mz}$ and $e^{\pm in\varphi}$ respectively. Further, $R(\rho)$ is solution of (8) which is Bessel's equation. Hence we get a solution of the form

$$R(\rho)e^{\pm mz \pm in\varphi} \quad \ldots (13)$$

Part (*iii*) Proceed like part (*i*) upto (11). We know that $Y_n(m\rho) \to \infty$ as $\rho \to 0$. Now on the axis (*i.e.*, z-axis), $\rho \to 0$ so that $Y_n(m\rho) \to \infty$ and hence we must take $F_{mn} = 0$ in (11) so that the solution may remain finite on the axis. Thus the resulting general solution is given by

$$u(\rho, \phi, z) = \sum_n \sum_m E_{mn} J_n(m\rho)\,(A_m e^{mz} + B_m e^{-mz})\;\;(C_n \cos n\phi + D_n \sin n\phi) \quad \ldots (14)$$

which is also expressed as

$$u(\rho, \phi, z) = \sum_n \sum_m E_{mn} J_n(m\rho)\, e^{\pm mz \pm in\phi} \quad \ldots (15)$$

Ex 2. *Show that solutions of Laplace's equation* $\nabla^2 u = 0$ *in cylindrical co-ordinates satisfying the conditions* (*i*) $u \to 0$ *as* $z \to \infty$. (*ii*) *u is finite as* $\rho \to 0$ *are of the form.*

$$u = \sum_m \sum_n G_{mn} J_n(m\rho) e^{-mz \pm in\phi}$$

Sol. Proceeds as in solved Ex. 1 upto equation (11)

Given that $u \to 0$ as $z \to \infty$, so we take $A_m = 0$ in (9).

Given that *u* is finite as $\rho \to 0$ so we take $F_{mn} = 0$ in (11), since $Y_n(m\rho) \to \infty$ as $\rho \to 0$.

Hence, required solution is of the form $u = \sum_m \sum_n B_m e^{-mz} (C_n \cos n\phi + D_n \sin n\phi)\, E_{mn} J_n(m\rho)$

or $u = \sum_m \sum_n G_{mn} J_n(m\rho) e^{-mz \pm in\phi}$, where $G_{mn} = B_m E_{mn}$

Ex 3. *Show that in cylindrical coordinates r, ϕ, z Laplace's equation has solutions of the form* $R(\rho)e^{\pm mz \pm in\phi}$, *where* $R(\rho)$ *is a solution of Bessel equation* $d^2R/d\rho^2 + (1/\rho)\,(dR/d\rho) + (m^2 - n^2/\rho^2)R = 0$. *If solution of Laplace's equation tends to zero as* $z \to \infty$ *and is finite when* $\rho = 0$, *show that, in general, in the usual notations for Bessel's functions, the appropriate solutions are made up of the terms of the form* $J_n(m\rho)e^{-mz \pm in\phi}$ **[Meerut 2005; Delhi Maths (Hons) 2002, 09]**

Sol. Proceed as in solved Ex. 1 upto equation (8). Then complete the solution as follows:

Re-writing (8), $d^2R/d\rho^2 + (1/\rho)\,(dR/d\rho) + (m^2 - n^2/\rho^2)R = 0$ ... (9)

Solutions of (4) and (7) can be written in the form $e^{\pm mz}$ and $e^{\pm im\phi}$ respectively. Hence (1) has a solution of the form $R(\rho)\,\Phi(\phi)\,Z(z)$ *i.e.*, $R(\rho)\,e^{\pm mz \pm in\phi}$, ... (10)

where $R(\rho)$ is a solution of Bessel's equation (9).

Second part. Solution of (9) is given by $R(\rho) = A\,J_n(m\rho) + BY_n(m\rho)$ ... (11)

Since $Y_n(m\rho) \to \infty$ as $\rho \to 0$ hence for the solution of (1) to tend to zero as $\rho \to 0$, we must take $B = 0$ in (11). Then (11) becomes

$$R(\rho) = A\,J_n(m\rho) \quad \text{... (12)}$$

Solution of (4) is $Z(z) = C\,e^{mz} + D\,e^{-mz}$ ... (13)

Since $e^{mz} \to \infty$ as $z \to \infty$, hence for the solution of (1) to tend to zero as $z \to \infty$, we must take $C = 0$ in (13). Hence (13) reduce to $Z(z) = D\,e^{-mz}$ ... (14)

Hence the solution of (1), which tends to zero when $z \to \infty$ and $\rho \to 0$, is given by help of (10), (12) and (4) in the form $J_n(m\rho)e^{-mz \pm in\phi}$.

Ex. 4. (*i*) *Obtain the axially summetrical solutions of the three dimensional Laplace equation.*

(*ii*) *Obtain the general solution of Laplace's equation in three dimensions which remains finite on the axis of z and is symmetrical about it.*

Sol. (*i*). Laplace's equation in cylindrical coordinates (ρ, ϕ, z) is

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{... (1)}$$

Since the required solution is symmetrical about the axis of z, we have $\partial^2 u / \partial \varphi^2 = 0$ so that (1) reduces to

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{... (2)}$$

Let a solution of (1) be of the form $u(\rho, z) = R(\rho)Z(z)$. ... (3)

Substituting this value of u in (2), we get

$$R''Z + \frac{1}{\rho} \times R'Z + R\,Z'' = 0 \quad \text{or} \quad \frac{1}{R}\left(R'' + \frac{R'}{\rho}\right) = -\frac{Z''}{Z}. \quad \text{... (4)}$$

Since ρ and z are independent, (4) can only be true if each side of equal to the same constant, say $-m^2$. Then (4) gives

$$Z'' - m^2 Z = 0 \qquad \ldots (5)$$

and $$R'' + (R'/\rho) = -m^2 R \qquad \text{or} \qquad \rho^2 R'' + \rho R' + m^2 \rho^2 R = 0 \qquad \ldots (6)$$

Solving (5), $Z(z) = A_m\, e^{mz} + B_m\, e^{-mz}$, A_m and B_m being arbitrary constants ... (7)

(6) is a Bessel's equation of zeroth order and its solution is

$$R(\rho) = C_m J_0(m\rho) + D_m Y_0(m\rho), \qquad \ldots (8)$$

where J_0 and Y_0 are functions of first and second kind respectively, From (7) and (8), a general axially symmetrical solution is given by

$$u(\rho, z) = \sum_m (A_m e^{mz} + B_m e^{-mz})[C_m J_0(m\rho) + D_m Y_0(m\rho)] \qquad \ldots (9)$$

Part (*ii*) First do like part (*i*) above. Since $Y_0 \to \infty$ as $\rho \to 0$, we take $D_m = 0$ in (9) in order that solution may remain finite on the axis (*i.e.*, axis of z where $\rho \to 0$ at origin). Hence taking $D_m = 0$ in (9) and writing $C_m A_m = E_m$ and $C_m B_m = F_m$, the required general solution is

$$u(\rho, z) = \sum_m J_0(m\rho)\,(E_m e^{mz} + F_m e^{-mz}).$$

Ex 5. *Obtain solution of Laplace's equation in cylindrical coordinates satisfying the following conditions:* (*i*) *Solution is symmetrical about the z-axis and remains finite along the line* $\rho = 0$.

(*ii*) Solution is symmetrical about the z-axis and tends to zero as $\rho \to 0$ and as $z \to \infty$.

Sol. (*i*) Proceed as in solved Ex. 4 upto equation (9).

$\therefore$ $$u(\rho, z) = \sum_m (A_m e^{mz} + B_m e^{-mz})[C_m J_0(m\rho) + D_m Y_0(m\rho)], \qquad \ldots (9)$$

which is solution of Laplace's equation when the solution is symmetrical about the z-axis. If the required solution remains finite along the line $\rho = 0$, we must take $D_m = 0$ because $Y_0(m\rho) \to \infty$ as $\rho \to 0$. So, (9) reduces to

$$u(\rho, z) = \sum_m \left(A_m e^{mz} + B_m e^{-mz}\right) C_m J_0(m\rho) = \sum_m (E_m e^{mz} + F_m e^{-mz}) J_0(m\rho), \qquad \ldots (10)$$

where E_m (= $A_m C_m$) and F_m (= $B_m C_m$) are new arbitrary constants.

Solution (10) can also be expressed as $$u(\rho, z) = \sum_m G_m J_0(m\rho) e^{\pm mz} \qquad \ldots (11)$$

(*ii*) This a particular case of part (*i*), Here, $u(\rho, z) \to 0$ as $z \to \infty$ and hence we must take $E_m = 0$ in (10). Then (10) reduces to

$$u(\rho, z) = \sum_m F_m e^{-mz} J_0(m\rho) \qquad \ldots (12)$$

Ex. 6. *Find the potential function* $\psi(\rho, z)$ *in the region* $0 \le \rho \le 1, z \ge 0$ *satisfying the conditions,* (*i*) $\psi \to 0$ as $z \to \infty$, (*ii*) $\psi = 0$ on $\rho = 1$ (*iii*) $\psi = f(\rho)$ on $z = 0$ for $0 \le \rho \le 1$.

Sol. We know that potential function $\psi(\rho, z)$ is solution of Laplace's equation

$$\partial^2\psi/\partial\rho^2 + (1/\rho) \times (\partial\psi/\partial\rho) + \partial^2\psi/\partial^2 z = 0 \qquad \ldots (1)$$

We are to solve (1) subject to the following boundary conditions :

$$\psi(\rho, z) \to 0, \qquad \text{as} \qquad z \to \infty, \qquad \ldots (2)$$

$$\psi(1, z) = 0 \qquad \ldots (3)$$

$$\psi(\rho, 0) = f(\rho), \qquad 0 \le p \le 1. \qquad \ldots (4)$$

Let a solution of (1) be of the form $\psi(\rho, z) = R(\rho)Z(z).$ ... (5)

Substituting this value of ψ in (1), we get

$$R''Z + \frac{1}{\rho}R'Z + R\,Z'' = 0 \qquad \text{or} \qquad \frac{1}{R}\left(R'' + \frac{1}{\rho}R'\right) = -\frac{Z''}{Z}$$

Since ρ and z are independent, the above equation is true if each side is equal to the same constant, say $-m^2$. Hence the above equation gives

$$Z'' - m^2 Z = 0$$

and $\quad R'' + (R''/\rho) = -m^2\rho \qquad$ or $\qquad \rho^2 R'' + \rho R' + m^2\rho^2 R = 0$

Solving these equations we obtain

$$Z(z) = A_m e^{mz} + B_m e^{-mz} \qquad \ldots (6)$$

and $$R(\rho) = C_m J_0(m\rho) + D_m Y_0(mp) \qquad \ldots (7)$$

We must take $A_m = 0$ in (6), since otherwise $Z(z) \to \infty$ as $z \to \infty$ and hence $\psi(\rho, z) \to \infty$ as $z \to \infty$ which contradicts boundary condition (2).

Hence (6) reduces to $Z(z) = B_m e^{-mz}$ ... (8)

Again we must take $D_m = 0$ in (7), since otherwise $R(\rho) \to \infty$ as $\rho \to 0$ ($\because Y_0(m\rho) \to \infty$ as $\rho \to 0$)

and hence $\psi(\rho, z) \to \infty$ as $\rho \to 0,$

which contradicts the fact that potential is finite on the axis of z.

Hence (7) reduces to $R(\rho) = C_m J_0(m\rho)$ ... (9)

Putting $\rho = 1$ in (5) and using (3), we get

$R(1)\,Z(z) = 0$ so that $R(1) = 0$ ... (10)

where we have taken $Z(z) \neq 0,$ since otherwsie $\psi(\rho, z) \equiv 0$ which does not satisfy (4).

Putting $\rho = 1$ in (9) and using (10), we get $J_0(m) = 0$ ... (11)

Let m_i (i = 1, 2, 3, ...) denote the ith positive root of (11). From (8), (9) and (5), the function $E_m J_0(m\rho)e^{-mz}$ is a solution of (1) satisfying (2) and (3). Here E_m (= $C_m B_m$) are new arbitrary constants. In view of (11), the functions

$$\psi(\rho, z) = F_i J_0(m_i\rho)\, e^{-m_i z},\ i = 1, 2, 3, \ldots$$

are solutions of (1), satisfying (2) and (3); here we write $E_{m_i} = F_i$. In order to obtain a solution also satisfying (4), we consider a more general solution by superposition

$$\psi(\rho, z) = \sum_{i=1}^{\infty} \psi_i(\rho, z) = \sum_{i=1}^{\infty} F_i J(m_i\rho)e^{-m_i z} \qquad \ldots (12)$$

Putting $z = 0$ in (12) and using (4), we get

$$f(\rho) = \sum_{i=1}^{\infty} F_i J_0(m_i\rho)$$

which is Fourier-Bessel series for $f(\rho)$ and so we get

$$F_i = \frac{2}{[J_1(m_i)]^2}\int_0^1 \rho\, f(\rho) J_0(m_i\rho)\, d\rho, \quad i = 1, 2, 3, \ldots \qquad \ldots (13)$$

Hence the required potential function $\psi(\rho, z)$ is given by (12) wherein F_i is given by (13).

EXERCISE 4(A)

1. Derive the solution of $\partial^2 V/\partial r^2 + (1/r) \times (\partial V/\partial r) + \partial^2 V/\partial z^2 = 0$ for the region $r \geq 0, z \geq 0,$ with boundary conditions: $V \to 0$ as $z \to \infty$ and $r \to \infty$. $V = f(r)$ on $z = 0, z \geq 0$.

2. Obtain the general solution of Laplace's equation in cylindrical coordinates for the case of axial symmetry, namely $\partial^2 u/\partial \rho^2 + (1/\rho) \times (\partial u/\partial \rho) + \partial^2 u/\partial z^2 = 0$ **[I.A.S. 1984]**

4.3. Circular harmonics

In general, the term harmonic is used to any solution of Laplace's equation. In practice, however, the term harmonic applies in a more restricted sense to mean a solution of Laplace's equation in a specified coordinate system.

Let us consider Laplace's equation in cylindrical coordinates and assume that u is independent of the coordinate *z*. Then, Laplace's equation takes the form

$$\partial^2 u/\partial r^2 + (1/r) \times (\partial u/\partial r) + (1/r^2) \times (\partial^2 u/\partial \theta^2) = 0 \quad \ldots (1)$$

Suppose that (1) has a solution of the form $u(r, \theta) = R(r)\, \Theta(\theta),$ ... (2)

where R and Θ are functions of r and θ respectively.

Substituting the value of u given by (2) in (1), we get

$$R''\Theta + (1/r) \times R'\Theta + (1/r^2) \times R\,\Theta'' = 0$$

or $$(r^2 R'' + rR')/R = -(\Theta''/\Theta), \quad \ldots (3)$$

where the dashes denote derivatives with respect to the relevant variables. Since r and θ are independent, (3) is true if each side is equal to the constant, which we have called n^2. Thus, (3) yields the following two differential equations:

$$r^2\,(dR/dr) + r\,(dR/dr) - n^2 R = 0 \quad \ldots (4)$$

and $$\Theta'' + n^2\Theta = 0 \quad \ldots (5)$$

Let $r = e^z$ and $D_1 \equiv r(d/dr) \equiv d/dz$. Then, (4) reduces to

$$\{D_1\,(D_1 - 1) + D_1 - n^2\}R = 0 \quad \text{or} \quad (D_1^2 - n^2)R = 0$$

Its auxiliary equation is $D_1^2 - n^2 = 0$ giving $D_1 = \pm n$. Hence solution of (4) is

$R(r) = A\, e^{nz} + Be^{-nz}$ or $R(r) = A\, r^n + Br^{-n}$, as $r = e^z$ where $n \neq 0$ ... (6)

Again, for $n = 0$, (4) reduces to $r^2(d^2R/dr^2) + r(dR/dr) = 0$ ... (7)

As before with $r = e^z$, (7) reduces to $\{D_1\,(D_1 - 1) + D_1\}\,R = 0$ or $D_1^2 R = 0$

Its auxiliary equation is $D_1^2 = 0$ giving $D_1 = 0, 0.$

Hence solution of (7) is $R(r) = (A_o + B_o z)\, e^{oz}$

or $R(r) = A_o + B_o \log r,$ as $r = e^z \Rightarrow z = \log r$

Again, solution of (5) is given by

$\Theta(\theta) = C_n \cos n\theta + D_n \sin n\theta,$ if $n \neq 0$ and $\Theta(\theta) = C_0\theta + D_0,$ if $n = 0,$

where $A_n, B_n, C_n, D_n, A_o, B_o, C_o, D_o$ are arbitrary constants.

The number n is known as the *degree of the harmonic*. The solutions of Laplace's equation in cylindrical coordinates when u is independent of z are called *circular harmonics*. Circular harmonics play a very important role in the solution of two-dimensional problems having cylindrical symmetry. The circular harmonics are given by

$$u_0(r, \theta) = (A_o + B_o \log r)(C_o\theta + D_o) \qquad \text{(degree zero)} \qquad \ldots (8a)$$

$$u_n(r, \theta) = (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \qquad \text{(degree } n\text{)} \qquad \ldots (8b)$$

In usual applications of circular harmonics to physical problems the function $u(r, \theta)$ is a single-valued function of θ. Since increasing θ by 2π, we arrive at the same point in xy-plane, it follows that for $u(r, \theta)$ to be single-valued, we must have

$$u_n(r, \theta + 2\pi) = u_n(r, \theta) \qquad \ldots (9)$$

From this it follows that n must take only integral values. By superposition principle, a general single-valued solution of Laplace's equation is obtained by summing the solutions (8a) and (8b) over all possible values of n in the form

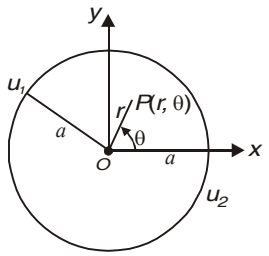
$$u(r,\theta) = a_0 \log r + \frac{1}{2}c_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) + \sum_{n=1}^{\infty} \frac{1}{r^n}(p_n \cos n\theta + q_n \sin n\theta), \qquad \ldots (10)$$

where a_o, c_o, a_n, b_n, p_n and q_n are arbitrary constants.

As an example of the application of circular harmonics, consider the following example:

Example. *Find the steady distribution of temperature inside a long circular cylinder of two halves maintained at temperature u_1 and u_2 respectively.* **[Delhi Physics (H) 1998]**

Solution. Let a very long circular cylinder of radius a be composed of two halves as shown in the adjoining figure. Let the two halves of the cylinder be thermally inslulated from each other, and the upper half of the cylinder be kept at temperature u_1, while the lower half be kept at temperature u_2. We are required to find the steady temperature in the region inside the cylinder. Let us suppose that the cylinder is so long that the temperature is independent of z. For complete solution, repeat matter of Art. 4.3 from equation (1) upto (10). To solve the above problem, we must solve (1) in region inside the cylinder and satisfy the boundary conditions



$$u(a,\theta) = \begin{cases} u_1, \text{ when } 0 < \theta < \pi \\ u_2, \text{ when } \pi < \theta < 2\pi \end{cases} \qquad \ldots (11)$$

In order to solve the problem under consideration, we take the general solution (10) satisfying the boundary conditions (11), In the first place, we observe that the temperature must be finite at $r = 0$. Hence we choose $a_0 = p_n = q_n = 0$ in (10). Therefore, the solution (10) takes the simplified form

$$u(r,\theta) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \qquad \ldots (12)$$

Putting $r = a$ in (12), we have

$$u(a,\theta) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} (a^n a_n \cos n\theta + a^n b_n \sin n\theta), \qquad \ldots (13)$$

which is usual expansion of $u(a, \theta)$ as Fourier series in $(0, 2\pi)$. Hence, we have (refer result II of Art. 3.1 of chapter 3).

$$c_0 = \frac{1}{\pi}\int_0^{2\pi} u(a,\theta)d\theta = \frac{1}{\pi}\left[\int_0^{\pi} u(a,\theta)d\theta + \int_{\pi}^{2\pi} u(a,\theta)d\theta\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} u_1 d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} u_2 d\theta \text{, using (1)}$$

$$= u_1 + u_2$$

$$a_n = \frac{1}{a^n\pi}\int_0^{2\pi} u(a,\theta)\cos n\theta d\theta = \frac{1}{a^n\pi}\left[\int_0^{\pi} u(a,\theta)\cos n\theta d\theta + \int_{\pi}^{2\pi} u(a,\theta)\cos n\theta\, d\theta\right]$$

$$= \frac{1}{a^n\pi}\int_0^{\pi} u_1 \cos n\theta d\theta + \frac{1}{a^n\pi}\int_{\pi}^{2\pi} u_2 \cos n\theta\, d\theta = \frac{u_1}{a^n\pi}\left[\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{u_2}{a^n\pi}\left[\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} = 0$$

$$b_n = \frac{1}{a^n\pi}\int_0^{2\pi} u(a,\theta)\sin n\theta\, d\theta = \frac{1}{a^n\pi}\left[\int_0^{\pi} u(a,\theta)\sin n\theta d\theta + \int_{\pi}^{2\pi} u(a,\theta)\sin n\theta\, d\theta\right]$$

$$= \frac{1}{a^n\pi}\int_0^{\pi} u_1 \sin n\theta\, d\theta + \frac{1}{a^n\pi}\int_{\pi}^{2\pi} u_2 \sin n\theta\, d\theta, \text{ using (11)}$$

$$= \frac{u_1}{a^n\pi}\left[-\frac{\cos n\theta}{n}\right]_0^{\pi} + \frac{u_2}{a^n\pi}\left[-\frac{\cos n\theta}{n}\right]_{\pi}^{2\pi} = \frac{u_1\{1-(-1)^n\}}{n\pi a^n} - \frac{u_2\{1-(-1)^n\}}{n\pi a^n}$$

$$= \{1/(n\,\pi\,a^n)\} \times (u_1 - u_2) \times \{1-(-1)^n\}$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ (even)}, m = 1, 2, 3, \ldots \\ 2(u_1 - u_2)/(2m-1)\pi a^{2m-1}, \text{ if } n = 2m-1 \text{ (odd)}, m = 1, 2, 3, \ldots \end{cases}$$

Substituting the above values of c_o, a_n and b_n in (12), the required solution is given by

$$u(r,\theta) = \frac{u_1 + u_2}{2} + \sum_{m=1}^{\infty} \frac{2r^{2m-1}(u_1 - u_2)}{(2m-1)\pi\, a^{2m-1}}\sin(2m-1)\theta$$

or

$$u(r,\theta) = \frac{u_1 + u_2}{2} + \frac{2(u_1 - u_2)}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1} \quad \ldots (14)$$

A particular case of the above example. *Find the steady temperature inside a long circular cylinder of radius a when the temperature of the upper half of the cylinder is kept at u_0 and the lower half is kept at temperature equal to zero.*

Hint. Proceed as in the above example by taking $u_1 = u_0$ and $u_2 = 0$. Then from (14), the required solution is given by

$$u(r,\theta) = \frac{u_0}{2} + \frac{2u_0}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1} \quad \ldots (15)$$

4.4 Solution of heat or (diffusion) equation.

Ex. 1. *Solve the heat (diffusion) equation in cylindrical coordinates.*

[Kurukshetra 2000; Delhi Maths (H) 1998]

OR

Obtain the solution of the diffusion equation $\nabla^2 u = (1/k)\times(\partial u/\partial t)$ in cylindrical coordinates (ρ, φ, z) in the form

$$\sum_{\lambda,\mu,\gamma} A_{\lambda\mu\gamma} J_{\pm\gamma}\{(\lambda^2+\mu^2)^{1/2}\rho\} \exp(\pm\mu z - \lambda^2 k t \pm i\gamma\phi).$$

Sol. Heat (diffusion) equation in cylindrical coordinates (ρ, ϕ, z) is

$$\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{k}\frac{\partial u}{\partial t} \quad \ldots (1)$$

Let a solution of (1) be of the form $u(\rho, \phi, z, t) = R(\rho)\Phi(\phi)Z(z)T(t)$ ... (2)

Substituting this value of u in (1) and then dividing by $R\Phi ZT$, we have

$$\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}+\frac{Z''}{Z}=\frac{1}{k}\frac{T}{T} \quad \ldots (3)$$

Since the right hand side of (3) depends only on t while the left hand side depends only on ρ, ϕ and z, (3) can only be true if each side is equal to the same constant, say $-\lambda^2$. We have chosen this special value of constant so that the resulting solution may tend to zero as $t\to\infty$ (Note that with $-\lambda^2$, (3) gives $T' = -k\lambda^2 T$ so that $T(t) = Ae^{-k\lambda^2 t}$ showing that $T(t)\to 0$ as $t\to\infty$ and hence $u\to 0$ at $t\to\infty$). Hence (3) reduces to

$$T' = -\lambda^2 kT \quad \ldots (4)$$

$$\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}+\lambda^2 = -\frac{Z''}{Z} \quad \ldots (5)$$

Since the right hand side of (5) depends on z only while the left hand side depends only on ρ and ϕ, (5) can only be true if each side is equal to the same constant, say $-\mu^2$ (as explained in Ex. 1 of Art. 4.2). So (5) gives

$$Z'' - \mu^2 Z = 0 \quad \ldots (6)$$

and

$$\rho^2\left(\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\lambda^2+\mu^2\right) = -\frac{\Phi''}{\Phi} \quad \ldots (7)$$

Since ρ and φ are independent, (7) can only be true if each side is equal to the same constant, say γ^2 (as in Ex. 1 of Art. 4.2). Hence (7) gives

$$\Phi'' + \gamma^2\Phi = 0 \quad \ldots (8)$$

and

$$(1/R)\times(\rho^2 R'' + \rho R') + (\lambda^2+\mu^2)\rho^2 = \gamma^2$$

i.e.,

$$\rho^2 R'' + \rho R' + \{(\lambda^2+\mu^2)\rho^2 - \gamma^2\}R = 0 \quad \ldots (9)$$

Solutions of (4), (6), (8) and (9) are given by

$$T_\lambda(t) = A_\lambda e^{-k\lambda^2 t} \quad \ldots (10)$$

$$Z_\mu(z) = B_\mu e^{\mu z} + C_\mu e^{-\mu z} \quad \ldots (11)$$

$$\Phi_\gamma(\phi) = D_\gamma \cos\gamma\phi + E_\gamma \sin\gamma\phi \quad \ldots (12)$$

and

$$R_{\lambda\mu\gamma}(\rho) = F_{\lambda\mu\gamma} J_\gamma\{(\lambda^2+\mu^2)^{1/2}\rho\} + G_{\lambda\mu\gamma} J_\gamma\{(\lambda^2+\mu^2)^{1/2}\rho\}. \quad \ldots (13)$$

General solution of (1) is given by $u(\rho, \phi, z, t) = \sum_{\lambda, \mu, \gamma} T_\lambda(t) Z_\mu(t) \Phi_\gamma(\phi) R_{\lambda\mu\gamma}(\rho)$

which can also be expressed as

$$u(\rho, \phi, z, t) = \sum_{\lambda, \mu, \gamma} A_{\lambda\mu\gamma} J_{\pm\gamma} \{(\lambda^2 + \mu^2)^{1/2} \rho\}\, e^{\pm\mu z - \lambda^2 kt \pm i\gamma\phi}$$

Ex. 2. *Determine the temperature distribution in the infinite cylinder* $0 \le \rho \le a$ *when the initial temperature is* $\theta(\rho, 0) = f(\rho)$ *and the surface* $\rho = a$ *is maintained at zero temperature.*

[Himanchal 2007, 08, 09; Delhi Maths (H) 2001, 2003, 05, 08]

OR

A circular plate of radius a has its plane faces insulated. If the initial temperature is $f(\rho)$ *and if the rim is kept at temperature zero, find the temperature of the plate at any time.*

[Delhi Physics (H) 1999]

Sol. Heat equation in cylindrical coordinates (ρ, φ, z) is given by

$$\frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\theta}{\partial\phi^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{k}\frac{\partial\theta}{\partial t} \quad \ldots (1)$$

Due to the physical conditions of the problem, we have

$\partial^2\theta/\partial\phi^2 = 0$ and $\partial^2\theta/\partial z^2 = 0$

Hence temperature $\theta(\rho, t)$ in the given problem is governed by

$$\frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} = \frac{1}{k}\frac{\partial\theta}{\partial t} \quad \ldots (2)$$

We are to solve (2) under the following conditions : $\theta(a, t) = 0$ ... (3)

and $\theta(\rho, 0) = f(\rho)$ ... (4)

Let a solution of (2) be of the form $\theta(\rho, t) = R(\rho)T(t)$ ... (5)

Substituting this value of θ in (2) and dividing by RT, we get

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} = \frac{1}{k}\frac{T'}{T} \quad \ldots (6)$$

Since ρ and t are independent, (6) can only be true if each side is equal to the same constant, say $-\lambda^2$. Then (6) gives

$$T' = -\lambda^2 kT \quad \ldots (7)$$

and $(R''/R) + (1/\rho) \times (R'/R) = -\lambda^2$ or $\rho^2 R'' + \rho R' + \lambda^2\rho^2 R = 0$ ... (8)

Solving (7), $T(t) = A_\lambda e^{-\lambda^2 kt}$ ... (9)

(8) is Bessel's equation of zeroth order and hence its solution is

$$R(\rho) = B_\lambda J_0(\lambda\rho) + C_\lambda Y_0(\lambda\rho) \quad .. (10)$$

We take $C_\lambda = 0$ in (10), since otherwise $R(\rho) \to \infty$ as $\rho \to 0$ ($\because$ $Y_0(\lambda\rho) \to \infty$ as $\rho \to 0$)

and hence $\theta(\rho, t) \to \infty$ as $\rho \to 0$

which contradicts the fact that temperature is finite on the axis of z. Hence (10) reduces to

$$R(\rho) = B_\lambda J_0(\lambda\rho) \quad \ldots (11)$$

Putting $\rho = a$ in (5) and using boundary condition (3), we get

$R(a)\, T(t) = 0$ so that $R(a) = 0$, ... (12)

where we have taken $T(t) \neq 0$, since otherwise $\theta(\rho, t) \equiv 0$ which does not satisfy (4).

Putting $\rho = a$ in (11) and using (12), we get

$$J_0(\lambda a) = 0, \qquad \text{as} \qquad B_\lambda \neq 0 \qquad \ldots (13)$$

Let λ_i (i = 1, 2, 3, ...) denote the ith positive roots of (13). From (5), (9) and (11), the function, $D_\lambda J_0(\lambda\rho)\, e^{-\lambda^2 kt}$, where $D_\lambda = A_\lambda B_\lambda$, is a solution of (2) satisfying (3). In view of (13), the functions

$$\theta_i(\rho, t) = E_i J_0(\lambda_i \rho) e^{-\lambda_i^2 kt}, i = 1, 2, 3, \ldots$$

are solutions of (2), satisfying (3). Here we write $D_{\lambda_i} = E_i$. In order to obtain a solution also satisfying (4), we consider a more general solution (by superposition)

$$\theta(\rho, t) = \sum_{i=1}^{\infty} \theta_i(\rho, t) = \sum_{i=1}^{\infty} E_t J_0(\lambda_i \rho) e^{-\lambda_1^2 kt} \qquad \ldots (14)$$

Putting $t = 0$ in (14) and using (4), we get

$$f(\rho) = \sum_{i=1}^{\infty} E_i J_0(\lambda_i \rho)$$

which is Fourier-Bessel series and hence

$$E_i = \frac{2}{a^2 [J_1(\lambda_i a)]^2} \int_0^a \rho\, f(\rho) J_0(\lambda_i \rho) d\rho, \; i = 1, 2, 3, \ldots \qquad \ldots (15)$$

Hence the required temperature distribution $\theta(\rho, t)$ is given by (14) wherein the coefficients E_i are determined by (15).

EXERCISE 4(B)

1. Show that in cylinderical coordinates Laplace's equation has solution of the form $R(\rho) e^{\pm mz \pm in\phi}$, where $R(\rho)$ is solution of Bessel's equation. Hence find the temperature distribution $\theta(\rho, \phi, z)$ in a cylindrical rod $0 \le \rho \le a, 0 \le z \le b$ with boundary conditions $\theta = 0$ at $0 \le \rho \le a$, $z = b$ and $\theta(a, \phi, z) = f(\phi, z)$.

2. A circular cylinder of radius a has its surface kept at a constant temperature θ_0. If the initial temperature is zero throughout the cylinder, prove that for $t > 0$.

$$\theta(r, t) = \theta_0 \left\{ 1 - \frac{2}{a} \sum_{i=1}^{\infty} \frac{J_0(\lambda_i a)}{\lambda_i J_1(\lambda_i a)} e^{-\lambda_i^2 k t} \right\}$$

where λ_i i = 1, 2, 3, ... are the positive roots of $J_0(\lambda a) = 0$ and k is the diffusivity.

3. An infinite homogeneous solid circular cylinder of radius a is lagged round the outside to prevent heat escape. At any time t, the temperature $u(r, t)$ at a distance r from the axis of symmetry $(0 \le r \le a)$ is given by the heat diffusion equation (axial symmetry).

$$\partial^2 u / \partial r^2 + (1/r) \times (\partial u / \partial r) = (1/k) \times (\partial u / \partial t), \; 0 \le r \le a; \quad t > 0$$

where k is a constant. At time $t = 0$, the intial temperature distribution at r from the axis is known to be $f(r)$. Find the temperature distribution at any subsequent time.

Ans. $u(r, t) = \sum_{i=1}^{\infty} A_i J_0(\lambda_i r) e^{-k\lambda_i^2 t}$, where $A_i = \dfrac{\int_0^a r f(r)\, J_0(\lambda_i r) dr}{\int_0^a r [J_0(\lambda_i r)]^2 dr}$

and λ_i, i = 1, 2, 3, ... are positive roots of $J_0(\lambda a) = 0$.

4.5 Solution of wave equation

Ex. 1. *Obtain a solution of wave equation in cylindrical coordinates by the method of separation variables.* **[Delhi Maths (H) 1998, 99, 2001]**

Sol. Wave equation in cylindrical coordinates (ρ, φ, z) is

$$\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \ldots (1)$$

Let a solution of (1) be of the form $u(\rho, \phi, z, t) = R(\rho)\Phi(\phi)Z(z)T(t)$ ...(2)

Substituting this value of u in (1) and then dividing $RZ\Phi T$, we obtain

$$\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}+\frac{Z''}{Z}=\frac{1}{c^2}\frac{T''}{T}$$

Since the right hand side depends only on t while the left hand side depends on ρ, ϕ and z, (3) can only be true if each side is equal to the same constant, say $-k^2$. Then (3) given

$$T''+c^2k^2T=0 \quad \ldots (4)$$

and

$$\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}+k^2=-\frac{Z''}{Z} \quad \ldots (5)$$

Since the right hand side of (5) depends on z only while the left hand side depends only on ρ and ϕ, (5) can only be true if each side is equal to the same constant, say γ^2. Then (5) gives

$$Z''+\gamma^2 Z=0 \quad \ldots (6)$$

and

$$\frac{1}{R}\left(R''+\frac{1}{\rho}R'\right)+k^2-\gamma^2=-\frac{1}{\rho^2}\frac{\Phi''}{\Phi} \quad \ldots (7)$$

i.e., $$\frac{1}{R}(\rho^2R''+\rho R')+(k^2-\gamma^2)\rho^2=-\frac{\Phi''}{\Phi}$$ *i.e.*, $$\frac{1}{R}(\rho^2R''+\rho R')+\omega^2\rho^2=-\frac{\Phi''}{\Phi}, \quad \ldots (8)$$

where $$\omega^2 = k^2-\gamma^2 \quad \ldots (8)'$$

Since ρ and φ are independent, (8) is true only if each side is equal to the same constant, say m^2. Then (8) gives

$$\Phi''+m^2\Phi=0 \quad \ldots (9)$$

and

$$\rho^2R''+\rho R'+(\omega^2\rho^2-m^2)R=0 \quad \ldots (10)$$

Solving (4), (6), (9) and (10), we get

$$T(t) = A_k \cos ckt + B_k \sin ckt \quad \ldots (11)$$

$$Z(z) = C_\gamma \cos\gamma z + D_\gamma \sin\gamma z \quad \ldots (12)$$

$$\Phi(\phi) = E_m \cos m\phi + F_m \sin m\phi \quad \ldots (13)$$

and

$$R(\rho) = G_{k\gamma m}J_m(\omega\rho) + H_{k\gamma m}Y_m(\omega\rho) \quad \ldots (14)$$

We choose $H_{k\gamma m} = 0$ in (14) so that $R(\rho)$ and hence $u(\rho, \varphi, z, t)$ may be finite at $r = 0$ (along the z-axis). Then (14) reduces to

$$R(\rho) = G_{k\gamma m} J_m(\omega\rho) \quad \ldots (15)$$

Hence the most general solution of (1) is

$$u(\rho,\phi,z,t) = \sum_{k,\gamma,m} G_{k\gamma m} J_m(\omega\rho)\,(A_k \cos ckt + B_k \sin ckt) \times (C_\gamma \cos\gamma z + D_\gamma \sin\gamma z)\,(E_m \cos m\phi + F_m \sin m\phi)$$

which can also be expressed as

$$u(\rho, \phi, z, t) = \sum_{k,\gamma,m} G_{k\gamma m} J_m(\omega\rho) e^{ikct - i\gamma z \pm im\phi}$$

Ex. 2. *Obtain the solution of the wave equation* $\partial^2 u/\partial r^2 + (1/r)(\partial u/\partial r) + (1/r^2)(\partial^2 u/\partial\theta^2) = (1/c^2)(\partial^2 u/\partial t^2)$ *by the method of separation of the variables in the form*

$$\sum_{m,k} A_{mk} J_m(kr)\, e^{\pm im\theta \pm kct}$$

Sol. Let a solution of the given equation be of the form $u(r,\theta,t) = R(r)\Theta(\theta)\,T(t)$ ... (1)

Substituting this is the given equation, we get

$$R''/R + (1/r)(R'/R) + (1/r^2)(\Theta''/\Theta) = (1/c^2)(T''/T) \quad \ldots (2)$$

Since r, θ and t are independent variables, (2) is true if its each side is equal to a constant, $-k^2$ (say). Thus, we get

$$T'' + k^2 c^2 T = 0 \quad \ldots (3)$$

and

$$R''/R + (1/r)(R'/R) + (1/r^2)(\Theta''/\Theta) = -k^2 \quad \ldots (4)$$

Re-writing (4),

$$(r^2/R)\{R'' + (1/r)R' + r^2 R\} = -(\Theta''/\Theta) \quad \ldots (5)$$

Since r and θ are independent variables, (5) is true if its each side is equal to a constant, say m^2. Then, we get

$$\Theta'' + m^2\Theta = 0 \quad \ldots (6)$$

and

$$r^2 R'' + \rho R' + (k^2 r^2 - m^2) R = 0 \quad \ldots (7)$$

Solving (3), (6) and (7), we have solutions of the forms

$$T(t) = e^{\pm ikct}, \qquad \Theta(\theta) = e^{\pm im\theta}$$

and

$$R(r) = A_{mk} J_m(kr) + B_{mk} Y_m(kr) \quad \ldots (9)$$

We choose $B_{mk} = 0$ in (9) so that $R(r)$ and hence $u(r, \theta, t)$ may be finite at $\rho = 0$. Then (9) reduces to

$$R(r) = A_{mk} J_m(kr) \quad \ldots (10)$$

Using (1), (8) and (10), a solution of the given equation is $A_{mk} J_m(kr) e^{\pm im\theta \pm ikct}$

By the principle of superposition, general solution is given by

$$u(r,\theta,t) = \sum_{m,k} A_{mk} J_m(kr) e^{\pm im\theta \pm ikct}$$

Ex. 3 (*a*) *Discuss the solution of tranverse vibrations of a thin membrane bounded by a circle of radius a in xy-plane described by the function z(x, y, t) satisfying the wave equation,*

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2},$$

with conditions $z = 0$ *on* $r = a$ *and* $z = f(r)$, $\partial z/\partial t = 0$ *at* $t = 0$.

[Delhi Maths (H) 1997, Meerut 1998, Pune 1995]

Sol. Due to axial symmetry, $\partial^2 z/\partial\theta^2 = 0$. Hence we determine deflection $z\,(r, t)$ satisfying

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \qquad \ldots (1)$$

which is subject to the following boundary conditions $z(a, t) = 0$ ... (2)

$$(\partial z/\partial t)_{t=0} = 0 \qquad \ldots (3)$$

and $z(r, 0) = f(r)$ ... (4)

Let a solution of (1) be of the form $z\,(r, t) = R\,(r)\,T(t)$ ... (5)

Substituting this value of z in (1) and dividing by RT, we get

$$R''/R+(R'/rR)=(T''/c^2T) \qquad \ldots (6)$$

Since r and t are independent, (6) can only be true if each side is equal to the same constant, say $-k^2$. Then (6) gives

$$T''+c^2k^2T=0 \qquad \ldots (7)$$

and $R''/R+(R'/rR)=-k^2$ *i.e.*, $r^2R''+rR'+k^2r^2R=0$ ... (8)

Solving (7) and (8), we have

$$T(t) = A_k \cos(ckt) + B_k \sin(ckt) \qquad \ldots (9)$$

$$R(r) = C_k J_0(kr) + D_k Y_0(kr) \qquad \ldots (10)$$

We choose $D_k = 0$ so that $R\,(r)$ and hence $z\,(r, t)$ may be finite at $r = 0$ (along the z-axis). Then (10) reduces to $R(r) = C_k J_0(kr)$ ... (11)

Differentiating with respect to t, (5) and (9) give

$$\partial z/\partial t = R(r)T'(t) \qquad \ldots (12)$$

and $T'(t)=-A_kck\sin(ckt)+B_kck\cos(ckt)$ ... (13)

Putting $t = 0$ in (12) and using (3), we get

$0 = R(r)T'(0)$ so that $T'(0) = 0$ ... (14)

where we have taken $R(r) \not\equiv 0$, since otherwise $z(r,t) \equiv 0$ which does not satisfy (4). Putting $t = 0$ in (13) and using (14), we get $0 = B_k ck$ so that $B_k = 0$.

Hence (9) reduces to $T(t) = A_k \cos(ckt)$ ... (15)

From (11) and (15), a solution of (1) is $A_k C_k J_0(kr) \cos(ckt)$

i.e., $E_k J_0(kr) \cos(ckt)$, ... (16)

where $E_k\,(= A_k C_k)$ is a new arbitrary constant.

Putting $r = a$ in (5) and using (2) we get $0 = R(a)\,T(t)$

so that $R(a) = 0$, as $T(t) \neq 0$

Putting $r = a$ in (11) and using $R(a) = 0$, we get

$$0 = C_k J_0(ka) \qquad \text{so that} \qquad J_0(ka) = 0 \quad \ldots (17)$$

where we have taken $C_k \neq 0$, since otherwise $R(r) \equiv 0$ so that $z(r,t) \equiv 0$ which does not satisfy (4).

Let k_i, $i = 1, 2, 3, \ldots$ be the positive roots of (17). Hence the functions [using (16)].

$$z_i(r, t) = F_i J_0(k_t r) \cos(ck_i t),\ i = 1, 2, 3, \ldots$$

are solutions of (1), satisfying (2) and (3). Here we write $E_{k_i} = F_i$. In order to obtain a solution also satisfying (4), we consider a more general solution (by superposition)

$$z(r,t) = \sum_{i=1}^{\infty} z_i(r,t) = \sum_{i=1}^{\infty} F_i J_0(k_i r)\cos(ck_i t) \quad \ldots (18)$$

Putting $t = 0$ in (18) an using (4), we get

$$f(r) = \sum_{i=1}^{\infty} F_i J_0(k_i r),$$

which is Fourier-Bessel series for $f(r)$ and so we get

$$F_i = \frac{2}{a^2[J_1(k_i a)]^2}\int_0^a r\, f(r) J_0(k_i r),\ i = 1, 2, 3, \ldots \quad \ldots (19)$$

Hence the required solution is given by (18) wherein F_i is given by (19).

Ex. 3(*b*) *Derive the equation of a thin uniform stretched membrane. Solve the above equation for a circular membrane with the boundary condition* $z\ (r = a,\ t) = 0$ *and initial condition* $z(r, t = 0) = 1 - (r^2/a^2)$. **[Delhi Physics (H) 1997]**

Sol. First part. Refer chapter 1

Second part. This example is a particular case of Ex. 3(*a*). For this problem (4) takes the form

$$z(r, 0) = f(r) = 1 - (r^2/a^2) \quad \ldots (4)'$$

Now proceed as in Ex. 3(*a*) upto equation (19). Using (4)′, (19) reduces to

$$F_i = \frac{2}{a^2\left\{J_1(k_i a)\right\}^2}\int_0^a r\left(1 - \frac{r^2}{a^2}\right) J_0(k_i r)dr = \frac{2}{a^2\left\{J_1(k_i a)\right\}^2} I, \quad \ldots (20)$$

where k_i, $(i = 1, 2, 3, \ldots)$ are positive roots of $J_0(ka) = 0$ ... (21)

and $$I = \int_0^a r\left(1 - \frac{r^2}{a^2}\right) J_0(k_i a)dr = \int_0^a r\, J_0(k_i r)dr - \int_0^a \frac{r^3}{a^2} J_0(k_i r)dr$$

or $$I = I_1 - I_2, \text{ say} \quad \ldots (22)$$

where $$I_1 = \int_0^a r\, J_0(k_i r)dr = \int_0^{k_i a} \frac{t}{k_i} J_0(t)\frac{dt}{k_i}, \text{ putting } \quad k_i r = t \quad \text{and} \quad dr = \frac{dt}{k_i}$$

$$= \frac{1}{k_i^2}\int_0^{k_i a} t\, J_0(t)\, dt = \frac{1}{k_i^2}\int_0^{k_i a} \frac{d}{dt}\left\{t\, J_1(t)\right\} dt \text{, using results of Bessel functions}$$

$$= \frac{1}{k_i^2}\left[t\, J_1(t)\right]_0^{k_i a} = \frac{1}{k_i^2} \times k_i a\, J_1(k_i a), \text{ since } J_1(0) = 0$$

Thus, $$I_1 = (a/k_i) \times J_1(k_i a) \quad \ldots (23)$$

From a recurrence relation of Bessel's function, we have

$$\frac{d}{dt}(t^n J_n) = t^n\, J_{n-1}(t) \qquad \text{and hence} \qquad \int t^n J_{n-1}(t)dt = t^n J_n(t) \quad \ldots (24)$$

Also, $I_2 = \int_0^a \frac{r^3}{a^2} J_0(k_i r)dr = \int_0^{k_i a} \frac{t^3}{a^2 k_i^3} J_0(t)\frac{dt}{k_i}$, putting $k_i r = t$ and $dr = \frac{dt}{k_i}$

$$= \frac{1}{a^2 k_i^4}\int_0^{k_i a} t^2 \{t\, J_0(t)\}dt = \frac{1}{a^2 k_i^4}\left\{\left[t^2 \cdot tJ_1(t)\right]_0^{k_i a} - \int_0^{k_i a} 2t \cdot t\, J_i(t)\, dt\right\}$$

[using integrating by parts and result (24) for $n = 1$]

$$= \frac{k_i^3 a^3 J_1(k_i a)}{a^2 k_i^4} - \frac{2}{a^2 k_i^4}\int_0^{k_i a} t^2 J_1(t)dt, \text{ as } J_1(0) = 0$$

$$= \frac{a}{k_i} J_1(k_i a) - \frac{2}{a^2 k_i^4}\left[t^2 J_2(t)\right]_0^{k_i a}, \text{ using result (24) for } n = 2$$

$$= (a/k_i) \times J_1(k_i a) - (2/a^2 k_i^4) \times k_i^2 a^2 J_2(k_i a), \text{ as } J_2(0) = 0$$

Thus, $$I_2 = (a/k_i) \times J_1(k_i a) - (2/k_i^2) \times J_2(k_i a) \quad \text{... (25)}$$

Then, (22), (23) and (25) $\Rightarrow$ $$I = (2/k_i^2) \times J_2(k_i a) \quad \text{... (26)}$$

Substituting the value of I given by (26) in (20), we get

$$F_i = \frac{2}{a^2\left\{J_i(k_i a)\right\}^2} \times \frac{2}{k_i^2} J_2(k_i a) = \frac{4J_2(k_i a)}{\left\{k_i a\, J_1(k_i a)\right\}^2} \quad \text{... (27)}$$

Substituting the value of F_i gievn by (27) in (18) the required solution is given by

$$z(r,t) = \frac{4}{a^2}\sum_{i=1}^{\infty} \frac{J_2(k_i a) J_0(k_i r)\cos(ck_i t)}{\left\{k_i J_1(k_i a)\right\}^2}$$

Ex. 4. *Consider vibrations of a circular membrane of radius a which is fixed along the bondary r = a. Let initial deflection be f(r) and initial velocity be g(r). Obtain radially symmetric solution u(r, t) of*

$$(\partial^2 u/\partial r^2) + (1/r) \times (\partial u/\partial r) = (1/c^2) \times (\partial^2 u/\partial t^2) \quad \text{... (1)}$$

with $$u(a, t) = 0 \quad \text{... (2)}$$

$$(\partial u/\partial t)_{t=0} = g(r) \quad \text{... (3)}$$

and $$u(r, 0) = f(r) \quad \text{... (4)}$$

[Delhi Physics Hons. 2001]

Sol. Let a solution of (1) be of the form $u(r, t) = R(r)\, T(t)$ ... (5)

Substituting this value of u in (1) and dividing by RT, we get

$$R''/R + R'/rR = T''/c^2 T \quad \text{... (6)}$$

Since r and t are independent, (6) can only be true if each side is equal to the same constant, say $-k^2$. Then (6) gives

$$T'' + c^2 k^2 T = 0 \quad \text{... (7)}$$

and $R''/R + R'/rR = -k^2$ or $r^2 R'' + r\,R' + k^2 rR = 0$ ... (8)

Solving (7) and (8), we have

$$T(t) = A_k \cos(ckt) + B_k \sin(ckt) \quad \text{... (9)}$$

and $$R(r) = C_k J_0(kr) + D_k Y_0(kr) \quad \text{... (10)}$$

We choose $D_k = 0$ so that $R(r)$ and hence $z(r, t)$ may be finite at $r = 0$ (along the z-axis).

$\therefore$ (10) reduces to $$R(r) = C_k J_0(kr) \quad \ldots (11)$$

From (9) and (11), a solution of (1) is

$[A_k \sin(ckt) + B_k \cos(ckt)]\, C_k J_0(kr)$ *i.e.*, $[E_k \sin(ckt) + F_k \cos(ckt)]\, J_0(kr)$, ... (12)

where E_k (= $A_k\, C_k$) and F_k (= $B_k\, C_k$) are new arbitrary constants.

Putting $r = a$ in (5) and using (2), we get

$0 = R(a)\, T(t)$ so that $R(a) = 0$, as $T(t) \neq 0$

Putting $r = a$ in (11) and using $R(a) = 0$, we get

$0 = C_k J_0(ka)$ so that $J_0(ka) = 0$ ... (13)

Let k_i, $i = 1, 2, 3, \ldots$ be the positive roots of (13). So by (12), the functions

$$[G_i \sin(c\, k_i\, t) + H_i \cos(c\, k_i\, t)]\, J_0(k_i r),\ i = 1, 2, 3, \ldots$$

are solutions of (1) satisfying (2). Here we have written $E_{k_i} = G_i$ and $F_{k_i} = H_i$. In order to obtain a solution also satisfying (3) and (4), we consider a more general solution

$$u(r,t) = \sum_{i=1}^{\infty} [G_i \sin(ck_i t) + H_i \cos(ck_i t)] J_0(k_i r) \quad \ldots (14)$$

Differentiating partially w.r.t. 't', (14) gives

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{\infty} [G_i c\, k_i \cos(c\, k_i t) - H_i c\, k_i \sin(c\, k_i t)] J_0(k_i r) \quad \ldots (15)$$

Putting t = 0 in (14) and (15) and using (3) and (4), we get

$$f(r) = \sum_{i=1}^{\infty} H_i J_0(k_i r) \quad \text{and} \quad g(r) = \sum_{i=1}^{\infty} G_i c\, k_i J_0(k_i r)$$

which are Fourier-Bessel series and hence

$$H_i = \frac{2}{a^2 [J_i(k_i a)]^2} \int_0^a r\, f(r)\, J_0(k_i r)\, dt \quad \ldots (16)$$

and

$$G_i c k_i = \frac{2}{a^2 [J_i(k_i a)]^2} \int_0^a r\, g(r)\, J_0(k_i r)\, dr$$

so that

$$G_i = \frac{2}{c k_i a^2 [J_1(k_i a)]^2} \int_0^a r\, g(r)\, J_0(k_i r)\, dr \quad \ldots (17)$$

Hence the required deflection $u(r, t)$ is given by (14) where G_i and H_i are given by (17) and (16) respectively.

Particular Case :— If initial velocity is zero so that $(\partial u / \partial t)_{t=0} = g(r) = 0$, then $G_i = 0$ from (17). Hence deflection is given by

$$u(r,t) = \sum_{i=1}^{\infty} H_i \cos(ck_i t) J_0(k_i r)$$

where H_i is given by (16). This has already been obtained in Ex. 3 in an alternative way.

Ex. 5. *Harmonic sound waves of period $2\pi/kc$ and small amplitude are propagated along a circular wave guide bounded by the cylinder $\rho = a$. Prove that solutions independent of the angle variable ϕ are the form $u = e^{i(kct - \gamma z)} J_0(\xi_n \rho / a)$, where ξ_n is a zero of $J_i(\xi)$ and $\gamma^2 = k^2 - (\xi_n^2 / a^2)$. Show that this mode is propagated only if $k > \xi_n / a$.*

Sol. The wave equation in cylindrical coordinates (ρ, ϕ, z) is

$$\partial^2 u/\partial\rho^2 + (1/\rho)\times(\partial u/\partial\rho) + (1/\rho^2)\times(\partial^2 u/\partial\phi^2) + \partial^2 u/\partial z^2 = (1/c^2)\times(\partial^2 u/\partial t^2)$$

Since u is independent of ϕ, this reduces to

$$\partial^2 u/\partial\rho^2 + (1/\rho)\times(\partial u/\partial\rho) + \partial^2 u/\partial z^2 = (1/c^2)\times(\partial^2 u/\partial t^2) \quad \ldots(1)$$

The boundary condition is that the velocity of the gas vanishes on the cylinder $\rho = a$ so that

$$(\partial u/\partial\rho)_{\rho=a} = 0 \quad \ldots(2)$$

Let a solution of (1) be of the form $u(\rho, z, t) = R(\rho)Z(z)T(t)$ ... (3)

Substituting this value of u in (1) and then dividing by $R(\rho)\,Z(z)\,T(t)$, we obtain

$$(1/R)\times R'' + (1/\rho)\times(R'/R) + Z''/Z = (1/c^2)\times(T''/T) \quad \ldots(4)$$

Since the R.H.S. of (4) depends on t while its L.H.S. depends on r and z, (3) can be true only if each side is equal to the same constant $-k^2$ (say). Then (4) gives

$$T'' + c^2k^2T = 0 \quad \ldots(5)$$

and $$R''/R + (1/\rho)\times(R'/R) + k^2 = -(Z''/Z) \quad \ldots(6)$$

Since the R.H.S. of (6) depends on z while its L.H.S. depends on ρ, (6) can be true only if each side is equal to the same constant, say γ^2, then (6) gives

$$Z'' + \gamma^2 Z = 0 \quad \ldots(7)$$

and $\rho^2R'' + \rho R' + (k^2 - \gamma^2)R = 0$ or $\rho^2R'' + \rho R' + w^2R = 0,$ ... (8)

where $w^2 = k^2 - \gamma^2$ or $\gamma^2 = k^2 - w^2$ ... (9)

For the present problem, solutions of (5) and (7) are taken in the form

$T(t) = e^{ikct}$ and $Z(z) = e^{-i\gamma z}$ ... (10)

Again, solution of Bessel's equation (8) is $R(\rho) = AJ_0(w\rho) + BY_0(w\rho)$ ... (11)

We know that $Y_0(w\rho) \to \infty$ as $\rho \to 0$. Hence for finite solution of (1), we choose B = 0 in (11). Then solution (8) is of the form

$$R(\rho) = J_0(w\rho) \quad \ldots(12)$$

From (10) and (12), a solution of (1) is of the form

$$u(\rho, z, t) = e^{ikct}e^{-i\gamma z}J_0(wp) = e^{i(kct-\gamma z)}J_0(w\rho) \quad \ldots(13)$$

Differentiating, (13) partially w.r.t. ρ and using boundary condition (2) we get

$[J_0'(w\rho)]_{\rho=a} = 0$ or $[-J_i(w\rho)]_{\rho=a} = 0$ $[\because J_0'(x) = -J_1(x)]$

or $$J_1(wa) = 0 \quad \ldots(14)$$

Let $\xi_n = wa$. Then (14) reduces to $J_1(\xi_n) = 0,$

i.e. $\xi_1, \xi_2, \ldots$ are the zeros of $J_1(\xi)$. Now, $\xi_n = wa$ gives $w = \xi_n/a$. Substituting this value of w in (13) and (9), we have

$$u(\rho, z, t) = e^{i(kct-\gamma z)}J_0(\xi_n\rho/a), \quad \ldots(15)$$

where $\gamma^2 = k^2 - (\xi_n^2/a^2)$ or $\gamma = [k^2 - (\xi_n^2/a^2)\}^{1/2}$ ... (16)

For the mode given by equation (15) to be propagated we must have γ real *i.e.*, $(k^2 - \xi_n^2/a^2) > 0$ *i.e.*, $k > \xi_n/a$ by (16)

EXERCISE 4(*C*)

1. Show that one possible type of vibration of a uniform circular membrane of radius a whose boundary is fixed is given by $u(r,\theta,t) = J_n(kr)\cos n\theta \cos ckt$ where k is a root of $J_n(ak) = 0$, and c is a constant depending on the membrane.

2. Find a solution of the equation

$$\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial\theta^2) = (1/c^2)\times(\partial^2 u/\partial t^2)$$

given $u(r,\theta,0) = f(r,\theta)$, $(\partial u/\partial t)_{t=0} = 0$, $u(1, \theta, t) = 0$.

3. Find the function $\psi(r,\theta,t)$ such that $\partial^2\psi/\partial r^2 + (1/r)\times(\partial\psi/\partial r) + (1/r^2)\times(\partial^2\psi/\partial\theta^2) = (1/c^2)\times(\partial^2\psi/\partial t^2)$ and (*i*) $\psi = 0$ on circle $r = a$ (ii) $\partial\psi/\partial t = 0$ at $t = 0$ **[Delhi Maths (H) 2008]**

[Hint. Proceed as in solved example 3 (a), page 4.15 by taking ψ in place of z and ignoring the condition $z = f(r)$ at $t = 0$. Then, we obtain the required solution (see equation (18) of example 3(a), page 4.15)] $\psi(r,\theta,t) = \sum_{i-1}^{\infty} F_i\ J_0(k_i r)\ \cos\ (c\,k_i t)$

MISCELLANEOUS PROBLEMS ON CHAPTER 4

1. Solve the diffusion equation $\partial\theta/\partial t = v\{\partial^2\theta/\partial r^2 + (1/r)\times(\partial\theta/\partial r)\}$ subject to

(*i*) when $r = 0$, u is finite for $t > 0$ (*ii*) when $r = a$, $u = 0$ for $t > 0$ and (*iii*) $u = (P/4\mu)\times(a^2 - r^2)$ when $t = 0$. **[Himanchal 2007]**

Ans. $\theta(r, t) = \dfrac{2Pa^2}{\mu}\sum_{m=1}^{\infty}\dfrac{1}{\lambda_m^3 J_1(\lambda_m)}\,J_0\left(\dfrac{\lambda_m r}{a}\right)e^{-(v\lambda_m^2 t/a^2)}$ where λ_m are the positive zeros of $J_0(\lambda)$

2. A homogeneous thermally conducting cylinder occupies the region $0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le z \le h$. The top $z = h$ and the lateral surface $r = a$ are held at 0°; while the base $z = 0$ is held at 100°. Assuning that there are no sources of heat generation within the cylinder, find the steady temperature distribution within the cylinder

Ans. $u(r,z) = 200\sum_{n=1}^{\infty}\dfrac{J_0(\xi_n r/a)\,\sinh\{(\xi_n/a)\times(h-z)\}}{\xi_n \sinh(\xi_n h/a)J_1(\xi_n)}$, when ξ_n are the positive zeros of $J_0(\xi)$.

3. Find the potential u inside the cylinder $0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le z \le h$, if potential on the top $z = h$, and on the lateral surface $r = a$ is held at zero, while on the base $z = 0$, the potantial is given by $u(r,\theta,0) = V_0(1 - r^2/a^2)$, where V_0 is a constant.

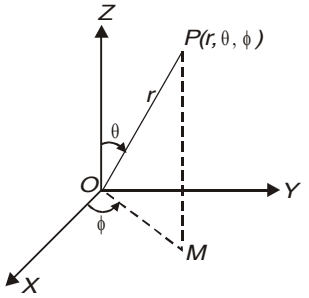
Ans. $u(r,z) = 8V_0\sum_{n=1}^{\infty}\dfrac{J_0(\xi_n\ r/a)\sinh(\xi(h-z)/a\}}{\xi_n^3 J_1(\xi_n)\sinh(\xi_n h/a)}$, where ξ_n are the positive zeros of $J_0(\xi)$

5

Boundary value problems in spherical coordinates

5.1. Introduction

While solving boundary value problems, appropriate choice of coordinate system is very useful. In physical problems that involve a spherical surface (for example, the problem of evaluating temperature in a solid sphere), it will be natural to make use of spherical coordinates.



Spherical coordinates (r, θ, ϕ) are defined by means of equations

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta, \qquad \ldots (1)$$

where $r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$. The Laplacian operator ∇^2 in spherical coordinates reduces to

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \qquad \ldots (2)$$

which may also be re-written as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \qquad \ldots (3)$$

We present some useful results for ready reference.

1. Laplace's equation $\nabla^2 \mathbf{u} = 0$

or
$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

or
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

2. Heat (Diffusion) equation $\nabla^2 \mathbf{u} = (\mathbf{1}/\mathbf{k}) \times (\partial \mathbf{u}/\partial \mathbf{t})$

or
$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

3. Wave equation $\nabla^2 \mathbf{u} = (\mathbf{1}/\mathbf{c^2}) \times (\partial^2 \mathbf{u}/\partial \mathbf{t^2})$

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

4. Table of some important equations and their solutions

Equation	Solution
4A. $\Theta'' + \cot\theta\,\Theta' + \{n(n+1) - m^2/\sin^2\theta)\}\Theta = 0$	$\Theta(\theta) = A_{mn}P_n^m(\cos\theta) + B_{mn}Q_n^m(\cos\theta)$
OR	OR
$(1-\mu^2)\dfrac{d^2\Theta}{d\mu^2} - 2\mu\dfrac{d\Theta}{d\mu} + \left\{n(n+1) - \dfrac{m^2}{1-\mu^2}\right\}\Theta = 0,$	$\Theta(\mu) = A_{mn}P_n^m(\mu) + B_{mn}Q_n^m(\mu),$
where $\mu = \cos\theta$. (Legendre's associated equation)	where P_m^m and Q_n^m are associated Legendre's functions of first and second kind respectively. Not that $Q_n^m \to \infty$ as $\theta \to 0$
4B. $\Theta'' + \cot\theta\,\Theta' + n(n+1)\Theta = 0$	$\Theta(\theta) = A_nP_n(\cos\theta) + B_nQ_n(\cos\theta)$
OR	OR
$(1-\mu^2)(d^2\Theta/d\mu^2) - 2\mu(d\Theta/d\mu) + n(n+1)\Theta = 0,$	$\Theta(\mu) = A_nP_n(\mu) + B_nQ_n(\mu),$
where $\mu = \cos\theta$. (Legendre's equation)	where P_n and Q_n are Legendre's functions of first and second kind respectively. Note that $Q_n \to \infty$ as $\theta \to 0$
4C. $r^2R'' + 2rR' + \{\lambda^2r^2 - n(n+1)\}R = 0$	$R(r) = (\lambda)^{-1/2}[A_nJ_{n+1/2}(\lambda r) + B_nJ_{-(n+1/2)}(\lambda r)],$ where $J_{\pm(n+1/2)}$ are spherical Bessel functions. Note that $J_{-(n+1/2)}(\lambda r) \to \infty$ as $r \to 0$
4D. $r^2R'' + 2rR' - n(n+1)R = 0$ (Homogeneous equation)	$R(r) = A_nr^n + B_n/r^{n+1}$, where A_n and B_n are arbitrary constants.

5. Legendre series Expansion of $f(\mu)$.

If $f(\mu) = \sum_{n=0}^{\infty} A_nP_n(\mu),$ then $A_n = \dfrac{2n+1}{2}\displaystyle\int_{-1}^{1} f(\mu)P_n(\mu)d\mu$

5.2 Solution of Laplace's equation

Ex. 1. (*i*) *Find the solution of Laplace's equation in spherical coordinates by the method of separation of variables.* **[Himanchal 1998, 99; Delhi Maths (H) 1994]**

(*ii*) *Show that in spherical polar coordinates Laplace's equation possesses solution of the form* $(Ar^n + B/r^{n+1})P_n^m(\cos\theta)\exp(\pm im\phi)$, *where* $P_n(\cos\theta)$ *i.e.*, $P_n(\mu)$ *satisfies the differential equation* $(1-\mu^2)\dfrac{d^2P_n}{d\mu^2} - 2\mu\dfrac{dP_n}{d\mu} + \left\{n(n+1) - \dfrac{m^2}{1-\mu^2}\right\}P_n = 0.$ **[Rajasthan 2001, Himanchal 1996, 99; 2002, Delhi Maths (H) 1998, 2001, 2003, 2008, 2010; Kurukshetra 2003]**

Sol. (*i*) Laplace's equation in spherical coordinates is

$$\frac{\partial^2u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2} = 0 \qquad \ldots (1)$$

Let a solution of (1) be of the form $u(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$ ... (2)

where R, Θ and Φ are functions of r, θ and ϕ respectively.

Substituting this value of u in (1) and dividing by $R\,\Theta\,\Phi$, we have

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} = 0$$

or
$$\left[\frac{1}{R}(R'' + \frac{2}{r}R') + \frac{1}{r^2\Theta}(\Theta'' + \cot\theta\,\Theta')\right] r^2\sin^2\theta = -\frac{\Phi''}{\Phi} \quad \text{... (3)}$$

Since the right hand side of (3) depends only on ϕ while the left hand side of (3) depends only on r and θ, (3) can only be true if each side is equal to the same constant, say m^2. In many physical problems, we require that solution should involve trignometric functions involving ϕ. Hence we have chosen m^2 as the constant of separation in (3). Thus (3) reduces to

$$\Phi'' + m^2\Phi = 0 \quad \text{... (4)}$$

and
$$\frac{1}{R}(r^2R'' + 2rR') + \frac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta') = \frac{m^2}{\sin^2\theta}$$

i.e.,
$$\frac{1}{\Theta}(\Theta'' + \cos\theta\,\Theta') - \frac{m^2}{\sin^2\theta} = -\frac{1}{R}(r^2R'' + 2rR') \quad \text{... (5)}$$

Since r and θ are independent, (5) can only be true if each side of (5) is equal to the same constant, say $-n\,(n+1)$. Then (5) gives

$$r^2R'' + 2rR' - n(n+1)R = 0 \quad \text{... (6)}$$

and
$$\Theta'' + \cot\theta\,\Theta' + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0 \quad \text{... (7)}$$

which is associated Legendre's equation.

Putting $\cos\theta = \mu$, (7) reduces to

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0 \quad \text{... (8)}$$

Solving (4), (6) and (8), we have

$$\Phi(\phi) = A_m\cos m\phi + B_m\sin m\phi \quad \text{... (9)}$$

$$R(r) = C_nr^n + D_n/r^{n+1} \quad \text{... (10)}$$

and
$$\Theta(\theta) = E_{mn}\,F_n^m(\cos\theta) + F_{mn}\,Q_n^m(\cos\theta) \quad \text{... (11)}$$

$$\therefore\ u_{mn}(r,\theta,\phi) = (A_m\cos m\phi + B_m\sin m\phi)\,(C_nr^n + D_n/r^{n+1})\,[E_{mn}P_n^m(\cos\theta) + F_{mn}Q_n^m(\cos\theta)] \quad \text{... (12)}$$

are solutions of (1). A more general solution of (1) is given by

$$u(r,\theta,\phi) = \sum_m\sum_n u_{mn}(r,\theta,\phi) \quad \text{... (13)}$$

where $u_{mn}(r, \theta, \phi)$ is given by (12)

Solution (13) can also be expressed as $\Sigma(C_nr^n + D_n/r^{n+1})\,\Theta(\cos\theta)e^{\pm im\phi}$

where $\Theta(\mu)$ is solution of associated Legendre's equation (7).

Part (*ii*) Do upto equation (12) as in part (*i*). In many real problems, we know that on physical grounds the functions Θ remains finite (and hence u remains finite) along the polar axis $\theta = 0$. But $\theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow Q_n^m(\cos\theta) \to \infty$. Hence we take $F_{mn} = 0$. Then solution (12) may be expressed in the desired form

$$(C_n r^n + D_n / r^{n+1})\, P_n^m(\cos\theta) \exp(\pm im\phi).$$

Remarks. A solution of (1) of the form $R(r)\Theta(\theta)\,\Phi(\phi)$ given by (12) is called a *spherical harmonic* and a solution of the form $\Theta(\theta)\,\Phi(\phi)$ is called a *surface harmonic*.

Ex. 2. *Obtain the solution of the Laplace's equation* $\partial^2 u/\partial r^2 + (2/r)(\partial u/\partial r) + (1/r^2)(\partial^2 u/\partial\theta^2) + (\cot\theta/r^2)(\partial u/\partial\theta) = 0$ *by the method of separation of variables in the form* $u(r,\theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n/r^n)\, P_n(\cos\theta)$, *where* $P_n(\mu)$ *are the Legendre's polynomials.*

Sol. Let a solution of the given equation be of the form $u(r,\theta) = R(r)\,\Theta(\theta)$. ... (1)

Substituting this in the given equation, we have

$$R''/R + (2/r)(R'/R) + (1/r^2)(\Theta''/\Theta) + (\cot\theta/r^2)(\Theta'/\Theta) = 0$$

or
$$-(1/R)(r^2R'' + 2rR') = (1/\Theta)(\Theta'' + \cot\theta\,\Theta') \quad \text{... (2)}$$

Since r and θ are independent variables, (2) is true if each side is equal to the same constant, say $-n(n+1)$. Then, (2) gives the differential equations.

$$r^2R'' + 2r\,R' - n(n+1)R = 0 \quad \text{... (3)}$$

and
$$\Theta'' + \cot\theta\,\Theta' + n(n+1)\,\Theta = 0 \quad \text{... (4)}$$

Putting $\mu = \cos\theta$, (5) reduces to Legendre's equation

$$(1-\mu^2)(d^2\Theta/\partial\mu^2) - 2\mu\,(d\Theta/d\mu) + n(n+1)\,\Theta = 0 \quad \text{... (5)}$$

Solving (3), $R(r) = A_n' r^n + B_n'/r^{n+1}$ ... (6)

Solving (5), $\Theta(\theta) = C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)$ ... (7)

We choose $D_n = 0$ if $\Theta(\theta)$ is to be finite on polar axis $\theta = 0$. Then (7) reduces to

$$\Theta(\theta) = C_n P_n(\cos\theta), \quad \text{... (8)}$$

where $P_n(\cos\theta)$ *i.e.*, $P_n(\mu)$ are the Legendre's polynomials.

Using (1), (6) and (8) a solution of the given equation is

$$\{C_n A_n'\, r^n + C_n B_n'(1/r^{n+1})\} P_n(\cos\theta) \quad \text{or} \quad (A_n r^n + B_n/r^{n+1}) P_n(\cos\theta),$$

where $A_n (= C_n A_n')$ and $B_n (= C_n\, B_n')$ are new arbitrary constants.

Using the principle of superposition, the required general solution is

$$u(r,\theta) = \sum_{n=0}^{\infty} (A_n r^u + B_n/r^{n+1}) P_n(\cos\theta). \quad \text{... (9)}$$

Ex. 3. *Potential* $u(r, \theta)$ *in the exterior of a unit sphere satisfies the equation*

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0.$$ *Obtain the expression for* $u(r, \theta)$ *if*

(*i*) $u(1, 0) = \cos 2\theta$. **[Nagpur 1999, 2001, 04]**

(*ii*) $u(1, \theta) = \cos 3\theta - 1$. **[Mumbai 2002, Kanpur 2001, 05]**

Sol. Part. (*i*) Re-writing given equation,

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} + \cot\theta \frac{\partial u}{\partial \theta} = 0. \quad \ldots (1)$$

We require potential $u(r, \theta)$ at a point (r, θ) outside the given unit sphere (*i.e.*, sphere of radius unity). On physical grounds, potential at infinity is zero.

$$\therefore \quad \lim_{r \to \infty} u(r, \theta) = 0. \quad \ldots (2)$$

Further, given $u(1, \theta) = \cos 2\theta$. ... (3)

We now solve (1) subject to the boundary conditions (2) and (3).

Let a solution of (1) be of the form $u(r, \theta) = R(r)\, \Theta(\theta)$ ... (4)

Substituting this value of u in (1) and dividing by $R\Theta$, we get

$$r^2 \frac{R''}{R} + 2r \frac{R'}{R} + \frac{\Theta''}{\Theta} + \cot\theta \frac{\Theta'}{\Theta} = 0. \quad \text{or} \quad \frac{1}{\Theta}(\Theta'' + \cot\theta\, \Theta') = -\frac{1}{R}(r^2 R'' + 2rR') \quad \ldots (5)$$

Since r and θ are independent, (5) can only be true if each side is equal to a constant, say $-n(n + 1)$. Then (5) gives the differential equations

$$r^2 R'' + 2rR' - n(n+1)\, R = 0 \quad \ldots (6)$$

and $$\Theta'' + \cot\theta\, \Theta' + n(n+1)\Theta = 0 \quad \ldots (7)$$

Putting $\mu = \cos\theta$, (7) reduces to

$$(1 - \mu^2)\,(d^2\Theta / d\mu^2) - 2\mu(d\Theta / d\mu) + n(n+1)\Theta = 0 \quad \ldots (8)$$

which is Legendre's equation.

Solution of (6) and (8) are $R(r) = A_n r^n + B_n / r^{n+1}$ ... (9)

and $$\Theta(\theta) = C_n P_n(\cos\theta) + D_n Q_n(\cos\theta) \quad \ldots (10)$$

We choose $D_n = 0$ if $\Theta(\theta)$ is to be finite along polar axis $\theta = 0$. Again, we take $A_n = 0$ in (9), since otherwise $R(r) \to \infty$ which contradicts (2). With these values of A_n and D_n, the functions

$$u_n(r, \theta) = E_n r^{-(n+1)} P_n(\cos\theta)$$

with $E_n = B_n C_n$, are solutions of (1), satisfying boundary condition (2).

In order to obtain a solution also satisfying (3), we consider a more general solution by superposition, namely,

$$u(r, \theta) = \sum_{n=0}^{\infty} u_n(r, \theta) = \sum_{n=0}^{\infty} E_n r^{-(n+1)} P_n(\cos\theta) \quad \ldots (11)$$

Putting $r = 1$ and (11) and using (3), we get

$$\cos 2\theta = \sum_{n=0}^{\infty} E_n P_n(\cos\theta) \quad \text{or} \quad 2\cos^2\theta - 1 = \sum_{n=0}^{\infty} E_n P_n(\cos\theta)$$

or $$2\mu^2 - 1 = \sum_{n=0}^{\infty} E_n P_n(\mu), \quad \text{taking} \quad \mu = \cos\theta \quad \ldots (12)$$

which is Legendre series expansion of $2\mu^2 - 1$. Hence, we have

$$E_n = \frac{2n+1}{2}\int_{-1}^{1}(2\mu^2 - 1)P_n(\mu)d\mu \qquad \ldots (13)$$

Putting $n = 0$ in (13) and noting that $P_0(\mu) = 1$, we get

$$E_0 = \frac{1}{2}\int_{-1}^{1}(2\mu^2 - 1)\,d\mu = \frac{1}{2}\times 2\int_{0}^{1}(2\mu^2 - 1)d\mu = \left[\frac{2\mu^3}{3} - \mu\right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3} \qquad \ldots (14)$$

Putting $n = 1$ in (13) and noting that $P_1(\mu) = \mu$, we get

$$E_1 = \frac{3}{2}\int_{-1}^{1}(2\mu^2 - 1)\mu d\mu = 0\text{, since } (2\mu^3 - \mu) \text{ is an odd function} \qquad \ldots (15)$$

Putting $n = 2$ in (13) and noting that $P_2(\mu) = (3\mu^2 - 1)/2$, we get

$$E_2 = \frac{5}{2}\int_{-1}^{1}\frac{(2\mu^2 - 1)(3\mu^2 - 1)}{2}d\mu = \frac{5}{4}\int_{-1}^{1}(6\mu^4 - 5\mu^2 + 1)d\mu = \frac{5}{4}\times 2\int_{0}^{1}(6\mu^4 - 5\mu^2 + 1)\,d\mu$$

Then, $$E_2 = (5/2)\times\left[(6/5)\times\mu^5 - (5/3)\times\mu^3 + \mu\right]_0^1 = 4/3 \qquad \ldots (16)$$

Again, we know that $$\int_{-1}^{1}\mu^m P_n(\mu)\,d\mu = 0 \text{ if } m < n \qquad \ldots (17)$$

and $$\int_{-1}^{1}P_n(\mu)\,d\mu = 0 \quad \text{for} \quad n \geq 1. \qquad \ldots (18)$$

where $P_n(\mu)$ is a polynomial of degree n. ... (19)

In view of (17), (18) and (19), (13) shows that $E_n = 0$ for each $n \geq 3$. ... (20)

Using (14), (15), (16) and (20), (11) reduces to

$$u(r,\theta) = \frac{E_0 P_0(\cos\theta)}{r} + 0 + \frac{E_2 P_2(\cos\theta)}{r^2} = -\frac{1}{3r} + \frac{4}{3r^3}\times\frac{3\cos^2\theta - 1}{2}$$

i.e., $u(r, \theta) = \{2(3\cos^2\theta - 1) - r^2\}/3r^2$, which is the required potential.

Part (*ii*) Left as an exercise for the reader.

Ex. 4. *Determine the potential outside and inside a spherical surface which is kept at a fixed distribution of electrical potential of the form* $u = F(\theta)$. It is assumed that the space inside and outside the surface is free of charges. **[Meerut 1998]**

Sol. We know that potential u satisfies a Laplace equation. Since we have a spherical surface, u is supposed to satisfy

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

Due to spherical symmetry of the problem, $\partial^2 u/\partial\phi^2 = 0$.

Hence we are to solve the resulting Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} = 0. \qquad \ldots (1)$$

Let a be the radius of the sphere. Then, by problem $u(a, \theta) = F(\theta)$. ... (2)

Again, the potential $u(r, \theta)$ is finite along the polar axis $\theta = 0$ so that

$$u(r, 0) = \text{finite quantity.} \quad \ldots (3)$$

Let a solution of (1) be of the form $u(r,\theta) = R(r)\,\Theta(\theta)$ ... (4)

Substituting this value of u in (1) and dividing by $R\,\Theta$, we have

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} = 0, \quad \text{or} \quad \frac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta') = -\frac{1}{R}(r^2R' + 2rR') \quad \ldots (5)$$

Since r and θ are independent, (2) can only be true it each side in equal to a constant, say $-n(n+1)$. Then (5) gives the differential equations

$$r^2R'' + 2rR' - n(n+1)R = 0 \quad \ldots (6)$$

and $$\Theta'' + \cot\theta\,\Theta' + n(n+1)\Theta = 0 \quad \ldots (7)$$

Putting $\mu = \cos\theta$, (7) reduces to

$$(1-\mu^2)(d^2\Theta/d\mu^2) - 2\mu(d\Theta/d\mu) + n(n+1)\Theta = 0, \quad \ldots (8)$$

which is Legendre's equation.

Solutions of (6) and (8) are

$$R(r) = A_n r^n + B_n/r^{n+1} \quad \ldots (9)$$

and $$\Theta(\theta) = C_nP_n(\cos\theta) + D_nQ_n(\cos\theta) \quad \ldots (10)$$

We choose $D_n = 0$ if $\Theta(\theta)$ is to be finite along the polar axis (*i.e.*, condition (3) is satisfied).

Hence (10) reduces to $$\Theta(\theta) = C_nP_n(\cos\theta) \quad \ldots (11)$$

Therefore, a general solution of (1) is of the form

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n\left(A_nr^n + B_n/r^{n+1}\right)P_n(\cos\theta) \quad \textit{i.e.,} \quad u(r,\theta) = \sum_{n=0}^{\infty}\left(E_nr^n + F_n/r^{n+1}\right)P_n(\cos\theta) \quad \ldots (12)$$

where E_n (= $C_n\,A_n$) and F_n (= $C_n\,B_n$) are new arbitrary constants. Two cases arise :

(*i*) Potential outside the sphere

Since the potential at $r = \infty$ is zero on physical grounds, we take $E_n = 0$ in (12). Hence the general solution of (1) for $r > a$ (*i.e.*, outside the sphere) is given by

$$u(r,\theta) = \sum_{n=1}^{\infty}\frac{F_n}{r^{n+1}}P_n(\cos\theta) \quad \ldots (13)$$

Putting $r = a$ in (13) and using (2), we get

$$F(\theta) = \sum_{n=0}^{\infty}\frac{F_n}{a^{n+1}}P_n(\cos\theta) \quad \text{or} \quad F(\cos^{-1}\mu) = \sum_{n=0}^{\infty}\frac{F_n}{a^{n+1}}P_n(\mu),$$

$[\because\ \mu = \cos\theta \Rightarrow \theta = \cos^{-1}\mu]$

which is Legendre's series expansion for $F(\cos^{-1}\mu)$ and hence

$$\frac{F_n}{a^{n+1}} = \frac{2n+1}{2}\int_{-1}^{1}F(\cos^{-1}\mu)P_n(\mu)\,d\mu \quad \text{so that} \quad F_n = \frac{2n+1}{2}a^{n+1}\int_{\pi}^{0}F(\theta)P_n(\cos\theta)\,(-\sin\theta\,d\theta)$$

$[\because\ \mu = \cos\theta \Rightarrow d\mu = -\sin\theta\,d\theta]$

or $$F_n = \frac{2n+1}{2}a^{n+1}\int_{0}^{\pi}F(\theta)P_n(\cos\theta)\sin\theta\,d\theta \quad \ldots (14)$$

Thus, the required potential is given by (13) wherein F_n is given by (14).

(*ii*) Potential inside the sphere.

Since the potential at $r = 0$ is finite on physical grounds, we take $F_n = 0$ in (12).

Hence the general solution of (1) for $r < a$ (*i.e.*, inside the sphere) is given by

$$u(r,\theta) = \sum_{n=0}^{\infty} E_n r^n P_n(\cos\theta) \qquad \ldots (15)$$

Putting $r = a$ in (15) and using (2), we get

$$F(\theta) = \sum_{n=0}^{\infty} E_n a^n P_n(\cos\theta) \qquad \text{or} \qquad F(\cos^{-1}\mu) = \sum_{n=0}^{\infty} E_n a^n P_n(\mu), \text{ as before}$$

which is Legendre series expansion for $F(\cos^{-1}\mu)$ and hence.

$$E_n a^n = \frac{2n+1}{2}\int_{-1}^{1} F(\cos^{-1}\mu)P_n(\mu)d\mu = \frac{2n+1}{2}\int_{\pi}^{0} F(\theta)P_n(\cos\theta)(-\sin\theta\, d\theta),$$

$$[\because\ \mu = \cos\theta \Rightarrow d\mu = -\sin\theta\, d\theta\ ]$$

or
$$E_n = \frac{2n+1}{2a^n}\int_0^{\pi} F(\theta)P_n(\cos\theta)\sin\theta\, d\theta \qquad \ldots (16)$$

Hence the required potential is given by (15) wherein E_n is given by (16).

Ex. 5(a). *Find the steady state temperature in a uniform solid sphere of radius a when its surface is maintained at temperature $F(\theta)$.*

Sol. Steady state temperature is governed by Laplace's equation (which we take in spherical coordinates due to the geometry of the present problem).

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

Due to spherical symmetry of the problem in hand, we must have $\partial^2 u/\partial\phi^2 = 0$ (*i.e.*, temperature u depends only on r and θ) and so temperature u satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} = 0 \qquad \ldots (1)$$

with boundary condition
$$u(a, \theta) = F(\theta) \qquad \ldots (2)$$

Again the temperature is finite along the polar axis $\theta = 0$ on physical grounds. Hence

$$u(r, 0) = \text{finite quantity} \qquad \ldots (3)$$

Let a solution of (1) be of the form
$$u(r,\theta) = R(r)\,\Theta(\theta) \qquad \ldots (4)$$

Substituting this value of u in (1) and dividing by $R\Theta$, we have

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} = 0 \qquad \text{or} \qquad \frac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta') = -\frac{1}{R}(r^2R'' + 2rR') \qquad \ldots (5)$$

Since r and θ are independent, (5) can only be true if each side is equal to a constant, say $-n(n+1)$. Then (5) gives the differential equations

$$r^2R'' + 2rR' - (n+1)R = 0 \qquad \ldots (6)$$

$$\Theta'' + \cot\theta\,\Theta' + n(n+1)\Theta = 0. \qquad \ldots (7)$$

Putting $\mu = \cos\theta$, (7) reduces to

$$(1-\mu^2)(d^2\Theta/d\mu^2) - 2\mu(d\Theta/d\mu) + n(n+1)\Theta = 0, \qquad \ldots (8)$$

which is Legendre's equation

Solutions of (6) and (8) are

$$R(r) = A_n r^n + B_n/r^{n+1} \qquad \ldots (9)$$

and
$$\Theta(\theta) = C_n P_n(\cos\theta) + D_n Q_n(\cos\theta) \qquad \ldots (10)$$

We choose $D_n = 0$ if $\Theta(\theta)$ is to be finite along the polar axis (*i.e.*, condition (3) is satisfied).

Hence (10) reduces to $$\Theta(\theta) = C_n P_n(\cos\theta) \quad \text{... (11)}$$

Hence a general solution of (1) is of the form

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n \left(A_n r^n + B_n / r^{n+1}\right) P_n(\cos\theta), \text{ i.e., } \quad u(r,\theta) = \sum_{n=0}^{\infty} (E_n r^n + F_n / r^{n+1}) P_n(\cos\theta) \quad \text{... (12)}$$

where E_n (= $C_n A_n$) and F_n (= $C_n B_n$) are new arbitrary constants.

Now, since the temperature at $r = 0$ is finite on physical grounds, we take $F_n = 0$ in (12) and obtain $$u(r,\theta) = \sum_{n=0}^{\infty} E_n r^n P_n(\cos\theta) \quad \text{... (13)}$$

Putting $r = a$ in (13) and using (2), we get

$$F(\theta) = \sum_{n=0}^{\infty} E_n a^n P_n(\cos\theta) \quad \text{or} \quad F(\cos^{-1}\mu) = \sum_{n=0}^{\infty} E_n a^n P_n(\mu),$$

$[\because\ \mu = \cos\theta \Rightarrow \theta = \cos^{-1}\mu]$

which is Legendre series expansion for $F(\cos^{-1}\mu)$ and hence

$$E_n a^n = \frac{2n+1}{2}\int_{-1}^{1} F(\cos^{-1}\mu) P_n(\mu)\, d\mu \quad \text{so that} \quad E_n = \frac{2n+1}{2a^n}\int_{\pi}^{0} F(\theta) P_n(\cos\theta)\,(-\sin\theta\, d\theta)$$

$[\because\ \mu = \cos\theta \Rightarrow d\mu = -\sin\theta\, d\theta]$

or $$E_n = \frac{2n+1}{2a^n}\int_0^{\pi} F(\theta) P_n(\cos\theta)\sin\theta\, d\theta. \quad \text{... (14)}$$

Hence the required temperature is given by (13) wherein E_n is given by (14).

Ex. 5(*b*) *Find the steady distribution of temperature inside a sphere subject to* $u(a, \theta) = T_0$, $0 < \theta < \pi/2$; $u(a, \theta) = 0$, $\pi/2 < \theta < \pi$. **[Delhi B.Sc (H) Physics 1997]**

Sol. This is a particular case of Ex. 5(*a*). Here in place (2) we are given

$$u(a,\theta) = F(\theta) = \begin{cases} T_0, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi \end{cases} \quad \text{... (2)}'$$

Taking (2)′ in place of (2) in Ex. 5(*a*) proceed upto equation (14). Now, we have

$$E_n = \frac{2n+1}{2a^n}\int_0^{\pi} F(\theta) P_n(\cos\theta)\sin\theta\, d\theta = \frac{2n+1}{2a^n}\left[\int_0^{\pi/2} F(\theta) P_n(\cos\theta)\sin\theta\, d\theta + \int_{\pi/2}^{\pi} F(\theta),\ P_n(\cos\theta)\sin\theta\, d\theta\right]$$

Thus, $$E_n = \frac{2n+1}{2a^n}\int_0^{\pi/2} T_0 P_n(\cos\theta)\sin\theta\, d\theta, \text{ using (2)} \quad \text{... (15)}$$

We know that $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$, $P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2$, $P_4(\cos\theta) = (35\cos^4\theta - 30\cos^2\theta + 3)/8$ and $P_5(\cos\theta) = (63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)/8$

Putting $n = 0, 1, 2, 3, 4, 5, \ldots$ in (15) and using the above values of $P_0(\cos\theta)$, $P_1(\cos\theta)$, $P_2(\cos\theta)$, $P_3(\cos\theta)$, $P_4(\cos\theta)$ and $P_5(\cos\theta)$, we have

$$E_0 = \frac{T_0}{2}\int_0^{\pi/2} \sin\theta\, d\theta = \frac{T_0}{2}\left[-\cos\theta\right]_0^{\pi/2} = \frac{T_0}{2}, \quad E_1 = \frac{3T_0}{2a}\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta = \frac{3T_0}{2a}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{3T_0}{4a},$$

$$E_2 = \frac{5T_0}{2a^2}\int_0^{\pi/2}\frac{(3\cos^2\theta - 1)}{2}\sin d\,\theta = \frac{5T_0}{4a^2}\left[3\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta - \int_0^{\pi/2}\sin\theta\, d\theta\right]$$

$$= \frac{5T_0}{4a^2}\left\{\left[-\frac{3\cos^3\theta}{3}\right]_0^{\pi/2} - \left[-\cos\theta\right]_0^{\pi/2}\right\} = 0,$$

$$E_3 = \frac{7T_0}{2a^3}\int_0^{\pi/2}\frac{5\cos^3\theta - 3\cos\theta}{2}\sin\theta d\theta = \frac{7T_0}{4a^3}\left[5\int_0^{\pi/2}\cos^3\theta\sin\theta\, d\theta - 3\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right]$$

$$= \frac{7T_0}{4a^3}\left\{\left[-\frac{5\cos^4\theta}{4}\right]_0^{\pi/2} - 3\left[-\frac{\cos^2\theta}{2}\right]_0^{\pi/2}\right\} = \frac{7T_0}{4a^3}\left(\frac{5}{4} - \frac{3}{2}\right) = -\frac{7T_0}{16a^3},$$

$$E_4 = \frac{9T_0}{2a^4}\int_0^{\pi/2}\frac{35\cos^4\theta - 30\cos^2\theta + 3}{8}\sin\theta\, d\theta = \frac{9T_0}{16a^4}\left\{35\left[-\frac{\cos^5\theta}{5}\right]_0^{\pi/2} - 30\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} + 3\left[-\cos\theta\right]_0^{\pi/2}\right\}$$

$$= (9T_0/16a^4) \times (7 - 10 + 3) = 0$$

and $$E_5 = \frac{11T_0}{2a^5}\int_0^{\pi/2}\frac{63\cos^5\theta - 70\cos^3\theta + 15\cos\theta}{8}\sin\theta\, d\theta$$

$$= \frac{11T_0}{16a^5}\left\{63\left[-\frac{\cos^6\theta}{6}\right]_0^{\pi/2} - 70\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/2} + 15\left[-\frac{\cos^2\theta}{2}\right]_0^{\pi/2}\right\}$$

$$= \frac{11T_0}{16a^5}\left(\frac{63}{6} - \frac{70}{4} + \frac{15}{2}\right) = \frac{11T_0}{16a^5}\left(\frac{21}{2} - \frac{35}{2} + \frac{15}{2}\right) = \frac{11T_0}{32a^5}$$

and so on. Substituting the above values in (13), the required solution is given by

$$u(r,\theta) = E_0P_0(\cos\theta) + E_1r\,P_1(\cos\theta) + E_2r^2P_2(\cos\theta) + E_3r^3P_3(\cos\theta) + E_4\,r^4P_4(\cos\theta) + E_5r^5P_5(\cos\theta) + ...$$

$$i.e.,\quad u(r,\theta) = T_0/2 + (3T_0r/4a)\times P_1(\cos\theta) - (7T_0r^3/16a^3)\times P_3(\cos\theta) + (11T_0r^5/32a^5)\times P_5(\cos\theta) + ... \quad ... (16)$$

Ex. 5 (c). *Find the steady state temperature in a uniform solid sphere of radius unity when one half of the surface is kept at the constant temperature* $0°C$ *and the other half at the constant temperature* $1°C$.

Sol. This is particular case of Ex. 5 (*b*). Here $a = 1$ and $T_0 = 1$. Hence, proceeding as in Ex. 5(*b*). The required solutions from (16) is given by

$u(r, \theta) = 1/2 + (3r/4) \times P_1(\cos\theta) - (7r^3/16) \times P_3(\cos\theta) + (11r^5/32) \times P_5(\cos\theta) + ...$

Ex. 6. *A rigid sphere of radius 'a' is placed in a stream of fluid whose velocity in the undisturbed state is V, determine the velocity of fluid at any point of the disturbed stream.*

Sol. Let z-axis be taken in the direction of the given velocity V. Let spherical polar coordinates of any point P of given fluid be (r, θ, ϕ) with origin at the centre of the fixed sphere of radius 'a'. We assume that there is perfect fluid having irrotational motion. Then know that the velocity $\boldsymbol{q}$ of fluid at P is given by* $$\boldsymbol{q} = -\text{grad}\,\psi = -\nabla\psi, \quad ... (1)$$

---

* Refer chapter 5 of Flud Dynamics by Dr. M.D. Raisinghania published by S. Chand & Co. Ltd. New Delhi.

where ψ is the velocity potential satisfying the following three conditions:

$$(i)\ \nabla^2\psi = 0 \quad i.e., \quad \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0 \quad \ldots (2)$$

$$(ii) \qquad (\partial\psi/\partial r)_{r=a} = 0, \quad \ldots (3)$$

i.e., the normal component of the velocity varishes on the boundary of the fixed sphere

(iii) $-\partial\psi/\partial z = V$, since z-axis is taken in the direction of velocity V

$$\Rightarrow \qquad \psi = -Vz = -Vr\cos\theta - = VrP_1(\cos\theta), \quad \ldots (4)$$

noting that $z = r\cos\theta$ in spherical polar coordinates and $P_1(\cos\theta) = \cos\theta$, P_1 being Legendre's polynomial.

Since the motion about the fixed sphere is axially summetric, it follows that velocity potential ψ must be independent of the variable angle ϕ, *i.e.*, $\partial^2\psi/\partial\phi^2 = 0$. Hence (2) reduces to

$$\partial^2\psi/\partial r^2 + (2/r)(\partial\psi/\partial r) + (1/r^2)(\partial^2\psi/\partial\theta^2) + (\cot\theta/r^2)(\partial\psi/\partial\theta) = 0 \quad \ldots (5)$$

Let a solution of (5) be of the form $\psi(r,\theta) = R(r)\,\Theta(\theta)$ ... (6)

Substituting (6) in (5), we have

$$R''/R + (2/r)(R'/R) + (1/r^2)(\Theta''/\Theta) + (\cot\theta/r^2)(\Theta'/\Theta) = 0$$

or $$-(1/R)(r^2R'' + 2rR') = (1/\Theta)(\Theta'' + \cot\theta\,\Theta') \quad \ldots (7)$$

Since r and θ are independent variables (7) is true if its each side is equal to the same constant (= $-n(n+1)$, say). Then (7) gives differential equations

$$r^2R'' + 2r\,R' - n(n+1)R = 0 \quad \ldots (8)$$

and $$\Theta'' + \cot\theta\,\Theta' + n(n+1)\,\Theta = 0 \quad \ldots (9)$$

Putting $\mu = \cos\theta$, (9) reduces to Legendre's equation

$$(1-\mu^2)(d^2\Theta/d\mu^2) - 2\mu(d\Theta/d\mu) + n(n+1)\,\Theta = 0 \quad \ldots (10)$$

Solutions of (8) and (10) are given by

$$R(r) = A_n\,r^n + B_n\,/r^{n+1} \quad \ldots (11)$$

and $$\Theta(\theta) = C_nP_n(\cos\theta) + D_nQ_n(\cos\theta) \quad \ldots (12)$$

We choose $D_n = 0$ in (12) if $\Theta(\theta)$ and hence ψ is be finite on z-axis (where $\theta = 0$). Hence (12) reduces to $$\Theta(\theta) = C_nP_n(\cos\theta) \quad \ldots (13)$$

Differentiating (6) partially w.r.t 'r', we get $\partial\psi/\partial r = R'(r)\,\Theta(\theta)$ ... (14)

Putting $r = a$ in (14) using (3), we have $R'(a) = 0$ ... (15)

Differentiating (11), $$R'(r) = nA_nr^{n-1} - (n+1)B_nr^{-n-2} \quad \ldots (16)$$

Putting $r = a$ in (16) and using (15), we get

$0 = nA_n\,a^{n-1} - (n+1)\,B_na^{-n-2}$ giving $B_n = \{n/(n+1)\}A_na^{2n+1}$

Substituting this value of B_n in (11), we get

$$R(r) = A_n\,\{r^n + na^{2n+1}/(n+1)\,r^{n+1}\} \quad \ldots (17)$$

Using (6), (13) and (17), a solution of (5) is

$A_n C_n \{r^n + na^{2n+1}/(n+1)\, r^{n+1}\}$ or $E_n\{r^n + n\, a^{2n+1}/(n+1)\, r^{n+1}\}$,

where E_n (= $A_n C_n$) are new arbitrary constants.

Now by principle of superposition, the general solution of (5) is

$$\psi(r,\theta) = \sum E_n \{r^n + na^{2n+1}/(n+1)r^{n+1}\} P_n(\cos\theta) \quad \ldots (18)$$

As $r \to \infty$, this velocity potential has the asymptotic form*

$$\psi \sim \sum E_n\, r^n P_n(\cos\theta) \quad \ldots (19)$$

Hence in order to satisfy the condition (4), we must take $E_1 = -V$ and all other E's zero. Thus, we take $n = 1$ in (18) and take the remaining E's equal to zero and obtain

$$\psi(r,\theta) = E_1(r + a^3/2r^2)P_1(\cos\theta) = -V(r + a^3/2r^2)\cos\theta,$$

giving the velocity potential of the fluid motion.

The components of the velocity are, therefore, given by

$$q_r = -\partial\psi/\partial r = V(1 - a^3/r^3)\cos\theta \qquad \text{and} \qquad q_\theta = -(1/r)\,(\partial\psi/\partial\theta) = -V(1 + a^3/2r^3)\sin\theta.$$

EXERCISE 5(A)

1. Find the potential $u(r, \theta)$ (*a*) interior to and (*b*) exterior to a hollow sphere of unit radius if half of its surface if charged to potential u_0 and the other half to potential zero.

Ans. (*a*) $u(r,\theta) = (u_0/2) \times [1 + (3r/2)\, P_1(\cos\theta) - (7/8)\, r^3 P_3(\cos\theta) + (1/16) r^5 P_5(\cos\theta) + ...]$

(*b*) $u(r, \theta) = (u_0/2r) \times [1 + (3/2r)\, P_1(\cos\theta) - (7/8r^3)\, P_3(\cos\theta) + (11/16r^5)\, P_5(\cos\theta) + ...]$

2. Show that a solution of Laplace's equation $\nabla^2 u = 0$ which is independent of ϕ is given by

$u(r,\theta) = (A_n r^n + B_n / r^{n+1})[C_n P_n(\mu) + D_n Q_n(\mu)]$, where $\mu = \cos\theta$.

3. Solve the equation ; $\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta \dfrac{\partial u}{\partial\theta}\right) = 0$, with the boundary conditions

$\partial u/\partial r = 0,\ r = a;\ \partial u/\partial r \to u_0\cos\theta,\ \ r \to \infty,$ where u_0 is a constant.

4. Show that a solution of a Laplace's equation in three dimensions $\nabla^2 u = 0,$ can be put in the form

$$u = \sum_{n=0}^{\infty} \left(A_n r^n + B_n / r^{n+1}\right) P_n(\cos\theta),$$

where A_n and B_n are constants, and deduce that if the potential has a constant value u_0 on a spherical surface of radius a, and the space inside and outside the surface is free of charge, then $u = u_0$ at all interior points and $u = u_0 a/r$ at each exterior point.

5. Find a solution of the equation $\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta \dfrac{\partial u}{\partial\theta}\right) = 0$

of the form $u = f(r)\cos\theta$ given that $-\partial u/\partial r = \nu\cos\theta$ when $r = a$; $-\partial u/\partial r = 0$ when $r = \infty$.

5.3 Solution of heat (or diffusion) equation.

Ex. 1. *Solve the heat (diffusion) equation in shperical polar coordinates.*

[Himanchal 2007; Kurukshetra 2001]

Sol. Heat (diffusion) equation in spherical coordinates is given by

---

* ~ is symbol which indicates that a quantity can be equated approximately (asymptotically)

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=\frac{1}{k}\frac{\partial u}{\partial t} \quad \ldots (1)$$

Let a solution of (1) be of the form $u(r, \theta, \phi, t) = R(r)\,\Theta(\theta)\,\Phi(\phi)\,T(t)$ ... (2)

Substituting this value of u in (1) and then dividing by $R\Theta\Phi T$, we have

$$\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+\frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta}+\frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi}=\frac{1}{k}\frac{T'}{T} \quad \ldots (3)$$

Since the right hand side of (3) depends only on t while the left hand side of (3) depends only on r, θ and ϕ, (3) can only be true if each side is equal to the same constant. On physical grounds temperature $u \to 0$ as $t \to \infty$ *i.e.*, $T(t) \to 0$ as $t \to \infty$. Accordingly, we take the constant of separation as $-\lambda^2$. Thus (3) gives the differential equations

$$T'' = -k\lambda^2 T \quad \ldots (4)$$

and
$$\frac{1}{R}\left(R''+\frac{2}{r}R'\right)+\frac{1}{r^2\Theta}\left(\Theta''+\cot\theta\,\Theta'\right)+\frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi}=-\lambda^2$$

i.e.
$$\left[\frac{1}{R}\left(R''+\frac{2}{r}R'\right)+\frac{1}{r^2\Theta}\left(\Theta''+\cot\Theta\,\Theta'\right)+\lambda^2\right]r^2\sin^2\theta=-\frac{\Phi''}{\Phi} \quad \ldots (5)$$

Since the right hand side of (5) depends only on ϕ while the left hand side of (5) depends only on r and θ, (5) can only be true if each side is equal to the same constant, say m^2. We choose m^2 as the constant of separation so that the resulting solution may involve trigonometric functions containing ϕ. Thus (5) gives the differential equations

$$\Phi'' + m^2\Phi = 0 \quad \ldots (6)$$

and
$$\frac{1}{R}\left(rR''+2rR'\right)+\frac{1}{\Theta}\left(\Theta''+\cot\theta\,\Theta'\right)+\lambda^2 r^2=\frac{m^2}{\sin^2\theta}$$

i.e.,
$$\frac{1}{\Theta}\left(\Theta''+\cot\theta\,\Theta'\right)-\frac{m^2}{\sin^2\theta}=-\frac{1}{R}\left(r^2R''+2rR'\right)-\lambda^2r^2 \quad \ldots (7)$$

Since r and θ are independent, (7) can only be true if each side is equal to the same constant, say $-n(n+1)$. Then (7) gives the differential equations

$$r^2R''+2rR'+\{\lambda^2r^2-n(n+1)\}R=0 \quad \ldots (8)$$

and
$$\Theta''+\cot\theta\,\Theta'+\left\{n(n+1)-\frac{m^2}{\sin^2\theta}\right\}\Theta=0 \quad \ldots (9)$$

Putting $\mu = \cos\theta$, (9) can be written as

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}\Theta=0, \quad \ldots (10)$$

which is associated Legendre's equations.

Solutions of (4), (6), (8) and (10) are

$$T_\lambda(t) = A_\lambda e^{-k\lambda^2 t} \quad \ldots (11)$$

$$\Phi_m(\phi) = B_m \cos m\phi + C_m \sin m\phi \quad \ldots (12)$$

$$R_{\lambda n}(r) = (\lambda r)^{-1/2}[D_{\lambda n} J_{n+1/2}(\lambda r) + E_{\lambda n} J_{-(n+1/2)}(\lambda r)] \quad \ldots (13)$$

$$\Theta_{mn}(\theta) = F_{mn} P_n^m(\cos\theta) + G_{mn} Q_n^m(\cos\theta) \quad \ldots (14)$$

A general solution of (1) is

$$u(r, \theta, \varphi, t) = \sum_n \sum_m \sum_\lambda T_\lambda(t) \Phi_m(\varphi) R_{\lambda n}(r)\, \Theta_{mn}(\theta),$$

where $T_{\lambda,} \Phi_m$, $R_{\lambda n}$ and Φ_{mn} are given by (11), (12), (13) and (14) respectively.

Ex. 2. *Obtain the solution of the three dimensional diffusion equation in spherical polar coordinates*

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \frac{1}{k}\frac{\partial\psi}{\partial t}$$

by the method of separation of variables in the form $r^{-1/2} J_{n+1/2}(\lambda r) P_n^m(\cos\theta) e^{\pm im\phi - \lambda^2 kt}$

Sol. On simplifying the given diffusion equation, we have

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \frac{1}{k}\frac{\partial\psi}{\partial t} \quad \ldots (1)$$

Proceed as in solved Ex. 1 upto equation (10). Now, in order to get the solution of the required form complete the solution after equation (10) as given below.

Solutions of (4), (6), (7) and (10) are of the forms

$$T(t) = e^{-k\lambda^2 t}, \qquad \Phi(\Phi) = e^{\pm im\phi}, \quad \ldots (12)$$

$$R(r) = r^{-1/2}\{A J_{n+1/2}(\lambda r) + B J_{-(n+1/2)}(\lambda r)\} \quad \ldots (13)$$

and

$$\Theta(\theta) = C P_n^m(\cos\theta) + D \Theta_n^m(\cos\theta) \quad \ldots (14)$$

We know that $J_{-(n+1/2)}(\lambda r) \to \infty$ as $r \to 0$ and $Q_n^m(\cos\theta) \to 0$ as $\theta \to 0$. Hence for getting a finite solution valid for $r \to 0$ and $\theta \to 0$, we must take $B = 0$ in (13) and $D = 0$ in (14). Thus,

$$R(r) = r^{-1/2} J_{n+1/2}(\lambda r) \quad \text{and} \quad \Theta(\theta) = P_n^m(\cos\theta) \quad \ldots (15)$$

Using (2), (12) and (15), a solution of (1) is of the form $r^{-1/2} J_{n+1/2}(\lambda r) P_n^m(\cos\theta) e^{\pm im\phi - \lambda^2 kt}$

Ex. 3. *Solve the diffusion equation in case of spherical symmetry.*

Sol. Diffusion (heat) equation in spherical co-ordinates (r, θ, ϕ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = \frac{1}{k}\frac{\partial u}{\partial t} \quad \ldots (1)$$

Due to spherical symmetry, u will be independent of θ and ϕ and hence (1) reduces to

$$\partial^2 u / \partial r^2 + (2/r)(\partial u / \partial r) = (1/k)(\partial u / \partial t) \quad \ldots (2)$$

Let a solution of (2) be of the form $u(r, t) = R(r)\, T(t)$. $\ldots (3)$

Substituting (3) in (2), we get

$$R'' T + (2/r) R' T = (1/k) R T'$$

or

$$\{R'' + (2/r)R'\}/R = (1/kT)T' \quad \ldots (4)$$

Since the L.H.S. of (4) depends only on r and the R.H.S. depends only on t, hence, the two sides of (4) must be equal to the same constant, μ say. On physical grounds we require $u \to 0$ as

$t \to \infty$, *i.e.*, $T(t) \to 0$ as $t \to \infty$. Accordingly, we take the constant $\mu = -\lambda^2$. Then, (4) leads to the following two ordinary differential equations

$$(1/kt)\,(dTdt) = -\lambda^2 \qquad \text{or} \qquad (1/T)\,dT = -\lambda^2 k\,dt \quad \ldots (5)$$

and $\{(d^2R/dr^2) + (2/r)(dR/dr)\}/R = -\lambda^2$ or $r^2(d^2R/dr^2) + 2r(dR/dr) + \lambda^2 r^2 R = 0.$ ... (6)

Solutions of (5) and (6) are $T(t) = A_\lambda e^{-\lambda^2 kt}$

and $$R(r) = (\lambda r)^{-1/2}\left[B_\lambda J_{1/2}(\lambda r) + C_\lambda J_{-1/2}(\lambda r)\right]$$

Hence, the most general solution of (2) is

$$u = \sum_\lambda (\lambda r)^{-1/2}\left[D_\lambda J_{1/2}(\lambda r) + E_\lambda J_{-1/2}(\lambda r)\right]e^{-\lambda^2 kt}, \quad \ldots (7)$$

where $D_\lambda\,(= A_\lambda B_\lambda)$ and $E_\lambda\,(= A_\lambda C_\lambda)$ are new arbitrary constants.

Ex. 4. *Solve the diffusion equation in case of a axial symmetry.*

Sol. Diffusion (heat) equation in spherical co-ordinates (r, θ, ϕ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{k}\frac{\partial u}{\partial t}. \quad \ldots (1)$$

Let u be symmetrical about z-axis. Then, u must be independent of ϕ and hence (1) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = \frac{1}{k}\frac{\partial u}{\partial t}. \quad \ldots (2)$$

Let a solution of (2) be of the form $u = R(r)\,\Theta(\theta)\,T(t).$ ... (3)

Substituting (3) and (2) and then dividing $R\,\Theta\,T$, we have

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta'}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} = \frac{1}{k}\frac{T'}{T}. \quad \ldots (4)$$

Since the L.H.S. of (4) depends only on r and θ whereas the R.H.S depends only on t, hence the two sides of (4) must be equal to the some constant, μ say. On physical grounds we require $u \to 0$ as $t \to \infty$ *i.e.*, $T(t) \to 0$ as $t \to \infty$. Accordingly, we take $\mu = -\lambda^2$. Then, (4) gives

$$T'' = -k\lambda^2 T \quad \ldots (5)$$

and $\dfrac{R''}{R} + \dfrac{2}{r}\dfrac{R'}{R} + \dfrac{1}{r^2}\dfrac{\Theta''}{\Theta} + \dfrac{\cot\theta}{r^2}\dfrac{\Theta'}{\Theta} = -\lambda^2$ *i.e.*, $\left(\dfrac{R''}{R} + \dfrac{2}{r}\dfrac{R'}{R} + \lambda^2\right)r^2 = -\dfrac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta')$ ... (6)

Since L.H.S. of (6) depends on r only and R.H.S. of (6) depends on θ only, the two-sides of (6) must be equal to the same constant, say $n\,(n+1)$. Then, (6) gives

$$r^2 R'' + 2rR' + \left\{\lambda^2 r^2 - n(n+1)\right\}R = 0. \quad \ldots (7)$$

and $$\Theta'' + \cot\theta\,\Theta' + n(n+1)\Theta = 0. \quad \ldots (8)$$

Solutions of (5), (7) and (8) are given by

$$T = A_\lambda e^{-k\lambda^2 t}, \qquad R = (\lambda r)^{-1/2}\left[B_{\lambda n} J_{n+1/2}(\lambda r) + C_{\lambda n} J_{-(n+1/2)}(\lambda r)\right]$$

and $$\Theta = D_n P_n(\cos\theta) + E_n Q_n(\cos\theta)$$

Hence, a solution of (1) in presence of axial symmetry is of the form

$$u = A_\lambda e^{-k\lambda^2 t}(\lambda r)^{-1/2}\left[B_{\lambda n}J_{n+1/2}(\lambda r) + C_{\lambda n}J_{-(n+1/2)}(\lambda r)\right]\times\left[D_n P_n(\cos\theta) + E_n Q_n(\cos\theta)\right]$$

Ex. 5. *Find temperature in a sphere of radius a when its surface is maintained at zero temperature and its initial temperature is f*(r, θ). **[Himachal 2008; Delhi Maths (H) 2003, 06]**

Sol. Heat equation in spherical coordinates (r, θ, ϕ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

Due to spherical symmetry in a sphere, $\partial^2 u/\partial\phi^2 = 0$ so that the resulting heat equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta^2} = \frac{1}{k}\frac{\partial u}{\partial t} \quad \ldots (1)$$

We are to solve (1) subject to the conditions :

Boundary Condition : $u(a, \theta, t) = 0$ ... (1)

Initial Condition : $u(r, \theta, 0) = f(r, \theta)$ ... (3)

Let a solution of (1) be of the form $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ ... (4)

Substitituting this value of u in (1) and then dividing by $R\Theta T$, we have

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} = \frac{1}{k}\frac{T'}{T} \quad \ldots (5)$$

Since the right hand side depends only on t while the left hand depends only on r and θ, (5) can only be true of each side is equal to the same constant, say $-\lambda^2$. Then (5) gives

$$T' = -k\lambda^2 T \quad \ldots (6)$$

$$\frac{1}{R}\left(R' + \frac{2}{r}R'\right) + \frac{1}{r^2\Theta}\left(\Theta'' + \cot\theta\,\Theta'\right) = -\lambda^2 \quad \text{or} \quad \frac{1}{\Theta}\left(\Theta'' + \cot\theta\,\Theta'\right) = -\frac{r^2}{R}\left(R'' + \frac{2}{R}R'\right) - \lambda^2 r^2 \quad \ldots (7)$$

Since r and θ are independent, (7) can only be true if each side is equal to the same constant, say $-n(n+1)$. Then (7) gives the differential equations

$$r^2R'' + 2rR' + \{n^2r^2 - n(n+1)\}R = 0 \quad \ldots (8)$$

and

$$\Theta'' + \cot\theta\,\Theta' + n(n+1)\Theta = 0 \quad \ldots (9)$$

Solutions of (6), (8) and (9) are

$$T(t) = A_\lambda e^{-k\lambda^2 t} \quad \ldots (10)$$

$$R(r) = (\lambda r)^{-1/2}[B_{\lambda n}J_{n+1/2}(\lambda r) + C_{\lambda n}J_{-(n+1/2)}(\lambda r)] \quad \ldots (11)$$

$$\Theta(\theta) = D_n P_n(\cos\theta) + E_n Q_n(\cos\theta) \quad \ldots (12)$$

In order to satisfy physical condition $u(r, \theta + 2\pi, t) = u(r, \theta, t)$, n must be an integer. We choose $C_{\lambda n} = 0$ in (11) so that $R(r)$ and hence $u(r,\theta,t)$, may remain finite at $r = 0$ (note that $J_{-(n+1/2)}(\lambda r) \to \infty$ as $r \to 0$).

Again, we choose $E_n = 0$ in (12) so that Θ and hence $u(r, \theta, t)$ may remain finite at $\theta = 0$ (polar axis) on physical grounds.

Using these values of $C_{\lambda n}$ and E_n, (11) and (12) reduce to

$$R(r) = B_{\lambda n}(\lambda r)^{-1/2} J_{n+1/2}(\lambda r) \quad \text{... (13)}$$

$$\Theta(\theta) = D_n P_n(\cos\theta) \quad \text{... (14)}$$

Putting $r = a$ in (4) and using boundary condition (2), we have

$$R(a)\Theta(\theta)\, T\,(t) = 0 \qquad \text{so that} \qquad R(a) = 0 \quad \text{... (15)}$$

where we have taken $\Theta(\theta) \not\equiv 0$ and $T(t) \not\equiv 0$, since otherwise $u \equiv 0$ which does not satisfy (3).

Now putting $r = a$ in (13) and using (15), we get

$$B_{\lambda n} J_{n+1/2}(\lambda a) = 0 \qquad \text{so that} \qquad J_{n+1/2}(\lambda a) = 0, \quad \text{... (16)}$$

where we have taken $B_{\lambda n} \neq 0$, since otherwise $R(r) \equiv 0$ so that $u \equiv 0$ which does not satisfy (3).

Let λ_{ni} (i = 1, 2, 3, ...) be the positive roots of (16). Then for each value of λ_{ni}, (10) and (13) may be written as

$$T(t) = F_i e^{-k\lambda_{ni}^2 t} \quad \text{... (17)}$$

and

$$R(r) = G_{ni}(\lambda_{ni} r)^{-1/2} J_{n+1/2}(\lambda_{ni} r) \quad \text{... (18)}$$

where F_i and G_{ni} are new arbitrary constants.

From (14), (17) and (18), the functions

$$u_{ni}(r,\theta,t) = H_{ni} P_n(\cos\theta) e^{-k\lambda_{ni}^2 t} (\lambda_{ni} r)^{-1/2} J_{n+1/2}(\lambda_{ni} r)$$

where $H_{ni} = D_n F_i G_{ni}$, are solutions of (1) satisfying (2). In order to obtain a solution also satisfying (3), we consider a more general solution of the form

$$u(r,\theta,t) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} H_{ni} P_n(\cos\theta) e^{-k\lambda_{ni}^2 t} (\lambda_{ni} r)^{-1/2} J_{n+1/2}(\lambda_{ni} r). \quad \text{... (19)}$$

Putting $t = 0$ in (19) and using B.C. (3), we have

$$f(r,\theta) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} H_{ni} P_n(\cos\theta)\, (\lambda_{ni} r)^{-1/2} J_{n+1/2}(\lambda_{ni} r) \quad \text{... (20)}$$

With the help of theory of Bessel functions and Legendre polynomials, (20) gives

$$H_{ni} = \frac{(2n+1)\lambda_{ni}^{1/2}}{a^2 [J'_{n+1/2}(\lambda_{ni} a)]^2} \int_0^a r^{3/2} J_{n+1/2}(r\lambda_{ni}) dr \int_{-1}^{1} P_n(\mu) f(r,\theta) d\mu, \text{ where } \mu = \cos\theta.$$

EXERCISE 5(B)

1. Obtain the solution of the diffusion equation $k\nabla^2\psi = \partial\psi/\partial t$ in sperical polar coordinates. Use your solution to find the temperature in a sphere of radius a when its surface is maintained at zero temperature and its initial temperature is $f(r, \theta)$.

2. Show that the solutions of the diffusion equation $k\nabla^2\psi = \partial\psi/\partial t$ are of the from

$$\sum_{m,n,\lambda} C_{mn\lambda} (\lambda r)^{-1/2} J_{n+1/2}(\lambda r) P_n^m(\mu) e^{\pm im\phi} e^{-\lambda^2 kt}, \text{ where } \mu = \cos\theta.$$

5.4. Solution of wave equation

Ex. 1. *Solve three dimensional wave equation in spherical polar coordinates.*

[Delhi Maths (H) 1995, 2000]

Sol. Wave equation in spherical polar coordinates (r, θ, ϕ) is

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \ldots (1)$$

Let a solution of (1) be of the form $u(r, \theta, \phi, t) = R(r)\Theta(\theta)\Theta(\phi)T(t)$ ... (2)

Substituting this value of u in (1) and then dividing by $R\Theta\Phi T$, we have

$$\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+\frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta}+\frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi}=\frac{1}{c^2}\frac{T''}{T} \quad \ldots (3)$$

Since the right hand side depends only on t while the left hand side depends only on r, θ and ϕ, (3) can only be true if each side is equal to the same constant, say $-k^2$. Then, (3) gives

$$T''+k^2c^2T=0 \quad \ldots (4)$$

and

$$\left[\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{1}{r^2\Theta}\left(\Theta''+\cot\theta\,\Theta'\right)+k^2\right]r^2\sin^2\theta=-\frac{\Phi''}{\Phi} \quad \ldots (5)$$

Since the right hand side depends only on ϕ while the left hand side depends only on r and θ, (5) can only be true if each side is equal to the same constant, say m^2. Then (5) gives

$$\Phi''+m^2\Phi=0 \quad \ldots (6)$$

and

$$\frac{1}{R}\left(r^2R''+2rR'\right)+k^2r^2+\frac{1}{\Theta}\left(\Theta''+\cot\theta\Theta''\right)=\frac{m^2}{\sin^2\theta}$$

i.e.,

$$\frac{1}{\Theta}\left(\Theta''+\cot\theta\Theta'\right)-\frac{m^2}{\sin^2\theta}=-\frac{1}{R}\left(r^2R''+2rR'\right)-k^2r^2 \quad \ldots (7)$$

Since r and θ are independent, (7) can only be true if each side is equal to the same constant, say $-n(n+1)$. Then (7) gives the differential equations

$$r^2R''+2rR'+\{k^2r^2-n(n+1)\}R=0 \quad \ldots (8)$$

and

$$\Theta''+\cot\theta\Theta'+\left\{n(n+1)-\frac{m^2}{\sin^2\theta}\right\}\Theta=0, \quad \ldots (9)$$

Putting $\mu = \cos\theta$, (9) may be re-written as

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}\Theta=0, \quad \ldots (10)$$

which is associated Legendre's equation.

Solution of (4), (6), (8) and (10) are

$$T_k(t) = A_k\cos(ckt) + B_k\sin(ckt), \quad \ldots (11)$$

$$\Phi_m(\phi) = C_m\cos m\phi + D_m\sin m\phi, \quad \ldots (12)$$

$$R_{kn}(r) = (kr)^{-1/2}\,[E_{kn}\,J_{n+1/2}(kr) + F_{kn}\,J_{-(n+1/2)}(kr)] \quad \ldots (13)$$

where $n + 1/2$ is neither zero nor an integer

and

$$\Theta_{mn}(\theta) = G_{mn}P_n^m(\cos\theta) + H_{mn}Q_n^m(\cos\theta) \quad \ldots (14)$$

A more general solution is given by

$$u(r,\theta,\varphi,t)=\sum_k\sum_m\sum_n T_k(t)\,\Phi_m(\varphi)\,R_{kn}(r)\,\Theta_{mn}(\theta),$$

where T_k, Φ_m, R_{kn} and Θ_{mn} are given by (11), (12), (13) and (14) respectively.

Ex. 2. *(a) Solve three dimensional wave equation in spherical polar co-ordinates.*

[Delhi Maths (H) 2005]

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \quad \ldots (1)$$

(*b*) *If u remains finite for* $\theta = 0$, *then find the solution of* (1).

(*c*) *If there is axial symmetry (taking z-axis as the axis of symmetry), then find the solution of (1) which remains finite for* $\theta = 0$.

(*d*) *If there is a spherical symmetry, then find the solution of* (1).

Sol. (*a*) For part (*a*) refer solved Ex. 1. Then the function

$$u = r^{-1/2}J_{\pm(n+1/2)}(kr)\left[G_{mn}P_n^m(\cos\theta)+H_{mn}Q_n^m(\cos\theta)\right]e^{\pm im\phi\pm ikct} \quad \ldots (2)$$

is a solution of the wave equation (1). If on physical grounds we require the above solution to have the symmetry properties

$u(r, \theta + \pi, \phi) = u(r, \theta, \phi)$ and $u(r, \theta, \phi + 2\pi) = u(r, \theta, \phi)$

then we must take *m* and *n* to be integers.

(*b*) If *u* remains finite for $\theta = 0$, we must take $H_{mn} = 0$ because $Q_n^m(\cos\theta) \to \infty$ as $\theta \to 0$. Hence, from (2) a solution of (1) is of the form

$$u = r^{-1/2}J_{\pm(n+1/2)}(kr)P_n^m(\cos\theta)e^{\pm im\phi\pm ikct} \quad \ldots (3)$$

(*c*) If *u* is symmetrical about *z*-axis, then *u* must be independent of ϕ and hence we must put $m = 0$ in solution (3) of (1). Then, noting that $P_n^m(\cos\theta) = P_n(\cos\theta)$, the required solution of (1) is of the form

$$u = r^{-1/2}J_{\pm(n+1/2)}(kr)P_n(\cos\theta)e^{\pm ickt}. \quad \ldots (4)$$

(*d*) When there is spherical symmetry, *u* will be independent of θ and ϕ both. Hence, we must put $m = 0$ and $n = 0$ in (3). Hence, a solution of (1) is of the form

$$u = r^{-1/2}J_{\pm(n+1/2)}(kr)e^{\pm ickt}$$

Ex. 3. *Using the method of separation of variables, solve wave equation in spherical polar coordinates, namely,*

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

What happens to the solution if on physical grounds, we require the solution of the above equation to have symmetric properties for θ *and* ϕ ? **[Delhi Maths (Hons) 2002]**

Sol. Proceed as in solved Ex. 1 upto equation (10). The complete the solution as follows:

Solutions of (4), (6), (8) and (10) are of the forms

$$T(t) = e^{\pm ickt},\ \Phi(\phi) = e^{\pm im\phi},\ R(r) = r^{-1/2}J_{\pm(n+1/2)}(kr) \quad \text{and} \quad \Theta(\theta) = AP_n^m(\cos\theta) + B\Theta_n^m(\cos\theta)$$

Since $\Theta_n^m(\cos\theta) \to \infty$ as $\theta \to 0$, hence for a solution to remain finite for $\theta = 0$, we take $B = 0$. Then the solution of wave equation can be taken as

$$u(r, \theta, \phi, t) = r^{-1/2}J_{\pm(n+1/2)}(kr)P_n^m(\cos\theta)e^{\pm im\phi\pm ikct} \quad \ldots (11)$$

Second part. We now find the required solution having symmetric properties for θ and ϕ. Then solution can be obtained from (11) by taking $m = 0$ and $n = 0$ in (11) and hence the required solution is given by $r^{-1/2} J_{\pm(n+1/2)}(kr) e^{\pm ikct}$.

Ex. 3. *Show that if there is spherical symmetry the solution of the wave equation can be obtained in the form* $\psi = (1/r) \times \{f(r - ct) + g(r + ct)\}$, *where f and g are arbitrary functions.*

[Delhi Maths (Hons.) 1996, 2004]

Sol. Wave equation in spherical polar co-ordinates (r, θ, ϕ) is given by

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \quad \ldots (1)$$

Since there is spherical symmetry, ψ must be independent of θ and ϕ both. Hence, ψ must satisfy the equation (1) in modified form

$$\partial^2 \psi / \partial r^2 + (2/r) \times (\partial \psi / \partial r) = (1/c^2) \times (\partial^2 \psi / \partial t^2). \quad \ldots (2)$$

We now re-write (2) by putting $\psi = \eta / r$. ... (3)

From (3), $\frac{\partial \psi}{\partial r} = -\frac{1}{r^2}\eta + \frac{1}{r}\frac{\partial \eta}{\partial r}$, $\frac{\partial^2 \psi}{\partial r^2} = \frac{2}{r^3}\eta - \frac{2}{r^2}\frac{\partial \eta}{\partial r} + \frac{1}{r}\frac{\partial^2 n}{\partial r^2}$, ... (4)

$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\eta}{r}\right) = \frac{1}{r}\frac{\partial \eta}{\partial t}$ and $\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial \eta}{\partial t}\right) = \frac{1}{r}\frac{\partial^2 \eta}{\partial t^2}$. ... (5)

Substituting the above values in (2), we get

$$\frac{2\eta}{r^3} - \frac{2}{r^2}\frac{\partial \eta}{\partial r} + \frac{1}{r}\frac{\partial^2 \eta}{\partial r^2} + \frac{2}{r}\left[-\frac{\eta}{r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r}\right] = \frac{1}{c^2 r}\frac{\partial^2 \eta}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 \eta}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 \eta}{\partial t^2}$$

or $(D^2 - c^2 D'^2)\eta = 0$, or $(D - cD')(D + cD')\eta = 0$ where $D \equiv \partial/\partial t$, $D' \equiv \partial/\partial r$

*whose general solution is $\eta = f(r - ct) + g(r + ct)$, where f and g are arbitrary functions.

or $\psi r = f(r - ct) + g(r + ct)$ or $\psi = (1/r) \times \{f(r - ct) + g(r + ct)\}$

Ex. 5. *A gas is contained in a rigid sphere of radius a. Show that if c is the velocity of sound in the gas, the frequencies of purely radical oscillations are* $c\xi_i/a$, *where* $\xi_1, \xi_2, \ldots$ *are the positive roots of the equation* $\tan \xi = \xi$. **[Delhi, Maths. (Hons.), 2003, 07, 11]**

Sol. The wave equation in spherical polar co-ordinates is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \quad \ldots (1)$$

The required wave function ψ for the present problem must have spherical symmetry, *i.e.*, ψ is a function of only r and t. Then (1) becomes

$$\partial^2 \psi / \partial r^2 + (2/r) \times (\partial \psi / \partial r) = (1/c^2) \times (\partial^2 \psi / \partial t^2). \quad \ldots (2)$$

---

*To understand this solution, refer chapter 4 of Part II of this book, namely, 'Partial Differential Equations'

Now the conditions to be satisfied by the wave function ψ are :

(*i*) ψ satisfies (2) (*ii*) ψ remains finite at the origin (*iii*) $u = \partial\psi / \partial r = 0$ at $r = a$.

In order to satisfy conditions (*i*) and (*ii*), we choose

$$\psi = A \times (1/r) \times \sin(kr)\, e^{ikct}, \text{ where } A \text{ is a constant.} \quad \ldots (3)$$

From (3), $$u = -\frac{\partial\psi}{\partial r} = A\left[\frac{k\cos(kr)}{r} - \frac{\sin(kr)}{r^2}\right]e^{ikct} \quad \ldots (4)$$

To satisfy condition (*iii*), putting $u = 0$ and $r = a$ in (4), we get

$$A\left[\frac{k\cos(ka)}{a} - \frac{\sin(ka)}{a^2}\right]e^{ikct} = 0 \qquad \text{so that} \qquad \tan(ka) = ka. \quad \ldots (5)$$

Hence, we have to choose k which satisfies (5).

The possible frequencies are therefore given by the expression $c\xi_i / a$ (i = 1, 2, 3, ...) where ξ_1, ξ_2, ... are the positive roots of the transcendental equation $\tan\xi = \xi$.

EXERCISE 5(C)

1. Prove that $\psi(r, \theta, \phi, t) = r^{-1/2} J_{\pm(n+1/2)}(kr) P_n^m(\cos\theta) e^{\pm im\phi \pm ickt}$ is solution of the wave equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

2. Show that the wave equation $\nabla^2\psi = \partial^2\psi / \partial t^2$ has solutions of the form $\psi = S(\theta, \phi)\, R(r, t)$ where r, θ, ϕ are spherical polar coordinates,

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2} + l(l+1)\right\}S = 0$$

and $$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}R = 0, \text{ where } l \text{ is a constant integer.}$$

MISCELLANEOUS PROBLEMS

1. Show that potential of a grounded conducting sphere in a uniform field given by $\nabla^2 u = 0$, $0 \le r < a$, $0 < \theta < \pi$, $0 \le \phi \le 2\pi$; $u(a, \theta) = 0$ and $u \to -E_0 r\cos\theta$ as $r \to \infty$ is given by $u(r, \theta) = -E_0\, r\cos\theta + (E_0 a^3 / r^2) \times \cos\theta$

2. A homogeneous solid sphere of radius a has the initial temperature distribution $f(r)$, $0 \le r \le a$, where r is the distance meansured from the centre. The surface temperature is maintained at 0°. Show that the temperature $u(r, t)$ in the sphere is the solution of $\partial u / \partial t = c^2\{\partial^2 u / \partial r^2 + (2/r) \times (\partial u / \partial r)\}$. Show also that the temperature in the sphere for $t > 0$ is given by

$$u(r, t) = (1/r) \times \sum_{n=1}^{\infty} B_n \sin(n\pi r / a) e^{-\lambda_n^2 t}, \text{ where } \lambda_n = (cn\pi) / a$$

MISCELLANEOUS PROBLEMS ON THIS PART OF THE BOOK

Ex.1. *Find the steady state temperature distribution in a thin plate bounded by x = 0, y = 0 and y = ∞ assuming that x = 0 and x = l being kept at temperature zero.*

(Madurai kamraj 2008)

Hint. Proceed as in Ex. 3(*a*) page 2.74 taking $a = l$. Here we shall not consider the condition $u(x, 0) = f(x)$. Also, we shall assume that $u(x, y) \to 0$ as $y \to \infty$. Then, from equation (18) of Ex. 3(*a*), the required temperature is given by (taking E_n, $n = 1, 2, 3$ ..... as arbitrery constants)

$$u(x, y) = \sum_{n-1}^{\infty} E_n \sin(n\pi x / l) e^{-(n\pi y / l)}$$

Ex. 2. *Solve* $\partial^2 V / \partial x^2 + \partial^2 V / dy^2 = \partial V / \partial t$, $V = 0$ *at* $t = \infty$, $x = 0$ or l and $y = 0$ or $y = l$.

(Madurai Kamraj 2008)

Sol. Proceed as in Art 2.7B, page 2.28, taking $a = b = l$ *and* $k = 1$. Here we shall not consider the condition $u(x, y, 0) = f(x, y)$. Then, from equation (21) of Art. 2.7B, we have

$$V(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} \sin \frac{m\pi x}{l} \sin \frac{n\pi y}{l} e^{-\lambda_{mn}^2 t},$$

where $\lambda_{mn}^2 = -(\pi^2 / l^2) \times (m^2 + n^2)$, by equation (20) of Art 2.7B, page 2.28

Ex. 3. *Solve* $\partial u / \partial t = \partial^2 u / \partial x^2, 0 < x < 6, t > 0$ *and* $u(0, t) = 0, u(6, t) = 0$. Also, given

$$u(x, 0) = \begin{cases} 1, \text{ if } 0 < x < 3 \\ 0, \text{ if } 3 < x < 6 \end{cases}$$ **(Madurai Kamraj 2008)**

Hint. Proceed as in Ex.2(b) page 2.8 by taking l =6 and $k = 1$. Solution is given by

$$u(x, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{n\pi}{4} \sin \frac{n\pi x}{6} e^{-(n^2\pi^2 t)/36}$$

Ex.4. The one-dimensional wave equation is $\partial^2 u / \partial x^2 =$ (*a*) $c^2 (\partial^2 u / \partial t^2)$

(*b*) $(1/c^2) \times (\partial^2 u / \partial t^2)$ (c) $\partial^2 u / \partial t^2$ (*d*) None of these **(Madurai Kamraj 2008)**

Sol. Ans. (b). Refer Art. 1.2

Ex. 5. Consider the wave equation $\partial^2 u / \partial t^2 = 4(\partial^2 u / \partial x^2)$, $0 < x < \pi, t > 0$ with $u(0, t) = u(\pi, t) = 0, u(x, 0) = \sin x$ and $\partial u / \partial t = 0$ at $t = 0$. Then, $u(\pi / 2, \pi / 2)$ is

(*a*) 2 (*b*) 1 (*c*) 0 (*d*) −1 **[GATE 2010]**

Sol. Ans. (*d*) Taking $c = 2$, $l = \pi$ and $y_0 = 1$ in Ex. 1 (c). page 2.40, get $u(x, t) = \sin x \cos 2t$ and so $u(\pi / 2, \pi / 2) = \sin(\pi / 2) \cos \pi = -1$.

Ex. 6. Write a short note on Dirichlet and Neumann problems. **[Meerut 2011]**

Laplace Transforms with Applications

Where is What

Chapters	*Pages*
1. The Laplace transform	1.3-1.54
2. The inverse Laplace transform	2.1-2.58
3. Applications to ordinary differential equations	3.1-3.38
4. Applications to integral equations	4.1-4.13
5. Applications to boundary value problems	5.1-5.26

1

The Laplace Transform

1.1. Introduction. The knowledge of "Integral Transforms" is an essential part of mathematical background required by scientists and engineers. This is because the transform methods provide an easy and effective means for the solutions of many problems arising in science and engineering. For example,the Laplace transformation replaces a given function F (t) by another function $f(s)$. Then Laplace transformation converts an ordinary differential equation with some given initial conditions into an algebraic equation in terms of $f(s)$ Finally, using inverse Laplace transformation we recover the original function F(t). Thus, the method of Laplace transformation is especially useful for initial value problems, as it enables us to solve the problem without the trouble of finding the general solution first and then evaluating the arbitrary constants. The use of Laplace transforms provide a powerful technique of solving differential and integral equations.

1.2. Laplace transform. Definition. **[Rohilkhand 2000, Meerut 2010]**

Given a function F(t) defined for all real $t \geq 0$, the Laplace transform of F (t) is a function of a new variable s given by

$$L\{F(t), s\} = L\{F(t)\} = f(s) = \overline{F}(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad \ldots (1)$$

The *Laplace transform* of F(t) is said to exist if the improper integral (1) converges for some value of s, otherwise it does not exist.

Remark 1. Some authors use new variable p in place of s. Therefore, corresponding changes may be done in proofs of articles and solutions of problems. Similarly, some authors use variable x in place of t.

Remark 2. Some authors use $f(t)$ in the place of F(t) and simultaneously use F(s) in place of $f(s)$ while defining Laplace transform.

Remark 3. L is called *Laplace transformation operator*.

Remark 4. The operation of multiplying F (t) by e^{-st} and then integrating between the limits 0 to ∞ is known as *Laplace transformation*.

Remark 5. Improper integral. Write $\int_0^\infty e^{-st} F(t)\, dt = \lim_{v \to \infty} \int_0^v e^{-st} F(t)\, dt$, where v is a positive real number. If the integral from 0 to v exists for each $v > 0$ and if the limit as $v \to \infty$ exists, then the improper integral is said to converge to that limiting value otherwise the integral is said to diverge or fail to exist. Note that with regard to upper limit, the Laplace integral $\int_0^\infty e^{-st} F(t)\, dt$ is to be understand as an improper one.

1.3. Working rule to find Laplace transform of F (t), $t \geq 0$ by using definition 1.2.

Step 1. By definition, $$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

Step 2. By definition of improper integral, we have

$$\int_0^\infty e^{-st} F(t)\, dt = \lim_{v \to \infty} \int_0^v e^{-st} F(t)\, dt$$

Step 3. Simplify e^{-st} F (t) and evaluate $\int_0^v e^{-st}F(t)\,dt$.

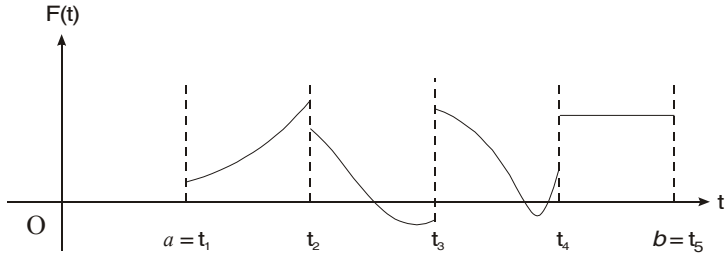
Step 4. Evaluate $\lim_{v\to\infty}\int_0^v e^{-st}F(t)\,dt$

Step 5. From steps 1 and 2, we have

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt = \lim_{v\to\infty}\int_0^v e^{-st}F(t)\,dt$$

1.4. Piecewise (or sectionally) continuous function. Definition.

A function F(t) is called piecewise continuous or sectionally continuous in the closed interval $[a, b]$ if there exist a finite number of points $t_1, t_2, \ldots\ldots, t_n$ $(a = t_1 < t_2 < t_3 < \cdots\cdots < t_n = b)$ such that F(t) is continuous in each of open subinterval $(t_i,\ t_{i+1})$ and has finite right limit F $(t_i + 0)$ and finite left limit F $(t_{i+1} - 0)$. Clearly F (t) need not necessarily be defined at the end points of the subintervals.



An example of a function F(t) which is sectionally continuous is shown in the above figure by taking particular case $n = 5$. We note that F (t) is continuous in subinterval (t_2, t_3) and has finite right and left limits F $(t_2 + 0)$ and F $(t_3 - 0)$ respectively. We also note that F (t) has discontinuities at t_2, t_3 and t_4.

1.5. Functions of exponential order. Definition. [Kanpur 2002]

If there exist a positive real constant m, a number σ and a finite number t such that for all $t \geq t_0$, $\left|e^{-\sigma t}F(t)\right| < m$ or $\left|F(t)\right| < m\,e^{\sigma t}$

we say that F (t) is a function of exponential order σ as $t \to \infty$.

Equivalently, we also write F (t) = O $(e^{\sigma t})$, as $t \to \infty$.

1.5 A. Solved examples based on functions of exponential order

Ex.1. *Show that* F *(t)* = t^2 *is of exponential order* 3.

Sol. We have

$$\lim_{t\to\infty} e^{-\sigma t}F(t) = \lim_{t\to\infty}\frac{t^2}{e^{\sigma t}} = \lim_{t\to\infty}\frac{2t}{\sigma e^{\sigma t}}, \text{ by L' Hospital's rule}$$

$$= \lim_{t\to\infty}\frac{2}{\sigma^2 e^{\sigma t}}, \text{ by L' Hospital's rule again}$$

$$= 0, \text{ provided } \sigma > 0$$

Hence F $(t) = t^2$ is of exponential order. Again, $|\,t^2\,| = t^2 < e^{3t}$ for all $t > 0$

Therefore t^2 is a function of exponential order 3.

Ex.2. *Show that* F *(t)* = t^n *is of exponential order as* $t \to \infty$, *n being any positive integer.*

[Kanpur 2004]

Sol. Assuming that $\sigma > 0$, we have

$$\lim_{t\to\infty} e^{-\sigma t} F(t) = \lim_{t\to\infty} \frac{t^n}{e^{\sigma t}} = \lim_{t\to\infty} \frac{n t^{n-1}}{\sigma e^{\sigma t}}, \text{ by L'Hospital's rule} = \lim_{t\to\infty} \frac{n(n-1)t^{n-2}}{\sigma^2 e^{\sigma t}}, \text{ by L'Hospital's rule again}$$

$$= \lim_{t\to\infty} \frac{n(n-1)(n-2)\cdots 2\cdot 1 t^{n-n}}{\sigma^n e^{\sigma t}}, \text{ after repeated use of L'Hospital's rule } (n-2) \text{ times more.}$$

$$= 0$$

Hence t^n is of exponential order σ, as $t \to \infty$.

Ex.3. *Show that the function* $F(t) = e^{t^2}$ *is not of exponential order as* $t \to \infty$.

Sol. We have, $\lim_{t\to\infty} e^{-\sigma t} F(t) = \lim_{t\to\infty} \left(e^{-\sigma t} e^{t^2}\right) = \lim_{t\to\infty} e^{t(t-\sigma)} = \infty$ for all values of σ.

Hence whatever be the value of σ, we cannot determine a real constant m such that

$$\left|e^{-\sigma t} F(t)\right| < m$$

Therefore, the given function is not of exponential order as $t \to \infty$.

1.6. Functions of class A. Definition.

A function $F(t)$ is said to belong class A if $F(t)$ is of some exponential order as $t \to \infty$ and is piecewise continuous over every finite interval of $t \geq 0$.

1.7. Sufficient conditions for the existence of Laplace transform

Theorem. *If F (t) is a function of class A, L {F (t)} exists.*

or If F (t) is of some exponential order as $t \to \infty$ *and is piecewise continuous over every finite interval of* $t \geq 0$*, then L {F (t)} exists.*

or If F(t) is a function which is piecewise continuous on every finite interval in the range $t \geq 0$ *and satisfies* $|F(t)| \leq me^{\sigma t}$ *for all* $t \geq 0$ *and for constants* σ *and m, then the Laplace transform of F(t) exists.* **[Osmania 2010; KU Kurukshetra 2004; Punjab 2005]**

Proof. Since $F(t)$ is of exponential order, say σ, we can find constants σ, m (> 0) and t_0 (> 0) such that

$$|F(t)| < m e^{\sigma t} \quad \text{for} \quad t \geq t_0. \qquad \ldots (1)$$

$$\text{Now,} \quad L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^{t_0} e^{-st} F(t)\, dt + \int_{t_0}^\infty e^{-st} F(t)\, dt = I_1 + I_2, \text{ say} \qquad \ldots (2)$$

Since $F(t)$ is piecewise continuous on every finite interval $0 \leq t \leq t_0$, I_1 exists.

$$\text{Again,} \quad |I_2| = \left|\int_{t_0}^\infty e^{-st} F(t)\, dt\right| \leq \int_{t_0}^\infty e^{-st} |F(t)|\, dt < m\int_{t_0}^\infty e^{-st} e^{\sigma t} dt, \text{ using (1)}$$

$$\text{Thus,} \quad |I_2| < m\left[-\frac{e^{-(s-\sigma)t}}{(s-\sigma)}\right]_{t_0}^\infty = \frac{m\, e^{-(s-\sigma)t_0}}{s-\sigma}. \qquad \ldots (3)$$

Now, when $s > \sigma$, then $e^{-(s-\sigma)t_0} \to 0$ as $t \to \infty$. Hence (3) shows that $|I_2|$ is finite for all $t_0 > 0$ when $s > \sigma$ and hence I_2 is also convergent. Then from (2), it follows that $L\{F(t)\}$ exists for all $s > \sigma$.

Remark: The conditions stated in the above theorem are sufficient to ensure the existence of $L\{F(t)\}$. These are not necessary conditions for the existence of $L\{F(t)\}$. In other words, if the

conditions of the above theorem are not satisfied, L{F(*t*)} may or may not exist as shown in the following example: Consider the function F (*t*) = $1/\sqrt{t}$.

As $t \to 0$, F (*t*) $\to \infty$. Hence *F* (*t*) is not piecewise continuous on every finite interval in the range $t \geq 0$. Now, by definition, we have

$$L\{1/\sqrt{t}\} = \int_0^\infty \overline{e}^{st}(1/\sqrt{t})\,dt = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}dx, \text{ putting } \sqrt{st} = x \text{ and } dt/\sqrt{t} = 2dx/\sqrt{s}\ ,\ s>0$$

$$= \frac{2}{\sqrt{s}} \times \frac{\sqrt{\pi}}{2} = \left(\frac{\pi}{s}\right)^{1/2},\ s>0. \qquad \text{as} \qquad \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Hence L $\{1/\sqrt{t}\}$ exists for $s > 0$ even if $1/\sqrt{t}$ is not piecewise continuous in the range $t \geq 0$.

1.8. Linearity property of Laplace transforms. If c_1 and c_2 be constants, then

$$L\{c_1 \mathrm{F}_1(t) + c_2 \mathrm{F}_2(t)\} = c_1 \mathrm{L}\{\mathrm{F}_1(t)\} + c_2 L\{\mathrm{F}_2(t)\}.$$

Proof. By definition, we have

$$L\{c_1 \mathrm{F}_1(t) + c_2 \mathrm{F}_2(t)\} = \int_0^\infty e^{-st}\{c_1F_1(t) + c_2F_2(t)\}dt = c_1\int_0^\infty e^{-st}F_1(t)\,dt + c_2\int_0^\infty e^{-st}F_2(t)\,dt$$

$$= c_1 L\{\mathrm{F}_1(t)\} + c_2 L\{\mathrm{F}_2(t)\}, \text{ by definition.}$$

1.9. Laplace transforms of some elementary functions

(*i*) *To find Laplace transform of F(t)* = 1.

[Kanpur 2003, Meerut 2004, M.S. Univ. T.N. 2007]

Sol. By definition of Laplace transform (see Art. 1.3), we have

$$L\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{v\to\infty}\int_0^v e^{-st}dt = \lim_{v\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^v$$

$$= -\frac{1}{s}\lim_{v\to\infty}\left(\frac{1}{e^{sv}} - 1\right) = -\frac{1}{s}(0-1) = \frac{1}{s}, \text{ provided } s > 0$$

(*ii*) *To find Laplace transform of the function* F *(t) = t^n, n being any real number greater than –1.* **[*Ranchi 2010; Purvanchal 2006*]**

Sol. By definition, $L\{\mathrm{F}(t)\} = \int_0^\infty e^{-st}F(t)\,dt.$

$$\therefore \qquad \mathrm{L}\{t^n\} = \int_0^\infty e^{-st}t^n dt = \int_0^\infty e^{-st}t^{(n+1)-1}dt. \qquad \ldots (i)$$

From the properties of 'Gamma function', we know that

$$\int_0^\infty e^{-ax}x^{m-1}dx = \frac{\Gamma(m)}{a^m}, \text{ if } a > 0 \text{ and } m > 0. \qquad \ldots (ii)$$

Replacing *a* by *s*, *m* by (*n* + 1) and *x* by *t* in (*ii*), we have

$$\int_0^\infty e^{-st}t^{(n+1)-1}dt = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } s > 0 \text{ and } n+1 > 0$$

$$\therefore \qquad (i) \Rightarrow \qquad \Gamma\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, s > 0 \text{ and } n > -1.$$

Thus, the condition $s > 0$ is necessary for the convergence of the integral (i).

(iii) *To find Laplace transform of the function* F $(t) = t^n$, *n being a positive integer.*

[Osmania 2004, Purvanchal 1994, Andhra 2000, 04, Meerut 2004]

Sol. By definition, $\quad L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt.$

$$\therefore \qquad L\{t^n\} = \int_0^\infty e^{-st} t^n dt = \int_0^\infty e^{-st} t^{(n+1)-1} dt. \qquad \ldots (i)$$

From the properties of 'Gamma function', we know that

$$\int_0^\infty e^{-ax} x^{m-1} dx = \frac{\Gamma(m)}{a^m}, \text{ if } a > 0 \text{ and } m > 0. \qquad \ldots (ii)$$

Replacing a by s, m by $(n + 1)$ and x by t in (ii), we have

$$\int_0^\infty e^{-st} t^{(n+1)-1} dt = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } s > 0 \qquad [\because\ n \text{ is a positive integer} \Rightarrow (n + 1) > 0]$$

$$\therefore \quad (i) \Rightarrow L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \text{ if } s > 0 \quad [\because n \text{ is a positive integer} \Rightarrow \Gamma\ (n + 1) = n!]$$

Thus, the condition $s > 0$ is necessary for the convergence of the integral (i).

(iv) *To find Laplace transform of the function F (t) =* e^{at}. **[Agra 2011; Meerut 2004]**

Sol. By definition of Laplace transform (see Art. 1.3), we have

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \lim_{v\to\infty} \int_0^v e^{-(s-a)t} dt = \lim_{v\to\infty} \left[-\frac{e^{-(s-a)t}}{s-a} \right]_0^v$$

$$= -\frac{1}{s-a} \lim_{v\to\infty} \left[e^{-(s-a)v} - 1 \right] = -\frac{1}{s-a}(0-1) = \frac{1}{s-0}, \text{ if } s > a.$$

(v) *To find Laplace transform of F(t) = sinh at* **[Meerut 2009; M.S. Univ. T.N. 2007]**

Sol. By definition of Laplace transform and the result $\sinh at = (e^{at} - e^{-at})/2$, we have

$$L\{\sinh at\} = \int_0^\infty e^{-st} \sinh at\, dt = \int_0^\infty e^{-st} \frac{e^{at} - e^{-at}}{2} dt = \frac{1}{2} \int_0^\infty \left\{ e^{-(s-a)t} - e^{-(s+a)t} \right\} dt$$

$$= \frac{1}{2} \lim_{v\to\infty} \int_0^v \left\{ e^{-(s-a)t} - e^{-(s+a)t} \right\} dt = \frac{1}{2} \lim_{v\to\infty} \left[-\frac{e^{-(s-a)t}}{s-a} + \frac{e^{-(s+a)t}}{s+a} \right]_0^v$$

$$= \frac{1}{2} \lim_{v\to\infty} \left[-\frac{1}{s-a} \left\{ e^{-(s-a)v} - 1 \right\} + \frac{1}{s+a} \left\{ e^{-(s+a)v} - 1 \right\} \right]$$

$$= \frac{1}{2} \left\{ -\frac{1}{s-a}(0-1) + \frac{1}{s+a}(0-1) \right\}, \text{ provided } s > a \text{ and } s > -a$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{1}{2} \frac{(s+a)-(s-a)}{(s-a)(s+a)} = \frac{a}{s^2 - a^2}, \text{ provided } |s| > a$$

(vi) *To find Laplace transform of F(t) = cosh at*

[Madras 2012; M.S. Univ. T.M. 2007, Osmania 2004; Sagar 2004]

Sol. By definition of Laplace transform and the result $\cosh at = (e^{at} + e^{-at})/2$, we get

$$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at\, dt = \int_0^\infty e^{-st} \frac{e^{at} + e^{-at}}{2} dt = \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} + e^{-(s+a)t}\right\} dt$$

$$= \frac{1}{2} \lim_{v\to\infty} \int_0^v \left\{e^{-(s-a)t} + e^{-(s+a)t}\right\} dt = \frac{1}{2} \lim_{v\to\infty} \left[-\frac{e^{-(s-a)}}{s-a} - \frac{e^{-(s+a)t}}{s+a} \right]_0^v$$

$$= \frac{1}{2} \lim_{v\to\infty} \left[-\frac{1}{s-a}\left\{e^{-(s-a)v} - 1\right\} - \frac{1}{s+a}\left\{e^{-(s+a)v} - 1\right\} \right]$$

$$= \frac{1}{2}\left\{ -\frac{1}{s-a}(0-1) - \frac{1}{s+a}(0-1) \right\}, \text{ provided } s > a \text{ and } s > -a$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a} \right] = \frac{s}{s^2 - a^2}, \text{ provided } |s| > a$$

(*vii*) To find Lapalce transform of F (*t*) = sin *at*

[M.S. Univ. 2007, Kanpur 1994, 95; Meerut 2005, 08, Kanpur 2004]

Sol. By definition of Laplace transform, we have

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt = \lim_{v\to\infty} \int_0^v e^{-st} \sin at\, dt$$

$$= \lim_{v\to\infty} \left[\frac{e^{-st}(-s\sin at - a\cos at)}{s^2 + a^2} \right]_0^v, \qquad \text{as} \qquad \int e^{ax} \sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$$

$$= -\frac{1}{s^2 + a^2} \lim_{v\to\infty} \left[\frac{s\sin av + a\cos av}{e^{sv}} - a \right] = -\frac{1}{s^2 + a^2}(0 - a), \text{ if } s > 0$$

Thus, $L\{\sin at\} = a/(s^2 + a^2)$, provided $s > 0$

(*viii*) *To find Laplace transform of F(t)* = cos *at* **[Kanpur 1995, M.S. Univ. T.N. 2007]**

Sol. By definition of Laplace transform, we have

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt = \lim_{v\to\infty} \int_0^v e^{-st} \cos at\, dt$$

$$= \lim_{v\to\infty} \left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2 + a^2} \right]_0^v, \qquad \text{as} \qquad \int e^{ax} \cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

$$= \frac{1}{s^2 + a^2} \lim_{v\to\infty} \left[\frac{-s\cos av + a\sin av}{e^{sv}} - (-s) \right] = \frac{1}{s^2 + a^2}(0 + s), \text{ if } s > 0$$

Thus, $L\{\cos at\} = s/(s^2 + a^2)$, provided $s > 0$

1.10. Table of Laplace transforms of some elementary functions. Commit to memory for direct applications. These have been evaluated in Art 1.9.

S.No.	*F(t)*	*L {F(t)} or f (s)*	
1	1	$1/s$,	$s > 0$
2	t^n, $n > -1$	$\Gamma(n+1)/s^{n+1}$,	$s > 0$
3	t^n, (n is a positive integer)	$n!/s^{n+1}$,	$s > 0$
4	e^{at}	$1/(s-a)$,	$s > a$
5	sinh at	$a/(s^2 - a^2)$,	$s > \lvert a \rvert$
6	cosh at	$s/(s^2 - a^2)$,	$s > \lvert a \rvert$
7	sin at	$a/(s^2 + a^2)$,	$s > 0$
8	cos at	$s/(s^2 + a^2)$,	$s > 0$

1.10 A. Solved examples based on Art.1.8 and Art. 1.9

Ex. 1. Find the Laplace transform of the function $F(t) = (e^{at} - 1)/a$.

[Purvanchal 2003; Meerut 1994]

Sol. $L\{F(t)\} = L\{(1/a) \times (e^{at} - 1)\} = (1/a) \times L\{e^{at} - 1\} = (1/a) \times [L\{e^{at}\} - L\{1\}]$

$$= \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{s(s-a)}, \text{ if } s > a \text{ and } s > 0.$$

Ex.2. *Find the Laplace transform of* $F(t) = (\sin t - \cos t)^2$. **[Vikram 2004]**

Sol. $L\{F(t)\} = L\{(\sin t - \cos t)^2\} = L\{\sin^2 t + \cos^2 t - 2 \sin t \cos t\}$

$$= L\{1 - \sin 2t\} = L\{1\} - L\{\sin 2t\} = \frac{1}{s} - \frac{2}{s^2 + 2^2}, \text{if } s > 0$$

$$= (s^2 - 2s + 4)/s(s^2 + 4), \text{ if } s > 0.$$

Ex.3. *Find the Laplace transform of* *(i)* $3e^{2t} + 4e^{-3t}$ *(ii)* $(t^2 + 1)^2$

Sol. (*i*) $L\{3e^{2t} + 4e^{-3t}\} = 3L\{e^{2t}\} + 4L\{e^{-3t}\} = 3 \times \frac{1}{s-2} + 4 \times \frac{1}{s+3}$, $s > 2$ and $s > -3$

$$= (7s + 1)/(s^2 + s - 6), \text{ if } s > 2$$

(*ii*) $L\{(t^2+1)^2\} = L\{t^4 + 2t^2 + 1\} = L\{t^4\} + 2L\{t^2\} + L\{1\} = \frac{4!}{s^5} + 2 \times \frac{2}{s^3} + \frac{1}{s}$, as $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s} = \frac{24 + 4s^2 + s^4}{s^5}, \text{provided } s > 0.$$

Ex.4. *Find (i)* $L\{6 \sin 2t - 5 \cos 2t\}$ *(ii)* $L\{\sin 5t + \cos 3t\}$
(iii) $L\{3 \cosh 5t - 4 \sinh 5t\}$ *(iv)* $\sin 2t \cos 3t$ **[Gulberga 2005]**

Sol. (*i*) $L\{6 \sin 2t - 5 \cos 2t\} = 6L\{\sin 2t\} - 5L\{\cos 2t\}$

$$= 6 \times \frac{2}{s^2 + 2^2} - 5 \times \frac{s}{s^2 + 2^2}, s > 0 = \frac{12 - 5s}{s^2 + 4}, s > 0$$

(*ii*) $L\{\sin 5t + \cos 2t\} = L\{\sin 5t\} + L\{\cos 2t\} = 5/(s^2 + 25) + s/(s^2 + 9)$, $s > 0$

(*iii*) $L\{3 \cosh 5t - 4 \sinh 5t\} = 3L\{\cosh 5t\} - 4L\{\sinh 5t\}$

$$= 3 \times \frac{s}{s^2 - 5^2} - 4 \times \frac{5}{s^2 - 5^2}, \ s > 5 = \frac{3s - 20}{s^2 - 25}, s > 5$$

(*iv*) We have, $\sin 2t \cos 3t = (1/2) \times (\sin 5t - \sin t)$.

$\therefore$ L $\{\sin 2t \cos 3t\}$ = L $\{(1/2) \times (\sin 5t - \sin t)\}$ = $(1/2) \times$ [L $\{\sin 5t\}$ – L $\{\sin t\}$]

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right],\ s>0 \qquad \text{as} \qquad \text{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$= \frac{1}{2}\,\frac{4s^2-20}{(s^2+25)\,(s^2+1)} = \frac{2(s^2-5)}{(s^2+25)\,(s^2+1)},\ s>0$$

Ex.5. *Find out the Laplace transform of (i) $\sin^2 at$*

(ii) $\cos^2$ at **[Kanpur 2005, Purvanchal 2002, 05]**

(iii) $\sin^2 t - t$ **[M.S. Univ. T.N. 2007]**

Sol. (*i*) $\text{L}\{\sin^2 at\} = \text{L}\{(1/2) \times (1 - \cos 2at)\} = (1/2) \times [L\,(1) - L\{\cos 2at\}]$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+(2a)^2}\right],\ s>0 = \frac{1}{2}\,\frac{4a^2}{s\,(s^2+4a^2)},\ s>0 = \frac{2a^2}{s\,(s^2+4a^2)},\ s>0$$

(*ii*) L $\{\cos^2 at\}$ = L $\{(1/2) \times (1 + \cos 2at)\} = (1/2) \times [L\{1\} + L\{\cos 2at\}]$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+(2a)^2}\right],\ s>0 = \frac{1}{2}\,\frac{2s^2+4a^2}{s\,(s^2+4a^2)},\ s>0 = \frac{s^2+2a^2}{s(s^2+4a^2)},\ s>0$$

(iii) $L\{\sin^2 t - t\} = L\{(1/2) \times (1 - \cos 2t) - t\} = (1/2) \times L\{1\} - (1/2) \times L\{\cos 2t\} - L\{t\}$

$$= \frac{1}{2} \times \frac{1}{s} - \frac{1}{2} \times \frac{s}{s^2+4} - \frac{1}{s^2},\ s>0 = \frac{2}{s(s^2+4)} - \frac{1}{s^2},\ s>0$$

Ex.6 (*a*). *Find the Laplace transform of $\sin^3 2t$.* **[CDLU 2004]**

Sol. Since $\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$, we get $\sin^3\theta = (1/4) \times (3 \sin\theta - \sin 3\theta)$.

$$\therefore \qquad \text{L}\{\sin^3 2t\} = \text{L}\left\{\frac{1}{4}(3\sin 2t - \sin 6t)\right\} = \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$$

$$= \frac{3}{4} \times \frac{2}{s^2+2^2} - \frac{1}{4} \times \frac{6}{s^2+6^2},\ s>0 = \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right],\ s>0 = \frac{48}{(s^2+4)\,(s^2+36)},\ s>0$$

Ex.6 (*b*). *Find L $\{\cos^3 x, p\}$.* **[Osmania 2004, Meerut 1998]**

Note: Here we have x and p in place of t and s. so corresponding changes have been made in solution as shown below

Sol. Since $\cos 3x = 4\cos^2 x - 3 \cos x$, we get $\cos^3 x = (\cos 3x + 3 \cos x)/4$.

$\therefore$ L $\{\cos^3 x, p\}$ = L $\{(\cos 3x + 3 \cos x)/4\}$ = $(1/4) \times$ L $\{\cos 3x\}$ + $(3/4) \times$ L $\{\cos x\}$

$$= \frac{1}{4} \times \frac{p}{p^2+3^2} + \frac{3}{4} \times \frac{p}{p^2+1} = \frac{p}{4}\left[\frac{1}{p^2+9} + \frac{3}{p^2+1}\right].$$

Ex.7. *Find out the Laplace transform of* *(i) $\sinh^2$ at* *(ii) $\cosh^2$ at.*

Sol. (*i*) since $\cosh 2\theta = 1 + 2 \sinh^2\theta$, we get $\sinh^2\theta = (1/2) \times (\cosh 2\theta - 1)$.

$$\therefore \text{L}\{\sinh^2 at\} = \text{L}\left\{\frac{1}{2}(\cosh 2at - 1)\right\} = \frac{1}{2}\left[L\{\cosh 2at\} - L(1)\right] = \frac{1}{2}\left[\frac{s}{s^2-(2a)^2} - \frac{1}{s}\right],\ s>0 \text{ and } s>|2a|$$

$$= (2a^2)/s(s^2-4a^2),\ s>2\,|\,a\,|.$$

(*ii*) Since $\cosh 2\theta = 2\cosh^2\theta - 1$, we get $\cosh^2\theta = (1/2)\times(\cosh 2\theta + 1)$.

$$\therefore\ L\{\cosh^2 at\} = L\left\{\frac{1}{2}(\cosh 2at + 1)\right\} = \frac{1}{2}\left[L\{\cosh 2at\} + L\{1\}\right] = \frac{1}{2}\left[\frac{s}{s^2-(2a)^2} + \frac{1}{s}\right],\ s>0 \text{ and } s>|2a|$$

$$= (s^2 - 2a^2)/s(s^2 - 4a^2),\ s > |2a|.$$

Ex.8. *Find (i)* $L\left\{e^{-2t} - e^{-3t}\right\}$ (*ii*) $L\{2e^{3t} - e^{-3t}\}$.

Sol. (*i*) $L\{e^{-2t} - e^{-3t}\} = L\{e^{-2t}\} - L\{e^{-3t}\} = \frac{1}{s+2} - \frac{1}{s+3}$, if $s > -2$ and $s > -3$

$$= 1/(s^2 + 5s + 6), \text{ if } s > -2.$$

(*ii*) $L\{2e^{3t} - e^{-3t}\} = 2L\{e^{3t}\} - L\{e^{-3t}\} = 2\times\frac{1}{s-3} - \frac{1}{s+3}$, if $s > 3$ and $s > -3$

$$= (s + 9)/(s^2 - 9),\ s > 3.$$

Ex.9. *Find the Laplace transform of sin t sin 2t sin 3t.*

Sol. $L\{\sin t \sin 2t \sin 3t\} = L\{(1/2)\times \sin t(2\sin 3t \sin 2t)\}$

$$= L\{(1/2)\times\sin t(\cos t - \cos 5t)\} = L(1/4)\times(2\sin t\cos t - 2\cos 5t\sin t)\} = (1/4)\times L\{\sin 2t - (\sin 6t - \sin 4t)\}$$

$$= \frac{1}{4}\left[\frac{2}{s^2+4} - \frac{6}{s^2+36} + \frac{4}{s^2+16}\right],\ s>0 = \frac{1}{2}\left[\frac{1}{s^2+4} - \frac{3}{s^2+36} + \frac{2}{s^2+16}\right],\ s>0$$

Ex.10. *Find* $L\{7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2\}$. **[Purvanchal 1994, Kanpur 2003]**

Sol. $L\{7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2\}$

$$= 7L\{e^{2t}\} + 9L\{e^{-2t}\} + 5L\{\cos t\} + 7L\{t^3\} + 5L\{\sin 3t\} + 2L\{1\}$$

$$= 7\times\frac{1}{s-2} + 9\times\frac{1}{s+2} + 5\times\frac{s}{s^2+1^2} + 7\times\frac{3!}{s^4} + 5\times\frac{3}{s^2+3^2} + \frac{2}{s} = \frac{4(4s-1)}{s^2-4} + \frac{5s}{s^2+1} + \frac{15}{s^2+9} + \frac{2(21+s^3)}{s^4}.$$

Ex.11. *Prove that* $L\{2^t\} = 1/(s - \log_e 2)$ **[Banglore 1997]**

Sol. We have, $2^t = e^{\log_e 2^t} = e^{t\log_e 2}$. Then, by definition, we have

$$L\{2^t\} = L\{e^{t\log_e 2}\} = \int_0^\infty e^{-st}e^{t\log_e 2}dt = \int_0^\infty e^{-(s-\log_e 2)t}dt = \left[-\frac{1}{s-\log\frac{2}{e}}e^{-(s-\log_e 2)t}\right]_0^\infty = \frac{1}{s-\log_e 2},$$

if $s > \log_e 2$.

Ex.12. *Find the Laplace transform of* $1/\sqrt{\pi t}$.

Sol. $L\left\{\frac{1}{\sqrt{(\pi t)}}\right\} = \frac{1}{\sqrt{\pi}}L\left\{t^{-\frac{1}{2}}\right\} = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}},\ s>0 \quad [\because \Gamma(t^n) = \Gamma(n+1)/s^{n+1}, s>0]$

$$= \frac{1}{\sqrt{\pi}}\frac{\Gamma(1/2)}{s^{1/2}} = \frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}} = \frac{1}{\sqrt{s}},\ s>0 \text{ as } \Gamma(1/2) = \sqrt{\pi}$$

Ex.13. *Show that* $L\left\{(5e^{2t} - 3)^2\right\} = \frac{25}{s-4} - \frac{30}{s-2} + \frac{9}{s}, s > 4.$

Sol. $L\{(5e^{2t} - 3)^2\} = L\{25e^{4t} - 30e^{2t} + 9\} = 25L\{e^{4t}\} - 30L\{e^{2t}\} + 9L\{1\}$

$$= \frac{25}{s-4} - \frac{30}{s-2} + \frac{9}{s},\ \text{if } s>4,\ s>2 \text{ and } s>0 \textit{ i.e.}, s>4.$$

Ex.14. *Find Laplace transform of the function F, where*

$$F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi. \end{cases}$$ **[Calicut 2004, Meerut 2009]**

Sol. $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$, by definition of Laplace transform

$= \int_0^\pi e^{-st} F(t)\, dt + \int_\pi^\infty e^{-st} F(t)\, dt = \int_0^\pi e^{-st} \sin t\, dt + \int_\pi^\infty e^{-st}(0)\, dt$, using the given values of F (t)

$= \left[\frac{e^{-st}}{s^2+1}(-s \sin t - \cos t)\right]_0^\pi$ as $\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}$

$= \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1}$ if $s > 0 = \frac{e^{-s\pi}+1}{s^2+1}$, if $s > 0$

Ex.15 (*a*). *Find Laplace transform of the following function:*

$$f(x) = \begin{cases} x/a, & 0 < x < a \\ 1, & x > a. \end{cases}$$ **[Meerut 1998, Manglore 1997, Rewa 1994]**

[**Note:** Here variable x has been used in place of t hence we shall make corresponding changes in our solution]

Sol. By definition of Laplace transform, we have

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx = \int_0^a e^{-sx} f(x)\, dx + \int_a^\infty e^{-sx} f(x)\, dx$$

$$= \int_0^a e^{-sx}\left(\frac{x}{a}\right) dx + \int_a^\infty e^{-sx} dx, \text{ using the given values of } f(x)$$

$$= \left[\left(\frac{x}{a}\right)\left(-\frac{1}{s}e^{-sx}\right) - \left(\frac{1}{a}\right)\left(\frac{1}{s^2}e^{-sx}\right)\right]_0^a + \left[-\frac{1}{s}e^{-sx}\right]_a^\infty, \; s > 0$$

$$= -(1/s)\, e^{-sa} - (1/as^2)\,(e^{-sa} - 1) + (1/s)\, e^{-sa} = (1 - e^{-sa})/as^2, \; s > 0$$

Ex.15 (*b*). *Find L {F (t)}, if* $F(t) = \begin{cases} 1, & 0 < t < 2 \\ t, & t > 2. \end{cases}$

Sol. $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$, by definition of Laplace transform

$= \int_0^2 e^{-st} F(t)\, dt + \int_2^\infty e^{-st} F(t)\, dt = \int_0^2 e^{-st}(1)\, dt + \int_2^\infty e^{-st} t\, dt$, using the given values of F (t)

$= \left[\frac{e^{-st}}{-s}\right]_0^2 + \left[t\left(\frac{e^{-st}}{-s}\right)\right]_2^\infty - \int_2^\infty (1)\left(\frac{e^{-st}}{-s}\right) dt$, on integration by parts the second integral

$$= -\frac{e^{-2s}}{s} + \frac{1}{s} - \frac{1}{s}\lim_{t\to\infty}\frac{t}{e^{st}} + \frac{2\,e^{-2s}}{s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_2^{\infty}, \; s > 0$$

$$\therefore \qquad \mathrm{L}\{\mathrm{F}(t)\} = -\frac{e^{-2s}}{s} + \frac{1}{s} - \frac{1}{s}\lim_{t\to\infty}\frac{t}{e^{st}} + \frac{2\,e^{-2s}}{s} + \frac{e^{-2s}}{s^2} \text{ if } s > 0 \qquad \ldots (1)$$

Now, $\lim_{t\to\infty}\frac{t}{e^{st}} \quad \left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{t\to\infty}\frac{1}{s\,e^{st}}, \text{ by L' Hospital's rule} = 0.$$

So $(1) \Rightarrow \quad \mathrm{L}\{\mathrm{F}(t)\} = (1/s)\times(1+e^{-2s}) + (1/s^2)\times e^{-2s}, \; s > 0.$

Ex.16. *Evaluate L {F (t)}. if F (t) = (t – 1)², t > 1 and F (t) = 0, 0 < t < 1.*

Sol. By definition, $\mathrm{L}\{\mathrm{F}(t)\} = \int_0^{\infty} e^{-st}F(t)\,dt = \int_0^1 e^{-st}F(t)\,dt + \int_1^{\infty} e^{-st}F(t)\,dt$

$= \int_0^1 e^{-st}\cdot(0)\,dt + \int_1^{\infty} e^{-st}(t-1)^2\,dt$, by given values of F (*t*)

$= \int_1^{\infty}(t-1)^2 e^{-st}dt = \int_0^{\infty} x^2 e^{-s(x+1)}dx$, putting $t - 1 = x$ and $dt = dx$

$= e^{-s}\int_0^{\infty} e^{-sx}x^{3-1}dx = e^{-s}\times\frac{\Gamma(3)}{s^3}, s > 0 \qquad \left[\because \int_0^{\infty} e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}, \text{ where } a > 0,\, n > 0\right]$

$= e^{-s}\times(2!/s^3) = (2e^{-s})/s^3, \; s > 0.$

Ex.17. *Find L {F (t)}, where F (t) = 0, if 0 < t < 1*
$= t$, if $1 < t < 2$
$= 0$, if $t > 2$ **[Rohilkhand 1997]**

Sol. $\mathrm{L}\{\mathrm{F}(t)\} = \int_0^{\infty} e^{-st}F(t)\,dt = \int_0^1 e^{-st}F(t)\,dt + \int_1^2 e^{-st}F(t)\,dt + \int_2^{\infty} e^{-st}F(t)\,dt$

$= \int_1^2 e^{-st}t\,dt$, using the given values of F (*t*)

$= \left[(t)\left(-\frac{1}{s}e^{-st}\right)\right]_1^2 - \int_1^2 1\times\left(-\frac{1}{s}e^{-st}\right)dt$, integrating by parts

$= -\left[\frac{t\,e^{-st}}{s}\right]_1^2 - \left[\frac{e^{-st}}{s^2}\right]_1^2 = -\frac{2e^{-2s} - e^{-s}}{s} - \frac{e^{-2s} - e^{-s}}{s^2}, \; s > 0$

Ex.18. *Find L {F (t)}, where* $\mathrm{F}(t) = \begin{cases} 0, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 2, & t > 2 \end{cases}$ **[S.V. University (A.P.) 1997]**

Sol. L {F (*t*)} = $\int_0^\infty e^{-st} F(t)\, dt = \int_0^1 e^{-st} F(t)\, dt + \int_1^2 e^{-st} F(t)\, dt + \int_2^\infty e^{-st} F(t)\, dt$

$= \int_0^1 e^{-st}(0)\, dt + \int_1^2 (e^{-st} \times 1)\, dt + \int_2^\infty (e^{-st} \times 2)\, dt$, by given values of F (*t*)

$= 0 + \left[-\frac{1}{s} e^{-st}\right]_1^2 + 2\left[-\frac{1}{s} e^{-st}\right]_2^\infty = -\frac{e^{-2s}}{s} + \frac{e^{-s}}{s} + 2\frac{e^{-2s}}{s}, s > 0$

$= (1/s) \times (e^{-s} + e^{-2s})$, $s > 0$.

Ex.19. *Find L {F (t)}, where* F (*t*) = $\begin{cases} t, & 0 \le t \le 1/2 \\ t-1, & 1/2 < t \le 1 \\ 0, & t \ge 1 \end{cases}$ **[S.V. University (A.P.) 1997]**

Sol. L {F (*t*)} = $\int_0^\infty e^{-st}\, F(t)\, dt = \int_0^{1/2} e^{-st} F(t)\, dt + \int_{1/2}^1 e^{-st} F(t)\, dt + \int_1^\infty e^{-st} F(t)\, dt$

$= \int_0^{1/2} e^{-st} t\, dt + \int_{1/2}^1 e^{-st}(t-1)\, dt + \int_1^\infty (e^{-st} \times 0)\, dt$, by the given values of F (*t*)

$= \left[\frac{t\, e^{-st}}{-s}\right]_0^{1/2} - \int_0^{1/2}\left[-\frac{e^{-st}}{s}\right] dt + \left[(t-1)\frac{e^{-st}}{-s}\right]_{1/2}^1 - \int_{1/2}^1 \left[-\frac{e^{-st}}{s}\right] dt$

$= -\frac{1}{2s} e^{-s/2} + \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]_0^{1/2} - \frac{1}{2s} e^{-s/2} + \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]_{1/2}^1, s > 0$

$= -\frac{1}{s} e^{-s/2} - \frac{1}{s^2} e^{-s/2} + \frac{1}{s^2} - \frac{1}{s^2} e^{-s} + \frac{1}{s^2} e^{-s/2}, s > 0$

$= -(1/s) \times e^{-s/2} + (1/s^2) \times (1 - e^{-s}), s > 0$

Ex.20. *Show that the Laplace transform of the function F (t) = t^n, – 1 < n < 0, exists, although it is not a function of class A.*

Sol. Here $\lim_{t\to\infty} \{e^{-\sigma t} F(t)\} = \lim_{t\to\infty} (e^{-\sigma t}\ t^n) = \lim_{t\to\infty} \frac{1}{e^{\sigma t} t^{-n}}$, where –1 < *n* < 0

= 0, provided σ > 0

Hence F (*t*) is of exponential order.

Since F (*t*) → ∞ as *t* → 0 from the right, it follows that F (*t*) is not piecewise continuous on every finite subinterval in the range *t* ≥ 0. Hence F (*t*) is not a function of class A.

Now, L {F (*t*)} = $\int_0^\infty e^{-st} F(t)\, dt = \int_0^\infty e^{-st} t^n dt = \int_0^\infty e^{-st} t^{(n+1)-1} dt$

$= \frac{\Gamma(n+1)}{s^{n+1}}$, if *s* > 0 and (*n* + 1) > 0 *i.e.*, *n* > – 1

$\left[\because \int_0^\infty e^{-ax} x^{m-1} dx = \frac{\Gamma(m)}{a^m}, \text{if } a > 0 \text{ and } m > 0\right]$

Thus we find that the Laplace transform of F $(t) = t^n$, $-1 < n < 0$ exists, although it is not a function of class A.

1.10B. Series expansion method.

Let the given function $F(t)$ be expressible as a power series

$F(t) = C_0 + C_1 t + C_2 t^2 + ... + C_n t^n + ...$, where C_0, C_1, C_2, ... are constants.

Then the $L\{F(t)\}$ is obtained by taking Laplace transform of each term in the series.

1.10C. Solved example based on Art. 1.10B.

Ex. 17. *Find L {sin* $\sqrt{t}$ *}.* **[Meerut 2008, 11, Kumaun 1998, Agra 2001, 02, 07; Kanpur 1999, 2004; Purvanchal 2006, Rohilkhand 2001]**

Sol. $\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \ldots = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \ldots.$

$\Rightarrow L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \frac{1}{7!}L\{t^{7/2}\} + \ldots$

$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!s^{5/2}} + \frac{\Gamma(7/2)}{5!s^{7/2}} - \frac{\Gamma(9/2)}{7!s^{9/2}} + \ldots., s > 0 \qquad [\because L\{t^n\} = \Gamma(n+1)/s^{n+1}, s > 0]$

$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\cdot\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \ldots\ldots., \text{ as } \Gamma(n+1) = n\Gamma(n) \text{ and } \Gamma(1/2) = \sqrt{\pi}$

$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \ldots\ldots\right\} = \frac{\sqrt{\pi}\, e^{-1/4s}}{2\, s^{3/2}}.$

Ex.1. *Show that* $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{s}\right)}\, e^{-1/4s}$ **[Agra 2008; Guwahati 2007, Meerut, 2002]**

Sol. $\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left\{1 - \frac{(\sqrt{t})^2}{2!} + \frac{(\sqrt{t})^4}{4!} - \frac{(\sqrt{t})^6}{6!} + \ldots.\right\} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \ldots..$

$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\{t^{-1/2}\} - \frac{1}{2!}L\{t^{1/2}\} + \frac{1}{4!}L\{t^{3/2}\} - \frac{1}{6!}L\{t^{5/2}\} + \ldots..$

$= \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \ldots., s > 0$; as $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, s > 0$

$= \frac{\sqrt{\pi}}{s^{1/2}} - \frac{\frac{1}{2}\cdot\sqrt{\pi}}{1\cdot 2}\frac{1}{s^{3/2}} + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{1\cdot 2\cdot 3\cdot 4}\frac{1}{s^{5/2}} - \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\frac{1}{s^{7/2}} + \ldots.$

$= \sqrt{\left(\frac{\pi}{s}\right)}\left[1 - \frac{(1/4s)}{1!} + \frac{(1/4s)^2}{2!} - \frac{(1/4s)^3}{3!} + \ldots.\right] = \sqrt{\left(\frac{\pi}{s}\right)}\, e^{-1/4s}$

EXERCISE 1(*A*)

1. *Find the Laplace transform of the following functions:*

(*i*) e^{4t} **[Bangalore 2005]** (*ii*) $\cosh 3t$ **[Kuvempa 2005]**

(*iii*) t^7 (*iv*) $t^{2/3}$ **[Purvanchal 2004]** (*v*) $t^{3/2}$ (*vi*) $t^{5/3}$

Ans. (*i*) $1/(s-4)$, $s>4$ (*ii*) $s/(s^2-9)$, $s>|3|$ (*iii*) $(5040)/s^8$, $s>0$

(*iv*) $\dfrac{2\Gamma(2/3)}{3s^{5/3}}$, $s>0$ (*v*) $\dfrac{3\sqrt{\pi}}{4s^{5/2}}$, $s>0$ (*vi*) $\dfrac{10\Gamma(2/3)}{9s^{8/3}}$, $s>0$

2. *Find the Laplace transform of the following functions*

(*i*) $F(t)=\begin{cases}e^t, & 0<t\le 1\\ 0, & t>1\end{cases}$ (*ii*) $F(t)=\begin{cases}t, & 0<t<4\\ 5, & t>4\end{cases}$ **[M.S. Univ. T.N. 2007]**

(*iii*) $F(t)=\begin{cases}\sin 3t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$ **[Bangalore 2005]** (*iv*) $F(t)=\begin{cases}-1, & 0\le t\le 4\\ 1, & t\ge 4\end{cases}$

Ans. (*i*) $\left\{1-e^{-(s-1)}\right\}\Big/(s-1)$, $s\neq 1$ (*ii*) $\left\{1+(s-1)e^{-4s}\right\}/s^2$, $s>0$

(*iii*) $(e^{1-s}-1+s)/(s-1)$, $s\neq 1$ (*iv*) $(2e^{-4s}-1)/s$, $s>0$

3. *Evaluate the following:*

(*i*) $L\{\sin 2t \sin 3t\}$ **[CDLU 2004]** (*ii*) $L\{2\sin t \sin 3t\}$ **[Gulberga 2005]**

(*iii*) $L\{t-\sinh 2t\}$ **[CDLU 2004]** (*iv*) $L\{\sin^2 4t\}$ **[Bangalore 2004]**

(*v*) $L\{\sin^2 2t\}$ **[Karnataka 2005]** (*vi*) $\cosh^2 2t$

(*vii*) $3t^3+4t^2-5t+7$ (*viii*) $4e^{-3t}-2\sin 5t+3\cos 2t-2t^3+3t^4$

Ans. (*i*) $(12s)/(s^2+1)(s^2+25)$, $s>5$ (*ii*) $(6s)/(s^2+4)(s^2+16)$, $s>4$

(*iii*) $(s^2+4)/s^2(4-s^2)$, $s>2$ (*iv*) $(s^2+32)/s(s^2-64)$, $s>0$

(*v*) $(s^2+8)/s(s^2+16)$, $s>0$ (*vi*) $(s^2-8)/s(s^2-16)$, $s>4$

(*vii*) $18/s^4+8/s^3-5/s^2+7/s$, $s>0$

(*viii*) $4/(s+3)-10/(s^2+25)+3s/(s^2+4)-12/s^4+72/s^5$, $s>0$

4. *Find the Laplace transform of the following functions:*

(*i*) $\cos^3 t$ (*ii*) $\cos(at+b)$ (*iii*) $\cos^3 4t+t$

(*iv*) $1+\sqrt{t}+3/\sqrt{t}$, $t>0$ (*v*) $\sin(at+b)$, $t\ge 0$ (*vi*) $\sin at \sin bt$

Ans. (*i*) $\dfrac{s(s^2+7)}{(s^2+9)(s^2+1)}$, $s>0$ (*ii*) $\dfrac{s\cos b-a\cos b}{s^2+a^2}$, $s>0$

(*iii*) $\dfrac{s(s^2+112)}{(s^2+16)(s^2+144)}+\dfrac{1}{s^2}$, $s>0$ (*iv*) $\dfrac{1}{s}+\dfrac{\sqrt{\pi}}{2s^{3/2}}+\dfrac{3\sqrt{\pi}}{s^{1/2}}$, $s>0$

(*v*) $\dfrac{a\cos b+s\sin b}{s^2+a^2}$, $s>0$ (*vi*) $\dfrac{2abs}{\{s^2+(a-b)^2\}\{s^2+(a+b)^2\}}$, $s>0$

1.11. First shifting (or first translation) theorem. [Agra 2012]

If $L\{F(t)\}=f(s)$, then $L\{e^{at}F(t)\}=f(s-a)$, where a is any real or complex constant

Or If $f(s)$ is the Laplace transform of $F(t)$, then $f(s-a)$ is the Laplace transform of $e^{at}F(t)$, where a is any real or complex number. **[Osmania 2004, Nagpur 2010, Purvanchal 2002]**

Proof. By definition, we have $f(s) = \mathrm{L}\{\mathrm{F}(t)\} = \int_0^\infty e^{-st} F(t)\, dt.$ ... (1)

Replacing s by $(s - a)$ on both sides of (1), we get

$$f(s-a) = \int_0^\infty e^{-(s-a)} F(t)\, dt = \int_0^\infty e^{-st} \{e^{at} F(t)\} dt = \mathrm{L}\{e^{at} F(t)\},$$ by definition of Laplace transform.

Remark From the above theorem, we have

(*i*) $\mathrm{L}\{\mathrm{F}(t)\} = f(s) \Rightarrow \mathrm{L}\{e^{at}\,\mathrm{F}(t)\} = f(s-a)$

(*ii*) $\mathrm{L}\{\mathrm{F}(t)\} = f(s) \Rightarrow \mathrm{L}\{e^{-at}\,\mathrm{F}(t)\} = f(s+a)$

1.11 A Solved Examples based on Art 1.11.

Ex.1. *Evaluate (i)* $\mathrm{L}\{e^{at} \cos bt\}$ **[Kanpur 2008]** *(ii)* $\mathrm{L}\{e^{-2t} \sin 4t\}$. **(Nagpur 2010)**

Sol. (*i*) Since $\mathrm{L}\{\cos bt\} = s/(s^2 + b^2) = f(s)$, (say) ... (1)

hence by first shifting theorm, we have

$$\mathrm{L}\{e^{at} \cos bt\} = f(s-a) = \frac{(s-a)}{(s-a)^2 + b^2}, \text{ using (1).}$$

(*ii*) Since $\mathrm{L}\{\sin 4t\} = 4/(s^2 + 16) = f(s)$, (say) ... (2)

hence by first shifting theorem, we have

$$\mathrm{L}\{e^{-2t} \sin 4t\} = f(s+2) = \frac{4}{(s+2)^2 + 16} = \frac{4}{s^2 + 4s + 20}, \text{ using (2)}$$

Ex.2. *Evaluate (i)* $\mathrm{L}\{e^{4t} \cosh 5t\}$. **[Meerut 1995]**

(*ii*) $\mathrm{L}\{e^{-4t} \cosh 2t\}$ (*iii*) $\mathrm{L}\{e^{at} \sinh bt\}$.

Sol. (*i*) Since $\mathrm{L}\{\cosh 5t\} = s/(s^2 - 5^2) = f(s)$, (say) ... (1)

hence by first shifting theorem, we have

$$\mathrm{L}\{e^{4t} \cosh 5t\} = f(s-4) = \frac{(s-4)}{(s-4)^2 - 5^2} = \frac{s-4}{s^2 - 8s - 9}$$

(*ii*) Proceed as in part (*i*). **Ans** $(s + 4)/(s^2 + 8s + 12)$.

(*iii*) Since $\mathrm{L}\{\sinh bt\} = b/(s^2 - b^2) = f(s)$, (say) ... (2)

hence by first shifting theorem, we have

$$\mathrm{L}\{e^{at} \sinh bt\} = f(s-a) = \frac{b}{(s-a)^2 - b^2}, \text{ using (2)}$$

Ex.3 (*i*). *Evaluate* (*i*) $\mathrm{L}\,[(t + 3)^2 e^t]$ **[Agra 1998; Rohilkhand 2004]**

(*ii*) $\mathrm{L}\{t + 2)^2 e^{6t}\}$ **[Meerut 2010]**

Sol. (*i*) we have, $\mathrm{L}\{(t+3)^2\} = \mathrm{L}\{t^2 + 6t + 9\} = \mathrm{L}\{t^2\} + 6\,\mathrm{L}\{t\} + 9\,\mathrm{L}\{1\}$

or $$\mathrm{L}\{(t+3)^2\} = \frac{2!}{s^3} + 6 \times \frac{1!}{s^2} + \frac{9}{s} = \frac{2 + 6s + 9s^2}{s^3} = f(s), \text{(say)} \quad \text{... (1)}$$

Hence by first shifting theorem, we have

$$\mathrm{L}\{e^t (t+3)^2\} = f(s-1) = \frac{2 + 6(s-1) + 9(s-1)^2}{(s-1)^3} = \frac{9s^2 - 12s + 5}{(s-1)^3}, \text{ using (1)}$$

(*ii*) Proceed as in part (*i*). **Ans.** $2(s^2 - 22s + 61)/(s - 6)^3$

Ex. 4(*a*) *Evaluate* $L\{(e^{-at}\, t^{n-1})/(n-1)!\}$.

Sol. Here, $\mathrm{L}\{t^{n-1}/(n-1)!\} = \dfrac{1}{(n-1)!} L\{t^{n-1}\} = \dfrac{1}{(n-1)!} \cdot \dfrac{(n-1)!}{s^n} = \dfrac{1}{s^n} = f(s), \text{(say)}$ ... (1)

Hence by first shifting theorem, we have

$$L\left\{e^{-at}\, \frac{t^{n-1}}{(n-1)!}\right\} = f(s+a) = \frac{1}{(s+a)^n}, \text{ using (1).}$$

Ex. 4(*b*) *If n is a positive integer, prove that* $L\{t^n e^{kt}\} = (n!)/(s-k)^n$ **[Kanpur 2005]**

Sol. Proceed as in Ex. 4(*a*).

Ex.5. *Evaluate* (i) $L\ \{e^{-t} \cos^2 t\}$ (*ii*) $L\ \{e^{-t} \sin^2 t\}$

(*iii*) $L\ \{e^{t} \sin^2 t\}$ (*iv*) $L\ \{e^{t} \cos^2 t\}$ **[Agra 2008]**

Sol. (*i*) we have, $L\ \{\cos^2 t\} = L\left\{(1/2) \times (1 + \cos 2t)\right\} = (1/2) \times \left[L\{1\} + L\{\cos\ 2t\}\right]$

$$\therefore \qquad L\{\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 2^2}\right] = \frac{s^2+2}{s(s^2+4)} = f(s), \text{ (say)} \qquad \text{... (1)}$$

Hence by first shifting theorem, we have

$$\mathrm{L}\ \{e^{-t} \cos^2 t\} = f(s+1) = \frac{(s+1)^2 + 2}{(s+1)\{(s+1)^2 + 4\}}, \text{ using (1)}$$

$$= \frac{s^2 + 2s + 3}{(s+1)\,(s^2 + 2s + 5)}$$

(*ii*) Here, $\mathrm{L}\ \{\sin^2 t\} = \mathrm{L}\ \left\{(1/2) \times (1 - \cos\ 2t)\right\} = (1/2) \times [\mathrm{L}\{1\} - \mathrm{L}\{\cos 2t\}]$

$$\therefore \qquad \mathrm{L}\ \{\sin^2 t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2 + 2^2}\right] = \frac{2}{s(s^2+4)} = f(s), \text{ (say)} \qquad \text{... (2)}$$

Hence by first shifting theorem, we have

$$\mathrm{L}\ \{e^{-t} \sin^2 t\} = f(s+1) = \frac{2}{(s+1)\{(s+1)^2 + 4\}}, \text{ using (1)}$$

$$= 2/(s+1)\,(s^2 + 2s + 5).$$

(*iii*) As in part (*ii*), $\mathrm{L}\ \{\sin^2 t\} = 2/s\,(s^2 + 4) = f(s)$, say

Hence by first shifting theorem, we have

$$\mathrm{L}\ \{e^{t} \sin^2 t\} = f(s-1) = \frac{2}{(s-1)\{(s-1)^2 + 4\}} = \frac{2}{(s-1)(s^2 - 2s + 5)}$$

(*iv*) As in part (*i*), $\mathrm{L}\ \{\cos^2 t\} = \dfrac{s^2+2}{s(s^2+4)} = f(s)$, (say)

Hence by first shifting theorem, we have

$$\mathrm{L}\ \{e^{t} \cos^2 t\} = f(s-1) = \frac{(s-1)^2 + 2}{(s-1)\ \{(s-1)^2 + 4\}} = \frac{s^2 - 2s + 3}{(s-1)\ (s^2 - 2s + 5)}$$

Ex.6. *If L {F (t)} = f (s), find L {F (t) cos wt}.*

Sol. Given that $L\{F(t)\} = f(s)$. ... (1)

From Trigonometry, we know that $\cos wt = (e^{iwt} + e^{-iwt})/2$.

$\therefore L\{F(t)\cos wt\} = L\{F(t) \times (1/2) \times (e^{iwt} + e^{-iwt})\} = (1/2) \times L\{e^{iwt} F(t)\} + (1/2) \times L\{e^{-iwt} F(t)\}$

$= (1/2) \times f(s - iw) + (1/2) \times f(s + iw)$, using first shifting theorem and (1)

Ex.7. *Find L {e^{-t} (3 sinh 2t – 5 cosh 2t)}.*

[Amaravati 2006, Bilaspur 2004, Gwalior 2003]

Sol. We have, $L\{3 \sinh 2t - 5 \cosh 2t\} = \frac{6}{s^2-4} - \frac{5s}{s^2-4} = \frac{6-5s}{s^2-4} = f(s)$, say

$\therefore \quad L\{e^{-t}(3 \sinh 2t - 5 \cosh 2t)\} = f(s+1) = \frac{6-5(s+1)}{(s+1)^2-4} = \frac{1-5s}{s^2+2s-3}$.

Ex.8. *Show that* $L\{(1 + te^{-t})^3\} = 1/s + 3/(s+1)^2 + 6/(s+2)^3 + 6/(s+3)^4$. **[Kanpur 2006]**

Sol. $L\{(1 + t e^{-t})^3\} = L\{1 + 3t e^{-t} + 3t^2 e^{-2t} + t^3 e^{-3t}\}$

$= L\{1\} + 3 L\{t e^{-t}\} + 3 L\{t^2 e^{-2t}\} + L\{t^3 e^{-3t}\}$ ... (1)

Now, $L\{t\} = (1!/s^2)$, $L\{t^2\} = (2!/s^3)$ and $L\{t^3\} = (3!/s^4)$

$\therefore$ By first shifting theorem, we set

$$L\{e^{-t}t\} = \frac{1!}{(s+1)^2}, \qquad L\{e^{-2t}\ t^2\} = \frac{2!}{(s+2)^3} \qquad \text{and} \qquad L\{e^{-3t}t^3\} = \frac{3!}{(s+3)^4}.$$

Putting these values in (1), gives $L\{(1 + t e^{-t})^3\} = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^2} + \frac{6}{(s+3)^4}$.

Ex.9. *Evaluate (i) L{e^{-t} (3 sin 2t – 5 cos 2t)}* **[Bhopal 2004]**

(ii) L{e^{-2t} sin t cos 3t} **[MDU Rohtak 2006]**

Sol. (*i*) We have, $L\{3 \sin 2t - 5 \cos 2t)\} = 3 L\{\sin 2t\} - 5 L\{\cos 2t\}$

$$= 3 \times \frac{2}{s^2+2^2} - 5 \times \frac{s}{s^2+2^2} = \frac{6-5s}{s^2+4} = f(s), \text{ say}$$

Hence, by first shifting theorem, we have

$$L\{e^{-t}(3\sin 2t - 5\cos 2t)\} = f(s+1) = \frac{6-5(s+1)}{(s+1)^2+4} = \frac{1-5s}{s^2+2s+5}$$

(*ii*) $L\{\sin t \cos 3t\} = L\{(\sin 4t - \sin 2t)/2\} = (1/2) \times \{L\{\sin 4t\} - L(\sin 2t)\}$

$$= \frac{1}{2} \times \frac{4}{s^2+4^2} - \frac{1}{2} \times \frac{2}{s^2+2^2} = \frac{s^2-8}{(s^2+16)(s^2+4)} = f(s), \text{ say}$$

Hence, by first shifting theorem, we have

$$L\{e^{-2t} \sin t \cos 3t\} = f(s+2) = \frac{(s+2)^2-8}{\{(s+2)^2+16\}\{(s+2)^2+4\}} = \frac{s^2+4s-4}{(s^2+4s+20)(s^2+4s+8)}$$

Ex. 10. *(a) If L{F (t)} = f(s), then show that L{(sinh at) F(t)} = {f (s – a) – f(s + a)}/2. Hence evaluate L{sinh 2t sin 3t}.*

(b) If L{F(t)} = f(s), then show that L{(cosh at) F(t)} = {f(s – a) + f(s + a)}/2. Hence evaluate L{cosh 3t cos 2t}

Sol. (*a*) Given that $L\{F(t)\} = f(s)$ ... (1)

We have, $L\{(\sinh at)\, F(t)\} = L\{(1/2) \times (e^{at} - e^{-at})\, F(t)\}$

$= (1/2) \times L\{e^{at} F(t)\} - (1/2) \times L\{e^{-at} F(t)\}$

$= (1/2) \times \{f(s-a) - f(s+a)\}$ by first shifting theorem and (1)

Now, $L\{\sin 3t\} = 3/(s^2 + 3^2) = 3/(s^2 + 9),\ s > 0$

Using the result just proved by taking $a = 2$ and $f(s) = 3/(s^2 + 9)$, we have

$$L\{\sinh 2t \sin 3t) = \frac{1}{2}\left[\frac{3}{(s-2)^2+9} + \frac{3}{(s+2)^2+9}\right],\ s-2>0, \text{ and } s+2>0$$

$$= \frac{3}{2}\left[\frac{1}{s^2-4s+13} - \frac{1}{s^2+4s+13}\right] = \frac{12s}{s^4+10s^2+169}, s>2$$

(*b*) Proceed as in part (*a*) **Ans.** $(s^3 - 5s)/(s^4 - 10s^2 + 169),\ s > 3$

Ex.11. *Prove that (i)* $L\{\sinh at \cos at\} = a\,(s^2 - 2a^2)/(s^4 + 4a^4)$. **[Meerut 2010]**

(*ii*) $L\{\sinh at \sin at\} = (2a^2 s)/(s^4 + 4a^4)$. **[Meerut 2010]**

Sol. We have $L\{\sinh at\} = a/(s^2 - a^2) = f(s)$, say ... (1)

$\therefore\ L\{e^{iat} \sinh at\} = f(s - ia)$, by first shifting theorem

$$= \frac{a}{(s-ia)^2 - a^2}, \text{ using (1)}$$

$$= \frac{a}{(s^2-2a^2) - 2ias} \times \frac{(s^2-2a^2)+2ias}{(s^2-2a^2)+2ias} = \frac{a(s^2-2a^2)+2ia^2 s}{(s^2-2a^2)^2 + 4a^2 s^2}.$$

$$\therefore\ L\{\sinh at\,(\cos at + i \sin at)\} = \frac{a(s^2-2a^2)+2ia^2 s}{s^4+4a^4}, \qquad \text{as } e^{i\theta} = \cos\theta + i\sin\theta$$

$$L\{\sinh at \cos at\} + i\, L\{\sinh at \sin at\} = \frac{a(s^2-2a^2)}{s^4+4a^4} + i\frac{2a^2 s}{s^4+4a^4}$$

Equating real and imaginary parts, we get

$$L\{\sinh at \cos at\} = \frac{a(s^2-2a^2)}{s^4+4a^4} \quad \text{and} \quad L\{\sinh at \sin at\} = \frac{2a^2 s}{s^4+4a^4}.$$

EXERCISE 1 (*B*)

Find the Laplace transform of the following functions.

1. $t^5 e^{3t}$ **[Kanpur 2004]** **2.** $e^{-2t} \sin 4t$ **(Ravishankar 2010)** **3.** $e^{3t} \cos 5t$.

4. $e^{-t}(3 \sin 2t - 5 \cosh 2t)$ **5.** $e^{-2t}(3 \cos 6t - 5 \sin 6t)$.

6. $e^t \{\cos 2t + (1/2) \times \sin 2t\}$ **[Kerala 2001]** **7.** $e^{2t}(t^2 + 3t + 4)$ **[Bangalore 2005]**

8. $e^{-3t} \sin 4t$ **[Gulberga 2005]** **9.** $e^{-3t}\{3\cos 6t - 5 \sin 6t\}$ **[Meerut 2009]**

10. $e^{2t}\{3\sin 4t - 4 \cos 4t\}$ **[Punjab 2005]** **11.** $\sinh t \cos t$.

12. $e^{-4t}\, t^{3/2}$ **13.** $\sinh 3t \cos^2 t$ **[Kurukshetra 2004]**

14. Prove that $L\{(at^2 + bt + c)e^{kt}\} = \frac{2a}{(s-k)^3} + \frac{b}{(s-k)^2} + \frac{c}{s-k},\ s > k$ **[Bangalore 2007]**

ANSWERS

1. $120/(s-3)^6$ **2.** $4/(s^2+4s+20)$ **3.** $(s-3)/(s^2-6s+34)$.

4. $\dfrac{6}{s^2+2s+5}-\dfrac{5(s+1)}{s^2+2s-3}$ **5.** $\dfrac{3(s-8)}{s^2+4s+40}$ **6.** $\dfrac{s}{s^2-2s+5}$

7. $(4s^2-13s+12)/(s-2)^3$ **8.** $4/(s^2+6s-7)$ **9.** $(3s-16)/(s^2+6s+45)$

10. $(20-4s)/(s^2-4s+20)$ **11.** $(s^2-2)/(s^2+4)$, $s>1$

12. $\dfrac{3\sqrt{\pi}}{4(s+4)^{5/2}}$, $s>-4$ **13.** $\dfrac{1}{2}\left[\dfrac{s^2-6s+11}{(s-3)(s^2-6s+13)}+\dfrac{s^2+6s+11}{(s+3)(s^2+6s+13)}\right]$, $s>3$

1.12. The unit step function or Heaviside's unit function. Definition.

The unit step function is denoted and defined by

$$u_a(t) = \mathrm{H}(t-a) = \begin{cases} 0, \text{ if } t<a \\ 1, \text{ if } t \geq a \end{cases} \qquad ...(1)$$

Note. $\mathrm{L}\{\mathrm{H}(t-a)\} = \int_0^\infty e^{-st}H(t-a)\,dt = \int_0^a e^{-st}H(t-a)dt + \int_a^\infty e^{-st}H(t-a)dt$, using (1).

$$= \int_a^\infty e^{-st}\cdot 1\,dt = \frac{e^{-as}}{s}.$$

1.13. Second shifting theorem or second translation theorem.

[Jabalpur 2004; Mysore 1997, Kanpur 2006, Agra 2005, Purvanchal 1994, 96]

If $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$ *and* $\mathrm{G}(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$

then $\mathrm{L}\{\mathrm{G}(t)\} = e^{-as}f(s)$.

Proof. By definition of Laplace transform, we have

$$\mathrm{L}\{\mathrm{G}(t)\} = \int_0^\infty e^{-st}G(t)\,dt = \int_0^a e^{-st}G(t)\,dt + \int_a^\infty e^{-st}G(t)\,dt, \text{ where } 0<a<\infty$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}F(t-a)\,dt, \text{ putting the given values of G}(t)$$

$$= 0 + \int_a^\infty e^{-st}F(t-a)\,dt = \int_0^\infty e^{-s(a+u)}F(u)\,du, \text{ [Putting } t-a=u \text{ and } dt=du.$$

Also note that when $t=a$, $u=0$ and when $t=\infty$, $u=\infty$]

$$= e^{-sa}\int_0^\infty e^{-su}F(u)\,du = e^{-sa}\int_0^\infty e^{-st}F(t)dt, \text{ using a property of definite integral}$$

$$= e^{-sa}\,\mathrm{L}\{\mathrm{F}(t)\}, \text{ by definition of Laplace transform}$$

$$= e^{-sa}f(s), \text{ using given result } \mathrm{L}\{\mathrm{F}(t)\} = f(s)$$

Another form of second shifting theorem or second translation theorem

If $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$ and $a>0$, then $\mathrm{L}\{\mathrm{F}(t-a)\,\mathrm{H}(t-a)\} = e^{-as}f(s)$,

where $\mathrm{H}(t-a) = \begin{cases} 0, \text{ if } t<a \\ 1, \text{ if } t>a. \end{cases}$

Proof. By definition, $\mathrm{L}\{\mathrm{F}(t-a)\,\mathrm{H}(t-a)\} = \int_0^\infty e^{-st}F(t-a)\,H(t-a)\,dt$

$$= \int_0^a e^{-st} F(t-a) H(t-a)\, dt + \int_a^\infty e^{-st} F(t-a)\ H(t-a)\, dt$$

$$= \int_0^a e^{-st} F(t-a) \cdot 0\ dt + \int_a^\infty e^{-st} F(t-a) \cdot 1\ dt, \text{ using the given value of H } (t-a)$$

$$= 0 + \int_a^\infty e^{-st} F(t-a)\, dt = \int_0^\infty e^{-s(a+u)} F(u)\ du$$

[putting $t - a = u$ and $dt = du$. Also note that when $t = a$, $u = 0$ and when $t = \infty$, $u = \infty$]

$$= e^{-sa} \int_0^\infty e^{-su} F(u)\, du = e^{-sa} \int_0^\infty e^{-st} F(t)\ dt\ = e^{-sa} L\{F(t)\} = e^{-sa} f(s).$$

1.13A Solved examples based on Art. 1.13

Ex.1. *Find L {G (t)}, where* $\mathrm{G}(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a. \end{cases}$

Sol. By second shifting theorem, if $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$ and $\mathrm{G}(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a. \end{cases}$,

then $$\mathrm{L}\{\mathrm{G}(t)\} = e^{-as} f(s).$$

On comparision with the given value of $G(t)$, here, we have $\mathrm{F}(t) = e^t$ and $a = a$.

Now, $f(s) = \mathrm{L}\{\mathrm{F}(t)\} = \mathrm{L}\{e^t\} = 1/(s-1),\ s > 1.$

$\therefore$ By second shifting theorem $\mathrm{L}\{G(t)\} = e^{-as} f(s) = e^{-as}/(s-1),\ s > 0$

Alternative Solution. By definition of Laplace transform, we have

$$\mathrm{L}\{\mathrm{G}(t)\} = \int_0^\infty e^{-st} G(t)\, dt\ = \int_0^a e^{-st} G(t)\, dt + \int_a^\infty e^{-st} G(t)\, dt = \int_0^a e^{-st} \cdot 0\ dt + \int_a^\infty e^{-st} e^{t-a} dt$$

[using the given values of G (t)]

$$= e^{-a} \int_a^\infty e^{-(s-1)t} dt = e^{-a} \left[-\frac{e^{-(s-1)t}}{s-1} \right]_a^\infty, s > 1\ = e^{-a} \cdot \frac{e^{-(s-1)a}}{s-1} = \frac{e^{-sa}}{s-1},\ s > 1.$$

Ex.2. *Find L {G (t)}, where* $\mathrm{G}(t) = \begin{cases} \sin(t - \pi/3), & t > \pi/3 \\ 0, & t < \pi/3. \end{cases}$ **[Rohilkhand 1997]**

Sol. By second shifting theorem, if $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$ and $\mathrm{G}(t) = \begin{cases} \mathrm{F}(t-a), & t > a \\ 0, & t < a \end{cases}$,

then $$\mathrm{L}\{\mathrm{G}(t)\} = e^{-as} f(s).$$

On comparison with the given value of $G(t)$, here $\mathrm{F}(t) = \sin t$, $a = \pi/3$.

Now, $f(s) = \mathrm{L}\{\mathrm{F}(t)\} = \mathrm{L}\{\sin t\} = 1/(s^2 + 1),\ s > 0$

$\therefore$ By second shifting theorem, $\mathrm{L}\{\mathrm{G}(t)\} = e^{-as} f(s) = \dfrac{e^{-\pi s/3}}{s^2 + 1}, s > 0$

Alternative solution. By definition of Laplace transform, we have

$$L\{G(t)\} = \int_0^{\infty} e^{-st} G(t)\, dt = \int_0^{\pi/3} e^{-st} G(t)\, dt + \int_{\pi/3}^{\infty} e^{-st} G(t)\, dt$$

$$= \int_0^{\pi/3} e^{-st} \cdot 0\, dt + \int_{\pi/3}^{\infty} e^{-st} \sin\left(t - \frac{\pi}{3}\right) dt, \text{ using given values of } G(t)$$

$$= 0 + \int_{\pi/3}^{\infty} e^{-st} \sin\left(t - \frac{\pi}{3}\right) dt = \int_0^{\infty} e^{-s(u+\pi/3)} \sin u\, du \quad [\text{putting } t - (\pi/3) = u \text{ and}$$

$dt = du$. Also note that when $t = \pi/3$, $u = 0$ and when $t = \infty$, $u = \infty$]

$$= e^{-s\pi/3} \int_0^{\infty} e^{-su} \sin u\, du = e^{-s\pi/3} \int_0^{\infty} e^{-st} \sin t\, dt$$

$$= e^{-s\pi/3} L\{\sin t\}, \text{ by definition of Laplace transform}$$

Thus, $L\{G(t)\} = e^{-s\pi/3} \times \{1/(s^2+1)\}, s > 0$

EXERCISE 2(*C*)

Ex.1. *Find L {G (t)}, where* $G(t) = \begin{cases} \sin(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3. \end{cases}$ **[Osmania 2004]**

Sol. Proceed as in Ex.2. Here $F(t) = \sin t$ and $a = 2\pi/3$. **Ans.** $e^{-2\pi s/3} \times \{1/(s^2+1)\}$, $s > 0$.

Ex. 2. *Find L{G (t)}, where* $G(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3. \end{cases}$ **Ans.** $\dfrac{se^{-2\pi s/3}}{s^2+1}$, $s > 0$

[Meerut 2008]

Ex. 3. *Find L {e^{t-1} H(t – 1)}.* **[Purvanchal 2004]**

[Sol. Let $F(t) = e^t$. Then, $L\{F(t)\} = 1/(s-1) = f(s)$, say.

Using second shifting theorem, we have

$$L\{e^{t-1} H(t-1)\} = e^{-s} f(s) = e^{-s}/(s-1)$$

$$[\because L\{F(t)\} = f(s) \quad \Rightarrow \quad L\{F(t-1) H(t-a)\} = e^{-as} f(s)]$$

Ex. 4. *Find the Laplace transform of the following functions:*

(*i*) $(t-1) H(t-1)$ (*ii*) $\cos t\, H(t-\pi)$

Ans. (*i*) e^{-s}/s^2, $s > 0$ (*ii*) $-(se^{-\pi s}) / (s^2+1)$, $s > 0$

1.14. Change of scale property.

[Agra 2007; Nagpur 2005, Osmania 2004, Lucknow 1996, Kanpur 2007]

If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = (1/a) \times f(s/a)$.

Proof. By definition, $\int_0^{\infty} e^{-st} F(t)\, dt = L\{F(t)\} = f(s)$ ... (1)

Now, $L\{F(at)\} = \int_0^{\infty} e^{-st} F(at)\, dt = \frac{1}{a} \int_0^{\infty} e^{-su/a} F(u)\, du$, putting $at = u$ and $a\, dt = du$

$$= \frac{1}{a} \int_0^{\infty} e^{-(s/a)t} F(t)\, dt = \frac{1}{a} f\left(\frac{s}{a}\right), \text{ using (1)}$$

1.14 A SOLVED EXAMPLES BASED ON ART. 1.14.

Ex.1. *If L {F (t)} = $(s^2 - s + 1)/(2s+1)^2 (s-1)$ prove that*

L {F (2t)} = $(s^2 - 2s + 4)/4 (s+1)^2 (s-2)$.

Sol. Given $\quad L\{F(t)\} = \dfrac{s^2 - s + 1}{(2s+1)^2(s-1)} = f(s)$, say

$\therefore$ By the change of scale property, we have

$$L\{F(2t)\} = \frac{1}{2} f\left(\frac{s}{2}\right) = \frac{1}{2} \frac{(s/2)^2 - (s/2) + 1}{[2 \times (s/2) + 1]^2 [(s/2) - 1]} = \frac{s^2 - 2s + 4}{4(s+1)^2 (s-2)}.$$

Ex.2. *If* $L\{F(t)\} = \dfrac{1}{s} e^{-1/s}$, *prove that* $L\{e^{-t} F(3t)\} = \dfrac{e^{-3/(s+1)}}{s+1}$.

Sol. Given $\quad L\{F(t)\} = (1/s)\, e^{-1/s} = f(s)$, say.

$\therefore$ By the change of scale property,

$$L\{F(3t)\} = \frac{1}{3} f\left(\frac{s}{3}\right) = \frac{1}{3} \times \frac{3}{s} \times e^{-3/s} = \frac{1}{s} e^{-3/s} = g(s), \text{ say} \qquad \ldots (1)$$

$\therefore$ By the first shifting theorem, $L\left\{e^{-t} F(3t)\right\} = g(s+1) = \dfrac{1}{s+1} e^{-3/(s+1)}$, using (1).

Ex.3. *Applying the change of scale property, find* (*i*) $L\{\sin 5t\}$ (*ii*) $L\{\sinh 5t\}$

Sol. (*i*). We have $\quad L\{\sin t\} = 1/(s^2 + 1) = f(s)$, say, $(s > 0)$

$\therefore$ By the change of scale property, we have

$$L\{\sin 5t\} = \frac{1}{5} f\left(\frac{s}{5}\right) = \frac{1}{5} \frac{1}{(s/5)^2 + 1} = \frac{5}{s^2 + 25}, s > 0$$

(*ii*) We have $\quad L\{\sinh t\} = 1/(s^2 - 1) = f(s)$, say

$\therefore$ By the change of scale property, $L\{\sinh 5t\} = \dfrac{1}{5} f\left(\dfrac{s}{5}\right) = \dfrac{1}{5} \dfrac{1}{(s/5)^2 - 1} = \dfrac{5}{s^2 - 25}$

EXERCISE 1 (D)

1. Apply the change of scale propert and evaluate

(*i*) $L\{\cos 4t\}$ **[Lucknow 1996]** (*ii*) $L\{\cosh 5t\}$

Ans. (*i*) $s/(s^2 + 16)$, $s > 0$ (*ii*) $s/(s^2 - 25)$, $|s| > 5$

2. (*i*) If $L\{(\sin t)/t\} = \tan^{-1}(1/s)$, prove that $L\{(\sin 3t)/t\} = \tan^{-1}(3/s)$ (*Agra 2009*)

(*ii*) If $L\{(\sin t)/t\} = \tan^{-1}(1/s)$, prove that $L\{(\sin 6t)/t\} = \tan^{-1}(6/s)$ **(Agra 2010)**

(*iii*) If $L\{(\sin t)/t\} = \tan^{-1}(1/s)$, prove that $L\{(\sin at)/t\} = \tan^{-1}(a/s)$ **(Meerut 2011)**

1.15. Laplace transforms of derivatives.

Theorem I. *Let F (t) be continuous for all* $t \geq 0$ *and be of exponential order* σ *as* $t \to \infty$ *and if F′ (t) is of class A, then Laplace transform of the derivative F′(t) exists when* $s > \sigma$ *and* $L\{F'(t)\} = sL\{F(t)\} - F(0)$. **[Bangalore 1997, Nagpur 2005, Madras 2005]**

Proof. We divide the proof in two parts.

Case I Let $F'(t)$ be continuous for all $t \geq 0$. Then

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt = \left[e^{-st} F(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} F(t)\, dt, \text{ integrating by parts}$$

$$= \lim_{t \to \infty} e^{-st} F(t) - F(0) + s \int_0^\infty e^{-st} F(t)\, dt$$

or $$\mathrm{L}\{\mathrm{F}'(t)\} = \lim_{t\to\infty} e^{-st}F(t) - F(0) + sL\{F(t)\}. \quad \ldots(1)$$

Since F (t) is of exponential order σ, we can find constant m (> 0) such that

$$|\mathrm{F}(t)| \le me^{\sigma t} \quad \text{for all } t \ge 0. \quad \ldots(2)$$

$\therefore$ $$|e^{-st}\mathrm{F}(t)| = e^{-st}|\mathrm{F}(t)| \le e^{-st}\, m\, e^{\sigma t}, \text{ using (2)}$$

$\therefore$ $$|e^{-st}\mathrm{F}(t)| \le me^{-(s-\mathrm{s})t} \quad \ldots(3)$$

Now, for $s > \sigma$, as $t \to \infty$, $me^{-(s-\sigma)t} \to 0$.

$\therefore$ From (3), $\lim_{t\to\infty} e^{-st}F(t) = 0$ for $s > \sigma$.

So when $s > \sigma$, from (1) we see that L {F′ (t)} exists and is given by

$$\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0). \quad \ldots(4)$$

Cast II. Let F′ (t) be piece-wise continuous. Now, by definition, we get

$$\mathrm{L}\{\mathrm{F}'(t)\} = \int_0^\infty e^{-st}F'(t)dt. \quad \ldots(5)$$

We now break up R.H.S. as the sum of integrals in different ranges from 0 to ∞ such that F′ (t) is continuous in each of these sub-intervals. We can now apply the procedure of case I in each such interval and finally obtain as in above case I,

$$\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0).$$

Remark 1. Suppose F (t) is not continuous at $t = 0$ but $\lim_{t\to 0+}$ F (t) *i.e.*, the right hand limit F (0 + 0) exists, then we can prove that

$$\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0+0).$$

Remark 2. Suppose F (t) is not continuous at $t = a$, $0 < a < \infty$ but right hand limit F (a + 0) and left hand limit F (a – 0) exist, then the result of the above theorem takes the form

$$\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0) - e^{-as}[\mathrm{F}(a+0) - \mathrm{F}(a-0)].$$

The quantity F (a + 0) – F (a – 0) is known as the jump at the discontinuity $x = a$.

Proof of result of remark 2:

$$\mathrm{L}\{\mathrm{F}'(t)\} = \int_0^\infty e^{-st}F'(t)dt = \int_0^a e^{-st}F'(t)dt + \int_a^\infty e^{-st}F(t)\,dt$$

$$= \left[e^{-st}F(t)\right]_0^a - \int_0^a (-s)e^{-st}F(t)dt + \left[e^{-st}F(t)\right]_a^\infty - \int_a^\infty (-s)e^{-st}F(t)\,dt$$

$$= e^{-as}\,\mathrm{F}(a-0) - \mathrm{F}(0) + \mathrm{s}\int_0^a e^{-st}F(t)\,dt + \lim_{t\to\infty} e^{-st}F(t) - e^{-as}F(a+0) + s\int_a^\infty e^{-st}F(t)dt$$

$$= \lim_{t\to\infty} e^{-st}F(t) + s\left[\int_0^a e^{-st}F(t)\,dt + \int_a^\infty e^{-st}F(t)\,dt\right] - F(0) - e^{-as}[F(a+0) - F(a-0)]$$

$$= 0 + s\int_0^\infty e^{-st}F(t)\,dt - F(0) - e^{-as}[F(a+0) - F(a-0)]$$

$$\left[\because \lim_{t\to\infty} e^{-st}F(t) = 0 \text{ for } s > \sigma, \text{ as proved in case I of above theorem}\right]$$

$\therefore$ $$\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0) - e^{-as}[F(a+0) - \mathrm{F}(a-0)].$$

Remark 3. If F (*t*) is discontinuous at $t = a_1, a_2, \ldots . a_n$, we break up the integral into ranges from 0 to ∞ as shown below:

$$L\{F'(t)\} = \int_0^{a_1} e^{-st} F'(t)\,dt + \int_{a_1}^{a_2} e^{-st} F'(t)\,dt + \ldots . + \int_{a_n}^{\infty} e^{-st} F'(t)\,dt$$

and then proceed as explained in remark 2.

Theorem II. *Let F (t) and F′ (t) be continuous functions for all $t \geq 0$ and F″ be of exponential order σ as $t \to \infty$ and if F″ (t) is of class A, then Laplace transform of F″ (t) exists when $s > \sigma$, and is given by* $L\{F''(t)\} = s^2 L\{F(t)\} - s F(0) - F'(0)$. **[Madras 2012]**

Proof. Let $G(t) = F'(t)$ ... (1)

so that $G'(t) = F''(t)$ and $G(0) = F'(0)$. ... (2)

Applying theorem I to function G (*t*), we get

$L\{G'(t)\} = s L\{G(t)\} - G(0)$ or $L\{F''(t)\} = s L\{F'(t)\} - F'(0)$, using (1) and (2), ... (3)

But by theorem I, $L\{F'(t)\} = s L\{F(t)\} - F(0)$. ... (4)

Using (4), (3) reduces to $L\{F''(t)\} = s[s L\{F(t)\} - F(0)] - F'(0)$

or $L\{F''(t)\} = s^2 L\{F(t)\} - s F(0) - F'(0)$.

Theorem III. *Let F (t), F′ (t) and be F″ (t) be continuous for all $t \geq 0$ and be of exponential order σ as $t \to \infty$ and if* $F'''(t)$ *(t) is of class A then Laplace transform of* $F'''(t)$ *(t) exists when $s > \sigma$ and is given by* $L\{F'''(t)\} = s^3 L\{F(t)\} - s^2 F(0) - s F'(0) - F''(0)$. **[Mysore 2004]**

Proof Let $G(t) = F''(t)$ ... (1)

so that $G'(t) = F'''(t)$ and $G(0) = F''(0)$. ... (2)

Applying theorem I to function G (*t*), we get $L\{G'(t)\} = s L\{G(t)\} - G(0)$

or $L\{F'''(t)\} = s L\{F''(t)\} - F''(0)$, using (1) and (2) ... (3)

But, from theorem II, we have $L\{F''(t)\} = s^2 L\{F(t)\} - s F(0) - F'(0)$ ... (4)

Using (4), (3) reduces to $L\{F'''(t)\} = s[s^2 L\{F(t)\} - s F(0) - F'(0)] - F''(0)$

or $L\{F'''(t)\} = s^3 L\{F(t)\} - s^2 F(0) - s F'(0) - F''(0)$.

Theorem IV. Laplace transform of the nth order derivative. General case.

Let F (t) and its derivative F′ (t), F″ (t) . . . . . , $F^{(n-1)}$ (t) be continuous for all $t \geq 0$ and be of exponential order σ as $t \to \infty$ and if $F^{(n)}$ (t) is of class A, then Laplace transform of $F^{(n)}$ (t) exists when $s > \sigma$, and is given by

$$L\{F^n(t)\} = s^n L\{F(t)\} - s^{n-1} F(0) - s^{n-2} F'(0) - \ldots . - F^{(n-1)}(0),$$

where $F^{(n)}(t) = d^n F(t)/dt^n$ etc.

Proof. We shall use the principle of mathematical induction to prove

$$L\{F^{(n)}(t)\} = s^n L\{F(t)\} - s^{n-1} F(0) - s^{n-2} F'(0) - \ldots . - F^{(n-1)}(0) \quad \ldots (1)$$

For $n = 1$, (1) reduces to $L\{F'(t)\} = s L\{F(t)\} - F(0)$, which is true by theorem I

Hence (1) is true for $n = 1$.

Assume that the result (1) is true for $n = k$, so that

$$L\{F^{(k)}(t)\} = s^k L\{F(t)\} - s^{k-1} F(0) - s^{k-2} F'(0) - \ldots . - F^{(k-1)}(0) \quad \ldots (2)$$

We shall now prove the result (1) for $n = k + 1$ *i.e.*, we shall prove that

$$L\{F^{(k+1)}(t)\} = s^{k+1} L\{F(t)\} - s^k F(0) - s^{k-1} F'(0) - \ldots . - F^{(k)}(0) \quad \ldots (3)$$

Let $G(t) = F^{(k)}(t)$ ... (4)

so that $G'(t) = F^{(k+1)}(t)$ and $G(0) = F^{(k)}(0)$ ... (5)

Applying theorem I to function G (*t*), we get

$$\mathrm{L}\{\mathrm{G}'(t)\} = s\,\mathrm{L}\{\mathrm{G}(t)\} - \mathrm{G}(0)$$

or $\mathrm{L}\{\mathrm{F}^{(k+1)}(t)\} = s\,\mathrm{L}\{\mathrm{F}^{(k)}(t)\} - \mathrm{F}^{(k)}(0)$, using (4) and (5)

$$= s\,[s^k\,\mathrm{L}\{\mathrm{F}(t)\} - s^{k-1}\,\mathrm{F}(0) - s^{k-2}\,\mathrm{F}'(0) - \ldots - \mathrm{F}^{(k-1)}(0)] - \mathrm{F}^{(k)}(0).$$

$$\therefore\ \mathrm{L}\{\mathrm{F}^{(k+1)}(t)\} = s^{k+1}\,\mathrm{L}\{\mathrm{F}(t)\} - s^k\,\mathrm{F}(0) - s^{k-1}\,\mathrm{F}'(0) - \ldots - \mathrm{F}^{(k)}(0).$$

This shows that the required result (1) is true for $n = k + 1$ whenever it is true for $n = k$. Hence, by the principle of mathematical induction, the required result (1) is true for all positive integers.

1.15A. SOLVED EXAMPLES BASED ON ART. 1.15

Ex.1. *If* $L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} = \frac{1}{s^{3/2}}$, *show that* $\frac{1}{s^{1/2}} = \mathrm{L}\left\{\frac{1}{\sqrt{(\pi t)}}\right\}$. **[Meerut 2000]**

Sol. Let $\mathrm{F}(t) = 2\sqrt{(t/\pi)}$. Then $\mathrm{F}(0) = 2\sqrt{(0/\pi)} = 0$. ... (1)

Again, $$\mathrm{F}'(t) = \frac{d}{dt}\left[\frac{2}{\sqrt{\pi}}\,t^{1/2}\right] = \frac{2}{\sqrt{\pi}} \times \frac{1}{2}t^{-1/2} = \frac{1}{\sqrt{(\pi t)}}.$$ ... (2)

Now, we know that (refer theorem I, Art. 1.15) $\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0)$. ... (3)

Substituting values of F′ (*t*), F (*t*) and F (0) in (3), we get

$$\mathrm{L}\left\{\frac{1}{\sqrt{(\pi t)}}\right\} = s\,L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} - 0 = s \times \frac{1}{s^{3/2}} = \frac{1}{s^{1/2}},$$ as given that $$L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} = \frac{1}{s^{3/2}}$$

Ex.2. *If* $L\left\{\sin\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2\,s^{3/2}}e^{-1/4s}$, *show* that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{s}\right)}e^{-1/4s}$. **[Meerut 2001, 05]**

Sol. Let $\mathrm{F}(t) = \sin\sqrt{t}$ so that $\mathrm{F}(0) = 0$. ... (1)

Again, $$\mathrm{F}'(t) = \frac{d}{dt}\sin t^{1/2} = \cos t^{1/2} \times \frac{1}{2}t^{-1/2} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$ ... (2)

Now, we know that $\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0)$. ... (3)

Substituting values of F′ (*t*), F (*t*) and F (0) in (3), we get

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,L\{\sin\sqrt{t}\} - 0 = s \times \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s},$$ as given that $$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$$

or $$\frac{1}{2}\,\mathrm{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{2\,s^{1/2}}e^{-1/4s}$$ or $$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{s}\right)}\,e^{-1/4s}.$$

Ex.3. *Find (i) L {F (t)} and (ii) L {F′ (t)}, where* $\mathrm{F}(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$

Sol. Given that $\mathrm{F}(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ t, & t > 1. \end{cases}$ ... (1)

(*i*) By definition of Laplace transform, we have

$$\mathrm{L}\{\mathrm{F}(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^1 e^{-st} F(t)\, dt + \int_1^\infty e^{-st} F(t)\, dt = \int_0^1 e^{-st}(2t)\, dt + \int_1^\infty e^{-st} t\, dt, \text{ using (1)}$$

$$= \left[(2t)\left(-\frac{1}{s}e^{-st}\right)\right]_0^1 - \int_0^1 2\left(-\frac{1}{s}e^{-st}\right) dt + \left[(t)\left(-\frac{1}{s}e^{-st}\right)\right]_1^\infty - \int_1^\infty (1)\left(-\frac{1}{s}e^{-st}\right) dt$$

[on integration by parts]

$$= -\frac{2}{s}e^{-s} + \frac{2}{s}\left[-\frac{1}{s}e^{-st}\right]_0^1 - \frac{1}{s}\lim_{t\to\infty} t\, e^{-st} + \frac{1}{s}e^{-s} + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_1^\infty$$

$$= -\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-s} + \frac{2}{s^2} - \frac{1}{s}\lim_{t\to\infty}\left(\frac{t}{e^{st}}\right) + \frac{1}{s}e^{-s} - \frac{1}{s^2}\lim_{t\to\infty} e^{-st} + \frac{1}{s^2}e^{-s}$$

$$= -\frac{1}{s}e^{-s} + \frac{2}{s^2} - \frac{1}{s^2}e^{-s}, \text{ as if } s>0, \text{ then } \lim_{t\to\infty} e^{-st} = 0 \text{ and } \lim_{t\to\infty}\frac{t}{e^{st}} = \lim_{t\to\infty}\frac{1}{s\,e^{st}}, \text{ by L Hospital's rule} = 0$$

$$\therefore \quad \mathrm{L}\{\mathrm{F}(t)\} = 2/s^2 - (1/s + 1/s^2)e^{-s},\ s > 0 \qquad \text{... (2)}$$

(*ii*) ***Method I for finding L {F′ (t)}***

We first examine F (*t*) for continuity at $x = 1$. We have

$$\mathrm{F}(1-0) = \text{Left hand limit (at } x = 1) = \lim_{h\to 0} F(1-h) = \lim_{h\to 0} 2(1-h) = 2 \qquad \text{... (3)}$$

and $$\mathrm{F}(1+0) = \text{Right hand limit (at } x = 1) = \lim_{h\to 0} \mathrm{F}(1+h) = \lim_{h\to 0}(1+h) = 1 \qquad \text{... (4)}$$

Since F (1 – 0) ≠ F (1 + 0), hence F (*t*) is discontinuous at $t = 1$. Therefore using the formula for Laplace transform of derivative of F (*t*) for the present case [Refer formula given in remark 2 of theorem I of Art 1.15 with $a = 1$]

$$\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0) - e^{-s}[\mathrm{F}(1+0) - \mathrm{F}(1-0)]$$
$$= s\{2/s^2 - (1/s + 1/s^2)e^{-s}\} - 0 - e^{-s}(1-2), \text{ by (1), (2), (3) and (4).}$$

$$\therefore \quad \mathrm{L}\{\mathrm{F}'(t)\} = \frac{2}{s} - e^{-s} - \frac{1}{s}e^{-s} + e^{-s} = \frac{2}{s} - \frac{e^{-s}}{s} = \frac{1}{s}(2 - e^{-s}).$$

Method II for finding L {F′ (t)}. From (1), we have

$$\mathrm{F}'(t) = \begin{cases} 2, & 0 < t < 1 \\ 1, & t > 1 \end{cases} \qquad \text{... (5)}$$

Using the definition of Laplace transform, we get

$$\mathrm{L}\{\mathrm{F}'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt = \int_0^1 e^{-st} F'(t)\, dt + \int_1^\infty e^{-st} F'(t)\, dt = \int_0^1 (e^{-st}\times 2)\, dt + \int_1^\infty (e^{-st}\times 1)\, dt, \text{using (5)}$$

$$= 2\left[-\frac{e^{-st}}{s}\right]_0^1 + \left[-\frac{e^{-st}}{s}\right]_1^\infty = -\frac{2}{s}e^{-s} + \frac{2}{s} - 0 + \frac{e^{-s}}{s},\ s > 0$$

$= (2/s) - (1/s) \times e^{-s} = (1/s) \times (2 - e^{-s}),\ s > 0$

Ex.4. *Find L {f (t)} if f ″ (t) + 3 f ′ (t) + 2 f (t) = 0, f (0) = 1, f ′ (0) = 2.*

[*S.V. University (A.P.) 1997*]

Sol. Given $f''(t) + 3f'(t) + 2f(t) = 0$ ... (1)

with $f(0) = 1$ and $f'(0) = 2$ ... (2)

Taking Laplace transform of both sides of (1), we get

$$L\{f''(t)\} + 3L\{f'(t)\} + 2L\{f(t)\} = 0$$

or $s^2 L\{f(t)\} - sf(0) - f'(0) + 3[sL\{f(t)\} - f(0)] + 2L\{f(t)\} = 0$

[Using results of theorems I and II of Art 1.15]

or $s^2 L\{f(t)\} - s - 2 + 3[sL\{f(t)\} - 1] + 2L\{f(t)\} = 0$, using (2)

or $(s^2 + 3s + 2)L\{f(t)\} = s + 5$ or $L\{f(t)\} = (s + 5)/(s^2 + 3s + 2)$.

Ex. 5. *Given $L\{t^3\} = 6/s^4$, evaluate $L\{t^6\}$*

Sol. Let $F(t) = t^6$ so that $F'(t) = 6t^5$, $F''(t) = 30t^4$, $F'''(t) = 120t^3$ ... (1)

Now, $L\{F'''(t)\} = s^3 L\{F(t)\} - s^2F(0) - sF'(0) - F''(0)$

$\Rightarrow L\{120t^3\} = s^3 L\{t^6\} - (s^2 \times 0) - (s \times 0) - 0$, using (1)

$\Rightarrow 120\, L\{t^3\} = s^3 L\{t^6\} \Rightarrow L\{t^6\} = (120/s^3) \times (6/s^4) = 720/s^7$, as $L\{t^3\} = 6/s^4$

EXERCISE 1 (E)

1. Using $L\{1\} = 1/s$, evaluate (*i*) $L\{t\}$ (*ii*) $L\{t^2\}$ **Ans.** (*i*) $1/s^2$ (*ii*) $2/s^3$

2. Given $L\{t^2\} = 2/s^3$, show that $L\{t^5\} = 120/s^6$

3. Show that (*i*) $L\{\sin at\} = \dfrac{a}{s^2 + a^2} \Rightarrow L\{\cos at\} = \dfrac{s}{s^2 + a^2}$

(*ii*) $L\{\cos at\} = \dfrac{s}{s^2 + a^2} \Rightarrow L\{\sin at\} = \dfrac{a}{s^2 + a^2}$

(*iii*) $L\{\sinh at\} = \dfrac{a}{s^2 - a^2} \Rightarrow L\{\cosh at\} = \dfrac{s}{s^2 - a^2}$

(*iv*) $L\{\cosh at\} = \dfrac{s}{s^2 - a^2} \Rightarrow L\{\sinh at\} = \dfrac{a}{s^2 - a^2}$

1.16. Multiplication by positive integral powers of *t*. Derivatives of Laplace transform.

Theorem I. *If $F(t)$ is a function of class A and if $L\{F(t)\} = f(s)$, then $L\{tF(t)\} = -f'(s)$.*

[Rohilkhand 1995, Osmania 2004, Purvanchal 1996 M.S. Univ. T.N. 2007]

or

Let $F(t)$ be a real function such that

(*i*) *$F(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$.*

(*ii*) *There exists constants σ and m such that $|F(t)| \leq me^{\alpha t}$ for $t \geq 0$*

then $L\{tF(t)\} = -f'(s)$, *where* $L\{F(t)\} = f(s)$.

Proof. By definition, $$f(s) = \int_0^\infty e^{-st} F(t)\, dt. \quad \ldots (1)$$

Since F (t) belongs to class A, Leibnitz's rule *for differentiation under the sign of integral is justified and so from (1), we get

$$\frac{d}{ds} f(s) = \int_0^\infty \frac{d}{ds}\left\{e^{-st} F(t)\right\} dt = \int_0^\infty (-t) e^{-st} F(t)\, dt = -\int_0^\infty e^{-st} \{tF(t)\}\, dt.$$

$\therefore$ $f'(s) = -\text{L}\ \{t\ \text{F}\ (t)\}$ or $\text{L}\ \{t\ \text{F}\ (t)\} = -f'(s)$.

Theorem II: (General case). *If F (t) is a function of class A and if* $\text{L}\ \{\text{F}\ (t)\} = f(s)$,

then $$\text{L}\ \{t^n\ \text{F}\ (t)\} = (-1)^n \frac{d^n f(s)}{ds^n},\ n = 1, 2, 3, \ldots.$$

[Meerut 2002, Kanpur 1997, 2008, Rohilkhand 1994, Nagpur 2005]

Proof. We use the principle of mathematical induction to prove that

$$\text{L}\ \{t^n\ \text{F}\ (t)\} = (-1)^n \frac{d^n f(s)}{d s^n}. \quad \ldots (1)$$

For $n = 1$, (1) reduces to $$\text{L}\ \{t\ \text{F}\ (t)\} = -\frac{d}{ds} f(s) = -f'(s),$$

which is true by theorem I. Hence (1) is true for $n = 1$

Assume that the result (1) is true for $n = k$ so that

$$\text{L}\ \{t^k\ \text{F}\ (t)\} = (-1)^k \frac{d^k f(s)}{d s^k}. \quad \ldots (2)$$

or $$\int_0^\infty e^{-st} \{t^k F(t)\} dt = (-1)^k \frac{d^k f(s)}{d s^k}. \quad \ldots (3)$$

Differentiating both sides of (3) w.r.t. 's' and applying the Leibnitz's rule* for differentiation under the integral sign, we get

$$\int_0^\infty \frac{\partial}{\partial s} \{e^{-st} t^k F(t)\} dt = (-1)^k \frac{d^{k+1} f(s)}{d s^{k+1}} \quad \text{or} \quad \int_0^\infty (-t) e^{-st} t^k F(t)\, dt = (-1)^k \frac{d^{k+1} f(s)}{d s^{k+1}}$$

or $$\int_0^\infty e^{-st} \{t^{k+1} F(t)\}\, dt = (-1)^{k+1} \frac{d^{k+1} f(s)}{d s^{k+1}}$$

or $$\text{L}\ \{\text{t}^{k+1}\ \text{F}\ (t)\} = (-1)^{k+1} \frac{d^{k+1} f(s)}{d s^{k+1}},\ \text{using definition of Laplace transform} \quad \ldots (4)$$

(2) and (4) show that the required result (1) is true for $n = k + 1$ whenever it is true for $n = k$. Hence, by the principle of mathematical induction, the required result (1) is true for all positive integers.

***Leibnitz's rule for differentiation under the integral sign.** Let a and b be constants.

Then $$f(s) = \int_a^b \text{F}(s,t) dt \Rightarrow \frac{d}{ds} f(s) = \int_a^b \frac{\partial}{\partial s} \text{F}(s,t) dt.$$

1.16A. Solved examples based on Art. 1.16

Ex.1. *Evaluate (i) L {t sin at} (ii) L {t cos at}* **[Kanpur 2004, Purvanchal 2002]**

Sol. (*i*) Since L {sin *at*} = $a/(s^2 + a^2)$, $s > 0$ hence

$$\text{L } \{t \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = -a\frac{d}{ds}(s^2+a^2)^{-1} = a(s^2+a^2)^{-2} \times 2s = 2as/(s^2+a^2)^2.$$

(*ii*) Since L {cos *at*} = $s/(s^2 + a^2)$, $(s > 0)$, hence

$$\text{L } \{t \cos at\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\frac{(s^2+a^2)\times 1 - s\times 2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Ex.2. Evaluate (*i*) *L {t sinh at}* **[Meerut 2006]** *(ii) L {t cosh at}.*

Sol. (*i*) Since L {sinh *at*} = $a/(s^2 - a^2)$, hence

$$\text{L } \{t \sinh at\} = -\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right) = -a\frac{d}{ds}(s^2-a^2)^{-1} = \frac{2as}{(s^2-a^2)^2}$$

(*ii*) Since L {cosh *at*} = $s/(s^2 - a^2)$, hence

$$\text{L } \{t \cosh at\} = -\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right) = -\frac{(s^2-a^2)\times 1 - s\times 2s}{(s^2-a^2)^2} = \frac{a^2+s^2}{(s^2-a^2)^2}.$$

Ex.3. *Evaluate L {t (3 sin 2t – 2 cos 2t)}.*

Sol. $\text{L } \{(3 \sin 2t - 2\cos 2t)\} = 3 \text{ L } \{\sin 2t\} - 2 \text{ L } \{\cos 2t\} = 3\left(\frac{2}{s^2+2^2}\right) - 2\left(\frac{s}{s^2+2^2}\right) = \frac{6-2s}{s^2+4}$

$$\therefore \text{ L } \{t(3\sin 2t - 2\cos 2t)\} = -\frac{d}{ds}\left(\frac{6-2s}{s^2+4}\right) = -\frac{(s^2+4)\times(-2) - (6-2s)\times 2s}{(s^2+4)^2} = \frac{8+12s-2s^2}{(s^2+4)^2}$$

Ex.4. (*a*) *Evaluate L {sin at – at cos at}*

(*b*) Prove that $\text{L}\{(\sin at - at\cos at)/2a^3\} = 1/(s^2 + a^2)^3$ **[KU Kurukshetra 2004]**

Sol. (*a*) We have, $\text{L } \{\sin at - at \cos at\} = \text{L } \{\sin at\} - a \text{ L } \{t \cos at\}$. ... (1)

Now, L {sin *at*} = $a/(s^2 + a^2)$. Again proceeding as in Ex.1, part (*ii*), we have

$$\text{L } \{t \cos at\} = (s^2 - a^2)/(s^2 + a^2)^2.$$

$$\therefore (1) \Rightarrow \quad \text{L } \{\sin at - at\cos at\} = \frac{a}{s^2+a^2} - a\times\frac{s^2-a^2}{(s^2+a^2)^2} = \frac{2a^3}{(s^2+a^2)^2}. \qquad ...(2)$$

(*b*) Re-writing, $(2) \Rightarrow \quad \frac{1}{2a^3}L\{(\sin at - at\cos at)\} = \frac{1}{(s^2+a^2)^3}$

or $\text{L } \{(\sin at - at\cos at)/2a^3\} = 1/(s^2 + a^2)^3$

Ex.5. *Evaluate (i) L {te^t}* **[Meerut 1995]** (*ii*) L $\{t^2 e^{2t}\}$ (*iii*) L $\{t^n e^{at}\}$. **[Purvanchal 2007]**

Sol. (*i*) Since L $\{e^t\} = 1/(s - 1)$, $s > 1$, hence

$$\text{L } \{te^t\} = -\frac{d}{ds}\frac{1}{s-1} = \frac{1}{(s-1)^2},\ s > 1$$

(*ii*) Since L $\{e^{2t}\} = 1/(s - 2)$, $s > 2$, hence

$$\text{L } \{t^2 e^{2t}\} = (-1)^2\frac{d^2}{ds^2}\frac{1}{s-2} = \frac{d}{ds}\left[\frac{d}{ds}(s-2)^{-1}\right] = \frac{d}{ds}\left[-(s-2)^{-2}\right] = (-1)(-2)(s-2)^{-3} = 2/(s-2)^3.$$

(*iii*) Since L $\{e^{at}\} = 1/(s-a)$, $s > a$, hence

$$\text{L}\{t^n e^{at}\} = (-1)^n \frac{d^n}{ds^n}\frac{1}{s-a} = (-1)^n \times \frac{(-1)^n n!}{(s-a)^{n+1}}, \text{ from Differential calculus}$$

$= n!/(s-a)^{n+1}$, $s > a$; noting that as $(-1)^{2n} = 1$.

Ex.6. *Evaluate (i) L $\{t^2 \sin at\}$* **[MDU Rohtak 2003]**

(ii) L $\{t^2 \sin 3t\}$

(iii) L $\{t^2 \cos at\}$ **[Osmania 2010, Punjab 2005, Agra 2012]**

Sol. (*i*) Since L $\{\sin at\} = a/(s^2 + a^2)$, hence

$$\text{L}\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = a\frac{d}{ds}\left[\frac{d}{ds}(s^2+a^2)^{-1}\right] = a\frac{d}{ds}\left[-(s^2+a^2)^{-2}\times 2s\right]$$

$$= -2a\frac{d}{ds}\frac{s}{(s^2+a^2)^2} = -2a\frac{(s^2+a^2)^2\times 1 - s\times 2(s^2+a^2)\times 2s}{(s^2+a^2)^4}$$

$$= -2a\frac{s^2+a^2-4s^2}{(s^2+a^2)^3} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}.$$

(*ii*) Proceed as in part (*i*): taking $a = 3$. **Ans.** $6(3s^2-9)/(s^2+9)^3$

(*iii*) Since L $\{\cos at\} = s/(s^2 + a^2)$, hence

$$\text{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right] = \frac{d}{ds}\left[\frac{(s^2+a^2)\times 1 - s\times 2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\frac{a^2-s^2}{(s^2+a^2)^2} = \frac{(s^2+a^2)^2\times(-2s) - (a^2-s^2)\times 2(s^2+a^2)\times 2s}{(s^2+a^2)^4}$$

$$= \frac{-2s(s^2+a^2) - 4s(a^2-s^2)}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}.$$

Ex.7. Show L $\{t^3 \cos t\} = (6s^4 - 36s^2 + 6)/(s^2+1)^4$.

Sol. Since L $\{\cos t\} = s/(s^2+1)$, $s > 0$, hence

$$\text{L}\{t^3 \cos t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right) = -\frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right] = -\frac{d^2}{ds^2}\left[\frac{(s^2+1)-s\times 2s}{(s^2+1)^2}\right]$$

$$= \frac{d^2}{ds^2}\frac{s^2-1}{(s^2+1)^2} = \frac{d}{ds}\left[\frac{(s^2+1)^2\times 2s - (s^2-1)\times 2(s^2+1)\times 2s}{(s^2+1)^4}\right] = \frac{d}{ds}\frac{6s-2s^3}{(s^2+1)^3}$$

$$= \frac{(6-6s^2)(s^2+1)^3 - 3(s^2+1)^2\times 2s\times(6s-2s^3)}{(s^2+1)^6} = \frac{6(s^4-6s^2+1)}{(s^2+1)^4}$$

Ex.8. *Evaluate L $\{(t^2 - 3t + 2)\sin 3t\}$.*

Sol. Here L $\{(t^2-3t+2)\sin 3t\}$ = L $\{t^2 \sin 3t\}$ − 3 L $\{t \sin 3t\}$ + 2 L $\{\sin 3t\}$ ... (1)

Since L $\{\sin 3t\} = 3/(s^2+3^2) = 3/(s^2+9)$, hence

$$\text{L}\{t \sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -3\frac{d}{ds}(s^2+9)^{-1} = 3(s^2+9)^{-2}\times 2s = \frac{6s}{(s^2+9)^2}.$$

and $\mathrm{L}\{t^2 \sin 3t\} = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{3}{s^2+9}\right) = 3\dfrac{d}{ds}\left[-(s^2+9)^{-2}\times 2s\right] = -6\dfrac{d}{ds}\dfrac{s}{(s^2+9)^2}$

$$= -6\,\frac{(s^2+9)^2\times 1 - s\times 2(s^2+9)\times 2s}{(s^2+9)^4} = -6\frac{s^2+9-4s^2}{(s^2+9)^3} = \frac{18s^2-54}{(s^2+9)^3}.$$

Substituting the above values in (1), we have

$$\mathrm{L}\{(t^2-3t+2)\sin 3t\} = \frac{18s^2-54}{(s^2+9)^3} - 3\left\{\frac{6s}{(s^2+9)^2}\right\} + 2\left(\frac{3}{s^2+9}\right)$$

$$= \frac{18s^2-54-18s(s^2+9)+6(s^2+9)^2}{(s^2+9)^3} = \frac{6s^4-18s^3+126s^2-16\,2s+432}{(s^2+9)^3}$$

Ex.9. *Evaluate (i) L {t cos² t}* **[Indore 2004, Jiwaji 2004]**

(ii) L {t sin² t}. **[Bhopal 2010; Mysore 1997]**

Sol. (*i*) Here $\mathrm{L}\{\cos^2 t\} = \mathrm{L}\{(1/2)\times(1+\cos 2t)\} = (1/2)\times[\mathrm{L}\{1\} + \mathrm{L}\{\cos 2t\}]$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+2^2}\right] = \frac{s^2+2}{s(s^2+4)}, s>0$$

$$\therefore \quad \mathrm{L}\{t\cos^2 t\} = -\frac{d}{ds}\left(\frac{s^2+2}{s^3+4s}\right) = -\frac{(s^3+4s)\times 2s - (s^2+2)\times(3s^2+4)}{(s^3+4s)^2} = \frac{s^4+2s^2+8}{s^2(s^2+4)^2}$$

(*ii*) Proceed as in part (*i*). **Ans.** $2(3s^2+4)/s^2(s^2+4)^2$

Ex.10. *Find the Laplace transform of t e^{-t} sin t* **[Kerala 2001]**

Sol. We have $\mathrm{L}\{\sin t\} = 1/(s^2+1) = f(s)$, say

Hence by first shifting theorem, we have

$\mathrm{L}\{e^{-t}\sin t\} = f(s+1) = 1/\{(s+1)^2+1\} = 1/(s^2+2s+2)$

$$\therefore \mathrm{L}\{t\,e^{-t}\sin t\} = -\frac{d}{ds}\frac{1}{s^2+2s+2} = \frac{2(s+1)}{(s^2+2s+2)^2}$$

EXERCISE 1 (F)

1. *Find the Laplace transform of the following functions for* $t \ge 0$

(*i*) te^{at} (*ii*) $at\cos 3t$ (*iii*) $t\cosh 3t$ (*iv*) $2t\sinh 2t$

Ans. (*i*) $1/(s-a)^2$ (*ii*) $a(s^2-9)/(s^2+9)^2$ (*iii*) $(s^2+9)/(s^2-9)^2$ (*iv*) $8s/(s^2-4)^2$

2. *Find the Laplace transform of the following functions for* $t \ge 0$.

(*i*) $at^2 e^{3t}$ (*ii*) $at^2\sinh 3t$ (*iii*) $at^2\cosh 7t$ (*iv*) $t^2\cos 4t$

Ans. (*i*) $\dfrac{2a}{(s-3)^3}$ (*ii*) $\dfrac{18a(s^2+3)}{(s^2-9)^3}$ (*iii*) $\dfrac{2as(s^2+147)}{(s^2-49)^2}$ (*iv*) $\dfrac{2s(s^2-48)}{(s^2+16)^3}$

3. *Find the Laplace transform of the following functions for* $t \ge 0$

(*i*) $t^3 e^{at}$ (*ii*) $t^3\sin 4t$ (*iii*) $t^3\sinh 4t$ (*iv*) $t^3\cosh(t/2)$

Ans. (*i*) $\dfrac{6}{(s-a)^4}$ (*ii*) $\dfrac{96s(s^2-16)}{(s^2+16)^4}$ (*iii*) $\dfrac{96s(s^2+16)}{(s^2-16)^4}$ (*iv*) $\dfrac{96(16s^4+24s^2+1)}{(4s^2-1)^4}$

4. *Find the Laplace transform of the following functions for* $t \geq 0$

(*i*) $t\,e^{3t} \sinh 3t$ (*ii*) $3e^{2t} \cosh 7t$

(*iii*) $t\,e^{at} \sin at$ **(Kanpur 2011)** (*iv*) $t\,e^{-t} \sin 3t$ **[Kurukshetra 2005]**

Ans. (*i*) $\dfrac{6(s-3)}{(s^2-6s)^2}$ (*ii*) $\dfrac{s^2-4s+53}{(s^2-4s-45)^2}$ (*iii*) $\dfrac{2a(s-a)}{(s^2-2as+2a^2)^2}$ (*iv*) $\dfrac{6(s+1)}{(s^2+2s+10)^2}$

5. *Prove that* $L\{(\sin at + at \cos at)/2a\} = s^2/(s^2 + a^2)^2$

1.17. Division by t. Theorem. *If* $L\{F(t)\} = f(s)$, *then* $L\left\{\dfrac{F(t)}{t}\right\} = \int_s^\infty f(s)\,ds$, *provided the integral exists.* **[Kanpur 2004]**

or

Let $F(t)$ *be a function of* t *such that*

(*i*) $F(t)$ *is piecewise continuous on every finite interval in the range* $t \geq 0$

(*ii*) *there exists constants* σ *and* m *such that* $|F(t)| \leq me^{\sigma t}$ *for* $t \geq 0$

(*iii*) $\lim\limits_{t \to o+} \{F(t)/t\}$ exists finitely,

then $L\left\{\dfrac{F(t)}{t}\right\} = \int_s^\infty f(s)\,ds, s > \sigma$; where $L\{F(t)\} = f(s)$

Proof. By definition, $$f(s) = \int_0^\infty e^{-st} F(t)\,dt. \quad \ldots (1)$$

Integrating (1) w.r.t. 's' from $s = s$ to $s = \infty$, we get

$$\int_s^\infty f(s)\,ds = \int_s^\infty \left\{\int_0^\infty e^{-st} F(t)\,dt\right\} ds. \quad \ldots (2)$$

Since s and t are independent variables, the order of integration in the double integral of R.H.S. of (2) can be interchanged. Hence, (2), may be re-written as

$$\therefore \quad \int_s^\infty f(s)ds = \int_0^\infty \left\{\int_s^\infty e^{-st} ds\right\} F(t)dt = \int_0^\infty \left[-\frac{e^{-st}}{t}\right]_s^\infty F(t)dt = \int_0^\infty e^{-st}\,\frac{F(t)}{t}\,dt$$

$$\therefore \quad \int_s^\infty f(s)ds = L\left\{\frac{F(t)}{t}\right\}, \text{ by definition of Laplace transform.} \quad \ldots (3)$$

Corollary. If $L\{F(t)\} = f(s)$, then $\int_0^\infty \dfrac{F(t)}{t}dt = \int_0^\infty f(s)\,ds$, *provided that the integral converges.*

Proof. From (3), we have

$$L\left\{\frac{F(t)}{t}\right\} = \int_0^\infty e^{-st}\,\frac{F(t)}{t}\,dt = \int_s^\infty f(s)\,ds = \int_s^\infty f(x)\,dx$$

or $$\int_0^\infty e^{-st}\,\frac{F(t)}{t}\,dt = \int_s^0 f(x)\,dx + \int_0^\infty f(x)\,dx = -\int_0^s f(x)\,dx + \int_0^\infty f(x)\,dx \quad \ldots (4)$$

Taking limit of both sides of (4) as $s \to 0+$ and assuming that the integral converges, we get

$$\int_0^\infty \frac{F(t)}{t}dt = \int_0^\infty f(x)\,dx = \int_0^\infty f(s)\,ds.$$

ILLUSTRATIVE EXAMPLE. Show that $\int_0^\infty \frac{F(t)}{t}dt = \int_0^\infty f(x)\,dx$, assuming that the integrals converge and $L\{F(t)\} = f(s)$ and hence *prove that* $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$. **[Purvanchal 1996]**

Sol. Use above corollary to prove that $\int_0^\infty \frac{F(t)}{t}dt = \int_0^\infty f(x)\,dx$ ... (1)

Let F $(t) = \sin t$ so that $f(s) = L\{\sin t\} = 1/(s^2 + 1)$.

$\therefore$ (1) gives $\int_0^\infty \frac{\sin t}{t}dt = \int_0^\infty \frac{1}{x^2+1}dx = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2}$.

1.17A. Solved examples based on Art. 1.17

Ex.1 (*a*). *Evalute L {(sin 2t)/t}* **[S.V. University (A. P.), 1997]**

Sol. Here $L\{\sin 2t\} = 2/(s^2 + 2^2) = f(s)$, say.

$$\therefore L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty f(s)\,ds = \int_s^\infty \frac{2}{s^2+2^2}ds = 2\times\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}\frac{s}{2} = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}(s/2), \text{ as } \tan^{-1} x + \cot^{-1} x = \pi/2$$

Ex.1 (*b*). *Prove L {(sin t)/t} = cot^{-1} s.* **[Guwahati 2004, Kumaun 2001]**

Hint: Proceed as in Ex. 1 (*a*)

Ex.2. *Find the Laplace transform of (sin at)/t. Does the Laplace transform of (cos at)/t exist?* **[Purvanchal 2003, Agra 1995, Kanpur 1994,Vikram 2004, Meerut 1996]**

Sol. Here $L\{\sin at\} = a/(s^2 + a^2) = f(s)$, say ... (1)

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty f(s)ds = \int_s^\infty \frac{a}{s^2+a^2}ds, \text{ using (1)}$$

$$= a\times\frac{1}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}\frac{s}{a} = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

Again, $L\{\cos at\} = s/(s^2 + a^2) = f(s)$, say

$$\therefore L\left\{\frac{\cos at}{t}\right\} = \int_s^\infty f(s)\,ds = \int_s^\infty \frac{s}{s^2+a^2}ds = \left[\frac{1}{2}\log(s^2+a^2)\right]_s^\infty = \frac{1}{2}\lim_{s\to\infty}\log(s^2+a^2) - \frac{1}{2}\log(s^2+a^2)$$

which does not exist since $\lim_{s\to\infty}\log(s^2+a^2)$ is infinite. Hence L {(cos *at*)/*t*} does not exist.

Ex. 3 (*a*). *Prove that* $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$. **[Meerut 1996, Purvanchal 2004]**

(*b*) $L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \frac{1}{2}\log\frac{s^2+9}{s^2+4}$ **[Kerala 2001]**

Sol. (*a*) Here $L\{\cos at - \cos bt\} = L\{\cos at\} - L\{\cos bt\}$.

$$\therefore \quad L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = f(s), \text{ say} \qquad \text{... (1)}$$

$$\therefore \text{ L}\left\{\frac{\cos at-\cos bt}{t}\right\}=\int_s^\infty f(s)ds = \int_s^\infty\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]ds, \text{ using (1)}$$

$$=\left[\frac{1}{2}\log(s^2+a^2)-\frac{1}{2}\log(s^2+b^2)\right]_s^\infty=\frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty=\frac{1}{2}\lim_{s\to\infty}\log\frac{s^2+a^2}{s^2+b^2}-\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$=\frac{1}{2}\lim_{s\to\infty}\log\frac{1+(a^2/s^2)}{1+(b^2/s^2)}+\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}=0+\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}=\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

(*b*) Here $a = 2$, $b = 3$. Do yourself

Ex.4. *Evaluate L {(sinh t)/t}*

Sol. Here $\text{L}\{\sinh t\} = 1/(s^2 - 1) = f(s)$, say.

$$\therefore \text{ L}\left\{\frac{\sinh t}{t}\right\}=\int_s^\infty f(s)\,ds=\int_s^\infty\frac{1}{s^2-1}ds=\left[\frac{1}{2}\log\frac{s-1}{s+1}\right]_s^\infty$$

$$=\frac{1}{2}\lim_{s\to\infty}\log\frac{s-1}{s+1}-\frac{1}{2}\log\frac{s-1}{s+1}=\frac{1}{2}\lim_{s\to\infty}\log\frac{1-(1/s)}{1+(1/s)}+\frac{1}{2}\log\frac{s+1}{s-1}=\frac{1}{2}\log\frac{s+1}{s-1}$$

Ex.5. *Show that* $L[(1 - e^t)/t] = \log\{(s - 1)/s\}$

Sol. Here $\text{L}\{1-e^t\} = \text{L}\{1\} - \text{L}\{e^t\} = \frac{1}{s}-\frac{1}{s-1} = f(s)$, say,

$$\therefore \text{ L}\left\{\frac{1-e^t}{t}\right\}=\int_s^\infty f(s)\,ds=\int_s^\infty\left[\frac{1}{s}-\frac{1}{s-1}\right]ds=\left[\log s-\log(s-1)\right]_s^\infty=\left[\log\frac{s}{s-1}\right]_s^\infty$$

$$=\lim_{s\to\infty}\log\frac{s}{s-1}-\log\frac{s}{s-1}=\lim_{s\to\infty}\log\frac{1}{1-(1/s)}+\log\frac{s-1}{s}=0+\log\frac{s-1}{s}=\log\frac{s-1}{s}.$$

Ex.6. *Find L {(2/t) × (1 – cosh 2t)}.* **[Lucknow 1998]**

Sol. Here $\text{L}(1 - \cosh 2t) = \text{L}\{1\} - \text{L}\{\cosh 2t\} = \frac{1}{s}-\frac{s}{s^2-2^2}=-\frac{4}{s^2-4}=f(s),\text{say}$ ... (1)

$$\therefore \text{ L}\{(2/t)\times(1-\cosh 2t)\}=2\text{ L}\left\{\frac{1-\cosh 2t}{t}\right\}=2\int_s^\infty f(s)\,ds=2\int_s^\infty\left(-\frac{4}{s^2-4}\right)ds\text{, by (1)}$$

$$=-8\times\frac{1}{(2\times2)}\left[\log\frac{s-2}{s+2}\right]_s^\infty=-2\left[\lim_{s\to\infty}\log\frac{s-2}{s+2}-\log\frac{s-2}{s+2}\right]$$

$$=-2\left[\lim_{s\to\infty}\log\frac{1-(2/s)}{1+(2/s)}-\log\frac{s-2}{s+2}\right]=2\log\frac{s-2}{s+2}$$

Ex.7 (*a*) *Prove* $\text{L}\left\{\frac{\sin^2 t}{t}\right\}=\frac{1}{4}\log\frac{s^2+4}{s^2}$.

[Agra 2009; Kumaun 1999, Meerut 2006, M.S. Univ. T.N. 2007]

Sol. We have, $\mathrm{L}\{\sin^2 t\} = \mathrm{L}\{(1/2) \times (1 - \cos 2t)\} = (1/2) \times [\mathrm{L}\{1\} - \mathrm{L}\{\cos 2t\}]$

or $$\mathrm{L}\{\sin^2 t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2 + 2^2}\right] = f(s), \text{ say} \qquad \ldots (1)$$

$\therefore$ $$\mathrm{L}\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty f(s)\, ds = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds, \text{ using (1)}$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log (s^2+4)\right]_s^\infty = \frac{1}{4}\left[\log \frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\lim_{s\to\infty} \log \frac{s^2}{s^2+4} - \frac{1}{4}\log \frac{s^2}{s^2+4} = \frac{1}{4}\lim_{s\to\infty} \log \frac{1}{1+(4/s^2)} + \frac{1}{4}\log \frac{s^2+4}{s^2}$$

$= 0 + (1/4) \times \log\{(s^2 + 4)/s^2\} = (1/4) \times \log\{(s^2 + 4)/s^2\}$.

Ex.7 (*b*). *Prove that* $\int_0^\infty \frac{\sin^2 t}{t^2} = \frac{\pi}{2}$.

Sol. Let $\mathrm{F}(t) = (\sin^2 t)/t$. Then proceed as in Ex. 7 (*a*) to obtain

$$\mathrm{L}\{F(t)\} = \mathrm{L}\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log \frac{s^2+4}{s^2} = f(s), \text{say} \qquad \ldots (1)$$

Then by Corollory of Art. 1.17, we have

$$\int_0^\infty \frac{\mathrm{F}(t)}{t} dt = \int_0^\infty f(s)\, ds \qquad \text{or} \qquad \int_0^\infty \frac{(\sin^2 t)/t}{t} dt = \frac{1}{4}\int_0^\infty \log \frac{s^2+4}{s^2} ds, \text{ by (1)} \qquad \ldots (2)$$

Let $$\mathrm{I} = \frac{1}{4}\int_0^\infty \log \frac{s^2+4}{s^2} ds. \qquad \ldots (3)$$

Put $s = 2 \tan \theta$ so that $ds = 2 \sec^2 \theta\, d\theta$. Then (3) gives

$$\mathrm{I} = \frac{1}{4}\int_0^{\pi/2} \log\left[\frac{4(\tan^2\theta + 1)}{4\tan^2\theta}\right] \times 2\sec^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \log\left(\frac{1}{\sin^2\theta}\right) \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[-\log(\sin\theta)^2\right] \sec^2\theta\, d\theta = -\int_0^{\pi/2} (\log \sin\theta) \sec^2\theta\, d\theta$$

$$= -\left[\left[(\log \sin\theta)(\tan\theta)\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{\sin\theta} \cos\theta \tan\theta\, d\theta\right]$$

[On integrating by parts taking $\sec^2\theta$ as the second function]

$\therefore$ $$\mathrm{I} = -\lim_{\theta\to\pi/2} (\log\sin\theta)(\tan\theta) + \lim_{\theta\to 0} (\log \sin\theta)(\tan\theta) + \int_0^{\pi/2} d\theta \qquad \ldots (4)$$

Now, $$\lim_{\theta\to\pi/2} (\log \sin\theta)(\tan\theta). \qquad [0 \times \infty]$$

$$= \lim_{\theta\to\pi/2} \frac{\log \sin\theta}{\cot\theta} \left[\frac{0}{0}\right] = \lim_{\theta\to\pi/2} \frac{(1/\sin\theta) \times \cos\theta}{-\mathrm{cosec}^2\theta} = 0, \text{ by L' Hospital's rule}$$

and $\lim_{\theta\to 0} (\log \sin\theta)(\tan\theta)\,[\infty \times 0]$

$$= \lim_{\theta\to 0} \frac{\log \sin\theta}{\cot\theta}\left[\frac{\infty}{\infty}\right] = \lim_{\theta\to 0} \frac{(1/\sin\theta)\times(\cos\theta)}{-\operatorname{cosec}^2\theta} = -\lim_{\theta\to 0} \sin\theta\cos\theta = 0.$$

Substituting the above values of limits in (4), we get

$$I = 0 + 0 + \int_0^{\pi/2} d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2} \qquad \ldots (5)$$

Then, (2), (3) and (5) $\Rightarrow$ $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = I = \frac{\pi}{2}$.

EXERCISE 1 (G)

1. *Find the Laplace transform of the following functions for* $t \geq 0$

(*i*) $(1 - e^{-t})/t$ **[Sagar 2004]** (*ii*) $(1 - e^{2t})/t$ **[Bangalore 2005]**

(*iii*) $(1 - \cos at)/t$ **[Indore 2004]** (*iv*) $2(1 - \cos at)/t$

(*v*) $2(e^t - \cos t)/t$ (*vi*) $(e^{-at} - e^{-bt})/t$

ANSWERS

(*i*) $\log\dfrac{s+1}{s}$ (*ii*) $\log\dfrac{s-2}{s}$ (*iii*) $\dfrac{1}{2}\log\dfrac{a^2+s^2}{s^2}$

(*iv*) $\log\dfrac{s^2+a^2}{s^2}$ (*v*) $\log\dfrac{s^2+1}{(s-1)^2}$ (*vi*) $\log\dfrac{s+b}{s+a}$

1.18. Laplace transforms of integrals.

Theorem. If $L\{F(t)\} = f(s)$, then $\displaystyle L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s}$.

[Lucknow 1997, Purvanchal 1995, Kanpur 1994, 95]

or

Let F(t) be a function of t such that (*i*) F(t) is piecewise continuous on every finite sub interval in the range $t \geq 0$ (*ii*) there exists contants σ and m such that $|F(t)| \leq me^{\alpha t}$ for $t \geq 0$, then the Laplace transform of $\int_0^t F(x)dx$ exist and is given by

$$L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s}, \text{ for } s > \sigma; \text{ where } L\{f(t)\} = f(s).$$

Sol. Let $G(t) = \displaystyle\int_0^t F(x)\,dx$. Then $G'(t) = F(t)$ and $G(0) = 0$ ... (1)

Now, $L\{G'(t)\} = s\,L\{G(t)\} - G(0) = s\,L\{G(t)\}$, using (1).

or $L\{F(t)\} = s\,L\{G(t)\}$ or $L\{G(t)\} = f(s)/s$

$[\because$ given that $L\{F(t)\} = f(s)]$

Thus, $\displaystyle L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s}$, using (1).

1.18A. Solved examples based on Art. 1.18

Ex.1. *Prove that* $L\left[\int_0^t \sin 2u\, du\right] = \dfrac{2}{s(s^2+4)}$. **[Bangalore 1997]**

Sol. Here $\quad L\{\sin 2t\} = 2/(s^2 + 2^2) = f(s)$, say $\quad \ldots (1)$

$\therefore \quad L\left[\int_0^t \sin 2u\, du\right] = \dfrac{f(s)}{s} = \dfrac{2}{s(s^2+4)}$, by (1)

Ex.2. *Evaluate* $L\left[\int_0^t \dfrac{\sin x}{x} dx\right]$.

[Kumaun 1997, Delhi 1997, K.V. Kurukshetra 2006; Purvanchal 2002]

Sol. Here $\quad L\{\sin t\} = 1/(s^2 + 1) = f(s)$, say. $\quad (1)$

$\therefore L\left\{\dfrac{\sin t}{t}\right\} = \int_s^\infty f(s)\, ds = \int_s^\infty \dfrac{1}{s^2+1} ds = \left[\tan^{-1} s\right]_s^\infty$, using (1)

$= \tan^{-1} \infty - \tan^{-1} s = (\pi/2) - \tan^{-1} s = \cot^{-1} s$, as $\tan^{-1} x + \cot^{-1} x = \pi/2$

$\therefore \quad L\left[\int_0^t \dfrac{\sin x}{x} dx\right] = \dfrac{\cot^{-1} s}{s}$.

Ex.3. If $L\{F(t), t \to s\} = f(s)$, show that $\quad L\left\{\int_0^t \dfrac{F(u)}{u} du, t \to s\right\} = \dfrac{1}{s}\int_s^\infty f(y)\, dy$.

Hence show that $\quad L\left\{\int_0^t \dfrac{\sin u}{u} du, t \to s\right\} = \dfrac{\cot^{-1} s}{s}$.

Sol. *Let G (t) = F (t)/t. Then by Art 1.17, we get*

$$L\{G(t)\} = L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\, ds = g(s), \text{ say}$$

where $\quad f(s) = L\{F(t)\}$.

$\therefore$ Using Art 1.18 for G (t), we get $\quad L\left\{\int_0^t G(u)\, du\right\} = \dfrac{g(s)}{s}$

or $\quad L\left\{\int_0^t \dfrac{F(u)}{u} du\right\} = \dfrac{1}{s}\int_s^\infty f(s)\, ds$, putting values of G ($u$) and g (s)

or $\quad L\left\{\int_0^t \dfrac{F(u)}{u} du,\ t \to s\right\} = \dfrac{1}{s}\int_s^\infty f(y)\, dy \quad \ldots (1)$

Second Part : Let $\quad F(t) = \sin t \quad$ so that $\quad f(s) = L\{\sin t\} = 1/(s^2 + 1)$. $\quad \ldots (2)$

$\therefore \ (1) \Rightarrow L\left\{\int_0^t \dfrac{\sin u}{u} du, t \to s\right\} = \dfrac{1}{s}\int_s^\infty \dfrac{dy}{y^2+1}$, using (2)

$= (1/s) \times \left[\tan^{-1} y\right]_s^\infty = (1/s) \times [(\pi/2) - \tan^{-1} s] = (1/s) \times \cot^{-1} s$.

Ex. 4. *Using* $L\{\sin bt\} = b/(s^2 + b^2)$, *find* $L\{\cos bt\}$. **[Kurukshetra 2004]**

Sol. Let $F(t) = \sin bt$. Then $\quad L\{F(t)\} = b/(s^2 + a^2) = f(s)$, say

Now, we have $L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s}$

$\therefore\ \mathrm{L}\left\{\int_0^t \sin bx\,dx\right\} = \frac{1}{s}\times\frac{b}{s^2+b^2} \quad \Rightarrow \quad L\left\{\left[-\frac{\cos bx}{b}\right]_0^t\right\} = \frac{b}{s(s^2+b^2)}$

or $\mathrm{L}\left\{\frac{1}{b}-\frac{\cos bt}{b}\right\} = \frac{b}{s(s^2+b^2)}$ or $\frac{1}{b}\mathrm{L}\{1\} - \frac{1}{b}L\{\cos bt\} = \frac{b}{s(s^2+b^2)}$

or $\frac{1}{b}\mathrm{L}\{\cos bt\} = \frac{1}{b}\times\frac{1}{s} - \frac{b}{s(s^2-b^2)} = \frac{s}{b(s^2+b^2)}$ or $\mathrm{L}\{\cos bt\} = \frac{s}{s^2+b^2}$

EXERCISE 1 (H)

1. Using $\mathrm{L}\{t\} = 1/s^2$, evaluate $\mathrm{L}\{t^2\}$ and $\mathrm{L}\{t^3\}$

2. Using $\mathrm{L}\{\cos at\} = s/(s^2 + a^2)$, evaluate $\mathrm{L}\{\sin at\}$

3. Using $\mathrm{L}\{\cosh at\} = s/(s^2 - a^2)$, evalaute $\mathrm{L}\{\sinh at\}$

4. Show that $\mathrm{L}\left\{\int_0^t e^{-x}\cos x\,dx\right\} = \frac{s+1}{s(s^2+2s+2)}$

5. Show that (*i*) $\mathrm{L}\left\{\int_0^t \frac{1-e^{-2x}}{x}dx\right\} = \frac{1}{s}\log\left(1+\frac{2}{s}\right)$ (*ii*) $\mathrm{L}\left\{\int_0^x \frac{e^x\sin x}{x}dx\right\} = \frac{\cot^{-1}(s-1)}{s}$

ANSWERS

1. $\mathrm{L}\{t^2\} = 2/s^3$, $\mathrm{L}\{t^3\} = 6/s^4$ **2.** $a/(s^2 + a^2)$ **3.** $a/(s^2 - a^2)$

1.19. Initial-value theorem and final-value theorem.

Theorem I, **Initial value theorem.** *Let F (t) be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if F′ (t) is of class A, then*

$$\lim_{t\to 0} F(t) = \lim_{s\to\infty} sL\{F(t)\}.$$

Proof. We know that $\mathrm{L}\{\mathrm{F}'(t)\} = s\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0)$

or $$\int_0^\infty e^{-st}\,\mathrm{F}'(t)\,dt = s\,\mathrm{L}\{F(t)\} - F(0). \quad \ldots(1)$$

Taking limit of both sides of (1) as $s \to \infty$, we get

$$\int_0^\infty \lim_{s\to\infty}[e^{-st}\,\mathrm{F}'(t)]dt = \lim_{s\to\infty}[sL\{F(t)\} - F(0)]. \quad \ldots(2)$$

Since F′(*t*) is sectionally continuous and of exponential order, we have

$$\int_0^\infty \lim_{s\to\infty}[e^{-st}F'(t)]\,dt = 0.$$

Then, (2) reduces to $0 = \lim_{s\to\infty} s\,\mathrm{L}\{F(t)\} - F(0).$ ... (3)

or $\lim_{t\to 0} F(t) = \lim_{s\to\infty} s\,\mathrm{L}\{F(t)\}$, since $\mathrm{F}(0) = \lim_{t\to 0}\mathrm{F}(t)$.

Theorem II. Final-Value Theorem. *Let F (t) be continuous for all $t \geq 0$ and be of exponential order and if F′ (t) is of class A, then*

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} s\,\mathrm{L}\{F(t)\}.$$ **[Osmania 2004]**

Proof. We know that $\mathrm{L}\{\mathrm{F}'(t)\} = s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0)$

or $$\int_0^\infty e^{-st} F'(t)\, dt = s\,\mathrm{L}\{F(t)\} - F(0). \quad \ldots (1)$$

Taking limit of both sides of (1) as $s \to 0$, we get

$$\lim_{s\to 0} \int_0^\infty e^{-st}\, \mathrm{F}'(t)\, dt = \lim_{s\to 0} [s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0)]$$

or $$\int_0^\infty \mathrm{F}'(t)\, dt = \lim_{s\to 0} s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0) \quad \text{or} \quad \big[\mathrm{F}(t)\big]_0^\infty = \lim_{s\to 0} s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0)$$

or $$\lim_{t\to\infty} \mathrm{F}(t) - \mathrm{F}(0) = \lim_{s\to 0} s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0) \quad \text{or} \quad \lim_{t\to\infty} \mathrm{F}(t) = \lim_{s\to 0} s\,\mathrm{L}\{\mathrm{F}(t)\}.$$

An illustrative example If $\mathrm{L}\{\mathrm{F}(t)\} = 1/s\,(s + a)$, find (*i*) $\lim_{t\to 0} \mathrm{F}(t)$ (*ii*) $\lim_{t\to\infty} \mathrm{F}(t)$

Sol. (*i*) Using initial value theorem, $\lim_{t\to 0} \mathrm{F}(t) = \lim_{s\to\infty} s\,\mathrm{L}\{\mathrm{F}(t)\} = \lim_{s\to\infty} 1/(s+a) = 0$

(*ii*) Using final value theorem, $\lim_{t\to\infty} \mathrm{F}(t) = \lim_{s\to 0} s\,\mathrm{L}\{\mathrm{F}(t)\} = \lim_{s\to 0} 1/(s+a) = 1/a$

1.20. Laplace transform of periodic functions. Theorem. *Let* F*(t) be a periodic function with period a. Then* $\mathrm{L}\{\mathrm{F}(t)\} = \dfrac{1}{1-e^{-sa}} \displaystyle\int_0^a e^{-st} F(t)\, dt.$ **[Kuvempa 2005, Ravishankar 2004]**

Proof. By definition of Laplace transform, we have

$$\mathrm{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^a e^{-st} F(t)\, dt + \int_a^\infty e^{-st} F(t)\, dt = \int_0^a e^{-st} F(t)\, dt + \int_0^\infty e^{-s(z+a)} F(z+a)\, dz$$

[In the second integral, put $t = z + a$ and $dt = dz$]

$$= \int_0^a e^{-st} F(t) dt + e^{-sa} \int_0^\infty e^{-st} F(t+a) dt = \int_0^a e^{-st} F(t) dt + e^{-sa} \int_0^\infty e^{-st} F(t) dt$$

[$F(t)$ is periodic with period $a \Rightarrow F(t + a) = F(t)$]

Thus, $\mathrm{L}\{F(t)\} = \displaystyle\int_0^a e^{-st} F(t) dt + e^{-sa} L\{F(t)\}$ or $(1 - e^{-sa})\,\mathrm{L}\{F(t)\} = \displaystyle\int_0^a e^{-st} F(t)\, dt$

or $$\mathrm{L}\{F(t)\} = \frac{1}{1-e^{-sa}} \int_0^a e^{-st} F(t) dt.$$

Note. A function $F(x)$, is said to be periodic if there exists a positive number T such that $F(x + T) = F(x)$ for all real numbers x and T. T is called period of $F(x)$.

1.20A. Solved examples based on Art. 1.20

Ex.1. *If* $F(t) = t^2$, $0 < t < 2$ *and* $F(t + 2) = F(t)$, *find* $\mathrm{L}\{F(t)\}$ **(Osmania 2010)**

Sol. $F(t + 2) = F(t) \Rightarrow F(t)$ is periodic with the period $= a = 2$.

$$\therefore \quad \mathrm{L}\{F(t)\} = \frac{1}{1-e^{-sa}} \int_0^a e^{-st} F(t)\, dt = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} t^2\, dt \quad \ldots (1)$$

Now, $\int_0^2 t^2 e^{-st}\,dt = \left[(t^2)\left(-\frac{1}{s}e^{-st}\right) - (2t)\left(\frac{1}{s^2}e^{-st}\right) + (2)\left(-\frac{1}{s^3}e^{-st}\right)\right]_0^2$

[Using chain rule of integration by parts]

$= (-4/s) \times e^{-2s} - (4/s^2) \times e^{-2s} - (2/s^3) \times e^{-2s} + (2/s^3) = \{2 - (4s^2 + 4s + 2)\, e^{-2s}\}/s^3.$

$\therefore$ From (1), $\quad L\{F(t)\} = \{2 - (4s^2 + 4s + 2)\, e^{-2s}\}/s^3 (1 - e^{-2s}).$

Ex.2. *Evaluate L {F(t)}, if F(t) has a period 2π and is given by*

$$F(t) = \begin{cases} \sin t, \text{ if } 0 < t < \pi \\ 0, \text{ if } \pi < t < 2\pi. \end{cases}$$

Sol. Since F (t) is periodic with the period = $a = 2\pi$ we have

$$L\{F(t)\} = \frac{1}{1 - e^{sa}} \int_0^a e^{-st} F(t)\, dt = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} F(t)\, dt \qquad \ldots (1)$$

Now, $\int_0^{2\pi} e^{-st} F(t)\,dt = \int_0^{\pi} e^{-st} F(t)\,dt + \int_{\pi}^{2\pi} e^{-st} F(t)\,dt = \int_0^{\pi} e^{-st} \sin t\, dt$, using the given value of $F(t)$

$$= \left[\frac{e^{-st}(-s\,\sin t - \cos t)}{s^2 + 1}\right]_0^{\pi} = \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} = \frac{1 + e^{-s\pi}}{1 + s^2}.$$

$$\therefore (1) \Rightarrow L\{F(t)\} = \frac{1 + e^{-s\pi}}{(1 + s^2)(1 - e^{-2\pi s})} = \frac{1 + e^{-s\pi}}{(1 + s^2)\{1 - (e^{-s\pi})^2\}} = \frac{1 + e^{-s\pi}}{(1 + s^2)(1 + e^{-s\pi})(1 - e^{-s\pi})}$$

$= 1/(1 + s^2)(1 - e^{-s\pi}).$

Ex.3. *Find the Laplace transform of a square wave' given by*

$$F(t) = \begin{cases} E, & \text{if } 0 \le t \le T/2 \\ -E, & \text{if } T/2 \le t < T \end{cases} \qquad \text{and} \qquad F(t + T) = F(t).$$

Sol. Since $F(t)$ is a periodic function with the period $a = T$, we have

$$L\{F(t)\} = \frac{1}{1 - e^{-sa}} \int_0^a e^{-st} F(t)\, dt = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} F(t)\, dt \qquad \ldots (1)$$

Now, $\int_0^T e^{-st} F(t)\,dt = \int_0^{T/2} e^{-st} F(t)\, dt + \int_{T/2}^T e^{-st} F(t)\,dt = \int_0^{T/2} e^{-st} E\, dt + \int_{T/2}^T e^{-st}(-E)\,dt$

[Using the given value of $F(t)$]

$= E\left[e^{-st}/(-s)\right]_0^{T/2} - E\left[e^{-st}/(-s)\right]_{T/2}^{T} = (E/s) \times (1 - e^{-sT/2}) - (E/s) \times (e^{-sT/2} - e^{-sT})$

$= (E/s) \times (1 - 2e^{-sT/2} + e^{-sT}) = (E/s) \times (1 - e^{-sT/2})^2$

Hence, (1) reduces to

$$L\{F(t)\} = \frac{E(1 - e^{-sT/2})^2}{s(1 - e^{-sT})} = \frac{E(1 - e^{-sT/2})^2}{s\{1 - (e^{-sT/2})^2\}} = \frac{E(1 - e^{-sT/2})^2}{s(1 - e^{-sT/2})(1 + e^{-sT/2})}$$

$$= \frac{E}{s} \times \frac{1 - e^{-sT/2}}{1 + e^{-sT/2}} = \frac{E}{s} \times \frac{e^{sT/4} - e^{-sT/4}}{e^{sT/4} + e^{-sT/4}} = \frac{E}{s} \tanh \frac{sT}{4}. \text{ s}$$

EXERCISE 1 (I)

1. Let F (t) be periodic with period 4, where F $(t) = \begin{cases} 3t, \text{ if } 0 < t < 2 \\ 6, \text{ if } 2 < t < 4, \end{cases}$

then prove that L {F (t)} = 3 $(1 - e^{-2s} - 2s\, e^{-4s})/s^2\,(1 - e^{-4s})$.

2. Find the Laplace transform of half-wave rectifier given by

$$F(t) = \begin{cases} \sin \omega t, \text{ if } 0 < t < \pi/\omega \\ 0, \text{ if } \pi/\omega < t < 2\pi/\omega \end{cases} \text{ and } F\left(t + \frac{2\pi}{\omega}\right) = F(t) \quad \textbf{[Madras 1997, Kuvempu 1997]}$$

Ans. L $\{f(t)\} = \omega/(1 - e^{-\pi s/\omega})\,(s^2 + \omega^2)$

3. Find the Laplace transform of the triangular wave of period $2a$, given by

$F(t) = t$, $0 < t < a$; $F(t) = 2a - t$, $a < t < 2a$. **[Delhi 1997, Madras 1997, Manglore 1997]**

Ans. $(1/s^2) \times \tanh(as/2)$.

1.21. Evaluation of integrals.

If L {F (t)} $= f(s)$, then $$\int_0^\infty e^{-st} F(t)\, dt = f(s). \quad \ldots (1)$$

Taking the limit of both sides of (1) as $s \to a$, we have

$$\int_0^\infty e^{-at} F(t)\, dt = f(a), \quad \ldots (2)$$

assuming the integral on R.H.S. of (2) to be convergent. The results (1) and (2) will be used in evaluation of various integrals as explained in the following solved examples.

1.21A. Solved examples based on Art. 1.21

Ex.1. *Evaluate* $\int_0^\infty t\, e^{-2t} \cos t\ dt$ **[Meerut 2000, 06]**

Sol. (*i*) Here L $\{\cos t\} = s/(s^2 + 1) = f(s)$, say. ... (1)

$$\therefore \quad \text{L } \{t \cos t\} = -\frac{d}{ds} f(s) = -\frac{d}{ds}\frac{s}{s^2+1} = -\frac{1 \times (s^2+1) - s \times 2s}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

or $$\int_0^\infty e^{-st} \{t \cos t\}\, dt = (s^2 - 1)/(s^2 + 1)^2 \text{ by definition of Laplace transform} \quad \ldots (2)$$

Taking limit of both sides of (2) as $s \to 2$, we get $$\int_0^\infty te^{-2t} \cos t\ dt = \frac{3}{25}$$

Ex.2. *Evaluate* $\int_0^\infty t^3 e^{-t} \sin t\ dt.$ **[Meerut 1997, Rohilkhand 1996, 97]**

Sol. Here L $\{\sin t\} = 1/(s^2 + 1) = f(s)$, say ... (1)

$$\therefore \quad \text{L } \{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3} f(s) = -\frac{d^3}{ds^3}(s^2+1)^{-1}, \text{ by (1)}$$

$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}(s^2+1)^{-1}\right] = -\frac{d^2}{ds^2}\left[-(s^2+1)^{-2} \times 2s\right] = 2\frac{d}{ds}\left[\frac{d}{ds}\frac{s}{(s^2+1)^2}\right] = 2\frac{d}{ds}\frac{1 \times (s^2+1)^2 - s \times 2(s^2+1) \times 2s}{(s^2+1)^4}$$

$$= 2\frac{d}{ds}\frac{1-3s^2}{(1+s^2)^3} = 2\frac{(-6s)\times(1+s^2)^3 - 3(1+s^2)^2\times 2s(1-3s^2)}{(1+s^2)^6} = -12s\frac{1+s^2+1-3s^2}{(1+s^2)^4} = \frac{24s(s^2-1)}{(1+s^2)^4}$$

or $\int_0^\infty e^{-st}\{t^3 \sin t\}\, dt = \frac{24s(s^2-1)}{(1+s^2)^4}$, by definition of Laplace transform ... (2)

Taking limit as $s \to 1$ on both sides of (2), we get $\int_0^\infty t^3 e^{-t} \sin t\ dt = 0.$

Ex.3. *Evaluate (i)* $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt$ *(ii)* $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}dt$ *(iii)* $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt.$

Sol. (*i*) Here $\mathrm{L}\{e^{-at}-e^{-bt}\} = \mathrm{L}\{e^{-at}\} - \mathrm{L}\{e^{-bt}\} = 1/(s+a) - 1(s+b) = f(s)$, say ... (1)

$$\therefore \quad \mathrm{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty f(s)\, ds = \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds, \text{ using (1)}$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty = \left[\log\frac{s+a}{s+b}\right]_s^\infty = \lim_{s\to\infty}\frac{s+a}{s+b} - \log\frac{s+a}{s+b} = \lim_{s\to\infty}\frac{1+(a/s)}{1+(b/s)} + \log\frac{s+b}{s+a} = \log\frac{s+b}{s+a}$$

or $\int_0^\infty e^{-st}\frac{e^{-at}-e^{-bt}}{t}dt = \log\frac{s+b}{s+a}$, by definition of Laplace transform ... (2)

Taking limit of both sides of (2) as $s \to 0$, we get $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt = \log\frac{b}{a}.$

(*ii*) Proceed as in part (*i*). Here $a = 3$, $b = 6$. **Ans.** log 2

(*iii*) Proceed as in part (*i*). Here $a = 1$, $b = 3$. **Ans.** log 3

Ex.4. *Show that (i)* $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}.$ **[Meerut 2002, Nagpur 2005, Purvanchal 2007]**

(*ii*) $\int_0^\infty e^{-t}\frac{\sin t}{t}dt = \frac{\pi}{4}$ **[Kurukshetra 2004, Kanpur 1997, Meerut 1998]**

Sol. Here $\mathrm{L}\{\sin t\} = 1/(s^2+1) = f(s)$, say. ... (1)

$$\therefore \quad \mathrm{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty f(s)\, ds = \int_s^\infty \frac{1}{s^2+1}ds\,, \text{ using (1)}$$

or $$\mathrm{L}\left\{\frac{\sin t}{t}\right\} = \left[\tan^{-1} s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s$$

or $$\int_0^\infty e^{-st}\frac{\sin t}{t}dt = \frac{\pi}{2} - \tan^{-1} s\,. \quad \text{... (2)}$$

Part (*i*) Taking limit of both sides of (2) as $s \to 0$, we get $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}.$

Part (*ii*) Taking limit of both sides of (2) as $s \to 1$, we get

$$\int_0^\infty e^{-t}\frac{\sin t}{t}dt = \frac{\pi}{2} - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Ex.5. *Prove that* $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt = \log\left(\frac{2}{3}\right)$.

Sol. Here L $\{\cos 6t - \cos 4t\}$ = L $\{\cos 6t\}$ – L$\{\cos 4t\}$= $s/(s^2 + 6^2) - s/(s^2 + 4^2) = f(s)$, say

$$\therefore \text{ L}\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \int_s^\infty f(s)\, ds = \int_s^\infty \left[\frac{s}{s^2+36} - \frac{s}{s^2+16}\right] ds$$

$$= \left[\frac{1}{2}\log(s^2+36) - \frac{1}{2}\log(s^2+16)\right]_s^\infty = \frac{1}{2}\left[\log\frac{s^2+36}{s^2+16}\right]_s^\infty$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{s^2+36}{s^2+16} - \frac{1}{2}\log\frac{s^2+36}{s^2+16} = \frac{1}{2}\lim_{s\to\infty}\frac{1+(36/s^2)}{1+(16/s^2)} + \frac{1}{2}\log\frac{s^2+16}{s^2+36}$$

or
$$\int_0^\infty e^{-st}\frac{\cos 6t - \cos 4t}{t} dt = \frac{1}{2}\log\frac{s^2+16}{s^2+36}, \qquad \ldots (1)$$

Taking limit of both sides of (1) as $s \to 0$, we get

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt = \frac{1}{2}\log\frac{16}{36} = \frac{1}{2}\log\left(\frac{2}{3}\right)^2 = \log\frac{2}{3}.$$

EXERCISE 1 (J)

Evaluate the following integrals:

$\int_0^\infty te^{-t}\sin t dt$

1. $\int_0^\infty e^{-3t}\sin t\, dt$ **[Purvanchal 2004]**

2. $\int_0^\infty e^{-2t}\cos 3t\, dt$ **[Purvanchal 2002]**

3. $\int_0^\infty t\, e^{-2t}\sin t\, dt$ **[KU Kurukshetra 2005]**

4. $\int_0^\infty t\, e^{-3t}\sin t\, dt$ **[Meerut 2000, Kanpur 2006]** **5.** $\int_0^\infty te^{-t}\sin t\, dt$ **[Madras 2012]**

Ans. 1. 1/10 **2.** 2/13 **3.** 4/25 **4.** 3/50 **5.** $1/2$

1.22. Laplace transforms of some special functions.

1. **The Error Function.** The error function, abbreviated as 'erf' or 'E' is defined as

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2} du$$

and erfc, read as 'the complement of error function' is defined as

$$\text{erfc}(t) = 1 - \text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-u^2} du.$$

To prove that L {erf ($\sqrt{t}$)} = 1/$s\sqrt{s+1}$ **[Kumaun 1996, Kanpur 1994]**

Proof. By definition, $\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\, du$

$$\therefore \ \operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \ldots\right) du$$

$$= \frac{2}{\sqrt{\pi}} \left[u - \frac{u^3}{3} + \frac{u^5}{5 \times 2!} - \frac{u^7}{7 \times 3!} + \ldots\right]_0^{\sqrt{t}} = \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5 \times 2!} - \frac{t^{7/2}}{7 \times 3!} + \ldots\right].$$

$$\therefore \ \mathrm{L}\{\operatorname{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}} \left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5 \times (2! s^{7/2})} - \frac{\Gamma(9/2)}{7 \times (3!\, s^{9/2})} + \ldots.\right\}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2} \times \frac{1}{s^{5/2}} + \frac{1 \times 3}{2 \times 4} \times \frac{1}{s^{7/2}} - \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \times \frac{1}{s^{9/2}} + \ldots$$

$$= \frac{1}{s^{3/2}} \left\{1 - \frac{1}{2} \times \frac{1}{s} + \frac{1 \times 3}{2 \times 4} \times \frac{1}{s^2} - \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \times \frac{1}{s^3} + \ldots.\right\}$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s^{3/2}} \left(\frac{s}{1+s}\right)^{1/2} = \frac{1}{s\sqrt{s+1}}.$$

1.22A. Solved examples based on error function

Ex.1. *Evaluate L {erf* $\sqrt{t}$ *} and hence prove that* $\mathrm{L}\{t \operatorname{erf}(2\sqrt{t})\} = (3s + 8)/s^2 (s + 4)^{3/2}$.

Sol. Proceed as in Art. 1.22 to show that $\mathrm{L}\{\operatorname{erf}\sqrt{t}\} = 1/s\sqrt{s+1}$. ... (1)

We know that $\mathrm{L}\{\mathrm{F}(t)\} = f(s) \Rightarrow \mathrm{L}\{\mathrm{F}(at)\} = (1/a) \times f(s/a)$. ... (2)

Now, $\mathrm{L}\{\operatorname{erf}(2\sqrt{t})\} = \mathrm{L}\{\operatorname{erf}\sqrt{4t}\} = \frac{1}{4} \times \frac{1}{(s/4) \times \sqrt{(s/4)+1}}$, using result (2)

or $$\mathrm{L}\{\operatorname{erf}(2\sqrt{t})\} = 2/s\sqrt{s+4} = \mathrm{g}(s), \text{ say} \quad \ldots (3)$$

$$\therefore \quad \mathrm{L}\{t \operatorname{erf}(2\sqrt{t})\} = -\frac{d}{ds} g(s) = -\frac{d}{ds} \frac{2}{s\sqrt{(s+4)}} = -2 \frac{d}{ds} \left[s^{-1}(s+4)^{-1/2}\right]$$

$$= -2 \left[-s^{-2}(s+4)^{-1/2} - \frac{1}{2}(s+4)^{-3/2} s^{-1}\right] = \frac{3s+8}{s^2(s+4)^{3/2}}$$

Ex.2. *Show that L {*e^{3t} *erf* $\sqrt{t}$ *}* $= 1/(s-3)\sqrt{s-2}$.

Sol. Proceed as in Art. 1.22 to show that $\mathrm{L}\{\operatorname{erf}\sqrt{t}\} = 1/s\sqrt{s+1}$. ... (1)

$\therefore$ by first shifting theorem we have

$$\mathrm{L}\{e^{3t} \operatorname{erf}\sqrt{t}\} = \frac{1}{(s-3)\sqrt{(s-3)+1}} = \frac{1}{(s-3)\sqrt{(s-2)}}.$$

II The sine integral. **[Kumaun 1997]**

The sine integral is denoted by $S_i(t)$ and is defined as $S_i(t) = \int_0^t \frac{\sin u}{u}\, du.$

To prove that $L\{S_i(t)\} = (1/s) \times \tan^{-1}(1/s).$

Proof. By definition, we have $S_i(t) = \int_0^t \frac{\sin u}{u} du$

$$\therefore \quad S_i(t) = \int_0^t \frac{1}{u}\left(u - \frac{u^3}{3!} + \frac{u^5}{3!} - \frac{u^7}{7!} + \ldots\right) du = \int_0^t \left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \frac{u^6}{7!} + \ldots\right)$$

$$= \left[u - \frac{u^3}{3\times 3!} + \frac{u^5}{5\times 5!} - \frac{u^7}{7\times 7!} + \ldots\right]_0^t = t - \frac{t^3}{3\times 3!} + \frac{t^5}{5\times 5!} - \frac{t^7}{7\times 7!} + \ldots.$$

$$\therefore \quad L\{S_i(t)\} = L\{t\} - \frac{1}{3\times 3!}L\{t^3\} + \frac{1}{5\times 5!}L\{t^5\} - \frac{1}{7\times 7!}L\{t^7\} + \ldots..$$

$$= \frac{1}{s^2} - \frac{1}{3\times 3!}\times\frac{3!}{s^4} + \frac{1}{5\times 5!}\times\frac{5!}{s^6} - \frac{1}{7\times 7!}\times\frac{7!}{s^8} + \ldots. = \frac{1}{s}\left[(1/s) - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \ldots..\right]$$

$= (1/s) \times \tan^{-1}(1/s)$, by Gregory's series.

III The cosine integral. The cosine integral is denoted by $C_i(t)$ and is defined as

$$C_i(t) = \int_t^\infty \frac{\cos u}{u}\, du.$$

To prove that $L\{C_i(t)\} = (1/2s) \times \log(s^2 + 1)$.

Proof. Let $F(t) = C_i(t) = \int_t^\infty \frac{\cos u}{u} du$ ... (*i*)

From (*i*), $F(t) = -\int_\infty^t \frac{\cos u}{u} du$ ... (*ii*)

From (*ii*), $F'(t) = -(\cos t)/t$ so that $t\,F'(t) = -\cos t.$

$$\therefore \quad L\{t\,F'(t)\} = -L\{\cos t\} \quad \text{or} \quad -\frac{d}{ds}L\{F'(t)\} = -\frac{s}{s^2+1}.$$

or $$\frac{d}{ds}[s\,f(s) - F(0)] = \frac{s}{s^2+1} \qquad [\text{Taking } f(s) = L\{F(t)\}]$$

or $$\frac{d}{ds}[s\,f(s)] = \frac{s}{s^2+1} \qquad [\because F(0) \text{ is constant}].$$

Integrating both sides w.r.t. '*s*', we get

$$s f(s) = \frac{1}{2}\int \frac{ds}{s^2+1} ds \quad \text{or} \quad s f(s) = \frac{1}{2}\log(s^2+1) + C. \quad \ldots (iii)$$

Now, by the final-value-theorem, we have $\lim_{s\to 0} s f(s) = \lim_{t\to\infty} F(t)$

Hence $\lim_{s\to 0}\left[\frac{1}{2}\log(s^2+1)+C\right]=\lim_{t\to\infty}\int_t^\infty \frac{\cos u}{u}du$, using (*i*) and (*iii*)

or $0 + C = 0$ so that $C = 0$. Then (*iii*) reduces to

$s f(s) = (1/2) \times \log (s^2 + 1)$ or $f(s) = (1/2s) \times \log (s^2 + 1)$

or $L\{F(t)\} = (1/2s) \times \log (s^2 + 1)$ or $L\{C_i(t)\} = (1/2s) \times \log (s^2 + 1)$, by (*i*)

IV The exponential integral. The exponential integral is denoted by $E_i(t)$ and is defined as

$$E_i(t) = \int_t^\infty \frac{e^{-u}}{u}du.$$

To prove that $\mathbf{L\{E_i(t)\} = (1/s) \times \log (1 + s).}$

Proof. Let $F(t) = E_i(t) = \int_t^\infty \frac{e^{-u}}{u}du.$ ... (*i*)

From (*i*), $F(t) = -\int_\infty^t \frac{e^{-u}}{u}du.$ ... (*ii*)

From (*ii*), $F'(t) = -(e^{-t}/t)$ so that $t F'(t) = -e^{-t}$.

$\therefore$ $L\{t F'(t)\} = -L\{e^{-t}\}$ or $-\frac{d}{ds}L\{F'(t)\} = -\frac{1}{s+1}$

or $\frac{d}{ds}[s f(s) - F(0)] = \frac{1}{s+1}$. [Taking $f(s) = L\{F(t)\}$]

or $\frac{d}{ds}[s f(s)] = \frac{1}{s+1}$ [$\because$ F (0) in constant]

Integrating, $s f(s) = \log (s + 1) + C$, C being an arbitrary constant ... (*iii*)

Now, by the Final-Value-theorem, $\lim_{s\to 0} s f(s) = \lim_{t\to\infty} F(t)$

Hence $\lim_{s\to 0}[\log(s+1)+C] = \lim_{t\to\infty}\int_t^\infty \frac{e^{-u}}{u}du$, using (*i*) and (*iii*)

Thus, $0 + C = 0$ or $C = 0$.

$\therefore$ from (*iii*), $s f(s) = \log (s + 1)$ or $f(s) = (1/s) \times \log (1 + s)$

or $L\{F(t)\} = (1/s) \times \log (1 + s)$ or $L\{E_i(t)\} = (1/s) \times \log (1 + s)$, by (*i*)

V Laguerre polynomial. Laguerre polynomial is defined by

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}t^n\right), \text{n} = 0, 1, 2, 3, \ldots\ldots$$

To prove that $\mathbf{L\{L_n(t)\} = (s-1)^n/s^{n+1}.}$

Proof. By definition, we have

$$L\{L_n(t)\} = \int_0^\infty e^{-st}\frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^n)\,dt = \frac{1}{n!}\int_0^\infty e^{-(s-1)t}\frac{d^n}{dt^n}(e^{-t}t^n)\,dt$$

$$= \frac{1}{n!}\left\{\left[e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)\right]_0^\infty - \int_0^\infty \frac{e^{-(s-1)t}}{-(s-1)}\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)\,dt\right\}$$

[on integrating by parts taking $d^n(e^{-t}t^n)/dt^n$ as second function]

$$= 0 + \frac{s-1}{n!}\int_0^\infty e^{-(s-1)}\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)\,dt, \text{ provided } s > 1$$

[Noting that each term in $\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)$ contains some integral power of t so that the first term vanishes in the limit as $t \to \infty$ and $t \to 0$]

$$= \frac{(s-1)^n}{n!}\int_0^\infty e^{-(s-1)t}(e^{-t}t^n)\,dt, \text{ after repeated use of integration by parts}$$

$$= \frac{(s-1)^n}{n!}\int_0^\infty e^{-st}t^n dt = \frac{(s-1)^n}{n!}L\{t^n\}, \text{ by definition}$$

$$= \frac{(s-1)^n}{n!}\times\frac{n!}{s^{n+1}} = \frac{(s-1)^n}{s^{n+1}}.$$

VI Dirac's delta function (or unit impulse function)

Consider a function $$\delta_\varepsilon(t) = \begin{cases} 0, \text{ if } t < 0 \\ 1/\varepsilon, \text{ if } 0 < t < \varepsilon \\ 0, \text{ if } t > \varepsilon \end{cases}$$

The Dirac's delta function is denoted by $\delta(t)$ and is defined as $\delta(t) = \lim_{\varepsilon\to 0}\delta_\varepsilon(t)$.

To prove that $$\mathrm{L}\{\delta(t)\} = 1.$$

Proof. We know that $$\delta(t) = \lim_{\varepsilon\to 0}\delta_\varepsilon(t). \quad \ldots(i)$$

where $$\delta_\varepsilon(t) = \begin{cases} 0, \text{ if } t < 0 \\ 1/\varepsilon, \text{ if } 0 < t < \varepsilon \\ 0, \text{ if } t > \varepsilon \end{cases} \quad \ldots(ii)$$

$$\therefore \mathrm{L}\{\delta_\varepsilon(t)\} = \int_0^\infty e^{-st}\,\delta_\varepsilon(t)\,dt = \int_0^\varepsilon e^{-st}\delta_\varepsilon(t)dt + \int_\varepsilon^\infty e^{-st}\delta_\varepsilon(t)dt = \int_0^\varepsilon e^{-st}\cdot\frac{1}{\varepsilon}dt, \text{ by } (ii)$$

$$= \frac{1}{\varepsilon}\left[-\frac{e^{-st}}{s}\right]_0^\varepsilon = \frac{1-e^{-s\varepsilon}}{s\varepsilon},$$

Thus, $$\int_0^\infty e^{-st}\,\delta_\varepsilon(t)\,dt = \frac{1}{s}\times\frac{1-e^{-s\varepsilon}}{\varepsilon} \quad \ldots(iii)$$

Taking limit as $\varepsilon \to 0$ on both sides of (*iii*), we get

$$\int_0^\infty e^{-st}\left\{\lim_{\varepsilon\to 0}\delta_\varepsilon(t)\right\}dt = \frac{1}{s}\lim_{\varepsilon\to 0}\frac{1-e^{-s\varepsilon}}{\varepsilon}$$

or $$\int_0^\infty e^{-st}\,\delta(t)\,dt = \frac{1}{s}\lim_{\varepsilon\to 0}\frac{se^{-s\varepsilon}}{1}, \text{ using (i) and L' Hospital's Rule}$$

Hence $\quad L\{\delta(t)\} = 1.$

VII The Bessel function. The Bessel function of order n is denoted by $J_n(x)$ and is defined by

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)}\left[1 - \frac{t^2}{2\times(2n+2)} + \frac{t^4}{2\times 4\times(2n+2)\,(2n+4)} - \ldots.\right]$$

or $$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}.$$

1.22 B. Solved examples based on Bessel function

Ex.1. *Prove that* $L\{J_0(t)\} = 1/(s^2+1)^{1/2}$ *and deduce that* **[Purvanchal 2003]**

(*i*) $L\{J_0(at)\} = 1/(s^2+a^2)^{1/2}$ **[Agra 2001, 02, Kanpur 2004, Rohilkhand 2001]**

(*ii*) $L\{t\,J_0(at)\} = s/(s^2+a^2)^{3/2}$ (*iii*) $L\{e^{-at}J_0(at)\} = 1/(s^2+2as+2a^2)^{1/2}$

(*iv*) $\int_0^{\infty} J_0(t)\,dt = 1$ (*v*) $\int_0^{\infty} te^{-3t}J_0(4t)\,dt = \frac{3}{125}$

Sol. We know that $$J_n(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}.$$

$$\therefore\quad J_0(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(r+1)}\left(\frac{t}{2}\right)^{2r} = \sum_{r=0}^{\infty}\frac{(-1)^r}{2^{2r}\cdot(r!)^2}t^{2r} = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \ldots.$$

$$\therefore\quad L\{J_0(t)\} = L\{1\} - \frac{1}{2^2}L\{t^2\} + \frac{1}{2^2\cdot 4^2}L\{t^4\} - \frac{1}{2^2\cdot 4^2\cdot 6^2}L\{t^6\} + \ldots$$

or $$L\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2}\cdot\frac{2!}{s^3} + \frac{1}{2^2\cdot 4^2}\cdot\frac{4!}{s^5} - \frac{1}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{6!}{s^7} + \ldots.$$

$$= \frac{1}{s}\left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{s^2}\right)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{s^2}\right)^3 + \ldots.\right\} = \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2} = \frac{1}{(1+s^2)^{1/2}}$$

Thus, $\quad L\{J_0(t)\} = 1/(1+s^2)^{1/2}$ ... (1)

Deduction (*i*): Using change of scale property, (1) gives

$$L\{J_0(at)\} = \frac{1}{a}\times\frac{1}{[1+(s/a)^2]^{1/2}} = \frac{1}{\sqrt{(s^2+a^2)}}.$$

Deduction (*ii*) : Using result of theorem I of Art. 1.16, we get

$$L\{t\,J_0(at)\} = -\frac{d}{ds}L\{J_0(at)\} = -\frac{d}{ds}\frac{1}{(s^2+a^2)^{1/2}}, \text{ by deduction } (i)$$

$$= s/(s^2+a^2)^{3/2}$$

Deduction (*iii*): Using first translation theorem and (1), we have

$$L\{e^{-at}J_0(at)\} = \frac{1}{[(s+a)^2+a^2]^{1/2}} = \frac{1}{(s^2+2sa+2a^2)^{1/2}}.$$

Deduction (*iv*): From (1), we get $$\int_0^\infty e^{-st} J_0(t)\, dt = \frac{1}{(1+s^2)^{1/2}}. \quad \text{... (2)}$$

Taking limit as $s \to 0$ in (2), we get $$\int_0^\infty J_0(t)\, dt = 1.$$

Deduction (*v*) : From deduction (*ii*), $$\int_0^\infty e^{-st}\, t\, J_0(at)\, dt = \frac{s}{(s^2+a^2)^{3/2}}. \quad \text{... (3)}$$

Putting $s = 3$ and $a = 4$, in (3), we get $$\int_0^\infty e^{-3t} t\, J_0(4t)\, dt = \frac{3}{(9+16)^{3/2}} = \frac{3}{125}.$$

Ex.2. *Prove that* (*i*) $\mathrm{L}\{\mathrm{J}_1(t)\} = 1 - \dfrac{s}{(s^2+1)^{1/2}}$ (*ii*) $\mathrm{L}\{t\,\mathrm{J}_1(t)\} = \dfrac{1}{(s^2+1)^{3/2}}$.

Sol. (*i*) we know that $$\mathrm{J}_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{t}{2}\right)^{n+2r}$$

$$\therefore \quad \mathrm{J}_1(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r+2)} \left(\frac{t}{2}\right)^{2r+1} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r+1)!} \left(\frac{t}{2}\right)^{2r+1} = \frac{t}{2} - \frac{1}{2^2}\cdot\frac{t^3}{4} + \frac{1}{2^2\cdot 4^2}\cdot\frac{t^5}{6} + \ldots$$

$$\therefore \mathrm{L}\{\mathrm{J}_1(t)\} = \frac{1}{2}\mathrm{L}\{t\} - \frac{1}{2^2\cdot 4}\mathrm{L}\{t^3\} + \frac{1}{2^2\cdot 4^2\cdot 6}\mathrm{L}\{t^5\} + \ldots. = \frac{1}{2}\cdot\frac{1!}{s^2} - \frac{1}{2^2\cdot 4}\cdot\frac{3!}{s^4} + \frac{1}{2^2\cdot 4^2\cdot 6}\cdot\frac{5!}{s^6} + \ldots.$$

$$= 1 - \left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{s^2}\right)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{s^2}\right)^3 + \ldots.\right\} = 1 - \left(1+\frac{1}{s^2}\right)^{-1/2} = 1 - \frac{s}{(s^2+1)^{1/2}}.$$

(*ii*) From part (*i*), $$\mathrm{L}\{\mathrm{J}_1(t)\} = 1 - \frac{s}{(1+s^2)^{1/2}}$$

$$\therefore \quad \mathrm{L}\{t\,\mathrm{J}_1(t)\} = -\frac{d}{ds}\mathrm{L}\{J_1(t)\} = -\frac{d}{ds}\left\{1 - \frac{s}{(1+s^2)^{1/2}}\right\} = \frac{1}{(s^2+1)^{3/2}}.$$

Ex.3. *Prove that* $L\{J_0(a\sqrt{t})\} = (1/s)\times e^{-(a^2/4s)}$

Sol. we know that $$\mathrm{J}_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \ldots\ldots \quad \text{...(1)}$$

Replacing t by a $\sqrt{t}$ in (*i*), $$\mathrm{J}_0(a\sqrt{t}) = 1 - \frac{a^2 t}{2^2} + \frac{a^4 t^2}{2^2\cdot 4^2} - \frac{a^6 t^3}{2^2\cdot 4^2\cdot 6^2} + \ldots..$$

$$\therefore \quad \mathrm{L}\{\mathrm{J}_0(a\sqrt{t})\} = \mathrm{L}\{1\} - \frac{a^2}{2^2}\mathrm{L}\{t\} + \frac{a^4}{2^2\cdot 4^2}\mathrm{L}\{t^2\} - \frac{a^6}{2^2\cdot 4^2\cdot 6^2}\mathrm{L}\{t^3\} + \ldots.$$

$$= \frac{1}{s} - \frac{a^2}{2^2}\cdot\frac{1!}{s^2} + \frac{a^4}{2^2\cdot 4^2}\cdot\frac{2!}{s^3} - \frac{a^6}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{3!}{s^4} + \ldots$$

$$= \frac{1}{s}\left[1 - \frac{(a^2/4s)}{1!} + \frac{(a^2/4s)^2}{2!} - \frac{(a^2/4s)^3}{3!} + \ldots..\right] = \frac{1}{s}e^{-(a^2/4s)}$$

1.23 Table of some Laplace Transform Theorems for ready reference

S.No.	Operation	$F(t)$	$L\{F(t)\}$ or $f(s)$ or $\bar{F}(s)$
1.	Linearity property	$c_1F_1(t) + c_2F_2(t)$	$c_1L\{F_1(t)\} + c_2L\{F_2(t)\}$
2.	First translation or shifting theorem	$e^{at}F(t)$	$f(s-a)$
3.	Second translation or shifting theorem	$G(t) = \begin{cases} F(t-a), t>0 \\ 0 \quad t<a \end{cases}$ or $F(t-a)\,H(t-a)$	$e^{-as}f(s)$
4.	Change of scale property	$F(at)$	$(1/a)\times f(s/a)$
5.	Differentiation theorem	$F'(t)$	$sL\{F(t)\} - F(0)$
		$F''(t)$	$s^2L\{F(t)\} - sF(0) - F'(0)$
		$F'''(t)$	$s^3L\{F(t)\} - s^2F(0) - sF'(0) - F''(0)$
6.	Multiplication theorem	$tF(t)$	$-f'(s)$
		$t^nF(t)$	$(-1)^n\frac{d^n}{ds^n}f(s)$
7.	Division theorem	$\frac{1}{t}F(t)$	$\int_s^\infty f(x)\,dx$
8.	Integral theorem	$\int_0^t F(x)\,dx$	$\frac{1}{s}f(s)$
9.	Initial value theorem	$\lim_{t\to 0}F(t) = \lim_{s\to\infty}sL\{F(t)\}$	
10.	Final Value theorem	$\lim_{t\to\infty}F(t) = \lim_{s\to 0}sL\{F(t)\}$	
11.	Fundamental theorem for a periodic function	$L\{F(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)dt$, where $F(t)$ is periodic function of period T *i.e.*,	$F(t+T) = F(t)$

OBJECTIVE PROBLEMS ON CHAPTER 1

Select (*a*), (*b*), (*c*) or (*d*) whichever is correct.

1. The value of $L\{t^2e^{-3t}\}$ is (*a*) $1/(s+3)^2$ (*b*) $(-2)/(s+3)^2$

(c) $2/(s+3)^3$ (d) $(-1)/(s+3)^2$ **[M.S. Univ. T.N. 2006]**

Sol. Ans. (c) Since $L\{t^2\} = 2/s^3$, so $L\{t^3e^{-3t}\} = 2/(s+3)^3$

2. $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$ is called (a) error function. (b) Dirac delta function

(c) Heaviside function (d) None of these **[Purvanchal 2003]**

Sol. Ans. (b) Refer Art. 1.12.

3. Let $F(t) = \begin{cases} t/a, & 0 < t < a \\ 1, & t > a \end{cases}$. Then $L\{F(t)\}$ equals

(a) $(1 - a^{-as})/as^3$ (b) $(1 + e^{-as})/as^3$ (c) e^{-as}/as^3 (d) e^{-as}/as^2

Sol. Ans. (a) Refer Ex. 15, Art. 1.10A

4. $L\{e^{at} \sinh bt\}$ equals (a) $b/\{(s-a)^2 + b^2\}$

(b) $a/\{(s-a)^2 - b^2\}$ (c) $b/\{(s-a)^2 - b^2\}$ (d) none of these

Ans. (c) Refer Ex. 2 (*iii*), Art. 1.11 A

5. $L\{(\sin 2t)/t\}$ equals (a) $\tan^{-1}(s/2)$ (b) $\cot^{-1}(s/2)$ (c) $\sin^{-1}(s/2)$ (d) $\cos^{-1}(s/2)$

Sol. Ans. (b). Refer Ex. 1, Art. 1.17 A

6. $L\{t^{1/2}\}$ is (a) $L^2/s^{3/2}$ (b) $\sqrt{\pi}/s^{3/2}$ (c) $\sqrt{\pi}/(2s^{3/2})$ (d) $1/(L^2s^{3/2})$

Sol. Ans. (c). Use result 2 of Art. 1.10 with $n = 1/2$ **[M.S. Univ. T.N. 2007]**

7. $L\{\cos(t/2)\}$ is (a) $4/(4s^2 - 1)$ (b) $4s/(4s^2 + 1)$ (c) $4s/(4s^2 - 1)$ (d) $4/(4s^2 + 1)$

Sol. Ans. (b). Use result 8 of Art. 1.10 with $a = 1/2$ **[M.S. Univ. T.N. 2007]**

8. $L\{1\}$ is (a) 0 (b) 1 (c) $1/s$ (d) s. **[M.S. Univ. T.N. 2007]**

Sol. Ans. (c) Refer result 1 of Art. 1.10.

9. The value of $L\{\cosh at\}$ is (a) $s/(s^2 + a^2)$ (b) $a/(s^2 + a^2)$ (c) $s/(s^2 - a^2)$ (d) $a/(s^2 - a^2)$.

Sol. Ans. (c). Use result 6 of Art. 1.10. **[M.S. Univ. T.N. 2007]**

10. If $F(t)$ is a function of class A and if $L\{F(t)\} = f(s)$ then $L\{t^3F(t)\}$ is equal to (a) $f'''(s)$ (b) $-f'''(s)$ (c) $\pm f'''(s)$ (d) None of these **[Kanpur 2004]**

Sol. Ans. (b). Refer theorem II, Art. 1.16.

11. If $L\{F(t)\} = f(p)$, then $L\{F(at)\}$ is

(a) $f(p/a)$ (b) $(1/a) \times f(p)$ (c) $(1/a) \times f(p/a)$ (d) None of these **(Agra 2008)**

Sol. Ans. (c). Refer Art. 1.14. Note that here we have 'p' in place of 's'.

12. Laplace transform of $t^{-1/2}$ is

(a) $\sqrt{\pi}/p$ (b) $\sqrt{p/\pi}$ (c) $\sqrt{\pi/p^3}$ (d) None of these **(Agra 2007)**

Sol. Ans. (a). Use formula 2 of table of Art. 1.10. Note that here we have 'p' in place or 's'.

MISCELLANEOUS PROBLEMS ON CHAPTER 1

Ex.1. *Prove that* $L\{t^2 e^t \sin 4t\} = 8(3s^2 - 6s - 13)/(s^2 - 2s + 17)^3$ **(Agra 2010)**

Sol. We have, $L\{\sin 4t\} = \dfrac{4}{s^2 + 16} \quad \Rightarrow \quad L\{e^t \sin 4t\} = \dfrac{4}{(s-1)^2 + 16} = \dfrac{4}{s^2 - 2s + 17}$

Hence, $L\{t^2 e^t \sin 4t\} = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{4}{s^2 - 2s + 17}\right) = 4\dfrac{d}{ds}\left[\dfrac{d}{ds}\dfrac{1}{s^2 - 2s + 17}\right]$ using theorem II, page 1.30

$$= -8\frac{d}{ds}\frac{s-1}{(s^2-2s+17)^2} = -8\times\frac{(s^2-2s+17)^2-4(s-1)^2\,(s^2-2s+17)}{(s^2-2s+17)^4}$$

$$= -8\times\frac{(s^2-2s+17)-4(s^2-2s+1)}{(s^2-2s+17)^3} = 8\times\frac{3s^2-6s-13}{(s^2-2s+17)^3}$$

Ex.2. Show that $L\left\{\sum_{n=0}^{\infty}u(t-n)\right\} = 1/\{s(1-e^{-s})\}$, where $L\{f(t)\}$ represents the Laplace transform of the function $f(t)$ and the function $u(t-a) = \begin{cases}0, \text{ for } t<n\\ 1, \text{for } t\geq n\end{cases}$ is the unit step function.

[Delhi B.Sc. (Hons.) II 2011]

Sol. Using the linearity property of Laplace transforms (see Art. 1.8), we have

$L\left(\sum_{n=0}^{\infty}u(t-n)\right) = \sum_{n=0}^{\infty}L\{u(t-n)\} = \sum_{n=0}^{\infty}(e^{-ns})/s$, using the note of Art 1.21.

$= (1/s)\times\{1+e^{-s}+(e^{-s})^2+....\} = (1/s)\times\{1/(1-e^{-s})\}$,

noting the fact that $1 + e^{-s}+\{e^{-s})^2+....$ is an infinite G.P. whose common ratio $= e^{-s} = 1/e^{s} < 1$.

Ex. 3. *Prove that* $L\{L\{F(t)\}\} = \int_0^{\infty}\frac{F(t)}{t+s}dt$ ***[Kanapur 2011]***

Ex. 4. If $L\{F(t)\} = f(s)$, prove that $L\{\int_0^t dt_1\int_0^{t_1}F(u)du\} = \frac{f(s)}{s^2}$

Ex.5. Find the Laplace transform of the function $Q(t.\ c)$ defined by

$$Q(t,c) = \begin{cases}1, \text{ if } 0<t<c\\ -1, \text{ if } c<t+<2c\end{cases} \text{ and } Q(t+2c,c) = Q(t,c)$$ **(Ranchi 2011)**

Hint: Proceed as in Ex.3, page 1.42. **Ans.** $\{(1-e^{-s})/s(1+e^{-s})\}$

2

The Inverse Laplace Transform

2.1. Introduction.

We have already said that the purpose of studying Laplace transform is to solve differential and integral equations. In next chapters, 3, 4 and 5, we shall observe that during the process of solving differential and integral equations we have to determine a function whose Laplace transform is already known. This is the reverse process of determining the Laplace transform. In this chapter, we propose to give methods of finding the function whose Laplace transform is already known.

2.2. Inverse Laplace Transform. [Osmania 2004, Purvanchal 1996]

Definition. If $f(s)$ be the Laplace transform of a function $F(t)$, *i.e.*, if $L\{F(t)\} = f(s)$, then $F(t)$ is called the *Inverse Laplace Transform* of the function $f(s)$ and is written as $F(t) = L^{-1}\{f(s)\}$, where L^{-1} is called the *Inverse Laplace Transformation operator.*

Remark. In view of the above definition, we note that $L\{F(t)\} = f(s) \Leftrightarrow L^{-1}\{f(s)\} = F(t)$.

For example, $$L\{e^{at}\} = \frac{1}{s-a} \Leftrightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$$

2.3. Null Function. [Osmania 2004]

Definition. If $N(t)$ is a function of t such that $\int_o^t N(t)\,dt = 0$, for all $t > 0$, then $N(t)$ is called a *Null Function.*

Example *The function* $$N(t) = \begin{cases} 1, & t = 1/2 \\ -1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$$

is a null function.

Also, we can show that $L\{N(t)\} = 0$

2.3A. Uniqueness of inverse Laplace Transforms. Lerch's Threorem

Since the Laplace Transform of a null function $N(t)$ is zero, hence

$$L\{f(t)\} = f(s) \quad \Rightarrow \quad L\{F(t) + N(t)\} = f(s),$$

showing that we can have two different functions with the same Laplace transform.

As an illustration, consider two functions

$$F_1(t) = e^{-3t} \quad \text{and} \quad F_2(t) = \begin{cases} 0, t = 1 \\ e^{-3t}, \text{otherwise} \end{cases}$$

Then we can easily show that both functions have the same Laplace transform, *i.e.* $1/(s+3)$.

Thus, if we allow null functions, then the inverse Laplace transform is not unique. Now null functions do not, in general, arise in problems of physical interest. Hence if null functions is not allowed, then the inverse Laplace transform of a function is unique. This result is stated in the following theorem.

Lerch's theorem. *If we restrict ourselves to functions F (t) which are sectionally continuous in every finite interval $0 \le t \le t_o$ and of exponential order for $t > t_o$, then the inverse of Laplace transform of f (s), i.e., $L^{-1}\{f(s)\} = F(t)$, is unique.*

While evaluating inverse Laplace transform of any function, we shall always assume such uniqueness unless otherwise stated.

2.4. Inverse Laplace Transform of Some Elementary Functions

The student in adviced to study Art. 1.9 of Chapter 1 before studying this article.

(i) *To find inverse Laplace transform of 1/s, s > 0.*

Proof. Since $L\{1\} = 1/s$, $s > 0$, *so* by definition $L^{-1}\{1/s\} = 1$, $s > 0$.

(ii) *To find inverse Laplace transform of $1/s^{n+1}$, n being any real number such that n > –1.*

Proof. Since $L\{t^n\} = \Gamma(n+1)/s^{n+1}$, $s > 0$, $n > -1$, so by definition

$$L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n \quad \text{or} \quad L^{-1}\left\{\frac{l}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}, n > -1, s > 0. \qquad \textbf{[Guwahati 2007]}$$

(iii) *To Find inverse Laplace tranform of $1/s^{n+1}$, n being a positive integer.*

Proof. Since $L\{t^n\} = n!/s^{n+1}$, $s>0$, so by definition

$$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \qquad \text{or} \qquad L^{-1}\left\{\frac{l}{s^{n+1}}\right\} = \frac{t^n}{n!}, s > 0.$$

(iv) *To find inverse Laplace transform of 1/(s – a), s > a.* **[Andhra 1990]**

Proof. Since $L\{e^{at}\} = 1/(s-a)$, $s > a$, so by definition

$$L^{-1}\left\{\frac{l}{s-a}\right\} = e^{at}, s > a$$

(v) *To find inverse Laplace transform of $1/(s^2 – a^2)$, s > |a|.*

Proof. Since L {sinh at} = $a/(s^2 - a^2)$, s > |a|, so by definition

$$L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at \qquad \text{or} \qquad L^{-1}\left\{\frac{l}{s^2-a^2}\right\} = \frac{\sinh at}{a}, s > |a|$$

(vi) *To find inverse Laplace transform of $s/(s^2 – a^2)$, s > |a|.*

Since L {cosh *at*} = $s/(s^2 - a^2)$, s > |a|, so by definition

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at, s > |a| \qquad \textbf{(Agra 2010)}$$

(vii) *To find inverse Laplace transform of $l/(s^2 + a^2)$, s > 0.* **[Meerut 2002]**

Proof. Since L{sin *at* } = $a/(s^2 + a^2)$, $s > 0$, so by definition

$$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at \qquad \text{or} \qquad L^{-1}\left\{\frac{l}{s^2+a^2}\right\} = \frac{\sin at}{a}, s > 0$$

(viii) *To find inverse Laplace transform of $s/(s^2 + a^2)$, s > 0.*

Proof. Since L{cos *at*} = $s/(s^2 + a^2)$, $s > 0$, so by definition

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at, s > 0$$

Remarks. Results of this article should be commited to memory. With these results, the inverse Laplace transform of all functions can be evaluated by using some general theorems, which will be presented at proper places in this chapter. For ready reference the results of this article are presented in the table given on next page.

S. No.	$f(s)$	$L^{-1}\{f(s)\} = F(t)$
1.	$1/s$	1
2.	$1/s^{n+1}$, $n > -1$, $s > 0$	$t^n/\Gamma(n+1)$
3.	$1/s^{n+1}$, n is positive integer, $s > 0$	$t^n/n!$
4.	$1/(s-a)$, $s > a$	e^{at}
5.	$\frac{1}{s^2-a^2}$, $s > \lvert a\rvert$	$\frac{\sinh at}{a}$
6.	$\frac{s}{s^2-a^2}$, $s > \lvert a\rvert$	$\cosh\ at$
7.	$\frac{1}{s^2+a^2}$, $s > 0$	$\frac{\sin at}{a}$
8.	$\frac{s}{s^2+a^2}$, $s > 0$	$\cos\ at$

2.5. Linearity property of inverse Laplace transform

Theorem. I*f f_1 (s) and f_2 (s) respectively be the Laplace transforms of functions F_1(t) and F_2 (t) respectively and C_1 and C_2* be two constants then

$$L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\} = C_1 F_1(t) + C_2 F_2(t)$$

i.e., $$L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\} = C_1 L^{-1}\{f_1(s)\} + C_2 L^{-1}\{f_2(s)\}$$

Proof. By linearity property of Laplace transforms, we have

$$L\{C_1 F_1(t) + C_2 F_2(t)\} = C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\} \quad \ldots (1)$$

But, given that $L\{F_1(t)\} = f_1(s)$ and $L\{F_2(t)\} = f_2(s)$ ... (2)

$\therefore$ From (1), we have $L\{C_1 F_1(t) + C_2 F_2(t)\} = C_1 f_1(s) + C_2 f_2(s)$

$\therefore$ By definition of inverse Laplace transform, we get

$$L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\} = C_1 F_1(t) + C_2 F_2(t) \quad \ldots (3)$$

Also from (2), $F_1(t) = L^{-1}\{f_1(s)\}$ and $F_2(t) = L^{-1}\{f_2(s)\}$

$\therefore$ (3) can be re-written as

$$L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\} = C_1 L^{-1}\{f_1(s)\} + C_2 L^{-1}\{f_2(s)\} \quad \ldots (4)$$

Relations (3) and (4) give the required results.

2.5A Solved examples based on Art. 2.4. and Art. 2.5.

Ex.1. *Evaluate the inverse Laplace transform of following functions:*

(*i*) $1/s^5$ (*ii*) $1/\sqrt{s}$ **(Agra 2007, Meerut 2009)** (*iii*) $1/(2s-5)$ **(Bhopal 2010)** (*iv*) $s/(s^2+16)$ **[Bangalore 2005]**

(*v*) $(2s-5)/(s^2-9)$ (*vi*) $(3s-16)/(s^2-16)$ **[M.S. Univ. T.N. 2007]**

Sol. (*i*) $L^{-1}\left\{\frac{1}{s^5}\right\} = L^{-1}\left\{\frac{1}{s^{4+1}}\right\} = \frac{t^4}{4!} = \frac{t^4}{24}$.

(*ii*) $L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{t^{(1/2)-1}}{\Gamma(1/2)}$, as $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$, $n > -1$ **[Meerut 2009]**

$$\therefore \qquad \mathrm{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1/\sqrt{t}}{\Gamma(1/2)} = \frac{1/\sqrt{t}}{\sqrt{\pi}} = \frac{1}{\sqrt{(\pi t)}}$$

(*iii*) $\mathrm{L}^{-1}\left\{\dfrac{1}{2s-5}\right\} = \mathrm{L}^{-1}\left\{\dfrac{1}{2(s-5/2)}\right\} = \dfrac{1}{2}\mathrm{L}^{-1}\left\{\dfrac{1}{s-5/2}\right\} = \dfrac{1}{2}e^{5t/2}$

(*iv*) $\mathrm{L}^{-1}\left\{\dfrac{s}{s^2+16}\right\} = L^{-1}\left\{\dfrac{s}{s^2+4^2}\right\} = \cos 4t$

(*v*) $\mathrm{L}^{-1}\left\{\dfrac{2s-5}{s^2-9}\right\} = \mathrm{L}^{-1}\left\{2\times\dfrac{s}{s^2-3^2} - 5\times\dfrac{1}{s^2-3^2}\right\} = 2\,\mathrm{L}^{-1}\left\{\dfrac{s}{s^2-3^2}\right\} - 5\,\mathrm{L}^{-1}\left\{\dfrac{1}{s^2-3^2}\right\}$

$= 2\cosh 3t - (5/3) \times \sinh 3t$

(*vi*) Proceed as in part (*v*). **Ans.** $3\cosh 4t - 4\sinh 4t$

Ex. 2. ***Find the Laplace transform of the following functions:***

(*i*) $\mathrm{L}^{-1}\left\{\dfrac{1}{s^{7/2}}\right\}$ (*ii*) $\mathrm{L}^{-1}\left\{\dfrac{3s-2}{s^{5/2}} - \dfrac{7}{3s+2}\right\}$ (*iii*) $\mathrm{L}^{-1}\left\{\dfrac{3s-8}{4s^2+25}\right\}$

Sol. (*i*) $\mathrm{L}^{-1}\left\{\dfrac{1}{s^{7/2}}\right\} = \dfrac{t^{(7/2)-1}}{\Gamma(7/2)} = \dfrac{t^{5/2}}{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \dfrac{8t^2}{15}\left(\dfrac{t}{\pi}\right)^{1/2}$

(*ii*) $\mathrm{L}^{-1}\left\{\dfrac{3s-2}{s^{5/2}} - \dfrac{7}{3s+2}\right\} = \mathrm{L}^{-1}\left\{3\times\dfrac{1}{s^{3/2}} - 2\times\dfrac{1}{s^{5/2}} + \dfrac{7}{3}\times\dfrac{1}{s+(2/3)}\right\}$

$= 3\times\dfrac{t^{1/2}}{\Gamma(3/2)} - 2\times\dfrac{t^{3/2}}{\Gamma(5/2)} - \dfrac{7}{3}e^{-(2t/3)}$ $= \dfrac{3\sqrt{t}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - 2\times\dfrac{t^{3/2}}{\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \dfrac{7}{3}e^{-(2t/3)}$

$= \dfrac{6\sqrt{t}}{\sqrt{\pi}} - \dfrac{8t\sqrt{t}}{3\sqrt{\pi}} - \dfrac{7}{3}e^{-(2t/3)} = 6\left(\dfrac{t}{\pi}\right)^{1/2} - \dfrac{8t}{3}\left(\dfrac{t}{\pi}\right)^{1/2} - \dfrac{7}{3}e^{-(2t/3)}$

(*iii*) $\mathrm{L}^{-1}\left\{\dfrac{3s-8}{4s^2+25}\right\} = \mathrm{L}^{-1}\left\{\dfrac{3}{4}\times\dfrac{s}{s^2+(5/2)^2} - 2\times\dfrac{1}{s^2+(5/2)^2}\right\}$

$= \dfrac{3}{4}\mathrm{L}^{-1}\left\{\dfrac{s}{s^2+(5/2)^2}\right\} - 2\,\mathrm{L}^{-1}\left\{\dfrac{1}{s^2+(5/2)^2}\right\} = \dfrac{3}{4}\cos\dfrac{5t}{2} - 2\times\dfrac{1}{(5/2)}\sin\dfrac{5t}{3} = \dfrac{3}{4}\cos\dfrac{5t}{3} - \dfrac{4}{5}\sin\dfrac{5t}{3}.$

Ex. 3. *Evaluate* $\mathrm{L}^{-1}\left\{\dfrac{6}{2s-3} - \dfrac{3+4s}{9s^2-16} + \dfrac{8-6s}{16s^2+9}\right\}$ **[Meerut 1996, Osmania 2004]**

Sol. Re-writing, the given expression

$= 6\,\mathrm{L}^{-1}\left\{\dfrac{1}{2s-3}\right\} - 3\,\mathrm{L}^{-1}\left\{\dfrac{1}{9s^2-16}\right\} - 4\,\mathrm{L}^{-1}\left\{\dfrac{s}{9s^2-16}\right\} + 8\,\mathrm{L}^{-1}\left\{\dfrac{1}{16s^2+9}\right\} - 6\,\mathrm{L}^{-1}\left\{\dfrac{s}{16s^2+9}\right\}$

$$=\frac{6}{2}\mathrm{L}^{-1}\left\{\frac{1}{s-(3/2)}\right\}-\frac{3}{9}\mathrm{L}^{-1}\left\{\frac{1}{s^2-(4/3)^2}\right\}-\frac{4}{9}\mathrm{L}^{-1}\left\{\frac{s}{s^2-(4/3)^2}\right\}+\frac{8}{16}\mathrm{L}^{-1}\left\{\frac{1}{s^2+(3/4)^2}\right\}-\frac{6}{16}\mathrm{L}^{-1}\left\{\frac{s}{s^2+(3/4)^2}\right\}$$

$$=3e^{3t/2}-\frac{1}{3}\times\frac{1}{(4/3)}\sinh\frac{4t}{3}-\frac{4}{9}\cosh\frac{4t}{3}+\frac{1}{2}\times\frac{1}{(3/4)}\sin\frac{3t}{4}-\frac{3}{8}\cos\frac{3t}{4}$$

$$=3e^{3t/2}-\frac{1}{4}\times\sinh\frac{4t}{3}-\frac{4}{9}\cosh\frac{4t}{3}+\frac{2}{3}\sin\frac{3t}{4}-\frac{3}{8}\cos\frac{3t}{4}.$$

Ex.4. *Evaluate* (*i*) $\mathrm{L}^{-1}\left\{\frac{3s-2}{s^{3/2}}+\left(\frac{\sqrt{s}-1}{s}\right)^2+\frac{4s-18}{9-s^2}\right\}$ **[Meerut 1997]**

(*ii*) $\mathrm{L}^{-1}\left\{\frac{5}{s^2}+\left(\frac{\sqrt{s}-1}{s}\right)^2-\frac{7}{3s+2}\right\}$.

Sol. (*i*) Re-writing, the given expression

$$=\mathrm{L}^{-1}\left\{\frac{3}{s^{1/2}}-\frac{2}{s^{3/2}}+\frac{s-2s^{1/2}+1}{s^2}-\frac{4s}{s^2-9}+\frac{18}{s^2-9}\right\}$$

$$=\mathrm{L}^{-1}\left\{\frac{3}{s^{1/2}}-\frac{2}{s^{3/2}}+\frac{1}{s}-\frac{2}{s^{3/2}}+\frac{1}{s^2}-\frac{4s}{s^2-9}+\frac{18}{s^2-9}\right\}$$

$$=5\mathrm{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\}-2\mathrm{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\}+\mathrm{L}^{-1}\left\{\frac{1}{s}\right\}+\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}-4\mathrm{L}^{-1}\left\{\frac{s}{s^2-3^2}\right\}+18\mathrm{L}^{-1}\left\{\frac{1}{s^2-3^2}\right\}$$

$$=5\times\frac{t^{-1/2}}{\Gamma(1/2)}-2\frac{t^{1/2}}{(1/2)\times\Gamma(1/2)}+1+\frac{t}{1!}-4\cosh 3t+18\times\frac{1}{3}\sinh 3t$$

$$=5\times\frac{1/\sqrt{t}}{\sqrt{\pi}}-\frac{4\sqrt{t}}{\sqrt{\pi}}+1+t-4\cosh 3t+6\sinh 3t=5/\sqrt{\pi t}-4(t/\pi)^{1/2}+1+t-4\cosh 3t+6\sinh 3t.$$

(*ii*) Proceed as in part (*i*) **Ans.** $6t+1-4\times(t/\pi)^{1/2}-(7/3)\times e^{-(2t/3)}$

Ex. 5. *Evaluate:* (*i*) $\mathrm{L}^{-1}\left\{\frac{3(s^2-1)^2}{2s^5}+\frac{4s-18}{9-s^2}+\frac{(s+1)\left(2-\sqrt{s}\right)}{s^{5/2}}\right\}$

(*ii*) $\mathrm{L}^{-1}\left\{\frac{3}{s^2-3}+\frac{3s+2}{s^3}-\frac{3s-27}{s^2+9}+\frac{6-30\sqrt{s}}{s^4}\right\}$

Sol. (*i*) Re-writing, the given expression $=\mathrm{L}^{-1}\left\{\frac{3\left(s^4-2s^2+1\right)}{2s^5}-\frac{4s-18}{s^2-9}+\frac{2s-s^{3/2}+2-s^{1/2}}{s^{5/2}}\right\}$

$$=\mathrm{L}^{-1}\left\{\frac{3}{2}\times\frac{1}{s}-\frac{3}{s^3}+\frac{3}{2}\times\frac{1}{s^5}-\frac{4s}{s^2-3^2}+\frac{18}{s^2-3^2}+\frac{2}{s^{3/2}}-\frac{1}{s}+\frac{2}{s^{5/2}}-\frac{1}{s^2}\right\}$$

$$= \frac{1}{2}\mathrm{L}^{-1}\left\{\frac{1}{s}\right\} - \mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\} - 3\mathrm{L}^{-1}\left\{\frac{1}{s^3}\right\} + \frac{3}{2}\mathrm{L}^{-1}\left\{\frac{1}{s^5}\right\} + 2\mathrm{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\} + 2\mathrm{L}^{-1}\left\{\frac{1}{s^{5/2}}\right\}$$

$$-4\mathrm{L}^{-1}\left\{\frac{s}{s^2-3^2}\right\} + 18\mathrm{L}^{-1}\left\{\frac{1}{s^2-3^2}\right\}$$

$$= \frac{1}{2}\times 1 - \frac{t}{1!} - 3\times\frac{t^2}{2!} + \frac{3}{2}\times\frac{t^4}{4!} + 2\times\frac{t^{1/2}}{\Gamma(3/2)} + 2\times\frac{t^{3/2}}{\Gamma(5/2)} - 4\cosh 3t + \frac{18}{3}\sinh 3t$$

$$= \frac{1}{2} - t + \frac{3}{2}t^2 + \frac{1}{16}t^4 + 2\frac{\sqrt{t}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} + 2\times\frac{t\sqrt{t}}{\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - 4\cosh 3t + 6\sinh 3t$$

$$= \frac{1}{2} - t + \frac{3t^2}{2} + \frac{t^4}{16} + \frac{4\sqrt{t}}{\sqrt{\pi}} + \frac{8t\sqrt{t}}{3\sqrt{\pi}} - 4\cosh 3t + 6\sinh 3t$$

$$= \frac{1}{2} - t + \frac{3t^2}{2} + \frac{t^4}{16} + 4\left(\frac{t}{\pi}\right)^{1/2} + \frac{8t}{3}\left(\frac{t}{\pi}\right)^{1/2} - 4\cosh 3t + 6\sinh 3t$$

(*ii*) Proceed as in part (*i*) **Ans.** $3t + t^2 + t^3 - 16t^2\left(t/\pi\right)^{1/2} + \sqrt{3}\sinh t\sqrt{3} - 3\cos 3t + 9\sin 3t$.

Ex. 6. *Show that* (*i*) $L^{-1}\left\{\frac{1}{s}cos\frac{1}{s}\right\} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots\ldots\ldots$

(*ii*) $\mathrm{L}^{-1}\left\{\frac{1}{s}\sin\frac{1}{s}\right\} = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \ldots\ldots\ldots$ (*iii*) $\mathrm{L}^{-1}\left\{\frac{1}{s^3+1}\right\} = \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^8}{8!} - \frac{t^{11}}{11!} + \ldots\ldots\ldots$

Sol. (*i*) L.H.S. $= \mathrm{L}^{-1}\left\{\frac{1}{s}\left(1 - \frac{(1/s)^2}{2!} + \frac{(1/s)^4}{4!} - \frac{(1/s)^6}{6!} + \ldots\right)\right\}$

$$= \mathrm{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2!}\mathrm{L}^{-1}\left\{\frac{1}{s^3}\right\} + \frac{1}{4!}\mathrm{L}^{-1}\left\{\frac{1}{s^5}\right\} - \frac{1}{6!}\mathrm{L}^{-1}\left\{\frac{1}{s^7}\right\} + \ldots$$

$$= 1 - \frac{1}{2!}\times\frac{t^2}{2!} + \frac{1}{4!}\times\frac{t^4}{4!} - \frac{1}{6!}\times\frac{t^6}{6!} + \ldots = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots$$

(*ii*) L.H.S. $= \mathrm{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s} - \frac{(1/s)^3}{3!} + \frac{(1/s)^5}{5!} - \frac{(1/s)^7}{7!} + \ldots\right)\right\} = \mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3!}\mathrm{L}^{-1}\left\{\frac{1}{s^4}\right\} + \frac{1}{5!}\mathrm{L}^{-1}\left\{\frac{1}{s^6}\right\} - \frac{1}{7!}\mathrm{L}^{-1}\left\{\frac{1}{s^8}\right\} + \ldots$

$$= \frac{t}{1!} - \frac{1}{3!}\times\frac{t^3}{3!} + \frac{1}{5!}\times\frac{t^5}{5!} - \frac{1}{7!}\times\frac{t^7}{7!} + \ldots = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \ldots$$

(*iii*) L.H.S. $= \mathrm{L}^{-1}\left\{\frac{1}{s^3+1}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{s^3(1+1/s^3)}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{s^3}\left(1+\frac{1}{s^3}\right)^{-1}\right\}$

$$= \mathrm{L}^{-1}\left\{\frac{1}{s^3}\left(1-\left(\frac{1}{s^3}\right)+\left(\frac{1}{s^3}\right)^2-\left(\frac{1}{s^3}\right)^3+\left(\frac{1}{s^3}\right)^4\ldots\right)\right\}, \text{by biomial theorem}$$

$$= \mathrm{L}^{-1}\left\{\frac{1}{s^3}-\frac{1}{s^6}+\frac{1}{s^9}-\frac{1}{s^{12}}+\ldots\right\} = \frac{t^2}{2!}-\frac{t^5}{5!}+\frac{t^8}{8!}-\frac{t^{11}}{11!}+\ldots$$

EX. 7. *Show that* $L^{-1}\left\{e^{-1/s}/s\right\} = J_0\left(2\sqrt{t}\right)$

Sol. L.H.S. $= \mathrm{L}^{-1}\left\{\frac{1}{s}\left(1-\frac{1}{s}+\frac{(1/s)^2}{2!}-\frac{(1/s)^3}{3!}+\frac{(1/s)^4}{4!}-\ldots\right)\right\}$

$$= \mathrm{L}^{-1}\left\{\frac{1}{s}\right\}-\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}+\frac{1}{2!}\mathrm{L}^{-1}\left\{\frac{1}{s^3}\right\}-\frac{1}{3!}\mathrm{L}^{-1}\left\{\frac{1}{s^4}\right\}+\frac{1}{4!}\mathrm{L}^{-1}\left\{\frac{1}{s^5}\right\}+\ldots..$$

$$= 1-\frac{t}{1!}+\frac{1}{2!}\times\frac{t^2}{2!}-\frac{1}{3!}\times\frac{t^3}{3!}+\frac{1}{4!}\times\frac{t^4}{4!}\ldots = 1-\frac{\left(2\sqrt{t}\right)^2}{2^2}+\frac{\left(2\sqrt{t}\right)^4}{2^2\times4^2}-\frac{\left(2\sqrt{t}\right)^6}{2^2\times4^2\times6^2}+\ldots$$

$= \mathrm{J}_0(2\sqrt{t})$ $\quad [\because \mathrm{J}_0(x) = 1 - x^2/2^2 + x^4/(2^2 \times 4^2) - x^6/(2^2 \times 4^2 \times 6^2) + \ldots .]$

Ex. 8. *Show that*

(*i*) $\int_0^\infty cos\left(tx^2\right)dx = \frac{1}{2}\left(\frac{\pi}{2t}\right)^{1/2}$ (*ii*) $\int_0^\infty \cos x^2 dx = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}$ **[MDU Rohtak 2005]**

(*iii*) $\int_0^\infty sin\left(tx^2\right)dx = \frac{1}{2}\left(\frac{\pi}{2t}\right)^{1/2}$ (*iv*) $\int_0^\infty \sin x^2 dx = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}$ **[Punjab 2005]**

(*v*) $\int_0^\infty e^{-tx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$ (*vi*) $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ **[Kanpur 1997]**

Sol. Parts (*i*) and (*ii*). Let $F(t) = \int_0^\infty \cos(tx^2)dx$ ... (1)

$$\therefore \quad \mathrm{L}\{F(t)\} = \int_0^\infty e^{-st}\,\mathrm{F}(t)dt = \int_0^\infty e^{-st}\left\{\int_0^\infty \cos(tx^2)\,dx\right\}dt, \text{ using (1)}$$

$$= \int_0^\infty\left[\int_0^\infty e^{-st}\cos(tx^2)\,dt\right]dx, \text{ changing the order of integration}$$

$$= \int_0^\infty \mathrm{L}\{\cos(tx^2)\}dx, \text{ by definition of Laplace transform}$$

$$= \int_0^\infty \frac{s}{s^2+x^4}dx, \qquad \text{as } \mathrm{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$= \frac{1}{2\sqrt{s}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{(\tan\theta)}}, \qquad \left[\begin{array}{l}\text{Put } x^2 = s\tan\theta \text{ so that } x = \sqrt{s\tan\theta}\\ \text{and } dx = (s\sec^2\theta d\theta)/2\sqrt{s\tan\theta}\end{array}\right]$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta d\theta = \frac{1}{2\sqrt{s}} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{1}{2\sqrt{s}} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{2} = \frac{1}{4\sqrt{s}} \times \frac{\pi}{\sin(\pi/4)}$$

$$\left[\because \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \text{where} 0 < p < 1\right]$$

Thus, $\mathrm{L}\{\mathrm{F(t)}\} = \pi / 2\sqrt{2s}$

$$\therefore \quad \mathrm{F}(t) = \mathrm{L}^{-1}\left\{\frac{\pi}{2\sqrt{(2s)}}\right\} = \frac{\pi}{2\sqrt{2}} \times \mathrm{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{\pi}{2\sqrt{2}} \times \frac{t^{(1/2)-1}}{\Gamma(1/2)} = \frac{\pi}{2\sqrt{2}} \times \frac{1}{\sqrt{\pi t}}$$

or $$\int_0^\infty \cos(tx^2)\,dx = \frac{1}{2}\left(\frac{\pi}{2t}\right)^{1/2}, \text{ using (1)} \qquad \ldots (2)$$

Putting $t = 1$ in (2), we get $$\int_0^\infty \cos x^2 dx = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} \qquad \ldots (3)$$

(2) and (3) give the required results of parts (*i*) and (*ii*) respectively

Parts (*iii*) and (*iv*) Proceed exactly as in parts (*i*) and (*ii*) above

Parts (*v*) and (*vi*). Let $F(t) = \int_0^\infty e^{-tx^2} dx \qquad \ldots (1)$

$$\therefore \mathrm{L}\{\mathrm{F}(t)\} = \int_0^\infty e^{-st} F(t)dt = \int_0^\infty e^{-st}\left\{\int_0^\infty e^{-tx^2} dx\right\} dt, \text{ by (1)}$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} e^{-tx^2} dt\right] dx, \text{ changing the order of integration}$$

$$= \int_0^\infty \mathrm{L}\left\{e^{-tx^2}\right\} dx, \text{by definition of Laplace transform}$$

$$= \int_0^\infty \frac{dx}{s+x^2} = \left[\frac{1}{\sqrt{s}} \tan^{-1}\frac{x}{\sqrt{s}}\right]_0^\infty = \frac{1}{\sqrt{s}} \times \frac{\pi}{2}$$

Thus, $\mathrm{L}\{\mathrm{F}(t)\} = \pi / 2\sqrt{s}$ so that $\mathrm{F}(t) = \mathrm{L}^{-1}\{\pi / 2\sqrt{s}\}$

$$\therefore \quad \mathrm{F}(t) = \frac{\pi}{2}\mathrm{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{\pi}{2} \times \frac{t^{(1/2)-1}}{\Gamma(1/2)} = \frac{\pi}{2} \times \frac{1}{\sqrt{t}\sqrt{\pi}}$$

$$\therefore \quad \int_0^\infty e^{-tx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}, \text{ using (1)} \qquad \ldots (2)$$

Putting $t = 1$ in (2), we get $$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}. \qquad \ldots (3)$$

(2) and (3) give the required results of parts (*v*) and (*vi*) respectively.

Ex. 9. *For a > 0, prove that L^{-1} {f (s)} = F (t) implies* $\mathrm{L}^{-1}\{f(as+b)\} = (1/a)\times e^{-bt/a}\mathrm{F}(t/a)$.

Sol. Given $\mathrm{L}^{-1}\{f(s)\} = \mathrm{F}(t)$ so that $f(s) = \mathrm{L}\{F(t)\}$

or $f(s) = \int_0^\infty e^{-st}\mathrm{F}(t)\,dt$, by definition of Laplace transform

$$\therefore\ f(as+b) = \int_0^\infty e^{-(as+b)t}\mathrm{F}(t)dt = \int_0^\infty e^{-ast}e^{-bt}\,\mathrm{F}(t)dt = \int_0^\infty e^{-sx}e^{-bx/a}\,\frac{1}{a}F\left(\frac{x}{a}\right)dx$$

[Putting $at = x$ and $a\,dt = dx$]

$$= \int_0^\infty e^{-st}\left\{\frac{1}{a}e^{-bt/a}\,\mathrm{F}\left(\frac{t}{a}\right)\right\}dt$$

$\therefore$ $f(as+b) = \mathrm{L}\{(1/a)\times e^{-bt/a}\,\mathrm{F}(t/a)\}$, by definition of Laplace transform

or $\mathrm{L}^{-1}\{f(as+b)\} = (1/a)\times e^{-bt/a}\,\mathrm{F}(t/a)$, by definition of inverse Laplace transform

EXERCISE 2 (A)

Evaluate the inverse Laplace transform of the following:

1. $\frac{s}{s^2+2}+\frac{6s}{s^2-16}+\frac{3}{s-3}$ **[Osmania 2004]**

2. $\frac{s}{s^2+2}+\frac{6s}{s^2-9}+\frac{5}{s-5}+\frac{7}{s^2-16}$

3. $\frac{7}{s^2+9}+\frac{5}{s-7}+\frac{1}{2s^{3/2}}$

4. $\frac{4}{s^2}+\left(\frac{\sqrt{s}-1}{s}\right)^2-\frac{5}{3s+4}$

5. $\frac{s}{s^2+14}$

6. $\mathrm{L}^{-1}\{1/s^4 - 3s/(s^2+16) + 5/(s^2+4)\}$ **[Ravishankar 2010]**

ANSWERS

1. $\cos(t\sqrt{2})+6\cosh 4t+3e^{3t}$

2. $\cos(t\sqrt{2})+6\cosh 3t+5e^{5t}+(7/4)\times\sinh 4t$

3. $(7/3)\times\sin 3t+5e^{7t}+(t/\pi)^{1/2}$

4. $5t+1-4(t/\pi)^{1/2}-(5/3)e^{-4t/3}$

5. $\cos(t\sqrt{14})$

6. $t^3/6 - 3\cos 4t + (5/2)\times\sin 2t$

2.6. Method of partial Fractions

Sometimes it is convenient to break a given expression into partial fractions and later on manipulate term by term. In this connection the student must consult any book of Algebra and revise partial fractions. However, here we wish to draw attention of students to two short cut methods to be used in two special situations as follows.

Situation I. Suppose that $f(x)/g(x)$ be a proper rational algebric function. Let $g(x) = (x-a)(x-b)(x-c)(x-d)$ so that we have non-repeated real linear factors. Suppose that

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-a)(x-b)(x-c)(x-d)} = \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\frac{D}{x-d} \qquad \ldots(1)$$

Then A, B, C, and D can be obtained by inspection as follows:

To obtain A, multiply both sides of (1) by $s - a$ and let $s \to a$; Then, we have

$$A = \lim_{s\to a}\frac{f(s)}{(s-b)(s-c)(s-d)} = \frac{f(a)}{(a-b)(a-c)(a-d)}$$

Similarly, B, C and D can be determined. This method in very useful whenever the denominator has distinct linear factors. However, if the denominator has repeated linear factors, we can also use the above method.

Suppose $$\frac{f(s)}{(s-a)(s-b)^3} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{(s-b)^2} + \frac{D}{(s-b)^3} \quad \ldots (2)$$

In this case, we can find A and D by above method. Multiply both sides of (2) by $s - a$ and let $s \to a$; then $$A = \lim_{s \to a} \frac{f(s)}{(s-b)^3} = \frac{f(a)}{(a-b)^3}$$

Again, multiply both sides of (2), by $(s - b)^3$ and let $s \to b$; then

$$D = \lim_{s \to b} \frac{f(s)}{(s-a)} = \frac{f(b)}{(b-a)}$$

Constants B and C cannot be obtained be the above procedure. So we put values of A and D in (2) and multiply both sides by $(s - a)\ (s - b)^3$. Then B and C are obtained by equating like powers as usual.

Situation II. Suppose that $f(x)/g(x)$ be a proper rational algebraic function. Let $g(x) = (x-1)^3(x-2)$ and $f(x) = x^2 + x + 1$ so that $g(x)$ possesses repeated linear factor $x - 1$. Then partial fractions can be obtained by well known **Long-Division-Method** as follows:

Rule: Let $x - 1 = y$ so that $x = y + 1$. Then, we have

$$\frac{f(x)}{g(x)} = \frac{x^2+x+1}{(x-1)^3(x-2)} = \frac{(y+1)^2+(y+1)+1}{y^3(y+1-2)} = \frac{1}{y^3} \times \frac{3+3y+y^2}{-1+y}, \quad \ldots (1)$$

where we have kept $1/y^3$ at extreme left and the numerator and denominator of the remaining fraction have been arranged in ascending powers of y. Now, we divide the numerator by denominator till the remainder contains powers of y less than 2 (note that power of y in $1/y^3$ is 3). Thus,

$$\begin{array}{r|l} & -3-6y-7y^2 \\ \hline -1+y & 3+3y+y^2 \\ & 3-3y \\ \hline & 6y+y^2 \\ & 6y-6y^2 \\ \hline & 7y^2 \\ & 7y^2-7y^3 \\ \hline & 7y^3 \end{array}$$

$$\therefore \ (1) \ \Rightarrow \quad \frac{x^2+x+1}{(x-1)^3(x-2)} = \frac{1}{y^3}\left[-3-6y-7y^2+\frac{7y^3}{-1+y}\right]$$

$$= -\frac{3}{y^3} - \frac{6}{y^2} - \frac{7}{y} + \frac{7}{y-1} = -\frac{3}{(x-1)^3} - \frac{6}{(x-1)^2} - \frac{7}{x-1} + \frac{7}{x-2}, \text{as } y = x-1$$

In what follows, we shall find partial fractions by the well known methods in addition to two special situations explained above.

2.6 A. Solved examples based on Art. 2.6

Ex.1. Find (*i*) $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$ **[Kanpur 2005]**

(*ii*) $L^{-1}\left\{\frac{s^2}{(s+1)(s+2)(s+3)}\right\}$ **[S.V. University (A.P.) 1997]**

(*iii*) $L^{-1}\left\{\frac{1+2s}{(s-1)(s-2)}\right\}$ **[M.S. Univ. T.N. 2007]**

(*iv*) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$ **[Kumaun 1997, Guwahati 2000]**

Sol. (*i*) Let $$\frac{3s+7}{s^2-2s-3}=\frac{3s+7}{(s-3)(s+1)}=\frac{A}{s-3}+\frac{B}{s+1} \quad \ldots (1)$$

Method 1. Multiply both sides of (1) by $s-3$ and let $s \to 3$; then

$$A = \lim_{s\to 3} \frac{3s+7}{s+1} = \frac{9+7}{3+1} = 4.$$

Multiply both sides of (1) by $s+1$ and let $s \to -1$ then

$$B = \lim_{s\to -1} \frac{3s+7}{s-3} = \frac{-3+7}{-1-3} = -1.$$

$\therefore$ From (1), $$\frac{3s+7}{s^2-2s-3} = \frac{4}{s-3}-\frac{1}{s+1}$$

$\therefore$ $$L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = 4L^{-1}\left\{\frac{1}{s-3}\right\}-L^{-1}\left\{\frac{1}{s+1}\right\} = 4e^{3t}-e^{-t}.$$

Method 2. Multiplying both sides of (1) by $(s-3)(s+1)$, we get

$$3s+7 = A(s+1)+B(s-3)$$

Equating coefficients of like powers on both sides, we get

$A+B=3$ and $A-3B=7$ so that $A=4$ and $B=-1$.

Rest of the solution is same as in Method 1.

(*ii*) Let $$\frac{s^2}{(s+1)(s+2)(s+3)} = \frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3} \quad \ldots (1)$$

Multiply both sides of (1) by $s+1$ and let $s \to -1$; then

$$A = \lim_{s\to -1} \frac{s^2}{(s+2)(s+3)} = \frac{(-1)^2}{(-1+2)(-1+3)} = \frac{1}{2}$$

Multiply both sides of (1) by $s+2$ and let $s \to -2$; then

$$B = \lim_{s\to -2} \frac{s^2}{(s+1)(s+3)} = \frac{(-2)^2}{(-2+1)(-2+3)} = -4$$

Multiply both sides of (1) by $s + 3$ and let $s \to -3$; then

$$C = \lim_{s\to -3} \frac{s^2}{(s+1)(s+2)} = \frac{(-3)^2}{(-3+1)(-3+2)} = \frac{9}{2}$$

$\therefore$ From (1), $$\frac{s^2}{(s+1)(s+2)(s+3)} = \frac{1}{2(s+1)} - \frac{4}{s+2} + \frac{9}{2(s+3)} \quad \ldots (2)$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s+1)(s+2)(s+3)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - 4L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{9}{2}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{e^{-t}}{2} - 4e^{-2t} + \frac{9e^{-3t}}{2}$$

(*iii*) **Ans.** $5e^{2t} - 3e^{t}$ (*iv*) **Ans.** $(e^{-at} - e^{-bt})/(b - a)$.

Ex.2. Find (*i*) $L^{-1}\left\{\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}\right\}$ **[Agra 2010; S.V. University (A.P.) 1997]**

(*ii*) $L^{-1}\left\{\frac{6s^2 + 22s + 18}{s^3 + 6s^2 + 11s + 6}\right\}$ (*iii*) $L^{-1}\left\{\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right\}$

Sol. (*i*) Here $s^3 - 6s^2 + 11s - 6 = s^2(s-1) - 5s(s-1) + 6(s-1) = (s-1)(s^2 - 5s + 6) = (s-1)(s-2)(s-3)$

$$\therefore \quad \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} \quad \ldots (1)$$

Let $$\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \quad \ldots (2)$$

Multiply both sides of (2) by $s - 1$ and let $s \to 1$; then

$$A = \lim_{s\to 1} \frac{2s^2 - 6s + 5}{(s-2)(s-3)} = \frac{(2\times 1) - (6\times 1) + 5}{(1-2)(1-3)} = \frac{1}{2}$$

Multiply both sides of (2) by $s - 2$ and let $s \to 2$; then

$$B = \lim_{s\to 2} \frac{2s^2 - 6s + 5}{(s-1)(s-3)} = \frac{(2\times 4) - (6\times 2) + 5}{(2-1)(2-3)} = -1$$

Multiply both sides of (2) by $s - 3$ and let $s \to 3$; then

$$C = \lim_{s\to 3} \frac{2s^2 - 6s + 5}{(s-1)(s-2)} = \frac{(2\times 9) - (6\times 3) + 5}{(3-1)(3-2)} = \frac{5}{2}$$

$\therefore$ (1) and (2) $\Rightarrow$ $$\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} = \frac{1}{2(s-1)} - \frac{1}{s-2} + \frac{5}{2(s-3)}$$

$$\therefore \quad L^{-1}\left\{\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{2}L^{-1}\left\{\frac{1}{s-3}\right\}, = \frac{e^{t}}{2} - e^{2t} + \frac{5e^{3t}}{3}$$

(*ii*) Proceed as in part (*i*) **Ans.** $e^{-t} + 2e^{-2t} + 3e^{-3t}$

(*iii*) Here the given fraction is a function of s^2. We reduce it to another fraction with linear factors by taking $s^2 = x$.

Then, $$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{x}{(x+a^2)(x+b^2)} \quad \ldots (1)$$

Let $$\frac{x}{(x+a^2)(x+b^2)} = \frac{\mathrm{A}}{x+a^2} + \frac{\mathrm{B}}{x+b^2} \quad \ldots (2)$$

Multiply both sides of (3) *by* $(x + a^2)$ and let $x \to -a^2$; then

$$\mathrm{A} = \lim_{x \to -a^2} \frac{x}{x+b^2} = \frac{-a^2}{-a^2+b^2} = \frac{a^2}{a^2-b^2}$$

Multiply both sides of (3) $x + b^2$ and let $x \to -b^2$; then

$$\mathrm{B} = \lim_{x \to -b^2} \frac{x}{x+a^2} = \frac{-b^2}{-b^2+a^2} = -\frac{b^2}{a^2-b^2}$$

(1) and (2) $\Rightarrow$ $$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{a^2}{(a^2-b^2)(x+a^2)} - \frac{b^2}{(a^2-b^2)(x+b^2)}$$

or $$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right], \text{ as } x = s^2 \quad \ldots (3)$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{a^2-b^2}\left(\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right)\right\}, \text{ by (3)}$$

$$= \frac{1}{a^2-b^2}\left[a^2\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} - b^2\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right] = \frac{1}{a^2-b^2}\left[a^2 \times \frac{1}{a}\sin at - b^2 \times \frac{1}{b}\sin bt\right] = \frac{a\sin at - b\,\sin bt}{a^2-b^2}$$

Ex.3. *Find* (*i*) $\mathrm{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$ (*ii*) $\mathrm{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right\}$

(*iii*) $\mathrm{L}^{-1}\{1/s\,(s^2+9)\}$ **[Kumaun 1997]**

(*iv*) $\mathrm{L}^{-1}\{1/s(s^2+a^2)\}$ **[Kerala 2001]**

Sol. (*i*). Let $$\frac{1}{(s+1)(s^2+1)} = \frac{\mathrm{A}}{s+1} + \frac{\mathrm{B}s+\mathrm{C}}{s^2+1} \quad \ldots (1)$$

Multiplying both sides of (1) by $(s+1)\,(s^2+1)$, we have

$$1 = \mathrm{A}(s^2+1) + (\mathrm{B}s+\mathrm{C})\,(s+1) \quad \ldots (2)$$

Equating coefficients of s^2, s and constant terms on both sides of (2), we have

$\mathrm{A} + \mathrm{B} = 0$, $\mathrm{C} + \mathrm{B} = 0$ and $\mathrm{A} + \mathrm{C} = 1$. ... (3)

Solving (3), we get $\mathrm{A} = 1/2$, $\mathrm{B} = -1/2$ and $\mathrm{C} = 1/2$.

$\therefore$ From (1), $$\frac{1}{(s+1)(s^2+1)} = \frac{1}{2(s+1)} - \frac{1}{2}\frac{s-1}{s^2+1}$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{2}\times\frac{1}{s+1}-\frac{1}{2}\times\frac{s}{s^2+1}+\frac{1}{2}\times\frac{1}{s^2+1}\right\}$$

$$= (1/2)\times e^{-t} - (1/2)\times\cos t + (1/2)\times\sin t = (1/2)\times(e^{-t}-\cos t+\sin t)$$

(*ii*) Let $$\frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2}+\frac{Cs+D}{s^2+b^2} \quad \ldots(1)$$

Multiplying both sides of (1) by $(s^2 + a^2)(s^2 + b^2)$, we have

$$s = (As + B)(s^2 + b^2) + (Cs + D)(s^2 + a^2) \quad \ldots(2)$$

Equating coefficient of s^2, s and constant terms on both sides of (2), we have

$$A + C = 0 \quad \ldots(3)$$
$$B + D = 0 \quad \ldots(4)$$
$$Ab^2 + Ca^2 = 1 \quad \ldots(5)$$
$$Bb^2 + Da^2 = 0 \quad \ldots(6)$$

Solving (3) and (5), $A = -1/(a^2 - b^2)$ and $C = 1/(a^2 - b^2)$

Solving (4) and (6), $B = D = 0$.

Substituting the values of A, B, C and D in (1), we get

$$\frac{s}{(s^2+a^2)(s^2+b^2)} = -\frac{1}{a^2-b^2}\times\frac{s}{s^2+a^2}+\frac{1}{a^2-b^2}\times\frac{s}{s^2+b^2}$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\} = -\frac{1}{(a^2-b^2)}L^{-1}\left\{\frac{s}{s^2+a^2}\right\} + \frac{1}{(a^2-b^2)}L^{-1}\left\{\frac{s}{s^2+b^2}\right\},$$

$$= -\frac{1}{a^2-b^2}\cos at + \frac{1}{a^2-b^2}\cos bt = \frac{1}{a^2-b^2}(\cos bt - \cos at)$$

(*iii*) Let $$\frac{1}{s(s^2+9)} = \frac{A}{s}+\frac{Bs+C}{s^2+9} \quad \ldots(1)$$

Multiplying both sides of (1) by $s(s^2 + 9)$ we have

$$1 = A(s^2 + 9) + (Bs + C)s \quad \ldots(2)$$

Equating coefficients of s^2, s and constant terms on both sides of (2), we have

$A + B = 0$, $C = 0$ and $9A = 1$ so that $A = 1/9$, $B = -1/9$, $C = 0$

$\therefore$ From (1), $$\frac{1}{s(s^2+9)} = \frac{1}{9s}-\frac{s}{9(s^2+9)} \quad \ldots(3)$$

$$\therefore \quad L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{9}L^{-1}\left\{\frac{s}{s^2+3^2}\right\}, \text{ using (3)}$$

$$= (1/9)-(1/9)\times\cos 3t = (1/9)\times(1-\cos 3t)$$

(*iv*) Do as in part (*iii*). **Ans.** $(1 - \cos at)/a^2$.

EXERCISE 2 (*B*)

Evaluate the inverse Laplace transform of the following:

1. $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$

2. $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$

3. $\dfrac{3s^2+5s-4}{s^3+s^2-2s}$

4. $\dfrac{3s+1}{(s-1)(s^2+1)}$ **[Purvanchal 2006]**

5. $\dfrac{s+5}{(s+1)(s^2+1)}$ **[Osmania 2004]** **6.** $\dfrac{s+1}{s(s^2+4s+8)}$ **(Meerut 2010)**

7. $\dfrac{1}{s^2(s-a)}$ **[Delhi B.Sc. (Hons) II 2011]** **8.** $\dfrac{s^2+2}{s^2(s+3)}$ **[Ravishankar 2005]**

9. $\dfrac{1}{s(s+1)(s+2)}$ **[Madras 2005, Madurai kamraj 2008]** **10.** $\dfrac{s+3}{s(s^2+1)}$ **[Bangalore 2005]**

ANSWERS

1. $(-1/6)\times e^{-t}-(4/3)\times e^{2t}+(7/2)\times e^{3t}$ **2.** $2+e^{t}-e^{-2t}$ **3.** $2+e^{t}-e^{-2t}$

4. $2e^{t}+\sin t-2\cos t$ **5.** $3e^{-t}+3\sin t-2\cos t$ **6.** $1/8-(1/8)\times e^{-2t}(\cos 2t-6\sin 2t)$

7. $(e^{at}-1-at)/a^2$ **8.** $(1/9)\times(6t+11-e^{-3t})$ **9.** $(1/2)-e^{-t}+(1/2)\times e^{-2t}$

10. $3-3\cos t+\sin t$

2.7. Heaviside's Expansion Theorem (or formula) [Jabalpur 2004; Meerut 2004, 06, 07]

Given that F(s) and G (s) are polynomials in s, the degree of F (s), being less than that of G(s) and and if G(s) = ($s-\alpha_1$) ($s-\alpha_2$) .... ($s-\alpha_n$) where α_1, α_2, ..., α_n are distinct constants, real or complex, then

$$L^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{r=1}^{n}\frac{F(\alpha_r)}{G'(\alpha_r)}e^{\alpha_r t}$$ **[Osmania 2010]**

Proof. Resolving into partial fractions, suppose that

$$\frac{F(s)}{G(s)} = \frac{A_1}{s-\alpha_1}+\frac{A_2}{s-\alpha_2}+\ldots+\frac{A_n}{s-\alpha_n} \qquad \ldots(1)$$

To compute A_r, multiplying both sides of (1) by $s-\alpha_r$ and taking limit as $s\to\alpha_r$, we get

$$A_r = \lim_{s\to\alpha_r}\frac{F(s)\times(s-\alpha_r)}{G(s)} = F(\alpha_r)\lim_{s\to\alpha_r}\frac{s-\alpha_r}{G(s)} = F(\alpha_r)\lim_{s\to\alpha_r}\frac{1}{G'(s)}$$, by L′ Hospital's rule

Thus, $A_r = F(\alpha_r)/G'(\alpha_r)$

Then, from (1), we have

$$L^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{r=1}^{n}L^{-1}\left\{\frac{A_r}{s-\alpha_r}\right\} = \sum_{r=1}^{n}A_r e^{\alpha_r t} = \sum_{r=1}^{n}\frac{F(\alpha_r)}{G'(\alpha_r)}e^{\alpha_r t}$$, putting value of A_r.

2.7A. Solved examples based on Art. 2.7.

Ex. 1. *Apply Heaviside expansion theorem to obtain* (*i*) $L^{-1}\left\{\dfrac{2s^2+5s-4}{s^3+s^2-2s}\right\}$

(*ii*) $L^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$ **[Agra 2009; Meerut 2002, Vikram 2004]**

Sol. (*i*) Here $F(s) = 2s^2+5s-4$ and $G(s) = s^3+s^2-2s = s(s-1)(s+2)$

$\therefore \quad G'(s) = 3s^2 + 2s - 2, \qquad$ and $\qquad \alpha_1 = 1, \qquad \alpha_2 = 1, \qquad \alpha_3 = -2.$

Using Heaviside's expansion formula, we have

$$\therefore L^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\} = \frac{F(0)}{G'(0)}e^{(0\times t)} + \frac{F(1)}{G'(1)}e^{t} + \frac{F(-2)}{G'(-2)}e^{-2t} = \frac{-4}{-2} + \frac{3}{3}e^{t} + \frac{-6}{6}e^{-2t} = 2 + e^{t} - e^{-2t}.$$

(*ii*) Here $\quad F(s) = 3s + 1 \qquad$ and $\qquad G(s) = (s-1)(s^2 - i^2) = (s-1)(s+i)(s-i)$

$\therefore \qquad G'(s) = 3s^2 - 2s + 1; \qquad$ and $\qquad \alpha_1 = 1, \qquad \alpha_2 = -i, \qquad \alpha_3 = i.$

$$\therefore \quad L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \frac{F(1)}{G'(1)}e^{t} + \frac{F(-i)}{G'(-i)}e^{-it} + \frac{F(i)}{G'(i)}e^{it} = \frac{4e^{t}}{2} + \frac{-3i+1}{-2+2i}e^{-it} + \frac{3i+1}{-(2+2i)}e^{it}$$

$$= 2e^{t} + \frac{(3i-1)(1+i)}{2(1-i)(1+i)}e^{-it} - \frac{(3i+1)(1-i)}{2(1+i)(1-i)}e^{it} = 2e^{t} + (1/2)\times(i-2)e^{-it} - (1/2)\times(i+2)e^{it}$$

$$= 2e^{t} - (i/2)\times(e^{it} - e^{-it}) - (e^{it} + e^{-it}) = 2e^{t} + \sin t - 2\cos t$$

Ex. 2. *Apply Heaviside expansion theorem to obtain* (*i*) $L^{-1}\left\{\frac{s+5}{(s+1)(s^2+1)}\right\}$

(*ii*) $L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$ **[S.V. University (A.P.) 1997, Meerut 2005]**

Sol. (*i*) Do as in Ex.1 (*i*). **Ans.** $2e^{-t} - 2\cos t + 3\sin t$.

(*ii*) Here $F(s) = 2s^2 - 6s + 5, \qquad G(s) = s^3 - 6s^2 + 11s - 6 = (s-1)(s-2)(s-3) \qquad \ldots (1)$

$\therefore \qquad G'(s) = 3s^2 - 12s + 11 \qquad$ and $\qquad \alpha_1 = 1, \qquad \alpha_2 = 2, \qquad \alpha_3 = 3, \qquad \ldots (2)$

Using Heaviside's expansion formula, we have

$$L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{F(1)}{G'(1)}e^{t} + \frac{F(2)}{G'(2)}e^{2t} + \frac{F(3)}{G'(3)}e^{3t} = \frac{e^{t}}{2} + e^{2t} + \frac{5e^{3t}}{2}, \text{ by (1) and (2)}$$

Ex.3. *Applying Heaviside's expansion formula, prove that*

$$L^{-1}\left\{1/(s^3+1)\right\} = (1/3)\times\left[e^{-t} - e^{t/2}\left\{\cos(\sqrt{3}\,t/2) - \sqrt{3}\sin(\sqrt{3}\,t/2)\right\}\right]$$

Sol. Here $\qquad F(s) = 1 \qquad$ and $\qquad G(s) = s^3 + 1 = (s+1)(s^2 - s + 1)$

$G(s) = 0 \quad \Rightarrow \quad s = -1 \quad$ or $\quad s^2 - s + 1 = 0 \quad \Rightarrow \quad s = -1 \quad$ or $\quad s = (1 \pm \sqrt{-3})/2$

$\therefore \quad \alpha_1 = -1, \qquad \alpha_2 = (1 + i\sqrt{3})/2, \qquad \alpha_3 = (1 - i\sqrt{3})/2. \qquad$ Also, $G'(s) = 3s^2$

Using Heaviside's expansion formula, we have

$$L^{-1}\left\{\frac{1}{s^3+1}\right\} = \frac{F(-1)}{G'(-1)}e^{-t} + \frac{F\{(1+i\sqrt{3}\}/2\}}{G'\{(1+i\sqrt{3})/2\}}e^{\frac{1}{2}(1+i\sqrt{3})t} + \frac{F\{(1-i\sqrt{3})/2\}}{G'\{(1-i\sqrt{3})/2\}}e^{\frac{1}{2}(1-i\sqrt{3})t}$$

$$= \frac{1}{3}e^{-t} + \frac{1}{(3/4)\times\left(1+i\sqrt{3}\right)^2}e^{\frac{1}{2}\left(1+i\sqrt{3}\right)t} + \frac{1}{(3/4)\times\left(1-i\sqrt{3}\right)^2}e^{\frac{1}{2}\left(1-i\sqrt{3}\right)t}$$

$$= \frac{1}{3}e^{-t} + \frac{4}{3}\times\frac{1}{-2+2i\sqrt{3}}e^{\frac{1}{2}\left(1+i\sqrt{3}\right)t} + \frac{4}{3}\times\frac{1}{-2-2i\sqrt{3}}e^{\frac{1}{2}\left(1-i\sqrt{3}\right)t}$$

$$= \frac{1}{3}e^{-t} - \frac{2}{3}\times\frac{1}{1-i\sqrt{3}}\times\frac{1+i\sqrt{3}}{1+i\sqrt{3}}e^{t/2}e^{\left(\sqrt{3}/2\right)ti} - \frac{2}{3}\times\frac{1}{1+i\sqrt{3}}\times\frac{1-i\sqrt{3}}{1-i\sqrt{3}}e^{t/2}e^{-\left(\sqrt{3}/2\right)ti}$$

$$= \frac{1}{3}e^{-t} - \frac{2}{3}\times\frac{1+i\sqrt{3}}{1+3}e^{t/2}e^{\left(\sqrt{3}/2\right)ti} - \frac{2}{3}\times\frac{1-i\sqrt{3}}{1+3}e^{t/2}e^{-\left(\sqrt{3}/2\right)ti}$$

$$= \frac{1}{3}e^{-t} - \frac{1}{6}e^{t/2}\left[\left(1+i\sqrt{3}\right)e^{\left(\sqrt{3}/2\right)ti} + \left(1-i\sqrt{3}\right)e^{-\left(\sqrt{3}/2\right)ti}\right]$$

$$= \frac{1}{3}e^{-t} - \frac{1}{6}e^{t/2}\left[e^{\left(\sqrt{3}/2\right)ti} + e^{-\left(\sqrt{3}/2\right)ti} + i\sqrt{3}\left\{e^{\left(\sqrt{3}/2\right)ti} - e^{-\left(\sqrt{3}/2\right)ti}\right\}\right]$$

$$= (1/3)\times e^{-t} - (1/6)\times e^{t/2}\left[2\cos\left(\sqrt{3}\,t/2\right) + i\sqrt{3}\times 2\,i\sin\left(\sqrt{3}\,t/2\right)\right]$$

$$= (1/3)\times\left[e^{-t} - e^{t/2}\left\{\cos\left(\sqrt{3}\,t/2\right) - \sqrt{3}\sin\left(\sqrt{3}\,t/2\right)\right\}\right]$$

EXERCISE 2 (*C*)

Using Heaviside's expansion formula, evaluate

(*i*) $\mathrm{L}^{-1}\left\{\dfrac{s^2-6}{s^3+4s^2+3s}\right\}$ **[Meerut 2006]** (*ii*) $\mathrm{L}^{-1}\left\{\dfrac{19s+37}{(s+1)(s-2)(s-3)}\right\}$

(*iii*) $\{1/(s^3-1)\}$

ANSWERS

(*i*) $-2+(5/2)\times e^{-t}+(1/2)\times e^{-3t}$ (*ii*) $-3e^{-t}+5e^{2t}-2e^{-3t}$

(*iii*) $(1/3)\times\left[e^{t}-e^{-t/2}\left\{\cos\left(\sqrt{3}\,t/2\right)+\sqrt{3}\sin\left(\sqrt{3}\,t/2\right)\right\}\right]$

2.8. First translation theorem or first shifting theorem

[Agra 2011; Ravishankar 2004, Kanpur 1995; Purvanchal 1993, 94]

Statement. If $\mathrm{L}^{-1}\{f(s)\} = \mathrm{F}(t)$, then $\mathrm{L}^{-1}\{f(s-a)\} = e^{at}\,\mathrm{F}(t) = e^{at}\,\mathrm{L}^{-1}\{f(s)\}$

Proof. By definition of Laplace transform, we have

$$f(s) = \int_0^\infty e^{-st}\mathrm{F}(t)\,dt \qquad \ldots(1)$$

Replacing s by $s-a$ on both sides of (1), we have

$$f(s-a) = \int_0^\infty e^{-(s-a)t}\,\mathrm{F}(t)dt = \int_0^\infty e^{-st}\left\{e^{at}\mathrm{F}(t)\right\}dt$$

or $f(s-a) = \mathrm{L}\left\{e^{at}\mathrm{F}(t)\right\}$, by definition of Laplace transform

Hence, by definition of inverse Laplace transform, we get

$$L^{-1}\{f(s-a)\} = e^{at}F(t) = e^{at}\,L^{-1}\{f(s)\} \qquad \ldots (2)$$

Remark 1. Replacing a by $-a$ in (2), we have

$$L^{-1}\{f(s+a)\} = e^{-at}F(t) = e^{-at}\,L^{-1}\{f(s)\}$$

Remark 2. In practice, generally, we apply the first shifting theorem in the following forms

$$L^{-1}\{f(s-a)\} = e^{at}\,L^{-1}\{f(s)\} \quad \text{and} \quad L^{-1}\{f(s+a)\} = e^{-at}\,L^{-1}\{f(s)\}$$

Commit to memory these both results for direct application.

Remark 3. The result of first shifting theorem can also be expressed in the following two forms: If $L\{F(t)\} = f(s)$, then $(i)\ L^{-1}\{f(s)\} = e^{-at}\,L^{-1}\{f(s-a)]$ $(ii)\ L^{-1}\{f(s)\} = e^{at}\,L^{-1}\{f(s+a)\}$

2.8A. Solved examples based on Art. 2.8

Ex. 1. *Evaluate the inverse Laplace transform of the following:*

$(i)\ \dfrac{s-a}{(s-a)^2+b^2}$ **[S.V. Univ. (A.P.) 1997]** $(ii)\ \dfrac{s}{(s+a)^2+b^2}$ **[M.S. Univ. (T.N). 2007]**

$(iii)\ \dfrac{s}{s^2+2s+5}$ **[Kuvempa 2005]** $(iv)\ \dfrac{s+3}{s^2+2s+2}$ **[M.S. Univ. (T. W.) 2007]**

$(v)\ s/(s+1)^5$ **[Osmania 2004]** $(vi)\ s/(s+1)^{5/2}$

Sol. $(i)\ L^{-1}\left\{\dfrac{s-a}{(s-a)^2+b^2}\right\} = e^{at}L^{-1}\left\{\dfrac{s}{s^2+b^2}\right\}$, by first shifting theorem

$$= e^{at}\cos bt$$

$$(ii)\ L^{-1}\left\{\frac{s}{(s+a)^2+b^2}\right\} = e^{-at}L^{-1}\left\{\frac{s-a}{s^2+b^2}\right\} = e^{-at}\left[L^{-1}\left\{\frac{s}{s^2+b^2}\right\} - aL^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right]$$

$$= e^{-at}\ \{\cos bt - (a/b) \times \sin bt\} = (1/b) \times e^{-at}\ (b\cos bt - a\sin bt)$$

$$(iii)\ L^{-1}\left\{\frac{s}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+2^2}\right\} = e^{-t}L^{-1}\left\{\frac{s-1}{s^2+2^2}\right\}$$

$$= e^{-t}\left[L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - L^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right] = e^{-t}\left(\cos 2t - \frac{\sin 2t}{2}\right)$$

(iv) Proceed as in part (iii) **Ans.** $e^{-t}(\cos t + 2\sin t)$

$$(v)\ L^{-1}\left\{\frac{s}{(s+1)^5}\right\} = e^{-t}L^{-1}\left\{\frac{s-1}{s^5}\right\} = e^{-t}\left[L^{-1}\left\{\frac{1}{s^4}\right\} - L^{-1}\left\{\frac{1}{s^5}\right\}\right] = e^{-t}\left(\frac{t^3}{3!} - \frac{t^4}{4!}\right) = \frac{t^3e^{-t}(4-t)}{24}$$

$$(vi)\ L^{-1}\left\{\frac{s}{(s+1)^{5/2}}\right\} = e^{-t}L^{-1}\left\{\frac{s-1}{s^{5/2}}\right\} = e^{-t}\left[L^{-1}\left\{\frac{1}{s^{3/2}}\right\} - L^{-1}\left\{\frac{1}{s^{5/2}}\right\}\right] = e^{-t}\left[\frac{t^{1/2}}{\Gamma(3/2)} - \frac{t^{3/2}}{\Gamma(5/2)}\right]$$

$$= e^{-t}\left[\frac{t^{1/2}}{(1/2)\times\sqrt{\pi}} - \frac{t^{3/2}}{(3/2)\times(1/2)\times\sqrt{\pi}}\right] = 2e^{-t}\left(\frac{t}{\pi}\right)^{1/2}\left(1 - \frac{2t}{3}\right)$$

Ex. 2. *If* $L\{F(t)\} = (s+1)/(s^2+2s-8)$, *find* $F(t)$. **[Bangalore 2005]**

Sol. By definition of Laplace inverse transform, we have

$$F(t) = L^{-1}\left\{\frac{s+1}{s^2+2s-8}\right\} = L^{-1}\left\{\frac{s+1}{(s+1)^2-3^2}\right\} = e^{-t}L^{-1}\left\{\frac{s}{s^2-3^2}\right\} = e^{-t}\cosh 3t$$

Ex. 3. *Evaluate* (*i*) $L^{-1}\left\{\frac{1}{s^2+8s+16}\right\}$ (*ii*) $L^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$

Sol. (*i*) $L^{-1}\left\{\frac{1}{s^2+8s+16}\right\} = L^{-1}\left\{\frac{1}{(s+4)^2}\right\} = e^{-4t}L^{-1}\left\{\frac{1}{s^2}\right\} = e^{-4t}t$

(*ii*) $L^{-1}\left\{\frac{1}{s^2-6s+10}\right\} = L^{-1}\left\{\frac{1}{(s-3)^2+1^2}\right\} = e^{3t}\,L^{-1}\left\{\frac{1}{s^2+1^2}\right\}$, by first shifting theorem

$$= e^{3t}\sin t.$$

Ex.4. *Evaluate the inverse Laplace transform of the following:*

(*i*) $\frac{1}{(s+2)(s-1)^2}$ (*ii*) $\frac{s^2-2s+3}{(s-1)^2(s+1)}$

(*iii*) $\frac{4s+5}{(s-1)^2(s+2)}$ **[Bangalore 2004, Purvanchal 2005]**

Sol. (*i*) *Let* $$\frac{1}{(s+2)(s-1)^2} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \quad \ldots (1)$$

Multiplying both sides of (1) by $(s+2)\,(s-1)^2$, we get

$1 = A(s-1)^2 + B(s-1)(s+2) + C(s+2)$ or $1 = A(s^2-2s+1) + B(s^2+s-2) + C(s+2)$

Equating coefficients of s^2, s and constant terms on both sides the above identity, we get

$A + B = 0$, $-2A + B + C = 0$ and $A - 2B + 2C = 1$

Solving these, $A = 1/9$, $B = -1/9$, $C = 1/3$

$\therefore$ From (1), $$\frac{1}{(s+2)(s-1)^2} = \frac{1}{9(s+2)} - \frac{1}{9(s-1)} + \frac{1}{3(s-1)^2}$$

$$\therefore L^{-1}\left\{\frac{1}{(s+2)(s-1)^2}\right\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{9}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s-1)^2}\right\},$$

$= (1/9) \times e^{-2t} - (1/9) \times e^{t} + (1/3) \times e^{t}\, L^{-1}\{1/s^2\}$, using first shifting theorem

$= (1/9) \times e^{-2t} - (1/9) \times e^{t} + (1/3) \times e^{t} \times (t/1!) = (1/9) \times [(3t-1)\,e^{t} + e^{-2t}]$

Ans. (*ii*) $(3/2) \times e^{-t} + (1/2) \times (2t-1)\,e^{t}$ (*iii*) $(1/3) \times (e^{t} - e^{-2t}) + 3t\,e^{t}$

Ex. 5. *Find* $L^{-1}\left\{\frac{1+2s}{(s+2)^2(s-1)^2}\right\}$ **[Kanpur 1995; Osmania 92, Purvanchal 2002]**

Sol. Let $$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} \quad \ldots (1)$$

Method.1. Multiplying both sides by $(s+2)^2\,(s-1)^2$, we get

$$1 + 2s = A(s+2)\,(s-1)^2 + B(s-1)^2 + C(s-1)\,(s+2)^2 + D(s+2)^2$$

or $1 + 2s = A(s + 2)(s^2 - 2s + 1) + B(s^2 - 2s + 1) + C(s - 1)(s^2 + 4s + 4) + D(s^2 + 4s + 4)$

Equating coefficients of like powers on both sides, we get

$A + C = 0$, $B + 3C + D = 0$, $-3A - 2B + 4D = 2$ and $2A + B - 4C + 4D = 1$

Solving these, $A = 0$, $B = -1/3$, $C = 0$, and $D = 1/3$.

$$\therefore \quad \text{From (1)}, \quad \frac{1+2s}{(s+2)^2(s-1)^2} = -\frac{1}{3(s+2)^2} + \frac{1}{3(s-1)^2} \qquad \ldots (2)$$

$$\therefore \quad L^{-1}\left\{\frac{1+2s}{(s+2)^2(s-1)^2}\right\} = -\frac{1}{3}L^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s-1)^2}\right\}, \text{ by (2)}$$

$$= -\frac{1}{3}e^{-2t}\,L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{3}e^{t}\,L^{-1}\left\{\frac{1}{s^2}\right\} = -\frac{1}{3}e^{-2t} \times \frac{t}{1!} + \frac{1}{3}e^{t} \times \frac{t}{1!} = \frac{t}{3}(e^{t} - e^{-2t})$$

Method. 2. Multiply both sides of (1) by $(s + 2)^2$ and let $s \to -2$, then

$$B = \lim_{s \to -2} \frac{1+2s}{(s-1)^2} = \frac{1-4}{(-2-1)^2} = -\frac{1}{3}$$

Multiply both sides of (1) by $(s - 1)^2$ and let $s \to 1$, then

$$D = \lim_{s \to 1} \frac{1+2s}{(s-2)^2} = \frac{1+2}{(1+2)^2} = \frac{1}{3}$$

$$\therefore \quad (1) \Rightarrow \frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} - \frac{1}{3(s+2)^2} + \frac{C}{s-1} + \frac{1}{3(s-1)^2} \qquad \ldots (3)$$

Multiply both sides of (3) by $(s+2)^2(s-1)^2$. Then, we obtain

$$1 + 2s = A(s+2)(s-1)^2 - (1/3) \times (s-1)^2 + C(s-1)(s+2) + (1/3) \times (s+2)^2 \qquad \ldots (4)$$

Putting $s = 0$ and $s = -1$ on both sides of the identity (4), we have

$1 = 2A - (1/3) - 2C + (4/3)$ and $-1 = 4A - (4/3) - 2C + (1/3)$

i.e., $A - C = 0$ and $2A - C = 0$, giving $A = C = 0$.

Rest of the procedure is same as in method 1.

Ex. 6. *Evaluae* $L^{-1}\left\{\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right\}$

Sol. Let $$\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3} \qquad \ldots (1)$$

Multiply both sides of (1) by $s + 1$ and let $s \to -1$; then

$$A = \lim_{s \to -1} \frac{5s^2 - 15s - 11}{(s-2)^3} = \frac{5 \times (-1)^2 - 15 \times (-1) - 11}{(-1-2)^3} = -\frac{1}{3}$$

Multiply both sides of (1) by $(s - 2)^3$ and let $s \to 2$; then

$$D = \lim_{s \to 2} \frac{5s^2 - 15s - 11}{(s+1)} = \frac{5 \times (2)^2 - 15 \times 2 - 11}{2+1} = -7$$

Substituting the values of A and D in (1), we have

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = -\frac{1}{3(s+1)}+\frac{\mathrm{B}}{s-2}+\frac{\mathrm{C}}{(s-2)^2}-\frac{7}{(s-2)^3} \quad \ldots (2)$$

To determine B and C we can substitute any two values of s (except $s = -1$ and $s = 2$ which make given fraction meaningless) in identity (2), say $s = 0$ and $s = 1$ by turn and obtain

$$\frac{11}{8} = -\frac{1}{3}-\frac{B}{2}+\frac{C}{4}+\frac{7}{8} \quad \text{and} \quad \frac{21}{2} = -\frac{1}{6}-\mathrm{B}+C+7$$

i.e. $3C - 6B = 10$ and $3C - 3B = 11$ giving $B = 1/3$ and $C = 4$.

$$\therefore \quad (2) \Rightarrow \quad \frac{5\mathrm{s}^2-15s-11}{(s+1)(s-2)^3} = -\frac{1}{3(s+1)}+\frac{1}{3(s-2)}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3} \quad \ldots (3)$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{5s^2-15s+11}{(s+1)\,(s-2)^3}\right\} = -\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}-7e^{2t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= -\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}\times\frac{t}{1!}-7e^{2t}\times\frac{t^2}{2!} = -\frac{1}{3}e^{-t}+e^{2t}\left(\frac{1}{3}+4t-\frac{7t^2}{2}\right)$$

Ex. 7. (*a*) *Find the inverse Laplace transform of* $1/s^2\,(s+1)^2$.

[Purvachal 2004; Kumaun 1997, Kanpur 1997]

Sol. Let $$\frac{1}{s^2(s+1)^2} = \frac{\mathrm{A}}{s}+\frac{\mathrm{B}}{s^2}+\frac{\mathrm{C}}{s+1}+\frac{\mathrm{D}}{(s+1)^2} \quad \ldots (1)$$

Multiply both sides of (1) by s^2 and let $s \to 0$; then

$$\mathrm{A} = \lim_{s\to 0}\frac{1}{(s+1)^2} = \frac{1}{(0+1)^2} = 1$$

Multiplty both sides of (1) by $(s+1)^2$ and let $s \to -1$; then

$$\mathrm{B} = \lim_{s\to -1}\frac{1}{s^2} = \frac{1}{(-1)^2} = 1$$

Then, from (1) $$\frac{1}{s^2(s+1)^2} = \frac{\mathrm{A}}{s}+\frac{1}{s^2}+\frac{\mathrm{C}}{s+1}+\frac{1}{(s+1)^2} \quad \ldots (2)$$

Putting $s = 1$ and $s = -2$ successively in identity (2), we get

$$\mathrm{A}+1+\frac{\mathrm{C}}{2}+\frac{1}{4} = \frac{1}{4} \quad \text{and} \quad -\frac{\mathrm{A}}{2}+\frac{1}{4}-\mathrm{C}+1 = \frac{1}{4}$$

$\Rightarrow$ $2\mathrm{A} + \mathrm{C} = -2$ and $\mathrm{A} + 2\mathrm{C} = 2$ so that $\mathrm{A} = -2$ and $\mathrm{C} = 2$

$$\therefore \quad (2) \Rightarrow \quad \frac{1}{s^2(s+1)^2} = -\frac{2}{s}+\frac{1}{s^2}+\frac{2}{s+1}+\frac{1}{(s+1)^2} \quad \ldots (3)$$

$$(3) \Rightarrow \mathrm{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = -2\,\mathrm{L}^{-1}\left\{\frac{1}{s}\right\}+\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}+2\,\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= -2+(t/1!)+2e^{-t}+e^{-t}\,\mathrm{L}^{-1}\{1/s^2\}\text{, using first shifting theorem}$$

$$= -2+t+2e^{-t}+e^{-t}t = t-2+e^{-t}(2+t)$$

Ex. 7. (*b*) *Evaluate* $\mathrm{L}^{-1}\left\{\dfrac{3s^3-3s^2-40s+36}{(s^2-4)^2}\right\}$

Sol. Here $$\frac{3s^3-3s^2-40s+36}{(s^2-4)^2} = \frac{3s^3-3s^2-40s+36}{(s-2)^2(s+2)^2}$$

Let $$\frac{3s^3-3s^2-40s+36}{(s-2)^2(s+2)^2} = \frac{\mathrm{A}}{s-2}+\frac{\mathrm{B}}{(s-2)^2}+\frac{\mathrm{C}}{s+2}+\frac{\mathrm{D}}{(s+2)^2}$$

Now proceed as in Ex.6. and get $A = 0$, $B = -2$, $C = 3$ and $D = 5$. Then, as usual rest of solution can be obtained. **Ans.** $(5t + 3)\, e^{-2t} - 2t\, e^{2t}$.

Ex. 8 (*a*). Find $\mathrm{L}^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\}$ **[Rohilkhand 1994]**

Sol. Let $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{\mathrm{A}}{s-1}+\frac{\mathrm{B}s+\mathrm{C}}{s^2+2s+5} \quad \ldots (1)$$

Multiplying both sides by $(s - 1)\ (s^2 + 2s + 5)$, we have

$$5s+3 = \mathrm{A}(s^2+2s+5)+(\mathrm{B}s+\mathrm{C})(s-1)$$

or $$5s+3 = \mathrm{A}(s^2+2s+5)+\mathrm{B}s^2+s(\mathrm{C}-\mathrm{B})-\mathrm{C} \quad \ldots (2)$$

Equating coefficients of s^2, s and constant terms, we get

$\mathrm{A} + \mathrm{B} = 0, \quad 2\mathrm{A} - \mathrm{B} + \mathrm{C} = 5, \quad 5\mathrm{A} - \mathrm{C} = 3 \quad \Rightarrow \quad \mathrm{A} = 1, \quad \mathrm{B} = -1, \quad \mathrm{C} = 2$

(1) $$\Rightarrow \frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1}+\frac{-s+2}{s^2+2s+5} = \frac{1}{s-1}-\frac{s-2}{s^2+2s+5}$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{s-1}\right\}-\mathrm{L}^{-1}\left\{\frac{s-2}{s^2+2s+5}\right\}$$

$$= e^{t} - \mathrm{L}^{-1}\left\{\frac{(s+1)-3}{(s+1)^2+4}\right\} = e^{t} - e^{-t}\,\mathrm{L}^{-1}\left\{\frac{s-3}{s^2+4}\right\}, \text{ using first shifting theorem}$$

$$= e^{t} - e^{-t}\left[\mathrm{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}-3\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right] = e^{t} - e^{-t}\left(\cos 2t - \frac{3}{2}\sin 2t\right)$$

Ex.8 (*b*). *Evaluate* $\mathrm{L}^{-1}\left\{\dfrac{s-1}{(s+3)(s^2+2s+1)}\right\}$

Sol. Proceed as in Ex. 8. **Ans.** $(1/5) \times e^{-t}\,(4 \cos t - 3 \sin t) - (4/5) \times e^{-3t}$

Ex. 9. *Evaluate* $\mathrm{L}^{-1}\left\{\dfrac{1}{(s^2+4)(s+1)^2}\right\}$

Sol. Let $$\frac{1}{(s^2+4)(s+1)^2} = \frac{\mathrm{A}}{s+1}+\frac{\mathrm{B}}{(s+1)^2}+\frac{\mathrm{C}s+\mathrm{D}}{s^2+4} \quad \ldots (1)$$

(1) $\Rightarrow$ $$1 = \mathrm{A}(s+1)(s^2+4)+\mathrm{B}(s^2+4)+(\mathrm{C}s+\mathrm{D})(s+1)^2 \quad \ldots (2)$$

Putting $s = -1$ in identity (2), we get $1 = 5B$ so that $B = 1/5$.

$$\therefore \quad (2) \Rightarrow \qquad 1 = A(s+1)(s^2+4) + (1/5)\times(s^2+4) + (Cs+D)(s^2+2s+1) \qquad \ldots (3)$$

Equating coefficients of s^3, s^2 and constant terms on both sides of (3), we have

$A + C = 0$, $A + (1/5) + D + 2C = 0$ and $4A + (4/5) + D = 1$

Solving these, $A = 2/25$, $C = -2/25$ and $D = -3/25$

$$\therefore (1) \Rightarrow \qquad \frac{1}{(s^2+4)(s+1)^2} = \frac{2}{25(s+1)} + \frac{1}{5(s+1)^2} - \frac{2s+3}{25(s^2+4)} \qquad \ldots (4)$$

$$(4) \Rightarrow \mathrm{L}^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\} = \frac{2}{25}\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{5}\mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \frac{2}{25}\mathrm{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{3}{25}\mathrm{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$= \frac{2}{25}e^{-t} + \frac{1}{5}e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{2}{25}\cos 2t - \frac{3}{25}\times\frac{\sin 2t}{2} = \frac{2}{25}e^{-t} + \frac{1}{5}e^{-t}t - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t$$

Ex. 10. *Show that* $\mathrm{L}^{-1}\left\{\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}\right\} = \dfrac{1}{2}\sin t \sinh t$ **[Meerut 2007]**

Sol. Resolving into partial fractions, by inspection, we have

$$\frac{s}{(s^2-2s+2)(s^2+2s+2)} = \frac{1}{4}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right]$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{s}{(s^2-2s+2)(s^2+2s+2)}\right\} = \frac{1}{4}\mathrm{L}^{-1}\left\{\frac{1}{s^2-2s+2}\right\} - \frac{1}{4}\mathrm{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$$

$$= \frac{1}{4}\mathrm{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} - \frac{1}{4}\mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$= (1/4) \times e^t\,\mathrm{L}^{-1}\{1/(s^2+1)\} - (1/4) \times e^{-t}\,\mathrm{L}^{-1}\{(1/(s^2+1)\}$, using first shifting theorem

$= (1/4) \times e^t \sin t - (1/4) \times e^{-t} \sin t = (1/2) \times \sin t \times \{(e^t - e^{-t})/2\} = (1/2) \times \sin t \sinh t$

Ex. 11. *Evaluate* $L^{-1}\{s/(s^4+s^2+1)\}$ **[Kanpur 2010; Purvanchal 2003; CDLU 2004]**

Sol. Here $$s^4 + s^2 + 1 = (s^2+1)^2 - s^2 = (s^2+1+s)(s^2+1-s) \qquad \ldots (1)$$

Now, $$\frac{s}{(s^2+1+s)(s^2+1-s)} = \frac{1}{2}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right], \text{ by inspection}$$

$$\Rightarrow \qquad \frac{s}{s^4+s^2+1} = \frac{1}{2}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right], \text{ using (1)}$$

$$\Rightarrow \qquad \mathrm{L}^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{1}{2}\mathrm{L}^{-1}\left\{\frac{1}{s^2+1-s}\right\} - \frac{1}{2}\mathrm{L}^{-1}\left\{\frac{1}{s^2+1+s}\right\}$$

$$= \frac{1}{2}\mathrm{L}^{-1}\left\{\frac{1}{(s-1/2)^2+(3/4)}\right\} - \frac{1}{2}\mathrm{L}^{-1}\left\{\frac{1}{(s+1/2)^2+(3/4)}\right\}$$

$$= \frac{1}{2} e^{t/2} \, \mathrm{L}^{-1}\left\{\frac{1}{s^2 + (\sqrt{3}/2)^2}\right\} - \frac{1}{2} e^{-t/2} \, \mathrm{L}^{-1}\left\{\frac{1}{s^2 + (\sqrt{3}/2)^2}\right\}, \text{ using first shifting theorem}$$

$$= \frac{1}{2} e^{t/2} \frac{1}{(\sqrt{3}/2)} \sin\left(\frac{1}{2}\sqrt{3}t\right) - \frac{1}{2} e^{-t/2} \frac{1}{(\sqrt{3}/2)} \sin\left(\frac{1}{2}\sqrt{3}t\right)$$

$$= \frac{2}{\sqrt{3}} \times \frac{e^{t/2} - e^{-t/2}}{2} \sin\left(\frac{1}{2}\sqrt{3}t\right) = \frac{2}{\sqrt{3}} \sinh\frac{t}{2} \sin\left(\frac{1}{2}\sqrt{3}t\right).$$

Ex. 12. *Evaluate* $\mathrm{L}^{-1}\left\{\dfrac{s}{s^4 + 4a^4}\right\}$ **[S.V. University (A.P) 1997]**

Sol. We have, $s^4 + 4a^4 = (s^2 + 2a^2)^2 - (2sa)^2 = (s^2 + 2a^2 + 2sa)\,(s^2 + 2a^2 - 2sa)$

$$\therefore \qquad \frac{s}{s^4 + 4a^4} = \frac{s}{(s^2 + 2a^2 + 2sa)(s^2 + 2a^2 - 2sa)}$$

or $$\frac{s}{s^4 + 4a^4} = \frac{1}{4a}\left[\frac{1}{s^2 + 2a^2 - 2sa} - \frac{1}{s^2 + 2a^2 + 2sa}\right], \text{ on resolving into partial fractions by inspection}$$

$$\Rightarrow \mathrm{L}^{-1}\left\{\frac{s}{s^4 + 4a^4}\right\} = \frac{1}{4a}\mathrm{L}^{-1}\left\{\frac{1}{s^2 - 2sa + 2a^2}\right\} - \frac{1}{4a}\mathrm{L}^{-1}\left\{\frac{1}{s^2 + 2sa + 2a^2}\right\}$$

$$= \frac{1}{4a}\mathrm{L}^{-1}\left\{\frac{1}{(s-a)^2 + a^2}\right\} - \frac{1}{4a}\mathrm{L}^{-1}\left\{\frac{1}{(s+a)^2 + a^2}\right\}$$

$$= \frac{1}{4a} e^{at} \, \mathrm{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} - \frac{1}{4a} e^{-at} \, \mathrm{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\}, \text{ using first shifting theorem}$$

$$= \frac{1}{4a} e^{at} \frac{\sin at}{a} - \frac{1}{4a} e^{-at} \frac{\sin at}{a} = \frac{1}{2a^2}\left[\frac{e^{at} - e^{-at}}{2}\right] \sin at = \frac{1}{2a^2} \sinh at \, \sin at$$

Ex. 13. *Prove* $L^{-1}\left\{\dfrac{s^2}{s^4 + 4a^4}\right\} = \dfrac{1}{2a}$ (*cosh at sin at + sinh at cos at*) **[Agra 2009]**

Sol. We have, $s^4 + 4a^4 = (s^2 + 2a^2)^2 - (2sa)^2 = (s^2 + 2a^2 + 2sa)(s^2 + 2a^2 - 2sa)$

$$\therefore \quad \frac{s^2}{s^4 + 4a^4} = s \times \frac{s}{(s^2 + 2a^2 + 2sa)(s^2 + 2a^2 - 2sa)} = s \times \frac{1}{4a}\left[\frac{1}{s^2 + 2a^2 - 2sa} - \frac{1}{s^2 + 2a^2 + 2sa}\right] \ldots (1)$$

(on resolving into partial fractions by inspection)

$$\therefore \qquad \mathrm{L}^{-1}\left\{\frac{s^2}{s^4 + 4a^4}\right\} = \frac{1}{4a}\mathrm{L}^{-1}\left\{\frac{s}{s^2 + 2a^2 - 2sa}\right\} - \frac{1}{4a}\mathrm{L}^{-1}\left\{\frac{s}{s^2 + 2a^2 + 2sa}\right\}$$

$$= \frac{1}{4a} \mathrm{L}^{-1}\left\{\frac{(s-a)+a}{(s-a)^2+a^2}\right\} - \frac{1}{4a} \mathrm{L}^{-1}\left\{\frac{(s+a)-a}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a} e^{at}\, \mathrm{L}^{-1}\left\{\frac{s+a}{s^2+a^2}\right\} - \frac{1}{4a} e^{-at}\, \mathrm{L}^{-1}\left\{\frac{s-a}{s^2+a^2}\right\}, \text{ using first shifting theorem}$$

$$= \frac{e^{at}}{4a}\left[\mathrm{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} + a\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right] - \frac{e^{-at}}{4a}\left[\mathrm{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} - a\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]$$

$$= \frac{e^{at}}{4a}(\cos at + \sin at) - \frac{e^{-at}}{4a}(\cos at - \sin at) = \frac{1}{2a}\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at + \frac{1}{2a}\left(\frac{e^{at}-e^{-at}}{2}\right)\cos at$$

$= (1/2a) \times (\cosh at \sin at + \sinh at \cos at)$

Ex.14(*a*). *Find* $L^{-1}\{(2s^3 + 2s^2 + 4s + 1)/(s^2 + 1)\,(s^2 + s + 1)\}$ **[Ravishankar 2004]**

Ans. $\cos t + 2\sin t + e^{-t/2}\,\{\cos(\sqrt{3}\,t/2) - \sqrt{3}\sin(\sqrt{3}\,t/2)\}$.

Ex. 14(*b*). *Evalute* $\mathrm{L}^{-1}\left\{(s^2+2s+3)/(s^2+2s+2)(s^2+2s+5)\right\}$

Sol. Let
$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5} \qquad \ldots (1)$$

Multiplying both sides of (1) by $(s^2+2s+2)(s^2+2s+5)$, we get

$$s^2 + 2s + 3 = (As + B)\,(s^2 + 2s + 5) + (Cs + D)\,(s^2 + 2s + 2)$$

or $\quad s^2 + 2s + 3 = (A + C)\,s^3 + (2A + B + 2C + D)\,s^2 + (5A + 2B + 2C + 2D)\,s + 5B + 2D.$

Equating coefficients of s^3, s^2, s and constant terms on both sides of the above identity, we have

$A + C = 0, \quad 2A + B + 2C + D = 1, \quad 5A + 2B + 2C + 2D = 2$ and $5B + 2D = 3.$

Solving these, $A = 0, \quad B = 1/3, \quad C = 0$ and $D = 2/3.$

$$\therefore (1) \Rightarrow \quad \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)} \qquad \ldots (2)$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = \frac{1}{3}\mathrm{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} + \frac{2}{3}\mathrm{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{2}{3}\mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{3}e^{-1}\mathrm{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} + \frac{2}{3}e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}, \text{ by first shifting theorem}$$

$= (1/3) \times e^{-t} \sin t + (2/3) \times e^{-t} \times (1/2) \times \sin 2t = (1/3) \times e^{-t} (\sin t + \sin 2t)$

Ex. 15(*a*). *Evaluate* $\mathrm{L}^{-1}\{3(s^2 + 2s + 3)/(s^2 + 2s + 2)\,(s^2 + 2s + 5)\}$

Sol. We can find partial fraction as in Ex. 14 (*b*). But here we can also use method of inspection to find partial fractions. Thus, we have

$$\frac{3(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)} = \frac{(s^2+2s+5)+2(s^2+2s+2)}{(s^2+2s+2)(s^2+2s+5)} = \frac{1}{s^2+2s+2} + \frac{2}{s^2+2s+5}$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{3(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} + 2\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\},$$

$$= \mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + 2\ \mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = e^{-t}\ \mathrm{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} + 2\,e^{-t}\ \mathrm{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

(using first shifting theorem)

$$= e^{-t}\sin t + 2e^{-t} \times (1/2) \times \sin 2t = e^{-t}(\sin t + \sin 2t)$$

Ex. 15(*b*). *Evaluate L^{-1} {(s + 1)/s (s^2 + 4s + 8)}*

Sol. Do yourself **Ans.** (1/8) × {1 – e^{-2t} (cos 2*t* – 3 sin 2*t*)

Ex. 16. *Find the inverse transform of 1 / [$(s-1)^5(s+2)$]*

[Purvanchal 2003; Kanpur 2009, 10]

Sol. Re-writing, $\mathrm{L}^{-1}\left\{\frac{1}{(s-1)^5(s+2)}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{(s-1)^5[(s-1)+3]}\right\} = e^{t}\ \mathrm{L}^{-1}\left\{\frac{1}{s^5}\times\frac{1}{(s+3)}\right\}$ ... (1)

(using first shifting theorem)

We now apply long division method to find partial fractions (refer Art. 2.6)

Now, we have $$\frac{1}{s^5(s+3)} = \frac{1}{s^5}\times\frac{1}{3+s}$$ ... (2)

We now divide 1 by 3 + *s* till power of *s* in the remainder is 5.

$$
\begin{array}{r}
1/3 - s/9 + s^2/27 - s^3/81 + s^4/243 \\
3+s\,\overline{)\,1} \\
\underline{1 + s/3} \\
-s/3 \\
\underline{-s/3 - s^2/9} \\
s^2/9 \\
\underline{s^2/9 + s^3/27} \\
-s^3/27 \\
\underline{-s^3/27 - s^4/81} \\
s^4/81 \\
\underline{s^4/81 + s^5/243} \\
-s^5/243
\end{array}
$$

$$\therefore\ (2) \Rightarrow \qquad \frac{1}{s^5}\times\frac{1}{3+s} = \frac{1}{s^5}\left[\frac{1}{3} - \frac{s}{9} + \frac{s^2}{27} - \frac{s^3}{81} + \frac{s^4}{243} + \frac{(-s^5/243)}{s+3}\right]$$

or $$\frac{1}{s^5(s+3)} = \frac{1}{3s^5} - \frac{1}{9s^4} + \frac{1}{27s^3} - \frac{1}{81s^2} + \frac{1}{243s} - \frac{1}{243(s+3)}$$ ... (3)

∴ From (1) and (3), we have

$$\mathrm{L}^{-1}\left\{\frac{1}{(s-1)^5(s+2)}\right\} = e^{t}\ \mathrm{L}^{-1}\left\{\frac{1}{3s^5} - \frac{1}{9s^4} + \frac{1}{27s^3} - \frac{1}{81s^2} + \frac{1}{243s} - \frac{1}{243(s+3)}\right\}$$

$$= e^t\left[\frac{1}{3}\times\frac{t^4}{4!}-\frac{1}{9}\times\frac{t^3}{3!}+\frac{1}{27}\times\frac{t^2}{2!}-\frac{1}{8!}\times\frac{t}{1!}+\frac{1}{243}-\frac{1}{243}e^{-3t}\right]= e^t\left[\frac{t^4}{72}-\frac{t^3}{54}+\frac{t^2}{54}-\frac{t}{81}+\frac{1}{243}\right]-\frac{1}{243}e^{-2t}$$

17. *Evaluate* $L^{-1}\{(s^2+s)/(s^2+1)(s^2+2s+2)\}$ **[KU Kurukshetra 2005]**

Sol. Let $$\frac{s^2+s}{(s^2+1)(s^2+2s+2)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2s+2} \quad \ldots(1)$$

$(1) \Rightarrow$ $$s^2+s=(As+B)(s^2+2s+2)+(Cs+D)(s^2+1) \quad \ldots(2)$$

Comparing the coefficients of s^3, s^2, s and constant terms in identity (2), we get

$$A+C=0 \quad \ldots(3)$$
$$2A+B+D=1 \quad \ldots(4)$$
$$2A+2B+C=1 \quad \ldots(5)$$
$$2B+D=0 \quad \ldots(6)$$

From (3), C = –A and from (6), D = –2B. Hence, (4) and (5) give

$2A - B = 1$ and $A + 2B = 1$ Solving these, $A = 3/5$, $B = 1/5$.

Thus, $C = -A = -3/5$ and $D = -2B = -2/5$

Substituting these values in (1), we have

$$\frac{s^2+s}{(s^2+1)(s^2+2s+2)}=\frac{3s+1}{5(s^2+1)}-\frac{3s+2}{5(s^2+2s+2)}$$

$$\therefore\ L^{-1}\left\{\frac{s^2+s}{(s^2+1)(s^2+2s+2)}\right\}=\frac{3}{5}L^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{5}L^{-1}\left\{\frac{1}{s^2+1}\right\}-\frac{3}{5}L^{-1}\left\{\frac{s+(2/3)}{s^2+2s+2}\right\}$$

$$=\frac{3}{5}L^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{5}L^{-1}\left\{\frac{1}{s^2+1}\right\}-\frac{3}{5}L^{-1}\left\{\frac{(s+1)-(1/3)}{(s+1)^2+1}\right\}$$

$$=\frac{3}{5}\cos t+\frac{1}{5}\sin t-\frac{3}{5}e^{-t}L^{-1}\left\{\frac{s-(1/3)}{s^2+1}\right\},\ \text{by first shifting theorem}$$

$$=\frac{3}{5}\cos t+\frac{1}{5}\sin t-\frac{3}{5}e^{-t}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\}-\frac{1}{3}L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$= (3/5) \times \cos t + (1/5) \times \sin t - (3/5) \times e^{-t}\{\cos t - (1/3) \times \sin t\}$

$= (3/5) \times (1 - e^{-t}) \cos t + (1/5) \times (1 + e^{-t}) \sin t$

EXERCISE 2 (D)

Evaluate the inverse Laplace transform of the following:

1. $1/(s-4)^3$ **[Gulberga 2005]** **2.** $(s+1)/(s^2+6s+25)$ **[Osmania 2010]**

3. $(s+2)/(s^2-2s+5)$ **[Kanpur 2008]** **4.** $(4s+5)/\{(s-4)^4(s+3)\}$ **[Bilaspur 2004]**

5. $(s+5)/\{(s+2)(s^2+4)\}$ **6.** $1/(2s+3)^{1/3}$ **[Meerut 1998]**

7. $s/(s+3)^{7/3}$ **[Kanpur 2010]** **8.** $1/(s-4)^5 + 5/\{(s-2)^2+5^2\}$

9. $1/(s+4)^{3/2}$ **[Sagar 2004]** **10.** $(s+2)/(s^2-4s+13)$ **[Ravishanker 2004]**

11. $1/(s^2-6s+10)$ **[Ravishanker 2004]** **12.** $9/(s^2+s+1/2)$

13. $\dfrac{3s+1}{(s+1)^4}$ **[Ranchi 2010]** **14.** $\dfrac{1}{s^2+4s+13}-\dfrac{s+4}{s^2+8s+97}+\dfrac{s+2}{s^2-4s+29}$

15. $(2s)/\{(s-1)(s+2)^3\}$ **[Gulberga 2005]** **16.** $1/\{(s-1)^2(s^2+1)\}$ **[Purvanchal 2005]**
17. $(3s-2)/(s^2-4s+20)$ **[Meerut 2011]**

ANSWERS

1. $(1/2)\times t^2 e^{4t}$ **2.** $e^{-3t}\{\cos 4t-(1/2)\times\sin 4t\}$

3. $e^t[\cos 2t+(3/2)\times\sin 2t]$ **4.** $3te^{4t}+(1/7)\times(e^{4t}-e^{-3t})$

5. $(1/8)\times(3e^{-2t}-3\cos 2t+7\sin 2t)$ **6.** $(1/2\pi t)^{1/2}e^{-3t/2}$

7. $[4e^{-3t}t^{3/2}(5-6t)]/15\sqrt{\pi}$ **8.** $(1/24)\times t^4e^{4t}+e^{2t}\sin 5t$

9. $2(t/\pi)^{1/2}e^{-4t}$ **10.** $e^{2t}\cos 3t+(4/5)\times e^{2t}\sin 3t$

11. $e^{3t}\sin t$ **12.** $18e^{-t/2}\sin(t/2)$

13. $(1/6)\times e^{-t}(9t^2-2t^3)$ **14.** $(1/3)\times e^{-2t}\sin 3t-e^{-4t}\cos 9t+e^{2t}\cos 5t+(4/5)\times e^{2t}\sin 5t$
15. $(2/27)\times e^{-2t}(18t^2-3t-1)+(2/27)\times e^t$ **16.** $2\cos t+(1/2)\times e^t(t-1)$
17. $e^{2t}(3\cos 4t+\sin 4t)$

2.9. Second shifting theorem or second translation theorem

[Purvanchal 1995; Kanpur 1994, 96; Meerut 1999]

Statement. If $\mathrm{L}^{-1}\{f(s)\}=\mathrm{F}(t)$, then $\mathrm{L}^{-1}\{e^{-as}f(s)\}=\mathrm{G}(t)$,

where $\mathrm{G}(t)=\begin{cases}F(t-a), t>a\\ 0 \quad\quad, t<a\end{cases}$ [**Meerut 2008**]

Proof. we have, by definition of Laplace transform

$$\mathrm{L}\{\mathrm{G}(t)\}=\int_0^\infty e^{-st}\,\mathrm{G}(t)\,dt=\int_0^a e^{-st}\,\mathrm{G}(t)\,dt+\int_a^\infty e^{-st}\,\mathrm{G}(t)\,dt=\int_0^a e^{-st}(0)dt+\int_a^\infty e^{-st}\,\mathrm{F}(t-a)dt,$$

[using the given value of $\mathrm{G}(t)$]

$$=\int_a^\infty e^{-st}\,\mathrm{F}(t-a)dt=\int_o^\infty e^{-s(a+x)}F(x)dx\text{, putting } t-a=x \text{ and } dt=dx$$

$$=\int_0^\infty e^{-as}e^{-sx}F(x)dx=e^{-as}\int_0^\infty e^{-st}\,\mathrm{F}(t)dt=e^{-as}\mathrm{L}\{\mathrm{F}(t)\}\text{, by definition}$$

Thus, $\mathrm{L}\{\mathrm{G}(t)\}=e^{-as}f(s)$, as $\mathrm{L}^{-1}\{f(s)\}=\mathrm{F}(t) \Rightarrow \mathrm{L}\{\mathrm{F}(t)\}=f(s)$.

Hence by the definition of inverse Laplace transform, we get

$$\mathrm{L}^{-1}\{e^{-as}f(s)\}=\mathrm{G}(t),\text{ as required}$$

Remark. The result of this theorem can also be expressed in the following two ways:

1. $\mathrm{L}^{-1}\{f(s)\}=\mathrm{F}(t)\Rightarrow\mathrm{L}^{-1}\{e^{-as}f(s)\}=\begin{cases}\mathrm{F}(t-a), t>a\\ 0 \quad\quad, t<a\end{cases}$

2. $\mathrm{L}^{-1}\{f(s)\}=\mathrm{F}(t)\Rightarrow\mathrm{L}^{-1}\{e^{-as}f(s)\}=\mathrm{F}(t-a)\,\mathrm{H}(t-a)$

where H $(t-a)$ is Heaviside unit step function which is defined as follows:

$$\mathrm{H}(t-a)=\begin{cases}1, \text{when } t>a\\ 0, \text{when } t<a\end{cases}$$

2.9A. Solved examples based on Art. 2.9

Ex. 1. *Evaluate* $L^{-1}\left\{e^{-4s}/(s-3)^4\right\}$ **[Agra 92]**

Sol. Let $f(s) = 1/(s-3)^4$ and $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\}$ ... (1)

$\therefore \quad \mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\left\{\dfrac{1}{(s-3)^4}\right\} = e^{3t}\,\mathrm{L}^{-1}\left\{\dfrac{1}{s^4}\right\}$, using first shifting theorem

or $$\mathrm{F}(t) = e^{3t} \times (t^3/3!) = (1/6) \times t^3 e^{3t} \quad \text{... (2)}$$

Hence by second shifting theorem, $\mathrm{L}^{-1}\left\{e^{-4s} f(s)\right\} = \begin{cases} \mathrm{F}(t-4), & t > 4 \\ 0, & t < 4 \end{cases}$

or $$\mathrm{L}^{-1}\left\{e^{-4s}\frac{1}{(s-3)^4}\right\} = \begin{cases} (1/6)\times(t-4)^3 e^{3(t-4)}, & t > 4 \\ 0, & t < 4 \end{cases}$$ [using (1) and (2)]

$= (1/6) \times (t-4)^3 e^{3(t-4)}\,\mathrm{H}(t-4)$, in terms of Heaviside unit step function.

Important note. The students should give only one form of the final answer. However, if a particular form is asked, then give the corresponding form of answer.

Ex. 2. *Evaluate* $\mathrm{L}^{-1}\left\{e^{4-3s}/(s+4)^{5/2}\right\}$

Sol. Let $f(s) = e^4/(s+4)^{5/2}$ and $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\}$ ... (1)

$\therefore \quad \mathrm{F}(t) = \mathrm{L}^{-1}\left\{\dfrac{e^4}{(s+4)^{5/2}}\right\} = e^4\,\mathrm{L}^{-1}\left\{\dfrac{1}{(s+4)^{5/2}}\right\} = e^4 e^{-4t}\,\mathrm{L}^{-1}\left\{\dfrac{1}{s^{5/2}}\right\}$

or $$\mathrm{F}(t) = e^{4(1-t)}\frac{t^{3/2}}{\Gamma(5/2)} = e^{4(1-t)} \cdot \frac{t^{3/2}}{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{4t^{3/2}e^{4(1-t)}}{3\sqrt{\pi}} \quad \text{... (2)}$$

Hence by second shifting theorem, $\mathrm{L}^{-1}\left\{e^{-3s} f(s)\right\} = \begin{cases} \mathrm{F}(t-3), & t > 3 \\ 0, & t < 3 \end{cases}$

or $$\mathrm{L}^{-1}\left\{e^{-3s}\frac{e^4}{(s+4)^{5/2}}\right\} = \begin{cases} \left(4/3\sqrt{\pi}\right)(t-3)^{3/2} e^{4[1-(t-3)]}, & t > 3 \\ 0, & t < 3 \end{cases}$$ [using (1) and (2)]

or $$\mathrm{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = \begin{cases} \left(4/3\sqrt{\pi}\right)(t-3)^{3/2} e^{-4(t-4)}, & t > 3 \\ 0, & t < 3 \end{cases}$$

$= \left(4/3\sqrt{\pi}\right)(t-3)^{3/2} e^{-4(t-4)}\,\mathrm{H}(t-3)$, in terms of Heaviside unit step function.

Ex. 3. *Evaluate* (*i*) $\mathrm{L}^{-1}\left\{\dfrac{e^{-s\pi}}{s^2+1}\right\}$ **[Kanpur 2008]** (*ii*) $\mathrm{L}^{-1}\left\{\dfrac{se^{-2\pi s/3}}{s^2+9}\right\}$ **[Purvanchal 1995]**

Sol. (*i*) Let $f(s) = 1/(s^2+1)$ and $\mathsf{F}(t) = \mathrm{L}^{-1}\{f(s)\}$ ... (1)

$\therefore$ $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\{1/(s^2+1)\} = \sin t$ ... (2)

Hence by second shifting theorem, we have

$$\mathrm{L}^{-1}\{e^{-s\pi} f(s)\} = \begin{cases} F(t-\pi), & t > \pi \\ 0 & , t < \pi \end{cases}$$

or $$\mathrm{L}^{-1}\left\{e^{-s\pi}\frac{1}{s^2+1}\right\} = \begin{cases} \sin(t-\pi), & t > \pi \\ 0 & , t < \pi \end{cases}$$ [using (1) and (2)]

$$= \begin{cases} -\sin t, & t > \pi \\ 0 & , t < \pi \end{cases}$$

$= -\sin t\ \mathrm{H}(t-\pi)$, in terms of Heaviside unit step function

(*ii*) **Ans.** $\cos 3t\ \mathrm{H}(t - 2\pi/3)$ or $\begin{cases} \cos 3t, & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$

Ex.4. Evaluate $\mathrm{L}^{-1}\left\{s\,e^{-as}/(s^2-\omega^2)\right\}$, $a > 0$

Sol. Let $f(s) = s/(s^2-\omega^2)$ ... (1)

$\therefore$ $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\{s/(s^2-\omega^2)\} = \cosh \omega t$ ... (2)

Hence by second shifting theorem, $\mathrm{L}^{-1}\{e^{-as} f(s)\} = \begin{cases} \mathrm{F}(t-a), & t > a \\ 0 & , t < a \end{cases}$

or $$\mathrm{L}^{-1}\left\{e^{-as}\frac{s}{s^2-\omega^2}\right\} = \begin{cases} \cosh\omega(t-a), & t > a \\ 0 & , t < a \end{cases}$$ [using (1) and (2)]

$= \cosh\omega(t-a)\,\mathrm{H}(t-a)$, in terms of Heaviside unit step funtion.

Ex. 5. *Evaluate $L^{-1}\{3(1 + e^{-s\pi})/(s^2 + 9)\}$.* **[KU Kurukshetra 2004]**

Sol. We have, $$\mathrm{L}^{-1}\left\{\frac{3(1+e^{-s\pi})}{s^2+9}\right\} = \mathrm{L}^{-1}\left\{\frac{3}{s^2+9}\right\} + \mathrm{L}^{-1}\left\{\frac{3e^{-s\pi}}{s^2+9}\right\}$$ ... (1)

Let $f(s) = 3/(s^2+9)$ ... (2)

$\therefore$ $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = 3 \times \frac{1}{3}\sin 3t = \sin 3t$ ... (3)

By second shifting theorem, we have $\mathrm{L}^{-1}\{e^{-s\pi} f(s)\} = \begin{cases} \mathrm{F}(t-\pi), & t > \pi \\ 0 & , t < \pi \end{cases}$

or $$\mathrm{L}^{-1}\left\{e^{-s\pi}\frac{3}{s^2+9}\right\} = \begin{cases} \sin 3(t-\pi), & t > \pi \\ 0 & , t < \pi \end{cases}$$ [using (1) and (2)]

$$= \begin{cases} -\sin 3t, & t > \pi \\ 0 & , t < \pi \end{cases}$$ or $-\sin 3t\ \mathrm{H}(t-\pi)$, ... (4)

where H($t - \pi$) is the Heaviside unit step function

Using (3)and (4), (1) to reduces $\mathrm{L}^{-1}\left\{\dfrac{3(1+e^{-s\pi})}{s^2+9}\right\} = \sin 3t - \sin 3t\ \mathrm{H}(t-\pi).$

Ex. 6. *Evaluate $L^{-1}\{(s + 1)\, e^{-\pi s}/(s^2 + s + 1)\}$* **[Agra 2009]**

Sol. Let $f(s) = (s + 1) / (s^2 + s + 1).$ ... (1)

$$\therefore \quad \mathrm{F}(t) = L^{-1}\{f(s)\} = \mathrm{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = \mathrm{L}^{-1}\left\{\frac{(s+1/2)+1/2}{(s+1/2)^2+3/4}\right\}$$

$$= e^{-t/2}\,\mathrm{L}^{-1}\left\{\frac{s+1/2}{s^2+(3/4)}\right\}, \text{ by first shifting theorem}$$

$$= e^{-t/2}\left[\mathrm{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{3}/2)^2}\right\}+\frac{1}{2}\mathrm{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{3}/2)^2}\right\}\right]$$

$$= e^{-t/2}\left[\cos\left(\frac{1}{2}\sqrt{3}t\right)+\frac{1}{2}\times\frac{1}{(\sqrt{3}/2)}\sin\left(\frac{1}{2}\sqrt{3}t\right)\right]$$

Thus, $\mathrm{F}(t) = (1/\sqrt{3})\times e^{-t/2}\left[\sqrt{3}\cos(t\sqrt{3}/2)+\sin(t\sqrt{3}/2)\right]$ ... (2)

Hence by second shifting theorem, we have $\mathrm{L}^{-1}\{e^{-\pi s} f(s)\} = \begin{cases} F(t-\pi), & t > \pi \\ 0, & t < \pi \end{cases}$

or $$\mathrm{L}^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+s+1}\right\} = \begin{cases} \dfrac{1}{\sqrt{3}}e^{-(t-\pi)/2}\left\{\sqrt{3}\cos\dfrac{\sqrt{3}(t-\pi)}{2}+\sin\dfrac{\sqrt{3}(t-\pi)}{2}\right\}, & t > \pi \\ 0, \quad t < \pi. \end{cases}$$

[using (1) and (2)]

$$= \frac{1}{\sqrt{3}}e^{(t-\pi)/2}\left\{\sqrt{3}\cos\frac{\sqrt{3}(t-\pi)}{2}+\sin\frac{\sqrt{3}(t-\pi)}{2}\right\}\mathrm{H}(t-\pi),$$

in terms of Heaviside unit step function.

Ex. 7. Find a function F(t) for which $\mathrm{F}(t) = \mathrm{L}^{-1}\left\{\dfrac{3}{s} - \dfrac{4e^{-s}}{s^2} + \dfrac{4e^{-3s}}{s^2}\right\}$

Sol. Re-writing $\mathrm{F}(t) = 3\,\mathrm{L}^{-1}\left\{\dfrac{1}{s}\right\} - 4\,\mathrm{L}^{-1}\left\{e^{-s}\times\dfrac{1}{s^2}\right\} + 4\,\mathrm{L}^{-1}\left\{e^{3s}\times\dfrac{1}{s^2}\right\}$ ... (1)

But $\mathrm{L}^{-1}\{1/s\} = 1$ and $\mathrm{L}^{-1}\{1/s^2\} = t/1! = t$

Hence by second shifting theorem, we have

$\mathrm{L}^{-1}\left\{e^{-s}\times\dfrac{1}{s^2}\right\} = (t-1)\,\mathrm{H}(t-1)$ and $\mathrm{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2}\right\} = (t-3)\,\mathrm{H}(t-3)$

$\therefore$ (1) $\Rightarrow$ $\mathrm{F}(t) = 3 - 4\,(t-1)\,\mathrm{H}\,(t-1) + 4\,(t-3)\,\mathrm{H}\,(t-3)$

EXERCISE 2 (*E*)

Find the inverse Laplace transform of the following functions

1. e^{-3s}/s^3 **2.** $e^{-5s}/(s-2)^4$ **[Vikram 2004]**

3. $e^{-s}/(s+1)^3$ **[A.M.I.E. 1997]** **4.** $e^{-3s}/(s-4)^2$ **[Agra 2005]**

5. e^{-2s}/s **[Mysore 2004]** **6.** $e^{-3s}/\{s(s^2+4)\}$ **[Mysore 2004]**

ANSWERS

1. $(1/2)\times(t-3)^2\,\mathrm{H}(t-3)$ **2.** $(1/6)\times(t-5)^3 e^{2(t-5)}\,\mathrm{H}(t-5)$

3. $(1/2)\times e^{-(t-1)}(t-1)^2\,\mathrm{H}(t-1)$ **4.** $(t-3)\ e^{4(t-3)}\,\mathrm{H}(t-3)$

5. $\begin{cases}1, & t>2\\ 0, & t<2\end{cases}$ **6.** $\begin{cases}[1-\cos(t-3)]/4, & t>3\\ 0, & t<3\end{cases}$

2.10. Change of scale property [Guwahati 2000, Osmania 2004; Rohilkhand 96]

Statement. If $\mathrm{L}^{-1}\{f(s)\}=\mathrm{F}(t)$, then $\mathrm{L}^{-1}\{f(as)\}=(1/a)\times\mathrm{F}(t/a)$

Proof. Given $\mathrm{L}^{-1}\{f(s)\}=\mathrm{F}(t)$ so that $f(s)=\mathrm{L}\{F(t)\}$.

$$\therefore \qquad f(s)=\int_0^\infty e^{-st}\,\mathrm{F}(t)\,dt, \text{ by definition} \qquad \ldots(1)$$

Replacing s by as on both sides of (1), we have

$$f(as)=\int_0^\infty e^{-ast}\,\mathrm{F}(t)\,dt=\frac{1}{a}\int_0^\infty e^{-sx}\,\mathrm{F}\left(\frac{x}{a}\right)dx, \text{ putting } at=x \text{ and } dt=(1/a)\,dx$$

$$=\frac{1}{a}\int_0^\infty e^{-st}\,\mathrm{F}\left(\frac{t}{a}\right)dt=\frac{1}{a}\mathrm{L}\left\{\mathrm{F}\left(\frac{t}{a}\right)\right\}, \text{ by definition of Laplace transform}$$

Thus, $f(as)=\mathrm{L}\left\{\frac{1}{a}\mathrm{F}\left(\frac{t}{a}\right)\right\}$ so that $\mathrm{L}^{-1}\{f(as)\}=\frac{1}{a}\mathrm{F}\left(\frac{t}{a}\right)$.

2.10 A. Solved examples based on Art. 2.10

Ex. 1. *If* $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t$, *find* $L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\}$ **[Meerut 2001]**

Sol. Given $$\mathrm{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t \qquad \ldots(1)$$

Writing as for s in (1) and using change of scale property, we have

$$\mathrm{L}^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\}=\frac{1}{2}\times\frac{1}{a}\times\frac{t}{a}\sin\frac{t}{a} \qquad \ldots(2)$$

For the present problem, we put $a=4$ in (2) and obtain

$$\mathrm{L}^{-1}\left\{\frac{4s}{(4^2s^2+1)^2}\right\}=\frac{1}{2}\times\frac{1}{4}\times\frac{t}{4}\sin\frac{t}{4} \qquad \text{or} \qquad 8\,\mathrm{L}^{-1}\left\{\frac{4s}{(16s^2+1)^2}\right\}=8\times\frac{1}{32}t\sin\frac{t}{4}$$

Thus, $$\mathrm{L}^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\}=\frac{1}{4}t\sin\frac{t}{4}.$$

Ex. 2. *If* $L^{-1}\left\{\dfrac{e^{-1/s}}{s^{1/2}}\right\}=\dfrac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$, *find* $L^{-1}\left\{\dfrac{e^{-a/s}}{s^{1/2}}\right\}$, *where* $a > 0$.

Sol. Given that $$\mathrm{L}^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\}=\frac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}} \quad \ldots (1)$$

Writing ks for s in (1) and using change of scale property, we have

$$\mathrm{L}^{-1}\left\{\frac{e^{-1/ks}}{(ks)^{1/2}}\right\}=\frac{1}{k}\times\frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t/k)}} \quad \text{or} \quad \frac{1}{\sqrt{k}}\mathrm{L}^{-1}\left\{\frac{e^{-1/ks}}{s^{1/2}}\right\}=\frac{1}{\sqrt{k}}\times\frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t)}}$$

For the present problem, we put $k = 1/a$ *and obtain* $$\mathrm{L}^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\}=\frac{\cos 2\sqrt{(ta)}}{\sqrt{(\pi t)}}$$

EXERCISE 2 (*F*)

1. If $\mathrm{L}^{-1}\left\{\dfrac{s}{s^2-16}\right\}=\cosh 4t$, prove that $\mathrm{L}^{-1}\left\{\dfrac{s}{2s^2-8}\right\}=\dfrac{1}{2}\cosh 2t$

2. If $\mathrm{L}^{-1}\left\{\dfrac{s^2-1}{(s^2+1)^2}\right\}=t\cos t$, prove that $\mathrm{L}^{-1}\left\{\dfrac{9s^2-1}{(9s^2+1)^2}\right\}=\dfrac{1}{9}t\cos\dfrac{t}{3}$

2.11. Inverse Laplace transform of derivatives [Agra 2012]

Theorem. If $\mathrm{L}^{-1}\{f(s)\}=\mathrm{F}(t)$, then $\mathrm{L}^{-1}\left\{\dfrac{d^n}{ds^n}f(s)\right\}=(-1)^n t^n\,\mathrm{F}(t), n=1,2,3\ldots$

Proof. We know that (Refer Art. 2.16, Chapter 1)

$$\mathrm{L}\{\mathrm{F}(t)\}=f(s) \quad \Rightarrow \quad \mathrm{L}\left\{t^n\,\mathrm{F}(t)\right\}=(-1)^n\frac{d^n f(s)}{ds^n} \quad \ldots (1)$$

[Result (1) must be proved in examination for complete proof]
Then by the definition of inverse Laplace transform, we have

$$\mathrm{L}^{-1}\left\{(-1)^n\frac{d^n f(s)}{ds^n}\right\}=t^n\,\mathrm{F}(t) \quad \text{or} \quad (-1)^n\,\mathrm{L}^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=t^n\,\mathrm{F}(t)$$

Multiplying both sides by $(-1)^n$ and noting that $(-1)^{2n} = 1$, we have

$$\mathrm{L}^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=(-1)^n t^n\,\mathrm{F}(t) \quad \ldots (2)$$

Note. The result (2) can also be written as $\mathrm{L}^{-1}\left\{\dfrac{d^n}{ds^n}f(s)\right\}=(-1)^n t^n\,\mathrm{L}^{-1}\{f(s)\}$

2.11A. Solved examples based on Art. 2.11.

Ex. 1. *Find* (*i*) $\mathrm{L}^{-1}\left\{s/(s^2+a^2)^2\right\}$ **[Agra 2005, 2012, Ravishanker 2005]**

(*ii*) $\mathrm{L}^{-1}\left\{s/(s^2+4)^2\right\}$ **[Guwahati 2000, Kumaun 1998, Purvanchal 2003]**

Sol. (*i*) Let $f(s) = 1/(s^2+a^2) = (s^2+a^2)^{-1}$ ... (1)

$\therefore \quad \dfrac{d}{ds} f(s) = -(s^2+a^2)^{-2} \times 2s = -\dfrac{2s}{(s^2+a^2)^2}$

Hence $\dfrac{s}{(s^2+a^2)^2} = -\dfrac{1}{2}\dfrac{d}{ds} f(s) = -\dfrac{1}{2}\dfrac{d}{ds}\left(\dfrac{1}{s^2+a^2}\right)$, using (1)

$\therefore \quad \mathrm{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = -\dfrac{1}{2}\mathrm{L}^{-1}\left\{\dfrac{d}{ds}\left(\dfrac{1}{s^2+a^2}\right)\right\} = -\dfrac{1}{2}\left[(-1)^1 t\ \mathrm{L}^{-1}\left\{\dfrac{1}{s^2+a^2}\right\}\right] = \dfrac{t}{2a}\sin at$

(*ii*) Proceed as in part (*i*) Here $a = 2$. **Ans.** $(t/4) \times \sin 2t$

Ex.2. Evaluate (*a*) $\mathrm{L}^{-1}\left\{(s+1)/(s^2+2s+2)^2\right\}$. **[Meerut 1998]**

(*b*) $\mathrm{L}^{-1}\{(s+2)/(s^2+4s+5)^2\}$ **[Kerala 2001]**

Sol. (*a*) Let $f(s) = 1/(s^2+2s+2) = (s^2+2s+2)^{-1}$ ... (1)

$\therefore \quad \dfrac{d}{ds} f(s) = -(s^2+2s+2)^{-2}(2s+2) = -2\left(\dfrac{s+1}{(s^2+2s+2)^2}\right)$

Hence $\dfrac{s+1}{(s^2+2s+2)^2} = -\dfrac{1}{2}\dfrac{d}{ds} f(s) = -\dfrac{1}{2}\dfrac{d}{ds}\left\{\dfrac{1}{s^2+2s+2}\right\}$, using (1)

$\therefore \quad \mathrm{L}^{-1}\left\{\dfrac{s+1}{(s^2+2s+2)^2}\right\} = -\dfrac{1}{2}\mathrm{L}^{-1}\left\{\dfrac{d}{ds}\left(\dfrac{1}{s^2+2s+2}\right)\right\} = -\dfrac{1}{2}\left[(-1)^1 t\ \mathrm{L}^{-1}\left\{\dfrac{1}{s^2+2s+2}\right\}\right]$

$= \dfrac{1}{2} t\,\mathrm{L}^{-1}\left\{\dfrac{1}{(s+1)^2+1^2}\right\} = \dfrac{t}{2} e^{-t}\,\mathrm{L}^{-1}\left\{\dfrac{1}{s^2+1^2}\right\} = \dfrac{1}{2} te^{-t}\sin t$

(*b*) Do yourself **Ans.** $(t/2) \times e^{-2t} \sin t$.

Ex. 3. *Evaluate* $L^{-1}\left\{log\dfrac{1+s}{s}\right\}$. **[S.V. University. (AP) 1997]**

Sol. Let $f(s) = \log\dfrac{1+s}{s} = \log(1+s) - \log s$ ... (1)

$\Rightarrow \quad \dfrac{d}{ds} f(s) = \dfrac{1}{s+1} - \dfrac{1}{s} \quad \Rightarrow \quad \mathrm{L}^{-1}\left\{\dfrac{d}{ds} f(s)\right\} = \mathrm{L}^{-1}\left\{\dfrac{1}{s+1}\right\} - \mathrm{L}^{-1}\left\{\dfrac{1}{s}\right\}$

$\Rightarrow \quad (-1)^1 t\ \mathrm{L}^{-1}\{f(s)\} = e^{-t} - 1 \ \Rightarrow\ \mathrm{L}^{-1}\{f(s)\} = (1-e^{-t})/t$

$\Rightarrow \quad \mathrm{L}^{-1}\left\{\log\dfrac{1+s}{s}\right\} = \dfrac{1-e^{-t}}{t}$, using (1).

Ex. 4. *Find (i)* $\mathrm{L}^{-1}\left\{\dfrac{s}{(s^2-a^2)^2}\right\}$ (*ii*) $\mathrm{L}^{-1}\left\{\dfrac{s}{(s^2-16)^2}\right\}$

Sol. (*i*) Let $f(s) = 1/(s^2-a^2) = (s^2-a^2)^{-1}$ ... (1)

$$\therefore \quad \frac{d}{ds}f(s) = -(s^2-a^2)^{-2} \times 2s = -2\left[\frac{s}{(s^2-a^2)^2}\right]$$

Hence $$\frac{s}{(s^2-a^2)^2} = -\frac{1}{2}\frac{d}{ds}f(s) = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2-a^2}\right), \text{ using (1).}$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = -\frac{1}{2}\mathrm{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2-a^2}\right)\right\} = -\frac{1}{2}\left[(-1)^1 t\,\mathrm{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\}\right] = \frac{t}{2a}\sinh at$$

(*i*) Proceed as in part (*i*) Note that here $a = 4$. **Ans.** $(t/8) \times \sinh 4t$.

Ex. 5. *Evaluate* $L^{-1}\{\log(1 - 1/s^2)\}$ **[Meerut 2004]**

Sol. (*i*) Let $f(s) = \log(1 - 1/s^2) = \log(s^2 - 1) - 2\log s.$ ... (1)

$$\therefore \quad \frac{d}{ds}f(s) = \frac{1}{s^2-1} \times 2s - \frac{2}{s} = -2\left(\frac{1}{s} - \frac{s}{s^2-1}\right)$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{d}{ds}f(s)\right\} = -2\left[\mathrm{L}^{-1}\left\{\frac{1}{s}\right\} - \mathrm{L}^{-1}\left\{\frac{s}{s^2-1}\right\}\right] \quad \text{or} \quad (-1)^1 t\,\mathrm{L}^{-1}\{f(s)\} = -2(t - \cosh t)$$

or $$\mathrm{L}^{-1}\{f(s)\} = \frac{2(1-\cosh t)}{t} \quad \text{or} \quad \mathrm{L}^{-1}\left\{\log\left(1-\frac{1}{s^2}\right)\right\} = \frac{2(1-\cosh t)}{t}, \text{ by (1)}$$

Ex. 6. Find the inverse Laplace transform of the function $\log(1 + w^2/s^2)$ **[Lucknow 1997]**

Sol. Let $f(s) = \log(1 + w^2/s^2) = \log(w^2 + s^2) - 2\log s$ ... (1)

$$\therefore \quad \frac{d}{ds}f(s) = \frac{2s}{w^2(e^{st} + s^2} - \frac{2}{-\frac{s}{e^{-5t}})/t} = -2\left(\frac{1}{s} - \frac{s}{s^2+w^2}\right)$$

$$\therefore \quad \mathrm{L}^{-1}\left\{\frac{d}{ds}f(s)\right\} = -2\left[\mathrm{L}^{-1}\left\{\frac{1}{s}\right\} - \mathrm{L}^{-1}\left\{\frac{s}{s^2+w^2}\right\}\right] = -2(t - \cos wt)$$

or $$(-1)^1 t^1\,\mathrm{L}^{-1}\{f(s)\} = -2(t - \cos wt) \quad \text{or} \quad \mathrm{L}^{-1}\{\log(1 + w^2/s^2)\} = 2(1 - \cos wt)t, \text{ by (1)}$$

EXERCISE 2 (*G*)

Find the inverse Laplace transform of the following functions:

1. $\log\{(s + a)/(s + b)\}$ **[Purvanchal 2004]** **2.** $\log(1 + 1/s^2)$

3. $\log[(s + 3)/(s + 2)\}$ **[Bhopal 2004]** **4.** $\log\{(s^2 + 1)/(s - 1)^2\}$ **[MDU Rohtak 2005]**

5. $\log(1 + 1/s)$ **[Nagpur 2010]** **Ans.** $(1 - \cos 2t)/t$

ANSWERS

1. $(e^{-bt} - e^{-at})/t$ 2. $2(1 - \cos t)/t$ 3. $(e^{-2t} - e^{-3t})/t$ 4. $2(e^t - \cos t)/t$

2.12. Inverse Laplace transform of integrals

Theorem. If $\mathrm{L}^{-1}\{f(s)\} = \mathrm{F}(t)$, then $$\mathrm{L}^{-1}\left\{\int_s^\infty f(s)\,ds\right\} = \frac{\mathrm{F}(t)}{t}$$

Proof. We know that (Refer Art. 1.17, Chapter 1)

$$\mathrm{L}\{\mathrm{F}(t)\} = f(s) \quad \Rightarrow \quad \mathrm{L}\left\{\frac{\mathrm{F}(t)}{t}\right\} = \int_s^\infty f(s)\,ds, \quad \text{... (1)}$$

provided the integral exists.

[Result (1) must be proved in examination for complete proof]

Hence by the definition of inverse Laplace transform, $$\mathrm{L}^{-1}\left\{\int_s^\infty f(s)\,ds\right\} = \frac{\mathrm{F}(t)}{t}$$

Illustrative Example. *Prove that* $L^{-1}\left\{\int_s^\infty \frac{1}{s(s+1)}ds\right\} = \frac{1-e^{-t}}{t}$

Sol. Let $f(s) = 1/s(s+1)$ and $\mathrm{L}^{-1}\{f(s)\} = \mathrm{F}(t)$.

Then, $\mathrm{F}(t) = \mathrm{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$, on resolving into partial fractions

Thus, $\mathrm{F}(t) = \mathrm{L}^{-1}\{1/s\} - \mathrm{L}^{-1}\{1/(s+1)\} = 1 - e^{-t}$.

Now, $\mathrm{L}^{-1}\left\{\int_s^\infty f(s)\,ds\right\} = \frac{\mathrm{F}(t)}{t} \quad \Rightarrow \quad \mathrm{L}^{-1}\left\{\int_s^\infty \frac{1}{s(s+1)}ds\right\} = \frac{1-e^{-t}}{t}$.

2.13. Multiplication by powers of *s* [Agra 2011]

Theorem I. If $\mathrm{L}^{-1}\{f(s)\} = \mathrm{F}(t)$ and $\mathrm{F}(0) = 0$, then $\mathrm{L}^{-1}\{sf(s)\} = \mathrm{F}'(t)$

Proof. We know that (Refer Art. 1.15, Chapter 1)

$$\mathrm{L}\{\mathrm{F}'(t)\} = sf(s) - \mathrm{F}(0) \qquad \ldots (1)$$

[Result (1) must be proved in examination for complete proof]

Since F(0) = 0 (given), (1) reduces to

$\mathrm{L}\{\mathrm{F}'(t)\} = s\,f(s)$ so that $\mathrm{L}^{-1}\{sf(s)\} = \mathrm{F}'(t)$

Theorem II. If $\mathrm{F}(0) = \mathrm{F}'(0) = \mathrm{F}''(0) = \ldots = F^{(n-1)}(0) = 0$, then $\mathrm{L}^{-1}\{s^n f(s)\} = F^{(n)}(t)$,

where $\mathrm{F}^{(n)}(t) = \frac{d^n}{dt^n}\mathrm{F}(t)$ and $\mathrm{F}^{(n-1)}(0) = \left[\frac{d^{n-1}}{dt^{n-1}}\mathrm{F}(t)\right]_{t=0}$

Proof. We know that (Refer Art. 1.15, chapter 1)

$$\mathrm{L}\{\mathrm{F}^{(n)}(t)\} = s^n f(s) - s^{n-1}\mathrm{F}(0) - s^{n-2}\mathrm{F}'(0) - \ldots - \mathrm{F}^{(n-1)}(0) \qquad \ldots (2)$$

[Result (2) must be proved in examination for complete proof]

Since $\mathrm{F}(0) = \mathrm{F}'(0) = \ldots \mathrm{F}^{(n-1)}(0) = 0$ (given), so (2) reduces to

$\mathrm{L}\{\mathrm{F}^{(n)}(t)\} = s^n f(s)$ so that $\mathrm{L}^{-1}\{s^n f(s)\} = \mathrm{F}^{(n)}(t)$

Illustrative Solved example. *Evaluate* $L^{-1}\{s^2/(s^2+4)^2\}$

Sol. Let $f(s) = 1/(s^2+4) = (s^2+4)^{-1}$. Then $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\} = (\sin 2t)/2$.

$$\therefore \quad \frac{d}{ds}f(s) = -(s^2+4)^{-2} \times 2s = (-2) \times \frac{s}{(s^2+4)^2}.$$

$$\therefore \ \mathrm{L}^{-1}\left\{\frac{d}{ds}f(s)\right\} = \mathrm{L}^{-1}\left\{(-2)\times\frac{s}{(s^2+4)^2}\right\} \quad \text{or} \quad (-1)^1 t\,\mathrm{F}(t) = -2\,\mathrm{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$

so that $$\mathrm{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{2}t\,\mathrm{F}(t) = \frac{1}{4}t\sin 2t \qquad \ldots (1)$$

Let $\mathrm{g}(s) = s/(s^2+4)^2$ and $\mathrm{G}(t) = (t/4) \times \sin 2t \quad \ldots (2)$

Here G (0) = 0. Hence we have

$$L^{-1}\{s\,g(s)\} = G'(t) \qquad \text{or} \qquad L^{-1}\left\{s \times \frac{s}{(s^2+4)^2}\right\} = \frac{d}{dt}\left(\frac{t}{4}\sin 2t\right), \text{ by (2).}$$

or $$L^{-1}\{s^2/(s^2+4)^2\} = (\sin 2t + 2t\cos 2t)/4.$$

2.14A. Division by powers of *s*.

Theorem I. *Let $L^{-1}\{f(s)\} = F(t)$. If $F(t)$ is sectionally continuous and of exponential order s such that $\lim_{t\to 0}\{F(t)/t\}$ exists, then for $s > \sigma$*

$$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(x)\,dx \qquad \text{or} \qquad L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(t)\,dt \qquad \textbf{[Kanpur 1997; Rohilkhand 1997]}$$

Proof. Let $$G(t) = \int_0^t F(x)\,dx \qquad \ldots (1)$$

From (1), $G'(t) = F(t)$ and $G(0) = 0$ ... (2)

Now, we know that (refer Art. 1.15) $L\{G'(t)\} = s\,L\{G(t)\} - G(0)$

or $L\{F(t)\} = s\,L\{G(t)\}$, using (1)

or $f(s) = s\;L\{G(t)\}$, as $L^{-1}\{f(s)\} = F(t) \Rightarrow L\{F(t)\} = f(s)$

or $$L\{G(t)\} = \frac{f(s)}{s} \qquad \text{or} \qquad L^{-1}\left\{\frac{f(s)}{s}\right\} = G(t) = \int_0^t F(x)dx$$

Theorem II. Let $L^{-1}\{f(s)\} = F(t)$, then

$$L^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^y F(x)\,dx\,dy \qquad \text{or} \qquad L^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^t F(t)\,dt^2 \qquad \textbf{[Meerut 2001]}$$

Proof. Let $$G(t) = \int_0^t \int_0^y F(x)\,dx\,dy \qquad \ldots (i)$$

$$\therefore \qquad G'(t) = \int_0^t F(x)dx \qquad \text{and} \qquad G''(t) = F(t) \qquad \ldots (ii)$$

Then, (i) and $(ii) \Rightarrow$ $G(0) = 0$ and $G'(0) = 0$ ... (iii)

Now, we know that (refer Art. 1.15) $L\{G''(t)\} = s^2 L\{G(t)\} - s\,G(0) - G'(0)$

or $L\{F(t)\} = s^2 L\{G(t)\}$, using (ii) and (iii)

or $f(s) = s^2 L\{G(t)\}$, as $L^{-1}\{f(s)\} = F(t) \Rightarrow L\{F(t)\} = f(s)$

or $$L\{G(t)\} = \frac{f(s)}{s^2} \qquad \text{or} \qquad L^{-1}\left\{\frac{f(s)}{s^2}\right\} = G(t)$$

or $$L^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^y F(x)\,dx\,dy, \text{ using } (i)$$

Theorem III. (General result) If $L^{-1}\{f(s)\} = F(t)$, then

$$L^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t \int_0^t \ldots\ldots \int_0^t F(t)\,t^n \qquad \textbf{[Meerut 2001]}$$

Proof. Left as an exercise.

2.14B. An important theorem. *To show that $L^{-1}\{f(s)\} = -(1/t) \times L^{-1}\{f'(s)\}$*

Proof. From Art. 1.16 of chapter 1, we have

$$L\{F(t)\} = f(s) \qquad \Rightarrow \qquad L\{t\,F(t)\} = -f'(s) \qquad \ldots (1)$$

In terms of inverse Laplace transforms, (1) reduces to

$t\,F(t) = L^{-1}\{-f'(s)\}$ or $t\,L^{-1}\{f(s)\} = -L^{-1}\{f'(s)\}$

Thus, $L^{-1}\{f(s)\} = -(1/t)\,L^{-1}\{f'(s)\}$.

2.14C. Solved examples based on Art. 2.14A and 2.14B

Ex.1. *Find* (*i*) $L^{-1}\{\log(1 + w^2/s^2)\}$

(*ii*) $L^{-1}\{\log(1 - 1/s^2)\}$ **[Meerut 2000]**

Sol. (*i*) Let $f(s) = \log(1 + w^2/s^2) = \log(s^2 + w^2) - \log s^2$.

$$\therefore \quad f'(s) = \frac{d}{ds}\left[\log(s^2 + w^2) - 2\log s\right] = \frac{2s}{s^2 + w^2} - \frac{2}{s}$$

We know that $L^{-1}\{f(s)\} = -(1/t) \times L^{-1}\{f'(s)\}$

$$\therefore \quad L^{-1}\left\{\log\left(1 + \frac{w^2}{s^2}\right)\right\} = -\frac{1}{t}L^{-1}\left\{2\left(\frac{s}{s^2 + w^2} - \frac{1}{s}\right)\right\} = -\frac{2}{t}(\cos wt - 1)$$

(*ii*) Proceed as in part (*i*). **Ans.** $(2/t) \times (1 - \cosh t)$

Ex.2. *Find* (*i*) $L^{-1}\{\log(1 + a/s)\}$ (*ii*) $L^{-1}\{\log[(s+1)/(s-1)]\}$

(*iii*) $L^{-1}\{\log[(s-1)/(s+1)]\}$ **[Banglore 2005]**

Sol. (*i*) Let $f(s) = \log(1 + a/s) = \log\{(s + a)/s\} = \log(s + a) - \log s$.

$\therefore \quad f'(s) = 1/(s + a) - 1/s$

We know that $L^{-1}\{f(s)\} = -(1/t) \times L^{-1}\{f'(s)\}$

$\therefore \quad L^{-1}\{\log(1 + a/s)\} = -(1/t) \times L^{-1}\{1/(s + a) - 1/s\} = -(1/t) \times (e^{-at} - 1)$

(*ii*) Let $f(s) = \log\{(s + 1)/(s - 1)\} = \log(s + 1) - \log(s - 1)$.

$\therefore \quad f'(s) = 1/(s + 1) - 1/(s - 1)$.

We know that $L^{-1}\{f(s)\} = -(1/t)\,L^{-1}\{f'(s)\}$

$\therefore \quad L^{-1}\{\log(s + 1)/(s - 1)\} = -(1/t) \times L^{-1}\{1/(s + 1) - 1/(s - 1)\}$

$= (-1/t) \times (e^{-t} - e^{t}) = (2/t) \times \{(e^{t} - e^{-t})/2\} = (2/t) \times \sinh t$.

(*iii*) Do like part (*ii*) **Ans.** $-(2/t) \times \sinh t$

Ex. 3(*a*). *Find* (*i*) $L^{-1}\{\tan^{-1}(1/s)\}$ (*ii*) $L^{-1}\{\tan^{-1}(2/s)\}$ **[Chennai 1997]**

Sol. (*i*) Let $f(s) = \tan^{-1}(1/s) = \tan^{-1}(s^{-1})$

$$\therefore \quad f'(s) = \frac{1}{1 + (s^{-1})^2}(-s^{-2}) = \frac{s^2(-s^{-2})}{1 + s^2} = -\frac{1}{1 + s^2}$$

We know that $L^{-1}\{f(s)\} = -(1/t) \times L^{-1}\{f'(s)\}$

$\therefore \quad L^{-1}\{\tan^{-1}(1/s)\} = -(1/t) \times L^{-1}\{(-1)/(1 + s^2)\} = (1/t) \times \sin t$.

(*ii*) Do yourself **Ans.** $(1/t) \times \text{sint } 2t$

Ex. 3(*b*). *Evaluate* (*i*) $L^{-1}\{\cot^{-1}(1 + s)\}$ **[A.M.I.E 1997]**

(*ii*) $L^{-1}\{\cot^{-1}(s/w)\}$ **[Mysore 2004]**

Sol. (*i*) Let $f(s) = \cot^{-1}(1 + s)$ so that $f'(s) = -1/\{1 + (s+1)^2\}$

We know that $L^{-1}\{f(s)\} = -(1/t) \times L^{-1}\{f'(s)\}$

$$\therefore \quad L^{-1}\{\cot^{-1}(1+s)\} = \frac{1}{t}L^{-1}\left\{\frac{1}{1 + (s+1)^2}\right\} = \frac{1}{t}e^{-t}L^{-1}\left\{\frac{1}{1 + s^2}\right\} = \frac{e^{-t}\sin t}{t}$$

(*ii*) Try yourself **Ans.** $(\sin \omega t)/t$

Ex. 3 (*c*) Prove that $L^{-1}\{\tan^{-1}(2/s^2)\} = (2/t) \times \sin t \sinh t$ **[Jabalpur 2004]**

Sol. Let $f(s) = \tan^{-1}(2/s^2)$ so that $f'(s) = \dfrac{1}{1+(4/s^4)} \times \left(-\dfrac{4}{s^3}\right) = -\dfrac{4s}{s^4+4}$

We know that $L^{-1}\{f(s)\} = -(1/t) \times L^{-1}\{f'(s)\}$

$$\therefore \quad L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\} = -\frac{1}{t}L^{-1}\left\{-\frac{4s}{s^4+4}\right\} = \frac{4}{t}L^{-1}\left\{\frac{s}{s^4+4}\right\} \quad \ldots (1)$$

Proceeding as in Ex. 12, Art. 2.8 with $a = 1$, prove that $L^{-1}\{s/(s^4 + 4)\} = (\sinh t \sin t)/2$

$\therefore$ (1) reduces to $L^{-1}\{\tan^{-1}(2/s^2)\} = (2/t) \times \sinh t \sin t.$

Ex. 3. (*d*) Show that $L^{-1}\{(s + a) / (s + b)\} = (e^{-bt} - e^{-at}) / t$ **[Mysore 1994]**

Ex. 4. *Evaluate* (*i*) $L^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$ **[Meerut 2001]** (*iii*) $L^{-1}\left\{\dfrac{1}{s}\log\dfrac{s+2}{s+1}\right\}$

(*ii*) $L^{-1}\left\{\dfrac{1}{s^4(s^2+1)}\right\}$ **[Meerut 1999; Rohilkhand 1998]**

Sol. (*i*) Here $L^{-1}\{1/(s^2 + 1)\} = \sin t.$

$$\therefore \quad L^{-1}\left\{\frac{1}{s} \times \frac{1}{(s^2+1)}\right\} = \int_0^t \sin x\, dx = \left[-\cos x\right]_0^t = 1 - \cos t.$$

$$\therefore \quad L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos x)\, dx = t - \sin t$$

$$\therefore \quad L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (x - \sin x)\, dx = \frac{1}{2}t^2 + \cos t - 1$$

(*ii*) First proceed as in part (*i*) and prove that $L^{-1}\left\{\dfrac{1}{s^3(s+1)}\right\} = \dfrac{1}{2}t^2 + \cos t - 1$

$$\text{This} \quad \Rightarrow \quad L^{-1}\left\{\frac{1}{s} \times \frac{1}{s^3(s+1)}\right\} = \int_0^t \left(\frac{1}{2}x^2 + \cos x - 1\right) dx = \frac{x^3}{6} + \sin x - x.$$

(*iii*) Let $f(s) = \log\{(s+2)/(s+1)\} = \log(s + 2) - \log(s + 1)$

$\therefore \quad f'(s) = 1/(s + 2) - 1/(s + 1)$

$\therefore \; L^{-1}\{f'(s)\} = e^{-2t} - e^{-t}$ or $-t\, L^{-1}\{f(s)\} = e^{-2t} - e^{-t}.$

$$\therefore \; L^{-1}\{f(s)\} = \frac{e^{-t} - e^{-2t}}{t} \quad \text{or} \quad L^{-1}\left\{\log\frac{s+2}{s+1}\right\} = \frac{e^{-t} - e^{-2t}}{t}$$

$$\therefore \quad L^{-1}\left\{\frac{1}{s}\log\frac{s+2}{s+1}\right\} = \int_0^t \frac{e^{-x} - e^{-2x}}{x}\, dx$$

Ex. 5. *Evaluate $L^{-1}\{1/s(s+1)^3\}$* **[Agra 2008; Rohilkhand 1997]**

Sol. We have, $$L^{-1}\left\{\frac{1}{s(s+1)^3}\right\}=L^{-1}\left\{\frac{1}{[(s+1)-1](s+1)^3}\right\}=e^{-t}\,L^{-1}\left\{\frac{1}{(s-1)s^3}\right\}, \quad \text{... (1)}$$

[Using first shifting theorem]

Now, $$L^{-1}\left\{\frac{1}{s-1}\right\}=e^{t}\Rightarrow L^{-1}\left\{\frac{1}{(s-1)s}\right\}=\int_0^t e^x\,dx=e^t-1$$

$\therefore$ $$L^{-1}\left\{\frac{1}{(s-1)s^2}\right\}=\int_0^t (e^x-1)\,dx=e^t-t-1$$

$\therefore$ $$L^{-1}\left\{\frac{1}{(s-1)s^3}\right\}=\int_0^t (e^x-x-1)\,dx=e^t-\frac{1}{2}t^2-t-1$$

Putting this value in (1), we get

$$L^{-1}\{1/s(s+1)^3\}=e^{-t}(e^t-t^2/2-t-1)=1-e^{-t}(1+t+t^2/2).$$

Ex. 6. *If* $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t$, *find* $L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

Sol. $L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}=L^{-1}\left\{\frac{1}{s}\times\frac{s}{(s^2+1)^2}\right\}=\int_0^t \frac{1}{2}\,x\sin x\,dx$, using the given result.

$$=\frac{1}{2}\left\{\left[x(-\cos x)\right]_0^t-\int_0^t 1\times(-\cos x)\,dx)\right\}=\frac{1}{2}(-t\cos t+\sin t)$$

Ex. 7. *Evaluate $L^{-1}\{(s+2)/s^2(s+3)\}$*

Sol. Here, re-writing, we have

$$L^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\}=L^{-1}\left\{\frac{(s+3)-1}{s^2(s+3)}\right\}=L^{-1}\left\{\frac{1}{s^2}\right\}-L^{-1}\left\{\frac{1}{s^2(s+3)}\right\} \quad \text{... (1)}$$

Now, $$L^{-1}\{1/s^2\}=t/1!=t. \quad \text{... (2)}$$

and $$L^{-1}[1/(s+3)]=e^{-3t} \quad \text{... (3)}$$

Then, (3) $\Rightarrow$ $$L^{-1}\left\{\frac{1}{s(s+3)}\right\}=\int_0^t e^{-3x}dx=-\frac{1}{3}\left[e^{-3x}\right]_0^t=\frac{1}{3}\left(1-e^{-3t}\right) \quad \text{... (4)}$$

$\therefore$ (4) $\Rightarrow$ $$L^{-1}\left\{\frac{1}{s^2(s+3)}\right\}=\frac{1}{3}\int_0^t(1-e^{-3x})dx=\frac{1}{3}\left[x+\frac{1}{3}e^{-3x}\right]_0^t$$

$$=(1/3)\times\{t+(1/3)\times e^{-3t}-1/3\}=t/3+(1/9)\times(e^{-3t}-1) \quad \text{... (5)}$$

With help of (2) and (5), (1) reduces to

$$L^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\}=t-\left[\frac{1}{3}t+\frac{1}{9}\left(e^{-3t}-1\right)\right]=\frac{2}{3}t-\frac{1}{9}\left(e^{-3t}-1\right)$$

Ex. 8. *Find* (*i*) $L^{-1}\left\{\log\left(1+1/s^2\right)\right\}$ (*ii*) $\mathrm{L}^{-1}\left\{(1/s)\times\log\left(1+1/s^2\right)\right\}$

Sol. (*i*) Let $f(s) = \log(1 + 1/s^2) = \log(s^2 + 1) - 2\log s.$

$$\therefore \frac{d}{ds}f(s) = \frac{2s}{s^2+1} - \frac{2}{s} \quad \text{so that} \quad \mathrm{L}^{-1}\left\{\frac{d}{ds}f(s)\right\} = \mathrm{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s}\right\}$$

or $$(-1)^1\, t\, \mathrm{L}^{-1}\{f(s)\} = 2\,\mathrm{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\,\mathrm{L}^{-1}\left\{\frac{1}{s}\right\} = 2\cos t - 2$$

or $$\mathrm{L}^{-1}\{f(s)\} = \frac{2(1-\cos t)}{t} \quad \text{or} \quad \mathrm{L}^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = \frac{2(1-\cos t)}{t}$$

(*ii*) First proceed as in part (*i*) to show that $\mathrm{L}^{-1}\left\{\log\left(1+1/s^2\right)\right\} = 2(1-\cos t)/t$ ... (1)

Then, with help of Art 2.14 A, we have $\mathrm{L}^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\} = \int_0^x \frac{2(1-\cos x)}{x}dx$

Ex. 9. *Without expressing (s – 1)/s² (s² + 1) into partial fractions, find its inverse Laplace transform.* **[Lucknow 1998]**

Sol. Here $$\mathrm{L}^{-1}\left\{\frac{s-1}{s^2+1}\right\} = \mathrm{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathrm{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \cos t - \sin t$$

$$\Rightarrow \mathrm{L}^{-1}\left\{\frac{1}{s}\frac{s-1}{s^2+1}\right\} = \int_0^t (\cos x - \sin x)dx = \left[\sin x + \cos x\right]_0^t = \sin t + \cos t - 1$$

$$\Rightarrow \mathrm{L}^{-1}\left\{\frac{1}{s^2}\frac{s-1}{s^2+1}\right\} = \int_0^t (\sin x + \cos x - 1)dx = \left[-\cos x + \sin x - x\right]_0^t$$

$$= -\cos t + \sin t - t - (-1) = 1 - t + \sin t - \cos t.$$

Ex. 10. *Prove that* (*i*) $\mathrm{L}^{-1}\left\{e^{-\sqrt{s}}\right\} = \left(2\sqrt{\pi}t^{3/2}\right)^{-1} e^{-1/4t}$ **(Agra 2007)**

(*ii*) $L^{-1}\{(1/s)\times e^{-\sqrt{s}}\} = \mathrm{erfc}\,(1/2\sqrt{t})$ **[Kanpur 1995, 1996]**

(*iii*) $L^{-1}\{(1/s)\times e^{-c\sqrt{s}}\} = \mathrm{erfc}\,(c/2\sqrt{t})$ **[Kanpur 1995, 1996]**

Sol. In what follows we shall use the following results:

$$\mathrm{L}^{-1}\{s^{n+1/2}\} = \frac{t^{-n-3/2}}{\Gamma(-n-1/2)} \text{ for } n = 0, 1, 2, 3... \quad ....1\,(A)$$

$$\mathrm{L}^{-1}\{s^{n+1/2}\} = \frac{(-1)^{n+1}}{\sqrt{\pi}}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{2n+1}{2}\right)t^{-n-3/2} \quad ...1\,(B)$$

and $$\mathrm{L}^{-1}\{s^n\} = 0,\ n \text{ being any integer.} \quad ... 1\,(C)$$

Part (*i*). $$\mathrm{L}^{-1}\left\{e^{-\sqrt{s}}\right\} = \mathrm{L}^{-1}\left\{1 - s^{1/2} + \frac{s}{2!} - \frac{s^{3/2}}{3!} + \frac{s^2}{4!} - \frac{s^{5/2}}{5!} + \ldots\right\}$$

$$= \frac{t^{-3/2}}{2\sqrt{\pi}} - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{t^{-5/2}}{3!\sqrt{\pi}} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{t^{-7/2}}{5!\sqrt{\pi}} + \ldots, \text{ using 1(A), 1(B), 1(C)}$$

$$= \frac{1}{2\sqrt{\pi}t^{3/2}}\left\{1 - \frac{(1/4t)}{1!} + \frac{(1/4t)^2}{2!} - \frac{(1/4t)^3}{3!} + \ldots\right\}$$

Thus, $$\mathrm{L}^{-1}\left\{e^{-\sqrt{s}}\right\} = \left(2\sqrt{\pi}t^{3/2}\right)^{-1} e^{-1/4t}. \qquad \ldots (2)$$

Part (*ii*). Proceed as in part (*i*) and prove result (2). Then, we have

$$\mathrm{L}^{-1}\left\{\frac{e^{-\sqrt{s}}}{s}\right\} = \int_0^t \frac{1}{2\sqrt{\pi}\ x^{3/2}} e^{-1/4x}\,dx, \text{ by Art. 2.14A}$$

$$= \frac{2}{\sqrt{\pi}}\int_{1/2\sqrt{t}}^{\infty} e^{-u^2}\,du, \text{ putting } x = (1/4u^2) \text{ so that } dx = -(1/2) \times u^{-3}\,du$$

$$\therefore \qquad \mathrm{L}^{-1}\{e^{-\sqrt{s}}/s\} = \operatorname{erfc}(1/2\sqrt{t}), \text{ refer Art. 1.22, chapter 1} \qquad \ldots (3)$$

Part (*iii*). Proceed as in part (*ii*) and prove result (3).

Writing $c^2 s$ for s in (3) and using change of scale property, we get

$$\mathrm{L}^{-1}\left\{\frac{e^{-\sqrt{c^2 s}}}{c^2 s}\right\} = \frac{1}{c^2}\operatorname{erfc}\left(\frac{1}{2\sqrt{(t/c^2)}}\right) \qquad \text{or} \qquad \mathrm{L}^{-1}\left\{\frac{e^{-c\sqrt{s}}}{s}\right\} = \operatorname{erfc}\left(\frac{c}{2\sqrt{t}}\right)$$

EXERCISE 2 (*H*)

1. Evaluate $\mathrm{L}^{-1}\{1/s^3(s+1)\}$

2. $\mathrm{L}^{-1}\{a^2/s(s+a)^2\}$

3. Evaluate (*i*) $\mathrm{L}^{-1}\left\{\log\frac{s+3}{s+2}\right\}$ (*ii*) $\mathrm{L}^{-1}\left\{\frac{1}{s}\log\frac{s+3}{s+2}\right\}$

ANSWERS

1. $1 - t + t^2/2 - e^{-t}$

2. $1 - (1+at)e^{at}$

3. (*i*) $\frac{1}{t}\left(e^{-2t} - e^{3t}\right)$ (*ii*) $\int_0^t \frac{1}{x}\left(e^{-2x} - e^{3x}\right)dx$

2.15. Convolution. Definition. Let F (*t*) and G (*t*) be two functions of class A, then the convolution of them is denoted and defined as

$$\mathrm{F}*\mathrm{G} = \int_0^t \mathrm{F}(u)\,\mathrm{G}(t-u)\,du \qquad \text{or} \qquad \mathrm{F}*\mathrm{G} = \int_0^t \mathrm{F}(t-u)\,\mathrm{G}(u)\,du$$

The relation F ∗ G, is also known as *falting or resultant* of F(*t*) and G(*t*).

Properties of convolution. **[Agra 2010]**

(*i*) F ∗ G = G ∗ F, *i.e. the commulative law for convolutions holds good.*

(*ii*) F ∗ (G + H) = F ∗ G + F ∗ H and (F+G) ∗ H = F ∗ H + G ∗ H.

i.e. the distributive law for convolutions holds good.

(*iii*) F ∗ (G ∗ H) = (F ∗ G) ∗ H. *i.e.* the associative law for convolutions holds good **[Agra 2010]**

Proof. (*i*) $F * G = \int_0^t F(u)\, G(t-u)\, du$, *by* definition of convolution

$$= \int_t^0 F(t-v)\, G(v)\, (-dv), \text{ putting } t-u=v \text{ and } du=-dv$$

$$= \int_0^t G(v)\, F(t-v)\, dv = G * F, \text{ by definition of the convolution}$$

(*ii*) $F*(G+H) = \int_0^t F(u)[G(t-u)+H(t-u)]\, du$, by definition of convolution

$$= \int_0^t F(u)\, G(t-u)\, du + \int_0^t F(u)\, H(t-u)\, du = F * G + F * H, \text{ by definition of convolution.}$$

Similarly, $(F+G) * H = F * H + G * H.$

(*iii*) Left as an exercise.

An illustrative example: Find the value of $e^{at} * e^{bt}$, $a \neq b$. **[Indore 2004]**

Solution. Let $F(t) = e^{at}$ and $G(t) = e^{bt}$, where $a \neq b$. Then by definition of convolution,

$$F * G = \int_0^t F(u) G(t-u)\, du \qquad \text{or} \qquad e^{at} * e^{bt} = \int_0^t e^{au} e^{b(t-u)} du$$

or $$e^{at} * e^{bt} = e^{bt} \int_0^t e^{(a-b)u} du = e^{bt} \left[\frac{1}{a-b} e^{(a-b)u} \right]_0^t = \frac{e^{bt}}{a-b} \left[e^{(a-b)t} - 1 \right] = \frac{1}{a-b} (e^{at} - e^{bt})$$

2.16. The Convolution Theorem (or Convolution property) [Agra 2011, 12 ; Kanpur 2009]

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then $L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du = F * G$

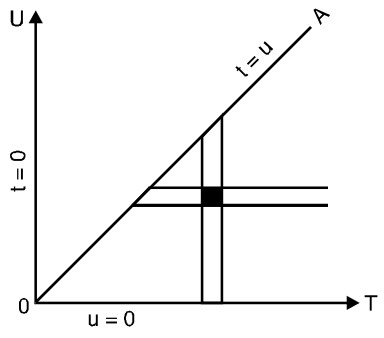
[Guwahati 2000, Osmania 2004; Meerut 2005, 08, 11; Nagpur 2005, 10; Purvanchal 2002]

Proof. Let $$H(t) = \int_{u=0}^t F(u)\, G(t-u)\, du = F * G \qquad \ldots (1)$$

Now, $$L\{H(t)\} = \int_{t=0}^{\infty} e^{-st}\, H(t)\, dt, \text{ by definition of Laplace transform}$$

or $$L\{H(t)\} = \int_{t=0}^{\infty} e^{-st} \left\{ \int_{u=0}^t F(u)\, G(t-u)\, du \right\} dt, \text{ using (1)} \qquad \ldots (2)$$

In (2), the region of integration in the double integral is the infinite area below the line OA (with equation $u = t$) and above the line OT (with equation $u = 0$). Here OT and OU are rectangular axes, t and u being measured along OT and OU respectively. In the double integral the strip is taken parallel to OU and if the order of integration is changed, the strip is taken parallel to OT and hence the new limits of t are from u to ∞ and those of u are from 0 to ∞. Hence changing the order of integration, (2) reduces to



$$L\{H(t)\} = \int_{u=0}^{\infty} F(u)\left\{\int_{t=u}^{\infty} e^{-st}\, G(t-u)\,dt\right\}du = \int_{u=0}^{\infty} e^{-su}\, F(u)\left\{\int_{t=u}^{\infty} e^{-s(t-u)}\, G(t-u)dt\right\}du$$

$$= \int_{u=0}^{\infty} e^{-su}\, F(u)\left\{\int_{v=0}^{\infty} e^{-sv}\, G(v)dv\right\}du \text{ , putting } t-u=v \text{ so that } dt=dv$$

$$= \int_{u=0}^{\infty} e^{-su}\, F(u)\, g(s)du = g(s)\int_{u=0}^{\infty} e^{-su}\, F(u)du \text{ , by definition}$$

$$\therefore \qquad L\{H(t)\} = g(s)\,f(s) \qquad \text{or} \qquad L^{-1}\{f(s)\,g(s)\} = H(t)$$

or
$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)du = F*G, \text{ by (1)}$$

Remark 1. The convolution theorem can be re-written as:

$$L\{\int_0^t F(u)\,G(t-u)du\} = L\{F(t)*G(t)\} = L\{F(t)\}\times L\{G(t)\} \qquad \textbf{[CDLU 2004]}$$

Remark 2. While using the convolution theorem, we use one of the following two forms. The particular choice will depend on the problem in hand keeping in mind the later work of integration involved in the later part of the solution.

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du \qquad \text{or} \qquad L^{-1}\{f(s)\,g(s)\} = \int_0^t G(u)\,F(t-u)\,du$$

2.16 A. Solved examples based on Art. 2.16.

Ex.1. *Use the convolution theorem to evaluate*:

(*i*) $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$ (*ii*) $L^{-1}\left\{\dfrac{1}{(s+1)(s-1)}\right\}$

(*iii*) $L^{-1}\left\{\dfrac{1}{(s-1)(s+2)}\right\}$ **[Bilaspur 2004, Meerut 2004, Kanpur 2004]**

Sol. (*i*) Let $f(s) = 1/(s+a)$ and $g(s) = 1/(s+b)$ ... (1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s+a)\} = e^{-at}$ ... (2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s+b)\} = e^{-bt}$ ... (3)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du$$

or
$$L^{-1}\left\{\frac{1}{(s+a)(s+b}\right\} = \int_0^t e^{-au}e^{-b(t-u)}du, \text{ using (1), (2) and (3)}$$

$$= e^{-bt}\int_0^t e^{(b-a)u}du = e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t = \frac{e^{-bt}}{b-a}\left[e^{(b-a)t}-1\right] = \frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$$

(*ii*) Let $f(s) = 1/(s+1)$ and $g(s) = 1/(s-1)$ ... (1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s+1)\} = e^{-t}$ ... (2)

and $$G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s-1)\} = e^t. \quad \ldots (3)$$

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$$

or $$L^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\} = \int_0^t e^{-u} e^{t-u}\, du = e^t \int_0^t e^{-2u}\, du, \text{ by (1), (2) and (3)}$$

$$= e^t \left[-\frac{1}{2} e^{-2u}\right]_0^t = -\frac{1}{2} e^t \left(e^{-2t} - 1\right) = \frac{1}{2}(e^t - e^{-t})$$

(*iii*) Proceed as in part (*ii*). **Ans.** $(e^t - e^{-2t})/3$.

Ex.2(*a*). *Evaluate* $L^{-1}\left\{1/s^2(s^2 - a^2)\right\}$ **[Purvanchal 1996, Meerut 1993]**

(*b*) $L^{-1}\{1/s^2(s^2 + a^2)\}$ **[MDU Rohtak 2005]**

Sol(*a*). Let $f(s) = 1/s^2$ and $g(s) = 1/(s^2 - a^2)$ ... (1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/s^2\} = t/1! = t$ ... (2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2 - a^2)\} = (1/a) \times \sinh at$ ... (3)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$$

or $$L^{-1}\left\{\frac{1}{s^2(s^2 - a^2)}\right\} = \int_0^t u \times \frac{1}{a} \times \sinh a(t-u)\, du, \text{ by (1), (2) and (3)}$$

$$= \frac{1}{a}\left\{\left[u \frac{\cosh a(t-u)}{-a}\right]_0^t - \int_0^t 1 \times \frac{\cosh a(t-u)}{-a}\, du\right\}$$

$$= \frac{1}{a}\left[-\frac{t}{a}\cosh 0 + 0 + \frac{1}{a}\int_0^t \cosh a(t-u)\, du\right] = -\frac{t}{a^2} + \frac{1}{a^2}\left[\frac{\sinh a(t-u)}{-a}\right]_0^t$$

$$= -(t/a^2) - (1/a^3) \times (0 - \sinh at) = (\sinh at - at)/a^3$$

(*b*) Proceed like part (*a*). **Ans.** $(at - \sin at)/a^3$

Ex.3. *Use the Convolution theorem to evaluate:*

(*i*) $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$ **[Osmania 2010]** (*ii*) $L^{-1}\left\{\frac{1}{(s+2)(s^2+4)}\right\}$

(*iii*) $L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}$

Sol. (*i*) Let $f(s) = 1/(s+1)$ and $g(s) = 1/(s^2+1)$ ... (1)

$\therefore F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s+1)\} = e^{-t}$, $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2+1)\} = \sin t$ ... (2)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t G(u)\, F(t-u)\, du$$

or $$L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \int_0^t \sin u\, e^{-(t-u)}\, du, \text{ using (1) and (2)}$$

$$= e^{-t}\int_0^t e^u \sin u\ du = e^{-t}\left[\frac{e^u}{1^2+1^2}(\sin u - \cos u)\right]_0^t, \text{ as} \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= (e^{-t}/2)\times\left[e^t(\sin t - \cos t) - e^0(0-1)\right] = (1/2)\times(\sin t - \cos t + e^{-t})$$

(*ii*) Let $f(s) = 1/(s+2)$ and $g(s) = 1/(s^2+4)$ ... (1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s+2)\} = e^{-2t}$ ... (2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2+2^2)\} = (1/2)\times\sin 2t$ ... (3)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t G(u)\,F(t-u)\,du$$

or $$L^{-1}\left\{\frac{1}{(s+2)(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\ e^{-2(t-u)}\,du, \text{ using (1), (2) and (3)}$$

$$= \frac{1}{2}e^{-2t}\int_0^t e^{2u}\sin 2u\,du = \frac{1}{2}e^{-2t}\left[\frac{e^{2u}}{2^2+2^2}(2\sin 2u - 2\cos 2u)\right]_0^t$$

$$= (1/16)\times e^{-2t}\left[2e^{2t}(\sin 2t - \cos 2t) + 2\right] = (1/8)\times(\sin 2t - \cos 2t + e^{-2t}).$$

(*iii*) Proceed as in part (*iii*) **Ans.** $(1/5)\times(e^{2t} - 2\sin t - \cos t)$

Ex. 4. *Find* (*i*) $L^{-1}\left\{\frac{1}{s(s+1)^3}\right\}$

(*ii*) $L^{-1}\{1/s\,(s+2)^3\}$ **[KU Kurukshetra 2006]**

(*iii*) $L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$ **[Kanpur 2011; Meerut 1997, Osmania 2004]**

Sol. (*i*) Let $f(s) = 1/s$ and $g(s) = 1(s+1)^3$ ... (1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/s\} = 1$ ... (2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s+1)^3\} = e^{-t}\,L^{-1}\{1/s^3\}$, using first shifting theorem

Thus, $G(t) = e^{-t}\times(t^2/2!) = (1/2)\times e^{-t}\,t^2$ ... (3)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t G(u)\,F(t-u)\,du$$

or $$L^{-1}\left\{\frac{1}{s(s+1)^3}\right\} = \int_0^t\left(\frac{1}{2}u^2e^{-u}\right)\times 1\,du, \text{ using (1) (2) and (3)}$$

$$= \left[(u^2/2)\times(-e^{-u})\right]_0^t - \int_0^t u\times(-e^{-u})\,du, \text{ integrating by parts taking } u^2/2 \text{ as first function}$$

$$= -\frac{t^2}{2}e^{-t} + \int_0^t u\,e^{-u}du = -\frac{t^2}{2}e^{-t} + \left[u(-e^{-u})\right]_0^t - \int_0^t 1\times\left(-e^{-u}\right)du$$

$$= -(t^2/2)\times e^{-t} - te^{-t} + \left[-e^{-u}\right]_0^t = 1 - e^{-t}(1+t+t^2/2)$$

(*ii*) Do like part (*i*) **Ans.** $(1/8) - (1/8)\times e^{-2t}(2t^2 + 2t + 1)$

(*iii*) Let $f(s) = 1/(s-2)$ and $g(s) = 1/(s+2)^2$. ... (1)

Then $F(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\{1/(s-2)\} = e^{2t}$ ... (2)

and $\mathrm{G}(t) = \mathrm{L}^{-1}\{g(s)\} = \mathrm{L}^{-1}\{1/(s+2)^2\} = e^{-2t}\mathrm{L}^{-1}\{1/s^2\}$, using first shifting theorem

Thus, $\mathrm{G}(t) = e^{-2t}\times(t/1!) = t\,e^{-2t}$ ... (3)

Now, using the convolution theorem we have

$$\mathrm{L}^{-1}\{f(s)\,g(s)\} = \int_0^t \mathrm{G}(u)\,\mathrm{F}(t-u)\,du$$

or $$\mathrm{L}^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\} = \int_0^t u\,e^{-2u}e^{2(t-u)}du, \text{ using (1), (2) and (3)}$$

$$= e^{2t}\int_0^t u\,e^{-4u}du = e^{2t}\left\{\left[(u)\left(\frac{1}{4}e^{-4u}\right)\right]_0^t - \int_0^t (1)\left(\frac{-1}{4}e^{-4u}\right)du\right\}, \text{ integrating by parts}$$

$$= e^{2t}\left\{-\frac{t}{4}e^{-4t} + \frac{1}{4}\left[-\frac{1}{4}e^{-4u}\right]_0^t\right\} = e^{2t}\left\{-\frac{t}{4}e^{-4t} - \frac{1}{16}e^{-4t} + \frac{1}{16}\right\} = \frac{1}{16}\{e^{2t} - (4t+1)e^{-2t}\}$$

Ex. 5. *Use the convolution theorem to evalute* $\mathrm{L}^{-1}\{1/s^2(s+1)^2\}$ **[Punjab 2005]**

Sol. Let $f(s) = 1/s^2$ and $g(s) = 1/(s+1)^2$ ... (1)

Then, $F(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\{1/s^2\} = t/! = t$ ... (2)

Also, $\mathrm{G}(t) = \mathrm{L}^{-1}\{1/(s+1)^2\} = \mathrm{e}^{-t}\,\mathrm{L}^{-1}\{1/s^2\}$, by first shifting theorem

or $G(t) = e^{-t}\times(t/!) = t\,e^{-t}$. ... (3)

Now, using the convolution theorem we have

$$\mathrm{L}^{-1}\{f(s)\,g(s)\} = \int_0^t \mathrm{G}(u)\,\mathrm{F}(t-u)du$$

or $$\mathrm{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t u\,e^{-u}(t-u)du, \text{ by (1),(2) and (3)}$$

$$= \int_0^t (tu-u^2)e^{-u}du = \left[(tu-u^2)(-e^{-u})\right]_0^t - \int_0^t (t-2u)(-e^{-u})\,du$$

$$= 0 + \int_0^t (t-2u)e^{-u}du = \left[(t-2u)(-e^{-u})\right]_0^t - \int_0^t (-2)(-e^{-u})\,du$$

$$= te^{-t} + t - 2[-e^{-u}]_0^t = te^{-t} + t - 2(-e^{-t}+1) = t - 2 + e^{-t}(2+t)$$

Ex. 6. *Use the convolution theorem to evalute* $L^{-1}\{s/(s^2+a^2)^2\}$. **[S.V. University (A.P.) 1997, Lucknow 1997; Meerut 2000, 09, ; Ravishankar 2005]**

Sol. Let $f(s) = s/(s^2 + a^2)$ and $g(s) = 1/(s^2 + a^2)$. ... (1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{(s/s^2 + a^2)\} = \cos at$ ... (2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2 + a^2)\} = (1/a) \times \sin at$ ... (3)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$$

or $L^{-1}\{s/(s^2 + a^2)^2\} = \int_0^t \cos au \times \frac{1}{a} \sin a(t-u)\, du$, by (1), (2) and (3)

$$= \frac{1}{a}\int_0^t \cos au(\sin at \cos au - \cos at \sin au)du = \frac{1}{2a}\sin at \int_0^t (1+\cos 2au)\, du - \frac{1}{2a}\cos at \int_0^t \sin 2au\, du$$

$$= \frac{\sin at}{2a}\left[u + \frac{\sin 2au}{2a}\right]_0^t - \frac{\cos at}{2a}\left[-\frac{\cos 2au}{2a}\right]_0^t = \frac{\sin at}{2a}\left[t + \frac{\sin 2at}{2a}\right] - \frac{\cos at}{2a}\left[\frac{1-\cos 2at}{2a}\right]$$

$$= \frac{\sin at}{2a}\left[t + \frac{\sin at \cos at}{a}\right] - \frac{\cos at}{2a} \times \frac{2\sin^2 at}{2a} = \frac{t \sin at}{2a}$$

Ex. 7(*a*). *Use the convolution theorem to evaluate*

(*i*) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$ **[Meerut 2006, Kanpur 2010]** (*ii*) $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$ **[Meerut 1995]**

Sol. (*i*) Let $f(s) = s/(s^2 + a^2)$ *and* $g(s) = s/(s^2 + a^2)$ ... (1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{s/(s^2 + a^2)\} = \cos at$ ... (2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{s/(s^2 + a^2)\} = \cos at$ ... (3)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$$

$\therefore$ $L^{-1}\{s^2/(s^2 + a^2)^2\} = \int_0^t \cos au \cos a(t-u)\, du$, by (1), (2) and (3)

$$= \int_0^t \cos au\,(\cos at \cos au + \sin at \sin au)\, du$$

$$= \frac{\cos at}{2}\int_0^t (1+\cos 2au)du + \frac{\sin at}{2}\int_0^t \sin 2au\, du = \frac{\cos at}{2}\left[u + \frac{\sin 2au}{2a}\right]_0^t + \frac{\sin at}{2}\left[-\frac{\cos 2au}{2a}\right]_0^t$$

$$= (1/2) \times \cos at\{t + (1/2a) \times \sin 2at\} + (1/4a) \times \sin at(1 - \cos 2at)$$

$$= (1/2) \times t \cos at + (1/4a) \times \sin at + (1/4a) \times (\sin 2at \cos at - \sin at \cos 2at)$$

$$= (1/2) \times t \cos at + (1/4a) \times \sin at + (1/4a) \times \sin(2at - at)$$

$$= (1/2) \times t \cos at + (1/4a) \times \sin at + (1/4a) \times \sin at = (at \cos at + \sin at)/2a$$

(*ii*) Do as in part (*i*) noting that $a = 2$ **Ans.** $(2t \cos 2t + \sin 2t)/4$.

Ex. 7(*b*) *Show that* $L\{t\cos at\} = (s^2 - a^2)/(s^2 + a^2)$ *and use this result to evaluate* $L^{-1}\{1/s^2 + a^2)^2\}$ **[Nagpur 2005]**

Solution. As in Ex. 1(*ii*), Ar. 1.16 A, to prove $\quad L\{t\cos at\} = (s^2 - a^2)/(s^2 + a^2)^2 \quad ... (1)$

$$(1) \Rightarrow \qquad t\cos at = L^{-1}\left\{\frac{s^2 - a^2}{(s^2 + a^2)^2}\right\} = L^{-1}\left\{\frac{s^2}{(s^2 + a^2)^2}\right\} - a^2 L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\}$$

$$\Rightarrow \qquad L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\} = \frac{1}{a^2} L^{-1}\left\{\frac{s^2}{(s^2 + a^2)^2}\right\} - \frac{t\cos at}{a^2} \qquad ... (2)$$

Proceed as in Ex. 7(*a*), to prove that $\quad L^{-1}\{s^2/(s^2 + a^2)^2\} = (at\cos at + \sin at)/2a \quad ... (3)$

From (2) and (3), we have

$$L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\} = \frac{at\cos at + \sin at}{2a^3} - \frac{t\cos at}{a^2} = \frac{\sin at - at\cos at}{2a^3}$$

Ex. 8. *Evaluate* $L^{-1}\{1/(s-1)^5 (s + 2)\}$

Sol. Let $\quad f(s) = 1/(s-1)^5 \quad$ and $\quad g(s) = 1/(s + 2) \quad ... (1)$

$\therefore \quad F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s-1)^5\} = e^t L^{-1}\{1/s^5\}$, by first shifting theorem

or $\qquad F(t) = e^t \times (t^4/4!) = (1/24) \times t^4 e^t. \qquad ... (2)$

Also $\qquad G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s + 2)\} = e^{-2t} \qquad ... (3)$

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$$

or $$L^{-1}\left\{\frac{1}{(s-1)^5 (s+2)}\right\} = \int_0^t \frac{1}{24} u^4 e^u e^{-2(t-u)}\, du = \frac{e^{-2t}}{24}\int_0^t u^4 e^{3u} du$$

$$= \frac{e^{-2t}}{24}\left[(u^4)\left(\frac{1}{3}e^{3u}\right) - (4u^3)\left(\frac{1}{9}e^{3u}\right) + (12u^2)\left(\frac{1}{27}e^{3u}\right) - (24u)\left(\frac{1}{8}e^{3u}\right) + (24)\left(\frac{1}{243}e^{3u}\right)\right]_0^t$$

[Using chain rule of integration by parts]

$= (1/24) \times e^{-2t}\{(1/3) \times t^4 e^{3t} - (4/9) \times t^3 e^{3t} + (4/9) \times t^2 e^{3t} - (8/27) \times te^{3t} - (8/81) \times e^{3t} - (8/81)\}$

$= e^t\{(1/72) \times t^4 - (1/54) \times t^3 + (1/54) \times t^2 - (1/81) \times t + (1/243)\} - (1/243) \times e^{-2t}.$

Ex. 9. *Evaluate* (*i*) $L^{-1}\{s/(s^2 + 4)^3\}$

(*ii*) $L^{-1}\{s/(s^2 + a^2)^3\}$ **[MDU Rohtak 2004]**

Sol. (*i*) Let $\quad f(s) = s/(s^2 + 4)^2 \quad$ and $\quad g(s) = 1/(s^2 + 4) \quad ... (1)$

Then, $\quad F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{s}{(s^2 + 4)^2}\right\} = L^{-1}\left\{-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2 + 4}\right)\right\}$

$$= -\frac{1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2 + 4}\right)\right\} = -\frac{1}{2}(-1)t\, L^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{t\sin 2t}{4} \qquad ... (2)$$

[using result of Art 2.11]

Also, $\quad G(t) = L^{-1}\left\{1/(s^2 + 2^2)\right\} = (1/2) \times \sin 2t \qquad ... (3)$

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\; G(t-u)\,du$$

or $$L^{-1}\left\{\frac{s}{(s^2+4)^3}\right\} = \int_0^t \frac{u}{4}\sin 2u \times \frac{\sin 2(t-u)}{2}\,du, \text{ by (1), (2) and (3)}$$

$$= \frac{1}{8}\int_0^t u\sin 2u(\sin 2t\cos 2u - \cos 2t\sin 2u)\,du = \frac{\sin 2t}{16}\int_0^t u\sin 4u\,du - \frac{\cos 2t}{16}\int_0^t u(1-\cos 4u)\,du$$

$$= \frac{\sin 2t}{16}\int_0^t u\sin 4u\,du + \frac{\cos 2t}{16}\int_0^t u\cos 4u\,du - \frac{\cos 2t}{16}\int_0^t u\,du$$

$$= \frac{\sin 2t}{16}\left\{\left[(u)\left(-\frac{1}{4}\cos 4u\right)\right]_0^t - \int_0^t 1\times\left(-\frac{1}{4}\cos 4u\right)du\right\} + \frac{\cos 2t}{16}\left\{\left[(u)\left(\frac{1}{4}\sin 4u\right)\right]_0^t\right.$$

$$\left. - \int_0^t 1\times\left(\frac{1}{4}\sin 4u\right)du\right\} - \frac{\cos 2t}{16}\left[\frac{1}{2}u^2\right]_0^t, \text{ integrating by parts}$$

$$= \frac{\sin 2t}{16}\left\{-\frac{t}{4}\cos 4t + \frac{1}{4}\left[\frac{1}{4}\sin 4u\right]_0^t\right\} + \frac{\cos 2t}{16}\left\{\frac{t}{4}\sin 4t - \frac{1}{4}\left[-\frac{1}{4}\cos 4u\right]_0^t\right\} - \frac{t^2\cos 2t}{32}$$

$$= \frac{\sin 2t}{16}\left(-\frac{t\cos 4t}{4} + \frac{\sin 4t}{16}\right) + \frac{\cos 2t}{16}\left(\frac{t\sin 4t}{4} + \frac{\cos 4t}{16} - \frac{1}{16}\right) - \frac{t^2\cos 2t}{32}$$

$$= (t/64)\times(\sin 4t\cos 2t - \cos 4t\sin 2t) + (1/256)\times(\cos 4t\cos 2t + \sin 4t\sin 2t)$$

$$-(1/256)\times\cos 2t - (t^2/32)\times\cos 2t$$

$$= (t/64)\times\sin 2t + (1/256)\times\cos 2t - (1/256)\times\cos 2t - (t^2 32)\times\cos 2t = (t/64)\times(\sin 2t - 2t\cos 2t)$$

(*ii*) Proceed as in part (*i*). Note that part (*i*) is a particular case of part (*ii*) with $a = 2$.

Ans. $t(\sin at - at\cos at)/8a^3$

Ex.10. *Evaluate* $L^{-1}\{1/s(s^2+4)^2\}$

Sol. Let $f(s) = 1/s^2$ and $g(s) = s/(s^2+4)^2$ ... (1)

Then, $F(t) = L^{-1}\{f(s)\} = L^{-1}(1/s^2) = t/1! = t$ ... (2)

Also, $G(t) = L^{-1}\{g(s)\} = L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = L^{-1}\left\{-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+4}\right)\right\}$

$$= -\frac{1}{2}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{(s^2+4)}\right)\right\} = -\frac{1}{2}(-1)^1 t\,L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{t\sin 2t}{4} \quad \text{... (3)}$$

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t G(u)\,F(t-u)\,du$$

$$\text{or } L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\} = \int_0^t \frac{1}{4}u\sin 2u \times (t-u)\,du = \int_0^t \left(\frac{(tu-u^2)}{4}\right) \times \sin 2u\,du, \text{ using (1), (2) and (3)}$$

$$= \left[\frac{tu-u^2}{4} \times \left(-\frac{\cos 2u}{2}\right)\right]_0^t - \int_0^t \frac{t-2u}{4}\left(-\frac{\cos 2u}{2}\right) du, \text{ integrating by parts}$$

$$= 0 + \int_0^t \frac{t-2u}{8} \times \cos 2u\,du = \left[\frac{t-2u}{8} \times \frac{\sin 2u}{2}\right]_0^t - \int_0^t \left(-\frac{2}{8}\right) \times \frac{\sin 2u}{2}\,du \text{, integrating by parts}$$

$$= -\frac{t\sin 2t}{16} + \frac{1}{8}\left[-\frac{1}{2}\cos 2u\right]_0^t = -\frac{t\sin 2t}{16} - \frac{\cos 2t}{16} + \frac{1}{16}$$

Ex 11. *Apply convolution theorem to evaluate* $L^{-1}\{s/(s^2+2s)^2\}$ **[MDU Rohtak 2006]**

Sol. Here
$$\frac{s}{(s^2+2s)^2} = \frac{1}{s} \times \frac{1}{(s+2)^2} = f(s)\,g(s) \quad \ldots (1)$$

where $f(s) = 1/s$ and $g(s) = 1/(s+2)^2$

Then $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\} = L^{-1}\{1/s\} = 1$ ... (2)

and $G(t) = \mathrm{L}^{-1}\{g(s)\} = \mathrm{L}^{-1}\{1/(s+2)^2\} = e^{-2t}\,\mathrm{L}^{-1}\{1/s^2\} = e^{-2t}\,t$ ... (3)

Now, using the convolution theorem, we have

$$\mathrm{L}^{-1}\{f(s)\,g(s)\} = \int_0^t \mathrm{F}(t-u)\,G(u)\,du$$

$$\text{or } \mathrm{L}^{-1}\{s/(s^2+2s)^2\} = \int_0^t (1 \times e^{-2u}u)du, \text{ by (1), (2) and (3)}$$

$$= \left[u\left\{-\frac{1}{2}e^{-2u}\right\}\right]_0^t - \int_0^t \left[1 \times \left\{-\frac{1}{2}e^{-2u}\right\}\right] du \text{ integrating by parts,}$$

$$= -\frac{1}{2}t\,e^{-2t} + \frac{1}{2}\left[-\frac{1}{2}e^{-2u}\right]_0^t = -\frac{1}{2}t\,e^{-2t} + \frac{1}{2}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right) = \frac{1}{4}(1 - 2te^{-2t} - e^{-2t})$$

Ex. 12. *Apply convolution theorem to show that* $\int_0^t \sin u \cos(t-u)du = \frac{1}{2}t\sin t$

Sol. By the convolution theorem (refer remark 1 of Art 2.16)

$$\mathrm{L}\left\{\int_0^t \mathrm{F}(u)\,\mathrm{G}(t-u)\,du\right\} = \mathrm{L}\{\mathrm{F}(t)\} \times \mathrm{L}\{\mathrm{G}(t)\} \quad \ldots (1)$$

Take F(t) = sin t and G(t) = cos$t$. Then (1) reduces to

$$\mathrm{L}\left\{\int_0^t \sin u \cos(t-u)\,du\right\} = \mathrm{L}\{\sin t\} \times \mathrm{L}\{\cos t\} = \frac{1}{s^2+1} \times \frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}$$

$$\therefore \qquad \int_0^t \sin u \cos(t-u)\,du = \mathrm{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \qquad \ldots (2)$$

Proceed as in Ex.6 with $a = 1$ and prove that

$$\mathrm{L}^{-1}\left\{s/(s^2+1)^2\right\} = (1/2)\times t\sin t \qquad \ldots (3)$$

Hence (2)and (3) $\Rightarrow$ $$\int_0^t \sin u\cos(t-u)\,du = \frac{1}{2}t\sin t$$

Ex. 13. *Apply the convolution theorem to prove that*

$$\mathrm{B}(m, n) = \int_0^t u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m>0, n>0.$$ **[[Meerut 2010]; Ravishanker 2004]**

[B(m,n) is called Beta function].

Sol. By the convolution theorem (refer remark 1 of Art 2.16)

$$\mathrm{L}\left\{\int_0^t \mathrm{F}(u)\,\mathrm{G}(t-u)\,du\right\} = \mathrm{L}\{\mathrm{F}(t)\}\times\mathrm{L}\{\mathrm{G}(t)\} \qquad \ldots (1)$$

Take $\mathrm{F}(t) = t^{m-1}$ and $\mathrm{G}(t) = t^{n-1}$. Then (1) reduces to

$$\mathrm{L}\left\{\int_0^t u^{m-1}(t-u)^{n-1}du\right\} = \mathrm{L}\left\{t^{m-1}\right\}\times\mathrm{L}\left\{t^{n-1}\right\} = \frac{\Gamma(m)}{s^m}\times\frac{\Gamma(n)}{s^n}$$

$$\therefore \qquad \int_0^t u^{m-1}(t-u)^{n-1}du = \mathrm{L}^{-1}\left\{\frac{\Gamma(m)\Gamma(n)}{s^{m+n}}\right\} = \Gamma(m)\Gamma(n)\times\frac{t^{m+n-1}}{\Gamma(m+n)}$$

Putting $t = 1$ in above result, we obtain

$$\int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \text{where } m>0, n>0.$$

Ex. 14. *Evaluate* $\int_0^t J_0(u)\,J_0(t-u)\,du$

Sol. By the convolution theorem (refer remark 1 of Art. 2.16)

$$\mathrm{L}\left\{\int_0^t \mathrm{F}(u)\,G(t-u)\,du\right\} = \mathrm{L}\{\mathrm{F}(t)\}\times\mathrm{L}\{\mathrm{G}(t)\} \qquad \ldots (1)$$

Take $\mathrm{F}(t) = \mathrm{J}_0(t)$ and $\mathrm{G}(t) = \mathrm{J}_0(t)$. Then (1) reduces to

$$\mathrm{L}\left\{\int_0^t \mathrm{J}_0(u)\,\mathrm{J}_0(t-u)\,du\right\} = \mathrm{L}\{\mathrm{J}_0(t)\}\times\mathrm{L}\{\mathrm{J}_0(t)\} = \left(1/\sqrt{s^2+1}\right)^2, \text{ as } \quad \mathrm{L}\{J_0(t)\} = 1/\sqrt{s^2+1},$$

$$\therefore \qquad \int_0^t \mathrm{J}_0(u)\,\mathrm{J}_0(t-u)\,du = \mathrm{L}^{-1}\left\{1/(1+s^2)\right\} = \sin t$$

Ex. 15. *Evaluate* $\mathrm{L}^{-1}\left\{1/[(s-1)\sqrt{s}\,]\right\}$ **[Kumaun 1998, Meerut 2001]**

Sol. Let $f(s) = 1/(s-1)$ and $g(s) = 1/\sqrt{s}$ ... (1)

Then $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\{1/(s-1)\} = e^t$. ... (2)

Also, $$G(t) = L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{t^{-1/2}}{\Gamma(1/2)} = \frac{1/\sqrt{t}}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi t}} \quad \ldots (3)$$

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t G(u)\,F(t-u)\,du$$

or $$L^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\} = \int_0^t \frac{1}{\sqrt{\pi}\sqrt{u}} e^{t-u}\,du, \text{ using (1), (2) and (3)}$$

$$= \frac{e^t}{\sqrt{\pi}}\int_0^t \frac{e^{-u}}{\sqrt{u}}\,du = \frac{e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} \frac{e^{-v^2}}{v} \times 2v\,dv \text{, putting } u = v^2 \text{ and } du = 2v\,dv$$

$$= \frac{2e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-v^2}\,dv = e^t\,erf(\sqrt{t}) \text{, by definition of error function (see chapter 1).}$$

Ex. 16. *Prove that $L^{-1}\{1/s\sqrt{s+4}\} = (1/2) \times erf(2\sqrt{t})$*

Sol. Let $f(s) = 1/s$ and $g(s) = 1/\sqrt{s+4}$ ... (1)

Then, $$F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/s\} = 1 \quad \ldots (2)$$

Also, $$G(t) = L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{(s+4)^{1/2}}\right\} = e^{-4t}\,L^{-1}\left\{\frac{1}{s^{1/2}}\right\}, \text{ using first shifting theorem}$$

or $$G(t) = e^{-4t} \times \frac{t^{-1/2}}{\Gamma(1/2)} = e^{-4t} \times \frac{1/\sqrt{t}}{\sqrt{\pi}} = e^{-4t}\,\frac{1}{\sqrt{\pi t}} \quad \ldots (3)$$

Now, using the convolution theorem we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t G(u)\,F(t-u)\,du$$

or $$L^{-1}\left\{\frac{1}{s\sqrt{s+4}}\right\} = \int_0^t e^{-4u}\,\frac{1}{\sqrt{\pi}\sqrt{u}}\,du, \text{ using (1), (2) and (3)}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-v^2}\,dv, \text{ putting } 4u = v^2 \text{ and } 4\,du = 2v\,dv, \textit{ i.e., } du/\sqrt{u} = dv$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-v^2}\,dv = \frac{1}{2}\,erf(2\sqrt{t}) \text{, by definition of error function}$$

Ex. 17. *Evaluate $L^{-1}\left\{1/\sqrt{s}(s-a)\right\}$ by the convolution theorem and deduce the value of $L^{-1}\left\{1/s\sqrt{s+a}\right\}$.*

Sol. Let $f(s) = 1/\sqrt{s}$ and $g(s) = 1/(s-a)$ ... (1)

Then, $\mathsf{F}(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\left\{\dfrac{1}{s^{1/2}}\right\} = \dfrac{t^{-1/2}}{\Gamma(1/2)} = \dfrac{1/\sqrt{t}}{\sqrt{\pi}} = \dfrac{1}{\sqrt{\pi t}}$ ... (2)

Also, $\mathrm{G}(t) = \mathrm{L}^{-1}\{g(s)\} = \mathrm{L}^{-1}\{1/(s-a)\} = e^{at}$ ... (3)

Now, using the convolution theorem, we have

$$\mathrm{L}^{-1}\{f(s)\,g(s)\} = \int_0^t \mathrm{F}(u)\,\mathrm{G}(t-u)\,du$$

or $$\mathrm{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a)}\right\} = \int_0^t \frac{1}{\sqrt{\pi}\sqrt{u}} e^{a(t-u)}\,du = \frac{e^{at}}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{u}} e^{-au}\,du$$

$$= \frac{e^{at}}{\sqrt{\pi}}\int_0^{\sqrt{at}} \frac{\sqrt{a}}{v} e^{-v^2} \times \frac{2v\,dv}{a}, \text{ putting } au = v^2 \text{ so that } du = \frac{2v\,dv}{a}$$

$$= \frac{e^{at}}{\sqrt{a}} \times \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{at}} e^{-v^2}\,dv = \frac{e^{at}\,erf(\sqrt{at})}{\sqrt{a}}, \text{ by def. of error function}$$

Deduction. We have proved that

$$\mathrm{L}^{-1}\left\{1/\sqrt{s}(s-a)\right\} = (e^{at}/\sqrt{a}) \times erf(\sqrt{at}) \quad \text{... (4)}$$

Now, $\mathrm{L}^{-1}\left\{\dfrac{1}{s\sqrt{s+a}}\right\} = \mathrm{L}^{-1}\left\{\dfrac{1}{[(s+a)-a]\sqrt{s+a}}\right\} = e^{-at}\,\mathrm{L}^{-1}\left\{\dfrac{1}{(s-a)\sqrt{s}}\right\}$

[Using first shifting theorem]

$$= e^{-at} \times (e^{at}/\sqrt{a}) \times erf(\sqrt{at}) = (1/\sqrt{a}) \times erf(\sqrt{at}), \text{using (4)}$$

Ex.18. *If $L^{-1}\{F(s)\} = f(t)$, then using the convolution theorem prove that*

$$\mathrm{L}^{-1}\left\{\frac{\mathrm{F}(s)}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{-at}\int_0^t f(\tau)\,e^{-a\tau}\sin b(t-\tau)\,d\tau \qquad \textbf{[Lucknow 1996]}$$

Sol. Let $\mathrm{G}(s) = 1/\{(s-a^2)+b^2\}$. ... (1)

Then $g(t) = \mathrm{L}^{-1}\{\mathrm{G}(s)\} = \mathrm{L}^{-1}\left\{\dfrac{1}{(s-a)^2+b^2}\right\} = e^{at}\,\mathrm{L}^{-1}\left\{\dfrac{1}{s^2+b^2}\right\}$, using first shifting theorem

$\therefore$ $\mathrm{G}(t) = (1/b) \times e^{at}\sin bt$ ... (2)

Now, using the convolution theorem we have

$$\mathrm{L}^{-1}\{\mathrm{F}(s)\,\mathrm{G}(s)\} = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

or $$\mathrm{L}^{-1}\left\{\frac{\mathrm{F}(s)}{(s-a)^2+b^2}\right\} = \int_0^t f(\tau)\frac{1}{b}e^{a(t-\tau)}\sin b(t-\tau)\,d\tau, \text{ by (1) and (2)}$$

$$= \frac{1}{b}e^{at}\int_0^t f(\tau)\,e^{-a\tau}\sin b(t-\tau)\,d\tau.$$

Ex. 19. *Prove that* $\int_0^t \int_0^t \ldots\ldots \int_0^t \mathrm{F}(t)dt^n = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} F(u)\,du$

Sol. Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Then, using the convolution theorem, we get

$$\mathrm{L}\left\{\int_0^t (t-u)^{n-1}\,\mathrm{F}(u)\,du\right\} = \mathrm{L}\left\{t^{n-1}\right\}\mathrm{L}\{\mathrm{F}(t)\} = \frac{(n-1)!}{s^n} f(s)$$

$$\therefore \qquad \mathrm{L}\left\{\int_0^t \frac{(t-u)^{n-1}}{(n-1)!}\mathrm{F}(u)\,du\right\} = \frac{f(s)}{s^n}$$

$$\therefore \qquad \int_0^t \frac{(t-u)^{n-1}}{(n-1)!}\mathrm{F}(u)\,du = \mathrm{L}^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t \int_0^t \ldots \int_0^t \mathrm{F}(t)\,dt^n\,,\ \text{by Art 2.14A}$$

Ex. 20. *Show that*

$$\underset{(n \text{ times})}{1 * 1 * 1 * \ldots * 1} = \frac{t^{n-1}}{(n-1)!}, \text{where } n = 1, 2, 3 \ldots\ldots \qquad \textbf{[Ravishanker 2004]}$$

Sol By the convolution theorem, $\mathrm{F} * \mathrm{G} = \int_0^t \mathrm{F}(x)\,\mathrm{G}\,(t-u)\,du$

Hence $$1 * 1 = \int_0^t (1 \cdot 1)\,du = [u]_0^t = t \qquad \ldots (1)$$

Using (1), $$1 * 1 * 1 = t * 1 = \int_0^t (u \cdot 1)\,du = \left[\frac{u^2}{2}\right]_0^t = \frac{t^2}{2} \qquad \ldots (2)$$

Using (2), $$1 * 1 * 1 * 1 = \frac{t^2}{2} * 1 = \int_0^t \left(\frac{u^2}{2} \times 1\right) du = \left[\frac{u^3}{6}\right]_0^t = \frac{t^3}{3!} \qquad \ldots (3)$$

Proceeding likewise, $$\underset{(n \text{ times})}{1 * 1 * 1 * \ldots * 1} = \frac{t^{n-1}}{(n-1)!},\ \text{where } n = 1, 2, 3, \ldots$$

EXERCISE 2 (I)

1. *Show that* (1) $1 * \sin at = (1 - \cos at)/2$, (*ii*) $\cos at * \cos at = (1/2a) \times (\sin at + at \cos at)$

2. *Use convolution to find the inverse Laplace transform of the following functions:*

(*i*) $\frac{1}{(s+3)(s-2)}$ **[Kurukshetra 2005]** (*ii*) $\frac{s}{(s^2+\pi^2)^2}$

(*iii*) $1/s^3(s^2+1)$ (*iv*) $4/(s^3+s^2+s+1)$

(*v*) $(s+3)/(s^2+6s+13)^2$ (*vi*) $1/(s^3-a^3)$

ANSWERS

2. (*i*) $(e^{2t} - e^{-3t})/5$ (*ii*) $(t \sin \pi t)/2\pi$ (*iii*) $\cos t - 1 + (t^2/2)$

(*iv*) $2(e^{-t} + \sin t + \cos t)$ (*v*) $(1/4) \times te^{-3t} \sin 2t$

(*vi*) $(1/3a^2) \times \left[e^{at} - e^{-at/2}\{\cos(at\sqrt{3}/2) + \sqrt{3}\sin(at\sqrt{3}/2)\}\right]$

3. *If* $L\{F(t)\} = f(s)$ and $L\{G(t)\} = g(s)$, *prove that* $f(s)\, g(s) = L\left\{\int_0^t F(t-u)\, G(u)\, du\right\}$

[Nagpur 2005]

4. *Verify convolution theorem for the functions* $F(t) = e^t$, $G(t) = \cos t$ **[Gulberga 2005]**

2.17 Table of some Inverse Laplace Transform Theorems for ready reference

	Operation	$f(s)$	$L^{-1}\{f(s)\} = F(t)$
1.	Linearity property	$c_1 f_1(s) + c_2 f_2(s)$	$c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$
2.	First translation or shifting theorem	$f(s-a)$	$e^{at} L^{-1}\{f(s)\}$
3.	Second translation or shifting theorem	$e^{-ap} f(s)$	$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$ or $F(t-a)\, H(t-a)$
4.	Change of scale property	$f(as)$	$(1/a) \times F(t/a)$
5.	Differentiation theorem	$f^{(n)}(s)$	$(-1)^n t^n F(t)$
6.	Integral theorem	$\int_s^\infty f(x)\, dx$	$\frac{1}{t} F(t)$
7.	Multiplication theorem	$sf(s)$ $s^n f(s)$	$F'(t)$ $F^{(n)}(t)$
8.	Division theorem	$\frac{f(s)}{s}$ $\frac{f(s)}{s^n}$	$\int_0^t F(t)\, dt$ $\int_0^t \int_0^t \ldots \int_0^t F(t)\, dt^n$
9.	Convolution theorem	$f(s)\, g(s)$	$F * G = \int_0^t F(u)\, G(t-u)\, du$ $= \int_0^t F(t-u)\, G(u)\, du$
10.	Heaviside's expansion theorem	$\frac{F(s)}{G(s)}$, where degree $F(s)$ < degree of $G(s)$	$\sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}$, where α_r are distinct roots of $G(s) = 0$, $r = 1, 2, 3, \ldots, n$
11.	An important theorem	$L^{-1}\{f(s)\} = -(1/t) \times L^{-1}\{f'(s)\}$	

OBJECTIVE PROBLEMS ON CHAPTER 2

Select (a), (b), (c) or (d) whichever is correct.

1. $L^{-1}\{1/(s-3)^4\}$ is (*a*) $(1/6) \times t^3 e^{3t}$ (*b*) $-(1/6) \times t^3 e^{3t}$ (*c*) $(1/6) \times t^3 e^{-3t}$ (*d*) None of these **[Kuvempa 2005]**

2. The value of $L^{-1}\{1/[(s+2)^2 + 16]\}$ is (*a*) $(1/4) \times e^{2t} \sin 4t$ (*b*) $(1/4) \times e^{-2t} \sin 4t$ (*c*) $(1/4) \times e^{-2t} \cos 4t$ (*d*) $(1/4) \times e^{2t} \cos 4t$ **[M.S. Univ. T.N. 2006]**

3. If $\int_0^t \mathrm{N}(t)\,dt = 0,$ for all $t > 0$ then $\mathrm{N}(t)$ is called

(a) unit function (b) Heaviside function (c) Null function (d) none these **[Purvanchal 2006]**

4. $\mathrm{L}^{-1}\{(3s + 7)/(s^2 - 2s - 3)\}$ is

(a) $4e^{-3t} + e^{-t}$ (b) $4e^{3t} - e^{-t}$ (c) $e^{3t} - 4\,e^{-t}$ (d) $e^{3t} + 4e^{t}$

5. $\mathrm{L}^{-1}\{s/(s + 5)^5\}$ is

(a) $t^3\,e^{-t}(4 - t)$ (b) $t^3\,e^{-t}(4 + t)$ (c) $\{t^3\,e^{-t}\,(4 - t)\}/4$ (d) $\{t^3\,e^{t}\,(4 - t)\}/4$

6. $\mathrm{L}^{-1}\{(s + a)/(s + b)$ is

(a) $(e^{bt} - e^{at})/t$ (b) $(e^{at} - e^{-bt})/2$ (c) $(e^{-bt} - e^{-at})/2$ (d) $(e^{-at} - e^{-bt})/2$

7. $\mathrm{L}^{-1}\{\tan^{-1}(1/s)\}$ is (a) $(\sin t)/t$ (b) $(\cos t)/t$ (c) $t/(\sin t)$ (d) $t/(\cos t)$

8. $\mathrm{L}\left\{\int_0^t (t-u)^2 e^u\,du\right\}$ is (a) $2/\{s^2\,(s + 1)\}$ (b) $2/\{s^3\,(s - 1)\}$

(c) $2/\{s^3\,(s + 1)\}$ (d) None of these **[Kuvempa 2005]**

Sol. Ans. (*b*) By convolution theorem $\mathrm{L}\left\{\int_0^t \mathrm{F}(t-u)\,\mathrm{G}(u)du\right\} = \mathrm{L}\{\mathrm{F}(t)\} \times \mathrm{L}\{\mathrm{G}(t)\}$

$\therefore\ \mathrm{L}\left\{\int_0^t (t-u)^2 e^u du\right\} = \mathrm{L}\{t\}^2 \times \mathrm{L}\{e^t\} = (2/s^3) \times \{1/(s-1)\},$ because here $\mathrm{F}(t) = t^2$ and $\mathrm{G}(t) = e^t$.

9. The value of $L^{-1}\{s/(2s^2 - 8)\}$ is

(a) $\cosh 2t$ (b) $\sinh 2t$ (c) $(1/2) \times \sinh 2t$ (d) $(1/2) \times \cosh 2t$ **[Agra 2009]**

Sol. Ans. (*d*) $L^{-1}\{s/(2s^2 - 8)\} = (1/2) \times L^{-1}\{s/(s^2 - 2^2)\} = (1/2) \times \cosh 2t$

ANSWERS

1. (a) **2.** (b) **3.** (c) **4.** (b) **5.** (c) **6.** (c) **7.** (a)

Miscellaneous examples on chapter 2

Ex :1. Evaluate $\{(2s - 3)/(s^2 + 6s + 25)\}$ **[Madurai Kamraj University 2008]**

Sol. $$L^{-1}\left[\frac{2s-3}{s^2+6s+25}\right] = L^{-1}\left\{\frac{2(s+3)-9}{(s+3)^2+4^2}\right\} = e^{-3t}L^{-1}\left\{\frac{2s-9}{s^2+4^2}\right\}$$

$$= 2L^{-1}\left\{s/(s^2+4^2)\right\} - 9\,L^{-1}\left\{1/(s^2+4^2)\right\} = 2\cos 4t - 9\sin 4t$$

Ex :2. Show that (i) $L^{-1}\{1/s(s^2 + 2s + 2)\} = \{1 - e^{-t}(\cos t + \sin t)\}/2$ **[Agra 2009]**

(ii) $L^{-1}\{(s+1)/s(s^2 + 4s + 8)\} = \{1 - e^{-2t}(\cos 2t + \sin 2t)\}/8$ **[Agra 2010]**

Ex.3. The value of t * t * t is

(a) $t^5/5!$ (b) $t^3/3!$ (c) $t^2/2!$ (d) $t^6/6!$ **[Agra 2009]**

Sol. Ans. (c). Take $n = 3$ in Ex. 2, page 2.55

Ex. 4. Prove that $B(m,n)=\int_0^1 u^{m-1}(1-u)^{n-1}du=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ and hence deduce that

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\ d\theta=\frac{1}{2}B(m,n)=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$ **[Meerut 2010]**

Hint : For first part, see Ex. 13, page 2.52

For second part, put $u=\sin^2\theta$ and $du=2\sin\theta\cos\theta$ and simply.

Ex. 5. Evaluate $L^{-1}\{1/(s^2+1)^2\}$. **[Kanpur 2011]**

Sol. Since $\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right)=-\dfrac{2s}{(s^2+1)^2}$, we get $L^{-1}\left\{-\dfrac{2s}{(s^2+1)^2}\right\}=L^{-1}\left\{\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right)\right\}$

$\Rightarrow -2\,L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}=-tL^{-1}\left\{\dfrac{1}{s^2+1}\right\}$, since $L^{-1}\left\{\dfrac{d}{ds}f(s)\right\}=-t\,L^{-1}\{f(s)\}$, by Art. 2.11

$\Rightarrow\ L^{-1}\left\{s/(s^2+1)^2\right\}=(t/2)\times\sin t=F(t)$, say

Now, $L^{-1}\left\{\dfrac{1}{s}\times\dfrac{s}{(s^2+1)^2}\right\}=\int_0^t F(x)\,dx$, as $L^{-1}\{f(s)\}=F(t)\Rightarrow L^{-1}\left\{\dfrac{f(s)}{s}\right\}=\int_0^t F(x)dx$

Thus, $L^{-1}\left\{1/(s^2+1)^2\right\}=\dfrac{1}{2}\int_0^t x\sin\ x\ dx=\dfrac{1}{2}\Big[x(-\cos x)-(1)\times\ (-\sin x)\Big]_0^t=\dfrac{\sin t-t\cos t}{2}$

Ex.6. Find $L^{-1}\{5\}$

Sol. $L^{-1}\{5\}=5\,L^{-1}\{1\}=5/s$, using table on page 2.3. **[Madras 2012]**

3

Applications to Ordinary Differential Equations

3.1. Solution of ordinary differential equation with constant coefficients.

Suppose we wish to solve the nth order ordinary differential equation with constant coefficients

$$d^n y/dt^n + a_1 (d^{n-1} y/d t^{n-1}) + a_2 (d^{n-2} y/d t^{n-2}) + \ldots + a_n y = \mathrm{F}(t), \qquad \ldots (1)$$

where $a_1, a_2, \ldots, a_n$ are constants, subject to the initial conditions

$$y(0) = k_0, \quad y'(0) = k_1, \quad y''(0) = k_2, \ldots, \quad y^{(n-1)}(0) = k_{n-1}, \qquad \ldots (2)$$

where $k_0, k_1, \ldots, k_{n-1}$ are constants.

On taking the Laplace transform of both sides of (1) and using (2), we obtain an algebraic equation (which is known as "subsidiary equation") for determination of $\mathrm{L}\{y(t)\}$. The required solution is then obtained by finding the inverse Laplace transform of $\mathrm{L}\{y(t)\}$. Since the formulas of Art 1.15, Chapter 1 depend on the initial conditions of the differential equation, these conditions are automatically contained in the final solution of the given differential equation when the inverse is found.

Very important note. Students are adviced to commit to memory all rules of finding Laplace transforms and inverse Laplace transforms discussed in chapters 1 and 2.

Notations. In what follows, we shall adopt the following notations:

(*i*) $dy/dt = \mathrm{D}y = y'(t) = y^{(1)}(t), \quad d^2y/dt^2 = \mathrm{D}^2 y = y''(t) = y^{(2)}(t), \ldots$

$d^n y/dt^n = \mathrm{D}^n y = y^{(n)}(t)$ etc.

(*ii*) At $t = 0$, we have $y(0) = y_0, \quad y'(0) = y_1, \quad y''(0) = y_2, \ldots y^{(n)}(0) = y_n$.

In some problems, the dependent variable may be x or z etc in place of y in (1). Then we make the corresponding changes in the notations which have just been discussed.

The whole procedure of solution will become clear by reading the following solved examples.

3.1A. Solved examples based on Art. 3.1

Ex. 1(a) *Solve $(D^2 + 4)y = t$ given that $y(0) = y'(0) = 0$ and $D \equiv d/dt$.*

[M.S.Univ. T.N. 2007]

Sol. Given $$y''(t) + 4y(t) = t \qquad \ldots (1)$$

with initial conditions: $y(0) = 0$ and $y'(0) = 0$ ... (2)

Taking Laplace transform of both sides of (1), we have

$\mathrm{L}\{y''(t)\} + 4\mathrm{L}\{y(t)\} = \mathrm{L}\{t\}$ or $s^2\,\mathrm{L}\{y\} - sy(0) - y'(0) + 4\mathrm{L}\{y\} = 1/s^2$

or $(s^2 + 4)\,\mathrm{L}\{y\} = 1/s^2$ or $\mathrm{L}\{y\} = 1/\{s^2(s^2 + 4)\}$, using (2) ...(3)

Taking inverse Laplace transform on both sides of (3), we get

$$y = \mathrm{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}\mathrm{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+4}\right\} = \frac{1}{4}\left(t - \frac{\sin 2t}{2}\right).$$

Ex. 1(*b*) *Using Laplace transform method solve* $(D + D^2)\,x = 2$, *if* $x = 3$ *and* $x' = 1$ *when* $t = 0$. **[Kanpur 2007]**

Sol. Re-writing, the given equation gives $x'(t) + x''(t) = 2$ ... (1)

Given boundary conditions are $x(0) = 3$ and $x'(0) = 1$ ... (2)

Taking Laplace transform of both sides of (1), we get

$\mathrm{L}\{x'(t)\} + \mathrm{L}\{x''(t)\} = 2\mathrm{L}\{1\}$ or $s\mathrm{L}\{x\} - x(0) + s^2\mathrm{L}\{x\} - sx(0) - x'(0) = 2/s$

or $s\mathrm{L}\{x\} - 3 + s^2\,\mathrm{L}\{x\} - 3s - 1 = 2/s$, using (2)

or $(s + s^2)\mathrm{L}\{x\} = \dfrac{2}{s} + 4 + 3s$ or $\mathrm{L}\{x\} = \dfrac{3s^2 + 4s + 2}{s^2(s+1)}$ ... (3)

Let $\dfrac{3s^2 + 4s + 2}{s^2(s+1)} = \dfrac{\mathrm{A}}{s+1} + \dfrac{\mathrm{B}}{s} + \dfrac{\mathrm{C}}{s^2}$ ... (4)

(4) $\Rightarrow$ $3s^2 + 4s + 2 = As^2 + Bs\,(s + 1) + C(s + 1)$ ... (5)

Equating coeffients of s, s^2 and constant terms on both sides of (5), we have

$\mathrm{A} + \mathrm{B} = 3$, $\mathrm{B} + \mathrm{C} = 4$ and $\mathrm{C} = 2$ $\Rightarrow$ $\mathrm{A} = 1$, $\mathrm{B} = 2$, $\mathrm{C} = 2$

$\therefore$ From (3), $\mathrm{L}\{x\} = 1/(s + 1) + 2/s + 2/s^2$ ... (6)

Taking inverse Laplace transform of both sides of (6), we get

$$x = \mathrm{L}^{-1}\{1/(s + 1)\} + 2\mathrm{L}^{-1}\{1/s\} + 2\mathrm{L}^{-1}\{1/s^2\} = e^{-t} + 2 + 2t$$

Ex.1(c). *Solve* $(D^2 - 1)\,y = a \cosh nt$, *if* $y = Dy = 0$ *when* $t = 0$.

Sol. Re-writing the given equation and conditions, we have

$$y'' - y = a \cosh nt, \quad \text{... (1)}$$

with initial conditions: $y(0) = 0$ and $y'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{y''\} - \mathrm{L}\{y\} = \mathrm{L}\{a \cosh nt\}$$

or $s^2\,\mathrm{L}\{y\} - s\,y(0) - y'(0) - \mathrm{L}\{y\} = as/(s^2 - n^2)$

or $(s^2 - 1)\,\mathrm{L}\{y\} - s \times 0 - 0 = as/(s^2 - n^2)$, using (2)

or $$\mathrm{L}\{y\} = \frac{as}{(s^2 - 1)(s^2 - n^2)} = \frac{as}{n^2 - 1}\left\{\frac{(s^2 - 1) - (s^2 - n^2)}{(s^2 - 1)(s^2 - n^2)}\right\}$$

or $$\mathrm{L}\{y\} = \frac{as}{n^2 - 1}\left[\frac{1}{s^2 - n^2} - \frac{1}{s^2 - 1}\right] = \frac{a}{n^2 - 1}\left[\frac{s}{s^2 - n^2} - \frac{s}{s^2 - 1}\right]$$

Taking inverse Laplace transform of both sides, we have

$$y = \frac{a}{n^2 - 1}\left[\mathrm{L}^{-1}\left\{\frac{s}{s^2 - n^2}\right\} - \mathrm{L}^{-1}\left\{\frac{s}{s^2 - 1}\right\}\right] = \frac{a}{n^2 - 1}\ (\cosh nt - \cosh t).$$

Ex.2. *Solve* $(D^2 + 1)y = 6 \cos 2t$, *if* $y = 3$, $Dy = 1$, *when* $t = 0$.

[GATE 2011; KU Kurukshetra 2005, Meerut 1993, Purvanchal 2002, 07]

Sol. Re-writing the given equation and conditions, we have

$$y'' + y = 6 \cos 2t, \quad \text{... (1)}$$

with initial conditions: $y(0) = 3$, and $y'(0) = 1$. ... (2)

Taking Laplace transform of both sides of (1), we get

$\mathrm{L}\{y''\} + \mathrm{L}\{y\} = 6\,\mathrm{L}\{\cos 2t\}$ or $s^2\,\mathrm{L}\{y\} - s\,y(0) - y'(0) + \mathrm{L}\{y\} = 6s/(s^2 + 2^2)$

or $(s^2 + 1)\ \mathrm{L}\{y\} - 3s - 1 = 6s/(s^2 + 4)$, using (2)

or $$\mathrm{L}\{y\} = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \frac{6s}{(s^2+1)(s^2+4)} = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + 2s\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$\therefore \quad \mathrm{L}\{y\} = 5 \times \frac{s}{s^2+1} + \frac{1}{s^2+1} - 2 \times \frac{s}{s^2+2^2}. \quad \ldots (3)$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = 5 \cos t + \sin t - 2 \cos 2t.$$

Ex.3. *Solve* $(D^2 + 4D + 8)\ y = 0,\ y(0) = 2,\ y'(0) = 2.$ **[Rohilkhand 1997]**

Sol. Re-writing the given equation and conditions, we have

$$y'' + 4y' + 8y = 0, \quad \ldots (1)$$

with initial conditions: $y(0) = 2$ and $y'(0) = 2$. ... (2)

Taking Laplace transform of both sides of (1), we get $\mathrm{L}\{y''\} + 4\,\mathrm{L}\{y'\} + 8\,\mathrm{L}\{y\} = 0$

or $s^2\,\mathrm{L}\{y\} - s\,y(0) - y'(0) + 4[s\,\mathrm{L}\{y\} - y(0)] + 8\,\mathrm{L}\{y\} = 0$

or $(s^2 + 4s + 8)\,\mathrm{L}\{y\} - 2s - 2 - (4 \times 2) = 0$, using (2)

or $$\mathrm{L}\{y\} = (2s + 10)/(s^2 + 4s + 8) \quad \ldots (3)$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = \mathrm{L}^{-1}\left\{\frac{2(s+2)+6}{(s+2)^2+4}\right\} = e^{-2t}\,\mathrm{L}^{-1}\left\{\frac{2s+6}{s^2+4}\right\}, \text{ using the first shifting theorem,}$$

$$= e^{-2t}\left[2\,\mathrm{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 6\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right]$$

$$= e^{-2t}\{2 \cos 2t + 6 \times (1/2) \times \sin 2t\} = e^{-2t}(2 \cos 2t + 3 \sin 2t).$$

Ex.4. *Find the general solution of* $(D^2 + k^2)\ y = 0$ *and verify it.*

Sol. Given that $y'' + k^2 y = 0$ ... (1)

Taking Laplace transform of both sides of (1), $\mathrm{L}\{y''\} + k^2\,\mathrm{L}\{y\} = 0$

or $s^2\,\mathrm{L}\{y\} - sy(0) - y'(0) + k^2\,\mathrm{L}\{y\} = 0$

or $(s^2 + k^2)\,\mathrm{L}\{y\} - As - B = 0$, where $y(0) = A$ and $y'(0) = B$, say

or $$\mathrm{L}\{y\} = \frac{As + B}{s^2 + k^2} = A \times \frac{s}{s^2 + k^2} + B \times \frac{1}{s^2 + k^2} \quad \ldots (2)$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = A \cos kt + (B/k) \times \sin kt = A \cos kt + C \sin kt, \quad \ldots (3)$$

where $C = B/k$. (3) is the general solution of given equation and A and C are arbitrary constants.

Verification: Differentiating (3) twice w.r.t. 't', we have

$y' = -Ak \sin kt + Ck \cos kt$ and $y'' = -Ak^2 \cos kt - Ck^2 \sin kt$ ... (4)

Substituting the values of y and y'' from (3) and (4) in (1) we get

$-Ak^2 \cos kt - Ck^2 \sin kt + k^2(A \cos kt + C \sin kt) = 0$ or $0 = 0$,

showing that (3) is a solution of (1).

Ex.5 (*a*). *Solve* $(D + 2)^2\ x = 4e^{-2t},\ x(0) = -1$ *and* $x'(0) = 4$ **[Agra 2008; Meerut 2001]**

Or Solve $x''(t) + 4x'(t) + 4x(t) = 4e^{-2t},\ x(0) = -1,\ x'(0) = 4$

Sol. Given $(D + 2)^2\ x = 4e^{-2t}$ or $(D^2 + 4D + 4)\ x = 4e^{-2t}$.

or $$x'' + 4x' + 4x = 4e^{-2t}, \quad \ldots (1)$$

with initial conditions: $x(0) = -1$ and $x'(0) = 4$. ... (2)

Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{x''\} + 4\,\mathrm{L}\{x'\} + 4\,\mathrm{L}\{x\} = 4\,\mathrm{L}\{e^{-2t}\}$$

or $$s^2\,\mathrm{L}\{x\} - s\,x(0) - x'(0) + 4[s\,\mathrm{L}\{x\} - x(0)] + 4\mathrm{L}(x) = 4/(s+2)$$

or $$(s^2 + 4s + 4)\,\mathrm{L}\{x\} + s - 4 + 4 = 4/(s+2), \text{ using (2)}$$

or $$(s+2)^2\,\mathrm{L}\{x\} = \frac{4}{s+2} - s = \frac{4 - s^2 - 2s}{s+2} \qquad \text{or} \qquad \mathrm{L}(x) = \frac{4 - s^2 - 2s}{(s+2)^3}.$$

Taking inverse Laplace transform of both sides, we get

$$x = \mathrm{L}^{-1}\left\{\frac{4 - (s+2)^2 + 2(s+2)}{(s+2)^3}\right\} = e^{-2t}\,\mathrm{L}^{-1}\left\{\frac{4 - s^2 + 2s}{s^3}\right\}$$

$$= e^{-2t}\left[4\mathrm{L}^{-1}\left\{\frac{1}{s^3}\right\} - L^{-1}\left\{\frac{1}{s}\right\} + 2\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}\right] = e^{-2t}\left[4 \times \frac{t^2}{2!} - 1 + 2 \times \frac{t}{1!}\right]$$

Thus, $x = e^{-2t}(2t^2 - 1 + 2t)$, which is the required solution.

Ex.5 (*b*). *Using Laplace transform, determine the solution of* $(d^2y/dt^2) + 3\,(dy/dt) + 2y = e^{-t}$, $y(0) = y'(0) = 0$. **[Lucknow 1998]**

Sol. Re-writing the given equation and conditions, we have

$$y'' + 3y' + 2y = e^{-t} \qquad \ldots (1)$$

with the initial conditions: $y(0) = 0$ and $y'(0) = 0$. ... (3)

Taking Laplace transform of both sides of (1), we have

$$\mathrm{L}\{y''\} + 3\,\mathrm{L}\{y'\} + 2\,\mathrm{L}\{y\} = \mathrm{L}\{e^{-t}\}$$

or $$s^2\,\mathrm{L}\{y\} - sy(0) - y'(0) + 3\{s\,\mathrm{L}\{y\} - y(0)\} + 2\,\mathrm{L}\{y\} = 1/(s+1)$$

or $$(s^2 + 3s + 2)\,\mathrm{L}\{y\} = 1/(s+1), \text{ using (2)}$$

or $$\mathrm{L}\{y\} = \frac{1}{(s^2 + 3s + 2)(s+1)} = \frac{1}{(s+2)(s+1)^2} = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

[On resolving into partial fractions]

Taking inverse transform of both sides of (3), we get

$$y = \mathrm{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-2t} - e^{-t} - e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

[Using first shifting theorem in the last term]

or $$y = e^{-2t} - e^{-t} - e^{-t} \times t. \qquad [\because\ \mathrm{L}^{-1}\{1/s^{n+1}\} = t^n/n!.]$$

Ex.6. *Solve* $(D^2 + D)\,x = 2$, *when* $x(0) = 3$, $x'(0) = 1$. **[Kanpur 1994, 96]**

Sol. Re-writing the given equation and conditions, $x'' + x' = 2$, ... (1)

with initial conditions: $x(0) = 3$ and $x'(0) = 1$. ... (2)

Taking Laplace transform of both sides of (1), $\mathrm{L}\{x''\} + \mathrm{L}\{x'\} = 2\,\mathrm{L}\{1\}$

or $$s^2\,\mathrm{L}\{x\} - s\,x(0) - x'(0) + s\,\mathrm{L}\{x\} - x(0) = 2 \times (1/s), \text{ using (2)}$$

or $$(s^2 + s)\,\mathrm{L}\{x\} - 3s - 1 - 3 = 2/s \qquad \text{or} \qquad (s^2 + s)\,\mathrm{L}\{x\} = 3s + 4 + 2/s$$

or $$\mathrm{L}\{x\} = \frac{3s^2 + 4s + 2}{s^2(s+1)} = \frac{s^2 + 2(s^2 + 2s + 1)}{s^2(s+1)} = \frac{1}{s+1} + \frac{2}{s} + \frac{2}{s^2} \qquad \ldots (3)$$

Taking inverse Laplace transform of both sides of (3), $x = e^{-t} + 2 + 2t$

Ex. 7(*a*) *Solve the differential equation using Laplace transforms* $y'' - 3y' + 2y = e^{3t}$ *given that* $y(0) = 0,\ y'(0) = 0.$ **[Bangalore 2005; Gulberga 2005]**

Sol. Given, $y''(t) - 3y'(t) + 2y(t) = e^{3t}$ ... (1)

with initial conditions $y(0) = 0$ and $y'(0) = 0$ ... (2)

Taking Laplace transform of both sides of (1), $\mathrm{L}\{y''(t)\} - 3\mathrm{L}\{y'(t)\} + 2\mathrm{L}\{y(t)\} = \mathrm{L}\{e^{3t}\}$

or $s^2\mathrm{L}\{y\} - sy(0) - y'(0) - 3[s\mathrm{L}\{y\} - y(0)] + 2\mathrm{L}\{y\} = 1/(s-3)$

$\therefore$ Using (2), $(s^2 - 3s + 2)\mathrm{L}\{y\} = 1/(s-3)$ or $\mathrm{L}\{y\} = 1/(s-3)\ (s^2 - 3s + 2)$

or $$\mathrm{L}\{y\} = \frac{1}{(s-3)(s-2)(s-1)} = \frac{1}{2(s-3)} - \frac{1}{s-2} + \frac{1}{2(s-1)} \quad \text{... (3)}$$

[On resolving into partial fractions]

Taking inverse Lapace transform on both sides of (3), we get

$$y(t) = (1/2) \times \mathrm{L}^{-1}\{1/(s-3)\} - \mathrm{L}^{-1}\ \{1/(s-2)\} + (1/2) \times \mathrm{L}^{-1}\ \{1/(s-1)\}$$

or $$y(t) = (1/2) \times e^{3t} - e^{2t} + (1/2) \times e^{t}$$

Ex. 7(*b*) *Using Laplace transform solve* $d^2y/dt^2 + 2(dy/dt) + 5y = 4e^{-t}$ *given that that* $y = 0 = dy/dt$ *when* $t = 0$ **[M.S. Univ. T.N. 2007]**

Sol. Given $y''(t) + 2y'(t) + 5y = 4e^{-t}$ ... (1)

with initial coditions : $y(0) = 0$ and $y'(0) = 0$ ... (2)

Taking Laplace transform of both sides of (1), $\mathrm{L}\{y''(t)\} + 2\mathrm{L}\{y'(t)\} + 5\mathrm{L}\{y\} = 4\mathrm{L}\{e^{-t}\}$

or $s^2\mathrm{L}\{y\} - sy(0) - y'(0) + 2[s\mathrm{L}\{y\} - y(0)] + 5\mathrm{L}\{y\} = 4/(s+1)$

or $(s^2 + 2s + 5)\ \mathrm{L}\{y\} = 4/(s + 1)$, using (2)

or $\mathrm{L}\{y\} = \dfrac{4}{(s+1)\ (s^2 + 2s + 5)}$ so that $y = \mathrm{L}^{-1}\left\{\dfrac{4}{(s+1)\ (s^2 + 2s + 5)}\right\}$ ... (3)

Let $$\frac{4}{(s+1)\ (s^2 + 2s + 5)} = \frac{\mathrm{A}}{s+1} + \frac{\mathrm{B}s + \mathrm{C}}{s^2 + 2s + 5} \quad \text{... (4)}$$

(4) $\Rightarrow$ $4 = \mathrm{A}(s^2 + 2s + 5) + (s + 1)\ (\mathrm{B}s + \mathrm{C})$ ... (5)

Equating coefficients of s^2, s and constant terms, (5) gives

$\mathrm{A} + \mathrm{B} = 0,\quad 2\mathrm{A} + \mathrm{C} + \mathrm{B} = 0$ and $5\mathrm{A} + \mathrm{C} = 4 \quad \Rightarrow \quad \mathrm{A} = 1,\quad \mathrm{B} = -1,\ \mathrm{C} = -1$

Hence, (4) yields $$\frac{4}{(s+1)\ (s^2 + 2s + 5)} = \frac{1}{s+1} - \frac{s+1}{s^2 + 2s + 5} \quad \text{... (6)}$$

Then, (3) and (6) $\Rightarrow$ $$y = \mathrm{L}^{-1}\left\{\frac{1}{s+1} - \frac{s+1}{(s+1)^2 + 2^2}\right\} = e^{-t}\left\{\frac{1}{s} - \frac{s}{s^2 + 2^2}\right\}$$

Thus, $y(t) = e^{-t}(1 - \cos 2t)$

Ex. 8. *Solve $(D^2 + 2D + 1)y = 3t\,e^{-t}$, subject to the conditions, $y = 4$, $Dy = 2$ when $t = 0$.*

Sol. Re-writing the given equation and conditions, we have

$$y'' + 2y' + y = 3t\,e^{-t} \quad \ldots (1)$$

with initial conditions: $y(0) = 4$ and $y'(0) = 2$ ... (2)

Taking Laplace transform of both sides of (1), we have

$$\mathrm{L}\{y''\} + 2\mathrm{L}\{y'\} + \mathrm{L}\{y\} = 3\mathrm{L}\{t\,e^{-t}\}$$

or $$s^2\mathrm{L}\{y\} - sy(0) - y'(0) + 2\,[sL\{y\} - y(0)] + L\{y\} = -3\frac{d}{ds}[L\{e^{-t}\}],$$

[using result of Art. 1.16, chapter 1]

or $$(s^2 + 2s + 1)\,\mathrm{L}\{y\} - 4s - 2 - (2 \times 4) = -3\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{3}{(s+1)^2}$$

or $$\mathrm{L}\{y\} = \frac{4s+10}{(s+1)^2} + \frac{3}{(s+1)^4} = \frac{4(s+1)+6}{(s+1)^2} + \frac{3}{(s+1)^2}$$

or $$\mathrm{L}\{y\} = \frac{4}{s+1} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4}. \quad \ldots (3)$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = \mathrm{L}^{-1}\left\{\frac{4}{s+1} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4}\right\} = e^{-t}\,\mathrm{L}^{-1}\left\{\frac{4}{s} + \frac{6}{s^2} + \frac{3}{s^4}\right\},$$ using first shifting theorem

$\therefore$ $$y = e^{-t}(4 + 6t + t^3/2),$$ which is the required solution.

Ex.9. *Solve $y'' + y = 8 \cos t$, if $y(0) = -1$, $y'(0) = 1$.* **[Rohilkhand 1993]**

Sol. Given $$y'' + y = 8 \cos t, \quad \ldots (1)$$

with initial conditions: $y(0) = -1$ and $y'(0) = 1$. ... (2)

Taking Laplace transform of both sides of (1), $\mathrm{L}\{y''\} + \mathrm{L}\{y\} = 8\,\mathrm{L}\{\cos t\}$

or $s^2\,\mathrm{L}\{y\} - s\,y(0) - y'(0) + \mathrm{L}\{y\} = 8s/(s^2 + 1^2)$

or $(s^2 + 1)\,\mathrm{L}\{y\} + s - 1 = 8s/(s^2 + 1)$, using (2)

or $$\mathrm{L}\{y\} = \frac{1}{s^2+1} - \frac{s}{(s^2+1)} + \frac{8s}{(s^2+1)^2}. \quad \ldots (3)$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = \mathrm{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} - L^{-1}\left\{\frac{s}{s^2+1^2}\right\} + 8\,\mathrm{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

or $$y = \sin t - \cot t + 8 \times \left(-\frac{1}{2}\right) \times \mathrm{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$

or $y = \sin t - \cos t - 4 \times (-1)^1\,t\,\mathrm{L}^{-1}\{1/(s^2 + 1^2)\}$, using result of Art. 2.15 of chapter 2

or $$y = \sin t - \cos t + 4t \sin t$$

Ex.10. *Solve* $(D^2 + 2D + 5)\, y = e^{-t} \sin t,\ y(0) = 0,\ y'(0) = 1.$

Sol. Re-writing the given equation and conditions, we have

$$y'' + 2y' + 5y = e^{-t} \sin t, \qquad \ldots (1)$$

with initial conditions: $y(0) = 0$ and $y'(0) = 1. \ldots (2)$

Taking Laplace transform of both sides of both sides of (1), we get

$$\mathrm{L}\{y''\} + 2\,\mathrm{L}\{y'\} + 5\,\mathrm{L}\{y\} = \mathrm{L}\{e^{-t} \sin t\}$$

or $$s^2\,\mathrm{L}\{y\} - s\,y(0) - y'(0) + 2[s\,\mathrm{L}\{y\} - y(0)] + 5\,\mathrm{L}\{y\} = f(s+1),$$

[Using first shifting theorem 1.11 and taking $f(s) = \mathrm{L}\{\sin t\} = 1/(s^2 + 1)$.]

or $$(s^2 + 2s + 5)\,\mathrm{L}\{y\} - 0 - 1 - (2 \times 0) = 1/[(s+1)^2 + 1], \text{ by (2)}$$

or $$\mathrm{L}\{y\} = \frac{1}{(s^2+2s+5)} + \frac{1}{(s^2+2s+5)\,[(s+1)^2+1]}$$

or $$y = \mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2+4} + \frac{1}{\{(s+1)^2+4\}\ \{(s+1)^2+1\}}\right\}$$

$$= e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+4} + \frac{1}{(s^2+4)\,(s^2+1)}\right\} = e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+4} + \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)\right\}$$

$$= e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{3}\times\frac{1}{s^2+1} + \frac{2}{3}\times\frac{1}{s^2+4}\right\} = e^{-t}\left[\frac{1}{3}\sin t + \frac{2}{3}\times\frac{1}{2}\sin 2t\right]$$

$\therefore$ $y = (1/3) \times e^{-t}(\sin t + \sin 2t)$ is required solution.

Ex.11. *Solve* $(D^2 + 1)\, y = \sin t \cos 2t,\ t > 0,$ *if* $y = 1,$ $\mathrm{D}y = 0$ when $t = 0$

Sol. Re-writing the given equation and conditions, we get

$$y'' + y = (1/2) \times (\cos t - \cos 3t), \qquad \ldots (1)$$

with initial conditions: $y(0) = 1$ and $y'(0) = 0. \ldots (2)$

Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{y''\} + \mathrm{L}\{y\} = (1/2) \times [\mathrm{L}\{\cos t\} - \mathrm{L}\{\cos 3t\}]$$

or $$s^2\,\mathrm{L}\{y\} - s\,y(0) - y'(0) + \mathrm{L}(y) = (1/2) \times \{s/(s^2+1) - s/(s^2+9)\}$$

or $$(s^2 + 1)\,\mathrm{L}\{y\} - s - 0 = s/2(s^2+1) - s/2(s^2+9), \text{ using (2)}$$

or $$\mathrm{L}\{y\} = \frac{s}{s^2+1} + \frac{1}{2}\frac{s}{(s^2+1)^2} - \frac{1}{2}\frac{s}{(s^2+1)\,(s^2+9)} = \frac{s}{s^2+1} + \frac{1}{2}\frac{s}{(s^2+1)^2} - \frac{s}{16}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

$\therefore$ $$\mathrm{L}\{y\} = \frac{15}{16}\times\frac{s}{s^2+1} + \frac{1}{2}\times\frac{s}{(s^2+1)^2} - \frac{1}{16}\times\frac{s}{s^2+9} \qquad \ldots (3)$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = \frac{15}{16}\mathrm{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} + \frac{1}{2}\mathrm{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} - \frac{1}{16}\mathrm{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$$

or $$y = \frac{15}{10}\cos t - \frac{1}{2}\times\frac{1}{2}\mathrm{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} - \frac{1}{16}\cos 3t \qquad \textbf{(Note)}$$

$= (15/16) \times \cos t - (1/4) \times (-1)^1\, t\,\mathrm{L}^{-1}\{1/(s^2+1)\} - (1/16) \times \cos 3t,$ using result of Art 2.11

Thus, $y = (15/16) \times \cos t + (1/4) \times t \sin t - (1/16) \times \cos 3t.$

Ex.12 (*a*). *Solve* $(D^2 + m^2)\, x = a \sin nt,\ t > 0$, *where* x, Dx *equal to* x_0 *and* x_1, *when* $t = 0$, $n \neq m$. **[Meerut 2000]**

(*b*) Solve $(D^2 + m^2)\, x = a \sin nt$, $t > 0$, where $x = 0$, $Dx = x_1$ when $t = 0$, $n \neq m$.

Sol. (*a*) Re-writing the given equation and conditions, we get

$$x'' + m^2 x = a \sin nt, \qquad \ldots (1)$$

with initial conditions: $x(0) = x_0$ and $x'(0) = x_1$. ... (2)

Taking Laplace transform of both sides of (1), we get

$$L\{x''\} + m^2 L\{x\} = a L\{\sin nt\}$$

or $$s^2 L\{x\} - s\,x(0) - x'(0) + m^2 L\{x\} = an/(s^2 + n^2)$$

or $$(s^2 + m^2) L\{x\} - s\,x_0 - x_1 = an/(s^2 + n^2), \text{ using (2)}$$

$$\text{or } L\{x\} = \frac{x_0 s}{s^2 + m^2} + \frac{x_1}{s^2 + m^2} + \frac{an}{(s^2 + m^2)(s^2 + n^2)} = \frac{x_0 s}{s^2 + m^2} + \frac{x_1}{s^2 + m^2} + \frac{an}{m^2 - n^2}\left[\frac{1}{s^2 + n^2} - \frac{1}{s^2 + m^2}\right].$$

Taking inverse Laplace transform of both sides, we get

$$x = x_0 \cos mt + \frac{x_1}{m} \sin mt + \frac{an}{m^2 - n^2}\left[\frac{1}{n}\sin nt - \frac{1}{m}\sin mt\right].$$

(*b*) Proceed as in part (*a*). Here $x(0) = x_0 = 0$.

$$\therefore \qquad x = \frac{x_1}{m}\sin mt + \frac{an}{m^2 - n^2}\left[\frac{1}{n}\sin nt - \frac{1}{m}\sin mt\right].$$

Ex.13 (*a*). *Solve* $(D^2 + m^2)\, x = a \cos nt,\ t > 0$, *if* x, Dx *equal to* x_0 *and* x_1, *when* $t = 0$, $n \neq m$.

(*b*) *Solve* $(D^2 + m^2)\, x = a \cos nt,\ t > 0$ *if* $x = Dx = 0$ *when* $t = 0$. **[Meerut 2010]**

(*c*) Solve $d^2y/dt^2 + 4y = A \cos kt$ given that $y = 0 = dy/dt$ when $t = 0$ using Laplace transform. **[M.S. Univ. T.M. 2007]**

Sol. (*a*) Re-writing the given equation and conditions, we get

$$x'' + m^2 x = a \cos nt, \qquad \ldots (1)$$

with the initial conditions: $x(0) = x_0$ and $x'(0) = x_1$. ... (2)

Taking Laplace transform of both sides of (1), $L\{x''\} + m^2 L\{x\} = a L\{\cos nt\}$.

or $$s^2 L\{x\} - s\,x(0) - x'(0) + m^2 L\{x\} = as/(s^2 + n^2)$$

or $$(s^2 + m^2) L\{x\} - s\,x_0 - x_1 = as/(s^2 + n^2), \text{ using (2)}$$

$$\text{or } L\{x\} = \frac{s x_0}{s^2 + m^2} + \frac{x_1}{s^2 + m^2} + \frac{as}{(s^2 + m^2)(s^2 + n^2)} = \frac{s x_0}{s^2 + m^2} + \frac{x_1}{s^2 + m^2} + \frac{a}{m^2 - n^2}\left[\frac{s}{s^2 + n^2} - \frac{s}{s^2 + m^2}\right].$$

Taking inverse Laplace transform of both sides, we get

$$x = x_0 \cos mt + \frac{x_1}{m}\sin mt + \frac{a}{m^2 - n^2}\left[\cos nt - \cos mt\right].$$

(*b*) Proceed as in part (*a*). Here $x_0 = 0$ and $x_1 = 0$ and hence solution is

$$y = a(\cos nt - \cos mt)/(m^2 - n^2).$$

(*c*) Proceed as in part (*a*). **Ans.** $y = A(\cos kt - \cos 2t)/(4 - k^2)$

Ex.14. *Solve* $y''(t) + y(t) = t$ *with* $y'(0) = 1$, $y(\pi) = 0$.

[Agra 2010; Kanpur 2008, Meerut 1995, S.V. Univ. (A.P.) 1997]

Sol. Given that $y''(t) + y(t) = t, \qquad \ldots (1)$

with initial conditions $y'(0) = 1 \qquad \ldots (2)$

and $y(\pi) = 0.$ ... (3)

Taking Laplace transform of both sides of (1), we get

$L\{y''\} + L\{y\} = L\{t\}$ or $s^2 L\{y\} - sy(0) - y'(0) + L\{y\} = 1/s^2$

or $(s^2 + 1) L\{y\} - As - 1 = 1/s^2$, taking $y(0) = A$ and using (2)

or $(s^2 + 1) L\{y\} = As + 1 + \frac{1}{s^2}$ or $L\{y\} = A\frac{s}{s^2+1} + \frac{1}{s^2}.$ ... (4)

Taking inverse Laplace transform of both sides of (4), we get $y(t) = A\cos t + t$ ... (5)

Keeping condition (3) in mind, we put $t = \pi$ in (5) and obtain

$y(\pi) = A\cos\pi + \pi$ or $0 = -A + \pi$, using (3). Hence $A = \pi$.

Hence from (5), the required solution is $y = \pi\cos t + t.$

Ex.15. *Solve $(D^2 + 9)\, y = \cos 2t,\ y(0) = 1,\ y(\pi/2) = -1.$*

[Agra 2011; Kumaun 1995, 98, Meerut 2007, Purvanchal 2005]

Sol. Re-writing the given equation and conditions, we have

$y'' + 9y = \cos 2t,$ ... (1)

with initial conditions: $y(0) = 1$ ... (2)

and $y(\pi/2) = -1.$ ... (3)

Taking Laplace transform of both sides of (1), we have

$L\{y''\} + 9L\{y\} = L\{\cos 2t\}$

or $s^2 L\{y\} - sy(0) - y'(0) + 9L\{y\} = s/(s^2 + 2^2)$

or $(s^2 + 9) L\{y\} - s - A = s/(s^2 + 4)$, taking $y'(0) = A$ and using (2)

or $L\{y\} = \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)} = \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{s}{5}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right]$

or $L\{y\} = \frac{4}{5}\times\frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{1}{5}\times\frac{s}{s^2+4}$ ... (4)

Taking inverse Laplace transform of both sides of (4), we get

$y(t) = (4/5) \times \cos 3t + (A/3) \times \sin 3t + (1/5) \times \cos 2t.$ ... (5)

Keeping condition (3) in mind, we put $t = \pi/2$ in (3) and obtain

$y\left(\frac{\pi}{2}\right) = \frac{4}{5}\cos\frac{3\pi}{2} + \frac{A}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\cos\pi$ or $-1 = 0 - \frac{A}{3} - \frac{1}{5}$, using (3).

Hence $A = 12/5$ and so from (5), the required solution is

$y = (4/5) \times \cos 3t + (4/5) \times \sin 3t + (1/5) \times \cos 2t.$

Ex.16. *Solve $(D^2 - D - 6)\, y = 2,\ t > 0$ if $y = 1,\ Dy = 0$ when $t = 0.$* **[Bhopal 2004]**

Sol. Re-writing the given equation and conditions, we get

$y'' - y' - 6y = 2$ ... (1)

with initial conditions: $y(0) = 1$ and $y'(0) = 0.$ ... (2)

Taking Laplace transform of both sides of (1), $L\{y''\} - L\{y'\} - 6L\{y\} = 2L\{1\}$

or $s^2 L\{y\} - sy(0) - y'(0) - [sL\{y\} - y(0)] - 6L\{y\} = 2/s$

or $(s^2 - s - 6) L\{y\} - s - 0 + 1 = 2/s$, using (2)

or $L\{y\} = \frac{s^2 - s + 2}{s(s^2 - s - 6)} = \frac{s^2 - s + 2}{s(s+2)(s-3)}$

$\therefore \quad y = L^{-1}\{(s^2 - s + 2)/s(s + 2)(s - 3)\}.$ ... (3)

Let $$\frac{s^2 - s + 2}{s(s+2)(s-3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-3}$$ ... (4)

Multiply both sides of (4) by s and let $s \to 0$, then

$$A = \lim_{s \to 0} \frac{s^2 - s + 2}{(s+2)(s-3)} = \frac{2}{(2) \times (-3)} = -\frac{1}{3}.$$

Multiply both sides of (4) by $(s + 2)$ and let $s \to -2$, then

$$B = \lim_{s \to -2} \frac{s^2 - s + 2}{s(s-3)} = \frac{(-2)^2 + 2 + 2}{(-2) \times (-5)} = \frac{4}{5}.$$

Multiply both sides of (4) by $(s - 3)$ and let $s \to 3$, then

$$C = \lim_{s \to 3} \frac{s^2 - s + 2}{s(s+2)} = \frac{3^2 - 3 + 2}{3 \times 5} = \frac{8}{15}.$$

$\therefore$ (4) $\Rightarrow$ $$\frac{s^2 - s + 2}{s(s+2)(s-3)} = -\frac{1}{3s} + \frac{4}{5(s+2)} + \frac{8}{15(s-3)}$$

Then, (3) $\Rightarrow$ $$y = -\frac{1}{3} L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{5} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{8}{15} L^{-1}\left\{\frac{1}{s-3}\right\}$$

or $$y = -(1/3) + (4/5) \times e^{-2t} + (8/15) \times e^{3t}.$$

Ex.17. *Solve* $(D^2 + 6D + 5)\, y = e^{-t}$, *if* $y(0) = 0$, $y'(0) = 1$.

Sol. Given $$y'' + 6y' + 5y = e^{-t}$$ ... (1)

with initial conditions: $y(0) = 0$ and $y'(0) = 1$. ... (2)

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} + 6L\{y'\} + 5L\{y\} = L\{e^{-t}\}$$

or $$s^2 L\{y\} - s\,y(0) - y'(0) + 6[s L\{y\} - y(0)] + 5 L\{y\} = 1/(s+1)$$

or $$(s^2 + 6s + 5) L\{y\} - 1 = 1/(1 + s), \text{ using (2)}$$

or $$(s+1)(s+5) L\{y\} = \frac{2+s}{s+1} \quad \text{or} \quad L\{y\} = \frac{s+2}{(s+1)^2 (s+5)}.$$

$\therefore \quad y = L^{-1}\{(s+2)/(s+1)^2 (s+5)\}.$ ... (3)

Let $$\frac{s+2}{(s+1)^2 (s+5)} = \frac{A}{s+5} + \frac{B}{s+1} + \frac{C}{(s+1)^2}.$$ ... (4)

Multiply both sides of (4) by $(s + 5)$ and let $s \to -5$, then

$$A = \lim_{s \to -5} \frac{s+2}{(s+1)^2} = \frac{-5+2}{(-5+1)^2} = -\frac{3}{16}.$$

Multiply both sides of (4) by $(s + 1)^2$ and let $s \to -1$, then

$$C = \lim_{s \to -1} \frac{s+2}{s+5} = \frac{-1+2}{-1+5} = \frac{1}{4}.$$

Then (4) $\Rightarrow$ $$\frac{s+2}{(s+1)^2(s+5)} = -\frac{3}{16(s+5)} + \frac{B}{s+1} + \frac{1}{4(s+1)^2}. \quad \ldots (5)$$

Putting $s = 0$ in identity (5), we obtain

$2/5 = -(3/80) + B + (1/4)$ so that $B = 3/16$.

$\therefore$ (5) $\Rightarrow$ $$\frac{s+2}{(s+1)^2(s+5)} = -\frac{3}{16(s+5)} + \frac{3}{16(s+1)} + \frac{1}{4(s+1)^2} \quad \ldots (6)$$

Then, (3) $\Rightarrow$ $$y = -\frac{3}{16}L^{-1}\left\{\frac{1}{s+5}\right\} + \frac{3}{16}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

or $$y = -\frac{3}{16}e^{-5t} + \frac{3}{16}e^{-t} + \frac{1}{4}e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}, \text{ using first shifting theorem}$$

or $$y = (3/16) \times (e^{-t} - e^{-5t}) + (1/4) \times t\, e^{-t}.$$

Ex.18. *Solve $(D + 1)^2 y = t$ given that $y = -3$, when $t = 0$ and $y = -1$ when $t = 1$.*

[Kumaun 1997, Meerut 1998]

Sol. Re-writing the given equation and conditions, we get

$(D^2 + 2D + 1)\, y = t$ or $y'' + 2y' + y = t$, ... (1)

with initial conditions: $y(0) = -3$ ... (2)

and $y(1) = -1$ ... (3)

Taking Laplace transform of both sides of (1), $L\{y''\} + 2L\{y'\} + L\{y\} = L\{t\}$

or $s^2 L\{y\} - s\,y(0) - y'(0) + 2[s\,L\{y\} - y(0)] + L\{y\} = 1/s^2$

$(s^2 + 2s + 1)\,L\{y\} + 3s - K - (2) \times (-3) = 1/s^2$, taking $y'(0) = K$ and using (2)

or $(s + 1)^2 L\{y\} = K - 6 - 3s + 1/s^2 = K + (1 - 6s^2 - 3s^3)/s^2$

or $$L\{y\} = \frac{K}{(s+1)^2} + \frac{1 - 6s^2 - 3s^3}{s^2(s+1)^2}$$

$\therefore$ $$y = K\,L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + L^{-1}\left\{\frac{1 - 6\,s^2 - 3s^3}{s^2(s+1)^2}\right\} \quad \ldots (4)$$

Let $$\frac{1 - 6\,s^2 - 3s^3}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} \quad \ldots (5)$$

Multiply both sides of (5) by s^2 and let $s \to 0$. Then

$$B = \lim_{s \to 0} \frac{1 - 6\,s^2 - 3s^3}{(s+1)^2} = \frac{1 - 0 - 0}{(0+1)^2} = 1$$

Multiply both sides of (5) by $(s + 1)^2$ and let $s \to -1$, then

$$D = \lim_{s \to -1} \frac{1 - 6\,s^2 - 3s^3}{s^2} = \frac{1 - 6 + 3}{(-1)^2} = -2.$$

Then, (5) $\Rightarrow$ $$\frac{1 - 6s^2 - 3s^3}{s^2(s+1)^2} = \frac{A}{s} + \frac{1}{s^2} + \frac{C}{s+1} - \frac{2}{(s+1)^2}. \quad \ldots (6)$$

Multiplying both sides of (6) by $s^2 (s + 1)^2$, we get

$$1 - 6s^2 - 3s^3 = As (s^2 + 2s + 1) + (s^2 + s + 1) + C s^2 (s + 1) - 2s^2 \quad \ldots (7)$$

Equating coefficients of s^3 and s^2 on both sides of above identity (7), we get

$A + C = -3$ and $2A + 1 + C - 2 = -6$ so that $A = -2$, and $C = -1$

Then, $(6) \Rightarrow \dfrac{1-6s^2-3s^3}{s^2(s+1)^2} = -\dfrac{2}{s} + \dfrac{1}{s^2} - \dfrac{1}{s+1} - \dfrac{2}{(s+1)^2}$.

$\therefore$ $(4) \Rightarrow y = K L^{-1} \{1/(s + 1)^2\} - 2L^{-1} \{1/s\} + L^{-1} \{1/s^2\} - L^{-1} \{1/(s + 1)\} - 2 L^{-1} \{1/(s + 1)^2\}$

$= (K - 2) L^{-1} \{1/(s + 1)^2\} - 2 + t - L^{-1} \{1/(s + 1)\} = (K - 2) e^{-t} L^{-1} \{1/s^2\} - 2 + t - e^{-t}$

Thus, $y(t) = (K - 2) e^{-t} t - 2 + t - e^{-t}$. ... (8)

Keeping condition (3) in mind, put $t = 1$ in (8) and obtain

$y(1) = (K - 2) e^{-1} - 2 + 1 - e^{-1}$ or $-1 = (K - 2) e^{-1} - 1 - e^{-1}$, by (3)

Thus, $(K - 2) e^{-1} = e^{-1}$ or $K - 2 = 1$ or $K = 3$.

Then, $(8) \Rightarrow y = e^{-t} t - 2 + t - e^{-t} = t - 2 + (t - 1) e^{-t}$.

Ex. 19. *Using transform method solve* $y'' + y = 3\cos 2t$ **[MDU Rohtak 2006]**

Sol. Given $y''(t) + y(t) = 3\cos 2t$ ... (1)

Taking Laplace transform of both sides of (1), we get

$L\{y''(t)\} + L\{y(t)\} = 3L\{\cos 2t\}$ or $s^2 L\{y\} - sy(0) - y'(0) + L\{y\} = (3s)/(s^2 + 1)$

or $(s^2 + 1) L\{y\} - As - B = (3s)/(s^2 + 1)$, where $A = y(0)$, $B = y'(0)$

or $L\{y\} = \dfrac{As}{s^2+1} + \dfrac{B}{s^2+1} + \dfrac{3s}{(s^2+1)(s^2+4)} = \dfrac{As}{s^2+1} + \dfrac{B}{s^2+1} + s\left(\dfrac{1}{s^2+1} - \dfrac{1}{s^2+4}\right)$

or $L\{y\} = (A+1) \times \dfrac{s}{s^2+1} + \dfrac{B}{s^2+1} - \dfrac{s}{s^2+2^2}$ ... (2)

Taking inverse Laplace transform of both sides of (2), we get

$y(t) = (A + 1) \cos t + B \sin t - \cos 2t$, A and B being arbitrary constants.

Ex.20. *Solve* $(D^2 + 6D + 9) y = \sin t$, *where* $y(0) = 1$, $y'(0) = 0$. **[Vikram 2004]**

Sol. Given that $y'' + 6y' + 9y = \sin t$ ... (1)

with initial conditions: $y(0) = 1$ and $y'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), $L\{y''\} + 6 L\{y'\} + 9 L\{y\} = L\{\sin t\}$

or $s^2 L\{y\} - sy(0) - y'(0) + 6 [s L\{y\} - y(0)] + 9 L\{y\} = 1/(s^2 + 1)$

or $(s^2 + 6s + 9) L\{y\} - s - 6 = 1/(s^2 + 1)$, using (2)

or $(s + 3)^2 L\{y\} = s + 6 + 1/(s^2 + 1) = (s + 3) + 3 + 1/(s^2 + 1)$

or $L\{y\} = \dfrac{1}{(s+3)} + \dfrac{3}{(s+3)^2} + \dfrac{1}{(s+3)^2(s^2+1)}$. ... (3)

Let $\dfrac{1}{(s+3)^2(s^2+1)} = \dfrac{A}{s+3} + \dfrac{B}{(s+3)^2} + \dfrac{Cs+D}{s^2+1}$. ... (4)

Multiply both sides of (4) by $(s + 3)^2$ and let $s \to -3$, then

$$B = \lim_{s \to -3} 1/(s^2 + 1) = 1/(9 + 1) = 1/10.$$

Then, (4) $\Rightarrow$ $$\frac{1}{(s+3)^2(s^2+1)} = \frac{A}{s+3} + \frac{1}{10(s+3)^2} + \frac{Cs+D}{s^2+1}. \quad \ldots (5)$$

Multiplying both sides of (5) by $(s + 3)^2 (s^2 + 1)$, we get

$$1 = A(s + 3)(s^2 + 1) + (1/10) \times (s^2 + 1) + (Cs + D)(s + 3)^2$$

or $$1 = A(s + 3)(s^2 + 1) + (1/10) \times (s^2 + 1) + (Cs + D)(s^2 + 6s + 9)$$

or $$1 = 3A + (1/10) + 9D + s(A + 9C + 6D) + s^2[3A + (1/10) + D + 6C] + s^3(A + C). \quad \ldots (6)$$

Equating the coefficients of s^3, s and constant terms on both sides of the identity (6),

$A + C = 0$, $A + 9C + 6D = 0$ and $3A + (1/10) + 9D = 1$.

Solving these, $A = 3/50$, $C = -3/50$, $D = 4/50$.

$$\therefore (5) \Rightarrow \frac{1}{(s+3)^2(s^2+1)} = \frac{3}{50(s+3)} + \frac{1}{10(s+3)^2} - \frac{3s-4}{50(s^2+1)}.$$

$$\therefore (3) \Rightarrow L\{y\} = \frac{1}{s+3} + \frac{3}{(s+3)^2} + \frac{3}{50(s+3)} + \frac{1}{10(s+3)^2} - \frac{3s-4}{50(s^2+1)}.$$

or $$L\{y\} = \frac{53}{50(s+3)} + \frac{31}{10(s+3)^2} - \frac{3s}{50(s^2+1)} + \frac{2}{25(s^2+1)}.$$

or $$y = \frac{53}{50}e^{-3t} + \frac{31}{10}e^{-3t} L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{3}{50}\cos t + \frac{2}{25}\sin t$$, using first shifting theorem

or $$y = (53/50) \times e^{-3t} + (31/10) \times t\, e^{-3t} - (3/50) \times \cos t + (2/25) \times \sin t$$

Ex.21. *Solve $d^2y/dt^2 + 2(dy/dt) - 3y = 2 \sin t$ using Laplace transform. Also use the initial conditions $y = dy/dt = 0$ when $t = 0$.* **[Kerala 2001]**

Sol. Re-writing the given equation, $y''(t) + 2y'(t) - 3y(t) = \sin t$ ... (1)

Re-writing the given initial conditions, $y(0) = y'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), we have

$$L\{y''(t)\} + 2L\{y'(t)\} - 3L\{y(t)\} = L\{\sin t\}$$

or $$s^2 L\{y\} - sy(0) - y'(0) + 2[sL\{y\} - y(0)] - 3L\{y\} = 1/(s^2 + 1)$$

or $$(s^2 + 2s - 3) L\{y\} = 1/(s^2 + 1)$$, using (2)

or $$L\{y\} = 1/(s^2 + 1)(s^2 + 2s - 3).$$ ... (3)

We now resolve R.H.S. of (3) into partial fractions.

Let $$\frac{1}{(s^2+1)(s^2+2s-3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s-3}. \quad \ldots (4)$$

Multiplying both sides of (4) by $(s^2 + 1)(s^2 + 2s - 3)$, we get

$$1 = (As + B)(s^2 + 2s - 3) + (Cs + D)(s^2 + 1) \quad \ldots (5)$$

Equating coefficients of s, s^2, s^3 and constant terms on both sides of the identity (5), we have

$-3A + 2B + C = 0$, $2A + B + D = 0$, $A + C = 0$, $-3B + D = 1$.

Solving these, $A = -1/10$, $B = -1/5$, $C = 1/10$, $D = 2/5$.

$\therefore$ (4) gives $$\frac{1}{(s^2+1)(s^2+2s-3)} = -\frac{s+2}{10(s^2+1)} + \frac{s+4}{10(s^2+2s-3)}$$

Then (3) becomes $\quad L\{y\} = \dfrac{(s+1)+3}{10\{(s+1)^2-4\}} - \dfrac{s+2}{10(s^2+1)}$

$$\therefore\ (6) \Rightarrow y(t) = \frac{1}{10} L^{-1}\left\{\frac{(s+1)+3}{(s+1)^2-4}\right\} - \frac{1}{10} L^{-1}\left\{\frac{s+2}{s^2+1}\right\} = \frac{1}{10} e^{-t}\, L^{-1}\left\{\frac{s+3}{s^2-2^2}\right\} - \frac{1}{10} L^{-1}\left\{\frac{s+2}{s^2+1}\right\}$$

[Using first shifting theorem in first term]

$$= \frac{e^{-t}}{10}\left[L^{-1}\left\{\frac{s}{s^2-2^2}\right\} + 3L^{-1}\left\{\frac{1}{s^2-2^2}\right\}\right] - \frac{1}{10}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} + 2L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

or $\quad y = (e^{-t}/10) \times \{\cosh 2t + (3/2) \times \sinh 2t\} + (1/10) \times (\cos t + 2 \sin t)$

Ex.22. *Solve $(D^2 + D)\, y = t^2 + 2t$. where $y(0) = 4$, $y'(0) = -2$.* **[Agra 2005]**

Sol. Given $\quad y'' + y' = t^2 + 2t \quad \ldots (1)$

subject to the initial conditions: $\quad y(0) = 4 \quad$ and $\quad y'(0) = -2. \quad \ldots (2)$

Taking Laplace transform of both sides of (1), $\quad L\{y''\} + L\{y'\} = L\{t^2\} + 2\, L\{t\}$

or $\quad s^2 L\{y\} - s\, y\{0\} - y'(0) + s\, L\{y\} - y(0) = 2!/s^3 + 2/s^2$

or $\quad (s^2 + s)\, L\{y\} - 4s + 2 - 4 = (2 + 2s)/s^3$, using (2)

or $\quad s(s+1)\, L\{y\} = 4s + 2 + 2(1+s)/s^3$

or $\quad L\{y\} = \dfrac{4s+2}{s(s+1)} + \dfrac{2}{s^4} = \dfrac{2s+2(s+1)}{s(s+1)} + \dfrac{2}{s^4} = \dfrac{2}{s+1} + \dfrac{2}{s} + \dfrac{2}{s^4}. \quad \ldots.(3)$

Taking inverse Laplace of both sides of (3), we get

$\therefore \quad y = 2\, e^{-t} + 2 + t^3/3.$

Ex.23. *Solve $(D^2 - D - 2)\, y = 20 \sin 2t$, if $y = -1$, $Dy = 2$ when $t = 0$.*

[Ravishanker 2005]

Sol. Given that $\quad y'' - y' - 2y = 20 \sin 2t \quad \ldots (1)$

subject to initial conditions: $\quad y(0) = -1 \quad$ and $\quad y'(0) = 2. \quad \ldots (2)$

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} - L\{y'\} - 2\, L\{y\} = 20\, L\{\sin 2t\}$$

or $\quad s^2 L\{y\} - sy(0) - y'(0) - [s\, L\{y\} - y(0)] - 2\, L\{y\} = 40/(s^2 + 4)$

or $\quad (s^2 - s - 2)\, L\{y\} + s - 2 - 1 = 40/(s^2 + 4)$, using (2)

or $\quad L\{y\} = \dfrac{3-s}{(s-2)(s+1)} + \dfrac{40}{(s-2)(s+1)(s^2+4)} \quad \ldots (3)$

As usual resolving into partial fractions, we get

$$\frac{3-s}{(s-2)(s+1)} = \frac{1}{3(s-2)} - \frac{4}{3(s+1)} \quad \ldots(4)$$

and $\quad \dfrac{40}{(s-2)(s+1)(s^2+4)} = \dfrac{5}{3(s-2)} - \dfrac{8}{3(s+1)} + \dfrac{s-6}{s^2+4} \quad \ldots (5)$

Using (4) and (5), (3) reduces to

$$L\{y\} = \frac{1}{3(s-2)} - \frac{4}{3(s+1)} + \frac{5}{3(s-2)} - \frac{8}{3(s+1)} + \frac{s-6}{s^2+4}$$

or $\quad L\{y\} = \dfrac{2}{s-2} - \dfrac{4}{s+1} + \dfrac{s}{s^2+4} - \dfrac{6}{s^2+4}. \quad \ldots (6)$

Taking inverse Laplace transform of both sides of (6), we get

or $\quad y = 2\, e^{2t} - 4\, e^{-t} + \cos 2t - 3 \sin 2t.$

Ex.24. *Solve $(D^4 - 1)\, y = 1$, if $y = Dy = D^2 y = D^3 y = 0$ when $t = 0$.*

Sol. Given $$y'''' - y = 1, \quad \ldots (1)$$

with initial conditions: $$y(0) = y'(0) = y''(0) = y'''(0) = 0. \quad \ldots (2)$$

Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{y''''\} - \mathrm{L}\{y\} = \mathrm{L}\{1\}$$

or $$s^4 \mathrm{L}\{y\} - s^3 y(0) - s^2 y'(0) - s\, y''(0) - y'''(0) - \mathrm{L}\{y\} = 1/s$$

or $$(s^4 - 1)\, \mathrm{L}\{y\} = 1/s, \text{ using (2)}$$

or $$\mathrm{L}\{y\} = \frac{1}{s(s^4 - 1)} = \frac{1}{s(s-1)(s+1)(s^2+1)}. \quad \ldots (3)$$

Let $$\frac{1}{s(s-1)(s+1)(s^2+1)} = \frac{\mathrm{A}}{s} + \frac{\mathrm{B}}{s-1} + \frac{\mathrm{C}}{s+1} + \frac{\mathrm{D}s + \mathrm{E}}{s^2+1}. \quad \ldots (4)$$

Multiply both sides of (4) by s and let $s \to 0$, then

$$\mathrm{A} = \lim_{s \to 0} \frac{1}{(s-1)(s+1)(s^2+1)} = \frac{1}{(-1) \times 1 \times 1} = -1.$$

Multiply both sides of (4) by $(s - 1)$ and let $s \to 1$, then

$$\mathrm{B} = \lim_{s \to 1} \frac{1}{s(s+1)(s^2+1)} = \frac{1}{1 \times 2 \times 2} = \frac{1}{4}.$$

Multiply both sides of (4) by $(s + 1)$ and let $s \to -1$, then

$$\mathrm{C} = \lim_{s \to -1} \frac{1}{s(s-1)(s^2+1)} = \frac{1}{(-1) \times (-2) \times (2)} = \frac{1}{4}.$$

$$\therefore \quad (4) \Rightarrow \quad \frac{1}{s(s-1)(s+1)(s^2+1)} = -\frac{1}{s} + \frac{1}{4(s-1)} + \frac{1}{4(s+1)} + \frac{\mathrm{D}s + \mathrm{E}}{s^2+1}$$

or $$\frac{1}{s(s^2-1)(s^2+1)} = -\frac{1}{s} + \frac{s}{2(s^2-1)} + \frac{\mathrm{D}s + \mathrm{E}}{s^2+1} \quad \ldots (5)$$

Multiplying both sides of (5) by $s\,(s^2 - 1)\,(s^2 + 1)$, we have

$$1 = -(s^2 - 1)(s^2 + 1) + (1/2) \times s^2 (s^2 + 1) + (\mathrm{D}s + \mathrm{E})\, s\, (s^2 - 1)$$

or $$1 = -(s^4 - 1) + (1/2) \times (s^4 + s^2) + (\mathrm{D}s + \mathrm{E})\, (s^3 - s). \quad \ldots (6)$$

Equating coefficients of s^4 and s on both sides of the identity (6) we have

$\mathrm{D} + (1/2) - 1 = 0$ and $-\mathrm{E} = 0$ so that $\mathrm{D} = 1/2$, and $\mathrm{E} = 0$

Now, (3) and (5) $\Rightarrow$ $$\mathrm{L}\{y\} = -\frac{1}{s} + \frac{s}{2(s^2-1)} + \frac{s}{2(s^2+1)}. \quad \ldots (7)$$

Taking inverse Laplace transform of both (7), we have

$\therefore$ $$y = -1 + (1/2) \times (\cosh t + \cos t).$$

Ex.25. *Solve $d^2y/dt^2 + y = t \cos 2t$, if $y = 0$, $dy/dt = 0$ when $t = 0$.*

[Kumaun 1999, Purvanchal 2006, Meerut 1997]

Sol. Re-writing the given equation and conditions, we have

$$y'' + y = t \cos 2t, \quad \ldots (1)$$

subject to initial conditions: $$y(0) = y'(0) = 0. \quad \ldots (2)$$

Taking Laplace transform of both sides of (1), $$\mathrm{L}\{y''\} + \mathrm{L}\{y\} = \mathrm{L}\{t \cos 2t\}$$

or $$s^2 \text{ L } \{y\} - s\,y\,(0) - y'\,(0) + \text{L } \{y\} = \text{L } \{t \cos 2t\} \quad \ldots (3)$$

Now, $$\text{L } \{t \cos 2t\} = -\frac{d}{ds}[\text{L}\{\cos 2t\}], \text{ by Art. 1.16 of chapter 1}$$

$$\therefore \quad \text{L } \{t \cos 2t\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\frac{(s^2+4)-(s\times 2s)}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2} \quad \ldots (4)$$

Using (2) and (4), (3) reduces to

or $$(s^2 + 1) \text{ L } \{y\} = \frac{s^2-4}{(s^2+4)^2} \qquad \text{or} \qquad \text{L } \{y\} = \frac{s^2-4}{(s^2+1)\,(s^2+4)^2}.$$

$$y = \text{L}^{-1} \{(s^2 - 4)/(s^2 + 1)\,(s^2 + 4)^2\} \quad \ldots (5)$$

Suppose that $s^2 = x$. Then, we have

$$\frac{s^2-4}{(s^2+1)(s^2+4)^2} = \frac{x-4}{(x+1)(x+4)^2} = \frac{\text{A}}{x+1} + \frac{\text{B}}{x+4} + \frac{\text{C}}{(x+4)^2}. \quad \ldots (6)$$

Multiply both sides of (6) by $x + 1$ and let $x \to -1$, then

$$\text{A} = \lim_{x\to -1} \frac{x-4}{(x+4)^2} = \frac{-1-4}{(-1+4)^2} = -\frac{5}{9}.$$

Multiplying both sides of (6) by $(x + 4)^2$ and let $x \to -4$, then

$$\text{C} = \lim_{x\to -4} \frac{x-4}{x+1} = \frac{-4-4}{-4+1} = \frac{8}{3}.$$

Then, (6) $\Rightarrow$ $$\frac{x-4}{(x+1)\,(x+4)^2} = -\frac{5}{9(x+1)} + \frac{\text{B}}{x+4} + \frac{8}{3(x+4)^2}. \quad \ldots (7)$$

To find B, put $x = 0$ in identity (7) and obtain $-(1/4) = -(5/9) + \text{B}/4 + 1/6$ and hence $\text{B} = 5/9$. Putting $\text{B} = 5/9$ and $x = s^2$ in (7), we get

$$\frac{s^2-4}{(s^2+1)(s^2+4)^2} = -\frac{5}{9(s^2+1)} + \frac{5}{9(s^2+4)} + \frac{8}{3(s^2+4)^2}.$$

$\therefore$ (5) $\Rightarrow$ $$y = \text{L}^{-1} \left\{-\frac{5}{9(s^2+1)} + \frac{5}{9(s^2+4)} + \frac{8}{3(s^2+4)^2}\right\}$$

or $$y = -\frac{5}{9}\sin t + \frac{5}{9}\times\frac{1}{2}\sin 2t + \frac{8}{3}\text{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} \quad \ldots (8)$$

Now, (4) $\Rightarrow$ $$\text{L } \{t \cos 2t\} = \frac{(s^2+4)-8}{(s^2+4)^2} = \frac{1}{s^2+4} - \frac{8}{(s^2+4)^2}.$$

$$\therefore \quad t \cos 2t = \text{L}^{-1}\left[\frac{1}{s^2+4} - \frac{8}{(s^2+4)^2}\right] = \frac{1}{2}\sin 2t - 8\,\text{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} \quad \ldots (9)$$

$\therefore$ $$\text{L}^{-1} \{1/(s^2 + 4)^2\} = (1/8) \times \{(1/2) \times \sin 2t - t \cos 2t\}$$

Using (9), (8) reduces to

$$y = -(5/9) \times \sin t + (5/18) \times \sin 2t + (1/3) \times \{(1/2) \times \sin 2t - t \cos 2t\}$$

or $$y = -(5/9) \times \sin t + (4/9) \times \sin 2t - (t/3) \times \cos 2t.$$

Ex.26. *$(D^3 - D^2 - D+1)\, y = 8t\, e^{-t}$, if $y = D^2 y = 0$, $Dy = 1$ when $t = 0$.*

[Jabalpur 2004]

Sol. Given $$y''' - y'' - y' + y = 8\ t\ e^{-t}, \qquad \ldots (1)$$

subject to initial conditions: $y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0.$ ... (2)

Taking Laplace transform of both sides of (1) we have

$$L\{y'''\} - L\{y''\} - L\{y'\} + L\{y\} = 8L\{te^{-t}\}$$

or $$s^3 L\{y\} - s^2 y(0) - s y'(0) - y''(0) - [s^2 L\{y\} - s y(0) - y'(0)]$$

$$- [s L\{y\} - y(0)] + L\{y\} = -8 \frac{d}{ds} [L\{e^{-t}\}], \text{ using result of Art 1.16}$$

or $$(s^3 - s^2 - s + 1) L\{y\} - s + 1 = -8 \frac{d}{ds}\left(\frac{1}{1+s}\right), \text{ using (2)}$$

or $$\{s^2 (s-1) - (s-1)\} L\{y\} = s - 1 + 8/(1+s)^2$$

or $$L\{y\} = \frac{1}{(s-1)(s+1)} + \frac{8}{(s-1)^2 (s+1)^3}. \qquad \ldots (3)$$

By inspection, $$\frac{1}{(s-1)(s+1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right). \qquad \ldots (4)$$

Let $$\frac{8}{(s-1)^2 (s+1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)^3}. \qquad \ldots (5)$$

Multiply both sides of (5) by $(s-1)^2$ and let $s \to 1$, then

$$B = \lim_{s \to 1} \frac{8}{(s+1)^3} = \frac{8}{(1+1)^3} = 1.$$

Multiply both sides of (5) by $(s+1)^3$ and let $s \to -1$, then

$$E = \lim_{s \to -1} \frac{8}{(s-1)^2} = \frac{8}{(-1-1)^2} = 2.$$

$\therefore$ $$\frac{8}{(s-1)^2 (s+1)^3} = \frac{A}{s-1} + \frac{1}{(s-1)^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} + \frac{2}{(s+1)^3} \qquad \ldots (6)$$

Multiplying both sides by $(s-1)^2 (s+1)^3$, we have

$$8 = A(s-1)(s^3+3s^2+3s+1) + (s^3+3s+3s+1) + C(s^4-2s^2+1) + D(s-1)(s^2-1) + 2(s^2-2s+1). \ldots (7)$$

Equating coefficients of s^4, s^3 and constant terms on both sides of the identity (7), we get

$A + C = 0$, $\quad 2A + 1 + D = 0$ and $\quad -A + 1 + C + D + 2 = 8.$

Solving these, $\quad A = -3/2, \quad C = 3/2, \quad D = 2.$

$\therefore$ $$\frac{8}{(s-1)^2 (s+1)^3} = -\frac{3}{2(s+1)} + \frac{1}{(s-1)^2} + \frac{3}{2(s+1)} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}.$$

Using the above result and (4), (3) reduces to

$$L\{y\} = \frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{3}{2(s-1)} + \frac{1}{(s-1)^2} + \frac{3}{2(s+1)} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

or $$L\{y\} = -\frac{1}{s-1} + \frac{1}{s+1} + \frac{1}{(s-1)^2} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}.$$

Taking inverse Laplace transform of both sides, we get

$$y = -e^t + e^{-t} + e^t\, L^{-t}\{1/s^2\} + 2e^{-t}\, L^{-1}\{1/s^2\} + 2e^{-t}\, L^{-1}\{1/s^3\}$$

(using first shifting theorem in the last three terms)

$\therefore \quad y = -e^t + e^{-t} + t\,e^t + 2t\,e^{-t} + e^{-t}\,t^2 = (1 + 2t + t^2)\,e^{-t} + (t-1)\,e^t.$

Ex. 27. *Using Laplace transforms find the solution of initial value problem* $y'' + 2y' + 2y = 2,\ y(0) = 0,\ y'(0) = 1$ **[Nagpur 2005]**

Sol. Given $y''(t) + 2y'(t) + 2y(t) = 2$ ... (1)

with initial conditions: $y(0) = 0$ and $y'(0) = 1$ ... (2)

Taking Laplace transform of both sides of (1), we get

$$s^2\,L\{y\} - sy(0) - y'(0) + 2[s\,L\{y\} - y(0)] + 2\,L\{y\} = 2L\{1\} \quad \text{... (3)}$$

Using (2), (3) gives $s^2\,L\{y\} - 1 + 2s\,L\{y\} + 2y = 2/s$

or $$L\{y\} = \frac{1}{s^2+2s+2} + \frac{2}{s(s^2+2s+2)} = \frac{1}{s^2+2s+2} + \frac{1}{s} - \frac{s+2}{s^2+2s+2}$$

[On resolving into partial fraction]

or $$L\{y\} = \frac{1}{s} + \frac{2}{s^2+2s+2} - \frac{s+1}{(s^2+2s+2)} = \frac{1}{s} + \frac{2}{(s+1)^2+1} - \frac{s+1}{(s+1)^2+1}$$

Taking inverse Laplace transform of both sides, we get

$$y(t) = L^{-1}\left\{\frac{1}{s}\right\} + 2L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} - L^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = 1 + 2e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\} - e^{-t}L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

or $$y(t) = 1 + 2e^{-t}\sin t - e^{-t}\cos t$$

Ex. 28. *Using Laplace transform method solve* $x''(t) + 2x'(t) = 8t,\ \ x(0) = x'(0) = 0.$ **[Mysore 2004]**

Sol. Given $x''(t) + 2x'(t) = 8t$ ... (1)

with initial conditions $x(0) = 0$ and $x'(0) = 0$ ... (2)

Taking Laplace transfofm of both sides of (1), we get

$$s^2 L\{x\} - s\,x(0) - x'(0) + 2[s\,L\{x\} - x(0)] = 8L\{t\} \quad \text{... (3)}$$

Using (2), (3) gives $(s^2 + 2s)\,L\{x\} = 8/s^2$ or $L\{x\} = 8/s^3(s+2)$

We shall use Long-Division method (see Art. 2.6 of chapter 2) to find the partial fractions

$$\begin{array}{r} 1/2 - s/4 + s^2/8 \\ 2 + s \;\overline{)\;\; 1} \\ \underline{1 + s/2} \\ -s/2 \\ \underline{-s/2 - s^2/4} \\ s^2/4 \\ \underline{s^2/4 + s^3/8} \\ -s^3/8 \end{array}$$

$$\therefore \qquad \bar{x} = \frac{8}{s^3} \times \frac{1}{s+2} = \frac{8}{s^3}\left[\frac{1}{2} - \frac{s}{4} + \frac{s^2}{8} + \frac{(s^3/8)}{s+2}\right]$$

or $$\bar{x} = 4 \times (1/s^3) - 2 \times (1/s^2) + (1/s) + 1/(s+2) \qquad \ldots (3)$$

Taking inverse Laplace transform of both sides of (3), we get

$$x = 4 \times (t^2/2!) - 2 \times (t/1!) + 1 - e^{-2t} = 2t^2 - 2t + 1 - e^{-2t}$$

Ex.29. *Solve $(D^3 - 2D^2 + 5D)\, x = 0$, if $x(0) = 0$, $x'(0) = 1$, $x(\pi/8) = 1$.* **[Agra 2009, 10]**

Sol. Given that $$x''' - 2x'' + 5x' = 0, \qquad \ldots (1)$$

subject to the initial conditions: $$x(0) = 0 \quad \text{and} \quad x'(0) = 1. \qquad \ldots (2)$$

and $$x(\pi/8) = 1. \qquad \ldots (3)$$

Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{x'''\} - 2\,\mathrm{L}\{x''\} + 5\,\mathrm{L}\{x'\} = 0$$

or $$s^3\,\mathrm{L}\{x\} - s^2 x(0) - sx'(0) - x''(0) - 2[s^2\,\mathrm{L}\{x\} - sx(0) - x'(0)] + 5[s\,\mathrm{L}\{x\} - x(0)] = 0$$

or $$(s^3 - 2s^2 + 5s)\,\mathrm{L}\{x\} - s - \mathrm{A} + 2 = 0, \text{ taking } x''(0) = \mathrm{A} \text{ and using (2)}$$

or $$\mathrm{L}\{x\} = \frac{s + (\mathrm{A}-2)}{s(s^2 - 2s + 5)} = \frac{1}{(s^2 - 2s + 5)} + \frac{\mathrm{A}-2}{s(s^2 - 2s + 5)} \qquad \ldots (4)$$

Let $$\frac{1}{s(s^2 - 2s + 5)} = \frac{\mathrm{B}}{s} + \frac{\mathrm{C}s + \mathrm{D}}{s^2 - 2s + 5}. \qquad \ldots (5)$$

$$(5) \Rightarrow \qquad 1 = \mathrm{B}(s^2 - 2s + 5) + s(\mathrm{C}s + \mathrm{D}).$$

Equating coefficients of s^2, s and constant terms on both sides of the above identity, we have

$$\mathrm{B} + \mathrm{C} = 0, \qquad -2\,\mathrm{B} + \mathrm{D} = 0 \qquad \text{and} \qquad 5\mathrm{B} = 1.$$

Solving these, $\mathrm{B} = 1/5$, $\mathrm{C} = -1/5$, $\mathrm{D} = 2/5$.

Then, $$(5) \Rightarrow \qquad \frac{1}{s(s^2 - 2s + 5)} = \frac{1}{5s} - \frac{s-2}{5(s^2 - 2s + 5)}.$$

Hence $$(4) \Rightarrow \qquad \mathrm{L}\{x\} = \frac{1}{s^2 - 2s + 5} + (\mathrm{A}-2)\left[\frac{1}{5s} - \frac{s-2}{5(s^2 - 2s + 5)}\right]$$

or $$\mathrm{L}\{x\} = \frac{1}{(s-1)^2 + 4} + \frac{\mathrm{A}-2}{5} \times \frac{1}{s} - \frac{\mathrm{A}-2}{5} \times \frac{(s-1)-1}{(s-1)^2 + 4}$$

Taking inverse Laplace transform of both sides, we have

$$x = \mathrm{L}^{-1}\left[\frac{1}{(s-1)^2 + 4}\right] + \frac{A-2}{5}\mathrm{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{A-2}{5}\mathrm{L}^{-1}\left\{\frac{(s-1)-1}{(s-1)^2 + 4}\right\}$$

or $$x = e^t\,\mathrm{L}^{-1}\left\{\frac{1}{s^2 + 2^2}\right\} + \frac{A-2}{5} - \frac{A-2}{5}e^t\,\mathrm{L}^{-1}\left\{\frac{s-1}{s^2 + 4}\right\}, \text{ using first shifting theorem}$$

or $$x = e^t \times \frac{1}{2}\sin 2t + \frac{A-2}{5} - \frac{A-2}{5}e^t\left[\mathrm{L}^{-1}\left\{\frac{s}{s^2 + 2^2}\right\} - \mathrm{L}^{-1}\left\{\frac{1}{s^2 + 2^2}\right\}\right]$$

or $$x = \frac{e^t \sin 2t}{2} + \frac{\mathrm{A}-2}{5} - \frac{\mathrm{A}-2}{5}e^t\left[\cos 2t - \frac{1}{2}\sin 2t\right] = \frac{\mathrm{A}-2}{5} - e^t\left[\left(\frac{\mathrm{A}-2}{5}\right)\cos 2t - \left(\frac{1}{2} + \frac{\mathrm{A}-2}{10}\right)\sin 2t\right]$$

or $$x(t) = \frac{A-2}{5} - e^t\left[\left(\frac{A-2}{5}\right)\cos 2t - \left(\frac{A+3}{10}\right)\sin 2t\right]. \quad \ldots (6)$$

Keeping condition (3) in mind, we put $t = \pi/8$ in (6) and obtain

or $$1 = \frac{A-2}{5} - e^{\pi/8} \times \frac{1}{\sqrt{2}}\left[\frac{A-2}{5} - \frac{A+3}{10}\right] \quad \text{or} \quad \frac{7-A}{5} - \frac{1}{\sqrt{2}} e^{\pi/8}\,\frac{(7-A)}{10} = 0$$

or $$(7-A)\,\{(1/5) - (1/10\sqrt{2})\,e^{\pi/8})\} = 0 \quad \text{so that} \quad A = 7.$$

So by (6), $x = 1 - e^t(\cos 2t - \sin 2t)$, which is the required solution.

Ex. 30. *Solve* $(D^4 + 2D^2 + 1)\,y = 0$, *where* $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$ and $y'''(0) = -3$. **[Meerut 1999]**

Sol. Given $$y'''' + 2y'' + y = 0 \quad \ldots (1)$$

with initial conditions: $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$, $y'''(0) = -3$. ... (2)

Taking Laplace transform of both sides of (1), we have

$$s^4 \mathrm{L}\{y\} - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) + 2\,[s^2 \mathrm{L}\{y\} - sy(0) - y'(0)] + \mathrm{L}\{y\} = 0$$

or $$(s^4 + 2s^2 + 1)\,\mathrm{L}\{y\} - s^2 - 2s + 3 + (2) \times (-1) = 0, \text{ using (2)}$$

or $$\mathrm{L}\{y\} = \frac{(s^2+1) + 2s - 2}{(s^2+1)^2} = \frac{1}{s^2+1} + \frac{2s}{(s^2+1)^2} - \frac{2}{(s^2+1)^2}$$

Taking inverse Laplace transfom of both sides, we get

$$y = \mathrm{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + 2\,\mathrm{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} - 2\,\mathrm{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}. \quad \ldots (3)$$

Now, $$\mathrm{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = -\frac{1}{2}\mathrm{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} = -\frac{1}{2}(-1)^1 t\,\mathrm{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}, \text{ using Art. 2.11}$$

$$\therefore \quad L^{-1}\{s/(s^2+1)^2\} = (t/2) \times \sin t \quad \ldots (4)$$

Let $f(s) = 1/(s^2+1)$ and $g(s) = 1/(s^2+1)$ ... (5)

Then, $\mathrm{F}(t) = \mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\{1/(1+s^2)\} = \sin t$

and $\mathrm{G}(t) = \mathrm{L}^{-1}\{g(s) = \mathrm{L}^{-1}\{1/(1+s^2)\} = \sin t$ ... (6)

Now, by the convolution theorem, $$\mathrm{L}^{-1}\{f(s)\,g(s)\} = \int_0^t \mathrm{F}(u)\,\mathrm{G}(t-u)\,du$$

or $$\mathrm{L}^{-1}\{1/(s^2+1)^2\} = \int_0^t \sin u \,\sin(t-u)\,du, \text{ using (5) and (6)}$$

$$= \frac{1}{2}\int_0^t [\cos(2u-t) - \cos t]\,du = \frac{1}{2}\left[\frac{1}{2}\sin(2u-t) - u\cos t\right]_0^t = \frac{1}{2}(\sin t - t\cos t). \quad \ldots (7)$$

Using (4) and (7), (3) reduces to

$$y = \sin t + 2\,[(1/2) \times t \sin t] - 2\,[(1/2) \times (\sin t - t\cos t)]$$

or $y = t(\sin t + \cos t)$, which is the required solution.

Ex.31. *Solve* $(D^2 + n^2)\, y = a \sin (nt + \alpha)$, *if* $y = Dy = 0$ *when* $t = 0$. **[Kumaun 1994]**

Sol. Given that $y'' + n^2 y = a \sin (nt + \alpha)$

i.e., $y'' + n^2 y = a (\sin nt \cos \alpha + \cos nt \sin \alpha)$ ... (1)

with initial conditions: $y(0) = 0$ and $y'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), we have

$$L\{y''\} + n^2 L\{y\} = a \cos \alpha\, L\{\sin nt\} + a \sin \alpha\, L\{\cos nt\}$$

or $s^2 L\{y\} - s\,y(0) - y'(0) + n^2 L\{y\} = (an \cos \alpha)/(s^2 + n^2) + (as \sin \alpha)/(s^2 + n^2)$

or $(s^2 + n^2)\, L\{y\} = (an \cos \alpha + as \sin \alpha)/(s^2 + n^2)$, by (2)

or $L\{y\} = (an \cos \alpha)/(s^2 + n^2)^2 + (as \sin \alpha)/(s^2 + n^2)^2$ ... (3)

Taking inverse Laplace transform of both sides of (3), we get

$$y = an \cos \alpha\, L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a \sin \alpha\, L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\}. \quad \text{... (4)}$$

Now, $L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} = -\frac{1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+n^2}\right)\right\} = -\frac{1}{2}(-1)^1 t\, L^{-1}\left\{\frac{1}{s^2+n^2}\right\}$, by Art. 2.11

Thus, $L^{-1}\{s/(s^2 + n^2)^2\} = (t/2n) \times \sin nt$ ... (5)

Let $f(s) = 1/(s^2 + n^2)$ and $g(s) = 1/(s^2 + n^2)$. ... (6)

Then, $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s^2+n^2)\} = (1/n) \times \sin nt$

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2+n^2)\} = (1/n) \times \sin nt$ ... (7)

Now, by the convolution theorem, $L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$

or $L^{-1}\left\{1/(s^2+n^2)^2\right\} = \int_0^t \left(\frac{\sin nu}{n} \times \frac{\sin n(t-u)}{n}\right) du$, by (6) and (7)

$$= \frac{1}{2n^2}\int_0^t [\cos n(t-2u) - \cos nt]\, du = \frac{1}{2n^2}\left[\frac{\sin n(t-2u)}{-2n} - u \cos nt\right]_0^t$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{2n} - t \cos nt + \frac{\sin nt}{2n}\right] = \frac{1}{2n^2}\left[\frac{\sin nt}{n} - t \cos nt\right]. \quad \text{... (8)}$$

Using (5) and (8), (4) reduces to

$$y = an \cos \alpha \times \frac{1}{2n^2}\left(\frac{\sin nt}{n} - t \cos nt\right) + a \sin \alpha \times \frac{1}{2n} t \sin nt$$

or $y = (a/2n^2) \times \cos \alpha \sin nt - (at/2n) \times (\cos \alpha \cos nt - \sin \alpha \sin nt)$

or $y = (a/2n^2) \times [\cos \alpha \sin nt - nt \cos (\alpha + nt)]$.

Ex.32. *Solve* $(d^2x/dt^2) + x = F(t)$, *if* $x = x' = 0$ *for* $t = 0$.

Sol. Given $x'' + x = F(t)$ ... (1)

with initial conditions: $x(0) = 0$ and $x'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), $L\{x''\} + L\{x\} = L\{F(t)\}$

or $s^2 L\{x\} - sx(0) - x'(0) + L\{x\} = f(s)$ ... (3)

where $L\{F(t)\} = f(s)$ so that $L^{-1}\{f(s)\} = F(t)$. ... (4)

Using (2), (3) $\Rightarrow$ $(s^2 + 1)\, L\{x\} = f(s)$ or $L\{x\} = f(s) \times [1/(s^2 + 1)]$

$\therefore \qquad x = \mathrm{L}^{-1}\{f(s) \times [1/(s^2+1)]\}.$ ... (5)

Let $g(s) = 1/(s^2+1)$ so that $\mathrm{G}(t) = \mathrm{L}^{-1}\{g(s)\} = \sin t$ ... (6)

Now, by the convolution theorem, $\mathrm{L}^{-1}\{f(s)\, g(s)\} = \int_0^t \mathrm{F}(u)\, \mathrm{G}(t-u)\, du$

or $\mathrm{L}^{-1}\{f(s) \times [1/(s^2+1)]\} = \int_0^t \mathrm{F}(u) \sin(t-u)\, du$, by (6)

$\therefore \qquad (5) \Rightarrow \qquad x = \int_0^t \mathrm{F}(u) \sin(t-u)\, du.$

Ex.33. Solve $y'' - 4y' + 3y = \mathrm{F}(t)$, if $y(0) = 1$, $y'(0) = 0$.

Sol. Given $y'' - 4y' + 3y = \mathrm{F}(t)$ ... (1)

with initial condition: $y(0) = 1$ and $y'(0) = 0.$ ... (2)

Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{y''\} - 4\,\mathrm{L}\{y'\} + 3\,\mathrm{L}\{y\} = \mathrm{L}\{\mathrm{F}(t)\}$$

or $s^2\,\mathrm{L}\{y\} - s\,y(0) - y'(0) - 4[s\,\mathrm{L}\{y\} - y(0)] + 3\,\mathrm{L}(y) = f(s),$ ... (3)

where $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$ so that $\mathrm{L}^{-1}\{f(s)\} = \mathrm{F}(t).$ ... (4)

Using (2), (3) $\Rightarrow$ $(s^2 - 4s + 3)\,\mathrm{L}\{y\} - s + 4 = f(s)$

or $$\mathrm{L}\{y\} = \frac{s-4+f(s)}{(s^2-4s+3)} = \frac{s-4}{(s-1)(s-3)} + \frac{f(s)}{(s-1)(s-3)}$$

or $$\mathrm{L}\{y\} = \frac{1}{2}\left[\frac{3}{s-1} - \frac{1}{s-3}\right] + \frac{1}{2}f(s)\left[\frac{1}{s-3} - \frac{1}{s-1}\right],$$ on resolving into partial fractions

$$\therefore \qquad y = \frac{3}{2}\mathrm{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathrm{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2}\mathrm{L}^{-1}\left\{f(s)\frac{1}{s-3}\right\} - \frac{1}{2}\mathrm{L}^{-1}\left\{f(s)\frac{1}{s-1}\right\}$$

or $y = (3/2)\times e^t - (1/2)\times e^{3t} + (1/2)\times \mathrm{L}^{-1}\{f(s)g(s)\} - (1/2)\times \mathrm{L}^{-1}\{f(s)h(s)\},$ ...(5)

where $g(s) = 1/(s-3)$ and $h(s) = 1/(s-1)$ ... (6)

so that $\mathrm{G}(t) = \mathrm{L}^{-1}\{g(s)\} = e^{3t}$ and $\mathrm{H}(t) = \mathrm{L}^{-1}\{h(s)\} = e^t.$ ... (7)

Now, by using the convulation theorem and (7), we have

$$\mathrm{L}^{-1}\{f(s)\, g(s)\} = \int_0^t \mathrm{F}(u)\, \mathrm{G}(t-u)\, du = \int_0^t F(u)\, e^{3(t-u)} du = e^{3t}\int_0^t F(u) e^{-3u} du \ldots (8)$$

and $$\mathrm{L}^{-1}\{f(s)\, h(s)\} = \int_0^t \mathrm{F}(u)\, \mathrm{H}(t-u)\, du = \int_0^t F(u)\, e^{t-u} du = e^t \int_0^t F(u)\, e^{-u}\, du.$$ ... (9)

Using (8) and (9), (5) reduces to

$$y = (1/2)\times(3e^t - e^{3t}) + (1/2)\times e^{3t}\int_0^t \mathrm{F}(u)\, e^{-3u} du - (1/2)\times e^t \int_0^t \mathrm{F}(u)\, e^{-u} du.$$

Ex.34 (*a*). *Find the general solution of the differential equation* $x''(t) + k^2 x(t) = F(t)$, *where* $x(0) = A$ *and* $x'(0) = B$. **[Meerut 2011]**

(*b*) *Find the general solution of* $(d^2x/dt^2) + k^2 x = F(t)$ *in terms of the constant* k *and the function* $F(t)$. **[Lucknow 1995]**

Sol (*a*). Given $x'' + k^2 x = \mathrm{F}(t)$ ... (1)

with initial conditions: $x(0) = \mathrm{A}$ and $x'(0) = \mathrm{B}.$ ... (2)

Taking Laplace transform of both sides of (1), $\mathrm{L}\{x''\} + k^2\,\mathrm{L}\{x\} = \mathrm{L}\{\mathrm{F}(t)\}$

or $s^2\,\mathrm{L}\{x\} - s\,x(0) - x'(0) + k^2\,\mathrm{L}\{x\} = f(s)$, ... (3)

where $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$ so that $\mathrm{L}^{-1}\{f(s)\} = \mathrm{F}(t)$. ... (4)

Using (2), $(3) \Rightarrow$ $(s^2 + k^2)\,\mathrm{L}\{x\} - \mathrm{A}s - \mathrm{B} = f(s)$

or $$\mathrm{L}\{x\} = \frac{\mathrm{A}s}{s^2 + k^2} + \frac{\mathrm{B}}{s^2 + k^2} + f(s)\frac{1}{s^2 + k^2}$$

$$\therefore \quad x = \mathrm{A}\,\mathrm{L}^{-1}\left[\frac{s}{s^2 + k^2}\right] + \mathrm{B}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\} + \mathrm{L}^{-1}\{f(s)\,g(s)\}, \quad \text{... (5)}$$

where $g(s) = 1/(s^2 + k^2)$ so that $\mathrm{G}(t) = \mathrm{L}^{-1}\{g(s)\} = (\sin kt)/k$ ... (6)

Now, using the convolution theorem, we have

$$\mathrm{L}^{-1}\{f(s)\,g(s)\} = \int_0^t \mathrm{F}(u)\,\mathrm{G}(t-u)\,du = \int_0^t \mathrm{F}(u)\frac{\sin k(t-u)}{k}\,du \text{ , using (4) and (6)}$$

$$\therefore \quad (5) \Rightarrow \quad x = \mathrm{A}\cos kt + \frac{\mathrm{B}}{k}\sin kt + \frac{1}{k}\int_0^t F(u)\sin k(t-u)\,du.$$

(*b*) Proceed exactly as in part (*a*) by taking $x(0) = \mathrm{A}$ and $x'(0) = \mathrm{B}$.

EXERCISE 3 (A)

Using Laplace transform method, solve the following equations:

1. $(\mathrm{D} + 1)\,y = 0$, $t > 0$, given that $y = y_0$, when $t = 0$.

2. $(\mathrm{D} + 1)\,y = 1$, given that $y = 2$ when $t = 0$.

3. $(\mathrm{D}^2 + 1)\,y = 0$, if $y = 1$, $dy/dt = 0$, when $t = 0$. **[Osmania 2010, Ravishankar 2010]**

4. $(\mathrm{D}^2 + 9)\,y = 18\mathrm{t}$, if $y(0) = 0$, $y(\pi/2) = 0$. **[Osmania 2004]**

5. $(\mathrm{D}^2 + 3\,\mathrm{D} + 2)\,y = 0$, $y = y_0$ and $\mathrm{D}y = y_1$ at $t = 0$. **[Rohilkhand 1994]**

6. $(\mathrm{D}^2 + 6\mathrm{D} + 25)\,y = 208\,e^{3t}$, $t > 0$, if $y = 1$, $\mathrm{D}y = 0$ when $t = 0$.

7. $(\mathrm{D}^3 + 1)\,y = 1$, $t > 0$ if $y = \mathrm{D}y = \mathrm{D}^2 y = 0$ when $t = 0$. **[Ravishankar 2004]**

8. $(\mathrm{D}^3 + \mathrm{D})\,y = e^{2t}$, $y(0) = y'(0) = y''(0) = 0$.

9. $(\mathrm{D}^3 - \mathrm{D})\,y = 2\cos t$, $y = 3$, $\mathrm{D}y = 2$, $\mathrm{D}^2 y = 1$ when $t = 0$.

10. $(\mathrm{D}^2 + 4\mathrm{D} + 4)\,x = \sin wt$, $t > 0$ with x_0 and x_1 for values of x and $\mathrm{D}x$ when $t = 0$.

11. $(\mathrm{D}^3 - \mathrm{D}^2 + 4\mathrm{D} - 4)\,y = 68\,e^t \sin 2t$, $y = 1$, $\mathrm{D}y = -19$, $\mathrm{D}^2 y = -37$, *at* $t = 0$.

12. $y''''(t) + y'''(t) = \cos t$, $y(0) = y'(0) = y'''(0) = 0$, $y''(0)$ arbitrary.

13. $y_2 + 4y_1 + 5y = (\cos t - \sin t)\,e^{-2t}$, $y(0) = 1$, $y'(0) = -3$. **[Agra 2005]**

14. $(\mathrm{D}^2 - 3\mathrm{D} + 2)y = 1 - e^{2t}$, $y(0) = 1$, $y'(0) = 1$. **[Osmania 2010; Bhopal 2004]**

15. $d^2x/dt^2 + x = t$, $x(0) = 1$, $x'(0) = -2$ **[Guwahati 2007]**

16. $y'' + 25y = 10\cos 5t$, $y(0) = 2$, $y'(0) = 0$ **[Kanpur 2006]**

17. $y'' + 5y' + 6y = 5e^{2t}$, $y(0) = 2$, $y'(0) = 1$ **[Purvanchal 2004]**

18. $y'' + 2y' - 3y = \sin t$, $y(0) = 0$, $y'(0) = 0$ **[MDU Rohtak 2005]**

19. $2y'' + 5y' + 2y = e^{-2t}$, $y(0) = y'(0) = 1$ **[Kanpur 2005]**

20. $x''(t) + 4x'(t) + 4x(t) = 4e^{-2t}$, $x(0) = -1$, $x'(0) = 4$ **[Kuvempa 2005]**

21. $y''(t) - 2y'(t) + y(t) = e^t$, $y(0) = 2$, $y'(0) = -1$ **[Bangalore 2005]**

22. $y''(t) + y(t) = 0,\ y(0) = 1,\ y'(0) = 1$ **[Mysore 2004]**

23. $y'''(t) + y''(t) - 4y'(t) - 4y(t) = \mathrm{F}(t)$, if $y(0) = y''(0) = 0\ y'(0) = 2$.

24. $y''(t) - k^2 y(t) = \mathrm{F}(t)$, if $y(0) = 0 = y'(0)$, $k \neq 0$.

25. Find the general solution of $d^2y/dt^2 - 2k\,(dy/dt) + k^2 y = \mathrm{F}(t)$.

26. $d^2y/dt^2 + a^2y = \mathrm{F}(t), y(0) = 1,\ y'(0) = -2$ **[Delhi Physics (H) 2005]**

ANSWERS

1. $y = y_0\, e^{-t}$. **2.** $y = 1 + e^{-t}$. **3.** $y = \cos t$. **4.** $y = 2t + \pi \sin 3t$.

5. $y = (2y_0 + y_1)\, e^{-t} - (y_0 + y_1)\, e^{-2t}$. **6.** $y = 4\, e^{3t} - (3/4) \times e^{-3t}\,(4 \cos 4t + 7 \sin 4t)$

7. $y = 1 - (1/3) \times e^{-t} - (2/3) \times e^{t/2} \cos(t\sqrt{3}/2)$. **8.** $y = -(1/2) + (1/10) \times e^{2t} + (1/5) \times (2\cos t - \sin t)$.

9. $y = \cosh t + 3 \sinh t - \sin t + 2$.

10. $y = e^{-2t}\left\{x_0(1-2t) + (x_1 + 4x_0) + \dfrac{\omega t}{4+\omega^2} + \dfrac{4\omega}{(4+\omega^2)^2}\right\} - \dfrac{4\omega}{(4+\omega^2)^2}\cos \omega t + \dfrac{4-\omega^2}{(4+\omega^2)^2}\sin \omega t.$

11. $y = (1/5) \times (e^t + 14 \cos 2t - 3 \sin 2t) - 2\, e^t\,(\cos 2t + 4 \sin 2t)$.

12. $y = t - 1 + (1/2) \times \mathrm{A}t^2 + (1/2) \times (e^{-t} + \cos t - \sin t)$, if $y''(0) = \mathrm{A}$.

13. $y = e^{-2t}\,[\cos t - (3/2) \times \sin t + (1/2) \times t \sin t + (1/2) \times t \cos t]$.

14. $y(t) = (1/2) \times (1 + e^{2t}) - te^{2t}$ **15.** $x(t) = \cos t - 3 \sin t + t$

16. $y(t) = (2 + t) \cos 5t - (1/5) \times \sin 5t$ **17.** $y(t) = (1/4) \times (23e^{-2t} + e^{2t}) - 4e^{-3t}$

18. $y(t) = (1/40) \times (e^{3t} - 5e^{-t}) + (\cos t - 2 \sin t)/2$

19. $y(t) = (20/9) \times e^{-t/2} + (1/9) \times (3t - 11)\, e^{-2t}$

20. $y(t) = (2t - 1 + 2t^2)\, e^{-2t}$ **21.** $y(t) = (2 - 3t + t^2/2)\, e^t$

22. $y(t) = e^t \cos t$ **23.** $\sin t + \cos t$

24. $y = \sinh 2t + \dfrac{1}{12}\displaystyle\int_0^t (e^{2u} + 3e^{-2u} - 4e^{-u}) F(t-u)\, du$

25. $y = \dfrac{1}{2k}\displaystyle\int_0^t (e^{ku} - e^{-ku}) F(t-u)\, du$ **26.** $y = \{(\mathrm{B} - \mathrm{A}k)t + \mathrm{A}\}e^{kt} + e^{kt}\displaystyle\int_0^t e^{-ku}(t-u)F(u)\, du$

3.2. Solution of ordinary differential equations with variable coefficients

Suppose the given differential equation contain a term of the form

$t^m\, y^{(n)}(t)$, *i.e.*, $t^m \dfrac{d^n y(t)}{dt^n}$ the Laplace transform of which is $(-1)^m \dfrac{d^m}{ds^m} \mathrm{L}\left\{y^{(n)}(t)\right\}$.

The method is illustrated in the following examples

3.2A. Solved examples based on Art 3.2.

Ex.1. *Solve* $[t D^2 + (1 - 2t) D - 2]\, y = 0,\ y(0) = 1,\ y'(0) = 2.$ **[Meerut 2000]**

Sol. Given $t y'' + y' - 2t y' - 2y = 0,$ ... (1)

with initial conditions: $y(0) = 1$ and $y'(0) = 2.$ ... (2)

Taking Laplace transform of both sides of (1), $\mathrm{L}\{t y''\} + \mathrm{L}\{y'\} - 2\mathrm{L}\{ty'\} - 2\mathrm{L}\{y\} = 0$

or $(-1)^1 \dfrac{d}{ds}\mathrm{L}\{y''\} + \mathrm{L}\{y'\} - 2(-1)^1 \dfrac{d}{ds}\mathrm{L}\{y'\} - 2\,\mathrm{L}\{y\} = 0$

or $-\dfrac{d}{ds}[s^2\,\mathrm{L}\{y\} - sy(0) - y'(0)] + s\,\mathrm{L}\{y\} - y(0) + 2\dfrac{d}{ds}[s\,\mathrm{L}\{y\} - y(0)] - 2\,\mathrm{L}\{y\} = 0$

or $$-\frac{d}{ds}[s^2\bar{y}-s-2]+s\bar{y}-1+2\frac{d}{ds}[s\bar{y}-1]-2\bar{y}=0, \text{ by (2)} \quad ...(3)$$

where we have written $$L\{y\} = \bar{y}(s). \quad ... (4)$$

From (3), we get $$-\left[s^2\frac{d\bar{y}}{ds}+2s\bar{y}-1\right]+s\bar{y}-1+2\left[s\frac{d\bar{y}}{ds}+\bar{y}\right]-2\bar{y}=0$$

or $$(s^2-2s)\frac{d\bar{y}}{ds}=s\bar{y} \quad \text{or} \quad \frac{d\bar{y}}{\bar{y}}+\frac{ds}{s-2}=0.$$

Integrating, $\log \bar{y} + \log (s-2) = \log C$ or $\bar{y} = C/(s-2)$.

$\therefore$ $L\{y\} = C/(s-2)$, using (4). ... (5)

Taking inverse transform of both sides of (5), we have

$$y(t) = C\,L^{-1}\{1/(s-2)\} = Ce^{2t}. \quad ... (6)$$

Putting $t = 0$ in (6), we get $y(0) = C$ so that $C = 1$, using (2).

Hence, from (6), $y = e^{2t}$, which is the required solution

Ex.2. *Solve $t\,y'' + y' + 4\,ty = 0$, if $y(0) = 3$, $y'(0) = 0$.*

Sol. Given $$t\,y'' + y' + 4\,t\,y = 0 \quad ... (1)$$

with initial conditions: $y(0) = 3$ and $y'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), $L\{ty''\} + L\{y'\} + 4\,L\{ty\} = 0$

or $$(-1)^1\frac{d}{ds}L\{y''\}+L\{y'\}+4(-1)^1\frac{d}{ds}L\{y\}=0$$

or $$-\frac{d}{ds}\left[s^2L\{y\}-sy(0)-y'(0)\right]+sL\{y\}-y(0)-4\frac{d}{ds}L\{y\}=0$$

or $$-\frac{d}{ds}[s^2L\{y\}-3s]+sL\{y\}-3-4\frac{d}{ds}L\{y\}=0, \text{ using (2)} \quad ... (3)$$

Let $$L\{y\} = \bar{y}(s). \quad ... (4)$$

Then, (3) $\Rightarrow$ $$-\frac{d}{ds}(s^2\bar{y}-3s)+s\bar{y}-3-4\frac{d\bar{y}}{ds}=0$$

or $$-\left[s^2\frac{d\bar{y}}{ds}+2s\bar{y}-3\right]+s\bar{y}-3-4\frac{d\bar{y}}{ds}=0$$

or $$(s^2+4)\frac{d\bar{y}}{ds}+s\bar{y}=0 \quad \text{or} \quad \frac{d\bar{y}}{\bar{y}}+\frac{1}{2}\times\frac{2s}{s^2+4}ds=0.$$

Integrating, $\log \bar{y} + (1/2)\log(s^2+4) = \log c$ or $\bar{y} = c/(s^2+4)^{1/2}$.

$\therefore$ $L\{y\} = c/(s^2+4)^{1/2}$, using (4). ... (5)

$$(5) \Rightarrow \quad y(t) = L^{-1}\left\{\frac{c}{(s^2+4)^{1/2}}\right\} = c\,L^{-1}\left\{\frac{1}{(s^2+2^2)^{1/2}}\right\} = c\,J_0(2t) \quad ... (6)$$

[$\because$ from Ex 1 (*i*), of Art. 1.22B, Chapter 1, $L\{J_0(at)\} = 1/(s^2+a^2)^{1/2}$]

Putting $t = 0$ in (6), we get $y(0) = C\,J_0(0)$ or $3 = C$, using (2) and noting that $J_0(0) = 1$.

Hence from (6), $y = 3\,J_0(2t)$, which is the required solution

Ex.3. *Solve* $(D^2 + tD - 1)\, y = 0$, *if* $y(0) = 0$, $y'(0) = 1$. **[Osmania 2010; Meerut 2004]**

Sol. Given $y'' + t\,y' - y = 0,$ ... (1)

with initial conditions $y(0) = 0$ and $y'(0) = 1.$ ... (2)

Let $\mathrm{L}\{y(t)\} = \bar{y}(s)$. Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{y''(t)\} + (-1)^1 \frac{d}{ds}[\mathrm{L}\{y'(t)\}] - \mathrm{L}\{y(t)\} = 0$$

or $$s^2\,\bar{y} - s\,y(0) - y'(0) - \frac{d}{ds}[s\bar{y} - y(0)] - \bar{y} = 0$$

or $s^2\bar{y} - 1 - \left[\bar{y} + s\frac{d\bar{y}}{ds}\right] - \bar{y} = 0$ or $$\frac{d\bar{y}}{ds} + \left(\frac{2}{s} - s\right) y = -\frac{1}{s},$$ by (2) ... (3)

which is linear differential equation whose

$$\text{integrating factor (I.F.)} = e^{\int \{(2/s) - s\}\, ds} = e^{2\log s - s^2/2} = e^{\log s^2} e^{-s^2/2} = s^2 e^{-s^2/2}$$

$\therefore$ Solution of (3) is $$\bar{y}\, s^2 e^{-s^2/2} = C + \int s^2 e^{-s^2/2}\left(-\frac{1}{s}\right) ds = C - \int s\, e^{-s^2/2} ds.$$ ... (4)

Putting $s^2/2 = z$ so that $sds = dz$, (4) reduces to

$$\bar{y} s^2 e^{-s^2/2} = C - \int e^{-z} dz = C + e^{-z} = C + e^{-s^2/2}$$

or $$\bar{y} = \frac{1}{s^2} + \frac{c}{s^2}\, e^{s^2/2} = \frac{1}{s^2} + \frac{\mathrm{C}}{s^2}\left[1 + \frac{s^2/2}{1!} + \frac{(s^2/2)^2}{2!} + \ldots.\right]$$

or $$\mathrm{L}\{y(t)\} = (\mathrm{C} + 1) \times \frac{1}{s^2} + \frac{1}{2}\mathrm{C} + \frac{1}{8}\mathrm{C}\, s^4 + \ldots$$ ... (5)

Taking inverse Laplace transform of both sides of (5), we get

$$y(t) = (\mathrm{C} + 1)\, \mathrm{L}^{-1}\{1/s^2\} + (1/2) \times \mathrm{C}\, \mathrm{L}^{-1}\{1\} + (1/8) \times \mathrm{C}\, \mathrm{L}^{-1}\{s^4\} + \ldots..$$

or $$y(t) = (\mathrm{C} + 1) \times (t/1!) + 0 + 0 + \ldots = (\mathrm{C} + 1)\, t.$$ ... (6)

[since $\mathrm{L}^{-1}\{s^n\} = 0$, if $n = 0, 1, 2, \ldots.$]

From (6), we have $y'(t) = \mathrm{C} + 1.$ ... (7)

Putting $t = 0$ in (7) and using $y'(0) = 1$, we get $1 = \mathrm{C} + 1$ so that $\mathrm{C} = 0$.

Then, from (6), the required solution is $y = t$.

Ex.4. *Solve* $y'' - ty' + y = 1$, *if* $y(0) = 1$, $y'(0) = 2$.

Sol. Given $y''(t) - ty'(t) + y(t) = 1$ ... (1)

with initial conditions: $y(0) = 1$ and $y'(0) = 2.$ ... (2)

Let $\mathrm{L}\{y(t)\} = \bar{y}(s)$ Taking Laplace transforms of both sides of (1), we get

$$\mathrm{L}\{y''(t)\} - \mathrm{L}\{t\,y'(t)\} + \mathrm{L}\{y(t)\} = \mathrm{L}\{1\}$$

or $$\mathrm{L}\{y''(t)\} - (-1)^1 \frac{d}{ds}[\mathrm{L}\{y'(t)\}] + \mathrm{L}\{y(t)\} = \frac{1}{s}$$

or $$s^2\,\bar{y} - sy(0) - y'(0) + \frac{d}{ds}[s\bar{y} - y(0)] + \bar{y} = \frac{1}{s}$$

or $$s^2\bar{y} - s - 2 + \frac{d}{ds}(s\bar{y} - 1) + \bar{y} = \frac{1}{s}$$ using (2)

or $$s^2\ \bar{y}\ - s - 2 + s\ \frac{d\,\bar{y}}{ds} + \bar{y} + \bar{y} = \frac{1}{s} \qquad \text{or} \qquad s\frac{d\,\bar{y}}{ds} + (s^2+2)\bar{y} = s + 2 + \frac{1}{s}$$

or $$\frac{d\,\bar{y}}{ds} + \left(s + \frac{2}{s}\right)\bar{y} = 1 + \frac{1}{s} + \frac{1}{s^2},\ \text{which is linear differential equation.} \qquad \ldots (3)$$

Its I.F. $= e^{\int (s + 2/s)\,ds} = e^{2\log s + s^2/2} = s^2\ e^{s^2/2}$ and solution of (3) is given by

$$\bar{y} s^2 e^{s^2/2} = c + \int \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{s^2/2} ds = c + \int (s^2 + 2s + 1) e^{s^2/2} ds$$

$$= c + \int (s^2 + 1) e^{s^2/2} ds + 2\int s e^{s^2/2} ds = c + \int (2q + 1) e^q \frac{dq}{\sqrt{2q}} + 2\int \sqrt{2q}\ e^q \frac{dq}{\sqrt{2q}}$$

[Putting $s^2/2 = q$ so that $ds = dq / \sqrt{2q}$]

$$= c + \int \sqrt{2q}\ e^q dq + \int \frac{e^q}{\sqrt{2q}} dq + 2\int e^q dq$$

$$= c + \sqrt{2q}\ e^q - \int \left\{\frac{1}{2}(2q)^{-1/2} \times 2 e^q\right\} dq + \int \frac{e^q}{\sqrt{2q}} dq + 2\ e^q$$

$\therefore$ $$s^2\ e^{s^2/2}\ \bar{y} = c + \sqrt{2q}\ e^q + 2 e^q = c + s\ e^{s^2/2} + 2\ e^{s^2/2},\ \text{as } q = s^2/2$$

or $$\bar{y} = \frac{c}{s^2} e^{-s^2/2} + \frac{1}{s} + \frac{2}{s^2} = \frac{c}{s^2}\left(1 - \frac{s^2/2}{1!} + \frac{s^4/4}{2!} - \ldots.\right) + \frac{1}{s} + \frac{2}{s^2}$$

$\therefore$ $$\mathrm{L}\ \{y\ (t)\} = \frac{1}{s} + \frac{c+2}{s^2} - \frac{c}{2} + \frac{c}{8} s^2 + \ldots.$$

Taking inverse Laplace transforms of both sides, we get

$y\ (t) = \mathrm{L}^{-1}\{1/s\} + (c + 2) \times \mathrm{L}^{-1}\ \{1/s^2\} + (c/8) \times \mathrm{L}^{-1}\ \{s^2\} + \ldots.$

or $y\ (t) = 1 + (c + 2)\ t,$ as $\mathrm{L}^{-1}\ \{s^n\} = 0,$ when $n = 0, 1, 2, 3, \ldots$ .... (3)

From (3), we have $y'\ (t) = c + 2$ ... (4)

Putting $t = 0$ in (4) and using $y'\ (0) = 2$, we get $2 = c + 2$ so that $c = 0.$

$\therefore$ From (3), $y\ (t) = 1 + 2t$, which is the required solution

Ex.5. *Solve [t D² + (t – 1) D – 1] y =0, if y (0) = 5, y(∞) = 0.*

Sol. Given $t\ y'' + t\ y' - y' - y = 0$ ... (1)

with initial conditions: $y\ (0) = 5$ and $y\ (\infty) = 0.$ ... (2)

Taking Laplace transform of both sides of (1), we have

$$\mathrm{L}\ \{t\ y''\} + \mathrm{L}\ \{t\ y'\} - \mathrm{L}\ \{y'\} - \mathrm{L}\ \{y\} = 0$$

or $$(-1)^1\ \frac{d}{ds}\left[\mathrm{L}\ \{y''\}\right] + (-1)^1 \frac{d}{ds}[\mathrm{L}\ \{y'\}] - \mathrm{L}\ \{y'\} - \mathrm{L}\ \{y\} = 0$$

$$-\frac{d}{ds}[s^2\ \mathrm{L}\{y\} - sy(0) - y'(0)] - \frac{d}{ds}[s\,\mathrm{L}\{y\} - y(0)] - [s\,\mathrm{L}\{y\} - y(0)] - \mathrm{L}\{y\} = 0$$

or $$-\frac{d}{ds}(s^2\bar{y} - 5s - \mathrm{A}) - \frac{d}{ds}(s\bar{y} - 5) - (s\bar{y} - 5) - \bar{y} = 0, \qquad \ldots (3)$$

where we have used (2) and taken $\mathrm{L}\ \{y\} = \bar{y}$ and $y'\ (0) = \mathrm{A}.$ ... (4)

Then, $(3) \Rightarrow \quad -\left[s^2 \frac{d\bar{y}}{ds} + 2s\bar{y} - 5\right] - \left(s\frac{d\bar{y}}{ds} + \bar{y}\right) - (s\bar{y} - 5) - \bar{y} = 0$

or $(s^2 + s)\frac{d\bar{y}}{ds} + (3s+2)\bar{y} = 10$ or $\frac{d\bar{y}}{ds} + \frac{3s+2}{s^2+s}\bar{y} = \frac{10}{s^2+s}$, ... (5)

which is a linear differential equation whose integrating factor (I.F.) is given by

$$\text{I.F.} = e^{\int \{(3s+2)/(s^2+s)\}ds} = \int e^{\{2/s + 1/(s+1)\}ds} = e^{2\log s + \log(s+1)}$$

$$= e^{\log s^2 + \log(s+1)} = e^{\log[s^2(s+1)]} = s^2(s+1)$$

Hence solution of (5) is given by

$$\bar{y}\, s^2(s+1) = c + \int \frac{10}{s(s^2+1)} \times s^2(s+1)\, ds = c + 10 \int s\, ds = c + 5s^2$$

$$\therefore \quad \bar{y} = \frac{c}{s^2(s+1)} + \frac{5}{s+1} = C\left(\frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}\right) + \frac{5}{s+1} \quad \text{... (6)}$$

[On resolving into partial fractions]

Taking inverse Laplace transform of both sides of (6) we get

$$y(t) = c\,(e^{-t} - 1 + t) + 5\, e^{-t}. \quad \text{... (7)}$$

Since given that $y(\infty) = 0$, (7) holds only if C = 0.

Then from (7), $y = 5\, e^{-t}$, which is the required solution.

EXERCISE 3 (*B*)

Solve the following differential equations:

1. $t\,y''(t) + y'(t) + t\,y(t) = 0, y(0) = 1, y'(0) = 0$. **(Agra 2005)**

2. $t\,y''(t) + (2t+3)\,y'(t) + (t+3)\,y(t) = a\,e^{-t}$ **[Lucknow 1994]**

3. $y''(t) + at\,y'(t) - 2ay(t) = 1, y(0) = y'(0) = 0, a > 0$

ANSWERS

1. $y = J_0(t)$ **2.** $y = [A + (1/3) \times at]\, e^{-t}$ **3.** $y = t^2/2$.

3.3. Solution of simultaneous ordinary differential equations

Simultaneous ordinary differential equations involve more than one dependent variable and Laplace transform is needed for each variable. The procedure is to solve the simultaneous algebraic equations which appear on taking Laplace transformation and finally invert to recover each dependent variable. We now explain this method in what follows.

3.3 A. Solved examples based on Art 3.3

Ex.1. *Solve dx/dt = 2x – 3y, dy/dt = y – 2x, if x (0) = 8 and y (0) = 3.*

Or Solve $(D - 2)x + 3y = 0$, $2x + (D - 1)y = 0$ if $x(0) = 8$ and $y(0) = 3$.

[Delhi Maths (H) 1995, Meerut 2001]

Sol. Given $x'(t) - 2x + 3y = 0$ ... (1A)

and $2x + y'(t) - y = 0$ ... (2A)

with initial conditions: $x(0) = 8$ and $y(0) = 3$. ... (3)

Let $L\{x(t)\} = \bar{x}(s)$ and $L\{y(t)\} = \bar{y}(s)$. Taking the Laplace transform of both sides of (1A) and (2A), we get

$$L\{x'(t)\} - 2L\{x(t)\} + 3L\{y(t)\} = 0 \quad \text{... (1B)}$$

and $$2L\{x(t)\} + L\{y'(t)\} - L\{y(t)\} = 0 \quad \text{... (2B)}$$

or $$s\,\bar{x} - x(0) - 2\,\bar{x} + 3\,\bar{y} = 0 \qquad \text{... (1C)}$$

and $$2\,\bar{x} + s\,\bar{y} - y(0) - \bar{y} = 0. \qquad \text{... (2C)}$$

Using (3), (1C) and (2C) reduce to

$$(s-2)\,\bar{x} + 3\,\bar{y} - 8 = 0 \qquad \text{... (1D)}$$

and $$2\,\bar{x} + (s-1)\,\bar{y} - 3 = 0. \qquad \text{... (2D)}$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$ by cross-multiplication, we get

$$\frac{\bar{x}}{-9+8(s-1)} = \frac{\bar{y}}{-16+3(s-2)} = \frac{1}{(s-1)(s-2)-6} \quad \text{or} \quad \frac{\bar{x}}{8s-17} = \frac{\bar{y}}{3s-22} = \frac{1}{s^2-3s-4}.$$

Hence $$\bar{x} = \frac{8s-1}{(s-4)(s+1)} = \frac{3}{s-4} + \frac{5}{s+1}. \qquad \text{... (1E)}$$

and $$\bar{y} = \frac{3s-22}{(s-4)(s+1)} = \frac{5}{s+1} - \frac{2}{s-4}. \qquad \text{... (2E)}$$

[on resolving into partial fractions]

Taking the inverse Laplace transform of (1E) and (2E), we get

$$x = 3\,\mathrm{L}^{-1}\left\{\frac{1}{s-4}\right\} + 5\,\mathrm{L}^{-1}\left\{\frac{5}{s+1}\right\} = 3e^{4t} + 5e^{-t}$$

and $$y = 5\,\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\,\mathrm{L}^{-1}\left\{\frac{1}{s-4}\right\} = 5e^{-t} - 2e^{4t}.$$

Thus, the required solutions are $x = 3e^{4t} + 5e^{-t}$ and $y = 5e^{-t} - 2e^{4t}$.

Ex.2. *Solve (D + 1) x + (D + 1) y = 1, – 2x + (D – 1) y = 0, if x (0) = 0, y (0) = 1.*

Sol. Given $$x'(t) + x + y'(t) + y = 1 \qquad \text{... (1A)}$$

and $$-2x + y'(t) - y = 0 \qquad \text{... (2A)}$$

with initial conditions: $x(0) = 0$ and $y(0) = 1$. ... (3)

Let $\mathrm{L}\{x(t)\} = \bar{x}(s)$ and $\mathrm{L}\{y(t)\} = \bar{y}(s)$. Taking the Laplace transform of both sides of (1A) and (2A), we get

$$\mathrm{L}\{x'(t)\} + \mathrm{L}\{x\} + \mathrm{L}\{y'(t)\} + \mathrm{L}\{y\} = \mathrm{L}\{1\} \qquad \text{... (1B)}$$

and $$-2\,\mathrm{L}\{x\} + \mathrm{L}\{y'(t)\} - \mathrm{L}\{y\} = 0 \qquad \text{... (2B)}$$

or $$s\,\bar{x} - x(0) + \bar{x} + s\,\bar{y} - y(0) + \bar{y} = 1/s \qquad \text{... (1C)}$$

and $$-2x + s\,\bar{y} - y(0) - \bar{y} = 0. \qquad \text{... (2C)}$$

Using (3), (1C) and (2C) reduce to

$$s(s+1)\,\bar{x} + s(s+1)\,\bar{y} - (s+1) = 0 \quad \text{or} \quad s\bar{x} + s\bar{y} - 1 = 0 \qquad \text{... (1D)}$$

and $$-2\,\bar{x} + (s-1)\,\bar{y} - 1 = 0 \qquad \text{... (2D)}$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$ by cross-multiplication, we get

$$\frac{\bar{x}}{-s+(s-1)} = \frac{\bar{y}}{2+s} = \frac{1}{s(s-1)+2s}.$$

Hence $$\bar{x} = -\frac{1}{s(s+1)} = -\left[\frac{1}{s} - \frac{1}{s+1}\right] = \frac{1}{s+1} - \frac{1}{s}. \quad \text{... (1E)}$$

and $$\bar{y} = \frac{2+s}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1}. \quad \text{... (2E)}$$

[On resolving into partial fractions]

Taking the inverse Laplace transform of (1E) and (2E), we get

$$x = \mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathrm{L}^{-1}\left\{\frac{1}{s}\right\} = e^{-t} - 1 \qquad \text{and} \qquad y = 2\,\mathrm{L}^{-1}\left\{\frac{1}{s}\right\} - \mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} = 2 - e^{-t}$$

Thus, the required solutions are $x = e^{-t} - 1$ and $y = 2 - e^{-t}$.

Ex.3. *Solve* $(D^2 + 2)\,x - Dy = 1$, $Dx + (D^2 + 2)\,y = 0$, *if* $x = Dx = y = Dy = 0$, *when* $t = 0$.

[Kumaun 1997, Meerut 1997, KU Kurukshetra 2004]

Sol. Given $$x''(t) + 2x(t) - y'(t) = 1 \quad \text{... (1A)}$$

and $$x(t) + y''(t) + 2y(t) = 0 \quad \text{... (2A)}$$

with initial conditions: $$x(0) = x'(0) = y(0) = y'(0) = 0. \quad \text{... (3)}$$

Let $\mathrm{L}\{x(t)\} = \bar{x}(s)$ and $\mathrm{L}\{y(t)\} = \bar{y}(s)$. Taking the Laplace transform of both sides of (1A) and (2A), we have

$$\mathrm{L}\{x''(t)\} + 2\,\mathrm{L}\{x(t)\} - \mathrm{L}\{y'(t)\} = \mathrm{L}\{1\} \quad \text{... (1B)}$$

and $$\mathrm{L}\{x(t)\} + \mathrm{L}\{y''(t)\} + 2\,\mathrm{L}\{y(t)\} = 0 \quad \text{... (2B)}$$

or $$s^2\bar{x} - sx(0) - x'(0) + 2\bar{x} - [s\bar{y} - y(0)] = 1/s \quad \text{... (1C)}$$

and $$\bar{x} + s^2\bar{y} - s\,y(0) - y'(0) + 2\bar{y} = 0. \quad \text{... (2C)}$$

Using (3), (1C) and (2C) reduce to

$$s(s^2 + 2)\,\bar{x} - s^2\bar{y} - 1 = 0 \quad \text{... (1D)}$$

and $$s\bar{x} + (s^2 + 2)\,\bar{y} - 0 = 0. \quad \text{... (2D)}$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$ by cross-multiplication, we get

$$\frac{\bar{x}}{s^2+2} = \frac{\bar{y}}{-s} = \frac{1}{s(s^2+2)^2 + s^3}$$

$$\Rightarrow \qquad \bar{x} = \frac{s^2+2}{s(s^2+1)\,(s^2+4)} = \frac{1}{2s} - \frac{s}{3(s^2+1)} - \frac{s}{6(s^2+4)}$$

and $$\bar{y} = -\frac{1}{s^4+5s^2+4} = -\frac{1}{(s^2+1)\,(s^2+4)} = \frac{1}{3(s^2+4)} - \frac{1}{3(s^2+1)}.$$

(on resolving into partial fractions)

Taking the inverse Laplace transform, we get

$x = (1/2) - (1/3) \times \cos t - (1/6) \times \cos 2t$ and $y = (1/6) \times \sin 2t - (1/3) \times \sin t$

Ex.4. *Solve* $(D^2 - 3)\,x - 4y = 0$, $x + (D^2 + 1)\,y = 0$, $t > 0$, *if* $x = y = Dy = 0$, $Dx = 2$ *when* $t = 0$.

Sol. Given $$x''(t) - 3x(t) - 4y(t) = 0 \quad \text{...(1A)}$$

and $$x(t) + y''(t) + y(t) = 0 \quad \text{... (2A)}$$

with initial conditions $x(0) = 0$, $y(0) = 0$, $y'(0) = 0$, $x'(0) = 2$. ... (3)

Let L $\{x(t)\} = \bar{x}(s)$ and L $\{y(t)\} = \bar{y}(s)$. Taking Laplace transforms of both sides of (1A) and (2A), we get.

$$\mathrm{L}\{x''(t)\} - 3\,\mathrm{L}\{x(t)\} - 4\mathrm{L}\{y(t)\} = 0 \quad \ldots (1B)$$

and $$\mathrm{L}\{x(t)\} + \mathrm{L}\{y''(t)\} + \mathrm{L}\{y(t)\} = 0 \quad \ldots (2B)$$

or $$s^2\,\bar{x} - s\,x(0) - x'(0) - 3\,\bar{x} - 4\,\bar{y} = 0. \quad \ldots (1C)$$

and $$\bar{x} + s^2\,\bar{y} - sy(0) - y'(0) + \bar{y} = 0 \quad \ldots (2C)$$

Using (3), (1C) and (2C) reduce to

$$(s^2 - 3)\,\bar{x} - 4\,\bar{y} = 2 \quad \ldots (1D)$$

and $$\bar{x} + (s^2 + 1) = 0. \quad \ldots (2D)$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{2(s^2+1)}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2} \quad \ldots (1E)$$

$$\bar{y} = -\frac{2}{(s+1)^2(s-1)^2} = \frac{1}{2}\left[-\frac{1}{s+1} + \frac{1}{s-1} - \frac{1}{(s+1)^2} - \frac{1}{(s-1)^2}\right]. \quad \ldots (2E)$$

[on resolving into partial fractions]

Taking inverse Laplace transform of (1E) and (2E), we get

$x(t) = \mathrm{L}^{-1}\{1/(s-1)^2\} + \mathrm{L}^{-1}\{1/(s+1)^2\}$ or $x = e^t\,\mathrm{L}^{-1}\{1/s^2\} + e^{-t}\,\mathrm{L}^{-1}\{1/s^2\} = e^t t + e^{-t} t$

Similarly, $y = (1/2) \times [-e^{-t} + e^t - e^{-t}\mathrm{L}^{-1}\{1/s^2\} - e^t \mathrm{L}^{-1}\{1/s^2\}] = (1/2) \times (-e^{-t} + e^t - te^{-t} - te^t)$

$\therefore$ $x = t(e^t + e^{-t})$ and $y = (1/2) \times [(1-t)e^t - (1+t)e^{-t}]$.

Ex.5. *Solve $(D^2 - 1)\,x + 5Dy = t$, $-2Dx + (D^2 - 4)\,y = -2$, if $y = x = 0 = Dx = Dy$ when $t = 0$.*

Sol. Given $$x''(t) - x(t) + 5y'(t) = t \quad \ldots (1A)$$

and $$-2x'(t) + y''(t) - 4y(t) = -2 \quad \ldots (2A)$$

with initial conditions $$x(0) = y(0) = x'(0) = y'(0) = 0 \quad \ldots (3)$$

Let L $\{x(t)\} = \bar{x}(s)$ and L $\{y(t)\} = \bar{y}(s)$. Taking Laplace transform of both sides of (1A) and (2A), we get

$$\mathrm{L}\{x''(t)\} - \mathrm{L}\{x(t)\} + 5\,\mathrm{L}\{y'(t)\} = \mathrm{L}\{t\} \quad \ldots (1B)$$

and $$-2\,\mathrm{L}\{x'(t)\} + \mathrm{L}\{y''(t)\} - 4\mathrm{L}\{y(t)\} = -2\mathrm{L}\{1\} \quad \ldots (2B)$$

or $$s^2\,\bar{x} - sx(0) - x'(0) - \bar{x} + 5\,[s\,\bar{y} - y(0)] = 1/s^2 \quad \ldots (1C)$$

and $$-2\,[s\,\bar{x} - x(0)] + s^2\,\bar{y} - sy(0) - y'(0) - 4\,\bar{y} = -2/s. \quad \ldots (2C)$$

Using (3), (1C) and (2C) reduce to

$$(s^2 - 1)\,\bar{x} + 5s\,\bar{y} = 1/s^2 \quad \ldots (1D)$$

and $$-2s\,\bar{x} + (s^2 - 4)\,\bar{y} = -2/s. \quad \ldots (2D)$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{11s^2 - 4}{s^2(s^2+1)(s^2+4)} \quad \text{and} \quad \bar{y} = \frac{4 - 2s^2}{s(s^2+1)(s^2+4)}.$$

Resolving into partial fractions, we obtain

$$\bar{x} = -\frac{1}{s^2} + \frac{5}{s^2+1} - \frac{4}{s^2+4} \quad \text{and} \quad \bar{y} = \frac{1}{s} - \frac{2s}{s^2+1} + \frac{s}{s^2+4}.$$

Taking inverse Laplace transforms, these give

$$x = 5 \sin t - 2 \sin 2t - t \quad \text{and} \quad y = 1 - 2 \cos t + \cos 2t.$$

Ex.6. *Solve (D – 2) x – (D + 1) y = 6 e^{3t}, (2D – 3) x + (D – 3) y = 6 e^{3t}, if x = 3, y = 0 when t = 0.*

Sol. Given $x'(t) - 2x(t) - y'(t) - y(t) = 6e^{3t}$ ... (1A)

and $2x'(t) - 3x(t) + y'(t) - 3y(t) = 6e^{3t}$ ... (2A)

with initial conditions $x(0) = 3$ and $y(0) = 0$. ... (3)

Let $L\{x(t)\} = \bar{x}(s)$ and $L\{y(t)\} = \bar{y}(s)$. Taking the Laplace transforms of both sides of (1A) and (2A), we get

$$L\{x'(t)\} - 2L\{x(t)\} - L\{y'(t)\} - L\{y(t)\} = 6L\{e^{3t}\} \quad \text{... (1B)}$$

and $$2L\{x'(t)\} - 3L\{x(t)\} + L\{y'(t)\} - 3L\{y(t)\} = 6L\{e^{3t}\} \quad \text{... (2B)}$$

or $$s\bar{x} - x(0) - 2\bar{x} - [s\bar{y} - y(0)] - \bar{y} = 6/(s-3) \quad \text{... (1C)}$$

and $$2[s\bar{x} - x(0)] - 3\bar{x} + s\bar{y} - y(0) - 3\bar{y} = 6/(s-3). \quad \text{... (2C)}$$

Using (3), (1C) and (2C) reduce to

$$(s-2)\bar{x} - (s+1)\bar{y} = 3 + \frac{6}{s-3} = \frac{3s-3}{s-3} \quad \text{... (1D)}$$

and $$(2s-3)\bar{x} + (s-3)\bar{y} = 6 + \frac{6}{s-3} = \frac{6s-12}{s-3}. \quad \text{... (2D)}$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{3s^2 - 6s - 1}{(s-1)(s-1)^2} \quad \text{and} \quad \bar{y} = \frac{5-3s}{(s-3)(s-1)^2}$$

Resolving into partial fractions, these give

$$\bar{x} = \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{2}{s-3} \quad \text{and} \quad \bar{y} = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{s-3}.$$

Taking the inverse Laplace transforms, these give

$$x(t) = e^t + 2e^t L^{-1}\{1/s^2\} + 2e^{3t} = e^t + 2e^t t + 2e^{3t}$$

and $$y(t) = e^t - e^t L^{-1}\{1/s^2\} - e^{3t} = e^t - e^t t - e^{3t}$$

Ex.7. *Solve Dx + Dy = t, D^2 x – y = e^{-t}, if x (0) = 3, x′ (0) = –2, y (0) = 0.*

Sol. Given $x'(t) + y'(t) = t$ ... (1A)

and $x''(t) - y(t) = e^{-t}$. ... (2A)

with initial conditions: $x(0) = 3$, $x'(0) = -2$, $y(0) = 0$. ... (3)

Let $L\{x(t)\} = \bar{x}$ and $L\{y(t)\} = \bar{y}$, Taking Laplace the transform of both sides of (1A) and (2A), we get

$$L\{x'(t)\} + L\{y'(t)\} = L\{t\} \quad \text{... (1B)}$$

and $$L\{x''(t)\} - L\{y(t)\} = L\{e^{-t}\} \quad \text{... (2B)}$$

or $$s\bar{x} - x(0) + s\bar{y} - y(0) = 1/s^2 \quad \text{... (1C)}$$

and $$s^2\bar{x} - sx(0) - x'(0) - \bar{y} = 1/(s+1). \quad \text{... (2C)}$$

Using (3), (1C) and (2C) reduce to

$$s\bar{x} + s\bar{y} = 3 + (1/s^2) = (3s^2+1)/s^2 \quad \text{... (1D)}$$

and $$s^2\bar{x} - \bar{y} = 3s - 2 + 1/(s+1) = (3s^2 + s - 1)/(s+1) \quad \text{... (2D)}$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{3s^2+1}{s^3(s^2+1)} + \frac{3s^2+s-1}{(s+1)(s^2+1)} \qquad \text{and} \qquad \bar{y} = \frac{3s^2+1}{s^2(s^2+1)} - \frac{3s^2+s-1}{(s+1)(s^2+1)}$$

Resolving into partial fractions, these give

$$\bar{x} = \frac{2}{s} + \frac{1}{s^3} + \frac{1}{2(s+1)} - \frac{3}{2(s^2+1)} + \frac{s}{2(s^2+1)}$$

and $$\bar{y} = \frac{1}{s} - \frac{1}{2(s+1)} + \frac{3}{2(s^2+1)} - \frac{s}{2(s^2+1)}.$$

Taking inverse Laplace transforms, these give

$$x = 2 + t^2/2 + (1/2) \times e^{-t} - (3/2) \times \sin t + (1/2) \times \cos t$$

and $$y = 1 - (1/2) \times e^{-t} + (3/2) \times \sin t - (1/2) \times \cos t$$

Ex.8. *Solve (D – 2) x – (D – 2) y = sin t, (D² + 1) x + 2Dy = 0, if x = 0 = x′ (0) = y (0).*

Sol. Given $$x'(t) - 2x(t) - y'(t) + 2y(t) = \sin t \quad \text{... (1A)}$$

and $$x''(t) + x(t) + 2y'(t) = 0. \quad \text{... (2A)}$$

with initial conditions: $x(0) = 0$, $x'(0) = 0$ and $y(0) = 0$. ... (3)

Let $\mathrm{L}\{x(t)\} = \bar{x}(s)$ and $\mathrm{L}\{y(t)\} = \bar{y}(s)$. Taking the Laplace transform of both sides of (1A) and (2A), we have

$$\mathrm{L}\{x'(t)\} - 2\,\mathrm{L}\{x\} - \mathrm{L}\{y'(t)\} + 2\,\mathrm{L}\{y\} = \mathrm{L}\{\sin t\} \quad \text{... (1B)}$$

and $$\mathrm{L}\{x''(t)\} + \mathrm{L}\{x\} + 2\mathrm{L}\{y'(t)\} = 0 \quad \text{... (2B)}$$

or $$s\bar{x} - x(0) - 2\bar{x} - [s\bar{y} - y(0)] + 2\bar{y} = 1/(s^2+1) \quad \text{... (1C)}$$

and $$s^2\bar{x} - sx(0) - x'(0) + \bar{x} + 2[s\bar{y} - y(0)] = 0. \quad \text{... (2C)}$$

Using (3), (1C) and (2C) reduce to

$$(s^2+1)(s-2)\bar{x} - (s^2+1)(s-2)\bar{y} - 1 = 0 \quad \text{... (2D)}$$

and $$(s^2+1)\bar{x} + 2s\bar{y} - 0 = 0. \quad \text{... (2D)}$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$ by cross-multiplication, we get

$$\frac{\bar{x}}{2s} = \frac{\bar{y}}{-(s^2+1)} = \frac{1}{2s(s^2+1)(s-2) + (s^2+1)^2(s-2)}$$

$$\Rightarrow \qquad \bar{x} = \frac{2s}{(s^2+1)(s-2)(2s+s^2+1)} = \frac{2s}{(s-2)(s+1)^2(s^2+1)}$$

and $$\bar{y} = -\frac{1}{(s-2)(2s+s^2+1)} = -\frac{1}{(s-2)(s+1)^2}.$$

Resolving these into partial fractions, we have

$$\bar{x} = \frac{1}{9(s+1)} + \frac{1}{3(s+1)^2} - \frac{4}{45(s-2)} - \frac{s+2}{5(s^2+1)} \quad \ldots (4)$$

and

$$\bar{y} = \frac{1}{9(s+1)} + \frac{1}{3(s+1)^2} - \frac{1}{9(s-2)}. \quad \ldots (5)$$

Taking inverse Laplace transform, we have

$$(4) \Rightarrow \quad \bar{x} = \frac{1}{9}\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}\mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \frac{4}{45}\mathrm{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{5}\mathrm{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{2}{5}\mathrm{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{9}e^{-t} + \frac{1}{3}e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{4}{45}e^{2t} - \frac{1}{5}\cos t - \frac{2}{5}\sin t \text{ , using first shifting theorem}$$

$$= (1/9) \times e^{-t} + (1/3) \times \mathrm{t}\, e^{-t} - (4/45) \times e^{2t} - (1/5) \times \cos t - (2/5) \times \sin t.$$

$$(5) \Rightarrow \quad \bar{y} = \frac{1}{9}\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}\mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \frac{1}{9}\mathrm{L}^{-1}\left\{\frac{1}{s-2}\right\} = \frac{1}{9}e^{-t} + \frac{1}{3}e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}e^{2t}$$

$$= (1/9)\times e^{-t} + (1/3)\times te^{t} - (1/9)\times e^{2t}.$$

Ex. 9. *Solve the following system of differential equations by the method of Laplace transforms: x′ + y = 2 cos t, x + y′ = 0, x(0) = 0, y(0) = 1.*

Sol. Given $x'(t) + y(t) = 2\cos t$ ... (1A)

and $x + y'(t) = 0$ ... (2A)

with initial coditions: $x(0) = 0$ and $y(0) = 1$ ... (3)

Let $\mathrm{L}\{x(t)\} = \bar{x}(s)$ and $\mathrm{L}\{y(t)\} = \bar{y}(s)$. Taking the Laplace transform of both sides of (1A) and (2A), we get

$\mathrm{L}\{x'(t)\} + \mathrm{L}\{y(t)\} = 2\mathrm{L}\{\cos t\}$ or $s\,\bar{x} - x(0) + \bar{y} = 2s/(s^2+1)$

and $\mathrm{L}\{x\} + \mathrm{L}\{y'(t)\} = 0$ or $x + s\,\bar{y} - y(0) = 0$

Using (3), the above equation reduce to

$$s\,\bar{x} + \bar{y} = 2s/(s^2+1) \quad \ldots (1B)$$

$$\bar{x} + s\,\bar{y} = 1 \quad \ldots (2B)$$

Solving (1B) and (2B) for $\bar{x}$ *and* $\bar{y}$, we have

$\bar{x} = 1/(1+s^2)$ and $\bar{y} = s/(1+s^2)$ ... (4)

Taking the inverse Laplace transform of (4), we get

$x = L^{-1}\{1/(1+s^2)\} = \sin t$ and $y = \mathrm{L}^{-1}\{s(1+s^2)\} = \cos t$

Ex. 10. *Solve the system of equation x′ + 5x – 2y = t, y′ + 2x + y = 0; x (0) = 0, y(0) = 2 by Laplace transform method.* **[S.V. Univ. A.P. 1997]**

Sol. Given $x'(t) + 5x(t) - 2y(t) = t$ ... (1A)

and $2x(t) + y'(t) + y(t) = 0$ ... (2A)

with initial conditions $x(0) = 0$ and $y(0) = 2$ ... (3)

Let $L\{x(t)\} = \bar{x}$ and $L\{y(t)\} = \bar{y}$. Taking the Laplace transform of both sides of (1A) and (2A), we get

$$\mathrm{L}\{x'\} + 5\mathrm{L}\{x\} - 2\mathrm{L}\{y\} = \mathrm{L}\{t\} \quad \text{or} \quad s\bar{x} - x(0) + 5\bar{x} - 2\bar{y} = 1/s^2$$

and $$2\mathrm{L}\{x\} + \mathrm{L}\{y'\} + \mathrm{L}\{y\} = 0 \quad \text{or} \quad 2\bar{x} + s\bar{y} - y(0) + \bar{y} = 0$$

Using (3), the above equation reduce to

$$(s+5)\bar{x} - 2\bar{y} - (1/s^2) = 0 \quad \text{... (1B)}$$

and $$2\bar{x} + (s+1)\bar{y} + 0 = 0 \quad \text{... (2B)}$$

Solving (1B) and (2B) for $\bar{x}$ and $\bar{y}$ by cross-multiplication, we set

$$\frac{\bar{x}}{0 + (s+1)/s^2} = \frac{\bar{y}}{-(2/s^2) - 0} = \frac{1}{(s+5)\,s+1) + 4}$$

Hence $$\bar{x} = \frac{s+1}{s^2(s^2+6s+9)} = \frac{s+1}{s^2(s+3)^2}, \qquad \bar{y} = -\frac{2}{s^2(s^2+6s+9)} = -\frac{2}{s^2(s+3)^2}$$

Resolving into partial fraction, we have

$$\bar{x} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}, \qquad \bar{y} = -\frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Taking the inverse Laplace transforms, we have

$$x = \frac{1}{27}\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{9}\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{27}\mathrm{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{2}{9}\mathrm{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$= 1/27 + t/9 - (1/27) \times e^{-3t} - (2/9) \times e^{-3t}\,\mathrm{L}\{1/s^2\}$$

$$= 1/27 + t/9 - (1/27) \times e^{-3t} - (2/9) \times e^{-3t}\,t$$

and $$y = -\frac{4}{27}L^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{9}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{4}{27}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{2}{9}L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$= -(4/27) - (2t/9) - (4/27) \times e^{-3}t - (2/9)\,e^{-3t}\,t, \text{ as before.}$$

Ex.11. *Solve x′ (t) + y′ (t) = 2z (t), y′ (t) + z′ (t) = 2x (t), z′ (t) + x′ (t) = 2y (t), given x (0) = a, y (0) = b, z (0) = c.*

Sol. Given that $x(0) = a$, $y(0) = b$ and $z(0) = c$ ...(1)

Let $\mathrm{L}\{x(t)\} = \bar{x}(s)$, $\mathrm{L}\{y(t)\} = \bar{y}(s)$ and $\mathrm{L}\{z(t)\} = \bar{z}(s)$.

Taking Laplace transform of first given equation, we get

$$\mathrm{L}\{x'(t)\} + \mathrm{L}\{y'(t)\} = 2\mathrm{L}\{z(t)\}.$$

or $$s\bar{x} - x(0) + s\bar{y} - y(0) = 2\bar{z} \quad \text{or} \quad s\bar{x} - a + s\bar{y} - b = 2\bar{z}, \text{ using (1)}$$

$$\therefore \quad s(\bar{x} + \bar{y}) - 2\bar{z} = a + b. \quad \text{... (2)}$$

Similarly, given second and third equations reduce to

$$s(\bar{y} + \bar{z}) - 2\bar{x} = b + c \quad \text{... (3)}$$

and $$s(\bar{z} + \bar{x}) - 2\bar{y} = c + a. \quad \text{... (4)}$$

Adding (2), (3) and (4), $$2s(\bar{x} + \bar{y} + \bar{z}) - 2(\bar{x} + \bar{y} + \bar{z}) = 2(a + b + c)$$

or $\bar{x}+\bar{y}+\bar{z}=d/(s-1)$, where $d=a+b+c$.

$\therefore$ $\bar{x}+\bar{y}=-\bar{z}+d/(s-1)$. ... (5)

Using it in (2), we get $s[-\bar{z}+d/(s-1)]-2\bar{z}=a+b=d-c$, as $d=a+b+c$

$\therefore$ $$\bar{z}=\frac{d}{(s-1)(s+2)}+\frac{c}{s+2}=\frac{d}{3}\left(\frac{1}{s-1}-\frac{1}{s+2}\right)+\frac{c}{s+2}.$$

Inverting, $z=(1/3)\times de^{t}-(c-d/3)\times e^{-2t}$

Similarly, we get $x=(1/3)\times d\,e^{t}+(a-d/3)\,e^{-2t}$ and $y=(1/2)\times d\,e^{t}+(b-d/3)\,e^{-2t}$:

EXERCISE (3*C*)

Solve the following system of differential equation by the method of Laplace transforms:

1. $Dx+Dy=t$, $D^2x-y=e^{-t}$, if $x(0)=y(0)=x'(0)=0$.

Ans. $x=(1/2)\times(t^2+e^{-t}+\cos t+\sin t)-1$, $y=1-(1/2)\times(e^{-t}+\cos t+\sin t)$.

2. $x'+x-4y=0,\ y'=3x-2y,\ x(0)=3, y(0)=4$ **Ans.** $x=4e^{2t}-e^{-5t}$, $y=3e^{2t}+e^{-5t}$

3. $x'+y=0,\ y'-x=0,\ x(0)=1, y(0)=0$ **Ans.** $x=\cos t$, $y=\sin t$

4. $x'+x-y=0,\ y'+x+y=0,\ x(0)=1, y(0)=0$ **Ans.** $x=e^{-t}\cos t$, $y=-e^{t}\sin t$

5. $x'=2x+4y,\ y'=x+2y,\ x(0)=-4, y(0)=-4$ **Ans.** $x=2-6e^{4t}$, $y=-1-3e^{4t}$

6. $3x'+y'+2x=1,\ x'+4y'+3y=0,\ x(0)=0=y(0)$ **[Rohtak 2004]**

Ans. $x=(1/10)\times(5-2e^{-t}-3e^{-6t/11})$, $y=(1/5)\times(e^{-t}-e^{-6t/11})$

7. $x'+y=3e^{2t},\ y'+x=0,\ x(0)=2, y(0)=0$

Ans. $x=2\,e^{2t}-(1/2)\times(e^{t}-e^{-t})$, $y=-e^{2t}+(1/2)\times(e^{t}+e^{-t})$

8. $x'=6x+9y,\ y'=x+6y,\ x(0)=3=y(0)$ **Ans.** $x=6e^{9t}-3e^{3t}$, $y=e^{3t}+2e^{9t}$

9. $2x'+y'-x-y=e^{-t},\ x'+y'+2x+y=e^{t},\ x(0)=2, y(0)=1$

Ans. $x=8\sin t+2\cos t$, $y=\cos t-1\ 3\sin t+(1/2)\times(e^{t}-e^{-t})$

10. $x'-y=e^{t},\ y'+x=\sin t,\ x(0)=1,\ y(0)=0$

Ans. $x=(e^{t}+2\sin t-t\cos t+\cos t)/2$, $y=(\cos t-e^{t}+t\sin t-\sin t)/2$

11. $x''-3x'+y'+2x-y=0,\ x'+y'-2x+y=0,\ x(0)=0, y(0)=-1,\ x'(0)=0$

Ans. $x=2e^{t}-e^{2t}-1$, $y=e^{t}-2$

12. $x''-x-3y=0,\ y''-4x+4e^{t}=0,\ x(0)=2,\ x'(0)=3,\ y(0)=1,\ y'(0)=2$

Ans. $x=e^{t}+e^{2t}$, $y=e^{2t}$

13. $x''+5x-2y=0,\ y''-2x+2y=0,\ x(0)=3,\ x'(0)=0, y(0)=1,\ y'(0)=0$

Ans. $x=\cos t+2\cos(t\sqrt{6}),\ y=2\cos t-\cos(t\sqrt{6})$

14. $d^2x/dt^2+dy/dt+3x=15e^{-t},\ d^2y/dt^2-4(dx/dt)+3y=15\sin 2t$, if $x(0)=35$, $x'(0)=-48$, $y(0)=27$ and $y'(0)=-55$.

Ans. $x=30\cos t-15\sin 3t-3\,e^{-t}+2\cos 2t$, $y=30\cos 3t-60\sin t-3\,e^{-t}+\sin 2t$

15. $Dx+2D^2y=e^{-t}$, $(D+2)x-y=1$ if $x(0)=y(0)=y'(0)=0$ **[Agra 2011]**

Ans. $x=1+e^{-t}-e^{-at}-e^{-bt}$, $y=1+e^{-t}-be^{-at}-ae^{-bt}$, where $a=(2-\sqrt{2})/2$ and $b=(2+\sqrt{2})/2$.

16. $x''+3x-2y=0,\ x''+y''-3x+5y=0,\ x(0)=0,\ x'(0)=3,\ y(0)=0,\ y'(0)=2$

Ans. $x = (11/4) \times \sin t + (1/12) \times \sin 3t$, $y = (11/4) \times \sin t - (1/4) \times \sin 3t$

17. $(D^2+2)x - Dy = 2t+5, (D-1)x + (D+1)y = -1-2t, x(0)=3,\ x'(0)=0,\ y(0)=-3,\ y'(0)=5$

Ans. $x = 2 + t + e^{-2t} + \sin t$, $y = 1 - t - 3\ e^{-2t} - \cos t$

18. *Solve for* $y(t)$, *using Laplace transform:* $x''(t) - x(t) + 5y'(t) = 1,\quad y''(t) - 4y(t) - 2x'(t) = -2,\quad x(0) = x'(0) = y(0) = y'(0) = 0$ **[Kuvempa 2005]**

Ans. $y(t) = (3/2) \times (\cos t - \cos 2t) + (1/3) \times (4 \sin t - 5 \sin 2t)$

19. *If* $x'(t) + y(t) = 0,\quad y'(t) - x(t) = 1$ *and* $x(0) = y(0) = 0$, *then* $y(t) =$

(*a*) $\sin t$ (*b*) $\cos t$ (*c*) $\sinh t$ (d) $\cosh t$ **[Kuvempa 2005]**

Ans. (*a*)

20. *Solve using Laplace transform :* $3(dx/dt) + dy/dt + 2x = 1$ and $dx/dt + 4\,(dy/dt) + 3y = 0$ given $x = y = 0$ when $t = 0$. **[M.S. Univ. T.N. 2007]**

Ans. $x = (1/2) - (1/5) \times e^{-t} - (3/10) \times e^{-(6t/11)}$, $y = (1/5) \times (e^{-t} - e^{-(6t/11)})$

21. Solve $dx/dt - y = e^t$, $dy/dt + x = \sin t$ given that $x(0) = 1$, $y(0) = 0$ using Laplace transform. **[M.S. Univ. T.N. 2007]**

Ans. $x = (1/2) \times (\cos t + 2 \sin t - t \cos t + e^t)$, $y = (1/2) \times (t \sin t - \sin t + \cos t - e^t)$

22. Using Laplace transform, solve the following equation

(*i*) $d^2y/dt^2 + 4\,(dy/dt) + 5y = (\cos t - \sin t)e^{-2t}$ subject to the boundary conditions

$y(0) = 1,\ y'(0) = -3$ **[Agra 2005]**

(*ii*) $(D^4 + 4D + 4)y = \sin t$, given $y(0) = 2$ and $y'(0) = 0$ **[Madurai Kamraj 2008]**

Ans. (*i*) $y = e^{-2t}(\cos t - \sin t)$ (*ii*) $y = (54/25 + 21t/5)e^{-2t} + (3\sin t - 4\cos t)/25$

Miscellaneous examples on chapter 3

Ex.1. *Let* y *be the solution of the initial value problem* $d^2y/dx^2 + y = 6\cos 2x,\quad y(0) = 3,\ y'(0) = 1$. *Let the Laplace transform of* y *be F(s). Then, the value of F(1) is*

(*a*) 17/5 (*b*) 13/5 (*c*) 11/5 (*d*) 9/5 **(GATE 2011)**

Sol. Ans. (*b*). Proceed upto equation (3) of Ex. 2 on page 3.2. Note that here independent variable is x in place of usual independent variable t. Also, note that here $L\{y\}$ = F(s). Hence, equation (3) takes the form

F(s) = (5s + 1)/(s^2 + 1) – (2s)/(s^2 + 4) and therefore F(1) = 3 – (2/5) = 13/5

Ex.2. *Solve the following differential equation by Laplace transform method.*

(*i*) $(D^2 - 2D + 2)\,y = 0$, where $D = d/dt$, $y(0) = y'(0) = 1$ **(Nagpur 2010)** **Ans.** $y = e^t \cos t$

(*ii*) $y'' + 2y' + 5y = 3e^{-x}\sin x, y(0) = 0,\ y'(0) = 3.$ **(Ranchi 2010) Ans**. $y = e^{-x}(\sin x + \sin 2x)$

(*iii*) $y''(t) + y(t) = 4e^t, y(0) = y'(0) = 0$ **(Kanpur 2010)** **Ans.** y = 2 (e^t – sin t – cos t)

(*iv*) $d^2y/dx^2 + y = \cos x, y(0) = y'(0) = 0$ **(Kanpur 2011)** **Ans.** $y = (x/2) \times \sin x$

Ex. 3. *The solution of the initial value problem* $y'' + 2y' + 10y = 6\delta(t),\ y(0) = 0,\ y'(0) = 0,$ *where* $\delta(t)$ *denotes the Dirac delta function is*

(*a*) $2e^t \sin 3t$ (*b*) $6e^t \sin 3t$ (c) $2e^{-t} \sin 3t$ (*d*) $6e^{-t} \sin 3t$ **[GATE 2012]**

Sol. Ans. (*c*). Given $y'' + 2y' + 10y = 6\delta(t)$ ... (1)

Taking Laplace transform of both sides of (1), we have

$$L\{y''\} + 2L\{y'\} + 10L\{y\} = 6L\{\delta(t)\}$$

or $s^2 L\{y\} - sy(0) - y'(0) + 2\{sL\{y\} - y(0)\} + 10\ L\{y\} = 6,$ since $L\{\delta(t)\} = 1$ (see page 1.49)

or $(s^2 + 2s + 10)L\{y\} = 6$ or $L\{y\} = 6/(s^2 + 2s + 10)$

Thus, $L\{y\} = \dfrac{6}{(s+1)^2 + 3^2}$ so that $y = L^{-1}\left\{\dfrac{6}{(s+1)^2 + 3^2}\right\} = 6e^{-t}\ L^{-1}\left\{\dfrac{1}{s^2 + 3^2}\right\}$

Hence, $y = 6e^{-t} \times \{(\sin 3t)/3\} = 2e^{-t} \sin 3t$, which is the required solution

Ex.4 Using Laplace transform, solve $(D^2 - 4D + 4)y = t, y(0) = 0, y'(0) = 1$ **(Madras 2012)**

Ans. $y = \{(5t - 1)e^{2t} + t + 1\}/4$

4

Applications to Integral Equations

4.1. Integral equations

Definition. *An integral equation is an equation in which an unknown function appears under integral sign.*

The most general type of linear integral equation is of the form

$$F(t) = G(t) + \lambda \int_a^b K(t,u)\, F(u)\, du, \qquad \ldots (1)$$

where the upper limit may be either variable or fixed. The function G (t) and K (t, u) are known functions, while F (t) is to be determined. The function K (t, u) is called the *kernel of the integral* equation.

If a and b are constants, the equation is known as *Fredholm integral equation*. If a is a constant while $b = t$, it is called *a Volterra integral equation*.

4.2. Some important results to be committed to memory

(i) **Convolution (or Falting).**

Definition. The convolution of F (t) and G (t) is denoted and defined as

$$F * G = \int_0^t F(x)\, G(t-x)\, dx \qquad \text{or} \qquad F * G = \int_0^t F(t-x)\, G(x)\, dx.$$

(ii) **Convolution theorem or convolution property.**

If $L\{F(t)\} = f(s)$ and $L\{G(t)\} = g(s)$, then $L^{-1}\{f(s)\, g(s)\} = \int_0^t F(x)\, G(t-x)\, dx = F * G$

or $$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(t-x)\, G(x)\, dx = F * G.$$

Moreover, $$L\{F * G\} = f(s)\, g(s) = L\{F(t)\} \times L\{G(t)\}.$$

i.e., $$L\left\{\int_0^t F(x)\, G(t-x)\, dx\right\} = L\left\{\int_0^t F(t-x)\, G(x)\, dx\right\} = f(s)\, g(s).$$

(iii) **Volterra Integral equation of first and second kinds.**

A linear integral equation of the form $$G(t) = \lambda \int_a^t K(t,u)\, F(u)\, du$$

is known as *Volterra integral equation of first kind.*

Again, a linear integral equation of the form $$F(t) = G(t) + \lambda \int_a^t K(t,u)\, F(u)\, du$$

is known as *Volterra integral equation of second kind.*

(iv) **Integral equation of convolution type.**

Definition. The integral equation $$y(t) = f(t) + \int_0^t K(t-x)\, y(x)\, dx,$$

in which the Kernel K ($t - x$) is a function of the difference $t - x$ only, is known as *integral equation of the convolution type*.

Note. Using the definition of convolution, we may re-write the above integral equation as

$$y(t) = f(t) + \mathrm{K}(t) * y(t).$$

4.3. Application of Laplace transform in solving Volterra integral equation with convolution type kernel. Working rule

(*i*) Consider the Volterra integral equation of the first kind

$$\mathrm{G}(t) = \lambda \int_0^t \mathrm{K}(t-u)\ \mathrm{F}(u)\,du. \qquad \text{or} \qquad \mathrm{G}(t) = \lambda\ \mathrm{K}(t) * \mathrm{F}(t). \qquad \ldots (1)$$

Let $\mathrm{L}\{\mathrm{G}(t)\} = g(s)$, $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$ and $\mathrm{L}\{\mathrm{K}(t)\} = k(s)$ ... (2)

Applying the Laplace transform of both sides of (1), we get

$$\mathrm{L}\{\mathrm{G}(t)\} = \lambda\ \mathrm{L}\{\mathrm{K}(t) * \mathrm{F}(t)\}$$

or $\mathrm{L}\{\mathrm{G}(t)\} = \lambda\, \mathrm{L}\{\mathrm{K}(t)\} \times \mathrm{L}\{\mathrm{F}(t)\}$, by the convolution theorem

or $g(s) = \lambda\, k(s) f(s)$ or $f(s) = g(s)/\lambda\, k(s)$, by (2) ... (3)

Applying the inverse Laplace transform of both sides of (3), we get

$$\mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\left\{\frac{g(s)}{\lambda k(s)}\right\} \qquad \text{or} \qquad \mathrm{F}(t) = \mathrm{L}^{-1}\left\{\frac{g(s)}{\lambda k(s)}\right\}.$$

(*ii*) Consider the Volterra integral equation of the second kind

$$\mathrm{F}(t) = \mathrm{G}(t) + \lambda \int_0^t \mathrm{K}(t-u)\ \mathrm{F}(u)\,du.$$

or $$\mathrm{F}(t) = \mathrm{G}(t) + \lambda\, \mathrm{K}(t) * \mathrm{F}(t). \qquad \ldots (1)$$

Let $\mathrm{L}\{\mathrm{G}(t)\} = g(s)$, $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$ and $\mathrm{L}\{\mathrm{K}(t)\} = k(s)$. ... (2)

Applying the Laplace transform of both sides of (1), we get

$$\mathrm{L}\{\mathrm{F}(t)\} = \mathrm{L}\{\mathrm{G}(t)\} + \lambda\, \mathrm{L}\{\mathrm{K}(t) * \mathrm{F}(t)\}$$

or $\mathrm{L}\{\mathrm{F}(t)\} = \mathrm{L}\{\mathrm{G}(t)\} + \lambda\, \mathrm{L}\{\mathrm{K}(t)\} \times \mathrm{L}\{\mathrm{F}(t)\}$, using the convolution theorem

or $f(s) = g(s) + \lambda\, k(s) f(s)$ or $[1 - \lambda\, k(s)] f(s) = g(s)$

or $$f(s) = g(s)/[1 - \lambda\, k(s)]. \qquad \ldots (3)$$

Applying the inverse Laplace transform of both sides of (3), we get

$$\mathrm{F}(t) = \mathrm{L}^{-1}\{g(s)/[1 - \lambda\, k(s)]\}.$$

4.4. Abel's Integral equation.

Definition. An integral equation of the form $\displaystyle\int_0^t \frac{\mathrm{F}(u)}{(t-u)^{\alpha}}\,du = \mathrm{G}(t)$

is known as *Abel's integral equation*, where F (t) is unknown function, G (t) is a known function and α is a constant such that $0 < \alpha < 1$.

4.5. Integro-differential equation.

Definition. It is an equation in which various derivatives of unknown function F (t) can also be present.

For example, $$\mathrm{F}'(t) = \mathrm{F}(t) + \mathrm{G}(t) + \int_0^t \mathrm{K}(t-u)\ \mathrm{F}(u)\,du$$

is an *integro-differential equation*, where F (t) is unknown function and G (t) and K ($t - u$) are known functions. The solution of such equation subject to given initial conditions can be easily obtained.

4.6. Solved examples based on integral equations

Ex.1. *Solve the integral equation* $F(t) = 1 + \displaystyle\int_0^t F(u)\ \sin(t-u)\,du$ *and verify your solution.*

[Osmania 2010; Kanpur 1995]

Sol. Given $$F(t) = 1 + \int_0^t F(u)\sin(t-u)\,du. \quad \ldots(1)$$

Using the definition of convolution, (1) can be written as

$$F(t) = 1 + F(t) * \sin t. \quad \ldots(2)$$

Let $L\{F(t)\} = f(s)$ Applying the Laplace transform of (2), we get

$$L\{F(t)\} = L\{1\} + L\{F(t) * \sin t\}$$

or $$L\{F(t)\} = (1/s) + L\{F(t)\} \times L\{\sin t\}, \text{ by the convolution theorem}$$

or $$f(s) = \frac{1}{s} + f(s) \times \frac{1}{s^2+1} \qquad \text{or} \qquad \left(1 - \frac{1}{s^2+1}\right) f(s) = \frac{1}{s}$$

or $$f(s) = (s^2+1)/s^3 = 1/s + 1/s^3. \quad \ldots(3)$$

Applying the inverse Laplace transform of both sides of (3), we get

$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^3}\right\} \qquad \text{or} \qquad F(t) = 1 + \frac{t^2}{2!} = 1 + \frac{t^2}{2}. \quad \ldots(4)$$

Verification of solution (4): We wish to show that (4) satisfies the given integral equation (1).
From (4), $F(u) = 1 + u^2/2$

$$\therefore \quad \text{R.H.S. of (1)} = 1 + \int_0^t \left(1 + u^2/2\right)\sin(t-u)\,du$$

$$= 1 + \left[\left(1 + u^2/2\right)\cos(t-u)\right]_0^t - \int_0^t u\cos(t-u)\,du, \text{ integrating by parts}$$

$$= 1 + 1 + \frac{t^2}{2} - \cos t - \left\{\left[-u\sin(t-u)\right]_0^t - \int_0^t 1\cdot\{-\sin(t-u)\}\,du\right\}$$

$$= 2 + \frac{t^2}{2} - \cos t - \int_0^t \sin(t-u)\,du.$$

$$= 2 + \frac{t^2}{2} - \cos t - \left[\cos(t-u)\right]_0^t = 2 + \frac{t^2}{2} - \cos t - (1 - \cos t)$$

$$= 1 + t^2/2 = F(t), \text{ using (4)}$$

$$= \text{L.H.S. of (4)}$$

Hence (4) is solution of given integral equation (1).

Ex.2. *Solve by the method of Laplace transform:* $3\sin 2x = y(x) + \int_0^x (x-t)y(t)\,dt.$

[Nagpur 2005; Lucknow 1994]

Sol. Note carefully that in the present problem we have x in place of t and t in place of u. So we shall make the corresponding changes in the usual solution and standard results.

Using the definition of convolution, the given equation can be written as

$$y(x) = 3\sin 2x - y(x) * x. \quad \ldots(1)$$

Let $L\{y(x)\} = f(s)$. Applying the Laplace transformation of both sides of (1), we have

$L\{y(x)\} = 3L\{\sin 2x\} - L\{y(x) * x\}$ or $f(s) = 3 \times \{2/(s^2+2^2)\} - L\{y(x)\} \times L\{x\}$

or $$f(s) = \frac{6}{s^2+4} - \frac{f(s)}{s^2} \qquad \text{or} \qquad f(s) = \frac{6s^2}{(s^2+1)(s^2+4)}$$

or $$f(s) = 2\{1/(s^2+1) - 1/(s^2+4)\} \quad \ldots (2)$$

Taking inverse Laplace transform of both sides of (2), we get

$$y(x) = 2\{\sin t - (\sin 2t)/2\} = 2 \sin t - \sin 2t.$$

Ex.3. *Solve the integral equation* $\mathrm{F}(t) = t + 2\int_0^t \mathrm{F}(u)\cos(t-u)\,du$ **[Kanpur 94]**

Sol. Given $$\mathrm{F}(t) = t + 2\int_0^t \mathrm{F}(u)\cos(t-u)\,du. \quad \ldots (1)$$

Using the definition of convolution, (1) can be re-written as

$$\mathrm{F}(t) = t + 2\,\mathrm{F}(t) * \cos t. \quad \ldots (2)$$

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Applying the Laplace transform of (1), we have

$$\mathrm{L}\{\mathrm{F}(t)\} = \mathrm{L}\{t\} + 2\,\mathrm{L}\{\mathrm{F}(t) * \cos t\}$$

or $\mathrm{L}\{\mathrm{F}(t)\} = (1/s^2) + 2\,\mathrm{L}\{\mathrm{F}(t)\} \times \mathrm{L}\{\cos t\}$, by the convolution theorem

or $$f(s) = \frac{1}{s^2} + 2\,f(s) \times \frac{s}{s^2+1} \qquad \text{or} \qquad f(s)\left[1 - \frac{2s}{s^2+1}\right] = \frac{1}{s^2}$$

or $$f(s) = \frac{s^2+1}{s^2(s-1)^2} = \frac{\mathrm{A}}{s} + \frac{\mathrm{B}}{s^2} + \frac{\mathrm{C}}{s-1} + \frac{\mathrm{D}}{(s-1)^2}. \quad \ldots (3)$$

Multiply both sides of (3) by s^2 and let $s \to 0$; then $\mathrm{B} = \lim_{s\to 0} \dfrac{s^2+1}{(s-1)^2} = \dfrac{0+1}{(0-1)^2} = 1.$

Multiply both siders of (3) by $(s-1)^2$ and let $s \to 1$; then $\mathrm{D} = \lim_{s\to 1} \dfrac{s^2+1}{s^2} = \dfrac{1+1}{1} = 2.$

Then, from (3), $$f(s) = \frac{s^2+1}{s^2(s-1)^2} = \frac{\mathrm{A}}{s} + \frac{1}{s^2} + \frac{\mathrm{C}}{s-1} + \frac{2}{(s-1)^2} \quad \ldots (4)$$

Multiplying both sides of (4) by $s^2(s-1)^2$, we have

$$s^2 + 1 = \mathrm{A}s(s-1)^2 + (s-1)^2 + \mathrm{C}\,s^2(s-1) + 2s^2.$$

or $$s^2 + 1 = \mathrm{A}s(s^2 - 2s + 1) + (s^2 - 2s + 1) + \mathrm{C}(s^3 - s^2) + 2s^2. \quad \ldots (5)$$

Equating coefficients of s^3 and s on both sides of (5), we get

$\mathrm{A} + \mathrm{C} = 0$ and $\mathrm{A} - 2 = 0$ so that $\mathrm{A} = 2$, $\mathrm{C} = -2$.

Then, (3) and (4) $\Rightarrow$ $$f(s) = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{2}{(s-1)^2}. \quad \ldots (6)$$

Taking inverse Laplace transform of both sides of (6), we get

or $\mathrm{F}(t) = 2 + t - 2e^t\,\mathrm{L}^{-1}\{1/s\} + 2e^t\,\mathrm{L}^{-1}\{1/s^2\}$, using first shifting theorem

or $\mathrm{F}(t) = 2 + t - 2e^t + 2e^t t = 2 + t - 2e^t(1-t)$

Ex.4. *Solve the integral equation:* $F(t) = t^2 + \int_0^t F(u)\sin(t-u)\,du$

Sol. Given $\mathrm{F}(t) = t^2 + \int_0^t \mathrm{F}(u)\sin(t-u)\,du$

or $\quad F(t) = t^2 + F(t) * \sin t$, using definition of convolution ... (1)

Let $L\{F(t)\} = f(s)$. Taking the Laplace transform of (1), we get

$$L\{F(t)\} = L\{t^2\} + L\{F(t) * \sin t\}$$

or $\quad f(s) = (2/s^3) + L\{F(t)\} \times L\{\sin t\}$, by the convolution theorem

or $\quad f(s) = \frac{2}{s^3} + f(s) \times \frac{1}{s^2+1}$ or $\quad f(s)\left[1 - \frac{1}{1+s^2}\right] = \frac{2}{s^3}$

or $\quad \frac{s^2 f(s)}{1+s^2} = \frac{2}{s^3}$ or $\quad f(s) = \frac{2(1+s^2)}{s^5} = \frac{2}{s^5} + \frac{2}{s^3}$.

Inverting, $\quad F(t) = 2 \times (t^2/2!) + 2 \times (t^4/4!) = t^2 + (t^4/12)$.

Ex.5. *Solve the following integral equations:*

(*a*) $F(t) = e^{-t} - 2\int_0^t F(u)\cos(t-u)\,du$. **[S.V. Univ. (A.P.) 1997]**

(*b*) $e^{-x} = y(x) + 2\int_0^x \cos(x-t)\, y(t)\,dt$.

Sol. (*a*) Given $\quad F(t) = e^{-t} - 2\int_0^t F(u)\cos(t-u)\,du$.

or $\quad F(t) = e^t - 2F(t) * \cos t$, using definition of convolution. ... (1)

Let $L\{F(t)\} = f(s)$. Taking the Laplace transform of both sides of (1), we get

$$L\{F(t)\} = L\{e^{-t}\} - 2L\{F(t) * \cos t\}$$

or $\quad f(s) = 1/(s+1) - 2L\{F(t)\} \times L\{\cos t\}$, by the convolution theorem

or $\quad f(s) = \frac{1}{s+1} - 2f(s) \times \frac{s}{s^2+1}$ or $\quad f(s)\left(1 + \frac{2s}{s^2+1}\right) = \frac{1}{s+1}$

or $\quad \frac{(s+1)^2}{s^2+1} f(s) = \frac{1}{s+1}$ or $\quad f(s) = \frac{s^2+1}{(s+1)^3}$

Inverting, $\quad F(t) = L^{-1}\left\{\frac{s^2+1}{(s+1)^3}\right\} = L^{-1}\left\{\frac{[(s+1)-1]^2+1}{(s+1)^3}\right\}$

$$= e^{-t}\, L^{-1}\left\{\frac{(s-1)^2+1}{s^3}\right\}, \text{ by first shifting theorem}$$

$$= e^{-t} L^{-1}\left\{\frac{s^2-2s+2}{s^3}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s} - \frac{2}{s^2} + \frac{2}{s^3}\right\}$$

$$= e^{-t}[1 - 2\times(t/1!) + 2\times(t^2/2!)] = e^{-t}(1 - 2t + t^2) = e^{-t}(1-t)^2.$$

(*b*) Here replace t by *x* and *u* by *t* in part (*a*) Also replace F (*t*) by *y* (*x*)

Ex.6. *Solve the integral equation* $F(t) = a \sin t - 2\int_0^t F(u)\cos(t-u)\,du$,

[Ravishanker 2004]

Sol. Given $\mathrm{F}(t) = a \sin t - 2\int_0^t \mathrm{F}(u)\cos(t-u)\,du$

or $\mathrm{F}(t) = a\sin t - 2\,\mathrm{F}(t) * \cos t$, by definition of convolution ... (1)

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Taking the Laplace transform of both sides of (1), we get

$$\mathrm{L}\{\mathrm{F}(t)\} = a\,\mathrm{L}\{\sin t\} - 2\,\mathrm{L}\{\mathrm{F}(t) * \cos t\}$$

or $f(s) = a/(1+s^2) - 2\,\mathrm{L}\{\mathrm{F}(t)\} \times \mathrm{L}\{\cos t\}$, by the convolution theorem

or $f(s) = \dfrac{a}{1+s^2} - 2f(s) \times \dfrac{s}{s^2+1}$ or $f(s)\left[1 + \dfrac{2s}{s^2+1}\right] = \dfrac{a}{1+s^2}$

or $\dfrac{(s+1)^2}{s^2+1} f(s) = \dfrac{a}{1+s^2}$ or $f(s) = \dfrac{a}{(s+1)^2}$.

Inverting, $\mathrm{L}^{-1}\{f(s)\} = a\,\mathrm{L}^{-1}\{1/(s+1)^2 = a\,e^{-t}\,\mathrm{L}^{-1}\{1/s^2\}$, using first shifting theorem

or $\mathrm{F}(t) = a\,e^{-t} \times (t/1!) = at\,\mathrm{e}^{-t}$.

Ex.7. *Solve the integral equation* $t = \int_0^t e^{t-u} F(u)\,du$.

Sol. Re-writing the given intergral equation, we get

or $t = e^t * \mathrm{F}(t)$, by definition of convolution. ... (1)

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Taking Laplace transform of both sides of (1), we get

$\mathrm{L}\{t\} = \mathrm{L}\{e^t * \mathrm{F}(t)\}$ or $\mathrm{L}\{t\} = \mathrm{L}\{e^t\} \times \mathrm{L}\{\mathrm{F}(t)\}$, using the convolution theorem

or $\dfrac{1}{s^2} = \dfrac{1}{s-1} f(s)$ or $f(s) = \dfrac{s-1}{s^2} = \dfrac{1}{s} - \dfrac{1}{s^2}$.

Inverting, $\mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\{1/s\} - \mathrm{L}^{-1}\{1/s^2\}$ or $\mathrm{F}(t) = 1 - t$.

Ex.8. *Solve the integral equation* $\int_0^t \mathrm{F}(u)\,\mathrm{F}(t-u)\,du = 16\sin 4t$.

[Kanpur 1995, S.V. Univ. (A.P.) 1997]

Sol. Re-writing the given integral equation, we have

$\mathrm{F}(t) * \mathrm{F}(t) = 16\sin 4t$, by definition of convolution ... (1)

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{\mathrm{F}(t) * \mathrm{F}(t)\} = 16\,\mathrm{L}\{\sin 4t\}$$

or $\mathrm{L}\{\mathrm{F}(t)\} \times \mathrm{L}\{\mathrm{F}(t)\} = 16\,\mathrm{L}\{\sin 4t\}$, using the convolution theorem

or $f(s) \times f(s) = 16\left(\dfrac{4}{s^2+4^2}\right)$ or $f(s) = \pm\dfrac{8}{(s^2+4^2)^{1/2}}$.

Inverting, $\mathrm{L}^{-1}\{f(s)\} = \pm 8\,\mathrm{L}^{-1}\left\{1/(s^2+4^2)^{1/2}\right\} = \pm 8\,\mathrm{J}_0(4t)$, as $\mathrm{L}\{\mathrm{J}_0(at)\} = 1/(s^2+a^2)^{1/2}$

Ex.9. *Solve* $\sin t = \int_0^t J_0(t-u)\,F(u)\,du$.

Sol. Re-writing the given integral equation, we have

or $\sin t = \mathrm{J}_0(t) * \mathrm{F}(t)$, by definition of convolution. ... (1)

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Taking the Laplace transform of both sides of (1), we get

$$\mathrm{L}\{\sin t\} = \mathrm{L}\{\mathrm{J}_0(t) * \mathrm{F}(t)\}$$

or $1/(1+s^2) = \mathrm{L}\{\mathrm{J}_0(t)\} \times \mathrm{L}\{\mathrm{F}(t)\}$, using the convolution theorem

or $$\frac{1}{1+s^2} = \frac{1}{(s^2+1)^{1/2}} f(s), \qquad \text{as} \qquad \mathrm{L}\{\mathrm{J}_0(at)\} = \frac{1}{(s^2+a^2)^{1/2}}$$

or $$f(s) = 1/(s^2+1)^{1/2}$$

Inverting, $$\mathrm{F}(t) = \mathrm{L}^{-1}\left\{1/(s^2+1)^{1/2}\right\} = \mathrm{J}_0(t).$$

Ex.10. *Solve the Abel's integral equation* $\int_0^t \frac{\mathrm{F}(u)du}{\sqrt{(t-u)}} = 1 + t + t^2$. **[Kanpur 1996]**

Sol. Given: $$\int_0^t (t-u)^{-1/2} F(u)\, du = 1 + t + t^2$$

or $$t^{-1/2} * \mathrm{F}(t) = 1 + t + t^2, \text{ by definition of convolution.} \qquad \ldots (1)$$

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Applying the Laplace transform to both sides of (1), we get

$$\mathrm{L}\{t^{-1/2} * \mathrm{F}(t)\} = \mathrm{L}\{1 + t + t^2\}$$

or $\mathrm{L}\{t^{-1/2}\} \times \mathrm{L}\{\mathrm{F}(t)\} = \mathrm{L}\{1\} + \mathrm{L}\{t\} + \mathrm{L}\{t^2\}$, using the convolution theorem

or $$\frac{\Gamma(1/2)}{s^{1/2}} f(s) = \frac{1}{s} + \frac{1!}{s^2} + \frac{2!}{s^3}$$

or $$f(s) = \frac{s^{1/2}}{\sqrt{\pi}}\left(\frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3}\right) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{s^{1/2}} + \frac{1}{s^{3/2}} + \frac{2}{s^{5/2}}\right), \qquad \text{as} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Applying the inverse Laplace transform to both sides, we get

$$\mathrm{F}(t) = \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\Gamma(1/2)} + \frac{t^{1/2}}{\Gamma(3/2)} + \frac{2\,t^{3/2}}{\Gamma(5/2)}\right] = \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\sqrt{\pi}} + \frac{t^{1/2}}{(1/2)\times\sqrt{\pi}} + \frac{2t^{3/2}}{(3/2)\times(1/2)\times\sqrt{\pi}}\right]$$

Thus, $$\mathrm{F}(t) = (1/\pi) \times [t^{-1/2} + 2\,t^{1/2} + (8/3) \times t^{3/2}].$$

Ex.11. *Solve the Abel's equation* $\int_0^t \frac{F(u)\,du}{(t-u)^{1/3}} = t(1+t)$.

Sol. Given $$\int_0^t (t-u)^{-1/3} \mathrm{F}(u)\, du = t + t^2.$$

or $$t^{-1/3} * \mathrm{F}(t) = t + t^2, \text{ by definition of convolution} \qquad \ldots (1)$$

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Taking the Laplace transform both sides of (1), we get

$$\mathrm{L}\{t^{-1/3} * \mathrm{F}(t)\} = \mathrm{L}\{t + t^2\}$$

or $\mathrm{L}\{t^{-1/3}\} \times \mathrm{L}\{\mathrm{F}(t)\} = \mathrm{L}\{t\} + \mathrm{L}\{t^2\}$, using the convolution theorem

or $$\frac{\Gamma(-1/3+1)}{s^{-(1/3)+1}} \times f(s) = \frac{1!}{s^2} + \frac{2!}{s^3} \qquad \text{or} \qquad \frac{\Gamma(2/3)}{s^{2/3}} f(s) = \frac{1}{s^2} + \frac{2}{s^3}$$

or $$f(s) = \frac{1}{\Gamma(2/3)}\left[\frac{s^{2/3}}{s^2} + \frac{s^{2/3}}{s^3}\right] = \frac{1}{\Gamma(2/3)}\left[\frac{1}{s^{4/3}} + \frac{2}{s^{7/3}}\right]. \qquad \ldots (2)$$

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathrm{F}(t) = \frac{1}{\Gamma(2/3)}\left[\mathrm{L}^{-1}\left\{\frac{1}{s^{4/3}}\right\} + 2\,\mathrm{L}^{-1}\left\{\frac{1}{s^{7/3}}\right\}\right] = \frac{1}{\Gamma(2/3)}\left[\frac{t^{(4/3)-1}}{\Gamma(4/3)} + 2\times\frac{t^{(7/3)-1}}{\Gamma(7/3)}\right]$$

$$= \frac{1}{\Gamma(2/3)}\left\{\frac{t^{1/3}}{(1/3)\Gamma(1/3)} + 2\times\frac{t^{4/3}}{(4/3)(1/3)\Gamma(1/3)}\right\}$$

$$= \frac{3t^{1/3}(1+3t/2)}{\Gamma(1/3)\Gamma(1-1/3)} = \frac{3t^{1/3}(1+3t/2)}{[\pi/\sin(\pi/3)]}, \qquad \text{as} \qquad \Gamma(n)\,\Gamma(1-n) = \pi/\sin(n\pi)$$

$$= \frac{3t^{1/3}(1+3t/2)}{(2/\sqrt{3})\pi} = \frac{3\sqrt{3}\,t^{1/3}(2+3t)}{4\pi}.$$

Ex.12. *Solve the Abel integral equation for F (t):* $G(t) = \int_0^t \frac{F(u)du}{(t-u)^{\alpha}}, 0 < \alpha < 1.$

Sol. Given $$\int_0^t (t-u)^{-\alpha} F(u)\,du = G(t)$$

or $$t^{-\alpha} * F(t) = G(t), \text{ by definition of convolution.} \qquad \ldots (1)$$

Let $L\{G(t)\} = g(s)$ and $L\{F(t)\} = f(s).$ ... (2)

Taking the Laplace transform of both sides of (1), we get

$$L\{t^{-\alpha} * F(t)\} = L\{G(t)\}$$

or $L\{t^{-\alpha}\} \times L\{F(t)\} = L\{G(t)\}$, using the convolution theorem

or $$\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} f(s) = g(\alpha) \qquad \text{or} \qquad f(\alpha) = \frac{s^{1-\alpha} g(\alpha)}{\Gamma(1-\alpha)}$$

or $$f(s) = \frac{s}{\Gamma(\alpha)\Gamma(1-\alpha)}\{\Gamma(\alpha)\, s^{-\alpha} g(s)\} = \frac{s}{(\pi/\sin\pi\alpha)}\left\{\frac{\Gamma(\alpha)}{s^{\alpha}} g(\alpha)\right\}$$

$$[\because \Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin\pi\alpha]$$

$$= \frac{s\sin\pi\alpha}{\pi} L\{t^{\alpha-1}\} \times L\{G(t)\}, \text{ as } L\{t^{a-t}\} = \frac{\Gamma(\alpha)}{s^{\alpha}}$$

$$= \frac{s\sin\pi\alpha}{\pi} L\{t^{\alpha-1} * G(t)\}, \text{ by the convolution theorem}$$

$$= \frac{s\sin\pi\alpha}{\pi} L\left\{\int_0^t (t-u)^{\alpha-1} G(u)\,du\right\}, \text{ by definition of convolution}$$

$$\therefore \qquad f(s) = \frac{\sin\pi\alpha}{\pi} s\, L\left\{\int_0^t (t-u)^{\alpha-1} G(u)\,du\right\}. \qquad \ldots (3)$$

Let $$H(t) = \int_0^t (t-u)^{\alpha-1} G(u)\,du. \qquad \ldots (4)$$

From (4), we have $$H(0) = 0. \qquad \ldots (5)$$

Now, $L\{H'(t)\} = s\,L\{H(t)\} - H(0) = s\,L\{H(t)\}$, using (5).

$$\therefore \qquad s\,L\{H(t)\} = L\left\{\frac{d}{dt}H(t)\right\}$$

or $$s\,L\left\{\int_0^t (t-u)^{\alpha-1} G(u)\,du\right\} = L\left\{\frac{d}{dt}H(t)\right\}, \text{ by (4)} \qquad \ldots (6)$$

Using (6), (3) reduces to $f(s) = \frac{\sin \pi\alpha}{\alpha} \mathrm{L}\left\{\frac{d}{dt}\mathrm{H}(t)\right\}$.

Inverting, $\mathrm{F}(t) = \frac{\sin \pi\alpha}{\alpha} \frac{d}{dt}\mathrm{H}(t) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dt}\left\{\int_0^t (t-u)^{\alpha-1}\,\mathrm{F}(u)\,du\right\}$

Ex.13. *Solve the integral equation* $\mathrm{F}'(t) = t + \int_0^t \mathrm{F}(t-u)\cos u\ du$, if $\mathrm{F}(0) = 4$.

[Kanpur 1997, KU Kurukshetra 2006]

Sol. Given $\mathrm{F}'(t) = t + \int_0^t \mathrm{F}(t-u)\cos u\ du$

or $\mathrm{F}'(t) = t + \mathrm{F}(t) * \cos t$, by definition of convolution. ... (1)

Also, given that $\mathrm{F}(0) = 4$. ... (2)

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{\mathrm{F}'(t)\} = \mathrm{L}\{t\} + \mathrm{L}\{\mathrm{F}(t) * \cos t\}$$

or $s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0) = \mathrm{L}\{t\} + \mathrm{L}\{\mathrm{F}(t)\} \times \mathrm{L}\{\cos t\}$

(using Art. 1.15 of chapter 1 and the convolution theorem)

or $s f(s) - 4 = (1/s^2) + f(s) \times \{s/(1+s^2)\}$, using (2)

or $\left(1 - \frac{1}{s^2+1}\right) s f(s) = 4 + \frac{1}{s^2}$ or $\frac{s^3}{s^2+1} f(s) = 4 + \frac{1}{s^2}$

or $f(s) = \frac{s^2+1}{s^3}\left(4 + \frac{1}{s^2}\right) = \frac{4(s^2+1)}{s^3} + \frac{s^2+1}{s^5} = \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}$. ... (3)

Taking the inverse Laplace transform of both sides of (3), we get

$$\mathrm{F}(t) = 4 + 5t^2/2 + t^4/24.$$

Ex.14. *Solve the integro-differential equation.* $F'(t) = \sin t + \int_0^t F(t-u)\cos u\ du$,

where F (0) = 0.

Sol. Given $\mathrm{F}'(t) = \sin t + \int_0^t \mathrm{F}(t-u)\cos u\ du$

or $\mathrm{F}'(t) = \sin t + \mathrm{F}(t) * \cos t$, by definition of convolution ... (1)

Also, given that $\mathrm{F}(0) = 0$. ... (2)

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Taking Laplace transform of both sides of (1), we get

$$\mathrm{L}\{\mathrm{F}'(t)\} = \mathrm{L}\{\sin t\} + \mathrm{L}\{\mathrm{F}(t) * \cos t\}$$

or $s\,\mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0) = 1/(s^2+1) + \mathrm{L}\{\mathrm{F}(t)\} \times \mathrm{L}\{\cos t\}$

(using Art. 1.15 of chapter 1 and the convolution theorem)

or $s f(s) = 1/(s^2+1) + f(s) \times [s/(s^2+1)]$, using (2)

or $\left(1 - \frac{1}{s^2+1}\right) s\, f(s) = \frac{1}{s^2+1}$ or $f(s) = \frac{1}{s^3}$.

Inverting, $\mathrm{L}^{-1}\{f(s)\} = \mathrm{L}^{-1}\left\{\frac{1}{s^3}\right\}$ or $\mathrm{F}(t) = \mathrm{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2!} = \frac{t^2}{2}$.

Ex.15. *Solve* $\int_0^t F'(t)\, F(t-u)\,du = 24t^3$, *if* $F(0) = 0$.

Sol. Given $$\int_0^t \mathrm{F}'(t)\,\mathrm{F}(t-u)\,du = 24\ t^3$$

or $$\mathrm{F}'(t) * \mathrm{F}(t) = 24\ t^3, \text{ by definition of convolution.} \quad \ldots(1)$$

Also, given $$\mathrm{F}(0) = 0. \quad \ldots(2)$$

Let $\mathrm{L}\{\mathrm{F}(t)\} = f(s)$. Taking the Laplace transform of both sides of (1), we get

$$\mathrm{L}\{\mathrm{F}'(t) * \mathrm{F}(t)\} = 24\ \mathrm{L}\{t^3\}$$

or $\mathrm{L}\{\mathrm{F}'(t)\} \times \mathrm{L}\{\mathrm{F}(t)\} = 24 \times (3!/s^4)$, using the convolution theorem

or $[s\ \mathrm{L}\{\mathrm{F}(t)\} - \mathrm{F}(0)] \times f(s) = 144/s^4$ or $s\,[f(s)]^2 = 144/s^4$, using (2)

or $[f(s)]^2 = 144/s^5$ or $f(s) = \pm\, 12/s^{5/2}$.

Inverting, $\mathrm{L}^{-1}\{f(s)\} = \pm\, 12\ \mathrm{L}^{-1}\left\{\dfrac{1}{s^{5/2}}\right\}$ or $\mathrm{F}(t) = \pm\, 12 \times \dfrac{t^{3/2}}{\Gamma(5/2)}$

$$\therefore \quad \mathrm{F}(t) = \pm\,\frac{12t^{3/2}}{(3/2)\times(1/2)\times\Gamma(1/2)} = \pm 12t^{3/2}\,\frac{1}{(3/4)\times\sqrt{\pi}} = \pm\frac{16t^{3/2}}{\sqrt{\pi}}.$$

Ex.16. *Show that the solution of integral equation* $F(t) = 4t - 3\int_0^t F(u)\,\sin(t-u)\,du$

is $F(t) = t + (3/2) \times \sin 2t$.

Sol. Given $$\mathrm{F}(t) = 4t - 3\int_0^t \mathrm{F}(u)\,\sin(t-u)\,du \quad \ldots(1)$$

and $$\mathrm{F}(t) = t + (3/2) \times \sin 2t. \quad \ldots(2)$$

We are to show that (2) is solution of (1).

Now, from (2), $$\mathrm{F}(u) = u + (3/2) \times \sin 2u. \quad \ldots(3)$$

$$\text{Then, R.H.S. of (1)} = 4t - 3\int_0^t \left(u + \frac{3}{2}\sin 2u\right)\sin(t-u)\,du = 4t - 3\int_0^t u\,\sin(t-u)\,du - \frac{9}{2}\int_0^t \sin 2u\,\sin(t-u)\,du$$

$$= 4t - 3\left[\left[u\cos(t-u)\right]_0^t - \int_0^t 1\cdot\cos(t-u)\,du\right] - \frac{9}{4}\int_0^t [\cos(3u-t) - \cos(t+u)]\,du$$

$$= 4t - 3\left\{t - \left[\frac{\sin(t-u)}{(-1)}\right]_0^t\right\} - \frac{9}{4}\left[\frac{1}{3}\sin(3u-t) - \sin(t+u)\right]_0^t = 4t - 3(t - \sin t) - \frac{9}{4}\left[\frac{1}{3}\sin 2t - \sin 2t + \frac{1}{3}\sin t + \sin t\right]$$

$= t + (3/2) \times \sin 2t =$ L.H.S. of (1)

Hence (2) is a solution of (1).

Ex.17. *Solve the integral equation.* $y + \int_0^t y\,dt = 1 - e^{-t}$.

Sol. Taking Laplace transform of both sides of the given equation, we have

$$\mathrm{L}\{y\} + \mathrm{L}\left\{\int_0^t y\,dt\right\} = \mathrm{L}\{1\} - \mathrm{L}\{e^{-t}\} \quad \ldots(1)$$

Let $\mathrm{L}\{y\} = \bar{y}$. Then, by Art. 2.18 of chapter 2, we have $\mathrm{L}\left\{\int_0^t y\,dt\right\} = \dfrac{\bar{y}}{s}$.

Hence (1) reduces to $\bar{y} + (\bar{y}/s) = (1/s) - 1/(s+1)$.

or $\dfrac{\bar{y}(s+1)}{s} = \dfrac{(s+1)-s}{s(s+1)}$ or $\bar{y} = \dfrac{1}{(s+1)^2}$ ... (2)

Taking inverse Laplace transformation of both sides of (2), we get

$$y = \mathrm{L}^{-1}\{1/(s+1)^2\} = e^{-t}\,\mathrm{L}^{-1}\{1/s^2\}, \text{ by first translation theorem}$$

or $$y = e^{-t} \times (t/1!) = t\,e^{-t}.$$

Ex.18. *Solve the integral equation* $\dfrac{dy}{dt} + 3y + 2\int_0^t y\,dt = t$, *with* $y(0) = 0$. **[Madras 1997]**

Sol. Taking the Laplace transform of the both sides of given equation,

$$s\,\bar{y} - y(0) + 3\,\bar{y} + 2\,(\bar{y}/s) = 1/s^2, \text{ where } \bar{y} = \mathrm{L}\{y\}$$

or $$\{s + 3 + (2/s)\}\ \bar{y} = (1/s^2), \text{ since } y(0) = 0 \text{ (given)}$$

or $$\bar{y} = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} \quad \text{... (1)}$$

Splitting the R.H.S. of (1) into partial fractions, we obtain

$$\bar{y} = \frac{1}{2}\times\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\times\frac{1}{s+2}$$

Taking the inverse transform of the two sides of (2), we obtain

$$\bar{y} = (1/2) - e^{-t} + (1/2) \times e^{-2t}.$$

Ex.19. *Solve* $\dfrac{dx}{dt} + x + 3\int_0^t y\,dt = \cos t + 3\sin t$, *and* $2(dx/dt) + 3(dy/dt) + 6y = 0$, *given that* $x(0) = -3$ *and* $y(0) = 2$.

Sol. Let $\mathrm{L}\{x\} = \bar{x}$ and $\mathrm{L}\{y\} = \bar{y}$. Then, taking the Laplace transform of both sides of the given first equation, we get

$$s\bar{x} - x(0) + \bar{x} + \frac{3\bar{y}}{s} = \frac{s}{s^2+1^2} + \frac{3}{s^2+1^2} \quad \text{or} \quad s\bar{x} + 3 + \bar{x} + \frac{3\bar{y}}{s} = \frac{s+3}{s^2+1}$$

or $$s(s+1)\,\bar{x} + 3\bar{y} = \frac{s(s+3)}{s^2+1} - 3s. \quad \text{... (1)}$$

Taking Laplace transform of both sides of given second equation, we get

$2\{s\bar{x} - x(0)\} + 3\{s\bar{y} - y(0)\} + 6\bar{y} = 0$ or $2s\bar{x} + 6 + 3s\bar{y} - 6 + 6\bar{y} = 0$

or $$\bar{x} = -\{3(s+2)/2s\}\,\bar{y}. \quad \text{... (2)}$$

Using (2), (1) becomes $$-\frac{3(s+1)(s+2)\bar{y}}{2} + 3\bar{y} = \frac{s(s+3)}{s^2+1} - 3s$$

or $$\frac{-3s(s+3)\bar{y}}{2} = \frac{s(s+3)}{s^2+1} - 3s \quad \text{or} \quad \bar{y} = -\frac{2}{3}\times\frac{1}{s^2+1} + \frac{2}{s+3}. \quad \text{... (3)}$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = -(2/3) \times \sin t + 2\,e^{-3t}. \quad \text{... (4)}$$

Using (3), (2) gives $\bar{x} = -\frac{3(s+2)}{2s}\left[-\frac{2}{3(s^2+1)}+\frac{2}{s+3}\right] = \frac{s+2}{s(s^2+1)}-\frac{3(s+2)}{s(s+3)}$

Splitting the R.H.S. into partial fractions, we get

$$\bar{x} = \frac{2}{s}+\frac{1-2s}{s^2+1}-\left[\frac{(2/3)}{s}+\frac{(1/3)}{s+3}\right] = \frac{2}{s}+\frac{1}{s^2+1}-\frac{2s}{s^2+1}-\frac{2}{3s}-\frac{1}{3(s+3)}$$

or $$\bar{x} = \frac{4}{3}\times\frac{1}{s}+\frac{1}{s^2+1}-2\times\frac{s}{s^2+1}-\frac{1}{3}\times\frac{1}{s+3}. \quad \ldots (5)$$

Taking inverse Laplace transform of both sides of (5), we get

$$x = 4/3 + \sin t - 2\cos t - (1/3)\times e^{-3t}.$$

EXERCISE

Solve the following integral equations:

1. $\mathrm{F}(t) = 1 + 2\int_0^t \mathrm{F}(t-u)\, e^{-2u}\, du.$ **Ans.** $\mathrm{F}(t) = 1 + 2t.$

2. $2\,\mathrm{F}(t) = 2 - t + \int_0^t \mathrm{F}(t-u)\,\mathrm{F}(u)\, du.$ **Ans.** $\mathrm{F}(t) = \pm 1.$

3. $\mathrm{F}(t) = 1 + 2\int_0^t \mathrm{F}(t-u)\cos u\, du.$ **Ans.** $\mathrm{F}(t) = 1 + 2\,t\,e^{t}.$

4. $\mathrm{F}(t) = 1 + \frac{1}{6}\int_0^t (t-u)^3\,\mathrm{F}(u)\, du.$ **Ans.** $\mathrm{F}(t) = \frac{1}{2}\,(\sinh t + \sin t).$

5. $\mathrm{F}(t) = t - 1 + \int_0^t \mathrm{F}(u)\sin(t-u)\, du.$ **Ans.** $\mathrm{F}(t) = t - 1 - t^2/2 + 1/6 + t^3/6.$

6. $y(t) = at + \int_0^t y(u)\,\sin(t-u)\, du.$ **(Bhopal 2010]** **Ans.** $y(t) = a\,(t + t^3/6).$

7. $y(t) = a\sin bt + c\int_0^t y(u)\,\sin b\,(t-u)\, du,$ when $b > c > 0$

Ans. $y = ab\,(b^2 - bc)^{-1/2}\,\sin\{t\,(b^2 - bc)^{1/2}\}.$

8. $\frac{dy}{dt} + 4y + 5\int_0^t y\, dt = e^{-t}$, with $y(0) = 0.$ **Ans.** $y = (1/2)\times[e^{-2t}(\cos t + 3\sin t) - e^{-3t}]$

9. $y + \int_0^t y(u)\, du = \sin t.$ **Ans.** $y = (1/2)\times(\sin t + \cos t - e^{-t})$

10. $y + \int_0^t y(u)\, du = e^{-t}$ **Ans.** $y = e^{-t}\,(1 - t).$

11. $\frac{dy}{dt} + y + \int_0^t y\,(u)\, du = e^{-t}.$ **Ans** $y = e^{-t/2}\left[\cos(t\sqrt{3}/2) + (1/\sqrt{3})\times\sin(t\sqrt{3}/2)\right] - e^{-t}$

12. $\int_0^t \frac{y(u)\,du}{(t-u)^{1/2}} = \sqrt{t}$ **[Sagar 2004]** **Ans.** $y(t) = 1/2$

13. $\int_0^t \frac{y(u)du}{(t-u)^{1/2}} = 1 + 2t - t^2$ **[M.D.U. Rohtak 2004; KU Kurukshetra 2004]**

Ans. $y(1) = (1/\pi) \times \{t^{-1/2} + 4t^{1/2} + (18/3) \times t^{3/2}\}$

14. $\mathrm{F}(t) = t + \int_0^t \mathrm{F}(t-u) \cos u \, du$ **[Kuvempa 2005]**

Ans. $\mathrm{F}(t) = 1 + t - e^{t/2} \{\cos(t\sqrt{3}/2) - (1/\sqrt{3}) \times \sin(t\sqrt{3}/2)\}$

15. $\mathrm{F}'(t) = t - \int_0^t \mathrm{F}(t-u) \cos u \, du, \mathrm{F}(0) = 1$ **[Punjab 2005, KU Kurukshetra 2006]**

Ans. $\mathrm{F}(t) = 1 + t^2 + t^4/24$

16. $y(t) = t + \int_0^t y(u) \sin(t-u) \, du$ **[Kurukshetra 2005]** **Ans.** $y(t) = t + t^3/6$

17. $f(t) = t - \int_0^t f(t-u) \cosh u \, du, f(0) = 1$ **[Punjab 2005]**

Ans. $f(t) = 1 + t - e^{-t/2} \{\cosh(t\sqrt{5}/2) - (1/\sqrt{5}) \times \sinh(t\sqrt{5}/2)\}$

5

Applications to Boundary Value Problems

5.1A. Initial and Boundary Conditions:

Definitions. The given partial differential equations are given along with certain prescribed conditions on the functions which arise from the physical situation. The conditions which are given for $t = 0$ are known as *initial conditions* whereas the conditions given at the boundary of the region or interval are called *boundary conditions*.

5.1B. Boundary Value Problems involving partial differential equations.

While formulation a probelm in science and engineering mathematically, we arrive at a partial differential equation involving one or more functions together with certain prescribed conditions on the functions which arise from the physical situation. The conditions are known as *boundary conditions* and the problem of getting solution to the partial differential equation which satisfy the boundary conditions is known as a *boundary value problem*.

5.1C. Working rule for solving boundary value problem involving partial differential equations.

Recall that in the case of an ordinary differential equations the Laplace transformation gives rise to an algebraic equation for the transform. We shall show that in the case of a partial differential equation in two independent variables, the Laplace transformation gives rise to an ordinary differential equation for the transform. Reader is advised to remember all results of the following theorem.

Theorem. Given the function $y(x, t)$ is defined for $a \le x \le b$, $t > 0$. Let $y(x, t)$, regarded as a function of t, is of some exponential order as $t \to \infty$ and is piecewise continuous over every finite interval of $t \ge 0$. Then using the following notations,

$$\mathrm{L}\{y(x, t)\} = \int_0^\infty e^{-st} y(x,t)\, dt = \bar{y}(x,s) = \bar{y}$$

$$\frac{\partial y}{\partial x} = y_x(x,t), \qquad \frac{\partial y}{\partial t} = y_t(x,t), \qquad \left(\frac{\partial y}{\partial x}\right)_{x=a} = y_x(a,t), \qquad \left(\frac{\partial y}{\partial t}\right)_{t=0} = y_t(x,0),$$

we have

(*i*) $\mathrm{L}\{\partial y / \partial t\} = s\,\bar{y}(x,s) - y(x,0)$. **[Meerut 2002, 04]**

(*ii*) $\mathrm{L}\{\partial^2 y / \partial t^2\} = s^2\,\bar{y}(x,s) - s\,y(x,0) - y_t(x,0)$.

(*iii*) $\mathrm{L}\{\partial y / \partial x\} = d\bar{y} / dx$ and $\mathrm{L}\{\partial^2 y / \partial x^2\} = d^2\bar{y} / dx^2$.

[Kanpur 2001, 05, Rajasthan 2001, 06, Rohilkhand 2000]

[Note that the Laplace transform $\bar{y}(x, s)$ is function not only of s but also of the "untransformed" variable x]

Proof. (*i*) $$\mathrm{L}\left\{\frac{\partial y}{\partial t}\right\} = \int_0^\infty e^{-st}\frac{\partial y}{\partial t}\,dt = \lim_{k\to\infty}\int_0^k e^{-st}\frac{\partial y}{\partial t}\,dt = \lim_{k\to\infty}\left[\left\{e^{-st}y(x,t)\right\}_0^k + s\int_0^k e^{-st}y(x,t)\,dt\right]$$

$$= -y(x,0) + s\int_0^\infty e^{-st}y(x,t)\,dt = s\,\bar{y}(x,s) - y(x,0).$$

(*ii*) Let $z = \partial y / \partial t$. Then, we have $\partial^2 y / \partial t^2 = \partial z / \partial t$.

$$\therefore \quad L\{\partial^2 y/\partial t^2\} = L\{\partial z/\partial t\} = s\bar{z}(x,s) - z(x,0), \text{ by part } (i)$$

$$= s\,L\{z\} - y_t(x,0) = s\,L\{\partial y/\partial t\} - y_t(x,0)$$

$$= s[s\bar{y}(x,s) - y(x,0)] - y_t(x,0), \text{ by part } (i)$$

$$= s^2\,\bar{y}(x,s) - sy(x,0) - y_t(x,0).$$

(*iii*) $$L\left\{\frac{\partial y}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial y}{\partial x}dt = \frac{d}{dx}\int_0^\infty e^{-st}y(x,t)\,dt$$

(By Leibnitz's rule for differentiation under the integral sign)

Thus, $L\{\partial / y / \partial x\} = d\bar{y}/dx.$

(*iv*) Let $u = \partial y / \partial x$ so that $\partial^2 y / \partial x^2 = \partial u / \partial x$. Then we have

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = L\left\{\frac{\partial u}{\partial x}\right\} = \frac{d\bar{u}}{dx}, \text{ by part } (iii)$$

$$= \frac{d}{dx}L\{u\} = \frac{d}{dx}L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d}{dx}\left(\frac{d\bar{y}}{dx}\right), \text{ by part } (iii)$$

Thus, $L\{\partial^2 y / \partial x^2\} = d^2\bar{y}/dx^2.$

5.2 Solved examples based on Art 5.1C

Ex.1. *Find the bounded solution of* $\partial^2 y/\partial x^2 - \partial^2 y/\partial t^2 = xt$, *when* $y = 0 = \partial y/\partial t$ *at* $t = 0$ *and* $y(0, t) = 0$. **[S.V. University (A.P.) 1997, Meerut 2001, 05]**

Sol. Given $\partial^2 y/\partial x^2 - \partial^2 y/\partial t^2 = xt.$ ... (1)

with $y(x, 0) = 0$ and $y_t(x, 0) = 0.$ ... (2)

Let $L\{y(x, t)\} = \bar{y}(x, s)$ Taking the Laplace transforms of both sides of (1), we get

$$L\{\partial^2 y/\partial x^2\} - L\{\partial^2 y/\partial t^2\} = x\,L\{t\}$$

or $$\frac{d^2\bar{y}}{dx^2} - [s^2\bar{y}(x,s) - sy(x,0) - y_t(x,0)] = \frac{x}{s^2} \quad \text{or} \quad \frac{d^2\bar{y}}{dx^2} - s^2\bar{y} = \frac{x}{s^2}, \text{ by (2)}$$

or $$(D^2 - s^2)\bar{y} = x/s^2, \quad \text{where} \quad D \equiv d/dx \quad \text{... (3)}$$

Auxiliary equation of (3) is $D^2 - s^2 = 0$ so that $D = \pm s.$

$\therefore$ Complementary function (C.F.) $= c_1 e^{cx} + c_2 e^{-cx}$, c_1, c_2 being arbitrary constants.

Again, Particular Integral (P.I.) $= \frac{1}{D^2 - s^2}\left(\frac{x}{s^2}\right) = \frac{1}{-s^4(1 - D^2/s^2)}x$

$$= -\frac{1}{s^4}\left(1 - \frac{D^2}{s^2}\right)^{-1} x = -\frac{1}{s^4}\left(1 + \frac{D^2}{s^2} + \ldots.\right)x = -\frac{x}{s^4}$$

$\therefore$ The general solution of (3) is $\bar{y}(x,s) = c_1 e^{sx} + c_2 e^{-sx} - (x/s^4).$ ... (4)

Since $u(x, t)$ is bounded for all x and t, so $\bar{u}(x, s)$ must be bounded, *i.e.*, $\bar{u}(x, s)$ is finite as $x \to \infty$. So we must take $c_1 = 0$ in (4). Then, we get

$$\bar{y}(x,s) = c_2\, e^{-sx} - (x / s^4). \qquad \ldots (5)$$

Given that $y(0, t) = 0.$ ... (6)

From (6), $L\{y(0, t)\} = 0$ or $\bar{y}(0, s) = 0$ ... (7)

Putting $x = 0$ in (5) and using (7), we get $c_2 = 0.$

$\therefore$ From (5), $$\bar{y}(x,s) = -x / s^4 \qquad \ldots (8)$$

Taking inverse Laplace transforms of both sides of (8), we get

$y(x, t) = -x \times (t^3 / 3!) = -(xt^3 / 6)$, which is the required solution.

Ex.2. *Solve $\partial y/\partial x = y + 2(\partial y/\partial t)$, $y(x, 0) = 6e^{-3x}$, which is bounded for $x > 0$, $t > 0$.*
[Delhi Maths (H) 1999; Meerut 1996]

Sol. Let $L\{y(x, t)\} = \bar{y}(x, s)$. Taking the Laplace transform of the given partial differential equation, we have

$$d\bar{y}/dx = \bar{y} + 2[s\bar{y} - y(x,0)$$

or $d\bar{y}/dx = \bar{y} + 2(s\bar{y} - 6e^{-3x})$, as given that $y(x, 0) = 6e^{-3x}$

or $$d\bar{y}/dx - (2s+1)\bar{y} = -12\, e^{-3x}. \qquad \ldots (1)$$

Its integrating factor = I.F. $= e^{-\int(2s+1)dx} = e^{-(2s+1)x}$.

$\therefore$ Solution of (1) is $$\bar{y}\, e^{-(2s+1)x} = c - 12\int \{e^{-3x} \times e^{-(2s+1)x}\}dx$$

or $$\bar{y}\, e^{-(2s+1)x} = c - 12\int e^{-2(s+2)x}dx = c + \frac{6}{s+2} \times e^{-2(s+2)x}$$

or $$\bar{y}(x,s) = c\, e^{(1+2s)x} + \{6/(s+2)\}e^{-3x}. \qquad \ldots (2)$$

Since $y(x, t)$ is bounded as $x \to \infty$, it follows that $\bar{y}(x, s)$ must be also bounded as $x \to \infty$. Using this fact in (2), we choose $c = 0$. So (2) reduces to

$$\bar{y}(x,s) = (6e^{-3x})/(s+2). \qquad \ldots (3)$$

Taking inverse Laplace transforms of both sides of (3), we get

$\therefore$ $y(x, t) = 6\, e^{-3x}\, L^{-1}\{1/(s + 2)\} = 6\, e^{-3x}\, e^{-2t} = 6\, e^{-(3x + 2t)}$.

Ex.3. *Solve $\partial y/\partial t = 2(\partial^2 y/\partial x^2)$ where $y(0, t) = y(5, t) = 0$ and $y(x, 0) = 10 \sin 4\pi x$.*
[Agra 2001, 04, G.N.D.U. Amritsar 2004; Kurukshatra 2004, Meerut 2000, 05, 06, 08, Rajasthan 2002, 06]

Sol. Let $L\{y(x, t)\} = \bar{y}(x, s)$. Taking the Laplace transform of the given partial differential equation, we have

$s\bar{y} - y(x,0) = 2(d^2\bar{y}/dx^2)$ or $d^2\bar{y}/dx^2 - (s/2)\bar{y} = -5 \sin 4\pi x$, as $y(x, 0) = 10 \sin 4\pi x$

or $$[D^2 - (s/2)]\,\bar{y} = -5\sin 4\pi x, \quad \text{where} \quad D \equiv d/dx \qquad \ldots (1)$$

Auxiliary equation of (1) is $D^2 - (s/2) = 0$ giving $D = \pm\sqrt{(s/2)}$

$\therefore$ Complementary function $= c_1\, e^{x\sqrt{(s/2)}} + c_2 e^{-x\sqrt{(s/2)}}$, c_1, c_2 being arbitrary constants.

And, P.I. $= \dfrac{1}{D^2-(s/2)}(-5\sin 4\pi x) = -5\dfrac{1}{-(4\pi)^2-(s/2)}\sin 4\pi x = \dfrac{10\sin 4\pi x}{32\pi^2+s}$

Hence, the general solution of (1) is

$$\bar{y}(x,s) = c_1 e^{x\sqrt{(s/2)}} + c_2 e^{-x\sqrt{(s/2)}} + (10\sin 4\pi x)/(32\pi^2+s) \quad \ldots (2)$$

Taking Laplace transform of given boundary conditions $y(0, t) = 0$ and $y(5, t) = 0$, we have

$$\bar{y}(0, s) = 0 \quad \ldots (3)$$

and

$$\bar{y}(5, s) = 0. \quad \ldots (4)$$

Putting $x = 0$ in (2) and using (3), we get $c_1 + c_2 = 0.$ ... (5)

Putting $x = 5$ in (2) and using (4), we get $c_1 e^{5\sqrt{(s/2)}} + c_2 e^{-5\sqrt{(s/2)}} = 0.$ ... (6)

Solving (5) and (6), $c_1 = c_2 = 0$ and so from (2), we get

$$\bar{y}(x,s) = (10\sin 4\pi x)/(32\pi^2+s). \quad \ldots (7)$$

Taking the inverse Laplace transforms of both sides of (7), we get

$$\therefore \quad y(x, t) = (10 \sin 4 \pi x)\ L^{-1}\{1/(32\pi^2+s)\} = 10e^{-32\pi^2 t}\ \sin 4\pi x.$$

Ex.4. (*a*) *Find the bounded solution of* $\partial y/\partial t = \partial^2 y/\partial x^2$ $x > 0$, $t > 0$ *given that* $y(0, t) = 1$ *and* $y(x, 0) = 0$. **[Delhi (H) 2001, 03, 05, 06, Meerut 2001, 07, 12; Nagpur 2006]**

Sol. As in Ex, 2, obtain $s\bar{y} - y(x,0) = d^2\bar{y}/dx^2$ or $(d^2\bar{y}/dx^2) - s\bar{y} = 0.$

whose solution is $\bar{y}(x,s) = c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}}$. c_1, c_2 being arbitrary constants. ... (1)

Since $y(x, t)$ must be bounded as $x \to \infty$, we must have $\bar{y}(x, s)$ also bounded as $x \to \infty$. It follows that we must choose $c_1 = 0$ provided $\sqrt{s} > 0$.

Then (1) reduces to $\bar{y}(x,s) = c_2 e^{-x\sqrt{s}}$ ... (2)

Now, $y(0, t) = 1 \Rightarrow L\{y(0, t)\} = L\{1\} \Rightarrow \bar{y}(0, s) = 1/s.$ ... (3)

Putting $x = 0$ in (2) and using (3), we get $c_2 = 1/s.$

$\therefore$ From (2), $\bar{y}(x, s) = (1/s) \times e^{-x\sqrt{s}}.$

$$\therefore \quad y(x,t) = L^{-1}\{(1/s) \times e^{-x\sqrt{s}}\} = erfc(x/2\sqrt{t}).$$ [Refer Ex. 10 (iii), Art 1.14C of Chapter 1]

Ex.4. (*b*) *Solve* $\partial^2 y/\partial t^2 = 9(\partial^2 y/\partial x^2)$, *where* $y(0, t) = 0$, $y(2, t) = 0$ *and* $y(x, 0) = 20 \sin 2\pi x$, $y_t(x, 0) = 0$

Sol. Given that $\partial^2 y/\partial t^2 = 9(\partial^2 y/\partial x^2),$ ... (1)

where $y(0, t) = 0$ and $y(2, t) = 0$ ... (2)

and $y(x, 0) = 20 \sin 2\pi x$ and $y_t(x, 0) = 0.$ ... (3)

Let $L\{y(x, t)\} = \bar{y}(x,s)$. Taking the Laplace transform of both sides of (1), we get

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = 9\,L\left\{\frac{\partial^2 y}{\partial x^2}\right\} \quad \text{or} \quad s^2\bar{y}(x,s) - sy(x,0) - y_t(x,0) = 9\frac{d^2\bar{y}}{dx^2}$$

or $$s^2\,\bar{y} - 20s\sin 2\pi x - 0 = 9(d^2\bar{y}/dx^2), \text{ using (3)}$$

or $$\frac{d^2\bar{y}}{dx^2} - \frac{s^2}{9}\bar{y} = -\frac{20}{9}s\,\sin 2\pi x \qquad \text{or} \qquad \left(D^2 - \frac{s^2}{9}\right)\bar{y} = -\frac{20}{9}s\,\sin\,2\pi x. \qquad \ldots (4)$$

Auxiliary equation of (4) is $D^2 - (s^2/9) = 0$ so that $D = \pm\, s/3$.

$\therefore$ Complementary function = C.F. = $c_1\, e^{(s/3)x} + c_2\, e^{-(s/3)x}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 - (s^2/9)}\left(-\frac{20}{9}s\,\sin\,2\pi x\right) = -\frac{20s}{9}\,\frac{1}{(-2\pi)^2 - (s^2/9)}\,\sin\,2\pi x = \frac{20s\sin 2\pi x}{36\pi^2 + s^2}$$

Hence the general solution of (4) is given by

$$\bar{y}(x,s) = c_1 e^{(s/3)x} + c_2 e^{-(s/3)x} + (20s\sin 2\pi x)/(36\pi^2 + s^2) \qquad \ldots (5)$$

Taking the Laplace transform of the boundary conditions (2), we have

$$\bar{y}(0,s) = 0 \qquad \text{and} \qquad \bar{y}(2,s) = 0. \qquad \ldots (6)$$

Putting $x = 0$ in (5) and using (6), we get $c_1 + c_2 = 0.$ ... (7)

Putting $x = 2$ in (5) and using (6), we get $c_1\, e^{2s/3} + c_2\, e^{-2s/3} = 0.$ ... (8)

Solving (7) and (8), we get $c_1 = c_2 = 0.$

$\therefore$ From (5), we get $\bar{y}\ (x, s) = (20s\,\sin 2\pi x)/(s^2 + 36\pi^2)$ ... (9)

Taking inverse Laplace transform on both sides of (9), we get

$$y(x,t) = 20\sin\,2\pi x\ L^{-1}\left\{\frac{s}{s^2 + (6\pi)^2}\right\} = 20\sin\,2\pi x\ \cos\,6\,\pi t.$$

Ex.5. *Solve* (*a*) $\dfrac{\partial y}{\partial t} = 3\dfrac{\partial^2 y}{\partial x^2}$, *where* $y\left(\dfrac{\pi}{2}, t\right) = 0, \left(\dfrac{\partial y}{\partial x}\right)_{x=0} = 0$ *and* $y\ (x,\ 0) = 30\ \cos\ 5x.$

[Delhi Physics (H) 2005; Meerut 1999, 2000, 06, 2012]

(*b*) $\dfrac{\partial y}{\partial t} = 3\dfrac{\partial^2 y}{\partial x^2}$, where $y\left(\dfrac{\pi}{2}, t\right) = 0, \left(\dfrac{\partial y}{\partial x}\right)_{x=0} = 0$ and $y\ (x,\ 0) = \cos\ 5x.$

Sol (*a*). Given that $\partial y/\partial t = 3\ (\partial^2 y/\partial x^2)$ ... (1)

where $y\ (\pi/2,\ t) = 0,$ ... (2)

$$\left(\partial y/\partial x\right)_{x=0} = 0. \qquad \ldots (3)$$

and $y\ (x,\ 0) = 30\ \cos\ 5x.$ ... (4)

Let L $\{y\ (x,\ t)\}$ = $\bar{y}\ (x,\ s)$. Taking the Laplace transforem of both sides of (1), we get

$$L\left\{\partial y/\partial t\right\} = 3\,L\left\{\partial^2 y/\partial x^2\right\} \qquad \text{or} \qquad s\,\bar{y}\,(x,s) - y\,(x,0) = 3(d^2\bar{y}/dx^2)$$

or $(d^2\bar{y}/dx^2) - (s/3)\bar{y} = -10\,\cos 5x$, using (4).

or $[D^2 - (s/3)]\bar{y}\ = -\ 10\ \cos\ 5x, \qquad$ where $\qquad D \equiv d/dx.$

Auxiliary equation of (5) is $D^2 - (s/3) = 0$ so that $D = \pm\ \sqrt{(s/3)}.$

$\therefore$ Complementary function = C.F. = $c_1\ e^{x\sqrt{(s/3)}} + c_2 e^{-x\sqrt{(s/3)}}$, c_1, c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^2-(s/3)}(-10\cos 5x) = -10\frac{1}{-5^2-(s/3)}\cos x = \frac{30\cos 5x}{75+s}.$$

Hence the general solution of (5) is given by

$$\bar{y}\ (x, s) = c_1 e^{x\sqrt{(s/3)}} + c_2\, e^{-x\sqrt{(s/3)}} + (30\cos 5x)/(75+s). \quad \ldots (6)$$

Taking the Laplace transform of conditions (2) and (3), we get

$$\bar{y}(\pi/2, s) = 0 \quad \ldots (7)$$

and

$$\left(d\,\bar{y}/d\,x\right)_{x=0} = 0. \quad \ldots (8)$$

Putting $x = \pi/2$ in (6) and using (7), we get $c_1 e^{(\pi/2)\sqrt{(s/3)}} + c_2 e^{-(\pi/2)\sqrt{(s/3)}} = 0.$ ... (9)

Differentiating both sides of (6) w.r.t. 'x', we have

$$\frac{d\,\bar{y}}{d\,x} = c_1\sqrt{(s/3)}\ e^{x\sqrt{(s/3)}} - c_2\sqrt{(s/3)}\ e^{-x\sqrt{(s/3)}} - \frac{150\sin 5x}{75+s} \quad \ldots (10)$$

Putting $x = 0$ in (6) and using (8), we get

$0 = \mathrm{C}_1\sqrt{(s/3)} - \mathrm{C}_2\sqrt{(s/3)}$ so that $\mathrm{C}_1 = \mathrm{C}_2.$ ... (11)

Solving (9) and (11), we have $C_1 = C_2 = 0$

$\therefore$ From (6), we get $\bar{y}\ (x, s) = (30\cos 5x)/(75 + s).$ ... (12)

Investing, $y\ (x, t) = 30\cos 5x\ \mathrm{L}^{-1}\{1/(75+s)\} = 30\cos 5x\ e^{-75t}$

(*b*) Proceed as in part (*a*). **Ans.** $y\ (x, t) = e^{-25t}\cos 5x.$

Ex.6. *Solve* $\partial y/\partial t = 2\ (\partial^2 y/\partial x^2)$, $y\ (0, t) = 0$, $y\ (5, t) = 0$, $y\ (x, 0) = 10\sin 4\pi x - 5\sin 6\pi x$. **[Meerut 2002, 04, 08]**

Sol. Given $\partial y/\partial t = 2\ (\partial^2 y/\partial x^2)$ ... (1)

with $y\ (0, t) = 0, \quad y\ (5, t) = 0$ ... (2)

and $y\ (x, 0) = 10\sin 4\pi x - 5\sin 6\pi x.$ ... (3)

Let $\mathrm{L}\ \{y\ (x, t)\} = \bar{y}\ (x, s)$. Taking the Laplace transform of both sides of (1), we have

$$\mathrm{L}\left\{\partial y/\partial t\right\} = 2\ \mathrm{L}\left\{\partial^2 y/\partial x^2\right\} \quad \text{or} \quad s\,\bar{y}(x,s) - y(x,0) = 2(d^2\bar{y}/dx^2)$$

or $s\,\bar{y}(x,s) - (10\sin 4\pi x - 5\sin 6\pi x) = 2(d^2\bar{y}/dx^2)$, using (3)

or $d^2\bar{y}/dx^2 - (s/2)\times\bar{y} = (1/2)\times(5\sin 6\pi x - 10\sin 4\pi x).$

or $[\mathrm{D}^2 - (s/2)]\,\bar{y} = (1/2)\times(5\sin 6\pi x - 10\sin 4\pi x), \quad$ where $\mathrm{D} \equiv d/dx.$ ... (4)

Auxiliary equation of (4) is $\mathrm{D}^2 - (s/2) = 0$ so that $\mathrm{D} = \pm\sqrt{(s/2)}$.

$\therefore$ Complementary function $= c_1 e^{x\sqrt{(s/2)}} + c_2\, e^{-x\sqrt{(s/2)}}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{\mathrm{D}^2-(s/2)}\,\frac{1}{2}\,(5\sin 6\pi x - 10\sin 4\pi x) = \frac{5}{2}\,\frac{1}{-(6\pi)^2-(s/2)}\sin 6\pi x - 5\frac{1}{-(4\pi)^2-(s/2)}\sin 4\pi x$$

Hence the general solution of (4) is

$$\bar{y}(x,s) = c_1 e^{x\sqrt{(s/2)}} + c_2\, e^{-x\sqrt{(s/2)}} - \frac{5\sin 6\pi x}{72\,\pi^2+s} + \frac{10\sin 4\pi x}{32\,\pi^2+s} \quad \ldots (5)$$

Taking the Laplace transform of both sides of conditions (2), we get

$$\bar{y}(0,s) = 0 \quad \text{and} \quad \bar{y}(5,s) = 0. \quad \ldots (6)$$

Putting $x = 0$ in (5) and using (6), we get $c_1 + c_2 = 0$ ... (7)

Putting $x = 5$ in (5) and using (6), we get $c_1 e^{5\sqrt{(s/2)}} + c_2 e^{-5\sqrt{(s/2)}} = 0.$ ... (8)

Solving (7) and (8), we have $c_1 = c_2 = 0.$

$\therefore$ From (5), we get $\bar{y}\ (x, s) = \dfrac{10 \sin 4\pi x}{32\ \pi^2 + s} - \dfrac{5 \sin 6\pi x}{72\,\pi^2 + s}.$

Inverting $y\ (x, t) = 10 \sin 4\pi x\ e^{-32\pi^2 t} - 5 \sin 6\pi x\ e^{-72\pi^2 t}.$

Ex.7. Solve $\partial y/\partial t = 3\ (\partial^2 y/\partial x^2)$, $y_x\ (0, t) = 0$, $y\ (\pi/2, t) = 0$ and $y\ (x, 0) = 20 \cos 3x - 5 \cos 9x$. **(Osmania 2004)**

Sol. Given $\partial y / \partial t = 3\ (\partial^2 y/\partial x^2)$ ... (1)

with $y_x\ (0, t) = \left(\partial y / \partial x\right)_{x=0} = 0, \qquad y(\pi/2, t) = 0$ ... (2)

and $y\ (x, 0) = 20 \cos 3x - 5 \cos 9x.$ ... (3)

Let L $\{y\ (x, t)\} = \bar{y}\ (x, s)$. Taking the Laplace transform of both sides of (1), we have

$$L\{\partial y / \partial t\} = 3L\{\partial^2 y / \partial x^2\} \quad \text{or} \quad s\,\bar{y}(x,s) - y(x,0) = 3(d^2\bar{y}/dx^2)$$

or $s\,\bar{y}\ (x, s) - (20 \cos 3x - 5 \cos 9x) = 3\ (d^2\,\bar{y}\ /dx^2)$, using (3)

or $(d^2\bar{y}/dx^2) - (s/3) \times \bar{y} = (1/3) \times (5 \cos 9x - 20 \cos 3x)$

or $[\mathrm{D}^2 - (s/3)]\ \bar{y}\ = (5/3) \times \cos 9x - (20/3) \times \cos 3x.$ ... (4)

Auxiliary equation of (4) is $\mathrm{D}^2 - (s/3) = 0$ so that $\mathrm{D} = \pm\ \sqrt{(s/3)}$.

$\therefore$ Complementary function $= c_1\, e^{x\sqrt{(s/3)}} + c_2 e^{-x\sqrt{(s/3)}}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{\mathrm{D}^2 - (s/3)}\left[\frac{5}{3}\cos 9x - \frac{20}{3}3x\right]$$

$$= \frac{5}{3}\,\frac{1}{-9^2 - (s/3)}\cos 9x - \frac{20}{3}\,\frac{1}{-3^2 - (s/3)}\ \cos 3x = \frac{20 \cos 3x}{27 + s} - \frac{5\cos 9x}{243 + s}.$$

Hence the general solution of (4) is given by

$$\bar{y}\ (x, s) = c_1\, e^{x\sqrt{(s/3)}} + c_2 e^{-x\sqrt{(s/3)}} + \frac{20\cos 3x}{27 + s} - \frac{5\cos 9x}{243 + s}. \quad \ldots (5)$$

Taking the Laplace transform of the conditions (2), we have

$$\left(d\,\bar{y}/dx\right)_{x=0} = 0 \quad \ldots (6)$$

and $\bar{y}\ (\pi/2, s) = 0.$ ... (7)

Putting $x = \pi/2$ in (5) and using (7), we have $c_1\, e^{(\pi/2)\sqrt{(s/3)}} + c_2\, e^{-(\pi/2)\sqrt{(s/3)}} = 0.$ ... (8)

Differentiations both sides of (5) w.r.t. 'x', we get

$$d\,\bar{y}/dx = c_1\sqrt{(s/3)}\ e^{x\sqrt{(s/3)}} - c_2\,\sqrt{(s/3)}\ e^{-x\sqrt{(s/3)}} - (60\sin 3x)/(27 + s) - (45\sin 9x)/(243 + s) \quad \ldots (9)$$

Putting $x = 0$ in (9) and using (6), we get

$$0 = c_1\sqrt{(s/3)} - c_2\sqrt{(s/3)} \qquad \text{so that} \qquad c_1 = c_2. \qquad \ldots (10)$$

Solving (8) and (10), we get $c_1 = c_2 = 0$.

$\therefore$ From (5), we get $\bar{y}(x, s) = (20\cos 3x)/(27+s) - (45\sin 9x)/(243+s)$.

Inverting, $y(x, t) = 20\cos 3x\ e^{-27t} - 5\cos 9x\ e^{-243t}$.

Ex.8. *Find the bounded solution y (x, t), 0 < x < 1, t > 0 of the boundary value problem* $(\partial y/\partial x) - (\partial y/\partial t) = 1 - e^{-t}, y(x,0) = x.$ **[Meerut 2007; S.V. University (A.P.) 1997]**

Sol. Given that $(\partial y/\partial x) - (\partial y/\partial t) = 1 - e^{-t}, \qquad \ldots (1)$

with $y(x, 0) = x. \qquad \ldots (2)$

Let $L\{y(x, t)\} = \bar{y}(x, s)$. Taking the Laplace transform of both sides of (1), we get

$$L\left\{\frac{\partial y}{\partial x}\right\} - L\left\{\frac{\partial y}{\partial t}\right\} = L\{1\} - L\{e^{-t}\} \qquad \text{or} \qquad \frac{d\bar{y}}{dx} - [s\bar{y}(x,s) - y(x,0)] = \frac{1}{s} - \frac{1}{s+1}$$

or $$\frac{d\bar{y}}{dx} - s\bar{y} + x = \frac{1}{s(s+1)} \qquad \text{or} \qquad \frac{d\bar{y}}{dx} - s\bar{y} = \frac{1}{s(s+1)} - x, \qquad \ldots (3)$$

which is linear equation. Its I.F. $= e^{-\int s\,dx} = e^{-sx}$. Hence the general solution of (3) is given by

$$\bar{y}\,e^{-sx} = \int \left[\frac{1}{s(s+1)} - x\right] e^{-sx}dx + c = \frac{1}{s(s+1)}\int e^{-sx}dx - \int x\,e^{-sx}dx + c$$

$$= \frac{1}{s(s+1)}\left[\frac{e^{-sx}}{-s}\right] - \left[x\left(\frac{e^{-sx}}{-s}\right) - \int 1\cdot\left(\frac{e^{-sx}}{-s}\right)dx\right] + c$$

$$\therefore \quad y\,e^{-sx} = -\frac{e^{-sx}}{s^2(s+1)} + \frac{x\,e^{-sx}}{s} - \frac{1}{s}\left[\frac{e^{-sx}}{-s}\right] + c, \ c \text{ being constant of integration}$$

or $$\bar{y}(x, s) = \frac{x}{s} - \frac{1}{s^2(s+1)} + \frac{1}{s^2} + c\,e^{sx}. \qquad \ldots (4)$$

Since we require bounded solution, it follows that $u(x, t)$ is bounded for all x and t and hence $\bar{u}(x, s)$ must be bounded, *i.e.*, $\bar{u}(x, s)$ is finite as $x \to \infty$. Hence we must take $c = 0$ in (4). Then, (4) reduces to

$$\bar{y}(x, s) = \frac{x}{s} + \frac{1}{s^2}\left[1 - \frac{1}{1+s}\right] = \frac{x}{s} + \frac{1}{s(s+1)} = \frac{x}{s} + \frac{1}{s} - \frac{1}{1+s}.$$

Inverting, $$y(x, t) = x\,L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{1+s}\right\} = x + 1 - e^{-t}.$$

Ex.9. *Solve the boundary value problem* $\partial u/\partial x + 4(\partial u/\partial t) = -8t$, $t > 0 > x > 0$, with the conditions $u = 0$, when $t = 0$, $x > 0$; $u = 2t^2$, when $x = 0$, $t > 0$.

Sol. given that $(\partial u/\partial x) + 4(\partial u/\partial t) = -8t, \qquad \ldots (1)$

with conditions: $u(x, 0) = 0 \qquad \ldots (2)$

and $u(0, t) = 2t^2. \qquad \ldots (3)$

Let L $\{u(x, t)\} = \bar{u}(x, s)$. Taking the Laplace transform of both sides of (1), we have

$$L\{\partial u/\partial y\} + 4\,L\{\partial u/\partial t\} = -8\,L\{t\}$$

or $$d\bar{u}/dx + 4[s\bar{u}(x,s) - u(x,0)] = -8/s^2$$

or $$(d\bar{u}/dx) + 4s\bar{u} = -8/s^2, \text{ which is linear equation.} \quad \ldots (3)$$

I.F. (integrating factor) of (3) $= e^{\int (4s)dx} = e^{4sx}$ and so solution of (3) is given by

$$\bar{u}\,e^{4sx} = \int (-8/s^2)e^{4sx}dx = c - \frac{2}{s^3}e^{4sx} \quad \text{or} \quad \bar{u}(x,s) = c\,e^{-4sx} - (2/s^3). \quad \ldots (4)$$

Now, (3) $\Rightarrow$ L $\{u(0, t)\} = 2\,L\{t^2\}$ or $\bar{u}(0, s) = 2 \times (2!/s^3) = 4/s^3$. ... (5)

Putting $x = 0$ in (4) and using (5), we get $4/s^3 = c - (2/s^3)$ so that $c = 6/s^3$.

Then, from (4), we get $\bar{u}(x, s) = (6/s^3) \times e^{-4sx} - (2/s^3)$.

Inverting, $$u(x, t) = 6L^{-1}\{(1/s^3) \times e^{-4sx}\} - 2\,L^{-1}\{1/s^3\}. \quad \ldots (6)$$

Now, $$L^{-1}\{(1/s^3)\} = t^2/2! = t^2/2. \quad \ldots (7)$$

Again by second shifting theorem (refer Art. 2.9 of chapter 2) we have

$L^{-1}\{(1/s^3) \times e^{-4sx}\} = (1/2) \times (t-4x)^2\,H(t-4x)$, where $H(t-4x)$ is Heaviside's unit step function ... (8)

Using (7) and (8), (6) reduces to $u(x, t) = 3\,(t-4x)^2\,H(t-4x) - t^2$.

Ex.10. *Solve the boundary value problem* $\partial^2 u/\partial t^2 = a^2\,(\partial^2 u/\partial x^2)$, $t > 0$, $x > 0$ *where* $u(x, 0) = 0$, $x \geq 0$, $u_t(x, 0) = 0$, $x > 0$ $u(0, t) = t$, $\lim_{x\to\infty} u(x,t) = 0, t \geq 0$. **[Osmania 2010]**

Sol. Given that $$\partial^2 u/\partial t^2 = a^2\,(\partial^2 u/\partial x^2), \quad \ldots (1)$$

with conditions: $$u(x, 0) = 0, \qquad u_t(x, 0) = 0, \quad \ldots (2)$$

$$u(x, t) = 0, \text{ as } x \to \infty \quad \ldots (3)$$

and $$u(0, t) = t. \quad \ldots (4)$$

Let L $\{u(x, t)\} = \bar{u}(x, s)$. Taking the Laplace transform of both sides of (1), we have

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = a^2\,L\left\{\frac{\partial^2 u}{\partial x^2}\right\} \quad \text{or} \quad s^2\,\bar{u}(x, s) - s\,u(x, 0) - u_t(x, 0) = a^2\,\frac{d^2\bar{u}}{dx^2}$$

or $(d^2\bar{u}/dx^2) - (s^2/a^2)\bar{u} = 0$ or $[D^2 - (s/a)^2]\bar{u} = 0$, by (2). ... (5)

Auxiliary equation of (5) is $D^2 - (s/a)^2 = 0$ so that $D = \pm s/a$. The general solution of (5) is

$$\bar{u}(x, s) = c_1\,e^{x(s/a)} + c_2\,e^{-x(s/a)}, \; c_1, c_2 \text{ being arbitrary constant.} \quad \ldots (6)$$

Now, (3) $\Rightarrow \bar{u}(x, s) = 0$ as $x \to \infty$. Hence we must take $c_1 = 0$ in (6).

$\therefore$ (6) gives $$\bar{u}(x, s) = c_2\,e^{-x(s/a)}. \quad \ldots (7)$$

Now, (4) $\Rightarrow$ L $\{u(0, t)\} = L\{t\}$ $\Rightarrow$ $\bar{u}(0, s) = 1/s^2$. ... (8)

Putting $x = 0$ in (7) and using (8), we get $\bar{u}(0, s) = c_2$ or $c_2 = 1/s^2$.

Then, from (7), we get $\bar{u}(x, s) = (1/s^2) \times e^{-(x/a)s}$.

Inverting, $$u(x, t) = L^{-1}\left\{(1/s^2) \times e^{-(x/a)s}\right\}. \quad \ldots (9)$$

Since $L^{-1}\{1/s^2\} = t$, so by second shifting theorem (Refer Art. 2.9 chapter 2), we get

$u(x, t) = (t - x/a)\,H(t - x/a)$, where $H(t - x/a)$ is Heaviside's unit step function

Ex.11. *Solve the boundary value problem, $\partial^2 u/\partial x^2 = (1/c^2)(\partial^2 u/\partial t^2)$, $t > 0$, $x > 0$ with the conditions (i) $u = E \sin at$, when $x = 0$, $t > 0$ (ii) u is finite when $x \to \infty$ (iii) $u = \partial u/\partial t = 0$, when $t = 0$, $x > 0$.*

Sol. Given that $\partial^2 u/\partial x^2 = (1/c^2)(\partial^2 u/\partial t^2)$, ... (1)

with conditions: (*i*) $u(0, t) = \mathrm{E} \sin at$ ... (2)

(*ii*) $u(x, t)$ is finite when $x \to \infty$ ... (3)

(*iii*) $u(x, 0) = 0$ and $u_t(x, 0) = 0$. ... (4)

Let $\mathrm{L}\{u(x, t)\} = \bar{u}(x, s)$. Taking the Laplace Transform of both sides of (1), we have

$$\mathrm{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{1}{c^2}\mathrm{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} \quad \text{or} \quad \frac{d^2\bar{u}}{dx^2} = \frac{1}{c^2}[s^2\bar{u}(x,s) - su(x,0) - u_t(x,0)]$$

or $(d^2\bar{u}/dx^2) - (s^2/c^2)\bar{u} = 0$ or $[\mathrm{D}^2 - (s/c)^2]\,\bar{u} = 0$, by (4) ... (5)

Auxiliary equation of (5) is $\mathrm{D}^2 - (s/c)^2 = 0$ so that $\mathrm{D} = \pm (s/c)$. The general solution of (5) is

$$\bar{u}(x,s) = c_1 e^{x(s/c)} + c_2 e^{-x(s/c)},\ c_1, c_2 \text{ being arbitrary constants.} \quad \text{... (6)}$$

Now, (3) $\Rightarrow$ $\bar{u}(x, s) \to$ finite quantity when $x \to \infty$.

Hence we must take $c_1 = 0$ in (6). Hence, from (6) $\bar{u}(x,s) = c_2 e^{-x(s/c)}$. ... (7)

Again, (2) $\Rightarrow$ $\mathrm{L}\{u(0, t)\} = \mathrm{E}\,\mathrm{L}\{\sin at\}$ $\Rightarrow$ $\bar{u}(0,s) = \mathrm{E}a/(s^2 + a^2)$. ... (8)

Putting $x = 0$ in (7) and using (8), we have $c_2 = (\mathrm{E}a)/(s^2 + a^2)$.

Hence, from (7), we get $\bar{u}(x,s) = \{Ea/(s^2 + a^2)\}e^{-(x/c)s}$.

Inverting, $$u(x, t) = \mathrm{E}a\,\mathrm{L}^{-1}\left\{\frac{1}{s^2 + a^2}\, e^{-(x/c)s}\right\}. \quad \text{... (9)}$$

Now, $\mathrm{L}^{-1}\{1/(s^2 + a^2)\} = (1/a) \times \sin at$.

Hence by second shifting theorem (refer Art.2.9 of chapter 2), (9) gives

$$u(x, t) = \mathrm{E} \sin a(t - x/c)\,\mathrm{H}(t - x/c).$$

Ex.12. *Solve the following boundary value problem $\partial^2 u/\partial t^2 = a^2(\partial^2 u/\partial x^2)$, $x > 0$, $t > 0$. $u_x(0, t) = A \sin wt$, $u(x, 0) = 0$, $u_t(x, 0) = 0$, $|u(x, t)| < M$.*

Sol. Given that $\partial^2 u/\partial t^2 = a^2(\partial^2 u/\partial x^2)$ ... (1)

with conditions $u(x, 0) = 0$, $u_t(x, 0) = 0$, ... (2)

$|u(x, t)| < M$ ... (3)

and $u_x(0, t) = (\partial u/\partial x)_{x=0} = \mathrm{A} \sin wt$. ... (4)

Let $\mathrm{L}\{u(x, t)\} = \bar{u}(x, s)$. Taking the Laplace transform of both sides of (1), we have

$$\mathrm{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = a^2\,\mathrm{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} \quad \text{or} \quad s^2\bar{u}(x,s) - u(x,0) - u_t(x,0) = a^2\frac{d^2\bar{u}}{dx^2}$$

or $(d^2\bar{u}/dx^2) - (s^2/a^2)\bar{u} = 0$ or $\left[\mathrm{D}^2 - (s^2/a^2)\right]\bar{u} = 0$, using (2) ... (5)

Its auxiliary equation is $\mathrm{D}^2 - (s/a)^2 = 0$ so that $\mathrm{D} = \pm s/a$. The general solution of (5) is

$$\bar{u}(x,s) = c_1 e^{x(s/a)} + c_2 e^{-x(s/a)},\ c_1, c_2 \text{ being arbitrary constants} \quad \text{... (6)}$$

Now, (3) $\Rightarrow$ $u(x, t)$ is bounded $\Rightarrow$ $\bar{u}(x, s)$ is also bounded

Thus, $\bar{u}(x,s) \to a$ finite quantity as $x \to \infty$. So take $c_1 = 0$ in (6). Then, (6) gives

$$\bar{u}(x, s) = c_2 e^{-x(s/a)}. \quad \ldots (7)$$

$$(4) \Rightarrow \quad \mathrm{L}\left\{\left(\frac{\partial u}{\partial x}\right)_{x=0}\right\} = \mathrm{A}\ \mathrm{L}\{\sin wt\} \quad \Rightarrow \quad \left(\frac{d\bar{u}}{dx}\right)_{x=0} = \frac{\mathrm{A}w}{s^2 + w^2} \quad \ldots(8)$$

Differentiating both sides of (7) w.r.t. 'x', we get $d\bar{u}/dx = -(s/a)c_2 e^{-x(s/a)}$. ... (9)

Putting $x = o$ in (9), $\left(\frac{d\bar{u}}{dx}\right)_{x=0} = -\frac{sc_2}{a}$ or $\frac{\mathrm{A}w}{s^2 + w^2} = -\frac{sc_2}{a}$, by (8)

$\therefore$ $c_2 = -(\mathrm{A}\ aw)/s\ (s^2 + w^2)$.

Then, from (7), we get $\bar{u}(x,s) = -\frac{\mathrm{A}aw}{s(s^2 + w^2)} e^{-x(s/a)}$.

Inverting, $u(x, t) = -\mathrm{A}aw\ \mathrm{L}^{-1}\left\{\frac{1}{s(s^2 + w^2)} e^{-(x/a)s}\right\}$. ... (10)

Now, $\mathrm{L}^{-1}\left\{\frac{1}{s(s^2 + w^2)}\right\} = \mathrm{L}^{-1}\left[\frac{1}{w^2}\left\{\frac{1}{s} - \frac{s}{s^2 + w^2}\right\}\right]$, on resolving into partial fractions

$$= \frac{1}{w^2}\left[\mathrm{L}^{-1}\left\{\frac{1}{s}\right\} - \mathrm{L}^{-1}\left\{\frac{s}{s^2 + w^2}\right\}\right] = \frac{1}{w^2}(1 - \cos wt).$$

So by second shifting theorem (refer Art. 2.9 of chapter 2), (10) gives

$$u(x, t) = -(\mathrm{A}a/w) \times \{1 - \cos w(t - x/a)\}\ \mathrm{H}(t - x a).$$

Ex.13. *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k) \times (\partial u/\partial t)$ *with the conditions* $u = 0$, *when* $x = \infty$, $u = 0$ *when* $t = 0$. **[Meerut 1999]**

Sol. Given that $\partial^2 u/\partial x^2 = (1/k) \times (\partial u/\partial t)$ ... (1)

with conditions $u(x, 0) = 0$ ... (2)

and $u(x, t) = 0$, as $x \to \infty$. ... (3)

Let $\mathrm{L}\{u(x, t)\} = \bar{u}(x,0)$. Taking the Laplace transform of both sides of (1), we have

$$\mathrm{L}\left\{\partial^2 u/\partial x^2\right\} = (1/k) \times \mathrm{L}\left\{\partial u/\partial t\right\} \quad \text{or} \quad d^2\bar{u}/dx^2 = (1/k) \times [s\bar{u}(x,s) - u(x,0]$$

or $(d^2\bar{u}/dx^2) - (s/k)\bar{u} = 0$ or $[\mathrm{D}^2 - (s/k)]\bar{u} = 0$, by (2). ... (4)

Auxiliary equation of (4) is $\mathrm{D}^2 - (s/k) = 0$ so that $\mathrm{D} = \pm\sqrt{(s/k)}$. Then, solution of (4) is

$$\bar{u}(x,s) = c_1 e^{x\sqrt{(s/k)}} + c_2 e^{-x\sqrt{(s/k)}},\ c_1, c_2 \text{ being arbitrary constants.} \quad \ldots (5)$$

From (3), $\bar{u}(x,s) = 0$ as $x \to \infty$. Hence, we must take $c_1 = 0$ in (5).

Then, (5) gives $\bar{u}(x,s) = c_2 e^{-x\sqrt{(s/k)}}$. ... (6)

Taking inverse Laplace transform of both sides of (6), we get

$$u(x,t) = c_2\, L^{-1}\left\{e^{-x\sqrt{(s/k)}}\right\} = c_2\ \mathrm{L}^{-1}\left\{1 - \frac{x}{\sqrt{k}}s^{1/2} + \frac{x^2}{2!k}s - \frac{x^3}{3!k^{3/2}}s^{3/2} + \frac{x^4}{4!k^2}s^2 - \frac{x^5}{5!k^{5/2}}s^{5/2} + \ldots.\right\}$$

$$= c_2\left[0 - \frac{x}{\sqrt{k}}\frac{(-1)^1}{\sqrt{\pi}}\frac{1}{2}t^{-3/2} + 0 - \frac{x^3}{3!k^{3/2}}\frac{(-1)^2}{\sqrt{\pi}}\frac{1}{2}\cdot\frac{3}{2}\cdot t^{-5/2} + 0 - \frac{x^5}{5!k^{3/2}}\cdot\frac{(-1)^3}{\sqrt{\pi}}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot t^{-7/2} + \ldots.\right]$$

$$\left[\because \mathrm{L}^{-1}\{s^{n+(1/2)}\} = \frac{(-1)^{n+1}}{\sqrt{\pi}}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\ldots.\left(\frac{2n+1}{2}\right)t^{-n-(3/2)}\right.$$

and $\mathrm{L}^{-1}\{s^n\} = 0$, if n is zero or a positive integer]

$$= \frac{c_2 x}{\sqrt{k}}\frac{1}{2\sqrt{(\pi t^3)}}\left[1 - \frac{(x^2/4kt)}{1!} + \frac{(x^2/4kt)^2}{2!} - \ldots.\right] = \frac{c_2 x}{\sqrt{k}} \times \frac{1}{2\sqrt{(\pi t^3)}}e^{-(x^2/4kt)}$$

EXERCISE 5(A)

Solve the following partial differential equation by Laplace transform method:

1. $\partial u/\partial t = (\partial^2 u/\partial x^2) - 4u$, given that $u(0, t) = 0$, $u(\pi, t) = 0$ and $u(x, 0) = 3 \sin x - 4 \sin 2x$.

[Delhi 2001, 06; Nagpur 2006]

2. $\partial y/\partial t = \partial^2 y/\partial x^2$, where $y(\pi/2, t) = 0$, $(\partial y/\partial x)_{x=0} = 0$ and $y(x, 0) = \cos 5x$.

[Agra 2001, 06; Kanpur 2001, 03, Nagpur 2001, 06]

3. $\partial u/\partial t = 2(\partial^2 u/\partial x^2)$, given that $u(0, t) = 0$, $y(5, t) = 0$ and $y(x,0) = \sin \pi x$

[Allahabad 2003, 06, Osmania 2006]

4. $\partial y/\partial t = 2(\partial^2 y/\partial x^2)$, given that $y(0, t) = 0$, $y(5, t) = 0$ and $y(x, 0) = 10 \sin 4\pi x - 5 \sin 6\pi x$

[Meerut 2002, 04]

5. $\partial u/\partial t = 9(\partial^2 u/\partial x^2)$ given that $u(0, t) = 0$, $u(2, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = 10 \sin 2\pi x - 20 \sin 5\pi x$. **[Aligarh 2001, 05, M.K.U. Tamil Nadu 2006]**

6. $\partial y/\partial t = \partial^2 y/\partial x^2$, $y(x, 0) = 3 \sin 2\pi x$, $y(0, t) = 0 = y(1, t)$, $t > 0$, $0 < x < 1$.

[Osmania 2010; Meerut 2006]

7. $\partial y/\partial t = (\partial^2 y/\partial x^2) - 4y$, $y(0, t) = 0$, $y(\pi, t) = 0$, $y(x, 0) = 6 \sin x - 4 \sin 2x$.

[Meerut 1995]

8. $\partial y/\partial t = 2(\partial^2 y/\partial x^2)$, $y(0, t) = 0$, $y(5, t) = 0$, $y(x, 0) = -5 \sin 6\pi x$.

ANSWERS

1. $u(x, t) = 3e^{-5t} \sin x - 4e^{-8t} \sin 2x$.

2. $y(x, t) = e^{-25t} \cos 5x$

3. $u(x,t) = e^{-2\pi^2 t} \sin \pi x$

4. $y(x,t) = 5e^{-72\pi^2 t} \sin 6\pi x + 10e^{-32\pi^2 t} \sin 4\pi x$

5. $u(x,t) = 10\cos 6\pi t \sin 2\pi x - 20\cos 15\pi t \sin 5\pi x$

6. $y(x, t) = 3e^{-4\pi^2 t} \sin 4\pi x$

7. $y(x, t) = 6e^{-5t} \sin x - 4e^{-8t} \sin 2x$

8. $y(x,t) = -5e^{-12\pi^2 t} \sin 6\pi x$

5.3. Summary of some important partial differential equations

(Commit to memory all equations)

1. One dimensional heat conduction or diffusion equation $\partial u/\partial t = k(\partial^2 u/\partial x^2),\ 0 < x < \infty,\ t > 0.$

Here $u(x, t)$ is the temperature in a solid at position x at time t. The constant k is called the diffusivity of the material of the solid. Again $k = K/\sigma\rho$, where the thermal conductivity K, the specific heat σ and the density ρ are assumed constant. The amount of heat per unit area per unit time conducted across a plane is given by $-K\, u_x(x, t)$.

2. One dimensional wave equation $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2),\ x > 0,\ t > 0.$

This equation is applicable to the small transverse viberations of a taut flexible string initially located on the x-axis and set into motion. Here $u(x, t)$ is the transverse displacement of the string at any time t. Again $c^2 = T/\rho$, where T is constant tension in the string and ρ is constant mass per unit length of the string.

3. Two dimensional heat conduction equation $\partial u/dt = k(\partial^2 u/dx^2 + \partial^2 u/\partial y^2),$

$u(x, y, t)$ being temperature at any point (x, y) at any time t.

4. Three dimensional heat conduction equation. $\partial u/\partial t = k(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2)$

5. Two dimensional wave equation. $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2),$

$u(x, y, t)$ being displacement at any point (x, y) at any time t.

6. Three dimensional wave equation. $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2),$

$u(x, y, z, t)$ being displacement at any point (x, y, z) at any time t.

7. Laplace's equation. Two dimensional : $\partial^2/\partial x^2 + \partial^2 u/\partial y^2 = 0$

Three dimensional : $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$

Remark. In steady state of heat conduction $\partial u/\partial t = 0$ and hence two dimensional and three dimensional heat conduction equation reduce to two dimensional and three dimensional Laplace's equation respectively.

5.4. Solution of one-dimensional heat conduction or diffusion equation.

Ex.1. *Find the temperature u (x, t) in a slab whose ends x = 0 and x = a are kept at temperature zero and whose initial temperature is sin (πx/a).*

Sol. We have to solve one-dimensional heat conduction equation

$$\partial u/\partial t = k\,(\partial^2 u/\partial x^2),\ 0 < x < a,\ t > 0, \qquad \ldots (1)$$

$u(x, t)$ being the temperature in the slab at any point x at any time t and k being the diffusivity of the material of the bar.

subject to the boundary conditions: $u(0, t) = 0$ and $u(a, t) = 0.$ ... (2)

and initial condition: $u(x, t) = \sin(\pi x/a).$ ... (3)

Let $L\{u(x, t)\} = \bar{u}(x, s)$. Taking the Laplace transform of both sides of (1), we have

$$L\left\{\frac{\partial u}{\partial t}\right\} = k\ L\left\{\frac{\partial^2 u}{\partial x^2}\right\} \qquad \text{or} \qquad s\,\bar{u}(x,s) - u(x,0) = k\,\frac{d^2\bar{u}}{dx^2}$$

$$\text{or} \qquad s\,\bar{u} - \sin\left(\frac{\pi x}{a}\right) = k\,\frac{d^2\bar{u}}{dx^2} \qquad \text{or} \qquad \left(D^2 - \frac{s}{k}\right)\bar{u} = -\frac{1}{k}\sin\frac{\pi x}{a}, \text{ by (3)} \qquad \ldots (4)$$

Auxiliary equation of (4) is $D^2 - (s/k) = 0$ so that $D = \pm \sqrt{(s/k)}$.

Hence C.F of (4) = $c_1 e^{x\sqrt{(s/k)}} + c_2 e^{-x\sqrt{(s/k)}}$, c_1 and c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^2-(s/k)}\left[-\frac{1}{k}\sin\frac{\pi x}{a}\right] = -\frac{1}{k}\,\frac{1}{(-\pi^2/a^2)-(s/k)}\sin\frac{\pi x}{a} = \frac{\sin(\pi x/a)}{(\pi^2 k/a^2)+s}$$

Hence the general solution of (4) is

$$\bar{u}(x,s) = c_1 e^{x\sqrt{(s/k)}} + c_2 e^{-x\sqrt{(s/k)}} + \frac{1}{s+(\pi^2 k/a^2)}\sin\frac{\pi x}{a}. \qquad \ldots (5)$$

Taking the Laplace transform of boundary conditions (2), we have

$$\bar{u}(0, s) = 0 \quad \text{and} \quad \bar{u}(a, s) = 0. \qquad \ldots (6)$$

Putting $x = 0$ in (5) and using (6), we get $c_1 + c_2 = 0.$ ... (7)

Putting $x = a$ in (5) and using (6), we get $c_1\, e^{a\sqrt{(s/k)}} + c_2 e^{-a\sqrt{(s/k)}} = 0.$ ... (8)

Solving (7) and (8), $c_1 = c_2 = 0$ and so (5) reduces to

$$\bar{u}(x,s) = \frac{\sin(\pi x/a)}{s+(\pi^2 k/a^2)} \quad \text{so that} \quad u(x, t) = \sin\frac{\pi x}{a}\; L^{-1}\left\{\frac{1}{s+(\pi^2 k/a^2)}\right\}$$

Thus, $u(x, t) = \sin(\pi x/a)\; e^{-(\pi^2 kt)/a^2}.$

Ex.2. *A semi-infinite solid $x > 0$ is initially at temperature zero. At time $t = 0$, a constant temperature $u_0 > 0$ is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid at later time $t > 0$.* **[Aligarh 2002, 06; Allahabad 2004; Himanchal 2001, 05, Kurukshetra 2003, Meerut 2001, Rajasthan 2001, 04, Delhi Physics (H) 2000]**

Sol. We have to solve one-dimensional heat equation

$$\partial u/\partial t = k\,(\partial^2 u/\partial x^2), \qquad 0 < x < \infty, \qquad t > 0, \qquad \ldots (1)$$

where $u(x, t)$ is the temperature in the solid at any point x at time t and k is called the diffusivity of the material of the bar.

subject to the boundary condition: $u(0, t) = u_0$ ... (2)

and the initial condition: $u(x, 0) = 0$ ... (3)

Let $L\{u(x, t)\} = \bar{u}(x, s)$. Applying Laplace transform to (1), we get

$$s\bar{u} - u(x,0) = k(d^2\bar{u}/dx^2) \quad \text{or} \quad (d^2\bar{u}/dx^2) - (s/k)\bar{u} = 0 \text{ whose general solution is}$$

$$\bar{u}(x.s) = c_1 e^{x\sqrt{(s/k)}} + c_2 e^{-x\sqrt{(s/k)}},\ c_1, c_2 \text{ being arbitrary constants} \qquad \ldots (4)$$

In the present physical problem, since the temperature $u(x, t)$ must be bounded for all x and t, so $\bar{u}(x,s)$ must also be bounded. Hence from (4), it follows that we must choose $c_1 = 0$.

Then, (4) reduces to $\bar{u}(x, s) = c_2 e^{-x\sqrt{(s/k)}}$ ... (5)

From (2), $u(0, t) = u_0 \Rightarrow L\{u(0, t)\} = L\{u_0\} \Rightarrow \bar{u}(0, s) = u_0/s$ ... (6)

Putting $x = 0$ in (5) and using (6), we get $c_2 = u_0/s$

$\therefore$ From (5), we get $\bar{u}(x, s) = u_0\,\{e^{-x\sqrt{(s/k)}}/s\}$

$\therefore$ $u(x, t) = u_0\, L^{-1}\{e^{-x\sqrt{(s/k)}}/s\} = u_0\ \text{erfc}\{(x/2)\times\sqrt{(kt)}\}.$

(Refer Ex. 10(iii), Art 1.14C of chapter 1)

Ex.3. *A semi-infinite solid $x > 0$ is initially at temperature zero, At time $t = 0$, a constant temperature $u_0 > 0$ is applied and maintained at the face $x = 0$. Show that the temperature at any point of the solid at $t = 1/4$ is given by u_0 erfc $(x/\sqrt{k})$.*

[**Hint:** Proceed as in Ex.2. and obtain $u(x, t) = u_0$ erfc $[(x/2)\times\sqrt{(kt)}]$.

Hence tempeature at $t = 1/4$ is given by $u(x, t) = u_0$ erfc $(x/\sqrt{k})$.]

Ex. 4. *A bar of lenght of l is at constant temperature u_0. At $t = 0$ the end $x = 1$ is suddenly given the constant temperature u_1 and the end $x = 0$ is insulated. Assuming that the surface of the bar is insulated, find the temperature at any point x of the bar at any time $t > 0$, given that*

$$L^{-1}\left\{\frac{\cosh x\sqrt{s}}{\cosh a\sqrt{s}}\right\} = 1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{(2n-1)^2\pi^2 t/4a^2}\cos\frac{(2n-1)\pi x}{2a}$$

Sol. We have to solve one-dimensional heat equation

$$\partial u/\partial t = k(\partial^2 u/\partial x^2),\ 0 < x < l,\ t > 0 \qquad \ldots (1)$$

subject to the boundary conditions: $u(l, t) = u_1, \quad u_x(0, t) = 0$ ... (2)

and initial condition : $u(x, 0) = u_0,$ ... (3)

Let $\mathrm{L}\{u(x,t)\} = \bar{u}(x,s)$. Applying Laplace transform to (1), we get

$$s\bar{u} - u(x,0) = k(d^2\bar{u}/dx^2) \quad \text{or} \quad \{D^2 - (s/k)\}\bar{u} = -u_0/k$$

Its auxiliary equation is $D^2 - (s/k) = 0$ giving $D = \pm\sqrt{(s/k)}$.

$\therefore$ C.F. $= c_1\cosh(x\sqrt{s/k}) + c_2\sin(x\sqrt{(s/k)}),$ c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-(s/k)}\left(-\frac{u_0}{k}\right) = -\frac{u_0}{k}\times\frac{1}{D^2-(s/k)}e^{0.x.} = -\frac{u_0}{k}\times\frac{1}{-(s/k)}e^{0.x} = \frac{u_0}{s}.$$

Hence the general solution is given by $\bar{u} = \text{C.F.} + \text{P.I.}$

or $$\bar{u}(x,s) = c_1\cosh(x\sqrt{s/k}) + c_2\sinh(x\sqrt{s/k}) + u_0/s \qquad \ldots (4)$$

Now, (2) $\Rightarrow$ $\mathrm{L}\{u(l, t)\} = \mathrm{L}\{u_1\}$ and $\mathrm{L}\{u_x(0, t)\} = \mathrm{L}\{0\}$.

$\therefore$ $$\bar{u}(l,s) = u_1/s \qquad \ldots (5)$$

and $$(d\bar{u}/dx)_{x=0} = 0. \qquad \ldots (6)$$

Then, (4) $\Rightarrow$ $d\bar{u}/dx = c_1\sqrt{s/k}\sinh(x\sqrt{s/k}) + c_2\sqrt{s/k}\cosh(x\sqrt{(s/k)})$ ... (7)

Putting $x = 0$ in (7) and using (6), we get $c_2 = 0.$

Putting $x = l$ and $c_2 = 0$ in (4) and using (5), we get $c_1 = \dfrac{u_1 - u_0}{s\cosh(l\sqrt{s/k})}.$

$\therefore$ From (4), we get $$\bar{u}(x,s) = \frac{u_0}{s} + (u_1 - u_0)\frac{\cosh(x\sqrt{s/k})}{s\cosh(l\sqrt{s/k})}. \qquad \ldots (8)$$

Taking inverse Laplace transforms of both side of (8), we get

$\therefore$ $$u = u_0\mathrm{L}^{-1}\left[\frac{1}{s}\right] + (u_1 - u_0)\mathrm{L}^{-1}\left\{\frac{\cosh x\sqrt{s/k}}{s\cosh l\sqrt{s/k}}\right\}$$

Replacing x by $x/\sqrt{k}$ and a by $1/\sqrt{k}$ in the given result, we get the value of inverse Laplace transform of the second term of R.H.S. Thus, we obtain

$$u = u_0 + (u_1 - u_0)\left[1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2 kt/4l^2}\cos\frac{(2n-1)\pi x}{2l}\right]$$

or
$$u = u_1 + \frac{4(u_1 - u_0)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2 kt/4l^2}\cos\frac{(2n-1)\pi x}{2l}$$

Ex. 5. *A semi-infinite solid $x > 0$ has its initial temperature equal to zero. A constant heat flux A is applied at the face $x = 0$ so that $-K\,u_x(0, t) = A$. Find the temperature at any point $x > 0$ of solid given that* $\mathrm{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s^{3/2}}\right\} = \left(\frac{t}{\pi}\right)^{1/2} e^{-x^2/4t^2} - \frac{x}{2}\,erfc\left(\frac{x}{2\sqrt{t}}\right)$

Sol. We have to solve one-dimensional heat equation

$$\partial u/\partial t = k(\partial^2 u/\partial x^2), \quad 0 < x < \infty, \quad t > 0 \qquad \ldots (1)$$

subject to the boundary condition: $u_x(0, t) = -\mathrm{A/K}$ ... (2)

and initial condition: $u(x, 0) = 0$ ... (3)

Hence k and K are thermal diffusivity and conductivity respectively of the material of the given solid. Let $\mathrm{L}\{u(x,t)\} = \bar{u}(x,s)$. Applying Laplace transform to (1), we get

$s\bar{u} - u(x,0) = k(d^2\bar{u}/dx^2)$ or $[\mathrm{D}^2 - (s/k)]\bar{u} = 0$, whose general solution is

$$\bar{u}(x,s) = c_1 e^{x\sqrt{(s/k)}} + c_2 e^{-x\sqrt{(s/k)}},\ c_1,\ c_2 \text{ being arbitrary constants.} \qquad \ldots (4)$$

In the present physical problem, since the temperature $u(x, t)$ must be bounded for all x and t, so $\bar{u}(x,s)$ must also be bounded. Hence from (4), it follows that we must choose $c_1 = 0$. Then, (4) reduces to

$$\bar{u}(x,s) = c_2 e^{-x\sqrt{(s/k)}} \qquad \ldots (5)$$

Now, (2) $\Rightarrow$ $\mathrm{L}\{u_x(0, t) = \mathrm{L}\ \{-\mathrm{A/K}\}$ $\Rightarrow$ $(d\bar{u}/dx)_{x=0} = -A/ks,$ ... (6)

From (5), we get $d\bar{u}/dx = -c_2\sqrt{s/k}\,e^{-x\sqrt{s/k}}$ ... (7)

Putting $x = 0$ in (7) and using (6), we get

$-(\mathrm{A/K}s) = c_2\,(s/k)^{1/2}$ so that $c_2 = (A\sqrt{k})/\mathrm{K}s^{3/2}$

$\therefore$ (5) gives $\bar{u}(x,s) = (A\sqrt{k}/\mathrm{K}s^{3/2})e^{-x\sqrt{s/k}}$

$$\therefore\ u(x,t) = \frac{\mathrm{A}\sqrt{k}}{\mathrm{K}}\mathrm{L}^{-1}\left\{\frac{e^{-x\sqrt{s/k}}}{s^{3/2}}\right\} = \frac{\mathrm{A}\sqrt{k}}{\mathrm{K}}\left[\left(\frac{t}{\pi}\right)^{1/2} e^{-x^2/4kt^2} - \frac{x}{2\sqrt{k}}\,erfc\left(\frac{x}{2\sqrt{kt}}\right)\right] \qquad \ldots(8)$$

[Replacing x by $x/\sqrt{k}$ in the give result]

Ex. 6. *The semi-infinite solid $x < 0$ has thermal diffusivity k_1 and conductivity K_1. It is in contact along the plane $x = 0$ with the semi-infinite solid $x > 0$ in which these quantities are k_2, K_2. If the initial temperature of the first solid is constant V_0 and that of the second solid is zero, show that the temperature at time t in the second solid is*

$$\frac{V_0}{1+\sigma} erfc\left\{\frac{x}{2\sqrt{(k_2 t)}}\right\} \text{ where } \sigma = \frac{K_2\sqrt{k_1}}{K_1\sqrt{k_2}}$$ **[C.S.I.R. (Net) 2005; I.A.S. 2003; I.F.S. 2001, 05; Osmania 2001, 04, 06; Meerut 1994, 96, 98]**

Sol. Let $u(x, t)$ and $v(x, t)$ be temperatures at any points of the two solids at any time $t > 0$. Then, we know that $u(x, t)$ and $v(x, t)$ satisfy the following one dimensional heat conduction equations:

$$\partial u / \partial t = k_1(\partial^2 u / \partial x^2), -\infty < x \le 0 \quad \text{... (1)}$$

and $$\partial v / \partial t = k_2(\partial^2 v / dx^2), 0 \le x < \infty. \quad \text{... (2)}$$

with boundary conditions : $$\lim_{x\to 0-} u(x, t) = \lim_{x\to 0+} v(x, t) \quad \text{... (3)}$$

$$u(x, 0) = V_0 \quad \text{... 4 (A)}$$

$$v(x, 0) = 0 \quad \text{... 4 (B)}$$

$$\lim_{x\to 0-} \{-K_1 u_x(x, t) = \lim_{x\to 0+} \{-K_2 v_x(x, t)\} \quad \text{... (5)}$$

$$\lim_{x\to -\infty} u(x,t) \text{ is finite} \quad \text{... 6 (A)}$$

and $$\lim_{x\to \infty} v(x, t) \text{ is finite} \quad \text{... 6 (B)}$$

Taking Laplace transform of both sides of (1), we get

$$L\left\{\frac{\partial u}{\partial t}\right\} = k_1 L\left\{\frac{\partial^2 u}{\partial x^2}\right\} \quad \text{or} \quad s\,\bar{u}(x,s) - u(x,0) = k_1 \frac{d^2\bar{u}}{dx^2}$$

or $$d^2\bar{u}/dx^2 - (s/k_1)\bar{u} = V_0/k \quad \text{or} \quad [D^2 - (s/k_1)]\bar{u} = -V_0/k_1. \quad \text{... (7)}$$

Auxiliary equation of (7) is $D^2 - (s/k_1) = 0$ so that $D = \pm\sqrt{(s/k_1)}$.

C.F. $= c_1 e^{x\sqrt{s/k_1}} + c_2 e^{-x\sqrt{s/k_1}}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 - (s/k_1)}\left(-\frac{V_0}{k_1}\right) = -\frac{V_0}{k_1}\frac{1}{D^2 - (s/k_1)} e^{0.x} = -\frac{V_0}{k_1} \times \frac{1}{-(s/k_1)} = \frac{V_0}{s}.$$

$\therefore$ The general solution of (7) is

$$\bar{u}(x.s) = c_1 e^{x\sqrt{s/k_1}} + c_2 e^{-x\sqrt{s/k_1}} + V_0/s \quad \text{... (8)}$$

(6A) gives $\bar{u}(x,s) \to$ finite as $x \to -\infty$. Hence take $c_2 = 0$ is (8). So from (8), we get

$$\bar{u}(x,s) = c_1 e^{x\sqrt{s/k_1}} + V_0/s \quad \text{... (9)}$$

Taking Laplace transforms of both sides of (2), we get

$$L\left\{\frac{\partial v}{\partial t}\right\} = k_2\left\{\frac{\partial^2 v}{\partial x^2}\right\} \quad \text{or} \quad s\,\bar{v}(x, s) - v(x, 0) = k_2 \frac{d^2\bar{v}}{dx^2}$$

or $$s\bar{v} = k_2(d^2\bar{v}/dx^2) \quad \text{or} \quad [D^2 - (s/k_2)]\,\bar{v} = 0 \quad \text{... (10)}$$

Its auxiliary equation is $D_2 - (s/k_2) = 0$ so that $D = \pm\sqrt{(s/k_2)}$.

So the solution of (10) is $\bar{v}(x, s) = c_3 e^{x\sqrt{s/k_2}} + c_4 e^{-x\sqrt{s/k_2}}$. ... (11)

Now, (6B) gives $\bar{v}(x, s) \to$ finite as $x \to \infty$.

So we must take $c_3 = 0$ in (11). So from (11), we get

$$\bar{v}(x, s) = c_4 e^{-x\sqrt{s/k_2}} \quad \text{... (12)}$$

Taking Laplace transforms of the conditions (3) and (5), we have

$$\lim_{x\to 0-} \bar{u}(x, s) = \lim_{x\to 0+} \bar{v}(x, s) \quad \text{... (13)}$$

and $$\lim_{x\to 0-} \{K_1 \bar{u}_x(x, s)\} = \lim_{x\to 0+} \{K_2 \bar{v}_x(x, s)\} \quad \text{... (14)}$$

From (9), (12) and (13), we get $c_2 + (V_0/s) = c_4$. ... (15)

From (9), we get $\bar{u}_x(x, s) = -c_2\sqrt{s/k_1}\, e^{-x\sqrt{s/k_1}}$ ... (16)

From (12), we get $\bar{v}_x(x, s) = -c_4\sqrt{s/k_2}\, e^{-x\sqrt{s/k_2}}$ ... (17)

From (16), (17) and (14), we get $K_1 c_2\sqrt{s/k_1} = -K_2 c_4\sqrt{s/k_2}$

$$\therefore \quad c_2 = \frac{K_2}{K_1}\times\frac{\sqrt{k_1}}{\sqrt{k_2}}c_4 = -\sigma c_4, \quad \text{as} \quad \sigma = \frac{K_2}{K_1}\times\frac{\sqrt{k_1}}{\sqrt{k_2}}$$

With this value of c_2, (15) gives $-\sigma c_4 + \frac{V_0}{s} = c_4$ so that $c_4 = \frac{V_0}{1+\sigma}\times\frac{1}{s}$

$\therefore$ From (12), we have $\bar{v}(x, s) = \frac{V_0}{1+\sigma}\frac{1}{s}e^{-x\sqrt{s/k_2}}$.

Taking inverse Laplace transforms, we get

$$v(x, t) = \frac{V_0}{1+\sigma} L^{-1}\left\{\frac{1}{s}e^{-x\sqrt{s/k_2}}\right\} \quad \text{or} \quad \bar{v}(x, s) = \frac{V_0}{1+\sigma} erfc\left\{\frac{x}{2\sqrt{k_2 t}}\right\}.$$

[$\because$ From Ex. 10 (*iii*), Art. 2.14 of Chapter 2, $L^{-1}\left\{(1/s)\times e^{-c\sqrt{s}}\right\} = erfc\left\{c/2\sqrt{t}\right\}$]

Ex. 7. *The initial temperature of a slab of homogeneous material bounded by the planes $x = 0$ and $x = l$ is u_0. Find the temperature in this solid after the face $x = 0$ is insulated and the temperature of face $x = l$ is reduced to zero*

[Kanpur 2007; Guwahati 2003, 2006, Kurukshetra 2003]

Sol. Here we have to solve one-dimensional heat equation

$$\partial u/\partial t = k(\partial^2 u/\partial x^2), \qquad 0 < x < \infty, \qquad t > 0 \quad \text{... (1)}$$

subject to the boundary conditons : $u_x(0, t) = 0, \quad u(l, t) = 0, \quad t > 0$ ... (2)

and the initial conditions : $u(x, 0) = u_0, \quad 0 < x < l$ ... (3)

Let $L\{u(x,t)\} = \bar{u}(x,s)\}$. Applying the Laplace transform of both sides of (1), we get

$$s\bar{u}(x,s) - u(x,0) = k(d^2\bar{u}/dx^2) \quad \text{or} \quad d^2\bar{u}/dx^2 - (s/k)\bar{u} = -u_0/k, \text{ using (3)}$$

or $$[D^2 - (s/k)]\bar{u} = -u_0/k, \quad \text{... (4)}$$

Auxiliary equation of (4) is $D^2 - (s/k) = 0$ giving $D = \pm\sqrt{s/k}$.

so C.F. $= c_1 \cosh(x\sqrt{s/k}) + c_2 \sinh(x\sqrt{s/k})$, c_1 and c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{\text{D}^2 - (s/k)}\left(-\frac{u_0}{k}\right) = -\frac{u_0}{k}\frac{1}{\text{D}^2-(s/k)}e^{0.x} = -\frac{u_0}{k}\times\frac{1}{-(s/k)}e^{0.x} = \frac{u_0}{s}.$$

Hence the general solution of (4) is given by

$$u(x,s) = c_1 \cosh(x\sqrt{s/k}) + c_2 \sinh(x\sqrt{s/k}) + u_0/s. \quad \text{... (5)}$$

Taking Laplace transforms of conditions (2), we get

$$\text{L}\{u_x(0,t)\} = 0 \qquad i.e., \qquad (d\bar{u}/dx)_{x=0} = 0 \quad \text{... (6)}$$

and $$\text{L}\{u(l,t) = 0 \qquad i.e., \qquad \bar{u}(l,s) = 0. \quad \text{... (7)}$$

From (5), $$d\bar{u}/dx = c_1\sqrt{s/k}\sinh(x\sqrt{s/k}) + c_2\sqrt{s/k}\cosh(x\sqrt{s/k}) \quad \text{... (8)}$$

Putting $x = 0$ in (8), we get $(d\bar{u}/dx)_{x=0} = c_2\sqrt{s/k}$ or $0 = c_2\sqrt{s/k}$, using (6)

Hence $c_2 = 0$ and so (5) becames $\bar{u}(x,s) = c_1\cosh(x\sqrt{s/k}) + u_0/s.$ ... (9)

Putting $x = l$ is (9) and using (7), we get

$$0 = c_1\cosh(l\sqrt{s/k}) + \frac{u_0}{s} \qquad \text{so that} \qquad c_1 = -\frac{u_0}{s\cosh(l\sqrt{s/k})}.$$

$\therefore$ From (9), we get $$\bar{u}(x,s) = \frac{u_0}{s} - \frac{u_0\cosh(x\sqrt{s/k})}{s\cosh(l\sqrt{s/k})}. \quad \text{... (10)}$$

Let $$\sqrt{s/k} = q. \quad \text{... (11)}$$

$\therefore$ From (10), $$\bar{u}(x,s) = \frac{u_0}{s} - \frac{u_0\cosh xq}{s\cosh lq} = \frac{u_0}{s} - \frac{u_0}{s}\times\frac{e^{xq}+e^{-xq}}{e^{lq}+e^{-lq}}$$

$$= \frac{u_0}{s} - \frac{u_0}{s}\times\frac{e^{xq}+e^{-xq}}{e^{lq}(1+e^{-2lq})} = \frac{u_0}{s} - \frac{u_0}{s}e^{-lq}(e^{xq}+e^{-xq})(1+e^{-2lq})^{-1}$$

$$= \frac{u_0}{s} - \frac{u_0}{s}\{e^{-q(x+l)} + e^{-q(-x+l)}\}\times\sum_{n=0}^{\infty}(-1)^n e^{-2nlq}$$

$\therefore$ $$\bar{u}(x,s) = \frac{u_0}{s} - \frac{u_0}{s}\sum_{n=0}^{\infty}(-1)^n[e^{-q(x+l+2nl)} + e^{-q(-x+1+2nl)}]. \quad \text{... (12)}$$

Let $$2n + 1 = m. \quad \text{... (13)}$$

Then, (12) may be re-written as $$\bar{u}(x,s) = \frac{u_0}{s} - \frac{u_0}{s}\sum_{n=0}^{\infty}(-1)^n\left[e^{-q(ml+x)} + e^{-q(ml-x)}\right]$$

or $$\bar{u}(x,s) = \frac{u_0}{s} - u_0\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{s}e^{-(ml+x)\sqrt{s/k}} + \frac{1}{s}e^{-(ml-x)\sqrt{s/k}}\right], \text{ by (11)}$$

Taking inverse Laplace transforms of both sides we get

$$u(x,t) = u_0 \mathrm{L}^{-1}\left\{\frac{1}{s}\right\} - u_0 \sum_{n=0}^{\infty}\left[\mathrm{L}^{-1}\left\{\frac{1}{s}e^{-\frac{ml+x}{\sqrt{k}}\sqrt{s}}\right\} + \mathrm{L}^{-1}\left\{\frac{1}{s}e^{-\frac{ml-x}{\sqrt{k}}\sqrt{s}}\right\}\right]$$

or
$$u(x,t) = u_0 - u_0 \sum_{n=0}^{\infty}\left[erfc\frac{ml+x}{2\sqrt{k}\sqrt{t}} - erfc\frac{ml-x}{2\sqrt{k}\sqrt{t}}\right]$$

$$\left[\because \text{ From Ex. 10 (iii) of Art. 2.14 } C \text{ of Chapter 2, } \mathrm{L}^{-1}\left\{(1/s)\times e^{-c\sqrt{s}}\right\} = erfc\,(c/2\sqrt{t})\right]$$

$$\therefore \quad u(x,t) = u_0 - u_0 \sum_{n=0}^{\infty}\left[erfc\frac{(2n+1)l+x}{2\sqrt{kt}} - erfc\frac{(2n+1)l-x}{2\sqrt{kt}}\right], \text{ by (13)}$$

Ex. 8. *Find the solution of the equation* $\partial u/\partial t = k(\partial^2 u/\partial x^2)$ *which tends to zero as* $x \to \infty$ *and which satisfies the conditions u = f(t) when x = 0, t > 0 and u = 0 when x > 0, t =* 0.

Sol. Given
$$\partial u/\partial t = k(\partial^2 u/\partial x^2) \quad \ldots (1)$$

with boundary conditions:
$$u(0,t) = f(t), \qquad \lim_{x\to\infty}(u,t) = 0 \quad \ldots (2)$$

and initial condition :
$$u(x,0) = 0. \quad \ldots (3)$$

Let $\mathrm{L}\{u(x,t) = \bar{u}\,(x,s)$. Taking Laplace transforms of both sides (1), we get

$s\bar{u}(x,s) - u(x,0) = k(d^2u/dx^2)$ or $s\,\bar{u}(x,s) - 0 = k(d^2\bar{u}/d\,x^2)$, using (3)

or
$$\{\mathrm{D}^2 - (s/k)]\,\bar{u} = 0, \qquad \text{where} \qquad D \equiv d/dx \quad \ldots (4)$$

whose auxiliary equation is $\mathrm{D}^2 - (s/k) = 0$ so that $\mathrm{D} = \pm\sqrt{s/k}$. The general solution of (4) is

$$\bar{u}\,(x,s) = c_1 e^{x\sqrt{s/k}} + c_2 e^{-x\sqrt{s/k}}, \ c_1, c_2 \text{ being arbitrary constants.} \quad \ldots (5)$$

Taking Laplace transforms of conditions (2), we get

$$\mathrm{L}\{u\,(0,t)\} = \mathrm{L}\,\{f(t)\} \qquad i.e., \qquad \bar{u}\,(0,s) = f(s), \text{ say} \quad \ldots (6)$$

and
$$\lim_{x\to\infty}\mathrm{L}\{u(x,t)\} = 0 \qquad i.e., \qquad \lim_{x\to\infty}\bar{u}\,(x,s) = 0 \quad \ldots (7)$$

In view of condition (7), we must take $c_1 = 0$ in (5) and hence (5) reduces to

$$\bar{u}\,(x,s) = c_2 e^{-x\sqrt{s/k}}. \quad \ldots (8)$$

Putting, $x = 0$ in (8) and using (6), we get $c_2 = f\ (s)$. Hence, (8) reduces to

$$\bar{u}(x,s) = f(s)e^{-x\sqrt{s/k}}.$$

Applying the complex inversion formula*, we get

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} f(s) e^{-x\sqrt{s/k}}\,ds.$$

---

* Refer Art. 4.18, Chapter 4 of Author's ''Integral Transforms'' published by S. Chand and Co, New Delh.

Ex. 9. *The faces x = 0 and x = 1 of a slab material for which k = 1 are kept at temperature 0 until the temperature distribution becomes u = x. After time t = 0 both faces are held at temperature 0. Determine the temperature formula, given*

$$\mathrm{L}^{-1}\left\{\frac{\sinh x\sqrt{s}}{s\sinh a\sqrt{s}}\right\}=\frac{x}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2t/a^2}\sin\left(\frac{n\pi x}{a}\right).$$

Sol. *We have to solve one-dimensional heat equation* (*noting that k = 1*)

$$\partial u/\partial t=\partial^2u/\partial x^2 \qquad \ldots(1)$$

subject to the boundary conditions: $u(0,t)=0, \quad u(1,t)=0$ ... (2)

and initial condition : $u(x,0)=x$ ... (3)

Let $\mathrm{L}\{u(x,t)\}=\bar{u}(x,s)$. Taking Laplace transforms of both sides of (1), we get

$$s\bar{u}(x,s)-u(x,0)=d^2\bar{u}/dx^2 \quad \text{or} \quad s\bar{u}(x,s)-x=d^2u/dx^2, \text{ using (3)}$$

or $$d^2\bar{u}/dx^2-s\bar{u}=-x \quad \text{or} \quad (\mathrm{D}^2-s)\,\bar{u}=-x,\ D\equiv d/dx \qquad \ldots(4)$$

Auxiliary equation of (4) is $\mathrm{D}^2-s=0$ giving $\mathrm{D}=\pm\sqrt{s}$.

$\therefore$ C.F. of (4) $=c_1\cosh x\sqrt{s}+c_2\sinh x\sqrt{s}$, c_1, c_2, being arbitrary constants.

And P.I. of (4) $=\dfrac{1}{\mathrm{D}^2-s}(-x)=\dfrac{1}{-s(1-\mathrm{D}^2/s)}(-x)=\dfrac{1}{s}\left(1-\dfrac{\mathrm{D}^2}{s}\right)^{-1}x=\dfrac{1}{s}\left(1+\dfrac{\mathrm{D}^2}{s}+..\right)x=\dfrac{x}{s}$

$\therefore$ The general solution of (4) is $\bar{u}(x,s)=c_1\cosh x\sqrt{s}+c_2\sinh x\sqrt{s}+x/s.$ ... (5)

Taking Laplace transforms of conditions (2), we get

$$\bar{u}(0,s)=0 \quad \text{and} \quad \bar{u}(1,s)=0. \qquad \ldots(6)$$

Putting $x = 0$ and $x = 1$ in (5) and using (6), we get

$$0=c_1 \quad \text{and} \quad 0=c_1\cosh\sqrt{s}+c_2\sinh\sqrt{s}+1/s$$

so that $$c_1=0 \quad \text{and} \quad c_2=-(1/s\sinh\sqrt{s})$$

$\therefore$ From (5), we get $\bar{u}(x,s)=(x/s)-(\sinh x\sqrt{s})/s\sinh\sqrt{s}\}.$

Taking inverse Laplace transforms of both sides, we get

$$u(x,t)=x\,\mathrm{L}^{-1}\{1/s\}-\mathrm{L}^{-1}\{\sinh x\sqrt{s}/s\sinh\sqrt{s}\}. \qquad \ldots(7)$$

Taking $a = 1$ in the given result, we get

$$\mathrm{L}^{-1}\left\{\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right\}=x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2t}\sin(n\pi x) \qquad \ldots(8)$$

Using (8), (7) reduces to

$$u(x,t)=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2t}\sin(n\pi x).$$

Ex. 10. *A semi-infinite solid $x > 0$ has its initial temperature equal to zero. A constant heat flux A is applied at the face $x = 0$. Show that the temperature at this face after time t is* $(A/K)\times\sqrt{kt/\pi}$.

Hint: Take $x = 0$ in equation (8) of solved Ex. 5

5.5. Solution of one-dimensional wave equation

Ex.1. *Solve the wave equation $\partial^2 u/\partial t^2 = c^2\,(\partial^2 u/\partial x^2)$, $x > 0$, $t > 0$, where $u(x, 0) = 0$, $u_t(x, 0) = 0$ $x > 0$ and $u(0, t) = F(t)$,* $\lim_{x\to\infty} u(x,t) = 0, t > 0$.

Sol. We have to solve one-dimensional wave equation

$$\partial^2 u/\partial t^2 = c^2\,(\partial^2 u/\partial x^2) \qquad \ldots (1)$$

subject to boundary conditions: $u(0, t) = F(t)$, $\lim_{x\to\infty} u(x,t) = 0$ ... (2)

and initial conditions: $u(x, 0) = 0$, $u_t(x, 0) = 0$. ... (3)

Let $L\{u(x, t)\} = \bar{u}(x, s)$. Applying Laplace transform to (1), we have

$$s^2\bar{u}(x,s) - su(x,0) - u_t(x,0) = c^2\frac{d^2\bar{u}}{dx^2} \quad \text{or} \quad \frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = 0, \text{ using (3)}$$

Its solution is $\bar{u}(x, s) = c_1 e^{sx/c} + c_2 e^{-sx/c}$, c_1, c_2 being arbitrary constants ... (4)

Now, $(2) \Rightarrow L\{u(0, t)\} = L\{F(t)\}$ and $\lim_{x\to\infty} L\{u(x,t)\} = 0$

$\Rightarrow$ $\bar{u}(0,s) = f(s)$, say and $\bar{u}(x, s) = 0$ as $x \to \infty$.

Since $\bar{u}(x, s) = 0$ as $x \to \infty$, we must choose $c_1 = 0$ in (4).

$\therefore$ (4) reduces to $\bar{u}(x, s) = c_2 e^{-sx/c}$ ... (5)

Putting $x = 0$ in (5) and using $\bar{u}(0, s) = f(s)$, we get $c_2 = f(s)$.

Then (5) reduces to $\bar{u}(x, s) = f(s)\, e^{-s(x/c)}$

$\therefore$ $u(x, t) = L^{-1}\{f(s)\, e^{-s(x/c)}\} = F(t - x/c)\, H(t - x/c)$,

where $H(t - x/c)$ is the Heaviside unit step function.

Ex.2. *A string is stretched between two fixed points (0, 0) and (a, 0). If it is displaced into the curve $u = b \sin(\pi x/a)$ and released from rest in that position at time $t = 0$, find its displacement at any time $t < 0$ and at any point $0 < x < a$.* **[Aligarh 2003, Lucknow B.Tech. 2005]**

Sol. We have to solve one-dimensional wave equation

$$\partial^2 u/\partial t^2 = c^2\,(\partial^2 u/\partial x^2) \qquad \ldots (1)$$

subject to boundary conditions: $u(0, t) = 0$, $u(a, t) = 0$ ... (2)

and initial conditions: $u(x, 0) = b \sin(\pi x/a)$, $u_t(x, 0) = 0$. ... (3)

Let $L\{u(x, t)\} = \bar{u}(x, s)$. Applying Laplace transform to (1), we get

$$s^2\,\bar{u}(x, s) - s\,u(x, 0) - u_t(x, 0), = c^2\,(d^2\bar{u}/dx^2)$$

or $d^2\bar{u}/dx^2 - (s^2/c^2)\bar{u} = -(bs/c^2)\times\sin(\pi x/a)$, using (3) ... (4)

or $[D^2 - (s^2/c^2)]\,\bar{u} = -(bs/c^2)\times\sin(\pi x/a)$, where $D \equiv d/dx$.

Its auxiliary equation is $D^2 - (s/c)^2 = 0$ giving $D = \pm(s/c)$

$\therefore$ C.F. $= c_1 e^{-sx/c} + c_2 e^{-sx/c}$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 - (s^2/c^2)}\left[-\frac{bs}{c^2}\sin\frac{\pi x}{a}\right] = -\frac{bs}{c^2}\,\frac{1}{-(\pi/a)^2 - (s/c)^2}\sin\frac{\pi x}{a}$$

$\therefore$ The general general solution of (4) is

$$\bar{u}\ (x, s) = c_1\, e^{sx/c} + c_2\, e^{-sx/c} + \frac{bs\,\sin(\pi x/a)}{s^2 + (\pi c/a)^2}. \qquad \ldots (5)$$

Now, (2) $\Rightarrow$ L $\{u\,(0, t) = 0$ and L $\{u\,(a, t)\} = 0$.

$\therefore$ $\bar{u}\ (0, s) = 0$ ... (6)

and $\bar{u}\ (a, s) = 0$ ... (7)

Putting $x = 0$ in (5) and using (6), we get $c_1 + c_2 = 0.$ ... (8)

Putting $x = a$ in (5) and using (7), we get $c_1\, e^{sa/c} + c_2\, e^{-sa/c} = 0.$ ... (9)

Solving (8) and (9), $c_1 = c_2 = 0$. Then (5) gives

$$\bar{u}\ (x, s) = b \sin\frac{\pi x}{a} \times \frac{s}{s^2 + (\pi c/a)^2}$$

Taking inverse Laplace transforms of both sides, we get

$$u\,(x, t) = b \sin\frac{\pi x}{a}\, \mathrm{L}^{-1}\left\{\frac{s}{s^2 + (\pi c/a)^2}\right\} = b \sin\frac{\pi x}{a}\cos\frac{\pi ct}{a}.$$

Ex.3. *Solve the boundary value problem: $\partial^2 u/\partial t^2 = a^2\,(\partial^2 u/\partial x^2) - g$, $x > 0$, $t > 0$ with the boundary conditions $u\,(x, 0) = 0 = u_t\,(x, 0)$, $x > 0$; $u\,(0, t) = 0$, $\lim_{x\to\infty} u_x(x,t) = 0, t \geq 0$.*

Sol. Given: $\partial^2 u/\partial t^2 = a^2\,(\partial^2 u/\partial x^2) - g,$ ... (1)

subject to boundary conditions: $u\,(0, t) = 0,$ $u_x\,(x, t) = 0,$ as $x \to \infty.$ ... (2)

and initial conditions: $u\,(x, 0) = u_t\,(x, 0) = 0.$ ... (3)

Let L $\{u\,(x, t)\} = \bar{u}\ (x, s)$. Applying Laplace transform to (1), we get

$$s^2\ \bar{u}\ (x, s) - su\,(x, 0) - u_t\,(x, 0) = a^2\,(d^2\ \bar{u}\ /dx^2) - (g/s)$$

or $\dfrac{d^2\bar{u}}{dx^2} - \dfrac{s^2}{a^2}\bar{u} = \dfrac{g}{a^2 s}$ or $\left(\mathrm{D}^2 - \dfrac{s^2}{a^2}\right)\bar{\mathrm{u}} = \dfrac{g}{a^2 s},$ by (3) ... (4)

Auxiliary equation is $\mathrm{D}^2 - (s^2/a^2) = 0$ giving $\mathrm{D} = \pm\, s/a.$

$\therefore$ Complementary Function $= c_1\, e^{sx/a} + c_2\, e^{-sx/a}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{\mathrm{D}^2 - (s^2/a^2)} \cdot \frac{g}{a^2 s} = \frac{g}{a^2 s} \cdot \frac{1}{(-s^2/a^2)(1 - \mathrm{D}^2 a^2/s^2)}.1$$

$$= -\frac{g}{s^3}\left(1 - \frac{\mathrm{D}^2 a^2}{s^2}\right)^{-1} \cdot 1 = -\frac{g}{s^3}\left(1 + \frac{a^2}{s^2}\mathrm{D}^2 + \ldots.\right)\cdot 1 = -\frac{g}{s^3}.$$

The solution of (4) is $\bar{u}\ (x, s) = c_1\, e^{sx/a} + c_2\, e^{-sx/a} - (g/s^3).$ ... (5)

$\therefore$ From (5), $\bar{u}_x\ (x, s) = c_1 \times (s/a) \times\ e^{sx/a} - c_2 \times (s/a) \times e^{-sx/a}.$ ... (6)

From (2), L $\{u\,(0, t)\} = 0$ and L $\{u_x\,(x, t)\} = 0$ as $x \to \infty$.

$\therefore$ $\bar{u}\ (0, s) = 0$ ... (7)

and $\bar{u}_x\ (x, s) = 0,$ as $x \to \infty.$ ... (8)

From (6) and (8) it follows that $c_1 = 0$. Putting $x = 0$ and $c_1 = 0$ in (5) and using (7), we get $c_2 = g/s^3$. Then (5) reduces to

$$\bar{u}\ (x, s) = g \times (1/s^3) \times e^{-s(x/a)} - g \times (1/s^3). \quad \text{... (9)}$$

Using $L^{-1}\ \{1/s^3\} = t^2/2$ and second shifting theorem in terms of the Heaviside unit step function, (9) gives

$$u\ (x, t) = (1/2) \times g\,(t - x/a)^2\, H(t - x/a) - (1/2) \times gt^2.$$

Ex.4. *An infinite long string having one end at x = 0 is initially at rest on the x-axis. The end x = 0 undergoes a periodic transverse displacement given by A_0 sin nt, t > 0. Find the displacement of any point on the string at t > 0.* **[Meerut 1998, Agra 1998, 2003; Himanchal 2004]**

Sol. We have to solve one-dimensional wave equation

$$\partial^2 u/\partial t^2 = c^2\ (\partial^2 u/\partial x^2) \quad \text{... (1)}$$

with boundary conditions: $u\ (0, t) = A_0 \sin nt,\ t > 0$ ... (2A)

and $$\lim_{x\to\infty} u(x,t) = \text{finite} \quad \text{... (2B)}$$

and initial conditions: $u\ (x, 0) = 0, \qquad u_t\ (x, 0) = 0.$ ... (3)

Let $L\ \{u\ (x, t)\} = \bar{u}\ (x, s)$. Taking Laplace transforms of (1), we get

$$s^2\ \bar{u}\ (x, s) - su\ (x, 0) - u_t\ (x, 0) = c^2\ (d^2\ \bar{u}\ /dx^2)$$

or $s^2\ \bar{u} = c^2\ (d^2\ \bar{u}\ /dx^2)$, by (3) or $\{D^2 - (s/c)^2\}\ \bar{u} = 0,\ D \equiv d/dx$

whose solution is $\bar{u}\ (x, s) = c_1\ e^{sx/c} + c_2\ e^{-sx/c}$, c_1, c_2 being arbitrary constants .... (4)

Taking Laplace transforms of (2A) and (2B), we get

$L\ \{u\ (0, t)\} = A_0\ L\ \{\sin nt\}$ or $\bar{u}\ (0, s) = (A_0\, n)/(s^2 + n^2)$ ... (5)

and $$\lim_{x\to\infty} L\,\{u(x,t)\} = \text{finite} \quad \text{or} \quad \lim_{x\to\infty} \bar{u}(x,s) = \text{finite.} \quad \text{... (6)}$$

In view of (6), we must taken $c_1 = 0$ in (4). Hence, $\bar{u}\ (x, s) = c_2\ e^{-sx/c}$. ... (7)

Putting $x = 0$ in (7) and using (5), we get $c_2 = (A_0\, n)/(s^2 + n^2)$

$\therefore$ From (7), we get $\bar{u}\ (x, s) = \{A_0 n / (s^2 + n^2)\} e^{-s\,x/c}.$

Taking inverse Laplace transforms, we get

$$u\ (x, t) = A_0\ n \times \frac{1}{n} \sin n\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right), \qquad \text{as} \qquad L^{-1}\left\{\frac{1}{s^2 + n^2}\right\} = \frac{\sin nt}{n}.$$

or $u\ (x, t) = A_0 \sin n\ (t - x/c)\ H\ (t - x/c)$, where $H\ (t - x/c)$ is the Heaviside unit step function.

Ex.5. *Solve the boundary value problems. $\partial^2 u/\partial t^2 = a^2\ (\partial^2 u/\partial x^2) + g$, $t > 0$, $0 < x < l$ with $u = \partial u/\partial t = 0$ when $t = 0$, $0 < x < l$; $u = 0$, when $x = 0$ and $\partial u/\partial x = 0$ when $x = l$, for $t > 0$ given*

$$L^{-1}\left\{\frac{\cosh sx}{s^3 \cosh sl}\right\} = \frac{t^2 + x^2 + l^2}{2} - \frac{16\,t^2}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos\frac{(2n-1)\pi x}{2l} \cos\frac{(2n-1)\pi t}{2l}.$$

Sol. Given $$\partial^2 u/\partial t^2 = a^2\,(\partial^2 u/\partial x^2) + g \quad \text{... (1)}$$

with boundary conditions: $u\ (0, t) = u_x\ (l, t) = 0$ ... (2)

and initial conditions: $u\ (x, 0) = u_t\ (x, 0) = 0.$ ... (3)

Let $L\ \{u\ (x, t)\} = \bar{u}\ (x, s)$. Taking Laplace transforms of (1), we get

$$L\ \{\partial^2 u/\partial t^2\} = a^2\ L\ \{\partial^2 u/\partial x^2\} + g\ \{1\}$$

or $$s^2\ \bar{u}\ (x, s) - su\ (x, 0) - u_t\ (x, 0) = a^2\ (d^2\bar{u}/dx^2) + g/s$$

or $s^2\bar{u} = a^2(d^2\bar{u}/dx^2) + g/s$ or $\{D^2 - (s^2/a^2)\}\,\bar{u} = -g/a^2 s$ ... (4)

whose anxiliary equation is $D^2 - s^2/a^2 = 0$ giving $D = \pm s/a$. and hence its

C.F. $= c_1 \cosh(sx/a) + c_2 \sinh(sx/a)$, c_1, c_2 being arbitrary constants.

and $$\text{P.I.} = -\frac{g}{a^2 s} \times \frac{1}{D^2 - (s^2/a^2)} e^{0.x} = -\frac{g}{a^2 s} \times \frac{1}{0 - (s^2/a^2)} = \frac{g}{s^3}.$$

$\therefore$ The general solution of (4) is

$$\bar{u}\ (x, s) = c_1 \cosh(sx/a) + c_2 \sinh(sx/a) + g/s^3 \qquad \ldots (5)$$

Taking Laplace transforms of (2), we get $$\bar{u}\ (0, s) = 0 \qquad \ldots (6)$$

and $$\left(d\bar{u}/dx\right)_{x=l} = 0. \qquad \ldots (7)$$

Putting $x = 0$ in (5) and using (6), $0 = c_1 + g/s^3$ so that $c_1 = -g/s^3$. ... (8)

Next, from (5), we get $$\frac{d\bar{u}}{dx} = \frac{c_1 x}{a} \sinh\left(\frac{sx}{a}\right) + \frac{c_2 x}{a} \cosh\left(\frac{sx}{a}\right).$$

Putting $x = l$ and using (7) and (8), we get

$$0 = -\frac{g}{s^3} \times \frac{l}{a} \sinh\left(\frac{sl}{a}\right) + \frac{c_2 l}{a} \cosh\frac{sl}{a} \qquad \text{giving} \qquad c_2 = \frac{g}{s^3} \tanh\left(\frac{sl}{a}\right).$$

Using (8) and (9) in (5), we get

$$\bar{u}\ (x, s) = \frac{g}{s^3} + \frac{g}{s^3}\left[-\cosh\left(\frac{sx}{a}\right) + \tanh\left(\frac{sl}{a}\right)\sinh\left(\frac{sx}{a}\right)\right]$$

$$= \frac{g}{s^3} - \frac{g}{s^3}\left[\cosh\frac{sx}{a} - \frac{\sinh(sl/a)\sinh(sx/a)}{\cosh(sl/a)}\right]$$

$$= \frac{g}{s^3} - \frac{g}{s^3} \times \frac{\cosh(sx/a)\cosh(sl/a) - \sinh(sl/a)\sinh(sx/a)}{\cosh(sl/a)}$$

$$= \frac{g}{s^3} - \frac{g}{s^3} \times \frac{\cosh\left(sl/a - sx/a\right)}{\cosh(sl/a)} = \frac{g}{s^3} - \frac{g}{s^3} \times \frac{\cosh\{s(l-x)/a\}}{\cosh(sl/a)}$$

$$\therefore \qquad u(x, t) = g\, L^{-1}\left\{\frac{1}{s^3}\right\} - g\, L^{-1}\left\{\frac{\cosh[s(l-x)/a]}{s^3 \cosh(sl/a)}\right\} \qquad \ldots (10)$$

Replacing x by $(l - x)/a$ and l by l/a in the given result, we get

$$L^{-1}\left\{\frac{\cosh[s(l-x)/a]}{s^3 \cosh(sl/a)}\right\} = \frac{1}{2}\left[t^2 + \left(\frac{l-x}{a}\right)^2 - \frac{l^2}{a^2}\right] - \frac{16(l/a)^2}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos\frac{(2n-1)\,\pi(l-x)}{a \times (2l/a)} \cos\frac{(2n-1)\,\pi t}{2 \times (l/a)}.$$

Using this value, (10) reduces to

$$u(x, t) = \frac{gt^2}{2} - \frac{g}{2}\left[t^2 + \left(\frac{l-x}{a}\right)^2 - \frac{l^2}{a^2}\right] + \frac{16 g l^2}{a^2 \pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos\frac{(2n-1)\,(l-x)\,\pi}{2l} \cos\frac{(2n-1)\,\pi t a}{2l}$$

or $$u(x, t) = -\frac{g}{2}\left\{\left(\frac{l-x}{a}\right)^2 - \frac{l^2}{a^2}\right\} + \frac{16\, g\, l^2}{a^2\, \pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos\frac{(2n-1)\,(l-x)\pi}{2l} \cos\frac{(2n-1)\pi t a}{2l}.$$

EXERCISE 5 (B)

1. A string is stretched between two fixed points (0, 0) and (c, 0). If it is displaced into the curve $y = b \sin(n\pi x/c)$ and released from rest in that position at time $t = 0$, solve the boundary value problem for the displacement $y(x, t)$.

Hint: Take wave equation as $\partial^2 u/\partial t^2 = a^2 (\partial^2 u/\partial x^2)$

2. A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$ the string in given shape defined by $u = \mu x (l - x)$ where μ is constant, and then released. Find the displacement of any point x of string at any time $t > 0$, given that

$$L^{-1}\left\{\frac{\cosh sx}{s^3 \cosh sa}\right\} = \frac{t^2 + x^2 - a^2}{2} - \frac{16a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi x}{2a}\cos\frac{(2n-1)\pi t}{2a}.$$

ANSWERS

1. $y(x, t) = b \cos(n\pi at/c) \sin(n\pi x/c)$.

2. $u(x, t) = \dfrac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3}\sin\dfrac{(2n-1)\pi x}{l}\cos\dfrac{(2n-1)\pi ct}{l}$

Miscellaneous problems on chapters 5

1. Using Laplace transform, solve the following boundary value problems :

(*i*) $u_t = u_{xx},\ 0 < x < 1,\ t > 0; u(0,t) = u(1,t) = 1,\ t > 0,\ u(x,0) = 1 + \sin \pi x,\ 0 < x < 1$

(*ii*) $u_{tt} = u_{xx},\ 0 < x < 1,\ t > 0; u(0,t) = u(1,t) = 0,\ t > 0,; u(x,0) = \sin \pi x$ and

$u_t(x,0) = -\sin \pi x,\ 0 < x < 1$

Ans. (*i*) $u(x,t) = 1 + (\sin \pi x) \times e^{-\pi^2 t}$ (*ii*) $u(x,t) = \sin \pi x \{\cos \pi t - (1/\pi) \times \sin \pi t\}$

2. Solve : $u_{xx} = (1/c^2) \times u_{tt} - \cos wt, 0 \le x < \infty,\ 0 \le t < \infty\ u(0,t) = 0$ and u is bounded as $x \to \infty.$. Also, $u(x, 0) = = u_t(x,0) = 0$

Ans. $u(x,t) = (c^2/w^2) \times [-\cos wt + \cos\{w(t - x/c)\} H(t - x/c)]$

3. Find the solution of $u_t = ku_{xx},\ 0 < x < \infty,\ t > 0$ subject to $u(x,0) \to 0\ as\ x \to \infty,$ $u(0,t) = f(t), u(x,0) = 0$ given that $L^{-1}\left[\exp\{-x(s/k)^{1/2}\}\right] = \left(x/2\sqrt{k\pi t^3}\right)\exp.(-x^2/4kt)$

Ans. $u(x, t) = \displaystyle\int_0^t \frac{x \exp\{-x^2/4k(t-u)\}}{2\sqrt{\pi k}\ (t-u)^{3/2}} f(u)\, du$

Fourier Transforms And Their Applications

Where is What

Chapters	*Pages*
1. Fourier integrals and Fourier transforms	1.3-1.50
2. Finite Fourier transforms	2.1-2.30

1

Fourier Integrals and Fourier Transforms

PART I : FOURIER INTEGRALS AND THEIR APPLICATIONS

1.1. Introduction.

Expansion of a function $f(x)$ of period $2l$ into a Fourier series is well known. The Fourier integral was first introduced by Fourier as an attempt to generalize his results from finite intervals to infinite intervals. In this chapter we propose to discuss Fourier integral representation and Fourier transforms, followed by applications involving the Fourier transforms.

1.2. Dirichlet's Conditions

Let $f(x)$ satisfy the following conditions:

(i) $f(x)$ is defined in the interval $-l < x < l$

(ii) $f(x)$ and $f'(x)$ are sectionally continuous in $-l < x < l$

(iii) $f(x)$ is periodic with period $2l$.

Then the above conditions are called *Dirichlet's conditions.*

1.3. Fourier integral theorem (or formula). Fourier integral representation

Let F (x) satisfy the following conditions:

(i) $f(x)$ satisfies the Dirichlet's conditions in every finite interval $-l \le x \le l$.

(ii) $\int_{-\infty}^{\infty} |f(x)|dx$ converges, *i.e.*, $f(x)$ is absolutely integrable in $-\infty < x < \infty$.

Then Fourier integral theorem states that

$$f(x) = \int_0^{\infty} \{A(s)\cos sx + B(s)\sin sx\}\, ds, \qquad \ldots (1)$$

where $$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos su\ du \quad \text{and} \quad B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin su\ du \qquad \ldots (2)$$

Fourier integral theorem can also be re-written as

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty} \int_{u=-\infty}^{\infty} f(u)\cos s(x-u)\ du\ ds. \qquad \ldots (3)$$

The result (1) holds if x is a point of continuity of $f(x)$. If x is a point of discontinuity, $f(x)$ is replaced by $\{f(x+0) + f(x-0)\}/2$. As for Fourier series, the above conditions are sufficient but not necessary.

The right side of (1) is sometimes called the *Fourier integral expansion* of $f(x)$, or briefly *Fourier integral.*

Remark. We have stated Fourier integral theorem and have not presented its rather lengthy proof. For detailed discussion, the reader should refer E.C. Titchmarsh Theory of Fourier integral, Oxford, Clarendon Press 1937.

1.4. Different forms of Fourier integral theorem (formula)

(i) ***General form:*** $$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_0^{\infty}\cos s(x-u)\, ds\right\} du. \qquad \ldots (1)$$

Proof. When x is a point of continuity of $f(x)$, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\ \cos s(x-u)\ du\right\} ds\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty}\cos s(x-u)\ ds\right\} du$$

[changing the order of integration]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left\{2\int_{0}^{\infty}\cos s(x-u)\, ds\right\} du\ ,\ \text{using a property of definite integral, namely,}$$

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx,\ \text{if}\ f(-x) = f(x)$$

$$= \frac{1}{\pi}\int_{u=-\infty}^{\infty} f(u)\left\{\int_{s=0}^{\infty}\cos s\ (x-u)\ ds\right\} du.$$

(*ii*) ***Another general form:*** If $f(x)$ is continuous at x, then

$$f(x) = \int_{0}^{\infty}[\mathrm{A}(s)\cos sx + \mathrm{B}(s)\sin sx]ds, \qquad \ldots (2)$$

where

$$\mathrm{A}(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\ \cos su\ du \qquad \ldots (3)$$

and

$$\mathrm{B}(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\ \sin su\ du. \qquad \ldots (4)$$

Proof. We have

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_{0}^{\infty}\cos s(x-u)ds\right\} du\ = \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\cos s(x-u)du\right\} ds$$

[changing the order of integration]

$$= \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\ (\cos sx\ \cos su + \sin sx\ \sin su)\ du\right\} ds$$

$$= \int_{0}^{\infty}\left[\cos sx\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos su\ du\right\}\ + \sin sx\ \left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin su\ du\right\}\right] ds$$

$$= \int_{0}^{\infty}\left[\mathrm{A}(s)\cos sx + \mathrm{B}(s)\sin sx\right]\mathrm{ds},\ \text{using (3) and (4)}$$

(*iii*) ***Fourier sine integral formula*:** If $f(x)$ is an odd function, then

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\sin sx\left\{\int_{0}^{\infty} f(u)\sin su\ du\right\} ds. \qquad \ldots (5)$$

Proof. Proceed yourself as in form (*ii*) and show that

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\cos sx\left\{\int_{-\infty}^{\infty} f(u)\cos su\ du\right\}\ +\ \sin sx\ \left\{\int_{-\infty}^{\infty} f(u)\sin su\ du\right\}\right] ds \qquad \ldots (6)$$

Since $f(x)$ an odd function, it follows that $f(u)\cos su$ is an odd function and $f(u)\sin su$ is an even function and so by a property of definite integrals, we have

$$\int_{-\infty}^{\infty} f(u)\cos su\ du = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(u)\sin su\ du = 2\int_{-\infty}^{\infty} f(u)\sin su\ du.$$

$$\therefore \qquad (6) \Rightarrow \qquad f(x) = \frac{2}{\pi}\int_0^{\infty} \sin sx\left\{\int_0^{\infty} f(u)\sin su\ du\right\} ds.$$

(*iv*) ***Fourier cosine integral formula:*** If $f(x)$ is an even function, then

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos sx\left\{\int_0^{\infty} f(u)\cos su\ du\right\} ds. \qquad \ldots (7)$$

Proof. Proceed yourself as in part (*ii*) and show that

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos sx\left\{\int_{-\infty}^{\infty} f(u)\cos su\,du\right\} + \sin sx\left\{\int_{-\infty}^{\infty} f(u)\sin su\,du\right\}\right]ds \qquad \ldots (8)$$

Since $f(x)$ is an even function, it follows that $f(u)\cos su$ is an even function and $f(u)\sin su$ is an odd function and hence we have

$$\int_{-\infty}^{\infty} f(u)\cos su\ du = 2\int_0^{\infty} f(u)\cos su\ du \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(u)\sin su\ du = 0.$$

$$\therefore \qquad (8) \Rightarrow \qquad f(x) = \frac{2}{\pi}\int_0^{\infty} \cos sx\int_0^{\infty} f(u)\cos su\ du.$$

(*v*) **Complex or exponential form of Fourier integral theorem**

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)\,e^{isu}du\right\}ds. \qquad \ldots (9)$$

Proof. From general form (*i*) of Fourier integral theorem, we have

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_0^{\infty}\cos s(x-u)ds\right\}du = \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\cos s(x-u)du\right\}ds, \text{ changing the order of integration}$$

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\,\frac{e^{is(x-u)}+e^{-is(x-u)}}{2}du\right\}ds$$

$$\therefore \quad f(x) = \frac{1}{2\pi}\int_0^{\infty} e^{isx}\left\{\int_{-\infty}^{\infty} f(u)\ e^{-isu}du\right\}ds + \frac{1}{2\pi}\int_0^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)\,e^{isu}du\right\}ds \qquad \ldots (10)$$

Substituting $s = -y$ and $ds = -dy$ in the first integral only, (10) reduces to

$$f(x) = \frac{1}{2\pi}\int_0^{-\infty} e^{-iyx}\left\{\int_{-\infty}^{\infty} f(u)\ e^{iyu}du\right\}(-dy) + \frac{1}{2\pi}\int_0^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)\ e^{isu}du\right\}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-iyx}\left\{\int_{-\infty}^{\infty} f(u)e^{iyu}du\right\}dy + \frac{1}{2\pi}\int_0^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)\,e^{isu}du\right\}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}du\right\}ds + \frac{1}{2\pi}\int_0^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)e^{isu}du\right\}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} f(u)\,e^{isu}du\right\}ds.$$

Note. In (10), substituting $s = -y$ and $ds = -dy$ in the second integral in place of first integral and proceeding as before, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left\{\int_{-\infty}^{\infty} f(u)e^{-isu}du\right\} ds, \qquad \ldots (11)$$

which is another form of complex form of Fourier integral formula.

1.5. Solved Examples based on Art. 1.4

Remember the following results for direct application.

$(i)\ \int_0^{\infty} e^{-ax}\cos bx\ dx = \frac{a}{a^2+b^2}$ $\qquad$ $(ii)\ \int_0^{\infty} e^{-ax}\sin bx\ dx = \frac{b}{a^2+b^2}$.

Ex.1. *Using Fourier cosine integral formula, prove that* $\int_0^{\infty}\frac{\cos \lambda x}{\lambda^2+1}d\lambda = \frac{\pi}{2}e^{-x}, x \geq 0.$

Sol. Fourier cosine integral formula is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos \lambda x\left\{\int_0^{\infty} f(u)\cos \lambda u\ du\right\} d\lambda \qquad \ldots (1)$$

Take $\qquad f(x) = e^{-x} \qquad$ so that $\qquad f(u) = e^{-u}. \qquad \ldots (2)$

Using (2), (1) reduces to

$$e^{-x} = \frac{2}{\pi}\int_0^{\infty}\cos \lambda x\left\{\int_0^{\infty} e^{-u}\cos \lambda u\ du\right\} d\lambda = \frac{2}{\pi}\int_0^{\infty}\cos \lambda x\cdot\frac{1}{1+\lambda^2}d\lambda, \quad \text{as} \int_0^{\infty} e^{-ax}\cos bx\ dx = \frac{a}{a^2+b^2}$$

$$\therefore \qquad \int_0^{\infty}\frac{\cos \lambda x}{1+\lambda^2}d\lambda = \frac{\pi}{2}e^{-x},$$

Ex.2. *Show that* $e^{-x}\cos x = \frac{2}{\pi}\int_0^{\infty}\frac{(u^2+2)\cos ux}{u^2+4}du, x > 0$ **[Kanput 2007]**

Sol. Fourier cosine integral formula is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos ux\left\{\int_0^{\infty} f(v)\cos uv\ dv\right\} du, \qquad \ldots (1)$$

Taking $f(x) = e^{-x}\cos x$ so that $f(v) = e^{-v}\cos v$, (1) becomes

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^{\infty}\cos ux\left\{\int_0^{\infty} e^{-v}\cos v\cos uv\ dv\right\} du = \frac{1}{\pi}\int_0^{\infty}\cos ux\left\{\int_0^{\infty} e^{-v}(2\cos uv\cos v)dv\right\} du$$

$$= \frac{1}{\pi}\int_0^{\infty}\cos ux\left[\int_0^{\infty} e^{-v}\{\cos(u+1)v + \cos(u-1)v\}dv\right] du$$

$$= \frac{1}{\pi}\int_0^{\infty}\cos ux\left[\int_0^{\infty} e^{-v}\cos(u+1)v\ dv + \int_0^{\infty} e^{-v}\cos(u-1)v\ dv\right] du$$

$$= \frac{1}{\pi}\int_0^{\infty}\cos ux\left[\frac{1}{1+(u+1)^2} + \frac{1}{1+(u-1)^2}\right] du, \qquad \text{as} \qquad \int_0^{\infty} e^{-ax}\cos bx\ dx = \frac{a}{a^2+b^2}$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{(u^2+2-2u)+(u^2+2+2u)}{(u^2+2+2u)\ (u^2+2-2u)}\cos ux\ dx = \frac{1}{\pi}\int_0^{\infty}\frac{2(u^2+2)\cos ux}{(u^2+2)^2-(2u)^2}dx = \frac{2}{\pi}\int_0^{\infty}\frac{(u^2+2)\cos ux\ dx}{u^4+4}.$$

Ex.3. *Using Fourier sine integral formula, show that*

$$e^{-ax} - e^{-bx} = \frac{2(b^2 - a^2)}{\pi} \int_0^\infty \frac{u \sin ux \, du}{(u^2 + a^2)(u^2 + b^2)}, \; a \geq 0, \; b \geq 0.$$

Sol. Fourier sine integral formula is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin ux \left\{ \int_0^\infty f(v) \sin uv \, dv \right\} du \qquad \ldots (1)$$

Taking $f(x) = e^{-ax} - e^{-bx}$ so that $f(v) = e^{-av} - e^{-bv}$, (1) reduces to

$$e^{-ax} - e^{-bx} = \frac{2}{\pi} \int_0^\infty \sin ux \left\{ \int_0^\infty (e^{-av} - e^{-bv}) \cos uv \, dv \right\} du = \frac{2}{\pi} \int_0^\infty \sin ux \left\{ \int_0^\infty e^{-av} \sin uv \, dv - \int_0^\infty e^{-bv} \sin uv \, dv \right\} du$$

$$= \frac{2}{\pi} \int_0^\infty \sin ux \left[\frac{u}{a^2 + u^2} - \frac{u}{b^2 + u^2} \right] du = \frac{2}{\pi} \int_0^\infty \frac{u(b^2 - a^2) \sin ux \, du}{(a^2 + u^2)(b^2 + u^2)}$$

$$\therefore \qquad e^{-ax} - e^{-bx} = \frac{2(b^2 - a^2)}{\pi} \int_0^\infty \frac{u \sin ux}{(a^2 + u^2)(b^2 + u^2)} du.$$

Ex.4. *Suppose a function f (x) is given by* $f(x) = \begin{cases} 0, \text{ when } x \leq 0 \text{ or } x \geq \pi \\ \sin x, \text{ when } 0 \leq x \leq \pi. \end{cases}$

Using Fourier integral formula, show that $f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos u(\pi - x) + \cos ux}{1 - u^2} du$

Sol. Fourier integral formula is given by

$$f(x) = \frac{1}{\pi} \int_0^\infty \{ f(v) \cos u (v - x) \, dv \} du, \quad -\infty < x < \infty. \qquad \ldots (1)$$

By given value of $f(x)$, we have $f(v) = \begin{cases} 0, v \leq 0 \;\; or \;\; \nu \geq \pi \\ \sin v, 0 \leq \nu \leq \pi \end{cases}$ ... (2)

$$\therefore \quad \int_{-\infty}^\infty f(v) \cos u(v - x) \, dv$$

$$= \int_{-\infty}^0 f(v) \cos u(v - x) \, dv + \int_0^\pi f(v) \cos u(v - x) \, dv + \int_\pi^\infty f(v) \cos u(v - x) \, dv$$

$$= \int_0^\pi \sin v \cos u(v - x) \, dv \,, \text{ using given values of } f(v) \text{ from (2)}$$

$$= \frac{1}{2} \int_0^\pi \left[\sin \{(u + 1)v - ux\} - \sin \{(u - 1)v - ux\} \right] dv, \text{as } 2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

$$= \frac{1}{2} \left[\frac{-\cos \{(u + 1)v - ux\}}{u + 1} + \frac{\cos \{(u - 1)v - ux\}}{u - 1} \right]_{v=0}^{v=\pi} = \frac{1}{2} \left[\frac{\cos C}{u - 1} - \frac{\cos D}{u + 1} \right]_{v=0}^{\pi},$$

[Taking C = $(u - 1)\, v - ux$ and D = $(u + 1)v - ux$]

$$= \frac{1}{2}\left[\frac{(u+1)\cos C-(u-1)\cos D}{(u-1)(u+1)}\right]_{v=0}^{\pi} = \left[\frac{u(\cos C-\cos D)+(\cos C+\cos D)}{2(u^2-1)}\right]_{v=0}^{\pi}$$

$$= \left[\frac{2u\sin\{(C+D)/2\}\sin\{(D-C)/2\}+2\cos\{(C+D)/2\}\cos\{(D-C)/2\}}{2(u^2-1)}\right]_{v=0}^{\pi}$$

$$= \{1/(u^2-1)\}\left[u\sin(v-x)\,u\sin v+\cos(v-x)\,u\cos v\right]_{v=0}^{\pi}, \text{ putting values of C and D}$$

$$= \left[\cos(\pi-x)\,u\cos\pi-\cos(-xu)\right]/(u^2-1) = [\cos u(\pi-x)+\cos ux]/(1-u^2)$$

Thus, $$\int_{-\infty}^{\infty} f(v)\cos u(v-x)\,dv = \frac{\cos u(\pi-x)+\cos ux}{1-u^2} \quad \ldots (3)$$

Using (4), (1) reduces to $$f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos u(\pi-x)+\cos ux}{1-u^2}\,du.$$

Ex.8. *Express the function* $f(x) = \begin{cases}1, \text{when } |x|\le 1\\ 0, \text{when } |x|>1\end{cases}$ *as a Fourier integral.* **[Kanpur 2005]**

Hence evaluate $\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda.$

Sol. Fourier integral formula is given by

$$f(x) = \frac{1}{\pi}\int_{s=0}^{\infty}\left\{\int_{u=-\infty}^{\infty} f(u)\cos s\,(u-x)\,du\right\}ds. \quad \ldots (1)$$

Re-writing the given function $f(x)$, we have

$$f(u) = \begin{cases}1, \text{when} -1\le u\le 1\\ 0, u<-1 \text{ or } u>1.\end{cases} \quad \ldots (2)$$

Now, $$\int_{-\infty}^{\infty} f(u)\cos s(u-x)\,du$$

$$= \int_{-\infty}^{-1} f(u)\cos s(u-x)\,du + \int_{-1}^{1} f(u)\cos s(u-x)\,du + \int_{1}^{\infty} f(u)\cos s(u-x)\,du$$

$$= 0+\int_{-1}^{1}\cos s(u-x)\,du+0, \text{using (2)}$$

$$= \left[\frac{\sin s(u-x)}{s}\right]_{u=-1}^{1} = \frac{\sin s(1-x)-\sin s(-1-x)}{s} = \frac{\sin s(1-x)+\sin s(1+x)}{s} = \frac{2\sin s\cos sx}{s} \quad \ldots (3)$$

Then using (3), (1) reduces to

$$f(x) = \frac{1}{\pi}\int_{s=0}^{\infty}\frac{2\sin s\cos sx}{s}ds = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda. \quad \ldots (4)$$

This is the required representation of $f(x)$ as Fourier integral.

To evaluate the given integral. Re-writing (4), we have

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda = \frac{\pi}{2}f(x) = \begin{cases}\pi/2, & |x|<1\\ 0, & |x|>1\end{cases}$$

[Using the given value of $f(x)$.]

Ex. 6. *Express* $f(x) = \begin{cases}1 & \text{for } 0\le x\le\pi\\ 0 & \text{for } x>\pi\end{cases}$ *as a Fourier sine integral and hence evaluate*

$$\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda\, d\lambda.$$ **[Madurai 1999, Mysore 1997]**

Sol. Re-writing the given function $f(x)$, we have $$f(u) = \begin{cases}1 & \text{for } 0\le u\le\pi\\ 0 & \text{for } u>\pi.\end{cases} \quad \ldots(1)$$

Using variable λ in place of s in form (*iii*) of Art. 1.4, Fourier sine integral formula is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left\{\int_0^\infty f(u)\sin\lambda u\, du\right\}d\lambda = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left\{\int_0^\pi f(u)\sin\lambda u\, du + \int_\pi^\infty f(u)\sin\lambda u\, du\right\}d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left\{\int_0^\pi \sin\lambda u\, du + \int_\lambda^\infty (0)\sin\lambda u\, du\right\}d\lambda, \text{ using (1)}$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[-\frac{\cos\lambda u}{\lambda}\right]_0^\pi d\lambda = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\, d\lambda.$$

Hence $$\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\, d\lambda = \frac{\pi}{2}f(x) = \begin{cases}\pi/2, & \text{for } 0\le x\le\pi\\ 0, & \text{for } x>\pi\end{cases}$$

[Using the given value of $f(x)$]

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi-0)+(\pi+0)}{2}\right] = \frac{\pi}{2}\times\frac{1+0}{2} = \frac{\pi}{4}.$$

EXERCISE 1(A)

1. *Using Fourier cosine integral formula, show that* $e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos ux}{u^2+a^2}dx,\ a>0,\ x\ge 0.$

2. *Show that* $e^{-x} - e^{-2x} = \frac{6}{\pi}\int_0^\infty \frac{u\sin ux\, du}{(u^2+1)(u^2+4)},\ x\ge 0.$

Sol. Proceed as in solved Ex. 3. Here $a = 1$, $b = 2$.

3. *Suppose a function f (x) is given by* $$f(x) = \begin{cases}0, \text{when } x\le 0 \text{ and } x\ge\pi\\ \cos x, \text{when } 0\le x\le\pi\end{cases}$$

Using Fourier integral formula, show that $$f(x) = \frac{1}{\pi}\int_0^\infty \frac{u[\sin ux - \sin u(\pi-x)]}{u^2-1}du.$$

Sol. proceed as in solved Ex.4 yourself.

4. *Find a Fourier cosine and Fourier sine integral representation of the function*

$$f(x)=\begin{cases}\cos x, & 0<x<\pi/2\\ 0, & x>\pi/2\end{cases}$$

Ans. $f(x)=\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\cos(\pi s/2)}{1-s^2}\cos sx\,ds;$ $\qquad f(x)=\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\frac{s-\sin(\pi s/2)}{s^2-1}\right)\sin sx\,dx$

5. If $f(x)=\begin{cases}0, & x<0\\ e^{-x}, & x>0\end{cases}$, (*a*) show that $f(x)=\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{\cos sx+s\sin sx}{s^2+1}ds$

(*b*) Verify directly that the above integral representation converges to the value 1/2 at $x = 0$.

PART II : (INFINITE) FOURIER TRANSFORMS

1.6. Fourier transform or Complex Fourier Transform. [Kanpur 2003; Purvanchal 2003]

Definition. Let F(x) be a function defined on ($-\infty$, ∞) and be piecewise continuous in each partial interval and absolutely integrable in ($-\infty$, ∞). Then the *Fourier transform* of F(x) is a function of a new variable s and it is denoted and defined as

$$\mathrm{F}\{\mathrm{F}(x)\}=\overline{\mathrm{F}}(s)=f(s)=\int_{-\infty}^{\infty}e^{isx}\,\mathrm{F}(x)\,dx \qquad \ldots(1)$$

The function F(x) is then called *inverse Fourier transform* of $\overline{\mathrm{F}}(s)$ or $f(s)$ and is denoted by

$$\mathrm{F}(x)=\mathrm{F}^{-1}\{\overline{\mathrm{F}}(s)\} \qquad \text{or} \qquad \mathrm{F}(x)=\mathrm{F}^{-1}\{f(s)\} \qquad \ldots(2)$$

Note: Some authors use p is place of s in (1) and (2). Also, they use f(x) and F(s) in place of F(x) and $f(s)$ respectively.

Inversion formula for Fourier Transform or Complex Fourier Transform

If $\overline{F}(s)$ is the Fourier transform of F(x) and if F(x) satisfies the Dirichlet conditions in every finite interval (–l, l) and further if $F(x)$ is *absolutely integrable in ($-\infty$, ∞), then at every point of continuity of F(x),*

$$\mathrm{F}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\mathrm{F}}(s)\,e^{-isx}ds.$$

Proof. Complex Fourier integral formula is given by

$$\mathrm{F}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-isx}\left\{\int_{-\infty}^{\infty}\mathrm{F}(u)\,e^{isu}du\right\}ds. \qquad \ldots(3)$$

Complex Fourier transform of F(x) is given by

$$\overline{\mathrm{F}}(s)=\int_{-\infty}^{\infty}e^{isx}\,\mathrm{F}(x)\,dx=\int_{-\infty}^{\infty}\mathrm{F}(u)\,e^{isu}\,du. \qquad \ldots(4)$$

Then, (3) and (4) $\Rightarrow$ $$\mathrm{F}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\mathrm{F}}(s)\,e^{-isx}ds, \qquad \ldots(5)$$

which is the required inversion formula for complex Fourier transform.

Thus, we have $$\mathrm{F}\{\mathrm{F}(x)\}=\overline{\mathrm{F}}(s)=f(s)=\int_{-\infty}^{\infty}e^{isx}F(x)\,dx \qquad \ldots 1(a)$$

and $\quad \mathrm{F}(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\,\overline{\mathrm{F}}(s)\,ds \qquad$ or $\qquad F(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds \qquad \ldots 1(b)$

Some authors define 1(*a*) and 1(*b*) in the following three additional forms also.

(*i*) $\qquad \mathrm{F}\{\mathrm{F}(x)\} = \overline{\mathrm{F}}(s) = f(s) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\,\mathrm{F}(x)\,dx \qquad \ldots 2(a)$

and $\quad \mathrm{F}(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\,\overline{\mathrm{F}}(s)\,ds \qquad$ or $\qquad f(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds \qquad \ldots 2(b)$

It is also known as symmetric form

(*ii*) $\qquad \mathrm{F}\{\mathrm{F}(x)\} = \overline{\mathrm{F}}\{F(s)\} = f(s) = \int_{-\infty}^{\infty} e^{-isx}\,\mathrm{F}(x)\,dx \qquad \ldots 3(a)$

and $\quad \mathrm{F}(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\,\overline{\mathrm{F}}(s)\,ds \qquad$ or $\qquad F(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx} f(s)\,ds \qquad \ldots 3(b)$

(*iii*) $\qquad \mathrm{F}\{\mathrm{F}(x)\} = \overline{\mathrm{F}}(s) = f(s) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\,\mathrm{F}(x)\,dx. \qquad \ldots 4(a)$

and $\quad \mathrm{F}(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\,\overline{\mathrm{F}}(s)\,ds \qquad$ or $\qquad F(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(s)\,ds \qquad \ldots 4(b)$

It is also known as symmetric form

Note. We shall generally use result 1(*a*) and 1(*b*). But in order to get a desired answer we have to use any one of the above four alternative forms for getting Fourier transform and inverse Fourier transform.

From (*i*), (*ii*) and (*iii*), we find that there is some variation as to which integral represents the transform and which one represents the inverse transform. In practice, of course, these differences are of little consequence but the user should be aware of them when solving question paper or consulting some other book on the same subject.

Theorem. *If f (s) is Fourier transform of F (x), then*

(*i*) $\mathrm{F}\{\mathrm{F}(-x)\} = f(-s)\}$ $\qquad$ (*ii*) $\mathrm{F}\{\overline{\mathrm{F}(-x)}\} = \overline{f(s)}$ $\qquad$ (*iii*) $\mathrm{F}\{\overline{\mathrm{F}(x)}\} = \overline{f(-s)}$,

where bar over a quantity represents its complex conjugate. **[Delhi Physics (H) 2001]**

Proof. By definition, $\qquad \mathrm{F}\{\mathrm{F}(x)\} = f(s) = \int_{-\infty}^{\infty} \mathrm{F}(x)\,e^{isx}\,dx. \qquad \ldots (1)$

(*i*) $\mathrm{F}\{\mathrm{F}(-x)\} = \int_{-\infty}^{\infty} \mathrm{F}(-x)e^{isx}dx = \int_{\infty}^{-\infty} \mathrm{F}(y)e^{-isy}(-dy)$, putting $y = -x$ and $dx = -dy$

$= \int_{-\infty}^{\infty} \mathrm{F}(x)e^{i(-s)x}dx = f(-s)$, by (1)

(*ii*) $\mathrm{F}\{\overline{\mathrm{F}(-x)}\} = \int_{-\infty}^{\infty} \{\overline{\mathrm{F}(-x)}\}\,e^{isx}dx = \int_{\infty}^{-\infty} \overline{\mathrm{F}(y)}\,e^{-isy}(-dy)$, putting $y = -x$ and $dx = -dy$

$= \int_{-\infty}^{\infty} \overline{\mathrm{F}(x)}\,e^{-isx}dx = \overline{\left\{\int_{-\infty}^{\infty} \mathrm{F}(x)\,e^{isx}dx\right\}} = \overline{f(s)}$, by (1)

(*iii*) Proceed as in part (*ii*) yourself.

1.7. Fourier sine transform or infinite Fourier sine transform.

[Kanpur 2006; Meerut 2002]

Definition. Let F(x) be a function defined on ($-\infty, \infty$) and be piecewise continuous in each partial interval and absolutely integrable in ($-\infty, \infty$). Then the infinite *Fourier sine transform* of F(x) is a function of a new variables s and it is denoted and defined as

$$F_s\{F(x)\} = \overline{F}_s(s) = f_s(s) = \int_0^\infty F(x)\sin sx\,dx \quad \ldots (1)$$

The function F(x) is then called *inverse Fourier sine transform* of $\overline{F}_s(s)$ or $f_s(s)$ and is denoted by

$$F(x) = F_s^{-1}\{\overline{F}_s(s)\} \quad \text{or} \quad F(x) = F_s^{-1}\{f_s(s)\} \quad \ldots (2)$$

Note: Some authors use p in place of s in (1) and (2).

Inversion formula for Inverse Fourier sine transform **[Bombay 2004]**

If $\overline{F}_s(s)$ is the infinite Fourier sine transform of F(x) and if F(x) satisfies the Dirichlet conditions in every finite interval (–l, l) and further if F(x) is absolutely integrable in ($-\infty, \infty$), then at every point of continuity of F(x),

$$F(x) = \frac{2}{\pi}\int_0^\infty \overline{F}_s(s)\sin sx\,ds$$

Proof. Fourier sine integral formula is given by

$$F(x) = \frac{2}{\pi}\int_0^\infty \sin sx\left\{\int_0^\infty F(u)\sin su\,du\right\}ds \quad \ldots (3)$$

Now, infinite Fourier sine transform of F(x) is given by

$$\overline{F}_s(s) = \int_0^\infty F(x)\sin sx\,dx = \int_0^\infty F(u)\sin su\,du \quad \ldots (4)$$

Then, (3) and (4) $\Rightarrow$ $F(x) = \dfrac{2}{\pi}\displaystyle\int_o^\infty \overline{F}_s(s)\sin sx\,ds$, ... (5)

which is the required inversion formula for infinite Fourier sine transform.

Thus, we have $F_s\{F(x)\} = \overline{F}_s(s) = f_s(s) = \displaystyle\int_0^\infty F(x)\sin sx\,ds$ ... 1(a)

and $F(x) = F_s^{-1}\{\overline{F}_s(s)\} = \dfrac{2}{\pi}\displaystyle\int_0^\infty \overline{F}_s(s)\sin sx\,ds$ ...1(b)

Some authors define 1(a) and 1(b) in the following symmetric form :

$$F_s\{F(x)\} = \overline{F}_s(s) = f_s(s) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty F(x)\sin sx\,dx \quad \ldots 2(a)$$

and $$F(x) = F_s^{-1}\{F_s^{-1}(s)\} = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \overline{F}_s(s)\sin sx\,ds \quad \ldots 2(b)$$

1.8. Fourier cosine transform or infinite Fourier cosine transform.

[Madras 2012; Kanpur 2006; Meerut 2002]

Definition. Let F(x) be a function defined on ($-\infty, \infty$) and be piecewise continuous in each partial interval and absolutely integrable in ($-\infty, \infty$). Then the infinite *Fourier cosine transform* of F(x) is a function of a new variable s and it is denoted and defined as

$$F_c\{F(x)\} = \bar{F}_c(s) = f_c(s) = \int_0^\infty F(x)\cos sx\, dx. \quad \ldots (1)$$

The function $F(x)$ is then called *inverse Fourier cosine transform* of $\bar{F}_c(s)$ or $f_c(s)$ and is denote by

$$F(x) = F_c^{-1}\{\bar{F}_c(s)\} \quad \text{or} \quad F(x) = F_c^{-1}\{f_c(s)\} \quad \ldots (2)$$

Note: Some authors use p in place of s in (1) and (2)

Inversion formula for inverse Fourier cosine transform **[Bombay 2004]**

If $\bar{F}_c(s)$ is the infinite Fourier cosine transform of F(x) and if F(x) satisfies the Dirichlet conditions in every finite interval (–l, l) and further if F(x) is absolutely integrable in $(-\infty, \infty)$, *then at every point of continuity of F(x),*

$$F(x) = \frac{2}{\pi}\int_0^\infty \bar{F}_c(s)\cos sx\, ds.$$

Proof. Fourier cosine integral formula is given by

$$F(x) = \frac{2}{\pi}\int_0^\infty \cos sx\left\{\int_0^\infty F(u)\cos su\, du\right\} ds \quad \ldots (3)$$

Now, infinite Fourier cosine transform of $F(x)$ is given by

$$\bar{F}_c(s) = \int_0^\infty F(x)\cos sx\, dx = \int_0^\infty F(u)\cos su\, du \quad \ldots (4)$$

Then, (3) and (4) $\Rightarrow$ $F(x) = \frac{2}{\pi}\int_0^\infty \bar{F}_c(s)\cos sx\, ds,$ ... (5)

which is the required inversion formula for infinite Fourier cosine transform.

Thus, we have $F_c\{F(x)\} = \bar{F}_c(s) = f_c(s) = \int_0^\infty F(x)\cos sx\, dx$ ... 1(*a*)

and $F(x) = F_c^{-1}\{\bar{F}_c(s)\} = \frac{2}{\pi}\int_0^\infty \bar{F}_c(s)\cos sx\, ds$ ... 1(*b*)

Some authors define 1 (*a*) and 1 (*b*) in the following symmetric form

$$F_c\{F(x)\} = \bar{F}_c(s) = f_c(s) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty F(x)\cos sx\, dx \quad \ldots 2(a)$$

and $F(x) = F_c^{-1}\{\bar{F}_c(s)\} = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \bar{F}_c(s)\cos sx\, dx.$ ... 2(*b*)

1.9. Relationship between Fourier and Laplace transforms.

[Purvanchal 1994, 96; Madurai Kamraj University 2008]

Let $F(t) = \begin{cases} e^{-xt}\, G(t), t > 0 \\ 0 \qquad\quad , t < 0. \end{cases}$

Then, $F\{F(t)\} = \int_{-\infty}^\infty e^{ist}\, F(t)dt$, by definition of Fourier transform

$$= \int_{-\infty}^{0} e^{ist}\, \mathrm{F}(t)dt + \int_{0}^{\infty} e^{ist} F(t)\, dt = \int_{0}^{\infty} e^{ist} e^{-xt}\, \mathrm{G}(t)dt = \int_{0}^{\infty} e^{-(x-is)t}\, \mathrm{G}(t)\, dt, \text{ by (1)}$$

Hence, $$F\{F(t)\} = \int_{0}^{\infty} e^{-pt}\, \mathrm{G}(t)\, dt, \text{ taking } x - is = p$$

$$= \mathrm{L}\{\mathrm{G}(t)\}, \text{ by the defintion of Laplace transform}$$

Thus $$\mathrm{F}\{\mathrm{F}(t)\} = \mathrm{L}\{\mathrm{G}(t)\},$$

which is the required relationship between Fourier and Laplace transforms.

1.10. Some useful results of Integral Calculus for direct applications.

1. $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}}{a^2+b^2}\ (a \sin bx - b \cos bx), \int e^{ax} \cos bx\, dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx).$

$$\int_{0}^{\infty} e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2} \qquad \text{and} \qquad \int_{0}^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$$

2. $$\frac{d^n}{dx^n}\left(\frac{x}{a^2+x^2}\right) = \frac{(-1)^n \times n!}{(a^2+x^2)^{(n+1)/2}} \cos\ \{(n+1)\tan^{-1}(a/x)\},$$

$$\frac{d^n}{dx^n}\left(\frac{a}{a^2+x^2}\right) = \frac{(-1)^n \times n!}{(a^2+x^2)^{(n+1)/2}} \sin\ \{(n+1)\tan^{-1}(a/x)\}.$$

3. $$\int_{0}^{\infty} \frac{\sin px}{x} dx = \begin{cases} \pi/2, \text{if } p > 0 \\ -\pi/2, \text{if } p < 0. \end{cases}$$

4. $$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \quad \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

5. $$\int_{0}^{\infty} \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} dx = \frac{1}{2}\tan\frac{a}{2}, \qquad \int_{0}^{\infty} \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} dx = \frac{1}{2}\sec\frac{a}{2}.$$

6. $$\int_{0}^{\infty} \frac{\cos px}{1+p^2} dp = \frac{\pi}{2} e^{-x}, \qquad \int_{0}^{\infty} \frac{p \sin px}{1+p^2} dp = \frac{\pi}{2} e^{-x}.$$

1.11. Linearity property of Fourier transforms.

If c_1 and c_2 be constants, then

(*i*) $\mathrm{F}\{c_1 \mathrm{F}_1(x) + c_2 \mathrm{F}_2(x)\} = c_1 \mathrm{F}\{F_1(x)\} + c_2 F\{F_2(x)\}$ **[Meerut 2004]**

(*ii*) $\mathrm{F}_s\{c_1 \mathrm{F}_1(x) + c_2 \mathrm{F}_2(x)\} = c_1 \mathrm{F}_s\{\mathrm{F}_1(x)\} + c_2 \mathrm{F}_s\{\mathrm{F}_2(x)\}$

(*iii*) $\mathrm{F}_c\{c_1 \mathrm{F}_1(x) + c_2 \mathrm{F}_2(x)\} = c_1 \mathrm{F}_c\{\mathrm{F}_1(x)\} + c_2 \mathrm{F}_c\{\mathrm{F}_2(x)\}$

Proof (*i*) By definition 1.6, we have

$$\mathrm{F}\{c_1 \mathrm{F}_1(x) + c_2 \mathrm{F}_2(x)\} = \int_{-\infty}^{\infty} e^{isx}\{c_1 \mathrm{F}_1(x) + c_2 \mathrm{F}_2(x)\} dx$$

$$= c_1 \int_{-\infty}^{\infty} e^{isx} \mathrm{F}_1(x)\, dx + c_2 \int_{-\infty}^{\infty} e^{isx} \mathrm{F}_2(x)\, dx = c_1 \mathrm{F}\{\mathrm{F}_1(x)\} + c_2 \mathrm{F}\{\mathrm{F}_2(x)\}, \text{ by definition}$$

(*ii*) By definition 1.7, we have

$$\mathrm{F}_s\{c_1 \mathrm{F}_1(x) + c_2 \mathrm{F}_2(x)\} = \int_{0}^{\infty} \{c_1 \mathrm{F}_1(x) + c_2 \mathrm{F}_2(x)\} \sin sx\, dx$$

$$= c_1 \int_{0}^{\infty} \mathrm{F}_1(x) \sin sx\, dx + c_2 \int_{0}^{\infty} \mathrm{F}_2(x) \sin sx\, dx = c_1 \mathrm{F}_s\{\mathrm{F}_1(x)\} + c_2 \mathrm{F}_s\{\mathrm{F}_2(x)\}, \text{ by definition}$$

(*iii*) Proceed as in part (*ii*) yourself. Use definition 1.8.

1.12. Change of scale property

(*i*) *If f(s) is the Fourier transform of F(x), then (1/| a |) f(s/a) is the Fourier transform of F(ax), where a ≠ 0.* **[Madras 2012; Kanpur 2001, 05; Purvanchal 1994]**

i.e., If $F\{F(x)\} = f(s)$, then $F\{F(ax)\} = (1/|a|) f(s/a)$.where $a \neq 0$. **[Osmania 2010]**

(*ii*) If $F_s\{F(x)\} = f_s(s)$, then $F_s\{F(ax)\} = (1/a) f_s(s/a)$. **[Meerut 2002]**

(*iii*) If $F_c\{F(x)\} = f_c(s)$, then $F_c\{F(ax)\} = (1/a) f_c(s/a)$.

Proof. (*i*) The following two cases arise :

Case I. Let $a > 0$. Then, by definition of Fourier transform, we get

$$F\{F(ax)\} = \int_{-\infty}^{\infty} e^{isx} F(ax)dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{i(s/a)t} F(t)\,dt$$

[Putting $ax = t$ so that $dx = (1/a)\,dt$]

$= (1/a) f(s/a)$, by definition of Fourier transform.

Case II. Let $a < 0$. Let $b > 0$ such that $a = -b$. Then, by definition, we have

$$F\{F(ax)\} = \int_{-\infty}^{\infty} e^{isx} F(ax)dx = \int_{-\infty}^{\infty} e^{isx} F(-bx)dx, \text{as } a = -b$$

$$= \int_{\infty}^{-\infty} e^{i(-s/b)t} F(t)\frac{dt}{(-b)}, \text{ putting } -bx = t \text{ and } dx = -(1/b)\,dt$$

$$= \frac{1}{b}\int_{-\infty}^{\infty} e^{i(s/a)t} F(t)dt = \frac{1}{(-a)} f(s/a), \text{ by definition}$$

Combining the above two cases, $F\{F(ax)\} = (1/|a|) f(s/a)$

(*ii*) By definition 1.7, $$\int_0^{\infty} F(x)\sin sx\,dx = f_s(s). \quad \ldots (1)$$

Now, $F_s\{F(ax)\} = \int_0^{\infty} F(ax)\sin sx\,dx$, by definition 1.7

$$= \frac{1}{a}\int_0^{\infty} F(t)\sin\frac{st}{a}dt, \text{ putting } ax = t \text{ so that } dx = \frac{dt}{a}$$

$$= \frac{1}{a}\int_0^{\infty} F(x)\sin\{(s/a)\times x\}\,dx = \frac{1}{a} f_s(s/a), \text{ on replacing } s \text{ by } s/a \text{ in (1)}$$

(*iii*) Proceed as in part (*ii*) yourself

Exercise. *If $f_c(s)$ is Fourier cosine transform of F(x), show that Fourier cosine transform of F(x/a) is a $f_c(as)$.* **[Purvanchal 1996; Rohilkhand 1995]**

Sol. By definition 1.8, $$\int_0^{\infty} F(x)\cos sx\,dx = f_c(s) \quad \ldots (1)$$

Now, $F_c\{F(x/a)\} = \int_0^{\infty} F(x/a)\cos sx\,dx$, by definition 1.8

$$= a\int_0^{\infty} F(t)\cos(ast)\,dt, \text{ putting } \frac{x}{a} = t \text{ so that } dx = a\,dt$$

$$= a\int_0^{\infty} F(x)\cos\{(as)x\}\,dx = a\,f_c(as), \text{ on replacing } s \text{ by } as \text{ in (1)}$$

1.13. Shifting property for Fourier Transform.

If $f(s)$ is the complex Fourier transform of $F(x)$, then complex Fourier transform of $F(x-a)$ is $e^{isa} f(s)$, i.e. if $\mathrm{F}\{F(x)\} = f(s)$, then $\mathrm{F}\{\mathrm{F}(x-a)\} = e^{\mathrm{isa}} f(s)$.

Proof By definition 1.6, we have $$\int_{-\infty}^{\infty} e^{isx}\,\mathrm{F}(x)\,dx = f(s). \qquad \ldots(1)$$

Now, $\mathrm{F}\{(x-a)\} = \int_{-\infty}^{\infty} e^{isx}\,\mathrm{F}(x-a)\,dx = \int_{-\infty}^{\infty} e^{is(t+a)}\,\mathrm{F}(t)\,dt$, putting $x-a=t$ and $dx=dt$

$$= e^{isa}\int_{-\infty}^{\infty} e^{ist}F(t)\,dt = e^{isa}\int_{-\infty}^{\infty} e^{isx}F(x)\,dx = e^{isa}f(s), \text{ using (1).}$$

1.14. Modulation theorem

(*i*) If $f(s)$ is the Fourier transform of the function $\mathrm{F}(x)$, then $[f(s+a)+f(s-a)]/2$ is the Fourier transform of $\mathrm{F}(x)\cos ax$. **[Osmania 2004; Purvanchal 1995]**

i.e., If $\mathrm{F}\{\mathrm{F}(x)\} = f(s)$, then $\mathrm{F}_c\{\mathrm{F}(x)\cos ax\} = [f(s+a)+f(s-a)]/2$

(*ii*) If $\mathrm{F}_s\{\mathrm{F}(x)\} = f_s(s)$, then $\mathrm{F}_s\{\mathrm{F}(x)\cos ax\} = [f_s(s+a)+f_s(s-a)]/2$

(*iii*) If $\mathrm{F}_s\{\mathrm{F}(x)\} = f_s(s)$, then $\mathrm{F}_c\{\mathrm{F}(x)\sin ax\} = [f_s(s+a)-f_s(s-a)]/2$

(*iv*) If $\mathrm{F}_c\{\mathrm{F}(x)\} = f_c(s)$, then $\mathrm{F}_s\{\mathrm{F}(x)\sin ax\} = [f_c(s-a)-f_c(s+a)]/2$

Prrof. (*i*) By definition 1.6, $$\int_{-\infty}^{\infty} e^{isx}\,\mathrm{F}(x)dx = f(s). \qquad \ldots(1)$$

Now, $\mathrm{F}\{\mathrm{F}(x)\cos ax\} = \int_{-\infty}^{\infty} e^{isx}\,\mathrm{F}(x)\cos ax$, by definition 1.6

$$= \int_{-\infty}^{\infty} e^{isx}\,\mathrm{F}(x)\frac{e^{iax}+e^{-iax}}{2}dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{i(s+a)x}\,\mathrm{F}(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{i(s-a)x}\,\mathrm{F}(x)\,dx$$

$$= (1/2)\times f(s+a) + (1/2)\times(s-a), \text{ using (1)}$$

$$= [f(s+a)+f(s-a)]/2$$

(*ii*) By definition 1.7, $$\int_0^{\infty} \mathrm{F}(x)\sin sx\,dx = f_s(s). \qquad \ldots(1)$$

Now, $\mathrm{F}_s\{\mathrm{F}(x)\}\cos ax\} = \int_0^{\infty} \mathrm{F}(x)\cos ax\sin sx\,dx$, by defintion 1.7

$$= \frac{1}{2}\int_0^{\infty}\mathrm{F}(x)(2\sin ax\cos ax)\,dx = \frac{1}{2}\int_0^{\infty}\mathrm{F}(x)[\sin(s+a)x+\sin(s-a)x]\,dx$$

$$= \frac{1}{2}\left[\int_0^{\infty}\mathrm{F}(x)\sin(s+a)x\,dx + \int_0^{\infty}\mathrm{F}(x)\sin(s-a)x\,dx\right] = \frac{1}{2}\left[f_s(s+a)+f_s(s-a)\right], \text{using (1).}$$

For parts (*iii*) and (*iv*), proceed as in part (*ii*) yourself

1.15 Solved examples based on (infinite) Fourier Transform

Ex.1. *Find the Fourier complex transform of f(x) if* $$f(x) = \begin{cases} e^{iwx}, & a < x < b \\ 0, & x < a \text{ or } x > b \end{cases}$$

Sol. By defintion 1.6, we have

$$\mathrm{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)dx = \int_{-\infty}^{a} e^{isx} f(x)dx + \int_{a}^{b} e^{isx} f(x)dx + \int_{b}^{\infty} e^{isx} f(x)dx$$

$= 0 + \int_a^b e^{isx} e^{iwx}\, dx + 0$, using the given values of $f(x)$

$$= \int_a^b e^{i(s+w)x} dx = \left[\frac{e^{i(s+w)x}}{i(s+w)}\right]_a^b = i\frac{e^{i(s+w)b} - e^{i(s+w)a}}{i^2(s+w)} = \frac{i}{s+w}[e^{i(s+w)a} - e^{i(s+w)b}], \text{as } i^2 = -1.$$

Ex.2. *Find the Fourier transform of* $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a. \end{cases}$ **[Kanpur 2008; Osmania 2010]**

Sol. By definition 1.6, we have

$$\mathrm{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}dx = \int_{-\infty}^{-a} f(x)e^{isx}dx + \int_{-a}^{a} f(x)e^{isx}dx + \int_{a}^{\infty} f(x)e^{isx}dx$$

$= \int_{-a}^{a} x\, e^{isx} dx$, using the given values of $f(x)$

$$= \left[x\frac{e^{isx}}{is}\right]_{-a}^{a} - \int_{-a}^{a} \frac{e^{isx}}{is} dx, \text{integrating by parts}$$

$$= \frac{a}{is}e^{isa} - \frac{(-a)}{is}e^{-isa} - \left[\frac{e^{isx}}{(is)^2}\right]_{-a}^{a} = \frac{a}{is}(e^{isa} + e^{-isa}) + \frac{1}{s^2}(e^{isa} - e^{-isa})$$

$$= -(ia/s) \times (2\cos sa) + (1/s^2) \times (2i \sin sa) = (2i/s^2) \times (\sin sa - as\cos sa).$$

Ex. 3. *Find the Fourier transform of F(x), if* $F(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a. \end{cases}$ **[Meerut 2000]**

Sol. By definition 1.6, we have

$$\mathrm{F}\{\mathrm{F}(x)\} = \int_{-\infty}^{\infty} \mathrm{F}(x)e^{isx}dx = \int_{-\infty}^{-a} \mathrm{F}(x)e^{isx}dx + \int_{-a}^{a} \mathrm{F}(x)e^{isx}dx + \int_{a}^{\infty} \mathrm{F}(x)e^{isx}dx$$

$= \int_{-a}^{a} x^2\, e^{isx} dx$, using the given values of $\mathrm{F}(x)$

$$= \left[x^2\frac{e^{isx}}{is}\right]_{-a}^{a} - \int_{-a}^{a} (2x)\frac{e^{isx}}{is} dx, \text{integrating by parts}$$

$$= \frac{a^2}{is}e^{isa} - \frac{a^2}{is}e^{-isa} - \frac{2}{is}\left\{\left[x\frac{e^{isx}}{is}\right]_{-a}^{a} - \int_{-a}^{a} 1 \cdot \frac{e^{isx}}{is} dx\right\} = \frac{a^2}{is}\left(e^{isa} - e^{-isa}\right) - \frac{2}{is}\left\{\frac{a}{is}e^{isa} + \frac{a}{is}e^{-isa} - \left[\frac{e^{isx}}{(is)^2}\right]_{-a}^{a}\right\}$$

$$= (a^2/is) \times (2i \sin sa) - (2a/i^2s^2) \times (e^{isa} + e^{-isa}) + (2/i^3s^3) \times (e^{isa} - e^{-isa})$$

$$= (2a^2/s)\sin\, sa + (2a/s^2)(2\cos sa) - (2/is^3)(2i \sin sa) = (4a/s^2)\cos sa + (2/s^3)(a^2s^2 - 2)\sin sa.$$

Ex. 4. (*a*) *Find the Fourier complex transform of* $F(x) = e^{-a|x|}$, *where* $a>0$ *and* x *belongs to* $(-\infty, \infty)$.

(*b*) *Find the Fourier transform of* $F(x) = e^{-|x|}$.

Sol. (*a*) By definition of the Fourier Complex transform, we have

$$\mathrm{F}\{\mathrm{F}(x)\} = \int_{-\infty}^{\infty} e^{isx}\,\mathrm{F}(x)\,dx = \int_{-\infty}^{\infty} e^{isx} e^{-a|x|}dx$$

$$= \int_{-\infty}^{0} e^{isx} e^{ax}dx + \int_{0}^{\infty} e^{isx} e^{-ax}dx, \quad \text{noting that } |x| = \begin{cases} x, x > 0 \\ -x, x < 0 \end{cases}$$

$$= \int_{-\infty}^{0} e^{(a+is)x}dx + \int_{0}^{\infty} e^{-(a-is)x}dx = \left[\frac{e^{(a+is)x}}{a+is}\right]_{-\infty}^{0} + \left[\frac{e^{-(a-is)x}}{-(a-is)}\right]_{0}^{\infty}$$

$$= 1/(a+is) + 1/(a-is) = 2a/(a^2+s^2)$$

(*b*) Proceed as in part (*a*) Note that here $a = 1$ **Ans.** $2/(1 + s^2)$.

Ex. 5. (*a*) *Show that the Fourier transform of* $e^{-x^2/2}$ is $\sqrt{2\pi}e^{-s^2/2}$. **[Kanpur 2011]**

(*b*) *Show that the Fourier transform of* $e^{-x^2/2}$ *is* $e^{-s^2/2}$ **[Calicut 1994; Meerut 1995]**

Sol. (*a*) Using the defintion of Fourier transform, we have

$$\mathrm{F}\{e^{-x^2/2}\} = \int_{-\infty}^{\infty} e^{isx} e^{-x^2/2}dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+isx+\frac{s^2}{2}} . e^{-\frac{s^2}{2}}dx = e^{-s^2/2}\int_{-\infty}^{\infty} e^{-(x^2-2isx+i^2s^2)/2}dx$$

$$= e^{-s^2/2}\int_{-\infty}^{\infty} e^{-(x-is)^2/2}dx = \sqrt{2}\,e^{-s^2/2}\int_{-\infty}^{\infty} e^{-y^2}dy \text{ putting } (x - is)/\sqrt{2} = y \text{ and } dx = \sqrt{2}\,dy$$

$$= \sqrt{2}\,e^{-s^2/2} \times \sqrt{\pi} = \sqrt{2\pi}\,e^{-s^2/2}, \qquad \text{as} \qquad \int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}.$$

(*b*) In order to get the desired result, we shall use the symmetric form of the definition of Fourier transform.

Thus, $\mathrm{F}\{e^{-x^2/2}\} = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}e^{-x^2/2}dx$

$= \left(1/\sqrt{2\pi}\right) \times \left(\sqrt{2\pi}\,e^{-s^2/2}\right)$, proceed as in part (*a*) and complete the solution.

$= e^{-s^2/2}$.

Ex. 6. *Find the inverse Fourier transform of* $f(s) = e^{-|s|y}$, *where y belongs to* $(-\infty, \infty)$.

Sol. By definition 1.6. of inverse Fourier transform, we have

$$\mathrm{F}^{-1}\{f(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}e^{-|s|y}ds$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-isx}e^{sy}ds + \int_{0}^{\infty} e^{-isx}e^{-sy}dy\right], \text{ as } |s| = \begin{cases} s, s \geq 0 \\ -s, s \leq 0 \end{cases}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(y-ix)s}dx + \frac{1}{2\pi}\int_{0}^{\infty} e^{-(y+ix)s}ds = \frac{1}{2\pi}\left[\frac{e^{(y-ix)s}}{y-ix}\right]_{-\infty}^{0} + \frac{1}{2\pi}\left[\frac{e^{-(y+ix)s}}{-(y+ix)}\right]_{0}^{\infty}$$

$$= \frac{1}{2\pi} \times \frac{1}{y-ix} + \frac{1}{2\pi} \times \frac{1}{y+ix} = \frac{1}{2\pi}\,\frac{(y+ix)+(y-ix)}{(y+ix)(y-ix)} = \frac{y}{\pi(x^2+y^2)}$$

Ex. 7. *Find the Fourier transform of F(x) defined by* $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence evalvate

[Meerut 1997, 99; Purvanchal 1996; S.N.University (A.P) 1997 Osmania 2004; Delhi Physics (Hons) 2005]

(*i*) $\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} dx$ **[Meerut 2004]**

(*ii*) $\int_{0}^{\infty} \frac{\sin s}{s} ds$ **[Osmania 2004; Manglore 1997]**

Sol. Since $|x| < a \Rightarrow -a < x < a$ and $|x| > a \Rightarrow x < -a$ or $x > a$, so

$$F(x) = \begin{cases} 1, \text{if} -a < x < a \\ 0, \text{if } x < -a \text{ or } x > a. \end{cases} \quad \ldots (1)$$

$\therefore$ $F\{F(x)\} = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx$, by definition of Fourier transform

$$= \int_{-\infty}^{-a} e^{isx} F(x) dx + \int_{-a}^{a} e^{isx} F(x) dx + \int_{a}^{\infty} e^{isx} F(x) dx = 0 + \int_{-a}^{a} e^{isx} dx + 0, \text{ by (1).}$$

$$= (1/is) \times \left[e^{isx} \right]_{-a}^{a} = (1/is) \times (e^{isa} - e^{-isa}) = (1/is) \times (2i \sin sa)$$

Thus, $F\{F(x)\} = \bar{F}(s) = (2 \sin sa)/s$ ... (2)

By the corresponding inversion formula,we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(s) e^{-isx} ds = F(x) \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{2 \sin sa}{s} e^{-isx} ds = 2\pi F(x), \text{ using (2)}$$

or $$2 \int_{-\infty}^{\infty} \frac{\sin sa}{s} (\cos sx - i \sin sx) ds = 2\pi F(x), \quad \text{as} \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

or $$\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} dx - i \int_{-\infty}^{\infty} \frac{\sin sa \sin sx}{s} dx = \pi F(x). \quad \ldots (3)$$

Equating real parts on both sides of (3),we get

$$\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} dx = \pi F(x) \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} dx = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}, \text{using (1)} \quad \ldots (4)$$

Putting $x = 0$ and $a = 1$ in (4), we get

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \quad \text{or} \quad 2 \int_{0}^{\infty} \frac{\sin s}{s} ds = \pi \quad \text{so that} \quad \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}.$$

Ex. 8. *Find the (complex) Fourier transform of* $F(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

[Kanpur 2010; Purvanchal 2003, 04; Bangalore 2004]

and hence evaluate $\int_{0}^{\infty} \left(\frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx$

[Madras 1997; Mysore 1997; Rohilkhand 1999; Warangal 1996]

Sol. $\mathrm{F}\{\mathrm{F}(x)\} = \overline{\mathrm{F}}(s) = \int_{-\infty}^{\infty} e^{isx}\,\mathrm{F}(x)\,dx$, by definition

$$= \int_{-\infty}^{-1} e^{isx}\,\mathrm{F}(x)dx + \int_{-1}^{1} e^{isx}\,\mathrm{F}(x)dx + \int_{1}^{\infty} e^{isx}\,\mathrm{F}(x)dx$$

$$= \int_{-\infty}^{-1} e^{isx}(1-x^2)\,dx, \text{ using given value of } \mathrm{F}(x)$$

$$= \left[(1-x^2)\frac{e^{isx}}{is}\right]_{-1}^{1} - \int_{-1}^{1}(-2x)\frac{e^{isx}}{is}dx, \text{ integrating by parts}$$

$$= \frac{2}{is}\left\{\left[x\times\frac{e^{isx}}{is}\right]_{-1}^{1} - \int_{-1}^{1} 1\times\frac{e^{isx}}{is}dx\right\} = -\frac{2}{s^2}(e^{is}+e^{-is}) + \frac{2}{is^3}(e^{is}-e^{-is})$$

$= -(2/s^2)\times(2\cos s) + (2/is^3)\times(2i\sin s) = -4(s\cos s - \sin s)/s^3$.

$\therefore$ By the corresponding inversion formula, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\mathrm{F}}(s)\,e^{-isx}ds = \mathrm{F}(x) \qquad \text{or} \qquad \int_{-\infty}^{\infty}\left\{-4\left(\frac{s\cos s-\sin s}{s^3}\right)\right\}e^{-isx}ds = 2\pi\,\mathrm{F}(x)$$

or $$\int_{-\infty}^{\infty}\left\{-4\left(\frac{s\cos s-\sin s}{s^3}\right)\right\}(\cos sx - i\sin sx)\,dx = 2\pi\,\mathrm{F}(x)$$

Equating real parts on both sides, we have

$$\int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}\cos sx\,ds = \frac{\pi}{2}\mathrm{F}(x)$$

or $$2\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}\cos sx\,ds = \begin{cases}(\pi/2)\times(1-x^2), & |x|\le 1\\ (\pi/2)\times 0, & |x|>1\end{cases} \qquad \ldots(1)$$

[using the given values of $\mathrm{F}(x)$]

Putting $x = 1/2$ in (1), we obtain

$$2\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}ds = \frac{\pi}{2}\left(1-\frac{1}{4}\right) \qquad \text{or} \qquad \int_{0}^{\infty}\frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx = -\frac{3\pi}{16}.$$

1.16 Solved Examples based on (infinite) Fourier sine transform and (infinite) Fourier Cosine Transform

Ex.1. *Find the sine transform of a function of x which is equal to sin x for 0 < x < a and o for x>a.* **[Rohilkhand 1995, Kanpur 2004]**

Sol. Let the given function of x be $\mathrm{F}(x)$, Then, we have

$$\mathrm{F}(x) = \begin{cases}\sin x, & 0<x<a\\ 0, & x>a.\end{cases} \qquad \ldots(1)$$

Then, $\mathrm{F_s}\{\mathrm{F}(x)\} = \int_{0}^{\infty}\mathrm{F}(x)\sin sx\,dx$, by definition 1.7

$$= \int_{0}^{a}\mathrm{F}(x)\sin sx\,dx + \int_{a}^{\infty}\mathrm{F}(x)\sin sx\,dx = \int_{0}^{a}\sin x\sin sx\,dx + 0, \text{using (1)}$$

$$= \frac{1}{2}\int_0^a [\cos(s-1)x - \cos(s+1)x]\,dx = \frac{1}{2}\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_0^a = \frac{1}{2}\left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right].$$

Ex. 2. *Find the cosine transform of a function of x which is unity for 0 < x < a and zero for x ≥ a. What is the function whose cosine transform is (sin as)/s?*

[Guwahati 2000]

Sol. Let the given function of x be F(x). Then, we have

$$\mathrm{F}(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x \geq a. \end{cases} \quad \ldots (1)$$

$$\therefore\ \mathrm{F_c}\{\mathrm{F}(x)\} = \int_0^\infty \mathrm{F}(x)\cos sx\,dx = \int_0^a \mathrm{F}(x)\cos sx\,dx + \int_a^\infty \mathrm{F}(x)\cos sx\,dx = \int_0^a \cos sx\,dx + 0, \text{ by (1)}$$

$$= \left[\frac{\sin sx}{s}\right]_0^a = \frac{\sin sa}{s}.$$

Let $f(x)$ be the function such that $\mathrm{F_c}\{f(x)\} = (\sin as)/s$.

Hence by the definition of inverse Fourier cosine transform, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin as}{s}\cos sx\,dx = \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)s + \sin(a-x)s}{s}\,dx = \frac{1}{\pi}\left[\int_0^\infty \frac{\sin(a+x)s}{s}\,ds + \int_0^\infty \frac{\sin(a-x)s}{s}\,ds\right]$$

$$= \begin{cases} (1/\pi)\times(\pi/2 + \pi/2), & x < a \\ (1/\pi)\times(\pi/2 - \pi/2), & x > a \end{cases} = \begin{cases} 1, & x < a \\ 0, & x > a \end{cases}, \text{ using result 3 of Art 1.10}$$

Ex. 5. *Find the Fourier sine and cosine transform of f(x), if*

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2. \end{cases}$$

[Kanpur 2007; 2011 Mysore 1997; Rohilkhand 1997]

Sol. By definition of Fourier sine transform, we have

$$\mathrm{F_s}\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx = \int_0^1 f(x)\sin sx\,dx + \int_1^2 f(x)\sin sx\,dx + \int_2^\infty f(x)\sin sx\,dx$$

$$= \int_0^1 x\sin sx\,dx + \int_1^2 (2-x)\sin sx\,dx + 0, \text{ by given values of } f(x)$$

$$= \left[(x)\left(-\frac{\cos sx}{s}\right)\right]_0^1 - \int_0^1 1\times\left(-\frac{\cos sx}{s}\right)dx + \left[(2-x)\left(-\frac{\cos sx}{s}\right)\right]_1^2 - \int_1^2 (-1)\left(-\frac{\cos sx}{s}\right)dx \text{ on integrating by parts}$$

$$= -\frac{\cos s}{s} + \frac{1}{s}\left[\frac{\sin sx}{s}\right]_0^1 + \frac{\cos s}{s} - \frac{1}{s}\left[\frac{\sin sx}{s}\right]_1^2 = \frac{\sin s}{s^2} - \frac{\sin 2s - \sin s}{s^2}$$

$= (1/s^2) \times (2 \sin s - 2 \sin s \cos s) = (2/s^2) \times \sin s\,(1 - \cos s)$

By definition of Fourier cosine transform, we have

$$\mathrm{F_c}\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx = \int_0^1 f(x)\cos sx\,dx + \int_1^2 f(x)\cos sx\,dx + \int_2^\infty f(x)\cos sx\,dx$$

$$= \int_0^\infty x \cos sx\, dx + \int_1^2 (2-x) \cos sx\, dx + 0, \text{ by given values of } f(x)$$

$$= \left[(x)\left(\frac{\sin sx}{s}\right) \right]_0^1 - \int_0^1 1 \times \left(\frac{\sin sx}{s}\right) dx + \left[(2-x)\left(\frac{\sin sx}{s}\right) \right]_1^2 - \int_1^2 (-1)\left(\frac{\sin sx}{s}\right) dx$$

$$= \frac{\sin s}{s} - \left[-\frac{\cos sx}{s^2} \right]_0^1 - \frac{\sin s}{s} + \left[-\frac{\cos sx}{s^2} \right]_1^2 = \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}$$

$= (1/s^2) \times \{2 \cos s - (1 + \cos 2s)\} = (1/s^2) \times (2 \cos s - 2 \cos^2 s) = (2/s^2) \times \cos s\, (1 - \cos s)$

Ex. 4. *Find the Fourier sine transform of e^{-x} and hence prove that*

$$\int_0^\infty \frac{t \sin tx}{1+t^2} dt = \frac{\pi e^{-x}}{2}. \qquad \textbf{[Kanpur 2009; Purvanchal 1994]}$$

Sol. By definition of Fourier sine transform of e^{-x}, we have

$$F_s\{e^{-x}\} = \int_0^\infty e^{-x} \sin sx\, dx = \frac{s}{s^2+1}, \qquad as \qquad \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}$$

Then by inversion formula for Fourier sine transform,we get

$$e^{-x} = \frac{2}{\pi} \int_0^\infty F_s\left\{e^{-x}\right\} \sin sx\, ds = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+1} \sin sx\, ds$$

or $$\frac{2}{\pi} \int_0^\infty \frac{t \sin tx}{1+t^2} dt = e^{-x} \qquad \text{so that} \qquad \int_0^\infty \frac{t \sin tx}{1+t^2} dt = \frac{\pi}{2} e^{-x}.$$

Ex.5. (*a*) *Find the Fourier cosine transform of* e^{-x^2} **[Kanpur 1996; Rohilkhand 1994]**

(*b*) *Find the Fourier sine transform of* $x e^{-x^2/2}$

Sol. (*a*) By definition, $$F_c\left\{e^{-x^2}\right\} = \int_0^\infty e^{-x^2} \cos sx\, dx = I, \text{ say} \qquad \ldots (1)$$

$\therefore$ $$\frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-x^2} \cos sx\right) dx = \int_0^\infty e^{-x^2} (-x \sin sx)\, dx$$

or $$\frac{dI}{ds} = \frac{1}{2} \int_0^\infty (\sin sx)(-2x e^{-x^2})\, dx. \qquad \ldots (2)$$

Putting $-x^2 = u$ so that $-2x dx = du$, we have

$$\int (-2x e^{-x^2})\, dx = \int e^u du = e^u = e^{-x^2}. \qquad \ldots (3)$$

Integrating by parts with $(-2x\ e^{-x^2})$ as the second function and using (3), (2) reducs to

$$\frac{dI}{ds} = \frac{1}{2}\left\{ \left[\sin sx\ e^{-x^2}\right]_0^\infty - \int_0^\infty s \cos sx . e^{-x^2} dx \right\} = -\frac{s}{2} \int_0^\infty e^{-x^2} \cos sx\, dx = -\frac{sI}{2}, \text{ using (1).}$$

Separating variables, $$(1/I)\, dI = -(s/2)\, ds$$

Integrating, $$\log I = -(s^2/4) + \log C \qquad \text{or} \qquad I = C e^{-s^2/4}, \qquad \ldots (4)$$

where C is constant of integration.

Now, (1) and (4) $\Rightarrow$ $\quad C\,e^{-s^2/4} = \int_0^\infty e^{-x^2} \cos sx\, dx.$ ... (5)

Putting $s = 0$ in (5), we get $\quad C = \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ ... (6)

$\therefore$ (1), (4) and (5) $\Rightarrow$ $\quad F_c\{e^{-x^2}\} = I = (\sqrt{\pi}/2)\, e^{-s^2/4}.$

(*b*) We have, $\quad F_s\{x e^{-x^2/2}\} = \int_0^\infty x\, e^{-x^2/2} \sin sx\, dx$, by definition

or $\quad F_s\{x e^{-x^2/2}\} = -\int_0^\infty \sin sx \times (-x\, e^{-x^2/2})\, dx.$ ... (1)

Putting $-x^2/2 = u$ so that $-x\, dx = du$, we have

$$\int (-x)\, e^{-x^2/2} dx = \int e^u du = e^u = e^{-x^2/2} \quad \text{... (2)}$$

Integrating by parts with $\left(-x\, e^{-x^2/2}\right)$ as the second function and using (2), (1) reduces to

$$F_s\{x\, e^{-x^2/2}\} = -\left\{\left[\sin sx \times e^{-x^2/2}\right]_0^\infty - \int_0^\infty s \cos sx\, e^{-x^2/2} dx\right\}$$

or $\quad F_s\{x\, e^{-x^2/2}\} = s\int_0^\infty \cos sx\ e^{-x^2/2} dx = s\,I,$ ... (3)

where $\quad I = \int_0^\infty \cos sx\ e^{-x^2/2} dx.$ ... (4)

From (4), $\quad \frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s}\left(\cos sx\ e^{-x^2/2}\right) dx = \int_0^\infty (-x \sin sx) e^{-x^2/2} dx$

or $\quad \frac{dI}{ds} = \int_0^\infty \sin sx \left(-x e^{-x^2/2}\right) dx = \left[\sin sx \times e^{-x^2/2}\right]_0^\infty - \int_0^\infty s \cos sx\, e^{-x^2/2} dx$

or $\quad \frac{dI}{ds} = -s\int_0^\infty \cos sx\ e^{-x^2/2} dx = -s\,I,$ using (4)

$\therefore$ $\quad (1/I)\, dI = -s\, ds$ $\quad$ so that $\quad \log I = -(s^2/2) + \log C$

$\therefore$ $\quad \log (I/C) = -(s^2/2)$ $\quad$ or $\quad I = C\,e^{-(s^2/2)}$ ... (5)

From (4) and (5), $\quad C\ e^{-(s^2/2)} = \int_0^\infty \cos sx\ e^{-x^2/2} dx$ ... (6)

Put $s = 0$ in (6), we get $\quad C = \int_0^\infty e^{-x^2/2} dx$ ... (7)

We know that $\quad \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ ... (8)

Putting $x = v/\sqrt{2}$ and $dx = dv/\sqrt{2}$, (8) reduces to

$$\int_0^\infty e^{-v^2/2}\frac{dv}{\sqrt{2}} = \frac{\sqrt{\pi}}{2} \qquad \text{so that} \qquad \int_0^\infty e^{-x^2/2}dx = \frac{\sqrt{2}\sqrt{\pi}}{2} \qquad \ldots (9)$$

Now, (7) and (9) $\Rightarrow$ $C = \sqrt{2}\sqrt{\pi}/2.$

Then, (4), (5) and (7) $\Rightarrow$ $I = (\sqrt{2}\sqrt{\pi}/2)e^{-s^2/2}.$

$\therefore$ (3) $\Rightarrow$ $F_s\left\{x e^{-x^2/2}\right\} = sI = \sqrt{(\pi/2)}\, s e^{-s^2/2}.$

Ex. 6. *Find Fourier sine transform of (1/x) × e^{-ax}*
[Meerut 2000; Kanpur 2005; Purvanchal 1995, 96; Bangalore 1994; Delhi Physics (Hons) 2005]

Sol. By definition, $$F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_0^\infty \frac{e^{-ax}}{x}\sin sx\, dx = 1, \text{ say.} \qquad \ldots (1)$$

$$\therefore \quad \frac{dI}{ds} = \int_0^\infty \frac{e^{-ax}}{x}(x\cos sx)\,dx = \int_0^\infty e^{-ax}\cos sx\, dx = \frac{a}{a^2+s^2}.$$

Thus, $dI = \{a/(a^2 + s^2)\}\, ds$

Integrating, $I = \tan^{-1}(s/a) + c$, c being an arbitrary constant ... (2)

For $s = 0$, (1) and (2) give $c + \tan^{-1} 0 = 0$ so that $c = 0$.

$\therefore$ From (1) and (2), $I = \tan^{-1}(s/a)$. or $F_s\{e^{-ax}/x\} = \tan^{-1}(s/a)$

Ex. 7. *Find Fourier cosine transform of $1/(1+x^2)$. Hence find Fourier sine transform of $x/(1 + x^2)$.* **[Osmania 2010; Kanpur 2007; Meerut 1998]**

Sol. By definition, $$F_c\left\{\frac{1}{1+x^2}\right\} = \int_0^\infty \frac{\cos sx}{1+x^2}dx = I, \text{ say} \qquad \ldots (1)$$

$$\therefore \quad \frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx\,dx}{1+x^2} = -\int_0^\infty \frac{x\sin sx}{1+x^2}dx \qquad \ldots (2)$$

or $$\frac{dI}{ds} = -\int_0^\infty \frac{x^2\sin sx\,dx}{x(1+x^2)} = -\int_0^\infty \frac{\left[(1+x^2)-1\right]\sin sx}{x(1+x^2)}dx = -\int_0^\infty \frac{\sin sx}{x}dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx$$

$$\therefore \quad \frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx, \quad \text{as} \quad \int_0^\infty \frac{\sin sx}{x}dx = \frac{\pi}{2} \qquad \ldots (3)$$

From (3), $$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}dx = \int_0^\infty \frac{\cos sx}{1+x^2}dx = I$$

or $(d^2I/ds^2) - I = 0$ or $(D^2 - 1)\, I = 0$, where $D \equiv d/ds$

whose solution is $I = Ae^{-s} + Be^{s}$, A and B being arbitrary constants ... (4)

From (4), $dI/ds = -Ae^{-s} + Be^{s}$ ... (5)

For $s = 0$, (1) and (4) give $\quad A + B = \int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2}.$ ... (6)

For $s = 0$, (3) and (5) give $\quad -A + B = \pi/2.$ ... (7)

Solving (6) and (7), A = π/2, B = 0. So from (1) and (4), we have

$$I = (\pi/2) \times e^{-s} \qquad \text{or} \qquad F_c\left\{1/(1+x^2)\right\} = (\pi/2) \times e^{-s} \quad \text{... (8)}$$

Second part. We have $\quad F_s\left\{\frac{x}{1+x^2}\right\} = \int_0^\infty \frac{x \sin sx}{1+x^2} dx$, by definition

$= -\, dI/ds$, from (2) of part (*a*)

$= (\pi/2) \times e^{-s}$, from (5) with A = π/2 and B = 0.

Ex. 8. (*a*) *Find the sine and cosine transform of* $x^n e^{-ax}$. **[Kanpur 1996]**

(*b*) *Find F(x) if* $\bar{F}_s(s) = s^n e^{-as}$ *and* $\bar{F}_c(s) = s^n e^{-as}$. **[Meerut 1994]**

Sol. (*a*) By definition $\quad F_s\{x^n e^{-ax}\} = \int_0^\infty x^n e^{-ax} \sin sx\, dx$ ... (1)

We have, $\quad \int_0^\infty e^{ax} \sin sx\, dx = \frac{s}{a^2 + s^2}.$ ... (2)

Differentiating (2) *n* times with respect to '*a*' we get

$$\int_0^\infty \left\{\frac{d^n}{da^n} e^{-ax}\right\} \times \sin sx\, dx = \frac{d^n}{da^n}\left(\frac{s}{a^2+s^2}\right)$$

or $\quad (-1)^n \int_0^\infty e^{-ax} x^n \sin sx\, dx = \frac{(-1)^n \times n!}{(a^2+s^2)^{(n+1)/2}} \sin\{(n+1)\tan^{-1}(s/a)\}$, using result 2, Art. 1.10

$\therefore$ (1) gives $\quad F_s\{x^n e^{-ax}\} = \frac{n! \sin\{(n+1)\tan^{-1}(s/a)\}}{(a^2+s^2)^{(n+1)/2}}$

Similarly, $\quad F_c\{x^n e^{-ax}\} = \frac{n! \cos\{(n+1)\tan^{-1}(s/a)\}}{(a^2+s^2)^{(n+1)/2}}.$

(*b*) Given $\bar{F}_s(s) = s^n e^{-as}$. Using inversion formula for sine transform, we get

$$F(x) = \frac{2}{\pi}\int_0^\infty \bar{F}_s(s) \sin sx \;\; ds = \frac{2}{\pi}\int_0^\infty s^n e^{-as} \sin sx\, ds \quad \text{... (1)}$$

We have, $\quad \int_0^\infty e^{-as} \sin sx\, ds = \frac{x}{a^2 + x^2}.$ ... (2)

Differentiating (2) *n* times with respect to '*a*', we get as in part (*a*)

$$(-1)^n \int_0^\infty e^{-as} s^n \sin sx\, ds = (-1)^n \frac{n! \sin\{(n+1)\tan^{-1}(x/a)\}}{(x^2+a^2)^{(n+1)/2}}$$

$\therefore$ From (1), $\quad F(x) = \frac{2}{\pi} \times \frac{n! \sin\{(n+1)\tan^{-1}(x/a)\}}{(x^2+a^2)^{(n+1)/2}}$

Similarly, when $\overline{F}_c(s) = s^n e^{-ax}$, we have $F(x) = \frac{2}{\pi} \times \frac{n! \cos\{(n+1)\tan^{-1}(x/a)\}}{(x^2+a^2)^{(n+1)/2}}$.

Ex. 9. *Find F(x) if its sine transform is (1/s) × e^{-as}. Hence deduce $F_s^{-1}\{1/s\}$*

Sol. Given $f_s(s) = (1/s) \times e^{-as}$. ... (1)

Using inversion formula for sine transform, we have

$F(x) = F_s^{-1}\{f_s(s)\} = \frac{2}{\pi}\int_0^\infty f_s(s)\sin sx\, ds$ or $F(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds$, using (1) ... (2)

Differentiating both sides of (2) w.r.t. 'x' we get

$\frac{dF}{dx} = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s} \times (s\cos sx)\, ds$ or $\frac{dF}{dx} = \frac{2}{\pi}\int_0^\infty e^{-as}\cos sx\, ds$

or $\frac{dF}{dx} = \frac{2}{\pi} \times \frac{a}{a^2+x^2}$ $\left[\because \int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}\right]$

or $dF = (2a/\pi) \times \{1/(a^2+x^2)\}\, dx$

Integrating, $F(x) = (2/\pi) \times \tan^{-1}(x/a) + A$, ... (3)

where A is arbitrary constant to be determined

From (2), we have $F(0) = 0$ ... (4)

Putting $x = 0$ in (3) and using (4), we get $0 = 0 + A$ so that $A = 0$.

$\therefore$ From (3), $F(x) = (2/\pi) \times \tan^{-1}(x/a)$. ... (5)

Deduction: From above, we have proved that

$$F_s^{-1}\{(1/s) \times e^{-as}\} = (2/\pi) \times \tan^{-1} \times (x/a) \quad \text{... (6)}$$

Putting $a = 0$ in (6), $F_s^{-1}\{(1/s)\} = (2/\pi) \times \tan^{-1}(\infty) = (2/\pi) \times (\pi/2) = 1$.

Ex. 10. *Find Fourier sine and cosine transform of F(x) = x.*

Sol. By definition, $f_c(s) = \int_0^\infty x\cos sx\, dx$ ... (1)

and $f_s(s) = \int_0^\infty x\sin sx\, dx$ ... (2)

(1) and (2) $\Rightarrow$ $f_c(s) - if_s(s) = \int_0^\infty x(\cos sx - i\sin sx)\, dx, = \int_0^\infty x e^{-isx} dx = \int_0^\infty \frac{y}{is} e^{-y} \frac{dy}{is}$

[Putting $isx = y$ so that $dx = dy/(is)$]

$\therefore$ $f_c(s) - if_s(s) = -\frac{1}{s^2}\int_0^\infty y^{2-1}e^{-y}dy = -\frac{\Gamma(2)}{s^2} = -\frac{1}{s^2}$, as $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx, n > 0$ *and* $\Gamma(2) = 1$

Equating real and imaginary parts, we get $f_c(s) = -1/s^2$ and $f_s(s) = 0$.

Ex. 11. *Find the Fourier sine transform of F(x) = 1/x (x^2 + a^2)*

Sol. By definition $F_s\left\{\frac{1}{x(x^2+a^2)}\right\} = \int_0^\infty \frac{1}{x(x^2+a^2)}\sin sx\, dx = I$, say ... (1)

From (1), $\frac{dI}{ds} = \int_0^\infty \frac{x\cos sx}{x(x^2+a^2)}dx = \int_0^\infty \frac{\cos sx}{x^2+a^2}dx$ ... (2)

From (2), $\dfrac{d^2\mathrm{I}}{ds^2} = -\int_0^\infty \dfrac{x\sin sx}{x^2+a^2}dx = -\int_0^\infty \dfrac{x^2\sin sx}{x(x^2+a^2)}dx = -\int_0^\infty \dfrac{(x^2+a^2)-a^2}{x^2+a^2}\dfrac{\sin sx}{x}dx$

$$= -\int_0^\infty \left(1-\frac{a^2}{x^2+a^2}\right)\frac{\sin sx}{x}dx = -\int_0^\infty \frac{\sin sx}{x}dx + a^2\int_0^\infty \frac{\sin sx\,dx}{x(x^2+a^2)}.$$

$\therefore$ $\dfrac{d^2\mathrm{I}}{ds^2} = -\dfrac{\pi}{2} + a^2\,\mathrm{I}$, using (1) and the result $\displaystyle\int_0^\infty \frac{\sin sx}{x}dx = \frac{\pi}{2}$

or $(d^2\mathrm{I}/ds^2) - a^2\,\mathrm{I} = -(\pi/2)$ or $(\mathrm{D}^2 - a^2)\mathrm{I} = -(\pi/2)$, where $\mathrm{D} \equiv (d/ds)$ ... (3)

Its auxiliary equation is $\mathrm{D}^2 - a^2 = 0$ giving $\mathrm{D} = a, -a$.

$\therefore$ C. F. $= \mathrm{A}e^{as} + \mathrm{B}e^{-as}$, A and B being arbitrary constants.

and P.I. $= \dfrac{1}{\mathrm{D}^2-a^2}\left(-\dfrac{\pi}{2}\right) = -\dfrac{\pi}{2}\dfrac{1}{\mathrm{D}^2-a^2}e^{0.s} = -\dfrac{\pi}{2}\dfrac{1}{0-a^2}e^{0.s} = \dfrac{\pi}{2a^2}$.

$\therefore$ Solution of (3) is $\mathrm{I} = \mathrm{A}e^{as} + \mathrm{B}e^{-as} + (\pi/2a^2)$. ... (4)

Putting $s = 0$ in (1) and (4), we get $\mathrm{I} = 0$ and $\mathrm{I} = \mathrm{A} + \mathrm{B} + \pi/(2a^2)$

$\therefore$ $\mathrm{A} + \mathrm{B} + (\pi/2a^2) = 0$ or $\mathrm{A} + \mathrm{B} = -(\pi/2a^2)$. ... (5)

From (4), $d\mathrm{I}/ds = \mathrm{A}a\,e^{as} - \mathrm{B}\,a\,e^{-as}$. ... (6)

Putting $s = 0$ in (2) and (6), we get

$\dfrac{d\mathrm{I}}{ds} = \displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty = \frac{\pi}{2a}$ and $\dfrac{d\mathrm{I}}{ds} = \mathrm{A}a - \mathrm{B}a$.

$\therefore$ $\mathrm{A}a - \mathrm{B}a = \pi/2a$ or $\mathrm{A} - \mathrm{B} = \pi/2a^2$.

Solving (5) and (7),we get $\mathrm{A} = 0$ and $\mathrm{B} = -(\pi/2a^2)$.

$\therefore$ From (4), $\mathrm{I} = -(\pi/2a^2)\times e^{-as} + (\pi/2a^2) = (\pi/2a^2)\times(1-e^{-as})$

or $\mathrm{F}_s\{1/x(x^2+a^2)\} = (\pi/2a^2)\times(1-e^{-as})$, by (1)

Ex. 12. *Find F(x) it is Fourier cosine transform is 1/(1+s²).*

or Compute $F_c^{-1}\{1/(1+s^2)\}$ **[Kanpur 1997; Meerut 1996; Purvanchal 1995]**

Sol. Let $\mathrm{F}(x) = \mathrm{F}_c^{-1}\{1/(1+s^2)\}$.

Then, by inversion formula, $\mathrm{F}(x) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{1}{1+s^2}\cos sx\,ds$ ... (1)

From (1), $\dfrac{d\mathrm{F}}{dx} = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{-s\sin sx}{1+s^2}ds = -\frac{2}{\pi}\int_0^\infty \frac{s^2\sin sx}{s(1+s^2)}ds = -\frac{2}{\pi}\int_0^\infty \frac{(1+s^2)-1}{1+s^2}\frac{\sin sx}{s}ds$

$$= -\frac{2}{\pi}\int_0^\infty \left(1-\frac{1}{1+s^2}\right)\frac{\sin sx}{s}ds = -\frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s}ds + \frac{2}{\pi}\int_0^\infty \frac{\sin sx\,ds}{s(1+s^2)}$$

$\therefore$ $\dfrac{d\mathrm{F}}{dx} = -\dfrac{2}{\pi}\times\dfrac{\pi}{2} + \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin sx\,ds}{s(1+s^2)}$, as $\displaystyle\int_0^\infty \frac{\sin sx}{s}ds = \frac{\pi}{2}$ ... (2)

From (2), $\dfrac{d^2\,\mathrm{F}}{dx^2} = \dfrac{2}{\pi}\int_0^\infty \dfrac{s\cos sx\,ds}{s(1+s^2)} = \dfrac{2}{\pi}\int_0^\infty \dfrac{\cos sx}{1+s^2}dx = \mathrm{F}(x)$, by (1)

or $(\mathrm{D}^2 - 1)\,\mathrm{F} = 0$, where $\mathrm{D} \equiv d/dx$ ... (3)

whose auxiliary equation is $\mathrm{D}^2 - 1 = 0$ giving $\mathrm{D} = 1, -1$.

$\therefore$ Solution of (3) is $\mathrm{F}(x) = \mathrm{A}\,e^{-x} + \mathrm{B}\,e^{x}$, *A, B* being arbitrary constants ... (4)

Putting $x = 0$ in (1) and (4), we get

$$\mathrm{F}(x) = \frac{2}{\pi}\int_0^\infty \frac{ds}{1+s^2} = \frac{2}{\pi}\Big[\tan^{-1} s\Big]_0^\infty = \frac{2}{\pi}\times\frac{\pi}{2} = 1 \quad \text{and} \quad \mathrm{F}(x) = \mathrm{A} + \mathrm{B}.$$

Thus, $\mathrm{A} + \mathrm{B} = 1$. ... (5)

From (4), $d\mathrm{F}/dx = -\mathrm{A}e^{-x} + \mathrm{B}\,e^{x}$. ... (6)

Putting $x = 0$ in (2) and (6), we get $d\mathrm{F}/dx = -1$ and $d\mathrm{F}/dx = -\mathrm{A} + \mathrm{B}$.

$\therefore$ $-\mathrm{A} + \mathrm{B} = -1$. ... (7)

Solving (5) and (7), A = 1 and B = 0, Hence, from (4), $\mathrm{F}(x) = e^{-x}$.

Ex. 13. *Find F(x) if its Fourier sine transform is $s/(1+s^2)$.*

or *Compute $F_s^{-1}\{s/(1+s^2)\}$* **[Meerut 2006; Rohilkhand 1993, 95]**

Sol. Let $\mathrm{F}(x) = \mathrm{F}_s^{-1}\{s/(1+s^2)\}$.

Then, by inversion formula, $\mathrm{F}(x) = \dfrac{2}{\pi}\int_0^\infty \dfrac{s}{1+s^2}\sin sx\,ds$ ... (1)

or $$\mathrm{F}(x) = \frac{2}{\pi}\int_0^\infty \frac{s^2\sin sx}{s(1+s^2)}ds = \frac{2}{\pi}\int_0^\infty \frac{(s^2+1)-1}{1+s^2}\frac{\sin sx}{s}ds$$

$$= \frac{2}{\pi}\int_0^\infty \left(1 - \frac{1}{1+s^2}\right)\frac{\sin sx}{s}ds = \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s}ds - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}ds$$

$$= \frac{2}{\pi}\times\frac{\pi}{2} - \frac{2}{\pi}\int_0^\infty \frac{\sin sx\,ds}{s(1+s^2)} \qquad \left[\because \int_0^\infty \frac{\sin sx}{s}ds = \frac{\pi}{2}, s > 0\right].$$

$\therefore$ $$\mathrm{F}(x) = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin sx\,ds}{s(1+s^2)}.$$ ... (2)

From (2), $\dfrac{d\,\mathrm{F}}{dx} = -\dfrac{2}{\pi}\int_0^\infty \dfrac{s\cos sx}{s(1+s^2)}ds = -\dfrac{2}{\pi}\int_0^\infty \dfrac{\cos sx}{1+s^2}ds$ ... (3)

From (3), $\dfrac{d^2\,\mathrm{F}}{dx^2} = -\dfrac{2}{\pi}\int_0^\infty \dfrac{(-s\sin sx)}{1+s^2}ds = \dfrac{2}{\pi}\int_0^\infty \dfrac{s\sin sx}{1+s^2}ds$ or $\dfrac{d^2F}{dx^2} = F$, by (1)

or $(\mathrm{D}^2 - 1)\,\mathrm{F} = 0$, $\mathrm{D} \equiv d/ds$; whose general solution is

$\mathrm{F}(x) = \mathrm{A}\,e^{x} + \mathrm{B}\,e^{-x}$, *A* and *B* being arbitrary constants ... (4)

Putting $x = 0$ in (2) and (4), we get $\mathrm{F}(x) = 1$ and $\mathrm{F}(x) = \mathrm{A} + \mathrm{B}$

$\therefore$ $\mathrm{A} + \mathrm{B} = 1$ ... (5)

Form (4), $d\mathrm{F}/dx = \mathrm{A}e^{x} - \mathrm{B}e^{-x}$ ... (6)

Putting $x = 0$ in (3) and (6), we get

$$\frac{d\,\mathrm{F}}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{ds}{1+s^2} = -\frac{2}{\pi}\left[\tan^{-1} s\right]_0^\infty = -\frac{2}{\pi}\times\frac{\pi}{2} = -1 \qquad \text{and} \qquad \frac{d\,\mathrm{F}}{dx} = \mathrm{A}-\mathrm{B}.$$

$$\therefore \qquad \mathrm{A}-\mathrm{B} = -1. \qquad \ldots (7)$$

Solving (5) and (7), A = 0, B = 1. So from (4), $\mathrm{F}(x) = e^{-x}$.

Ex. 14. *Find the Fourier sine transform of $x/(1+x^2)$.*

Sol. By the definition of Fourier sine transform, we have

$$\mathrm{F}_s\left\{\frac{x}{1+x^2}\right\} = \int_0^\infty \frac{x}{1+x^2}\sin sx\,dx = \mathrm{I}, \text{ say.} \qquad \ldots (1)$$

Now, $$\mathrm{I} = \int_0^\infty \frac{(x^2+1)-1}{x(x^2+1)}\sin sx\,dx = \int_0^\infty \frac{\sin sx}{x}dx - \int_0^\infty \frac{\sin sx}{x(x^2+1)}dx$$

or $$\mathrm{I} = \frac{\pi}{2} - \int_0^\infty \frac{\sin sx}{x(x^2+1)}dx, \qquad \text{as} \qquad \int_0^\infty \frac{\sin sx}{x}dx = \frac{\pi}{2} \qquad \ldots (2)$$

Differentiating both sides of (2) *w.r.t.* '*s*' we have

$$\frac{d\,\mathrm{I}}{ds} = -\int_0^\infty \frac{\partial}{\partial s}\left\{\frac{\sin sx}{x(x^2+1)}\right\}dx = -\int_0^\infty \frac{x\cos sx}{x(x^2+1)}dx = -\int_0^\infty \frac{\cos sx}{x^2+1}dx \qquad \ldots (3)$$

Differentiatiang again both sides of (3) *w.r.t.* '*s*', we have

$$\frac{d^2\,\mathrm{I}}{ds^2} = -\int_0^\infty \frac{\partial}{\partial s}\left(\frac{\cos sx}{x^2+1}\right)dx = \int_0^\infty \frac{x\sin sx}{x^2+1}dx = \mathrm{I}, \text{ by (1)}$$

or $$(\mathrm{D}^2-1)\,\mathrm{I} = 0, \qquad \text{where} \qquad \mathrm{D} \equiv d/ds. \qquad \ldots (4)$$

Auxiliary equation of (4) is $\mathrm{D}^2 - 1 = 0$ so that $\mathrm{D} = 1, -1$.

So solution of (4) is $\mathrm{I} = \mathrm{A}\,e^{s} + \mathrm{B}\,e^{-s}$, *A, B* being arbitrary constants ... (5)

When $s = 0$, from (2), $\mathrm{I} = (\pi/2) - 0 = \pi/2$. ... (6)

When $s = 0$, from (3), $$\frac{d\mathrm{I}}{ds} = -\int_0^\infty \frac{dx}{x^2+1} = -\left[\tan^{-1} x\right]_0^\infty = -\frac{\pi}{2}. \qquad \ldots (7)$$

Differentiating both sides of (5) *w.r.t.* '*s*', $d\mathrm{I}/ds = \mathrm{A}\,e^{s} - \mathrm{B}e^{-s}$. ... (8)

Putting $s = 0$ in (5) and using (6), we have $\mathrm{A} + \mathrm{B} = \pi/2$ ... (9)

Putting $s = 0$ in (8) and using (7), we have $\mathrm{A} - \mathrm{B} = -\pi/2$ ... (10)

Solving (9) and (10), we have A = 0 and B = $\pi/2$

$\therefore$ (1) and (5) $\Rightarrow$ $\mathrm{F}_s\{x/(1+x^2)\} = \mathrm{I} = (\pi/2)\times e^{-s}$

EXERCISE 1 (B)

1. Find the Fourier cosine transform of $\mathrm{F}(x) = \begin{cases}\cos x, & 0 < x < a\\ 0, & x > a\end{cases}$ **(Osmania 2010)**

Ans. $\{(1-s)\sin(1+s)a + (1+s)\sin(1-s)a\}/2(1-s^2)$

2. Find the Fourier sine and cosine transorm of F(x), if

$$\mathrm{F}(x) = \begin{cases}1, & 0 \le x < 1\\ 0, & x > 1\end{cases}$$ **[Guwahati 2000]**

Ans. $F_s\{F(x)\} = (1-\cos s)/s$; $F_c\{F(x)\} = (1/s)\times\sin s$

3. Find F(x) if its cosine transform is given by

$$\overline{F}_c(s) = \begin{cases} (1/2\pi) \times (a - s/2), & s < 2a \\ 0, & s \geq 2a \end{cases}$$

[Rohilkhand 1996]

Ans. $(1 - \cos 2xa)/2\pi^2x^2$

4. Find the Fourier cosine transform of F(x), where $F(x) = \begin{cases} 0, & 0 < x < a \\ x, & a \leq x < b \\ 0, & x > b \end{cases}$

Ans. $(a \cos sa - b \cos sb)/s + (\sin sb - \sin sa)/s^2$

5. Find F(x) if $\overline{F}_c(s) = (\pi/2) \times e^{-s}$ **[Meerut 1995]** **Ans.** $1/(1 + s^2)$

6. Find the sine and cosine transform of $2e^{-5x} + 5e^{-2x}$. **[S.V. Univ. (A.P.) 1997]**

Ans. $(2s)/(s^2 + 25) + (5s)/(s^2 + 4)$

7. Find the sine and cosine transform of $(e^{ax} + e^{-ax}) / (e^{\pi x} - e^{-\pi x})$.

Ans. $(\sinh s)/2(\cos a + \cosh s)$; $\{\cos(a/2) \cosh(s/2)\}/(\cos a + \cosh s)$

8. Find the Fourier sine and cosine transform of x^{m-1}.

Ans. $F_s\{x^{m-1}\} = \{\Gamma(m)/s^m\} \times \sin(m\pi/2)$; $F_c\{x^{m-1}\} = \{\Gamma(m)/s^m\} \times \cos(m\pi/2)$

9. Find the cosine tranform of $1/(x^2 + a^2)$. **Ans.** $(\pi/2) \times e^{-as}$

10. Find F(x) if its sine transform is $\pi/2$. **Ans.** $\pi/2x$

11. Find the sine transform of $1/(e^{\pi x} - e^{-\pi x})$ and deduce that

$F_s(\text{cosech } \pi x) = (1/\sqrt{2\pi}) \times \tanh(s/2)$.

12. Find the Fourier sine transform of $F(x) = 1/x$. **[Meerut 2001; Kanpur 2002]**

Ans. $\pi/2$

13. Find Fourier sine and cosine transform of e^{-x} and using the inversion formula recover the original functions in both the cases. **Ans.** $F_s\{e^{-x}\} = s/(1+s^2)$; $F_c\{e^{-x}\} = 1/(1+s^2)$

14. Find the Fourier transform of $f(x)$, if $f(x) = \begin{cases} \sqrt{2\pi}/2\varepsilon, & |x| < \varepsilon \\ 0, & |x| > \varepsilon. \end{cases}$

[**Hint:** Use symmetric form of definition given Art. 1.6] **Ans.** $(\sin s\ \varepsilon)/(s\ \varepsilon)$]

15. Find $f(x)$ if (*i*) its sine transform is e^{-as} (*ii*) its cosine transform is e^{-as}

Ans. (*i*) $F_s^{-1}\left[e^{-as}\right] = 2x/\pi(a^2 + x^2)$ (*ii*) $F_s^{-1}\left[e^{-as}\right] = 2a/\pi(a^2 + x^2)$

16. Find Fourier Complex transform of $(\sin ax)/x$. **Ans.** π, if $|s| < a$; 0, if $|s| > a$.

17. Find (*i*) $F_c\{x^{-n}\}$ (*ii*) $F_s\{x^{-n}\}$.

Ans. (*i*) $[\pi s^{n-1}/2\Gamma(n)]\sec(n\pi/2)$ (*ii*) $[\pi s^{n-1}/2\Gamma(n)]\text{cosec}(n\pi/2)$

18. Show that $F\left\{\dfrac{\cosh ax}{\cosh \pi x}\right\} = \dfrac{2\cos(a/2)\cosh(s/2)}{\cos a + \cosh s}, -\pi < a < \pi.$

1.17. Solution of integral equations with help of Fourier sine and cosine transforms.

For definition and more details about integral equations, please refer chapter 4 of part IV–A of this book.

1.18 Solved examples based on Art. 1.17

Ex.1. *Solve the integral equation* $\int_0^\infty F(x)\cos sx\,dx = \begin{cases}1-s, 0\le s\le 1\\ 0, s>1\end{cases}$

[Osmania 2010; Purvanchal 1996; S.V. University (A.P) 1997]

Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}.$

Sol. Let $\int_0^\infty F(x)\cos dx = f_c(s)$ ... (1)

where $f_c(s) = \begin{cases}1-s, 0\le s\le 1\\ 0, s>1.\end{cases}$ ... (2)

Then, by defintion, $f_c(s)$ is Fourier cosine transform of $F(x)$. In order to solve the given integral equotion, we must obtain value of $F(x)$.

Now, by inversion formula for Fourier cosine transform, we get

$$F(x) = F_c^{-1}\{f_c(s)\} = \frac{2}{\pi}\int_0^\infty f_c(s)\cos sx\,ds$$

$$= \frac{2}{\pi}\left[\int_0^1 f_c(s)\cos sx\,ds + \int_1^\infty f_c(s)\cos sx\,ds\right] = \frac{2}{\pi}\left[\int_0^1 (1-s)\cos sx\,ds + 0\right], \text{ using (2)}$$

$$= \frac{2}{\pi}\left\{\left[(1-s)\frac{\sin sx}{x}\right]_0^1 - \int_0^1 (-1)\frac{\sin sx}{x}ds\right\} = \frac{2}{\pi}\left\{0 + \frac{1}{x}\left[-\frac{\cos sx}{x}\right]_0^1\right\} = \frac{2}{\pi x^2}(1-\cos x). \quad \text{... (3)}$$

Deduction. From (1), (2) and (3), we have

$$\int_0^\infty \frac{2}{\pi x^2}(1-\cos x)\cos sx\,dx = \begin{cases}1-s, 0\le s\le 1\\ 0, s>1\end{cases} \quad \text{... (4)}$$

Taking limit as $s \to 0$ of both sides of (4), we have

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}dx = 1 \quad \text{or} \quad \frac{2}{\pi}\int_0^\infty \frac{2\sin^2(x/2)}{x^2}dx = 1 \quad \text{... (5)}$$

Putting $x = 2t$ so that $dx = 2dt$, (5) reduces to

$$\frac{4}{\pi}\int_0^\infty \frac{\sin^2 t}{4t^2}(2dt) = 1 \quad \text{or} \quad \int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}.$$

Ex. 2. *Solve the integral equation* $\int_0^\infty f(x)\cos \lambda x\,dx = e^{-\lambda}.$

Sol. Let $\int_0^\infty f(x)\cos \lambda x\,dx = f_c(\lambda),$ ... (1)

where $f_c(\lambda) = e^{-\lambda}.$ ... (2)

[Here λ has been used in place of the usual variable s]

Then, by definition, $f_c(\lambda)$ is Fourier cosine transform of $f(x)$. In order to solve the given integral equation, we must obtain value of $f(x)$.

Now, by inversion formula for Fourier cosine transform, we get

$$f(x) = \mathrm{F}_c^{-1}\{f_c(\lambda)\} = \frac{2}{\pi}\int_0^\infty f_c(\lambda)\cos\lambda x\, d\lambda = \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\, d\lambda, \text{ using (2)}$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda x + x\sin\lambda x)\right]_{\lambda=0}^{\infty} = \frac{2}{\pi(1+x^2)}.$$

Ex. 3. *Solve for F(x) the integral equotion:* $\int_0^\infty F(x)\sin xt\, dx = \begin{cases}1, 0 \le t < 1\\ 2, 1 \le t \le 2\\ 0, t \ge 2.\end{cases}$

Sol. Let
$$\int_0^\infty \mathrm{F}(x)\sin xt\, dx = f_s(t), \quad \ldots (1)$$

where
$$f_s(t) = \begin{cases}1, 0 \le t < 1\\ 2, 1 \le t \le 2\\ 0, t \ge 2.\end{cases} \quad \ldots (2)$$

[Here t has been used in place of the usual variable s]

Then, by definition, $f_c(t)$ is Fourier sine transform of $\mathrm{F}(x)$.

Now, by inversion formula for Fourier sine transform, we have

$$\mathrm{F}(x) = \mathrm{F}_s^{-1}\{f_s(t)\} = \frac{2}{\pi}\int_0^\infty f_s(t)\sin tx\, dt = \frac{2}{\pi}\left[\int_0^1 f_s(t)\sin tx\, dt + \int_1^2 f_s(t)\sin tx\, dt + \int_2^\infty f_s(t)\sin tx\, dt\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 \sin tx\, dt + \int_1^2 (2\times\sin tx)\, dt + 0\right] = \frac{2}{\pi}\left\{\left[-\frac{\cos tx}{x}\right]_0^1 + 2\left[-\frac{\cos tx}{x}\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left[-\frac{\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right] = \frac{2}{\pi x}(1 + \cos x - 2\cos 2x).$$

Ex. 4. *Show that* $\int_0^\infty \frac{\cos\lambda x}{\lambda^2+1} d\lambda = \frac{\pi}{2}e^{-x}, x \ge 0$ **[Purvanchal 1995]**

Sol. We know that
$$\int_0^\infty e^{-x}\cos\lambda x\, dx = \frac{1}{1+\lambda^2} \quad \ldots (1)$$

Re-writing (1),
$$\int_0^\infty e^{-x}\cos\lambda x\, dx = f_c(\lambda), \quad \ldots (2)$$

where
$$f_c(\lambda) = 1/(1+\lambda^2). \quad \ldots (3)$$

[Here λ has been used in place of the usual variable s]

Then, by definition, $f_c(\lambda)$ is Fourier cosine transform of e^{-x}. So, by inversion formula for Fourier cosine transform, we have

$$e^{-x} = \mathrm{F}_c^{-1}\{f_c(\lambda)\} = \frac{2}{\pi}\int_0^\infty f_c(\lambda)\cos\lambda x\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{1+\lambda^2} d\lambda, \text{ using (1)}$$

$$\therefore \quad \int_0^\infty \frac{\cos\lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2}e^{-x}.$$

1.19. Convolution or Falting. The convolution of two functions F(x) and G (x), where $-\infty < x < \infty$, is denoted and defined as

$$F * G = \int_{-\infty}^{\infty} F(u)\,G(x-u)\,du. \qquad \text{or} \qquad F*G = \int_{-\infty}^{\infty} G(u)\,F(x-u)\,du.$$

1.20. The convolution or Falting theorem for Fourier transforms.

[Maduraj Kamraj University 2008; Delhi Physics (Hons.) 1998, 99]

The Fourier transform of the convolution of F(x) and G(x) is the product of the Fourier transforms of F(x) and G(x), i.e., $F\{F * G\} = F\{F(x)\}\,F\{G(x)\}$.

Proof. L.H.S. $= F\left\{\int_{-\infty}^{\infty} F(u)\,G(x-u)du\right\}$, by definition of convolution

$$= \int_{-\infty}^{\infty} e^{isx}\left\{\int_{-\infty}^{\infty} F(u)\,G(x-u)du\right\}dx\,, \text{ by definition of Fourier transform}$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} e^{isx}\,G(x-u)\,dx\right\}du\,, \text{ changing the order of integration}$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} e^{is(u+v)}\,G(v)\,dv\right\}du, \text{ putting } x-u=v \text{ so that } dx=dv$$

$$= \int_{-\infty}^{\infty} e^{isu}\,F(u)\left\{\int_{-\infty}^{\infty} e^{isv}\,G(v)\,dv\right\}du = \int_{-\infty}^{\infty} e^{isu}\,F(u)\,F\{G(x)\}du\,, \text{ by definition}$$

$$= F\{G(x)\}\int_{-\infty}^{\infty} e^{isu}\,F(u)du = F\{G(x)\}\,F\{F(x)\}, \text{ by definition again}$$

1.21. Parseval's identity for Fourier transform. Rayleigh's Theorem or Plancherel's theorem. *If f (s) and g(s) are the complex Fourier transforms of F(x) and G(x) respectively, then*

$$(i)\ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\overline{g(s)}\,ds = \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,dx \qquad (ii)\ \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|^2\,ds = \int_{-\infty}^{\infty} |F(x)|^2\,dx,$$

where bar signifies the complex conjugate. **[Madurai Kamraj University 2008]**

Proof. (*i*) Using the inversion formula for Fourier transform, we get

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\,e^{-isx}\,ds \qquad \ldots (1)$$

Taking complex conjugates on both sides, (1) gives

$$\overline{G(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\,e^{isx}\,ds \qquad \ldots (2)$$

$$\therefore \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,dx = \int_{-\infty}^{\infty} F(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\,e^{isx}ds\right\}dx, \text{ using (2)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\left\{\int_{-\infty}^{\infty} F(x)\,e^{isx}dx\right\}ds, \text{ changing the order of integration}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(s)}\,f(s)\,ds, \text{ by definition of Fourier transform}$$

(*ii*) Taking G(*x*) = F(*x*) in part (*i*), we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\overline{f(s)}ds = \mathrm{F}(x)\overline{\mathrm{F}(x)}\,dx \qquad \text{or} \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}|f(s)|^2\,ds = \int_{-\infty}^{\infty}|\mathrm{F}(x)|^2\,dx.$$

Note. The reader can prove as above the following Parseval's identities for Fourier cosine and sine transforms:

$$(i)\ \frac{2}{\pi}\int_0^{\infty} f_c(s)g_c(s)ds = \int_0^{\infty}\mathrm{F}(x)\mathrm{G}(x)dx. \qquad (ii)\ \frac{2}{\pi}\int_0^{\infty} f_s(s)g_s(s)ds = \int_0^{\infty}\mathrm{F}(x)\mathrm{G}(x)dx.$$

$$(iii)\ \frac{2}{\pi}\int_0^{\infty}\left[f_c(s)\right]^2 ds = \int_0^{\infty}[\mathrm{F}(x)]^2\,dx. \qquad (iv)\ \frac{2}{\pi}\int_0^{\infty}\left[f_s(s)^2\right]ds = \int_0^{\infty}[\mathrm{F}(x)]^2\,dx.$$

1.21A. Examples based on Art. 1.21.

Ex.1. *Find Fourier transform of F(x) defined by* $F(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$ *and hence prove that*

$$\int_0^{\infty}\frac{\sin^2 ax}{x^2}dx = \frac{\pi a}{2}$$ **[Kanpur 2008; Purvanchal 1994]**

Sol. Given that $$\mathrm{F}(x) = \begin{cases} 1, -a < x < a \\ 0, x < -a \text{ or } x > a. \end{cases} \quad \ldots(1)$$

Let $f(s)$ be Fourier transform of F(x). Then, by definition

$$f(s) = \int_{-\infty}^{\infty} e^{isx}\mathrm{F}(x)\,dx = \int_{-\infty}^{-a} e^{isx}\mathrm{F}(x)\,dx + \int_{-a}^{a} e^{isx}\mathrm{F}(x)\,dx + \int_{a}^{\infty} e^{isx}\mathrm{F}(x)\,ds = \int_{-a}^{a} e^{isx}dx, \text{ by (1)}$$

$$= \left[(1/is)\times e^{isx}\right]_{-a}^{a} = (1/is)\times(e^{isa} - e^{-isa}) = (2/s)\times\sin sa. \quad \ldots(2)$$

Using Parseval's identity for Fourier transform, $$\int_{-\infty}^{\infty}\left[\mathrm{F}(x)\right]^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f(s)\right]^2 ds$$

or $$\int_{-\infty}^{-a}\left[\mathrm{F}(x)\right]^2 dx + \int_{-a}^{a}\left[\mathrm{F}(x)\right]^2 dx + \int_{a}^{\infty}\left[\mathrm{F}(x)\right]^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f(s)\right]^2 ds$$

or $$0 + \int_{-a}^{a}(1^2)\,dx + 0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin sa}{s}\right)^2 ds$$, putting ralues of F(x) and $f(s)$ from (1) and (2)

or $$2a = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 ax}{x^2}dx = \frac{2}{\pi}\times 2\int_0^{\infty}\frac{\sin^2 ax}{x^2}dx,$$ as $\frac{\sin^2 ax}{x^2}$ is an even function of x

Thus, $$\int_0^{\infty}\frac{\sin^2 ax}{x^2}dx = \frac{\pi a}{2}.$$

Ex. 2. *If* $F(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$ *and* $f(s) = \frac{2\sin sa}{s}, s \neq 0$, *then prove that* $\int_0^{\infty}\frac{\sin^2 ax}{x}dx = \frac{\pi a}{2}$, *where f(s) is Fourier transform of F(x).*

Sol. Left as an exercise.

Ex. 3. *Use Parseval's identities to prove that*

$$(i)\ \int_0^{\infty}\frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)} \qquad (ii)\ \int_0^{\infty}\frac{\sin at}{t(a^2+t^2)}dt = \frac{\pi}{2}\times\frac{1-e^{-a^2}}{a^2}$$

Sol. Left as exercises **[Madurai Kamraj University 2008]**

1.22. Fourier transforms of the derivatives of a function. Derivative Theorems.

Theorem 1. *If $F\{F(x)\} = f(s)$, then $F\{F'(x)\} = -is\,f(s)$.*

Or If F(x) has the Fourier transform f(s), then theFourier transform of F′(x), the derivative of F(x) is –is F{F(x)} **[Rohilkhand 1994; Kanpur 2004]**

Theorem 2. *$F\{F^{(n)}(x)\} = (-is)^n\, \mathrm{F}\{\mathrm{F}(x)\}$, i.e., the Fourier transform of the function $d^n F/dx^n$ is $(-is)^n$ times the Fourier transform of the function F(x) provided that the first (n – 1) derivatives of F(x) vanish as $x \to \pm\infty$.*

Theorem 3. Let $f_c^n(s)$ and $f_s^n(s)$ be the Fourier cosine and sine transform of the function $\mathrm{F}^{(n)}(x), i.e., d^n\,\mathrm{F}/dx^n$, then

$$(i)\ f_c^{2n}(s) = -\sum_{r=0}^{n-1}(-1)^r\,\alpha_{2n-2r-1}\ s^{2r} + (-1)^n s^{2n} f_c(s)$$

$$(ii)\ f_c^{2n+1}(s) = -\sum_{r=0}^{n}(-1)^r\,\alpha_{2n-2r}\ s^{2r} + (-1)^n s^{2n+1} f_s(s)$$

$$(iii)\ f_s^{2n}(s) = -\sum_{r=0}^{n}(-1)^r\,\alpha_{2n-2r}\ s^{2r-1} + (-1)^{n+1} s^{2n} f_s(s)$$

$$(iv)\ f_s^{2n+1}(s) = -\sum_{r=0}^{n}(-1)^r\,\alpha_{2n-2r+1}\ s^{2r-1} + (-1)^{n+1} s^{2n+1} f_c(s),$$

provided that the first $(n-1)$ derivatives of $f(x)$ vanish as $x \to \infty$ and $F^{(n-1)}(x) \to \alpha_{n-1}$ etc. as $x \to 0$

Theorem 4. *To show that $F\left\{\int_a^x F(x)dx\right\} = \dfrac{f(s)}{(-is)}$, where f(s) is Fourier transform of F(x).*

Proofs of theorems 1, 2, 3 and 4 are left as exercises for the reader.

1.23. Multiple Fourier transforms. The theory of Fourier transforms of functions of a single variable can be extended to functions of several variables. Let F(*x, y*) be a function of two variables. To begin with, we treat it temporarily as a function of *x*, then

$$\overline{\mathrm{F}}_x(p,y) = \int_{-\infty}^{\infty} \mathrm{F}(x,y)e^{ipx}dx,$$

where $\overline{\mathrm{F}}_x(p,y)$ stands for Fourier complex transform of *F*(*x, y*) with respect to *x*. Now, regarding $\overline{\mathrm{F}}_x(p,y)$ as a function of y, its complex Fourier transform $\overline{\mathrm{F}}(p,q)$ is given by

$$\overline{\mathrm{F}}(p,q) = \int_{-\infty}^{\infty} \overline{\mathrm{F}}_x(p,y)\,e^{iqy}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{F}(x,y)\,e^{i(px+qy)}dx\,dy$$

and the corresponding inversion formula is given by

$$\mathrm{F}(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\mathrm{F}}(p,q)\ e^{-i(px+qy)}dp\,dq$$

Note. When F(*x, y*) is a function of two variables we shall use new variables *p* and *q* while finding Fourier transform *w.r.t.* '*x*' and *'y'* respectively in place of single variable *s* as before.

PART III : APPLICATIONS OF INFINITE FOURIER TRANSFORMS TO BOUNDARY VALUE PROBLEMS

1.24. Some Useful results for direct applications

We first discuss the use of the (infinite) sine and cosine transform and later on examples will be presented in which the complex Fourier transform has to be used.

The (infinite) sine and cosine transforms can be applied when the range of the variable selected for exclusion (in a given boundary value problem) is 0 to ∞. The choice of sine and cosine transform is decided by the form of the boundary conditions at the lower limit of the variable selected for exclusion. In this connection note the following:

$$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin sx\,dx = \left[\frac{\partial u}{\partial x}\sin sx\right]_0^\infty - s\int_0^\infty \frac{\partial u}{\partial x}\cos sx\,dx, \text{ integrating by parts}$$

$$= -s\int_0^\infty \frac{\partial u}{\partial x}\cos sx\,dx, \text{if } \frac{\partial u}{\partial x}\to 0 \quad as \quad x\to\infty \text{ which usually happnes in physical problems}$$

$$= -s\left\{\left[u\cos sx\right]_0^\infty + s\int_0^\infty u\sin sx\,dx\right\} = s(u)_{x=0} - s^2\overline{u}_{s,} \text{ assuming that } u\to 0 \text{ as } x\to\infty$$

Thus
$$F_s\left\{\partial^2 u/\partial x^2\right\} = s\,u\,(0,\,t) - s^2\overline{u}_s(s,t). \qquad \ldots (1)$$

where $u\,(x,\,t)$ is a function of two variables x and t and $\overline{u}_s(s,t)$ is the Fourier sine transform of $u\,(x,\,t)$ with respect to x.

$$\text{Next, } F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx\,dx = \left[\frac{\partial u}{\partial x}\cos sx\right]_0^\infty + s\int_0^\infty \frac{\partial u}{\partial x}\sin sx\,dx, \text{ integrating by parts}$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\int_0^\infty \frac{\partial u}{\partial x}\sin sx\,dx, \text{ assuming that } \frac{\partial u}{\partial x}\to 0 \text{ as } x\to\infty$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\left\{\left[u\sin sx\right]_0^\infty - s\int_0^\infty u\cos sx\,dx\right\} = \left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2\int_0^\infty u(x,t)\cos sx\,dx,$$

assuming that $u\to 0$ as $x\to\infty$

$$\therefore \qquad F_c\{\partial^2 u/\partial x^2\} = -(\partial u/\partial x)_{x=0} - s^2\,\overline{u}_c(s,t), \qquad \ldots (2)$$

where $\overline{u}_c(s,t)$ is the Fourier cosine transform of $u\,(x,\,t)$ with respect to x.

From (1) and (2), note carefully that the successful use of a sine transform in removing a term $\partial^2 u/\partial x^2$ from a partial differential equation requires a knowledge of $u\,(0,\,t)$ *i.e.* u when $x = 0$, while the use of a cosine transform for the same purpose requires $u_x(0,t)$ *i.e.*, $\partial u/\partial x$ when $x = 0$.

Remark 1. A term $\partial u/\partial x$ or any partial derivative of odd order cannot be removed with the help of sine or cosine transforms.

Remark 2. When one of the variables in a differential equations ranges from $-\infty$ to ∞ then that variable can be excluded by the help of complex Fourier transforms.

1.25. Solved examples based on applications of infinite Fourier transform to boundary value problems.

Ex. 1. *Solve* $\partial u/\partial t = \partial^2 u/\partial x^2, x > 0, t > 0,$ *subject to conditions*

$u(0, t) = 0$, $u(x, 0) = \begin{cases} 1, 0 < x < 1 \\ 0, x > 1 \end{cases}$, $u(x, t)$ *is bounded*

[Osmania 2010; Kerala 2000, 03, 05, Meerut 1995]

Give a possible physical interpretation of the problem.

Sol. Since $u(0, t)$ *i.e.* $(u)_{x=0}$ is given, taking the Fourier sine transform of both sides of the given partial differential equations, we have

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx\, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx\, dx$$

or
$$\frac{d}{dt}\int_0^\infty u \sin sx\, dx = \left[\frac{\partial u}{\partial x}\sin sx\right]_0^\infty - s\int_0^\infty \frac{\partial u}{\partial x}\cos sx\, dx$$

or
$$\frac{d\bar{u}_s}{dt} = -s\int_0^\infty \frac{\partial u}{\partial x}\cos sx\, dx, \text{if } \frac{\partial u}{\partial x} \to 0 \quad \text{as} \quad x \to \infty$$

[Here $\bar{u}_s(s,t)$ is the Fourier sine transform of $u(x, t)$]

or
$$\frac{d\bar{u}_s}{dt} = -s\left\{\left[u\cos sx\right]_0^\infty + s\int_0^\infty u \sin sx\, dx\right\} = su(0,t) - s^2\bar{u}_s, \text{if } u \to 0 \text{ as } x \to \infty$$

or $d\bar{u}_s/dt = -s^2\bar{u}_s$ or $(1/\bar{u}_s)d\bar{u}_s = -s^2 dt,$ as $u(o, t) = 0$

whose solution is $\bar{u}_s(s,t) = ce^{-s^2 t}$, c being an arbitrary constant ... (1)

Putting $t = 0$ in (1), we have $c = \bar{u}_s(s,o)$

$$\therefore \quad c = \bar{u}_s(s,0) = \int_0^\infty u(x,0)\sin sx\, dx = \int_0^1 u(x,0)\sin sx\, dx + \int_1^\infty u(x,0)\sin sx\, dx$$

$$= \int_0^1 \sin sx\, dx, \text{ using the given value of } u(x, 0).$$

$$\therefore \quad c = \left[(-1/s)\times\cos sx\right]_0^1 = (1-\cos s)/s$$

$\therefore$ (1) becomes $\bar{u}_s(s,t) = \{(1-\cos s)/s\}e^{-s^2 t}$

Now, taking the inverse Fourier sine transform, we get

$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos s}{s} e^{-s^2 t}\sin sx\, ds$, which is the required solution.

Physical interpretation of the given boundary value problem. $u(x,t)$ represents the temperature at any point x at any time t in a solid $x > 0$. The condition $u(0, t) = 0$ expresses the fact that end $x = 0$ is kept at temperature zero, while the value of $u(x, 0)$ expresses the initial temperature as a function of x. Finally $u(x, t)$ is bounded shows that physically temperature of given solid can never be infinite.

Ex. 2. *Solve* $\partial U/\partial t = 2(\partial^2 U/\partial x^2)$, *if* $U(0, t) = 0$, $U(x,0) = e^{-x}$, $x > 0$, $U(x, t)$ *in bounded where* $x > 0$, $t > 0$. **[Patna 2003, 06, Meerut 2004]**

Solution. Given $\partial U/\partial t = 2(\partial^2 U/\partial x^2),\ x > 0,\ t > 0,$ ... (1)

with the boundary conditions: $U(0, t) = 0,$ $U(x, t)$ is bounded ... (2)

and initial condition: $U(x, 0) = e^{-x},\ x > 0$ ... (3)

Since $U(0, t)$ is given , we take the Fourier sine transform of both sides of (1) and obtain.

$$\int_0^\infty \frac{\partial U}{\partial t}\sin sx\,dx = 2\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin sx\,dx$$

or $$\frac{d}{dt}\int_0^\infty U(x,t)\sin sx\ dx = 2\left\{\left[\frac{\partial U}{\partial x}\sin sx\right]_0^\infty - \int_0^\infty \frac{\partial U}{\partial x}(s\cos sx)\ dx\right\},$$ integrating by parts

or $$\frac{d\overline{U}_s}{dt} = -2s\int_0^\infty \frac{\partial U}{\partial x}\cos sx\,dx,$$ if $\frac{\partial U}{\partial x} \to 0$ as $x \to \infty$, assuming $\overline{U}_s(s,t) = \int_0^\infty U(x,t)\sin sx\,dx$

or $$\frac{d\overline{U}_s}{dt} = -2s\left\{[U(x,t)\cos sx]_0^\infty - \int_0^\infty U(x,t)\cdot(-s\sin sx)dx\right\}$$

or $$\frac{d\overline{U}_s}{dt} = -2s\left\{0 - U(0,t) + s\int_0^\infty U(x,t)\sin sx\,dx\right\},$$ since given that $U(x,t) \to 0$ as $x \to \infty$

or $$d\overline{U}_s/dt = 2sU(0,t) - 2s^2\overline{U}_s = -2s^2\overline{U}_s,$$ as from (2), $U(0, t) = 0$

or $$(1/\overline{U}_s)d\overline{U}_s = -2s^2dt.$$

Integrating, $\log \overline{U}_s - \log c = -2s^2t$ or $\overline{U}_s(s,t) = c\,e^{-2s^2t}$. ... (4)

Taking the Fourier sine transform of both sides of (3), we get

$$\int_0^\infty U(x,0)\sin sx\,dx = \int_0^\infty e^{-x}\sin sx\,dx$$

or $$\overline{U}_s(s,0) = \left[\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right]_0^\infty = \frac{s}{1+s^2}$$ ... (5)

Putting $t = 0$ in (4) and using (5), we get $c = s/(1 + s^2)$.

$\therefore$ From (4), $\overline{U}_s(s,t) = \{s/(1+s^2)\}e^{-2s^2t}$.

Now, taking the inverse Fourier sine transform, we get

$$U(x,t) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}e^{-2s^2t}\sin sx\,ds.$$

Ex. 3. *Using the Fourier sine transform, solve the partial differential equation* $\partial V/\partial t = k(\partial^2 V/\partial x^2)$ *for* $x > 0$, $t > 0$, *under the boundary conditions* $V = V_0$ *when* $x = 0$, $t > 0$ *and the initial condition* $V = 0$, *when* $t = 0$, $x > 0$.

[Himanchal 2001, 02, 05, Rohilkhand 1998, 2001]

Sol. Given $\partial V/\partial t = k(\partial^2 V/\partial x^2),\ x > 0,\ t > 0,$ ... (1)

with the boundary condition: $V(0, t) = V_0,$ ... (2)

and initial condition : $V(x, 0) = 0.$ ... (3)

Taking the Fourier sine transform of both sides of (1), we get

$$\int_0^\infty \frac{\partial V}{\partial t}\sin sx\,dx = k\int_0^\infty \frac{\partial^2 V}{\partial x^2}\sin sx\,dx$$

or $$\frac{d}{dt}\int_0^\infty V(x,t)\sin sx\,dx = k\left[\left\{\frac{\partial V}{\partial x}\sin sx\right\}_0^\infty - \int_0^\infty \frac{\partial V}{\partial x}s\cos sx\,dx\right],$$ integrating by parts

or $$\frac{d\overline{V}_s}{dt} = -ks\int_0^\infty \frac{\partial V}{dx}\cos sx\,dx, \text{if } \frac{\partial V}{\partial x}\to 0,\ \text{as } x\to\infty, \text{ assuming } \overline{V}_s(s,t) = \int_0^\infty V(x,t)\sin sx\,dx$$

or $$\frac{d\overline{V}_s}{dt} = -ks\left\{[V(x,t)\cos sx]_0^\infty - \int_0^\infty V(x,t).(-s\sin sx)\,dx\right\}$$

or $$\frac{d\overline{V}_s}{dt} = -ks\left\{0 - V(0,t) + s\int_0^\infty V(x,t)(-s\sin sx)\,dx\right\}$$ assuming that $V(x, t) \to 0$ as $x \to \infty$

or $d\overline{V}_s/dt = ks\,V_0 - ks^2\,\overline{V}_s$ or $d\overline{V}_s/dt + ks^2\,\overline{V}_s = ks\,V_0,$ by (2)

which is linear differential equation whose I.F. $= e^{\int ks^2 dt} = e^{ks^2 t}$

and solution is $$e^{ks^2 t}\,\overline{V}_s(s,t) = \int (ks\,V_0)e^{ks^2 t}dt = \frac{ks\,V_0}{ks^2}e^{ks^2 t} + c$$

or $$\overline{V}_s(s,t) = V_0/s + ce^{-ks^2 t}.$$ ... (4)

Taking the Fourier sine transform of condition (3), we have

$$\int_0^\infty V(x,0)\sin sx\,dx = 0 \quad \text{or} \quad \overline{V}_s(s,0) = 0.$$ ... (5)

Putting $t = 0$ in (4) and using (5), $0 = V_0/s + c$ so that $c = -V_0/s.$

$\therefore$ From (4), $\overline{V}_s(s,t) = (V_0/s)\ (1 - e^{-ks^2 t})$

Now, taking the inverse Fourier sine transform, we get

$$V(x, t) = \frac{2}{\pi}\int_0^\infty \frac{V_0}{s}(1 - e^{-ks^2})\sin sx\,ds = \frac{2V_0}{\pi}\left[\int_0^\infty \frac{\sin sx}{s}ds - \int_0^\infty \frac{1}{s}e^{-ks^2 t}\sin sx\,ds\right]$$

$$= \frac{2V_0}{\pi}\left[\frac{\pi}{2} - \int_0^\infty \frac{1}{s}e^{-ks^2 t}\sin sx\,ds\right], \quad \text{as} \quad \int_0^\infty \frac{\sin sx}{s}dx = \frac{\pi}{2}$$

$\therefore$ $$V(x, t) = V_0\left[1 - \frac{2}{\pi}\int_0^\infty \frac{1}{s}e^{-ks^2 t}\sin sx\,ds\right]$$ ... (7)

Remark. The result (7) can also be expressed in terms of *error function* or *complementary error function* (For their definition, refer chapter 1 of part IV-A of this book) as follows.

We know that $$\int_0^\infty e^{-\alpha^2 x^2} \cos 2\beta x \, dx = \frac{\sqrt{\pi}}{2\alpha} e^{-\beta^2/\alpha^2} \quad \ldots (8)$$

Integrating both sides of (8) w.r.t. β from 0 to β, we get

$$\int_0^\beta \left[\int_0^\infty e^{-\alpha^2 x^2} \cos 2\beta \, dx \right] d\beta = \frac{\sqrt{\pi}}{2\alpha} \int_0^\beta e^{-\beta^2/\alpha^2} d\beta$$

or $$\int_0^\infty e^{-\alpha^2 x^2} \left[\int_0^\beta \cos 2\beta x \; d\beta \right] dx = \frac{\sqrt{\pi}}{2\alpha} \int_0^{\beta/\alpha} e^{-u^2} \alpha \, du$$

[on R.H.S putting $\beta/\alpha = u$ so that $d\beta = \alpha \, du$ and changing the order of integration on L.H.S.]

or $$\int_0^\infty e^{-\alpha^2 x^2} \left[\frac{\sin 2\beta x}{2x} \right]_0^\beta dx = \frac{\sqrt{\pi}}{2} \int_0^{\beta/\alpha} e^{-u^2} du \quad \text{or} \quad \int_0^\infty \frac{e^{-\alpha^2 x^2} \sin 2\beta x}{x} dx = \frac{\pi}{2} \times \frac{2}{\sqrt{\pi}} \int_0^{\beta/\alpha} e^{-u^2} du$$

or $$\int_0^\infty \frac{e^{-\alpha^2 x^2} \sin 2\beta x}{x} dx = \frac{\pi}{2} \times erf\,(\beta/\alpha), \qquad \text{as} \qquad erf\,(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$$

or $$\int_0^\infty \frac{e^{-\alpha^2 s^2} \sin 2\beta s}{s} ds = \frac{\pi}{2} erf\left(\frac{\beta}{\alpha}\right) \qquad \text{or} \qquad \int_0^\infty \frac{e^{-kts^2} \sin sx}{s} ds = \frac{\pi}{2} erf\left(\frac{x}{2\sqrt{kt}}\right),$$

taking $\alpha = \sqrt{kt}$ and $\beta = x/2$

Hence (7) becomes $$V(x,t) = V_0[1 - erf\,(x/2\sqrt{kt})] \quad \ldots (9)$$

or $$V(x, t) = V_0 \, erfc\;(x/2\sqrt{kt}) \quad \ldots (10)$$

(9) and (10) are modified forms of solution (7) in terms of error function and complementary error function respectively.

Ex. 4. *Show that the solution of Laplace equation for V inside the semi-infiinite strip x >0. 0 < y < b, such that V = f (x), when y = 0, 0 < x < ∞ ; V = 0, when y = b, 0 < x < ∞;*

V = 0, when x = 0, 0 < *y* < *b*, is given by $V = \frac{2}{\pi} \int_0^\infty f(u)\,du \int_0^\infty \frac{\sinh(b-y)s}{\sinh sb} \sin xs \; \sin us \, ds.$

Sol. Here we are to solve two-dimensional Laplace equation

$$\partial^2 V / \partial x^2 + \partial^2 V / \partial y^2 = 0, \qquad 0 < x < \infty, \qquad 0 < y < b \quad \ldots (1)$$

with the boundary conditions: $$V(x, 0) = f(x), \; 0 < x < \infty, \quad \ldots (2)$$

$$V(x, b) = 0, \; 0 < x < \infty \quad \ldots (3)$$

and $$V(0, y) = 0, \; 0 < y < b. \quad \ldots (4)$$

[Here we have variable *y* in place of *t*. We shall take Fourier sine transform w.r.t. *x* and keep *y* unchanged as we used to do with variable *t*]

Taking the Fourier sine transform of (1), we get

$$\int_0^\infty \frac{\partial^2 V}{\partial x^2} \sin sx \, dx + \int_0^\infty \frac{\partial^2 V}{\partial y^2} \sin sx \, dx = 0$$

$$\left[\frac{\partial V}{\partial x} \sin sx \right]_0^\infty - \int_0^\infty \frac{\partial V}{\partial x} s \cos sx \, dx + \frac{d^2}{dy^2} \int_0^\infty V(x,y) \sin sx \, dx = 0, \text{ integrating by parts}$$

or $$-s \int_0^\infty \frac{\partial V}{\partial x} \cos sx \, dx + \frac{d^2 \overline{V}_s}{dy^2} = 0, \text{if } \frac{\partial V}{\partial x} \to 0 \text{ as } x \to \infty, \text{ assuming } \overline{V}_s(s,y) = \int_0^\infty V(x,y) \sin sx \, dx$$

or $$-s \left\{ \left[V(x,y) \cos sx \right]_0^\infty - \int_0^\infty V(x,y)(-s \sin sx)\, dx \right\} + \frac{d^2 \overline{V}_s}{dy^2} = 0, \text{ integrating by parts}$$

or $$-s\left\{-\mathrm{V}(0,y)+s\,\overline{\mathrm{V}}_s\right\}+(d^2\,\overline{\mathrm{V}}_s/\,dy^2)=0, \text{if } \mathrm{V}\to 0 \text{ as } x\to\infty$$

or $$-\,s^2+(d^2\,\overline{\mathrm{V}}\,s/\,dy^2)=0, \text{ using (4)}$$

or $$(\mathrm{D}^2-s^2)\,\overline{\mathrm{V}}_s=0,\ \mathrm{D}\equiv d/dy\text{, whose general solution is}$$

$\overline{\mathrm{V}}_s(s,y)$ = A cosh *sy* + B sinh *sy*, where *A* and *B* are arbitrary constants ... (5)

Taking the Fourier sine transform of (2) and (3), we get

$$\int_0^\infty \mathrm{V}(x,0)\sin sx\,dx=\int_0^\infty f(x)\sin sx\,dx \qquad \text{and} \qquad \int_0^\infty \mathrm{V}(x,b)\sin sx\,dx=0$$

i.e. $$\overline{\mathrm{V}}_s(s,0)=\int_0^\infty f(u)\sin su\,du \qquad \text{... (6)}$$

and $$\overline{\mathrm{V}}_s(s,b)=0 \qquad \text{... (7)}$$

Putting $y = 0$ in (5) and using (6), we get $$\int_0^\infty f(u)\sin su\,du=\mathrm{A}. \qquad \text{... (8)}$$

Putting $y = b$ in (5) and using (7) and (8), we get

$$0=\cosh sb\int_0^\infty f(u)\sin su\,du+\mathrm{B}\sinh sb \quad\Rightarrow\quad B=-\frac{\cosh sb}{\sinh sb}\int_0^\infty f(u)\sin su\,du \qquad \text{... (9)}$$

Putting values of A and B in (5), we get

$$\overline{\mathrm{V}}_s(s,y)=\cosh sy\int_0^\infty f(u)\sin su\,du-\frac{\sinh sy\cosh sb}{\sinh sb}\int_0^\infty f(u)\sin su\,du$$

$$\therefore\ \overline{\mathrm{V}}_s(s,y)=\int_0^\infty f(u)\sin su\left[\cosh sy-\frac{\sinh sy\cosh sb}{\sinh sb}\right]du=\int_0^\infty f(u)\sin su\,\frac{\sinh(b-y)s}{\sinh sb}\,du.$$

Now, taking the inverse Fourier sine transform, we get

$$\mathrm{V}(x,y)=\frac{2}{\pi}\int_0^\infty\overline{\mathrm{V}}_s(x,y)\sin sx\,ds=\frac{2}{\pi}\int_0^\infty\sin sx\left[\int_0^\infty f(u)\,\frac{\sin su\sinh(b-y)s}{\sinh sb}\,du\right]ds$$

$$\therefore\ V(x,y)=\frac{2}{\pi}\int_0^\infty f(u)\,du\int_0^\infty\frac{\sinh(b-y)s}{\sinh sb}\sin su\sin sx\,ds\text{, on changing the order of integration}$$

Ex. 5. *The temperature U in the semi-infinite rod $0\le x<\infty$ is determined by the equation $\partial U/\partial t = k\,(\partial^2 U/\partial x^2)$ subject to the conditions.* (*i*) U = 0 when $t = 0$, $x \ge 0$

(*ii*) $\partial\mathrm{U}/\partial x = -\mu$ (*a constant*), *when* $x = 0$ *and* $\mathrm{t} > 0$.

Making use of cosine transform, show that $$\mathrm{U}(x,t)=\frac{2\mu}{\pi}\int_0^\infty\frac{\cos sx}{s^2}(1-e^{-ks^2t})\,ds$$

Sol. Given $$\partial\mathrm{U}/\partial t = k\,(\partial^2\mathrm{U}/\partial x^2) \qquad \text{... (1)}$$

with the boundary condition: $$\left(\partial\,\mathrm{U}/\,\partial x\right)_{x=0}=-\mu \qquad \text{... (2)}$$

and initial conditions: $$\mathrm{U}(x,0)=0, x\ge 0. \qquad \text{... (3)}$$

Taking the Fourier cosine transforms of both sides of (1), we get

$$\int_0^\infty\frac{\partial\,\mathrm{U}}{\partial t}\cos sx\,dx=k\int_0^\infty\frac{\partial^2\,\mathrm{U}}{\partial x^2}\cos sx\,dx$$

or $$\frac{d}{dt}\int_0^\infty \mathrm{U}(x,t)\cos sx\,dx = k\left\{\left[\frac{\partial \mathrm{U}}{\partial x}\cos sx\right]_0^\infty - \int_0^\infty \frac{\partial \mathrm{U}}{\partial x}(-s\sin sx)\,dx\right\}$$, integrating by parts

or $$\frac{d\,\overline{\mathrm{U}}_c}{dt} = -k\left(\frac{\partial \mathrm{U}}{\partial x}\right)_{x=0} + ks\int_0^\infty \frac{\partial \mathrm{U}}{\partial x}\sin sx\,dx, \text{if } \frac{\partial \mathrm{U}}{\partial x} \to 0 \text{ as } x \to \infty$$, assuming $\overline{\mathrm{U}}_c = \int_0^\infty \mathrm{U}(x,t)\cos sx\,dx$

or $$\frac{d\,\overline{\mathrm{U}}_c}{dt} = k\mu + ks\left\{\left[\mathrm{U}(x,t)\sin sx\right]_0^\infty - \int_0^\infty \mathrm{U}(x,t)\,s\cos sx\,dx\right\}$$, by (2)

or $$d\,\overline{\mathrm{U}}_c/dt = k\mu + ks(0 - s\,\overline{\mathrm{U}}_c), \text{ if } \mathrm{U} \to 0 \text{ as } x \to \infty$$

or $$(d\,\overline{\mathrm{U}}_c/dt) + ks^2\,\overline{\mathrm{U}}_c = k\mu, \text{ which is a linear equation} \qquad \ldots (4)$$

I.F. of (4) = $e^{\int ks^2 dt} = e^{ks^2 t}$ and so general solution of (4) is

$$e^{ks^2 t}\,\overline{\mathrm{U}}_c(s,t) = \int (k\mu)\,e^{ks^2 t}dt + c = k\mu \times \frac{e^{ks^2 t}}{ks^2} + c.$$

$\therefore$ $$\overline{\mathrm{U}}_c(s,t) = \mu/s^2 + c\,e^{-ks^2 t}. \qquad \ldots (5)$$

Taking the Fourier cosine transform of (3), we get

$$\int_0^\infty \mathrm{U}(x,0)\cos sx\,dx = 0 \quad \text{or} \quad \overline{\mathrm{U}}_c(s,0) = 0. \qquad \ldots (6)$$

Putting $t = 0$ in (5) and using (6), $0 = \mu/s^2 + c$ so that $c = -\mu/s^2$.

$\therefore$ From (5), $\overline{\mathrm{U}}_c(s,t) = (\mu/s^2) \times (1 - e^{-ks^2 t})$.

$\therefore$ Taking the inverse Fourier cosine transform, we get

$$\mathrm{U}(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\mu}{s^2}(1 - e^{-ks^2 t})\cos sx\,ds = \frac{2\mu}{\pi}\int_0^\infty \frac{\cos sx}{s^2}(1 - e^{-ks^2 t})\,ds$$

Ex. 6. *Solve* $\partial u/\partial t = \partial^2 u/\partial x^2$, $x > 0, t > 0$, *subject to conditions* $u_x(0, t) = 0$, $u(x, t)$ *is bounded and* $u(x,0) = \begin{cases} x, 0 \le x \le 1 \\ 0, x > 1 \end{cases}$

Sol. Since $u_x(0, t)$ is given, taking the Fourier cosine transform of both sides of the given partial differential equation, we have

$$\int_0^\infty \frac{\partial u}{\partial t}\cos sx\,dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx\,dx$$

or $$\frac{d}{dt}\int_0^\infty u\cos sx\,dx = \left[\frac{\partial u}{\partial x}\cos sx\right]_0^\infty + s\int_0^\infty \frac{\partial u}{\partial x}\sin sx\,dx$$, integrating by parts

or $$\frac{d\overline{u}_c}{dt} = -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\left[\left[u\sin sx\right]_0^\infty - s\int_0^\infty u\cos sx\,dx\right], \qquad \ldots (1)$$

where we have assumed that $\partial u/\partial x \to 0$ as $x \to \infty$ and $\overline{u}_c(s,t)$ is the Fourier cosine transform of $u(x, t)$.

$\therefore$ (1) $\Rightarrow d\,\bar{u}_c / dt = -s^2 \bar{u}_c$ or $(1/\bar{u}_c)d\,\bar{u}_c = -s^2 dt.$

[$\because$ given $u_x(0, t) = 0$ and let $u \to 0$ as $x \to \infty$]

Integrating, $\log \bar{u}_c - \log A = -s^2 t$ or $\bar{u}_c(s,t) = A\,e^{-s^2 t}$. ... (2)

Putting $t = 0$ in (2), we get $A = \bar{u}_c(s,0)$. Hence, using definition, we have

$$A = \bar{u}_c(s,0) = \int_0^\infty u(x,0)\cos sx\,dx = \int_0^1 u(x,0)\cos sx\,dx + \int_1^\infty u(x,0)\cos sx\,dx = \int_0^1 x\cos sx\,dx$$

[using the given value of u $(x, 0)$]

$$= \left[x\frac{\sin sx}{s}\right]_0^1 - \int_0^1 1\times\frac{\sin sx}{s}dx = \frac{\sin s}{s} + \left[\frac{\cos sx}{s^2}\right]_0^1 = \frac{\sin s}{s} + \frac{\cos s - 1}{s^2}$$

$\therefore$ (2) becomes $\bar{u}_c(s,t) = \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right)e^{-s^2 t}.$

Now, taking the inverse Fourier cosine transforms, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right)e^{-s^2 t}\cos sx\,dx$$

Ex. 7. *Use a cosine transform to show that the steady temperature in the semi-infinite solid $y > 0$ when the temperature on the surface $y = 0$ is kept at unity over the strip $|x| < a$ and at zero outside the strip, is* $\frac{1}{\pi}\left[\tan^{-1}\left(\frac{a+x}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right)\right]$.

The result $\int_0^\infty e^{-sx}x^{-1}\sin rx\,dx = \tan^{-1}\frac{r}{s}$, *$r > 0$, $s > 0$ may be assumed.*

[Allahabad 2003; Himanchal 2003; Meerut 2005; Nagpur 2001]

Sol. We know that the steady temperature in the semi-infintie solid is represented by two-dimensional Laplace equation

$$\partial^2 U/\partial x^2 + \partial^2 U/\partial y^2 = 0 \qquad 0 < y < \infty, \qquad -\infty < x < \infty \qquad \ldots (1)$$

subject to the boundary conditions: $U(x, 0) = 1, \quad |x| < a \quad i.e. \quad -a < x < a$ ... (2)

and $U(x, 0) = 0, \quad x < -a$ or $x > a.$ ... (3)

[Here we have variable y in place of t. We shall take Fourier cosine transforms w.r.t. x and keep y unchanged as we used to do with the variable t]

Taking the Fourier cosine transform of (1), we get

$$\int_0^\infty \frac{\partial^2 U}{\partial x^2}\cos sx\,dx + \int_0^\infty \frac{\partial^2 U}{\partial y^2}\cos sx\,dx = 0$$

or $\left[\frac{\partial U}{\partial x}\cos sx\right]_0^\infty - \int_0^\infty \frac{\partial U}{\partial x}(-s\sin sx)\,dx + \frac{d^2}{dy^2}\int_0^\infty U(x,y)\cos sx\,dx = 0$ integrating by parts

or $s\int_0^\infty \frac{\partial U}{\partial x}\sin sx\,dx + \frac{d^2\bar{U}_c}{dy^2} = 0$, where $\bar{U}_c(s,y) = \int_0^\infty U(x,y)\cos sx\,dx$

[$\because$ due to symmetry, $\partial U/\partial x \to 0$ as $x \to \infty$ and $\partial U/\partial x \to 0$ as $x \to 0$.]

or $s\left\{\left[\mathrm{U}(x,y)\sin sx\right]_0^\infty - \int_0^\infty \mathrm{U}(x,y)\, s\cos sx\, dx\right\} + (d^2\,\overline{\mathrm{U}}_c / dy^2) = 0$, integrating by parts again,

or $$-s^2\,\overline{\mathrm{U}}_c + (d^2\,\overline{\mathrm{U}}_c / dy^2) = 0, \text{ if } \mathrm{U}(x,y) \to 0 \text{ as } x \to \infty$$

or $$(\mathrm{D}^2 - s^2)\,\overline{\mathrm{U}}_c = 0, \quad \text{where} \quad \mathrm{D} \equiv d/dy.$$

whose general solution is $\overline{\mathrm{U}}_c(s,y) = \mathrm{C}_1 e^{sy} + \mathrm{C}_2 e^{-sy}$, C_1, C_2 being arbitrary constants ... (4)

Since $\overline{\mathrm{U}}_c(s,y)$ is finite, we must take $\mathrm{C}_1 = 0$ in (4),otherwise $\overline{\mathrm{U}}_c(s,y)$ would beecome infinite as $y \to \infty$.

$\therefore$ (4) reduces to $$\overline{\mathrm{U}}_c(s,y) = \mathrm{C}_2 e^{-sy} \quad \text{... (5)}$$

Again, $$\int_0^\infty \mathrm{U}(x,0)\cos sx\, dx = \int_0^a \mathrm{U}(x,0)\cos sx\, dx + \int_a^\infty \mathrm{U}(x,0)\cos sx\, dx$$

or $$\overline{\mathrm{U}}_c(s,0) = \int_0^a \cos sx\, dx + 0, \text{ by (2) and (3)}$$

or $$\overline{\mathrm{U}}_c(s,0) = \left[(1/s)\times\sin sx\right]_0^a = (1/s)\times\sin sa. \quad \text{... (6)}$$

Putting $y = 0$ in (5) and using (6), we get $\mathrm{C}_2 = (\sin sa)/s$.

$\therefore$ From (5), $$\overline{\mathrm{U}}_c(s,y) = \{(1/s)\times\sin sa\}e^{-sy}.$$

$\therefore$ Taking the inverse Fourier cosine transform, we get

$$\mathrm{U}(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\sin sa}{s} e^{-sy}\cos sx\, ds = \frac{1}{\pi}\int_0^\infty \frac{e^{-sy}}{s}[\sin(a+x)s + \sin(a-x)s]\, ds$$

$$= \frac{1}{\pi}\left[\int_0^\infty e^{-sy}s^{-1}\sin(a+x)s\, ds + \int_0^\infty e^{-sy}s^{-1}\sin(a-x)s\, ds\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{a+x}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right)\right], \text{ as } \int_0^\infty e^{-sx}x^{-1}\sin rx\, dx = \tan^{-1}\frac{r}{s}.$$

Ex. 8(a). *If the flow of heat is linear so that the variation of* θ *(temperature) with z and y may be neglected and if it is assumed that no heat is generated in the medium, then solve the differential equation* $\partial\theta/\partial t = k(\partial^2\theta/\partial x^2)$ *where* $-\infty < x < \infty$ *and* $\theta = f(x)$ *when* $t = 0$, $f(x)$ *being a given function of x.* **[C.S.I.R. (Net) 2002; I.F.S. 2005]**

Sol. Given $$\partial\theta/\partial t = k(\partial^2\theta/\partial x^2), -\infty < x < \infty, \quad \text{... (1)}$$

with the initial condition: $$\theta(x,0) = f(x). \quad \text{... (2)}$$

Taking the complex Fourier transform of (1), we get

$$\int_{-\infty}^\infty \frac{\partial\theta}{\partial t}e^{isx}dx = k\int_{-\infty}^\infty \frac{\partial^2\theta}{\partial x^2}e^{isx}dx$$

or $$\frac{d}{dt}\int_{-\infty}^\infty \theta(x,t)e^{isx}dx = k(-is)^2\,\mathrm{F}\{\theta(x,t)\}, \text{ using theorem 2, Art. 1.22}$$

or $$\frac{d\overline{\theta}}{dt} = -ks^2\,\overline{\theta}, \quad \text{where} \quad \overline{\theta}(s,t) = \mathrm{F}\{\theta(x,t)\} = \int_{-\infty}^\infty \theta(x,t)e^{isx}dx$$

whose general solution is $\overline{\theta}(s,t) = \mathrm{A}\,e^{-ks^2t}$, A being an arbitrary constant ... (3)

Taking the complex Fourier transform of both sides of (2), we get

$$\int_{-\infty}^{\infty} \theta(x,0)\, e^{isx}\, dx = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \qquad \text{or} \qquad \bar{\theta}(s,0) = \bar{f}(s) \qquad \ldots (4)$$

Putting $t = 0$ in (3) and using (4), we get $\bar{f}(s) = \text{A}.$

$\therefore$ From (3), $\bar{\theta}(s,t) = \bar{f}(s) e^{-ks^2 t}$

Taking the inverse Fourier transform, we get

$$\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{-ks^2 t}\, e^{-isx}\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{-ks^2 t - isx}\, ds,$$

where $$\bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

Ex. 8(b). *If the function $U(x, y)$ is determined by equation $\partial U/\partial x = \partial^2 U/\partial y^2$ for $x \geq 0, -\infty < y < \infty$ and $U = f(y)$ when $x = 0$, show that*

$$U(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s) e^{-s^2 x - isy}\, ds, \text{ where } \bar{f}(s) \text{ is the Fourier transform of } f(y).$$

Sol. Proceed as in Ex. 8(a) by replacing k by 1, θ by U, t by x and x by y:

Ex. 9. *Solve $\partial^4 V/\partial x^4 + \partial^2 V/\partial y^2 = 0, -\infty < x < \infty, y \geq 0$ satisfying the conditions. (i) V and its partial derivatives tend to zero as $x \to \pm\infty$ and (ii) $V = f(x)$, $\partial V/\partial y = 0$ on $y = 0$.*

[Bangalore 2001; Meerut 1997; Kanpur 2005; Rajasthan 2003; Rewa 2006]

Sol. Given $$\partial^4 \text{V}/\partial x^4 + \partial^2 \text{V}/\partial y^2 = 0 \qquad \ldots (1)$$

with the boundary conditons:

$$\text{V} \to 0, \quad \partial \text{V}/\partial x \to 0, \quad \partial^2 \text{V}/\partial x^2 \to 0, \quad \partial^3 \text{V}/\partial x^3 \to 0, \quad \text{as} \quad x \to \pm\infty, \ldots (2)$$

$$\text{V}(x, 0) = f(x) \qquad \ldots (3)$$

and $$(\partial \text{V}/\partial y)_{y=0} = 0 \qquad \ldots (4)$$

Taking the complex Fourier transform of (1), we get

$$\int_0^{\infty} \frac{\partial^4 \text{V}}{\partial x^4} e^{isx}\, dx + \int_0^{\infty} \frac{\partial^2 \text{V}}{\partial y^2} e^{isx}\, dx = 0$$

or $$(-is)^4 F\{V(x,y)\} + \frac{d^2}{dy^2}\int_0^{\infty} \text{V}(x,y) e^{isx}\, dx = 0, \text{ by theorem 2 of Art. 6.14.}$$

or $$s^4 \bar{\text{V}} + \frac{d^2 \bar{\text{V}}}{dy^2} = 0, \qquad \text{where} \qquad \text{F}\{\text{V}(x,y)\} = \bar{\text{V}}(s,y) = \int_0^{\infty} \text{V}(x,y) e^{isx}\, dx.$$

or $$(\text{D}^2 + s^4)\, \bar{\text{V}} = 0, \qquad \text{where} \qquad \text{D} \equiv d/dy,$$

whose general solution is given by $$\bar{\text{V}}(s,y) = \text{C}_1 \cos s^2 y + \text{C}_2 \sin s^2 y \qquad \ldots (5)$$

From (5), $$d\bar{\text{V}}/dy = -C_1 s^2 \sin s^2 y + \text{C}_2\, s^2 \cos s^2 y. \qquad \ldots (6)$$

Taking complex Fourier transform of (4), we get

$$\int_0^\infty \left(\frac{\partial \mathrm{V}}{\partial y}\right)_{y=0} e^{isx}dx = 0 \qquad \text{or} \qquad \left(\frac{d\,\overline{\mathrm{V}}}{dy}\right)_{y=0} = 0 \qquad \ldots (7)$$

Putting $y = 0$ in (6) and using (7), we get $0 = \mathrm{C}_2 s^2$ so that $\mathrm{C}_2 = 0$.

$\therefore$ From (5), $\overline{\mathrm{V}}(s, y) = \mathrm{C}_1 \cos s^2 y.$ ...(8)

Taking complex Fourier transform of (3), we get

$$\int_0^\infty \mathrm{V}(x,t)e^{isx}dx = \int_0^\infty f(x)\,e^{isx}dx \qquad \text{or} \qquad \overline{\mathrm{V}}(s,0) = \overline{f}(s), \qquad \ldots (9)$$

where $$\overline{f}(s) = \int_0^\infty f(x)e^{isx}dx \qquad \ldots (10)$$

Putting $y = 0$ in (8) and using (9), we get $\overline{f}(s) = \mathrm{C}_1$

$\therefore$ From (8), we have $\overline{\mathrm{V}}(s, y) = \overline{f}(s)\cos s^2 y.$

Taking the inverse Fourier transform, we get

$$\mathrm{V}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(s)\cos s^2 y\, e^{-isx}\,ds \text{ , where } \bar{f}(s) \text{ is given by (10).}$$

Ex. 10. *Use the complex form of the Fourier transform to show that*

$$\mathrm{V} = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(u)\,exp\left\{-\frac{(x-u)^2}{4t}\right\}du$$

is the solution of the boundary value problem

$\partial \mathrm{V}/\partial t = \partial^2 \mathrm{V}/\partial x^2$, $-\infty < x < \infty$, $t > 0$; $V = f(x)$; *when* $t = 0$.

[Allahabad 1997; I.F.S. 2004; Kurukshetra 2002, 05; Meerut 1997, 98, 99, 2003]

Sol. Taking the complex Fourier transform of both sides of the given partial differential equation, we have

$$\int_{-\infty}^{\infty} \frac{\partial \mathrm{V}}{\partial t}e^{isx}dx = \int_{-\infty}^{\infty} \frac{\partial^2 \mathrm{V}}{\partial x^2}e^{isx}dx$$

or $$\frac{d}{dt}\int_{-\infty}^{\infty} \mathrm{V}\,e^{isx}dx = (-is)^2\,\mathrm{F}\{\mathrm{V}(x,t)\}, \text{ by theorem 2 of Art. 1.22}$$

or $$\frac{d\,\overline{\mathrm{V}}}{dt} = -s^2\,\overline{\mathrm{V}}, \qquad \text{where} \qquad \overline{\mathrm{V}}(s,t) = \int_{-\infty}^{\infty} \mathrm{V}(x,t)\,e^{isx}dx \qquad \ldots (1)$$

or $$(1/\overline{\mathrm{V}})\,d\,\overline{\mathrm{V}} = -s^2 dt \qquad \text{so that} \qquad \overline{\mathrm{V}}(s,t) = \mathrm{A}\,e^{-s^2 t}. \qquad \ldots (2)$$

Putting $t = 0$ in (1), we get $\mathrm{A} = \overline{\mathrm{V}}\ (s, 0)$

$\therefore$ $$\mathrm{A} = \overline{\mathrm{V}}(s,0) = \int_{-\infty}^{\infty} \mathrm{V}(x,0)e^{isx}dx = \int_{-\infty}^{\infty} f(x)\,e^{isx}dx = \int_{-\infty}^{\infty} f(u)e^{isu}du$$

$[\because\ \mathrm{V} = f(x)$ when $t = 0 \Rightarrow \mathrm{V}(x, 0) = f(x)]$

$\therefore$ From (2) $$\overline{\mathrm{V}}(s,t) = e^{-s^2 t}\int_{-\infty}^{\infty} f(u)\,e^{isu}du.$$

Taking the inverse Fourier transform, we have

$$V(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{-s^2t}\int_{-\infty}^{\infty} f(u)e^{isu}du\right]e^{-isx}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-s^2t-is(x-u)}ds\right]du$$

[changing the order of integration]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t[s^2+is(x-u)/t]}ds\right]du = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t[\{s+i(x-u)/2t\}^2+(x-u)^2/4t^2]}ds\right]du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/4t}\left[\int_{-\infty}^{\infty} e^{-t\{s+i(x-u)/2t\}^2}ds\right]du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{t}} f(u)e^{-(x-u)^2/4t}\left[\int_{-\infty}^{\infty} e^{-y^2}dy\right]du$$

[Putting $\sqrt{t}\,\{s+i(x-u)/2t\} = y$ so that $ds = dy/\sqrt{t}$]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{t}} f(u)\,e^{-(x-u)^2/4t}\sqrt{\pi}\,du, \qquad as \qquad \int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}$$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(u)\exp\left\{-\frac{(x-u)^2}{4t}\right\}du.$$

Ex. 11. *Use the method of Fourier transform to determine the displacement y(x, t) of an infinite string, given that the string is initially at rest and that the initial displacement is f(x), $-\infty < x < \infty$. Show that solution can also be put in the form* $y(x, t) = [f(x+ct)+f(x-ct)]/2$.

[Allahabad 2003; Kolkata 2003, 05, 06; Kurukshetra 1997; Meerut 2006, 2012]

Sol. Here we are to solve one-deminsional wave equation

$$\partial^2 y/\partial t^2 = c^2(\partial^2 y/\partial x^2), \quad -\infty < x < \infty, \quad t > 0, \qquad \ldots (1)$$

subject to the following initial conditions:

$$y(x, 0) = \text{Initial displacement} = f(x) \qquad \ldots (2)$$

and

$$y_t(x, 0) = \text{Initial velocity} = 0 \qquad \ldots (3)$$

Taking the complex Fourier transform of (1), we get

$$\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial t^2}e^{isx}dx = c^2\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial x^2}e^{isx}dx$$

or

$$\frac{d^2}{dt^2}\int_{-\infty}^{\infty} ye^{isx}dx = (-is)^2c^2\,F\{y(x,t)\}, \text{ by theorem 2 of Art. 1.22}$$

or

$$\frac{d^2\bar{y}}{dt^2}+c^2s^2\bar{y} = 0, \qquad \text{where} \qquad F\{y(x,t)\} = \bar{y}(s,t) = \int_{-\infty}^{\infty} y(x,t)e^{isx}dx$$

or

$$(D^2+c^2s^2)\bar{y} = 0, \text{ where } D \equiv d/dt. \text{ Its solution is given by}$$

$$\bar{y}(s,t) = A\cos(cst) + B\sin(cst), \text{ where } A \text{ and } B \text{ are arbitrary constants} \qquad \ldots (4)$$

From (4), $\bar{y}_t(s,t) = \partial\bar{y}/\partial t = -Acs\,\sin(cst)+Bcs\,\cos(cst).$ ... (5)

Taking infinite Fourier transform, (2) and (3), we have

$$\bar{y}(s,0) = \int_{-\infty}^{\infty} f(x)e^{isx}dx = \bar{f}(s) \text{ say} \qquad \ldots (6)$$

and $$\bar{y}_t(s,0) = \int_{-\infty}^{\infty} 0.e^{isx}dx = 0. \quad \text{... (7)}$$

Putting $t = 0$ in (5) and using (7), $0 = B\,cs$ so that $B = 0$.

Putting $t = 0$ in (4) and using (6) and B = 0, we get $\overline{f}(s) = A$

$\therefore$ From (4), $\bar{y}(s,t) = \overline{f}(s)\cos(cst).$

Taking the inverse Fourier transform, we have

$$y(x.t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(s)\cos cst\, e^{-isx}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(s)\left(\frac{e^{icst}+e^{-icst}}{2}\right)e^{-isx}ds$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(s)\,e^{-is(x+ct)}ds + \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(s)e^{-is(x-ct)}ds\right]$$

$$= \left[f(x+ct)+f(x-ct)\right]/2, \text{ by inversion formula for Fourier transform.}$$

Ex. 12. *A thin membrane of great extent is released from rest in position z = f(x, y). Show that displacement at any subsequent time is given by*

$$z(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{F}(p,q)\cos\{ct(p^2+q^2)^{1/2}\}e^{-i(px+qy)}\,dp\,dq,$$

where F(p, q) is double Fourier transform of f (x, y).

Or

The variation of a function z over the xy-place and for t ≥ 0 is determined by the equation.

$$\nabla^2 z = (1/c^2)\times(\partial^2 z/\partial t^2), \qquad i.e., \qquad \partial^2 z/\partial t^2 = c^2(\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2)$$

where $c^2 = T/\rho$. *If when* $t = 0$, $z = f(x, y)$ *and* $\partial z/\partial t = 0$, *show that, at any subsequent time,*

$$z(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p.q)\cos\{ct(p^2+q^2)^{1/2}\}e^{-i(px+qy)}dp\,dq,$$

where $$F(p,q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i(px+qy)}dx\,dy \qquad \text{and} \qquad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial x$$

[Lucknow (B.Tech) 2005]

Sol. We know that the displacement of the membrane satisfies the two-dimensional wave equation $$\partial^2 z/\partial t^2 = c^2\left(\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2\right) \quad \text{... (1)}$$

with given boundary conditions: $$z\,(x,\ y,\ 0) = f\,(x,\ y) \quad \text{... (2)}$$

and $$\left(\partial z/\partial t\right)_{t=0} = 0 \quad \text{... (3)}$$

Taking double Fourier transform (refer Art. 1.23) of (1) we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 z}{\partial t^2}e^{i(px+qy)}dx\,dy = c^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)e^{i(px+qy)}dx\,dy$$

or $$\frac{d^2\bar{z}}{dt^2} = c^2(-p^2-q^2)\bar{z} \qquad \text{or} \qquad [\mathrm{D}^2 + c^2(p^2+q^2)]\bar{z} = 0, \quad \mathrm{D} \equiv d/dt \quad \text{... (4)}$$

where $$\bar{z}(p,q,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(x,y,t)e^{i(px+qy)}dx\,dy \quad \ldots (5)$$

is the two-dimensional Fourier transform of $z(x, y,t)$.

Auxiliary equation of (4) is $D^2 + c^2(p^2 + q^2) = 0$ giving $D = \pm ic(p^2 + q^2)^{1/2}$.

Hence solution of (4) is given by

$$\bar{z}(p,q,t) = C_1 \cos\{ct(p^2+q^2)^{1/2}\} + C_2 \sin\{ct(p^2+q^2)^{1/2}\},\ C_1, C_2 \text{ being arbitrary constants} \quad \ldots (6)$$

Taking the double Fourier transform of conditions (2) and (3), we get

$$\bar{z}(p,q,0) = F(p, q) \quad \ldots (7)$$

and $$(d\bar{z}/dt)_{t=0} = 0, \quad \ldots (8)$$

where $$F(p,q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{i(px+qy)}dx\,dy \quad \ldots (9)$$

From (6), $$d\bar{z}/dt = -C_1 c(p^2+q^2)^{1/2} \sin\left\{ct(p^2+q^2)^{1/2}\right\} + C_2 c(p^2+q^2)^{1/2} \cos\left\{ct(p^2+q^2)^{1/2}\right\}. \quad \ldots (10)$$

Putting $t = 0$ in (10) and using (8), we get $0 = C_2 c(p^2+q^2)^{1/2}$ so that $C_2 = 0$.

$\therefore$ From (6), $$\bar{z}(p,q,t) = C_1 \cos\{ct(p^2+q^2)^{1/2}\}. \quad \ldots (11)$$

Putting $t = 0$ in (11) and using (7), we get $C_1 = F(p,q)$.

From (11), $$\bar{z}(p,q,t) = F(p,q)\cos\left\{ct(p^2+q^2)^{1/2}\right\}.$$

By applying inversion theorem for double Fourier transform, we get

$$z(x,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p,q)\cos\left\{ct(p^2+q^2)^{1/2}\right\}e^{-i(px+qy)}dp\,dq.$$

Miscellaneous problems on chapter 1

Ex.1. Under the usual meanings of the notations the relation between the Fourier and Laplace transform is $F\{F(t)\} =$ (*a*) $L\{F(t)\}$ (*b*) $L\{G(t)\}$ (*c*) $L\{F\{G(t)\}\}$ (d) None of these

Sol. Ans. (b). Refer Art. 1.9, page 1.13 **[Madurai Kamraj Univerisity 2008]**

Ex. 2. Evaluate $F_s^{-1}\{e^{-ap}/p\}$ **[Madurai Kamraj Univerisity 2008]**

Hint. Refer Ex. 9, page 1.26. Here we have p in place of s.

Ex. 3. Find the Fourier cosine and sine transforms of e^{-bx} and show that

(*i*) $\int_0^{\infty}\frac{\cos sx}{s^2+b^2}dx = \frac{\pi}{2b}e^{-bx}$ (*ii*) $\int_0^{\infty}\frac{s\sin sx}{s^2+b^2}dx = \frac{\pi}{2}e^{-bx}$

Ex. 4. Solve $k(\partial^2 u/\partial x^2) = \partial u/\partial t, -\infty < x < \infty, t > 0$ subject to $u(x, t)$ and $u_x(x, t)$ both tend to zero as $|x| \to \infty$ and $u(x, 0) = f(x)$, $-\infty < x < \infty$ **Ans.** $u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(s)e^{-\{(x-s)^2/4kt\}}ds$

Ex.5. Obtain the temperature distribution in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial temperature is $f(x)$.

Hint. We are to solve $\partial u/\partial t = k(\partial^2 u/\partial x^2),\ 0 < x < \infty,\ t > 0$ subject to $u(0,t) = 0,\ t > 0$ and $u(x,0) = f(x),\ 0 < x < \infty$. Using Fourier sine transform, the required solution is given by

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)\ e^{-ks^2 t}\sin sx\ ds,\ \text{where } \bar{f}_s(s) = \int_0^\infty f(x)\sin sx\,dx$$

Ex. 6. Determine the solution of free vibrations of a semi-infinite string described by $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2),\ 0 < x < \infty,\ t > 0$ subject to $u(x,0) = f(x)$ and $u_t(0,t) = g(x)$.

Ans. $u(x,t) = (1/2)\times\{f(x+ct) + f(x-ct)\} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\ ds,\ \text{where } g(s) = \int_0^\infty g(x)\sin sx\,dx$

Ex. 7 (*a*) Solve the following boundary value problem in the half-plane y > 0, described by $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0,\ -\infty < x < \infty,\ y > 0$ subject to $u(x,0) = f(x),\ -\infty < x < \infty$, u is bounded as $y \to \infty,\ u$ and $\partial u/\partial x$ both vamish as $|x| \to \infty$

(*b*) Solve the following boundary value problem in the half-plane $x > 0$, described by $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0,\ -\infty < y < \infty,\ x > 0$ subject to $u(0,y) = f(y),\ -\infty < y < \infty,\ u$ is bounded as $x \to \infty,\ u$ and $\partial u/\partial y$ both vanish as $|y| \to \infty$.

Ans. (*a*) $u(x,y) = \dfrac{y}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{f(\xi)}{(\xi - x)^2 + y^2}\,d\xi$ (*b*) $u(x,y) = \dfrac{x}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{f(\xi)}{x^2 + (y-\xi)^2}\,d\xi$

Ex. 8. If $\bar{f}(s)$ is the complex Fouier transform of *f*(*x*), show that the complex Fourier transform of $f(ax)$ is $(1/a)\times\bar{f}(s/a)$ **[Osmania 2010]**

Hint. Refer Art 1.12 , page 1.15.

Ex. 9 *Find the Fourier tranform of* $f(x)$ defined by

$$f(x) = \begin{cases} 0, & x < a \\ 1, & a < x < b \\ 0, & x < b \end{cases}$$ **(Madras 2012)**

Sol. By definition, $F\{f(x)\} = \displaystyle\int_{-\infty}^{\infty} f(x)e^{isx}dx = \int_{-\infty}^{a} f(x)e^{isx}\,dx + \int_a^b f(x)e^{isx}dx + \int_b^\infty f(x)e^{isx}dx$

or $$F(x) = \int_a^b e^{isx}dx = (1/is)\times[e^{isx}]_a^b = (1/is)\times(e^{isb} - e^{isa}),$$

where we used the given value of *f*(*x*)

2

Finite Fourier Transforms

PART I : FINITE FOURIER TRANSFORMS

2.1. Introduction.

So far in this part of the book use has only been made of intergal transforms in which the range of integration has been infinite. In our solutions of boundary-value problems we have thus only been able to exclude a variable with with range $(0, \infty)$ or $(-\infty, \infty)$. It would be useful to be able to employ the same technique in many physical problems of great interest in which such a condition does not hold, that is, to use transforms to exclude a variable whose range is finite. In order to do this, we propose to study the so called finite Fourier transforms in this chapter. The use of finite transforms does not solve problems which are incapable of solution by the classical methods of Fourier series. It does, however, facilitate their solution in that the same "drill" as has already been given for transforms with infinite range of integration can be used. In this, the use of finite Fourier transforms appears to have a distinct advantage over the classical methods, which often require great ingenuity in assuming at the outset the correct form of the solution.

2.2. The finite Fourier sine transform. Definition. [Kanpur 2006]

Let F (x) be defined in (o, l) and satisfy Dirichlet's conditions in that finite interval. Then the *finite Fourier sine transform* of F (x), o $< x < l$ is denote and defined as

$$F_s\{F(x)\} = \overline{F}_s(s) = f_s(s) = \int_o^l F(x) \sin \frac{s\pi x}{l} dx , \qquad \ldots (1a)$$

where s a positive integer.

Remark. 1. In many physical situations, the choice of π as the upper limit of integration in these transforms is found more convenient. Forthermore, by a proper choice of substitution, the upper limit can be arranged as π in place of l. Hence taking $l = \pi$ in $(1a)$, we define the *finite Fourier transform as*

$$F_s\{F(x)\} = \overline{F}_s(s) = f_s(s) = \int_o^\pi F(x) \sin sx \, dx. \qquad \ldots (1b)$$

Remark. 2. Note that $F_s\{F(x)\}$ is always zero when $s = 0$. So we take $s = 1, 2, 3,...$ in $1(a)$ and $1(b)$.

If $\overline{F}_s(s)$ or $f_s(s)$ is the finite Fourier sine transform of $F(x)$, then the function $F(x)$ is called the *inverse finite Fourier sine transform* of $\overline{F}_s(s)$ or $f_s(s)$ and is denoted by

$$F(x) = F_s^{-1}\{\overline{F}_s(s)\} = F_s^{-1}\{f_s(s)\}.$$

Inversion formula for finite Fourier sine transform.

If $f_s(s)$ or $\overline{F}_s(s)$ is the finite Fourier sine transform of F(x) over the interval (o, l), then the inverse finite Fourier sine transform is given by

$$F(x) = \frac{2}{l} \sum_{s=1}^{\infty} f_s(s) \sin \frac{s\pi x}{l} \qquad \text{or} \qquad F(x) = \frac{2}{l} \sum_{s=1}^{\infty} \overline{F}_s(s) \sin \frac{s\pi x}{l}.$$

Proof. Here F(x) is defined in the interval (o, l) and satisfy Dirichlet's conditions in that interval. Define F (x) in the interval ($-l$, 0) by the equation F($-x$) = –F(x). The function thus defined becomes an odd function for ($-l$,l) and satisfy Dirichlet's conditions in ($-l$, l). In this case *Fourier series contains sine termes only and is given by

$$\mathrm{F}(x) = \sum_{s=1}^{\infty} b_s \sin\frac{s\pi x}{l}, \qquad \ldots (2)$$

where $$b_s = \frac{2}{l}\int_o^l \mathrm{F}(x)\sin\frac{s\pi x}{l}dx, \qquad s = 1, 2, 3, \ldots \qquad \ldots (3)$$

Using 1(a), $(3) \Rightarrow$ $b_s = (2/l) \times f_s(s)$ or $b_s = (2/l) \times \overline{\mathrm{F}}_s(s)$

$\therefore$ $(2) \Rightarrow$ $\mathrm{F}(x) = \frac{2}{l}\sum_{s=1}^{\infty} f_s(s)\sin\frac{s\pi x}{l}$ or $\mathrm{F}(x) = \frac{2}{l}\sum_{s=1}^{\infty} \overline{\mathrm{F}}_s(s)\sin\frac{s\pi x}{l}$. ... (2$a$)

As before if we take $l = \pi$, then 2(a) reduces to

$\mathrm{F}(x) = \frac{2}{\pi}\sum_{s=1}^{\infty} f_s(s)\sin sx$ or $\mathrm{F}(x) = \frac{2}{\pi}\sum_{s=1}^{\infty} \overline{\mathrm{F}}_s(s)\sin sx$. ... (2$b$)

Remark. Students should note that for $o < x < l$, finite Fourier sine transform and its inverse finite Fourier sine transform are given by 1(a) and 2(a) respectively. For $o < x < \pi$, results 1(b) and 2(b) will be used. Commit to memory all these four results.

2.3. The finite Fourier cosine transform. Definition [Kanpur 2006, Meerut 2002]

Let F(x) be defined in (o, l) and satisfy Dirichlet's conditions in that finite interval. Then the *finite Fourier cosine transform* of F(x), $o < x < l$, is denoted and defined as

$$\mathrm{F}_c\{\mathrm{F}(x)\} = \overline{\mathrm{F}}_c(s) = f_c(s) = \int_o^l \mathrm{F}(x)\cos\frac{s\pi x}{l}dx, \qquad \ldots (1a)$$

where s is positive integer or zero.

As a particular case, if $l = \pi$, 1(a) reduces to

$$\mathrm{F}_c\{\mathrm{F}(x)\} = \overline{\mathrm{F}}_c(s) = f_c(s) = \int_0^{\pi} \mathrm{F}(x)\cos sx\,dx. \qquad \ldots (1b)$$

If $\overline{\mathrm{F}}_c(s)$ or $f_c(s)$ is the finite Fourier cosine transform of F(x), then the function F(x) is called the *inverse Fourier cosine transform* of $\overline{\mathrm{F}}_c(s)$ or $f_c(s)$ and is denoted by

$$\mathrm{F}(x) = \mathrm{F}_c^{-1}\{\overline{\mathrm{F}}_c(s)\} = \mathrm{F}_c^{-1}\{f_c(s)\}.$$

Inversion formula for finite Fourier cosine transform.

If $f_c(s)$ or $\overline{F}_c(s)$ is the finite Fourier cosine transform of F(x) over the interval (o, l), then the inverse Fourier cosine transform is given by

$$\mathrm{F}(x) = \frac{1}{l}\overline{\mathrm{F}}_c(o) + \frac{2}{l}\sum_{s=1}^{\infty}\overline{\mathrm{F}}_c(s)\cos\frac{s\pi x}{l} \quad \text{or} \quad \mathrm{F}(x) = \frac{1}{l}f_c(o) + \frac{2}{l}\sum_{s=1}^{\infty} f_c(s)\cos\frac{s\pi x}{l}.$$

where $$\overline{\mathrm{F}}_c(o) = f_c(o) = \int_o^l \mathrm{F}(x)\,dx.$$

*For details of Fourier series, refer chapter 1 of author's "Integral transforms" published by S.Chand & Co. New Delhi

Proof. Here F(x) is defined in the interval (o, l) and satisfy Dirichlet's conditions in that interval. Define F(x) in the interval ($-l$, 0) by the equation F($-x$) = F(x). The function thus defined becomes an even function for ($-l$, l) and satisfy Dirichlet's conditions in ($-l$, l). In this case* Fourier series contains cosine terms only and is given by

$$\mathrm{F}(x) = \frac{a_0}{2} + \sum_{s=1}^{\infty} a_s \cos\frac{s\pi x}{l}, \qquad \ldots (2)$$

where
$$a_s = \frac{2}{l}\int_o^l \mathrm{F}(x)\cos\frac{s\pi x}{l}\,dx. \qquad \ldots (3)$$

From (3), for $s = 0$,
$$a_o = \frac{2}{l}\int_o^l \mathrm{F}(x)\,dx. \qquad \ldots (4)$$

Using 1(a), (3) $\Rightarrow$ $a_s = (2/l)\times f_c(s)$ or $a_s = (2/l)\times \overline{\mathrm{F}}_c(s)$ ... (5)

Also, for $s = 0$, $a_o = (2/l)\times f_c(o)$ or $a_o = (2/l)\times \overline{\mathrm{F}}_c(o)$ ... (6)

Using (5) and (6), (2) $\Rightarrow$
$$\mathrm{F}(x) = \frac{1}{l}\overline{\mathrm{F}}_c(o) + \frac{2}{l}\sum_{s=1}^{\infty}\overline{\mathrm{F}}_c(s)\cos\frac{s\pi x}{l}$$

or
$$\mathrm{F}(x) = \frac{1}{l}f_c(o) + \frac{2}{l}\sum_{s=1}^{\infty} f_c(s)\cos\frac{s\pi x}{l}. \qquad \ldots (2a)$$

Also, (4) and (6) $\Rightarrow$
$$\overline{\mathrm{F}}_c(o) = f_c(o) = \int_o^l \mathrm{F}(x)\,dx$$

As a particular case, when $l = \pi$, 2(a) reduces to

$$\mathrm{F}(x) = \frac{1}{\pi}\bar{f}_c(o) + \frac{2}{\pi}\sum_{s=1}^{\infty}\overline{\mathrm{F}}_c(s)\cos sx$$

or
$$\mathrm{F}(x) = \frac{1}{\pi}f_c(o) + \frac{2}{\pi}\sum_{s=1}^{\infty} f_c(s)\cos sx, \qquad \ldots (2b)$$

where
$$\overline{\mathrm{F}}_c(o) = f_c(o) = \int_o^{\pi}\mathrm{F}(x)\,dx.$$

Remark. Students should note that for $o < x < l$, finite Fourier cosine transform and its inverse finite Fourier sine transforms are given by 1(a) and 2(a) respectively. For $o < x < \pi$, results 1(b) and 2(b) will be used. Commit to memory all these four results.

2.4. Some useful results for direct applications

(i) If n is an integer, then $\sin n\pi = 0$, $\cos n\pi = (-1)^n$,

$\sin\{(2n+1)\times(\pi/2)\} = (-1)^n$ and $\cos\{(2n+1)\times(\pi/2)\} = 0$.

(ii) $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$, $\quad\int_0^{\infty} e^{-ax}\sin bx\,dx = \dfrac{b}{a^2+b^2}$.

(iii) $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$, $\quad\int_0^{\infty} e^{-ax}\cos bx\,dx = \dfrac{a}{a^2+b^2}$.

(iv) $\sinh x = (e^x - e^{-x})/2$, $\quad\cosh = (e^x + e^{-x})/2$.

---

* For Fourier series, refer author's "Integral Transforms" published by S. Chand and Co. New Delhi.

(*v*) **The method of successive integration by parts or chain rule of integration.**

Read the solution of the following problem carefully.

$$\int x^3 \cos 2x\, dx = (x^3)\left(\frac{\sin 2x}{2}\right) - (3x^2)\left(-\frac{\cos 2x}{4}\right) + (6x)\left(-\frac{\sin 2x}{8}\right) - (6)\left(\frac{\cos 2x}{16}\right).$$

Rule: As usual we regard x^3 as the first function for differentiation and regard cos 2x as the second function for integration. We write successive terms with alternately positive and negative signs. We write x^3 as such in first term and write down its successive differential coefficients in first brackets of successive terms. Also we integrate cos 2x successively and write down the integration within second brackets of successive terms.

Note: We can evaluate the above integral by the usual rule of integration by parts three times step by step. But the above rule is short cut method to get the result in one step.

2.5. Solved examples Based on Finite Fourier Sine and Cosine Transforms

Important Note: While dealing with finite Fourier sine and cosine transform, unless stated otherwise, the upper limit will be taken as $l = \pi$.

Ex. 1. *Find finite Fourier sine and cosine transforms of f(x)=1.* **[Madras 2012]**

Sol. Since the range of x is not given, we assume that $f(x)$ is defined in (o, π). By definition of finite Fourier sine transform, we have

$$F_s\{f(x)\} = \int_0^\pi f(x)\sin sx\, dx = \int_0^\pi \sin sx\, dx = -\left[\frac{\cos sx}{s}\right]_0^\pi = \frac{1-\cos s\pi}{s} = \frac{1-(-1)^s}{s}, \text{ if } s = 1, 2, 3\ldots$$

By definition of finite Fourier cosine transform, we have

$$F_c\{f(x)\} = \int_0^\pi f(x)\cos sx\, dx = \int_0^\pi \cos sx\, dx = \left[\frac{\sin sx}{s}\right]_0^\pi = 0, \text{ if } s = 1, 2, 3\ldots$$

If $s = 0$, then $$F_c\{f(x)\} = \int_0^\pi f(x)\, dx = \int_0^\pi dx = \pi.$$

Thus, $F_s\{f(x)\} = \{1-(-1)^s\}/s$, if $s = 1, 2, 3\ldots$ and $F_c\{f(x)\} = 0$, if $s = 1, 2, 3$; $F_c\{f(x)\} = \pi$, if $s = 0$.

Ex. 2. *Find the (a) finite Fourier sine transform and (b) finite cosine transform of the function f(x)= 2x, o < x <4.* **[Meerut 1999, 2004]**

Sol. (*a*) By definition of finite Fourier sine transform, we have

$$F_s\{f(x)\} = \int_0^4 f(x)\sin\frac{s\pi x}{4}dx = \int_0^4 (2x)\sin\frac{s\pi x}{4}dx, \text{ as } f(x) = 2x$$

$$= \left[(2x)\times\frac{-\cos\{(s\pi x)/4\}}{(s\pi/4)}\right]_0^4 - \int_0^4 2\times\frac{-\cos\{(s\pi x)/4\}}{(s\pi/4)}dx, \text{ integrating by parts}$$

$$= -\frac{8}{s\pi}\left[x\cos\frac{s\pi x}{4}\right]_0^4 + \frac{8}{s\pi}\left[\frac{\sin\{(s\pi x)/4\}}{(s\pi/4)}\right]_0^4 = -\frac{32}{s\pi}(-1)^s$$

$\therefore$ $$F_s\{f(x)\} = -(32/s\pi)\times(-1)^s, \text{ if } s = 1, 2, 3. \ldots\ldots$$

(*b*) By definition of finite Fourier cosine transform, we have

$$F_c\{f(x)\} = \int_0^4 f(x)\cos\frac{s\pi x}{4}dx = \int_0^4 (2x)\cos\frac{s\pi x}{4}dx,\ s = 1, 2, 3, \ldots.$$

$$= \left[(2x)\times\frac{\sin\{s\pi x/4\}}{(s\pi/4)}\right]_0^4 - \int_0^4 2\times\sin\frac{\sin\{(s\pi x)/4\}}{(s\pi/4)}dx \;=\; \frac{8}{s\pi}\left[x\sin\frac{s\pi x}{4}\right]_0^4 - \frac{8}{s\pi}\left[\frac{-\cos\{(s\pi x/4)\}}{(s\pi/4)}\right]_0^{\pi}$$

$$= 0+\frac{32}{s^2\pi^2}\left[\cos\frac{s\pi x}{4}\right]_0^4 = \frac{32}{s^2\pi^2}\left[\cos s\pi - 1\right] = \frac{32[(-1)^s-1]}{s^2\pi^2}$$

If $\quad s=0, \mathrm{F_c}\{f(x)\} = \int_0^4 f(x)\,dx = \int_0^4 (2x)dx = \left[x^2\right]_0^4 = 16$

$\therefore \qquad \mathrm{F_c}\{f(x)\} = (32/s^2\pi^2)\times[(-1)^s-1], \quad$ if $s = 1, 2, 3...; \quad \mathrm{F}_c\{f(x)\}=16$, if $s = 0$.

Ex. 3. *Find finite Fourier sine and cosine transform of f(x) = x^2, o<x <4.*

[Purvanchal 2002; Kanpur 2004]

Sol. $\mathrm{F}_s\{f(x)\} \;=\; \int_0^4 f(x)\sin\frac{s\pi x}{4}dx \;=\; \int_0^4 x^2\sin\frac{s\pi x}{4}dx$, if $s = 1, 2, 3...$

$$= \left[\left(x^2\right)\times\left\{-\frac{\cos(s\pi x)/4}{(s\pi)/4}\right\}\right]_0^4 - \int_0^4 (2x)\times\left\{-\frac{\cos(s\pi x)/4}{(s\pi)/4}\right\}dx, \text{ integrating by parts}$$

$$= -\frac{64\cos s\pi}{s\pi} + \frac{8}{s\pi}\int_0^4 x\cos\frac{s\pi x}{4}dx \;=\; -\frac{64(-1)^s}{s\pi} + \frac{8}{s\pi}\left\{\left[x\times\frac{\sin(s\pi x)/4}{(s\pi)/4}\right]_0^4 - \int_0^4 1\times\frac{\sin(s\pi x)/4}{(s\pi)/4}dx\right\}$$

$$= \frac{64(-1)^{s+1}}{s\pi} - \frac{32}{s^2\pi^2}\left[-\frac{\cos(s\pi x)/4}{(s\pi)/4}\right]_0^4 \;=\; \frac{64(-1)^{s+1}}{s\pi} + \frac{128}{s^3\pi^3}\left[(-1)^s-1\right].$$

$$\mathrm{F}_c\{f(x)\} \;=\; \int_0^4 f(x)\cos\frac{s\pi x}{4}dx = \int_0^4 x^2\cos\frac{s\pi x}{4}dx, \; \textit{if } s = 1,2,3..$$

$$= \left[x^2\times\frac{\sin(s\pi x)/4}{(s\pi)/4}\right]_0^4 - \int_0^4 (2x)\times\left\{\frac{\sin(s\pi x)/4}{(s\pi)/4}\right\}dx \;=-\;\frac{8}{s\pi}\int_0^4 x\sin\frac{s\pi x}{4}dx$$

$$= -\frac{8}{s\pi}\left\{\left[(x)\times\frac{-\cos(s\pi x)/4}{(s\pi)/4}\right]_0^4 - \int_0^4 1\times\frac{-\cos(s\pi x)/4}{(s\pi)/4}dx\right\}, \text{ integrating by parts}$$

$$= -\frac{8}{s\pi}\left[-\frac{16\cos s\pi}{s\pi} + \frac{4}{s\pi}\int_0^4\cos\frac{s\pi x}{4}dx\right] = \frac{128(-1)^s}{s^2\pi^2} - \frac{32}{s^2\pi^2}\left[\frac{\sin(s\pi x)/4}{(x\pi)/4}\right]_0^4 = \frac{128(-1)^s}{s^2\pi^2}$$

If $s = 0$, then $\qquad \mathrm{F}_c\{f(x)\} = \int_0^4 f(x)\,dx = \int_0^4 x^2dx = \left[\frac{1}{3}x^3\right]_0^4 = \frac{64}{3}.$

Ex. 4. *Find the finite Fourier sine and cosine trasnsform of f(x) = x^2 , if o < x<π.*

[Osmania 2010; Nagpur 2005]

Sol. $\mathrm{F}_s\{x^2\} = \int_0^{\pi} x^2\sin sx\,dx$, s = 1, 2, 3, ...

$$= \left[x^2\left(-\frac{\cos sx}{s}\right)\right]_0^{\pi} - \int_0^{\pi}(2x)\left(-\frac{\cos sx}{s}\right)dx = -\frac{\pi^2\cos s\pi}{s} + \frac{2}{s}\left\{\left[x\times\frac{\sin sx}{s}\right]_0^{\pi} - \int_0^{\pi}1\times\frac{\sin sx}{s}dx\right\}$$

$$= -\frac{\pi^2(-1)^s}{s} + \frac{2}{s}\left\{0 - \frac{1}{s}\left[-\frac{\cos sx}{s}\right]_0^{\pi}\right\}, \text{ integrating by parts}$$

$$= \frac{\pi^2(-1)^{s+1}}{s} + \frac{2}{s^3}(\cos s\pi - 1) = \frac{\pi^2(-1)^{s+1}}{s} + \frac{2[(-1)^s - 1]}{s^3}.$$

$$\mathrm{F}_c\{x^2\} = \int_0^{\pi}x^2\cos sx\,dx = \left[x^2\times\frac{\sin sx}{s}\right]_0^{\pi} - \int_0^{\pi}(2x)\times\frac{\sin sx}{s}dx$$

$$= -\frac{2}{s}\left\{\left[x\left(-\frac{\cos sx}{s}\right)\right]_0^{\pi} - \int_0^{\pi}1\times\left(-\frac{\cos sx}{s}\right)dx\right\}, \text{ integrating by parts}$$

$$= \frac{2\pi\cos s\pi}{s^2} - \frac{2}{s^2}\left[\frac{\sin sx}{s}\right]_0^{\pi} = \frac{2\pi(-1)^s}{s^2}, \text{ if } s = 1, 2, 3, \ldots\ldots$$

If $s = 0$, $\quad \mathrm{F}_c\{x^2\} = \int_0^{\pi}x^2dx = \left[(x^3/3)\right]_0^{\pi} = \pi^3/3.$

Ex. 5. *Find the finite Fourier sine and cosine transform of f(x)* $= (\pi/3) - x + (x^2/2\pi)$. **[Meerut 2001, 02]**

Sol. Since the range of x is not given, we take $o < x < \pi$.

Now, by the definition of finite Fourier sine transform, we get

$$\mathrm{F}_s\left\{\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right\} = \int_0^{\pi}\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\sin sx\,dx, s = 1, 2, 3\ldots$$

$$= \left[\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\left(\frac{-\cos sx}{s}\right)\right]_0^{\pi} - \int_0^{\pi}\left(-1 + \frac{x}{\pi}\right)\left(-\frac{\cos sx}{s}\right)dx = -\left(\frac{\pi}{3} - \pi + \frac{\pi}{2}\right)\frac{\cos s\pi}{s} + \frac{\pi}{3} + \int_0^{\pi}\left(\frac{x}{\pi} - 1\right)\frac{\cos sx}{s}dx$$

$$= \frac{\pi(-1)^s}{6s} + \frac{\pi}{3} + \left[\left(\frac{x}{\pi} - 1\right)\left(\frac{\sin sx}{s^2}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{1}{\pi}\right)\left(\frac{\sin sx}{s^2}\right)dx$$

$$= \frac{\pi(-1)^s}{6s} + \frac{\pi}{3} - \frac{1}{\pi s^2}\left[-\frac{\cos sx}{s}\right]_0^{\pi} = \frac{\pi(-1)^s}{6s} + \frac{\pi}{3} + \frac{1}{\pi s^3}[\cos s\pi - 1]$$

$$= (\pi/6s)\times\left\{(-1)^s + 2\right\} + (1/\pi s^3)\times\left\{(-1)^s - 1\right\}, s = 1, 2, 3\ldots$$

Now, by, the definition of finite Fourier cosine transform, we get

$$\mathrm{F}_c\left\{\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right\} = \int_0^{\pi}\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\cos sx\,dx, \; s = 1, 2, 3 \ldots$$

$$= \left[\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\left(\frac{\sin sx}{s}\right)\right]_0^{\pi} - \int_0^{\pi}\left(-1 + \frac{x}{\pi}\right)\left(\frac{\sin sx}{s}\right)dx = -\frac{1}{s}\left\{\left[\left(\frac{x}{\pi} - 1\right)\left(-\frac{\cos sx}{s}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{1}{\pi}\right)\left(-\frac{\cos sx}{s}\right)dx\right\}$$

$$= (1/s^2) - (1/\pi s^2) \times \left[(1/s) \times \sin sx\right]_0^{\pi} = (1/s^2), s = 1, 2, 3...$$

$$\text{If } s = 0,\ \mathrm{F}_c\left\{\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right\} = \int_0^{\pi}\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)dx = \left[\frac{\pi x}{3} - \frac{x^2}{2} + \frac{x^3}{6\pi}\right]_0^{\pi} = \frac{\pi^2}{3} - \frac{\pi^2}{2} + \frac{\pi^2}{6} = 0$$

Ex. 6. *Find the finite Fourier cosine transform of f(x), if f(x)* $= \begin{cases} 1, \text{when } 0 < x < \pi/2 \\ -1, \text{when } \pi/2 < x < \pi \end{cases}$

[Meerut 2000]

Sol. By definition of finite Fourier cosine transform, for s = 1, 2, 3, . .

$$\mathrm{F}_c\{f(x)\} = \int_0^{\pi} f(x)\cos sx\,dx = \int_0^{\pi/2} f(x)\cos sx\,dx + \int_{\pi/2}^{\pi} f(x)\cos sx\,dx \ \ldots$$

$$= \int_0^{\pi/2} 1 \times \cos sx\,dx + \int_{\pi/2}^{\pi} (-1) \times \cos sx\,dx, \text{ using the given values of } f(x)$$

$$= \left[\frac{\sin sx}{s}\right]_0^{\pi/2} - \left[\frac{\sin sx}{s}\right]_{\pi/2}^{\pi} = \frac{1}{s}\sin\frac{s\pi}{2} - \left(-\frac{1}{s}\sin\frac{s\pi}{2}\right) = \frac{2}{s}\sin\frac{s\pi}{2}$$

$$= \begin{cases} o, \text{if } s = 2m, \text{where } m = 1, 2, 3.... \\ \dfrac{2}{2m-1}\sin\dfrac{(2m-1)\pi}{2}, \text{if } s = 2m-1, \text{where } m = 1, 2, 3... \end{cases}$$

$$= \begin{cases} o, \text{if } s = 2m, \text{ where, } m = 1, 2, 3.... \\ 2(-1)^{m-1}/(2m-1), \text{if } s = 2m-1, \text{where } m = 1, 2, 3... \end{cases}$$

$$\text{If } s = 0 \qquad \mathrm{F}_c\{f(x)\} = \int_0^{\pi} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx$$

$$= \int_0^{\pi/2} (1)\,dx + \int_{\pi/2}^{\pi} (-1)dx = \left[x\right]_0^{\pi/2} - \left[x\right]_{\pi/2}^{\pi} = \frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right) = 0.$$

Ex. 7. *Find the finite Fourier sine transform of f(x), if f(x)* $= \begin{cases} x, o \le x \le \pi/2 \\ \pi - x, \pi/2 \le x \le \pi. \end{cases}$

Sol. By the defintion of finite Fourier sine transform, we have

$$\mathrm{F}_s\{f(x)\} = \int_0^{\pi} f(x)\sin sx\,dx = \int_0^{\pi/2} f(x)\sin sx\,dx + \int_{\pi/2}^{\pi} f(x)\sin sx\,dx$$

$$= \int_o^{\pi/2} x\sin sx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin sx\,dx, \text{ using the given values of } f(x)$$

$$= \left[(x)\left(-\frac{\cos sx}{s}\right)-(1)\left(-\frac{\sin sx}{s^2}\right)\right]_0^{\pi/2}+\left[(\pi-x)\left(-\frac{\cos sx}{s}\right)-(-1)\left(-\frac{\sin sx}{s^2}\right)\right]_{\pi/2}^{\pi}$$

[Using the chain rule of integration by parts, refer Art. 2.4]

$$= -\frac{\pi}{2s}\cos\frac{s\pi}{2}+\frac{1}{s^2}\sin\frac{s\pi}{2}+\frac{\pi}{2s}\cos\frac{s\pi}{2}+\frac{1}{s^2}\sin\frac{s\pi}{2} = \frac{2}{s^2}\sin\frac{s\pi}{2}$$

$$= \begin{cases} o, \text{if } s = 2m, \text{ where } m = 1,2,3... \\ \dfrac{2}{(2m-1)^2}\sin\dfrac{(2m-1)\pi}{2}, \text{if } s = 2m-1, \text{ where } m = 1,2,3... \end{cases}$$

$$= \begin{cases} o, \text{if } s = 2m, \text{ where } m = 1,2,3.... \\ 2(-1)^{m-1}/(2m-1)^2, \text{if } s = 2m-1, \text{ where } m = 1,2,3.... \end{cases}$$

Ex. 8. *Find the finite Fourier sine transform of f(x), if f(x) = sin nx, where n is an integer.*

Sol. Since the range of x is not given, we take $o < x < \pi$.

Now, by the difinition of the finite Fourier sine transform, we get

$$F_s\{\sin nx\} = \int_o^{\pi}\sin sx\sin nx\,dx = \frac{1}{2}\int_0^{\pi}\{\cos(n-s)x-\cos(n+s)x\}dx = \frac{1}{2}\left[\frac{\sin(n-s)x}{n-s}-\frac{\sin(n+s)x}{n+s}\right]_0^{\pi} = 0, \text{if } s \neq n.$$

$$\text{If } s = n,\ F_s\{\sin nx\} = \int_0^{\pi}\sin nx\sin sx\,dx = \int_0^{\pi}\sin^2 sx\,dx = \frac{1}{2}\int_0^{\pi}(1-\cos 2sx)\,dx$$

$$= (1/2)\times\left[x-(1/2s)\times\sin 2sx\right]_0^{\pi} = \pi/2$$

Ex. 9. *Find finite Fourier cosine transform of sin nx, n being an integer.*

Sol. Since the range of x is not given, we take $o < x < \pi$.

Then by the definition of the finite Fourier cosine transform, for $s = 1, 2, 3, \ldots$

$$F_c\{\sin nx\} = \int_o^{\pi}\sin nx\cos sx\,dx = \frac{1}{2}\int_0^{\pi}[\sin(n+s)x+\sin(n-s)x]\,dx = \frac{1}{2}\left[-\frac{\cos(n+s)x}{n+s}-\frac{\cos(n-s)x}{n-s}\right]_0^{\pi}$$

$$\therefore\ F_c\{\sin nx\} = \frac{1}{2}\left[-\frac{\cos(n+s)\pi}{n+s}-\frac{\cos(n-s)\pi}{n-s}+\frac{1}{n+s}+\frac{1}{n-s}\right]_0^{\pi},\ \text{if } s \neq n \qquad \ldots (1)$$

Now, the following two cases arise:

Case. I. Let $n - s$ be even. Then $n + s = (n - s) + 2s$ = even + even = even. Thus we see that $(n - s)$ and $(n + s)$ are both even. Hence, $\cos(n + s)\pi = (-1)^{n+s} = 1$ and $\cos(n - s)\pi = (-1)^{n-s} = 1$.

$$\text{Then, } (1) \Rightarrow \qquad F_c\{\sin nx\} = \frac{1}{2}\left[-\frac{1}{n+s}-\frac{1}{n-s}+\frac{1}{n+s}+\frac{1}{n-s}\right] = 0.$$

$$\text{When } s = n,\ F_c\{\sin nx\} = \int_0^{\pi}\sin nx\cos nx\,dx = \frac{1}{2}\int_0^{\pi}\sin 2sx\,dx = -\frac{1}{2}\left[\frac{\cos 2sx}{2s}\right]_0^{\pi} = 0$$

Case. II. Let $n - s$ be odd. Then $n + s = (n - s) + 2s$ = odd + even = odd. Thus we see that $(n - s)$ and $(n + s)$ are both odd. Hence, $\cos(n + s)\pi = (-1)^{n+s} = -1$, $\cos(n - s)\pi = (-1)^{n-s} = -1$.

$$\therefore (1) \Rightarrow \quad F_c\{\sin nx\} = \frac{1}{2}\left[\frac{1}{n+s}-\frac{1}{n-s}+\frac{1}{n+s}+\frac{1}{n-s}\right]=\frac{1}{n+s}+\frac{1}{n-s}=\frac{2n}{n^2-s^2}$$

Thus, $$F_c\{\sin nx\} = \begin{cases}0, \text{if } (n-s) \text{ is even or } s=n, \text{ where } s=1,2,3,....\\ 2n/(n^2-s^2), \text{if } (n-s) \text{ is odd, where } s=1,2,3,...\end{cases}$$

For $s = 0$, $$F_c\{\sin nx\}=\int_0^{\pi}\sin nx\,dx=\left[-\frac{\cos nx}{n}\right]_0^{\pi}=\frac{1}{n}\left[1-(-1)^n\right].$$

Ex. 10. *Find finite Fourier sine transform of cos kx, if k is any real number.*

Sol. Since the range of x is not given, we take o < x < π.

Then by the definition of the finite Fourier sine transform, we have

$$F_s\{\cos kx\} = \int_0^{\pi}\cos kx\sin sx\,dx=\frac{1}{2}\int_0^{\pi}\{\sin(s+k)x-\sin(s-k)x\}\,dx$$

$$=\frac{1}{2}\left[-\frac{\cos(s+k)x}{s+k}-\frac{\cos(s-k)x}{s-k}\right]_0^{\pi}=\frac{1}{2}\left[-\frac{\cos(s+k)\pi}{s+k}-\frac{\cos(s-k)\pi}{s-k}+\frac{1}{s+k}+\frac{1}{s-k}\right],\ s\neq k$$

$$=\{1/2(s^2-k^2)\}[s-k+s+k-(s-k)\cos(s+k)\pi-(s+k)\cos(s-k)\pi]$$

$$=\{1/2(s^2-k^2)\}[2s-(s-k)(\cos s\pi\cos k\pi-\sin s\pi\sin k\pi)\ -(s+k)(\cos s\pi\cos k\pi+\sin s\pi\sin k\pi)]$$

$$=\{1/2(s^2+k^2)\}[2s-(s-k)(-1)^s\cos k\pi-(s+k)(-1)^s\cos k\pi]$$

$$[\because\ s \text{ is an integer} \Rightarrow \cos s\pi=(-1)^s \quad \text{and} \quad \sin s\pi=0]$$

$$=\{1/2(s^2-k^2)\}[2s-2s(-1)^s\cos k\pi]=\{s/(s^2-k^2)\}[1-(-1)^s\cos k\pi], s\neq k$$

Also, when $s = k$, $$F_s\{\cos kx\}=\frac{1}{2}\int_0^{\pi}\sin 2sx\,dx=\frac{1}{2}\left[-\frac{\cos 2sx}{2s}\right]_0^{\pi}=0$$

Ex. 11. *Find the finite Fourier sine transform of f(x) if f(x)* $= [\sin k(\pi-x)]/\sin k\pi, k \neq 0$, *1, 2, 3...*

Sol. Since the range of x is not given, we take $o < x < \pi$. Then by the definition of finite Fourier sine transform, we get

$$F_s\{f(x)\} = \int_0^{\pi} f(x)\sin sx\,dx=\int_0^{\pi}\frac{\sin k(\pi-x)}{\sin k\pi}\sin sx\,dx\ ,\ s = 1, 2, 3, \ldots$$

$$= \frac{1}{2\sin k\pi}\int_0^{\pi}[\cos\{sx-k(\pi-x)\}-\cos\{sx+k(\pi-x)\}]\,dx$$

$$= \frac{1}{2\sin k\pi}\left[\frac{\sin\{sx-k(\pi-x)\}}{s+k}-\frac{\sin\{sx+k(\pi-x)\}}{s-k}\right]_0^{\pi}, s\neq k$$

$$= \frac{1}{2\sin k\pi}\left[-\frac{\sin(-k\pi)}{s+k}+\frac{\sin k\pi}{s-k}\right], \text{as } \sin s\pi = 0 \text{ since } s \text{ is an integer}$$

$$= \frac{1}{2\sin k\pi}\left[\frac{\sin k\pi}{s+k}+\frac{\sin k\pi}{s-k}\right]=\frac{s}{s^2-k^2}, k\neq 0,1,2,3...$$

Ex. 12. *Find finite Fourier cosine transform of f(x), if f(x) = cosh {c(π – x)} / sinh πc*

Sol. Since the range of x is not given, we take $o < x < \pi$. Then, by the definition of finite Fourier cosine transform, we get

$$F_c\{f(x)\} = \int_0^\pi f(x)\cos sx\,dx = \int_0^\pi \frac{\cosh\{c(\pi-x)\}}{\sinh \pi c}\cos sx\,dx,\ s = 1, 2, 3, \ldots$$

$$= \frac{1}{\sinh \pi c}\int_0^\pi \frac{1}{2}\left[e^{c(\pi-x)} + e^{-c(\pi-x)}\right]\cos sx\,dx = \frac{1}{2\sinh \pi c}\left[e^{c\pi}\int_0^\pi e^{-cx}\cos sx\,dx + e^{-c\pi}\int_0^\pi e^{cx}\cos sx\,dx\right]$$

$$= \frac{1}{2\sinh \pi c}\left[e^{c\pi}\frac{e^{-cx}}{c^2+s^2}\left[-c\cos sx + s\sin sx\right] + e^{-c\pi}.\frac{e^{cx}}{c^2+s^2}(c\cos sx + s\sin sx)\right]_0^\pi$$

$$= \frac{1}{2(c^2+s^2)\sinh \pi c}\left[-e^{c\pi}e^{-c\pi}c\cos s\pi + e^{-c\pi}e^{c\pi}c\cos s\pi + ce^{c\pi} - ce^{-c\pi}\right]$$

$$= \frac{c(e^{c\pi}-e^{-c\pi})}{(c^2+s^2)(e^{c\pi}-e^{-c\pi})} = \frac{c}{c^2+s^2}, \qquad \text{as} \qquad \sinh \pi c = \frac{e^{c\pi}-e^{-c\pi}}{2}$$

If $s = 0$, $\quad F_c\{f(x)\} = \int_0^\pi \frac{\cosh\{c(\pi-x)\}}{\sinh \pi c}dx = \left[\frac{\sinh\{c(\pi-x)\}}{-c\sinh \pi c}\right]_0^\pi = \frac{1}{c}$

Ex. 13. *Find f (x) if* $\overline{f_s}(s) = \frac{\cos(2\pi s/3)}{(2s+1)^2}, o < x < 1$ **(Osmania 2010)**

Sol. For $o < x < l$, the inverse finite Fourier sine transform is given by

$$f(x) = \frac{2}{l}\sum_{s=1}^{\infty}\bar{f}_s(s)\sin\frac{s\pi x}{l}, \text{ where } l = 1 \qquad \text{so that} \qquad f(x) = 2\sum_{s=1}^{\infty}\frac{\cos(2\pi s/3)}{(2s+1)^3}\sin s\pi x.$$

Ex. 14. *If* $o < x < \pi$, *prove that* $F_s^{-1}\left\{(1-\cos s\pi)/s^3\right\} = x(\pi - x)/2.$

Sol. To prove that required result, we first prove that

$$F_s\{(x/2)\times(\pi - x)\} = (1 - \cos s\pi)/s^3. \qquad \ldots (1)$$

Now, by definition of finite Fourier sine transform, we have

$$F_s\{(1/2)\times(\pi x - x^2)\} = \int_0^\pi \frac{1}{2}(\pi x - x^2)\sin sx\,dx$$

$$= \frac{1}{2}\left[(\pi x - x^2)\left(-\frac{\cos sx}{s}\right) - (\pi - 2x)\left(-\frac{\sin sx}{s^2}\right) + (-2)\left(\frac{\cos sx}{s^3}\right)\right]_0^\pi$$

[Using chain rule of integration by parts, refer Art.2.4]

$$= (1/2)\times[-(2/s^3)\cos s\pi + 2/s^3] = (1-\cos s\pi)/s^3$$

Hence by definition of inverse finite Fourier sine transform, we get

$$F_s^{-1}\{(1-\cos s\pi)/s^3\} = x(\pi - x)/2.$$

Ex. 15. *Find f(x) if* $\overline{f_c}(s) = [1/(2s+1)^2]\cos(2s\pi/3)$ *if* $o < x < 1$.

Sol. For $o < x < l$, the inverse finite Fourier cosine transform $f(x)$ of $\overline{f_c}(s)$ is given by

$$f(x) = \frac{1}{l}\overline{f_c}(o) + \frac{2}{l}\sum_{s=1}^{\infty}\overline{f_c}(s)\cos\frac{s\pi x}{l}. \qquad \ldots (1)$$

Here $l = 1$ Also $\overline{f_c}(o) = \cos o / 1 = 1$, by the given value of $\bar{f}_c(s)$.

Then, (1) $\Rightarrow$ $$f(x) = 1 + 2\sum_{s=1}^{\infty} \frac{\cos(2s\pi/3)}{(2s+1)^2} \cos(s\pi x)$$

Ex. 16. *Find f(x) if* $\overline{f_c}(s) = \{6/\pi(2s+1)\} \times [\sin(s\pi/2) - \cos s\pi]$ for $s = 1, 2, 3\ldots$ and $\overline{f_c}(s) = 2/\pi$ *for* $s = 0$, *where* $o < x < 4$.

Sol. We know that for $o < x < l$, the inverse finite Fourier cosine transform $f(x)$ is given by

$$f(x) = \frac{1}{l}\overline{f_c}(s) + \frac{2}{l}\sum_{s=1}^{\infty} \overline{f_c}(s) \cos\frac{s\pi x}{l}. \quad \ldots (1)$$

Here $l = 4$. Also given that $\overline{f}_c(0) = 2/\pi$

Then, (1) $\Rightarrow$ $$f(x) = \frac{1}{4} \times \frac{2}{\pi} + \frac{2}{4}\sum_{s=1}^{\infty} \frac{6[\sin(s\pi/2) - \cos s\pi]}{(2s+1)\pi} \cos\frac{s\pi x}{4}$$

or $$f(x) = \frac{1}{2\pi} + \frac{3}{\pi}\sum_{s=1}^{\infty} \frac{[\sin(s\pi/2) - \cos s\pi]}{(2s+1)} \cos\frac{s\pi x}{4}.$$

EXERCISE 2(A)

1. Find the finite Fourier sine and cosine transform of $F(x) = x$. **(Osmania 2010)**

Ans. $F_s\{F(x)\} = (\pi/s) \times (-1)^{s+1}$, $s = 1, 2, 3, \ldots$ $F_c\{F(x)\} = (1/s^2) \times \{(-1)^s - 1\}$, for $s = 1, 2, 3, \ldots$; $F_c\{F(x)\} = \pi^2/2$ for $s = 0$.

2. Find the finite Fourier sine transform of $1 - (x/\pi)$ and $x/4\pi$.

Ans. $F_s\{(1 - x/\pi)\} = 1/s$, $s = 1, 2, 3, \ldots$, $F_s\{x/4\pi\} = (-1)^{s+1}/(4s)$, $s = 1, 2, 3, \ldots$

3. Find the Fourier cosine transform of $1 - (x/\pi)$ and $x/4\pi$.

Ans. $F_c\{(1 - x/\pi)\} = \{1 - (-1)^s\}/\pi s^2$, $s = 1, 2, 3, \ldots$, $F_c\{(1 - x/\pi)\} = \pi/2$, if $s = 0$

Ans. $F_c\{x/4\pi\} = \{(-1)^s - 1\}/(4\pi s^2)$, $s = 1, 2, 3, \ldots$; $F_c\{x/4\pi) = \pi/8$, if $s = 0$

4. Find the finite Fourier cosine transform of $(1 - x/\pi)^2$. **[Kanpur 1995; Meerut 1994]**

Ans. $F_c\{(1 - x/\pi)^2\} = 2/\pi s^2$, if $s = 1, 2, 3, \ldots$; $F_c\{(1 - x/\pi)^2\} = \pi/3$ if $s = 0$

5. Find the finite Fourier sine transform of x^3. **[Meerut 1997, 2006]**

Ans. $F_s\{x^3\} = \pi(-1)^s (6/s^3 - \pi^2/s)$, $s = 1, 2, 3, \ldots$.

6. Find the finite Fourier sine transform of (*i*) $x(\pi - x)$ and (*ii*) $x(\pi^2 - x^2)$.

Ans. (*i*) $(2/s^3) \times \{1 - (-1)^s\}$, $s = 1, 2, 3, \ldots$. (*ii*) $(6\pi/s^3) \times (-1)^{s+1}$, $s = 1, 2, 3, \ldots$.

7. Find the finite Fourier sine transform of $f(x)$, if $f(x) = e^{ax}$. **[Rohilkhand 1994]**

Ans. $F_x\{e^{ax}\} = s\{1 - (-1)^s e^{a\pi}\}/(a^2 + s^2)$

8. Find the finite Fourier sine transform of $F(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases}$ where $0 \le c \le \pi$.

Ans. $(\pi \cos cs)/s^2$, $s = 1, 2, 3, \ldots$

9. Find the finite Fourier cosine transform of $F(x)$, if $F(x) = -\{\cos k(\pi - x)\}/k \sin k\pi$, $k \neq 0, 1, 2, 3, \ldots$

Ans. $F_c(F(x)\} = 1/(s^2 - k^2)$, $s = 1, 2, 3, \ldots$, $F_c\{F(x)\} = -(1/k^2)$, if $s = 0$

10. Show that the finite Fourier sine transform of x/π is $(1/s)\times(-1)^{s+1}$

11. Find $f(x)$, if its finite Fourier sine transform is given by $\overline{f_s}(s) = [2\pi(-1)^{s-1}]/s^3, s = 1, 2, 3...$, where $o < x < \pi$. **Ans.** $f(x) = 4\sum_{s=1}^{\infty}(1/s^3)(-1)^{s-1}\sin sx$

12. Find $f(x)$, if $\overline{F}_c(s) = (1/2s)sin(s\pi/2), s = 1, 2, 3,...$ and $\overline{F}_c(o) = \pi/4$, where $o < x < 2\pi$. **Ans.** $f(x) = \frac{1}{8} + \frac{1}{2s}\sum_{s=1}^{\infty}\frac{\sin(s\pi/2)}{s}\cos\left(\frac{sx}{2}\right)$.

13. Prove that $F_c\{3x^2\} = \pi^3$, if $s = 0$; $F_c\{3x^2\} = [6\pi(-1)^s]/s^2$, if $s = 1,2,3...$

14. Prove that $F_c\{x^2/2\pi - \pi/6\} = 0$, if $s = 0$; $F_c\{x^2/2\pi - \pi/6\} = (-1)^s/s^2$, of $s = 1,2,3, ...$

15. Find the finite Fourier sine transform of $f(x)$, if

$$f(x) = \frac{\pi\sin kx}{2k\sin^2 k\pi} - \frac{x\cos k(\pi - x)}{2k\sin k\pi},\ k \neq 0, 1, 2, 3,...$$ **[Meerut 2002]**

Ans. $F_s\{f(x)\} = s/(s^2 - k^2)^2$, $s = 1, 2, 3, ...$

16. Find the finite Fourier sine transform of $F(x)$, if

(*i*) $F(x) = (2/\pi)\times\tan^{-1}\{(b\sin x)/(1 - b\cos x)\}$ (*ii*) $F(x) = (2/\pi)\times\tan^{-1}\{(2b\sin x)/(1 - b^2)\}$

Ans. (*i*) b^2/s, if $|b| \le 1$ (*ii*) 0, if s is even integer, $2b^s/s$ if s is an odd integer

2.6. Relation between the finite Fourier transforms of the derivatives of a function

Theorem I. Let $F(x)$ be continuous and $F'(x)$ be sectionally continuous on the interval $o \le x \le \pi$, then

(*i*) $F_s\{F'(x)\} = -sF_c\{F(x)\}, s = 1,2,3...$

(*ii*) $F_c\{F'(x)\} = sF_s\{F(x)\} - F(o) + (-1)^s F(\pi), s = 0, 1, 2, 3...$

Theorem II. Let $F(x)$ and $F'(x)$ be continuous and $F''(x)$ be sectionally continuous on the interval, $0 \le x \le \pi$. Then (*i*) $F_s\{F''(x)\} = -s^2 F_s\{F(x)\} + s[F(0) - (-1)^s F(\pi)]$.

(*ii*) $F_c\{F''(x)\} = -s^2 F_c\{F(x)\} - F'(0) + (-1)^s F'(\pi)$.

Theorem III. Let $F''(x)$ and $F'''(x)$ be continuous and $F''''(x)$ be sectionally continuous on the interval $0 \le x \le \pi$. Then.

(*i*) $F_s\{F''''(x)\} = s^4 F_s\{F(x)\} - s^3[F(0) - (-1)^s F(\pi)] + s[F''(0) - (-1)^s F''(\pi)]$

(*ii*) $F_c\{F''''(x)\} = s^4 F_c\{F(x)\} + s^2[F'(0) - (-1)^s F'(\pi)] - F'''(0) + (-1)^s F'''(\pi)$.

Proofs of theorems I, II and III are left as exercises

2.7. Multiple finite Fourier transforms.

Let $f(x, y)$ be a function of two independent variables x and y, defined in the square $0 \le x \le \pi$, $0 \le y \le \pi$.

For the time being, let us treat $f(x, y)$ as function of x. Let $\overline{f_s}(p, y)$ be the finite Fourier sine transform of $f(x, y)$ with respect to x, Then, we have

$$\overline{f_s}(p, y) = \int_0^{\pi} f(x, y)\sin px\,dx \quad ... (1)$$

Let $\overline{f_s}(p,q)$ be the finite Fourier sine transform of $\overline{f_s}(p,y)$ with respect to y. Then,

$$\overline{f_s}(p,q) = \int_0^{\pi} \overline{f_s}(p,y) \sin qy\, dy = \int_0^{\pi} \sin qy \left[\int_0^{\pi} f(x,y) \sin px\, dx \right] dy, \text{ using (1)}$$

$$\therefore \qquad \overline{f_s}(p,q) = \int_0^{\pi} \int_0^{\pi} f(x,y) \sin px \sin qy\, dx\, dy. \qquad \ldots (2)$$

Equation (2) defines the double finite Fourier sine transform of the function $f(x, y)$ for the region $0 \le x \le \pi$, $0 \le y \le \pi$,

Using inversion theorem,we easily obtain

$$\overline{f_s}(p,y) = \frac{2}{\pi} \sum_{q=1}^{\infty} \overline{f_s}(p,q) \sin qy \qquad \ldots (3)$$

and

$$f(x, y) = \frac{2}{\pi} \sum_{p=1}^{\infty} \overline{f_s}(p,y) \sin px \qquad \ldots (4)$$

Using (3), (4) reduces to $$f(x, y) = \frac{2}{\pi} \sum_{p=1}^{\infty} \left[\frac{2}{\pi} \sum_{q=1}^{\infty} \overline{f_s}(p,q) \sin qy \right] \sin px$$

or

$$f(x, y) = \frac{4}{\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \overline{f_s}(p,q) \sin px \sin qy \qquad \ldots (5)$$

Equation (5) defines the inversion for double finite Fourier sine transform $\overline{f_s}(p,q)$ of $f(x,y)$.

Proceeding as above, we get $$\overline{f_c}(p,q) = \int_0^{\pi} \int_0^{\pi} f(x,y) \cos px \cos qy\, dx\, dy, \quad \ldots (6)$$

which defines the *double finite Fourier cosine transform* of $f(x, y)$ for the region $0 \le x \le \pi, 0 \le y \le \pi$.

As before, we can easily get the corresponding inversion formula

$$f(x,y) = \frac{4}{\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \overline{f_c}(p,q) \cos px \cos qy. \qquad \ldots (7)$$

PART II : APPLICATIONS OF FINITE FOURIER SINE AND COSINE TRANSFORMS TO BOUNDARY VALUE PROBLEMS

2.8. Some useful results for direct applications.

The finite Fourier transform can be applied when the range of one of the variables is finite, say $0 < x < l$. The choice of sine or cosine transform is decided by the form of the given boundary conditions. In this connection note the following. By, definition, we have

(*i*) $$\mathrm{F}_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^l \frac{\partial^2 u}{\partial x^2} \sin \frac{s\pi x}{l}\, dx = \left[\frac{\partial u}{\partial x} \sin \frac{s\pi x}{l}\right]_0^l - \frac{s\pi}{l} \int_0^l \frac{\partial u}{\partial x} \cos \frac{s\pi x}{l} dx, \text{ integrating by parts}$$

$$= -\frac{s\pi}{l} \left\{ \left[u(x,t) \cos \frac{s\pi x}{l} \right]_0^l + \frac{s\pi}{l} \int_0^l u(x,t) \sin \frac{s\pi x}{l} dx \right\}, \text{integrating by parts}$$

$$= -(s\pi/l) \times \left[(-l)^s\, u(l,t) - u(0,t) \right] - (s^2\pi^2/l^2) \times \overline{u}_s(s,t), \qquad \ldots (1)$$

where $u(x, t)$ is a function of two variables x and t and $\overline{u}_s(s,t)$ is the finite Fourier sine transform of $u(x, t)$ with respect to x.

$$(ii)\ \mathrm{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^l \frac{\partial^2 u}{\partial x^2}\cos\frac{s\pi x}{l}dx = \left[\frac{\partial u}{\partial x}\cos\frac{s\pi x}{l}\right]_0^l + \frac{s\pi}{l}\int_0^l \frac{\partial u}{\partial x}\sin\frac{s\pi x}{l}dx, \text{ integrating by parts}$$

$$= \left[u_x(x,t)\cos\frac{s\pi x}{l}\right]_0^l + \frac{s\pi}{l}\left\{\left[u(x,t)\sin\frac{s\pi x}{l}\right]_0^l - \frac{s\pi}{l}\int_0^l u(x,t)\cos\frac{s\pi x}{l}dx\right\}$$

$$= u_x(l,t)\cos s\pi - u_x(o,t) - (s^2\pi^2/l^2)\times\overline{u}_c(s,t) \qquad \ldots (2)$$

where $\overline{u}_c(s,t)$ is finite Fourier cosine transform of $u(x, t)$ with respect to x.

Choice of finite Fourier sine and cosine transforms :

From (1) and (2), note carefully that the successful use of a finite sine transform in removing a term $\partial^2 u/\partial x^2$ from a partial differential equation requires a knowledge of $u(l, t)$ and $u(0, t)$ while the use of finite cosine transform for the same purpose requires $u_x(l, t)$ and $u_x(0, t)$.

Remark. A term $\partial u/\partial x$ or any partial derivative of odd order cannot be removed with help of finite Fourier transforms.

2.9. Solved examples based on applications of finite Fourier sine and cosine transforms to boundary value problems.

Ex. 1. *Solve* $\partial u/\partial t = \partial^2 u/\partial x^2, 0 < x < 4, t > 0$, *subject to conditions* $u(0, t) = 0$, $u(4, 0) = 0$, $u(x,0) = 2x$. *Give a physical interpretation of the problem.*

Sol. Given $$\partial u/\partial t = \partial^2 u/\partial x^2, 0 < x < 4, t > 0, \qquad \ldots (1)$$

subject to boundary conditions: $$u(0,t) = 0, \quad \mathbf{u}(4,t) = 0 \qquad \ldots (2)$$

and initial conditon : $$u(x,0) = 2x. \qquad \ldots (3)$$

Since $u(0, t)$ and $u(4, t)$ are given, taking the finite Fourier sine transform (with $l = 4$ in this problem) of both sides of (1), we have

$$\int_0^4 \frac{\partial u}{\partial t}\sin\frac{s\pi x}{4}dx = \int_0^4 \frac{\partial^2 u}{\partial x^2}\sin\frac{s\pi x}{4}dx$$

or $$\frac{d}{dt}\int_0^4 u(x,t)\sin\frac{s\pi x}{4}dx = \left[\frac{\partial u}{\partial x}\sin\frac{s\pi x}{4}\right]_0^4 - \frac{s\pi}{4}\int_0^4 \frac{\partial u}{\partial x}\cos\frac{s\pi x}{4}dx, \text{ integrating by parts}$$

or $$\frac{d\overline{u}_s}{dt} = -\frac{s\pi}{4}\left\{\left[u(x,t)\cos\frac{s\pi x}{4}\right]_0^4 + \frac{s\pi}{4}\int_0^4 u(x,t)\sin\frac{s\pi x}{4}dx\right\}, \text{ taking } \overline{u}_s = \overline{u}_s(s,t) = \int_0^4 u(x,t)\sin\frac{s\pi x}{4}dx$$

or $$\frac{d\overline{u}_s}{dt} = -\frac{s\pi}{4}\left[u(4,t)\cos s\pi - u(0,t)\right] - \frac{s^2\pi^2}{16}\overline{u}_s = -\frac{s^2\pi^2}{16}\overline{u}_s, \text{ using (2)}$$

or $$(1/\overline{u}_s)d\,\overline{u}_s = -(s^2\pi^2/16)dt, \text{ whose solution is}$$

$$\overline{u}_s(s,t) = ce^{-s^2\pi^2 t/16}, \ c \text{ being an arbitrary constant} \qquad \ldots (4)$$

Putting $t = 0$ in (4), we have $$c = \overline{u}_s(s,0)$$

Now, $$\overline{u}_s(s,0) = \int_0^4 u(x,0)\sin\frac{s\pi x}{4}dx, \text{ by definition}$$

$$\therefore \qquad c = \int_0^4 u(x,0)\sin\frac{s\pi x}{4}du = \int_0^4 2x\sin\frac{s\pi x}{4}dx, \text{ using (3)}$$

$$= \left[(2x)\times\left(-\frac{4}{s\pi}\cos\frac{s\pi x}{4}\right)-(2)\times\left(\frac{-16}{s^2\pi^2}\sin\frac{s\pi x}{4}\right)\right]_0^4 = \frac{32(-1)^{s+1}}{s\pi}, \text{as } \cos s\pi = (-1)^s$$

$\therefore$ From (4), $\qquad \bar{u}_s(s,t) = (32/s\pi)(-1)^{s+1}e^{-s^2\pi^2t/16}$ ... (5)

Now, taking the inverse finite Fourier sine transform, we get

$$u(x,t) = \frac{2}{4}\sum_{s=1}^{\infty}\bar{u}_s(s,t)\sin\frac{s\pi x}{4} = \frac{16}{\pi}\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s}e^{-s^2\pi^2t/16}\sin\frac{s\pi x}{4}, \text{ by (5)}$$

Physical interpretation of the given problem. Physically $u(x,t)$ represents the temperature at any point x at any time t in a solid bounded by the planes $x = 0$ and $x = 4$. The conditions $u(0, t) = u(4, t) = 0$ express the fact that ends $x = 0$ and $x = 4$ are kept at temperature zero, while $u(x, 0) = 2x$ expresses the initial temperature as a function of x.

Ex. 2. *Solve* $\partial u/\partial t = \partial^2 u/\partial x^2, 0 < x < 6, t > 0,$ *subject to conditions* $u_x(0, t) = u_x(6, t) = 0$, $u(x, 0) = 2x$. **[Meerut 1996, 2003]**

Sol. Since $u_x(0, t)$ and $u_x(6, t)$ are given, taking the finite Fourier cosine transform (with $l = 6$) of both sides of the given equation, we have

$$\int_0^6 \frac{\partial u}{\partial t}\cos\frac{s\pi x}{6}dx = \int_0^6 \frac{\partial^2 u}{\partial x^2}\cos\frac{s\pi x}{6}dx$$

or $$\frac{d}{dt}\int_0^6 u(x,t)\cos\frac{s\pi x}{6}dx = \left[\frac{\partial u}{\partial x}\cos\frac{s\pi x}{6}\right]_0^6 + \frac{s\pi}{6}\int_0^6\frac{\partial u}{\partial x}\sin\frac{s\pi x}{6}dx \text{ , integrating by parts}$$

or $$\frac{d\bar{u}_c}{dt} = u_x(6,t)\cos s\pi - x_x(0,t) + \frac{s\pi}{6}\left\{\left[u(x,t)\sin\frac{s\pi x}{6}\right]_0^6 - \frac{s\pi}{6}\int_0^6 u(x,t)\cos\frac{s\pi x}{6}dx\right\}$$

or $$d\bar{u}_c/dt = -(s^2\pi^2/36)\bar{u}_c \qquad \text{as,} \quad \text{given} \quad u_x(6, t) = u_x(0, t) = 0$$

whose solution is $\bar{u}_c(s,t) = ce^{-s^2\pi^2t/36}$, c being an arbitrary constant ... (1)

Putting $t = 0$ in (1), we have $\qquad c = \bar{u}_c(s,0)$

$$\therefore \qquad c = \bar{u}_c(s,0) = \int_0^6 u(x,0)\cos\frac{s\pi x}{6}dx = \int_0^6 2x\cos\frac{s\pi x}{6}dx \qquad [\because u(x, 0) = 2x]$$

or $$c = \left[(2x)\left(\frac{6}{s\pi}\sin\frac{s\pi x}{6}\right)-(2)\left(-\frac{36}{s^2\pi^2}\right)\cos\frac{s\pi x}{6}\right]_0^6, \text{ integrating by parts}$$

$$= (72/s^2\pi^2)\times(\cos s\pi - 1) = (72/s^2\pi^2)\times[(-1)^s - 1]$$

$\therefore$ (1) reduces to $\qquad \bar{u}_c(s,t) = (72/s^2\pi^2)[(-1)^s - 1]e^{-s^2\pi^2t/36}, s = 1,2,3,\ldots$ ... (2)

Also, $\qquad \bar{u}_c(0,t) = \int_0^6 u(x,0)dx = \int_0^6 2x\,dx = \left[x^2\right]_0^6 = 36$ ... (3)

Taking the inverse finite Fourier cosine transform, we get

$$u(x,t) = \frac{1}{6}\bar{u}_c(0,t) + \frac{2}{6}\sum_{s=1}^{\infty}\bar{u}_c(s,t)\cos\frac{s\pi x}{6} = 6 + \frac{24}{\pi^2}\sum_{s=1}^{\infty}\frac{(-1)^s - 1}{s^2}e^{-s^2\pi^2t/36}\cos\frac{s\pi x}{6}, \text{ using (2) and (3)}$$

Ex. 3. *Solve* $\partial u/\partial t = \partial^2 u/\partial x^2$, $0 < x < \pi$, $t > 0$, *where* $u(o, t) = 0$, $u(\pi, t) = 0$, $u(x, 0) = 2x$. **[Meerut 1989, 92]**

Sol. Given $$\partial u/\partial t = \partial^2 u/\partial x^2, \quad o < x < \pi, \quad t > 0 \qquad \ldots (1)$$

subject to boundary conditions : $$u(0,t) = u(\pi,t) = 0 \qquad \ldots (2)$$

and initial condition : $$u(x,0) = 2x \qquad \ldots (3)$$

Since $u(o, t)$ and $u(\pi, t)$ are given, taking the finite Fourier sine transform (with $l = \pi$ in this problem) of both sides of (1), we get

$$\int_0^\pi \frac{\partial u}{\partial t}\sin\frac{s\pi x}{\pi}dx = \int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin\frac{s\pi x}{s}dx \quad \text{or} \quad \frac{d}{dt}\int_0^\pi u(x,t)\sin sx\,dx = \left[\frac{\partial u}{\partial x}\sin sx\right]_0^\pi - s\int_0^\pi \frac{\partial u}{\partial x}\cos sx\,dx$$

[On integrating by parts]

or $$\frac{d\bar{u}_s}{dt} = -s\left\{\left[u(x,t)\cos sx\right]_0^\pi - \int_0^\pi (-s)\sin sx\ u(x,t)\,dx\right\}$$

or $$\frac{d\,\bar{u}_s}{dt} = s[u(\pi,t)\cos s\pi - u(0,t)] - s^2\int_0^\pi u(x,t)\sin sx\ dx$$

or $$\frac{d\bar{u}_s}{dt} = -s^2\bar{u}_s, \text{ using (2)} \qquad \left[\text{Here } \bar{u}_s = \bar{u}_s(s,t) = \int_0^\pi u(x,t)\sin sx\,dx\right]$$

Separating variables, $$(1/\bar{u}_s)d\,\bar{u} = -s^2 dt, \quad \text{whose solution is}$$

$$\bar{u}_s(s,t) = c\,e^{-s^2 t}, \ c \text{ being an arbitrary constant} \qquad \ldots (4)$$

Putting $t = 0$ in (4), we get $$c = \bar{u}_s(s,0). \qquad \ldots (5)$$

$$\therefore \quad c = \bar{u}_s(s,0) = \int_0^\pi u(x,0)\sin\frac{s\pi x}{\pi}dx = \int_0^\pi (2x)\sin\frac{s\pi x}{\pi}dx \text{, using (3) and (5)}$$

$$= \left[(2x)\left(-\frac{1}{s}\cos sx\right) - (2)\left(-\frac{1}{s^2}\sin sx\right)\right]_0^\pi \text{, using chain rule of integration by parts}$$

Thus, $$c = -(2\pi/s)\cos s\pi = -(2\pi/s)(-1)^s = (2\pi/s)(-1)^{s+1}.$$

$\therefore$ From (4), $$\bar{u}_s(s,t) = 2\pi(-1)^{s+1}(1/s)e^{-s^2 t}. \qquad \ldots (6)$$

Now, taking the inverse finite Fourier sine transform, we get

$$u(x,\ t) = \frac{2}{\pi}\sum_{s=1}^{\infty}\bar{u}_s(s,t)\sin\frac{s\pi x}{\pi} = \frac{2}{\pi}\sum_{s=1}^{\infty}\frac{2\pi(-1)^{s+1}}{s}e^{-s^2 t}\sin sx, \text{ using (6)}$$

or $$u(x,\ t) = 4\sum_{s=1}^{\infty}(1/s)(-1)^{s+1}e^{-s^2 t}\sin sx$$

Ex. 4. *Find the temperature* $\theta(x,\ t)$ *in a slab* $o < x < \pi$, *initially at temperature* $\theta = 1$, *its faces being kept at temperature zero and* k = *diffusivity* = 1.

Sol. Temperature $\theta(x, t)$ in the given slab $o < x < \pi$ is governed by the heat conduction equation

$$\partial\theta/\partial t = k(\partial^2\theta/\partial x^2) \quad \text{or} \quad \partial\theta/\partial t = \partial^2\theta/\partial x^2, \text{as } k = 1 \text{ (given).} \ldots (1)$$

subject to the boundary conditons $$\theta(0,\ t) = \theta\ (\pi,\ t) = 0 \qquad \ldots (2)$$

and initial condition: $\theta(x, 0) = 1$ ... (3)

Taking finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial\theta}{\partial t}\sin sx\,dx = \int_0^\pi \frac{\partial^2\theta}{\partial x^2}\sin sx\,dx \quad \text{or} \quad \frac{d}{dt}\int_0^\pi \theta(x,t)\sin sx\,dx = \left[\frac{\partial\theta}{\partial x}\sin sx\right]_0^\pi - s\int_0^\pi \frac{\partial\theta}{\partial x}\cos sx\,dx$$

[On integrating by parts]

$$\text{or} \quad \frac{d\bar{\theta}_s}{dt} = -s\left\{\left[\theta(x,t)\cos sx\right]_0^\pi + s\int_0^\pi \theta(x,t)\sin sx\,dx\right\} = -s\left\{\theta(\pi,t)(-1)^s - \theta(0,t) + s\bar{\theta}_s\right\} = -s^2\bar{\theta}_s, \text{ by (2)}$$

$$\therefore \quad (1/\bar{\theta}_s)\,d\bar{\theta}_s = -s^2 dt \qquad \text{so that} \qquad \bar{\theta}_s(s,t) = c\,e^{-s^2t}. \quad \text{... (4)}$$

Putting $t = 0$ in (4), we have $c = \bar{\theta}_s(s,0)$.

$$\therefore \quad c = \bar{\theta}_s(s,0) = \int_0^\pi \theta(x,0)\sin sx\,dx = \int_0^\pi \sin sx\,dx, \text{ using (3)}$$

$$= \left[-\frac{\cos sx}{s}\right]_0^\pi = -\frac{1}{s}\left[(-1)^s - 1\right] = \frac{1-(-1)^s}{s}.$$

So from (4), $\bar{\theta}_s(s,t) = (1/s)[1-(-1)^s]e^{-s^2t}$ ... (5)

Now, taking the inverse finite Fourier sine transform, we get

$$\theta(x,\,t) = \frac{2}{\pi}\sum_{s=1}^{\infty}\bar{\theta}_s(s,t)\sin sx = \frac{2}{\pi}\sum_{s=1}^{\infty}\frac{1-(-1)^s}{s}e^{-s^2t}\sin sx, \text{by (5)}$$

Ex. 5. *Solve $\partial U/\partial t = \partial^2 U/\partial x^2$, $0 < x < 6, t > 0$ subject to the conditions $U(o, t) = 0$, $U(6,t) = 0$, $U(x, 0) = \begin{cases} 1, \text{if } 0 < x < 3 \\ 0, \text{if } 3 < x < 6 \end{cases}$ and interpret physically.* **(Kanpur 2011)**

Sol. Given $\partial U/\partial t = \partial^2 U/\partial x^2$ ... (1)

with the boundary conditions: $U(0,t) = 0, \quad U(6,t) = 0$ ... (2)

and initial condition: $U(x, 0) = \begin{cases} 1, \text{if } 0 < x < 3 \\ 0, \text{if } 3 < x < 6 \end{cases}$ ... (3)

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^6 \frac{\partial U}{\partial t}\sin\frac{s\pi x}{6}dx = \int_0^6 \frac{\partial^2 U}{\partial x^2}\sin\frac{s\pi x}{6}dx$$

$$\text{or} \quad \frac{d}{dt}\int_0^6 U(x,t)\sin\frac{s\pi x}{6}dx = \left[\frac{\partial U}{\partial x}\sin\frac{s\pi x}{6}\right]_0^6 - \frac{s\pi}{6}\int_0^6 \frac{\partial U}{\partial x}\cos\frac{s\pi x}{6}dx, \text{ integrating by parts}$$

$$\text{or} \quad \frac{d\bar{U}_s}{dt} = -\frac{s\pi}{6}\int_0^6 \frac{\partial U}{\partial x}\cos\frac{s\pi x}{6}dx \text{ , taking} \qquad \bar{U}_s = \bar{U}_s(s,t) = \int_0^6 U(x,t)\sin\frac{s\pi x}{6}dx$$

$$\text{or} \quad \frac{d\bar{U}_s}{dt} = -\frac{s\pi}{6}\left\{\left[U(s,t)\cos\frac{s\pi x}{6}\right]_0^6 + \frac{s\pi}{6}\int_0^6 U(x,t)\sin\frac{s\pi x}{6}dx\right\}, \text{ integrating by parts}$$

or $\dfrac{d\,\overline{\mathrm{U}}_s}{dt} = -\dfrac{s\pi}{6}\left\{\mathrm{U}(6,t)\cos s\pi - \mathrm{U}(0,t) + \dfrac{s\pi}{6}\overline{\mathrm{U}}_s\right\}$ or $\dfrac{d\,\overline{\mathrm{U}}_s}{dt} = -\dfrac{s^2\pi^2}{36}\overline{\mathrm{U}}_s$, by(2)

whose general solution is $\overline{\mathrm{U}}_s(s,t) = C e^{-s^2\pi^2 t/36}$, C being an arbitrary constant ... (4)

Putting $t = 0$ in (4), we have $C = \overline{\mathrm{U}}_s(s,0)$.

$\therefore\ C = \overline{\mathrm{U}}_s(s,0) = \displaystyle\int_0^6 \mathrm{U}(x,0)\sin\frac{s\pi x}{6}dx$, by definition

$= \displaystyle\int_0^3 \mathrm{U}(x,0)\sin\frac{s\pi x}{6}dx + \int_3^6 \mathrm{U}(x,0)\sin\frac{s\pi x}{6}dx = \int_0^3 \sin\frac{s\pi x}{6}dx$, using (3)

$= \dfrac{6}{s\pi}\left[-\cos\dfrac{s\pi x}{6}\right]_0^3 = \dfrac{6}{s\pi}\left(1-\cos\dfrac{s\pi}{2}\right)$ ... (5)

$\therefore$ From (4), $\overline{\mathrm{U}}_s(s,t) = (6/s\pi)[1-\cos(s\pi/2)]e^{-s^2\pi^2 t/36}$

Taking the inverse finite Fourier sine transform, we get

$$\mathrm{U}(x,t) = \frac{2}{6}\sum_{s=1}^{\infty}\overline{\mathrm{U}}_s(s,t)\sin\frac{s\pi x}{6} = \frac{2}{\pi}\sum_{s=1}^{\infty}\frac{1}{s}\left(1-\cos\frac{s\pi}{2}\right)e^{-s^2\pi^2 t/36}\sin\frac{s\pi x}{6}, \text{ by (5)}$$

Physical Interpretation. Physically, U (x, t) represents the temperature at any point x at any time t in a solid bounded by the planes $x = 0$ and $x = 6$. The conditions U(0, t) = 0 and U (6, t) = 0 express the fact that the ends are kept at temperature zero. The expression for U (x, 0) shows that initially the temperature in the half solid from $x = 0$ to $x = 3$ is constant equal to 1 unit while the second half of the same solid from $x = 3$ to $x = 6$ is at zero temperature. Equivalently, the solid can be replaced by a bar on the x axis with end points at $x = 0$ and $x = 6$ whose surface is insulated.

Ex. 6. *Using the finite cosine transform to solve one dimensional heat equation* $\partial u/\partial t = k(\partial^2 u/\partial x^2)$ *with the boundary conditions* $\partial u/\partial x = 0$, *when* $x = 0$ *and* $x = \pi$, $t > 0$ *and the initial condition* $u = f(x)$, *when* $t = 0$, $0 < x < \pi$.

Sol. We are to solve $\partial u/dt = k(\partial^2 u/\partial x^2)$ ... (1)

with the boundary conditions: $u_x(0,t) = u_x(\pi,t) = 0$ ... (2)

and initial conditon: $u(x,0) = f(x)$ ... (3)

Taking the finite Fourier cosine transform of (1), we get

$$\int_0^\pi \frac{\partial u}{\partial t}\cos sx\,dx = k\int_0^\pi \frac{\partial^2 u}{\partial x^2}\cos sx\,dx \text{ or } \frac{d}{dt}\int_0^\pi u(x,t)\cos sx\,dx = k\left\{\left[\frac{\partial u}{\partial x}\cos sx\right]_0^\pi + s\int_0^\pi \frac{\partial u}{\partial x}\sin sx\,dx\right\}$$

[On integrating by parts]

or $\dfrac{d\bar{u}_c}{dt} = k[u_x(\pi,t)\cos s\pi - u_x(0,t)] + ks\left\{\left[u\sin sx\right]_0^\pi - s\displaystyle\int_0^\pi u\cos sx\,dx\right\} = -ks^2\bar{u}_c$, using (2)

or $(1/\bar{u}_c)\,d\bar{u}_c = -ks^2 dt$, whose solution is

$\bar{u}_c(s,t) = c e^{-ks^2 t}$, c being an arbitrary constant ... (4)

Putting $t = 0$ in (4), we have $c = \bar{u}_c(s,0)$.

$\therefore\ c = u_c(s,0) = \displaystyle\int_0^\pi u(x,0)\cos sx\,dx = \int_0^\pi f(x)\cos sx\,dx$, by (3)

$$\therefore \qquad c = \int_0^{\pi} f(y)\cos sy\,dy, \qquad \text{... (5)}$$

where the new variable y has been introduced to avoid confusion later on when we apply inverse Fourier transform.

$\therefore$ From (4) and (5), $$\bar{u}_c(s,t) = e^{-ks^2t}\int_0^{\pi} f(y)\cos sy\,dy \qquad \text{... (6)}$$

Taking the inverse finite Fourier cosine transform, we get

$$u(x,\,t) = \frac{1}{\pi}\bar{u}_c(0,t) + \frac{2}{\pi}\sum_{s=1}^{\infty}\bar{u}_c(s,t)\cos sx, \qquad \text{... (7)}$$

where $$\bar{u}_c(0,t) = \int_0^{\pi} u(x,0)\,dx = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(y)\,dy. \qquad \text{... (8)}$$

Using (6) and (8), (7) becomes

$$u(x,\,t) = \frac{1}{\pi}\int_0^{\pi} f(y)dy + \frac{2}{\pi}\sum_{s=1}^{\infty}\left[e^{-ks^2t}\cos sx\int_0^{\pi} f(y)\cos sy\,dy\right]$$

Ex. 7. *Solve the boundary value problem* $\partial^2 U/\partial t^2 = 9(\partial^2 U/\partial x^2)$ *subject to the conditions* $U(0,t) = 0$, $U(2, t) = 0$, $U(x, 0) = (0.05)x(2 - x)$ *and* $U_t(x,0) = 0$, *when* $0 < x < 2$, $t > 0$.

[Nagpur 1997, 2003]

Sol. Given $$\partial^2 U/\partial t^2 = 9(\partial^2 U/\partial x^2) \qquad \text{... (1)}$$

with the boundary conditions $$U(0, t) = 0, \qquad U(2, t) = 0 \qquad \text{... (2)}$$

and initial conditions: $$U(x, 0) = (0.05)x(2 - x) \qquad \text{... (3}a\text{)}$$

and $$U_t(x, 0) = 0. \qquad \text{... (3}b\text{)}$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^2 \frac{\partial^2 U}{\partial t^2}\sin\frac{s\pi x}{2}dx = 9\int_0^2 \frac{\partial^2 U}{\partial x^2}\sin\frac{s\pi x}{2}dx$$

or $$\frac{d^2}{dt^2}\int_0^2 U(x,t)\sin\frac{s\pi x}{2}dx = 9\left\{\left[\frac{\partial U}{\partial x}\sin\frac{s\pi x}{2}\right]_0^2 - \frac{s\pi}{2}\int_0^2\frac{\partial U}{\partial x}\cos\frac{s\pi x}{2}dx\right\}, \text{ integrating by parts}$$

or $$\frac{d^2\bar{U}_s}{dt^2} = -\frac{9s\pi}{2}\int_0^2\frac{\partial U}{\partial x}\cos\frac{s\pi x}{2}dx, \text{ taking } \quad \bar{U}_s = \bar{U}_s(s,t) = \int_0^2 U(x,t)\sin\frac{s\pi x}{2}dx$$

or $$\frac{d^2\bar{U}_s}{dt^2} = -\frac{9s\pi}{2}\left\{\left[U(x,t)\cos\frac{s\pi x}{2}\right]_0^2 + \frac{s\pi}{2}\int_0^2 U(x,t)\sin\frac{s\pi x}{2}dx\right\}, \text{ integrating by parts}$$

or $$\frac{d^2\bar{U}_s}{dt^2} = -\frac{9s\pi}{2}\left\{U(2,t)\cos s\pi - U(0,t) + \frac{s\pi}{2}\bar{U}_s\right\} \qquad \text{or} \qquad \frac{d^2\bar{U}_s}{dt^2} = -\frac{9s^2\pi^2}{4}\bar{U}_s, \text{ by (2)}$$

or $$(D^2 + 9s^2\pi^2/4)\bar{U}_s = 0, \qquad \text{where} \quad D \equiv d/dt \qquad \text{...(4)}$$

whose auxiliary equation is $D^2 + (9s^2\pi^2/4) = 0$ giving $D = \pm(3is\pi/2)$

and so the general solution of (4) is

$$\bar{U}_s(s,t) = c_1\cos(3\pi st/2) + c_2\sin(3\pi st/2), \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.} \qquad \text{... (5)}$$

Taking finite Fourier sine transform of (3b), we get

$$(d\bar{U}_s/dt)_{t=0} = 0 \qquad \text{... (6)}$$

From (5), $(d\,\overline{U}_s/dt) = -(3s\pi c_1/2)\sin(3\pi st/2) + (3\pi s c_2/2)\cos(3\pi st/2)$

Putting $t = 0$ in above result and using (6), we get $c_2 = 0$.

Then, from (5), $\overline{U}_s(s,t) = c_1\cos(3\pi st/2)$ ... (7)

Putting $t = 0$ in (7), we get $c_1 = \overline{U}_s(s,0)$

$$\therefore\ c_1 = \overline{U}_s(s,0) = \int_0^2 U(x,0)\sin\frac{s\pi x}{2}dx = (0.05)\int_0^2 (2x-x^2)\sin\frac{s\pi x}{2}dx, \text{ by}(3a)$$

$$= (0.05)\left\{\left[(2x-x^2)\times\frac{2}{s\pi}\cos\frac{s\pi x}{2}\right]_0^2 + \frac{2}{s\pi}\int_0^2 (2-2x)\cos\frac{s\pi x}{2}dx\right\}$$

$$= (0.05)\times\frac{4}{s\pi}\int_0^2 (1-x)\cos\frac{s\pi x}{2}dx = \frac{0.2}{s\pi}\left\{\left[(1-x)\frac{\sin(s\pi x/2)}{-(s\pi/2)}\right]_0^2 - \int_0^2 1\times\frac{\sin(s\pi x/2)}{-(s\pi/2))}dx\right\}$$

$$= \frac{0.4}{s^2\pi^2}\int_0^2 \sin\frac{s\pi x}{2}dx = \frac{0.4}{s^2\pi^2}\left[-\frac{2}{s\pi}\cos\frac{s\pi x}{2}\right]_0^2 = \frac{0.8}{s^3\pi^3}\left[1-(-1)^s\right]$$

$\therefore$ From (7), $\overline{U}_s(s,t) = (0.8/s^3\pi^3)\{1-(-1)^s\}\cos(3\pi st/2)$. ... (8)

Taking the inverse finite fourier sine transform, we get

$$U(x,\ t) = \frac{2}{2}\sum_{s=1}^{\infty}\overline{U}_s(s,t)\sin\frac{s\pi x}{2} = \frac{0.8}{\pi^3}\sum_{s=1}^{\infty}\frac{1-(-1)^s}{s^3}\cos\frac{3\pi st}{2}\sin\frac{s\pi x}{2}, \text{ by (8)} \quad ...(9)$$

Now, terms vanish if $s = 2n$ (even) and non-zero terms correspond to $s = 2n + 1$ (odd) where $n = 0, 1, 2, 3...$. If n is odd, then $1 - (-1)^s = 2$. Hence (9) reduces to

$$U(x,\ t) = \frac{1.6}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos\frac{3(2n+1)\pi t}{2}\sin\frac{(2n+1)\pi x}{2},$$

Ex. 8. *Using finite sine transform, find solution of the wave equation $\partial^2U/\partial t^2 = 4(\partial^2U/\partial x^2)$ subject to the conditions $U(0, t) = 0$, $U(\pi, t) = 0$, $U(x, 0) = (0.1)\sin x + (0.01)\sin 4x$ and $U_t(x, 0) = 0$ for $0 < x < \pi$, $t > 0$.* **[Meerut 2007]**

Sol. Given $\partial^2U/\partial t^2 = 4(\partial^2U/\partial y^2)$ ... (1)

with the boundary conditons: $U(0,t) = U(\pi,t) = 0$ ... (2)

and initial conditions $U(x, 0) = (0.1)\sin x + (0.01)\sin 4x$ ... (3*a*)

and $U_t(x, 0) = 0,\ 0 < x < \pi,\ t > 0.$ ... (3*b*)

Taking the finite fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial^2 U}{\partial t^2}\sin sx\ dx = 4\int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin sx\ dx \quad \text{or} \quad \frac{d^2}{dt^2}\int_0^\pi U(x,t)\sin sx\,dx = 4\left\{\left[\frac{\partial U}{\partial x}\sin sx\right]_0^\pi - s\int_0^\pi \frac{\partial U}{\partial x}\cos sx\,dx\right\}$$

or $$\frac{d^2\,\overline{U}_s}{dt^2} = -4s\int_0^\pi \frac{\partial U}{\partial x}\cos sx\,dx, \text{ taking } \quad \overline{U}_s = \overline{U}_s(s,t) = \int_0^\pi U(x,t)\sin sx\,dx$$

or $$\frac{d^2\,\overline{U}_s}{dt^2} = -4s\left\{\left[U(x,t)\cos sx\right]_0^\pi + s\int_0^\pi U(x,t)\sin sx\,dx\right\}, \text{ integrating by parts}$$

or $$\frac{d^2\overline{U}_s}{dt^2} = -4s\left\{U(\pi,t)\cos s\pi - U(0,t) + s\overline{U}_s\right\} = -4s^2\overline{U}_s, \text{ by (2)}$$

or $d^2\overline{U}_s/dt^2 = -4s^2\overline{U}_s$, or $(D^2+4s^2)\overline{U}_s = 0$, where $D \equiv d/dt$

whose general solution is $\overline{U}_s(s,t) = C_1\cos 2st + C_2\sin 2st.$ ... (4)

Taking finite Fourier sine transform of (3*b*), we get

$$(d\overline{U}_s/dt)_{t=0} = 0. \quad \text{... (5)}$$

From (4), $d\overline{U}_s/dt = -2C_1 s\sin 2st + 2C_2 s\cos 2st$ ... (6)

Putting $t = 0$ in (6) and using (5), we get $C_2 = 0.$

$\therefore$ From (4), $\overline{U}_s(s,t) = C_1\cos 2st.$ ... (7)

Putting $t = 0$ in (7), we get $C_1 = \overline{U}_s(s,0).$

$$\therefore \quad C_1 = \int_0^\pi U(x,0)\sin sx\,dx = \int_0^\pi [(0.1)\sin x + (0.01)\sin 4x]\sin sx\,dx, by\,(3a) \quad \text{... (8)}$$

Using (8), (7) $\Rightarrow$ $\overline{U}_s(s,t) = [(0.1)\int_0^\pi \sin x\sin sx\,dx + (0.01)\int_0^\pi \sin 4x\sin sx\,dx]\cos 2st,$

Taking the inverse Fourier sine transform, we get

$$U(x,t) = \frac{2}{\pi}\sum_{s=1}^{\infty}\overline{U}_s(s,t)\sin sx = \frac{2}{\pi}\sum_{s=1}^{\infty}\left[(0,1)\int_0^\pi \sin x\sin sx\,dx + (0.01)\int_0^\pi \sin 4x\sin sx\,dx\right]\cos 2st\sin sx,$$

$$= \frac{(0.1)}{\pi}\sum_{s=1}^{\infty}\left\{\int_0^\pi 2\sin x\sin sx\,dx\right\}\cos 2st\sin sx + \frac{(0.01)}{\pi}\sum_{s=1}^{\infty}\left\{\int_0^\pi 2\sin 4x\sin sx\,dx\right\}\cos 2st\sin sx \quad \text{... (9)}$$

Now, $$\int_0^\pi 2\sin x\sin sx\,dx = \int_0^\pi [\cos(s-1)x - \cos(s+1)x]\,dx = \left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_0^\pi = 0, \text{if } s \neq 1.$$

When $s = 1$, $\int_0^\pi 2\sin x\sin sx\,dx = \int_0^\pi 2\sin^2 x\,dx = \int_0^\pi (1-\cos 2x)\,dx = [x - (1/2)\sin 2x]_0^\pi = \pi.$

$$\therefore \quad \int_0^\pi 2\sin x\sin sx\,dx = \begin{cases} 0, \text{if } s \neq 1 \\ \pi, \text{if } s = 1. \end{cases} \quad \text{... (10)}$$

Similarly, $$\int_0^\pi 2\sin 4x\sin sx\,dx = \begin{cases} 0, \text{if } s \neq 4 \\ \pi, \text{if } s = 4 \end{cases} \quad \text{... (11)}$$

Using results (10) and (11) in (9), we get

$$U(x,t) = (0.1/\pi)\times\pi\cos 2t\sin x + (0.01/\pi)\times\pi\cos 8t\sin 4x = (0.1)\cos 2t\sin x + (0.01)\cos 8t\sin 4x.$$

Ex. 9. *Solve* the boundary value problem $\partial U/\partial t = \partial^2 U/\partial x^2$, $U(0,t) = 1$, $U(\pi, t) = 3$, $U(x, 0) = 1$, *where* $0 < x < \pi$, $t > 0$. Give the physical interpretation of the problem.

Sol. Given $\partial U/\partial t = \partial^2 U/\partial x^2$ ... (1)

with boundary conditons: $U(0, t) = 1$, $U(\pi, t) = 3$ ... (2)

and initial condition $U(x, 0) = 1$ ... (3)

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial U}{\partial t}\sin sx\,dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin sx\,dx \quad \text{or} \quad \frac{d}{dt}\int_0^\pi U(x,t)\sin sx\,dx = \left[\frac{\partial U}{\partial x}\sin sx\right]_0^\pi - s\int_0^\pi \frac{\partial U}{\partial x}\cos sx\,dx$$

[On integrating by parts]

or $$\frac{d\bar{U}_s}{dt} = -s\int_0^\pi \frac{\partial U}{\partial x}\cos sx\,dx, \qquad \text{where} \qquad \bar{U}_s = \bar{U}_s(s,t) = \int_0^\pi U(x,t)\sin sx\,dx$$

or $$\frac{d\bar{U}_s}{dt} = -s\left\{\left[U(x,t)\cos sx\right]_0^\pi + s\int_0^\pi U(x,t)\sin sx\,dx\right\}, \text{ integrating by parts}$$

$$d\bar{U}_s/dt = -s\left\{U(\pi,t)\cos s\pi - U(0,t) + s\bar{U}_s\right\} = -s\{3\cos s\pi - 1 + s\bar{U}_s\}, \text{by (2)}$$

or $$d\bar{U}_s/dt + s^2\bar{U}_s = s(1-3\cos s\pi), \qquad \text{which is linear equation.}$$

Its I.F. $= e^{\int s^2 dt} = e^{s^2 t}$ and its general solution is

$$\bar{U}_s(s,t)e^{s^2 t} = \int s(1-3\cos s\pi)e^{s^2 t}dt = \text{s}\,(1 - 3\,\cos\,s\pi) \times (1/s^2) \times\ e^{s^2 t} + \text{C}$$

or $$\bar{U}_s(s,t) = \text{C}e^{-s^2 t} + (1-3\cos s\pi)/s. \qquad \ldots (4)$$

Putting $t = 0$ is (4), we get $\bar{U}_s(s,0) = \text{C} + (1 - 3\,\cos\,s\pi)/s.$ ... (5)

Now, $$\bar{U}_s(s,0) = \int_0^\pi U(x,0)\sin sx\,dx = \int_0^\pi \sin sx\,dx \text{ using (3)}$$

$$= \left[(-1/s)\times\cos sx\right]_0^\pi = (1-\cos s\pi)/s.$$

$\therefore$ (5) gives, $(1-\cos s\pi)/s = \text{C} + (1-3\cos s\pi)/s$ giving $\text{C} = (2/s)\times\cos\,s\pi$

$\therefore$ (4) gives, $\bar{U}_s(s,t) = (2/s)\times\cos s\pi\ e^{-s^2 t} + (1-3\cos s\pi)/s$ ... (6)

Taking the inverse finite Fourier sine transform, we get

$$U(x,\,t) = \frac{2}{\pi}\sum_{s=1}^{\infty}\bar{U}_s(s,t)\sin sx = \frac{2}{\pi}\sum_{s=1}^{\infty}\left[\frac{2\cos\,s\pi}{s}e^{-s^2 t} + \frac{1-3\cos\,s\pi}{s}\right]\sin\,sx$$

or $$U\,(x,\,t) = \frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\cos\,s\pi}{s}e^{-s^2 t}\sin sx + \frac{2}{\pi}\sum_{s=1}^{\infty}\frac{1-3\cos s\pi}{s}\sin sx \qquad \ldots (7)$$

But $$F_s\,\{1\} = \int_0^\pi (1\times\sin sx)\,dx = \left[-\frac{1}{s}\cos sx\right]_0^\pi = \frac{1-\cos s\pi}{s}.$$

$\therefore$ Taking the inverse Fourier sine transform, we get

$$1 = \frac{2}{\pi}\sum_{s=1}^{\infty}\frac{1-\cos s\pi}{s}\sin sx. \qquad \ldots (8)$$

Also, we have

$$F_s\{x\} = \int_0^\pi x\sin sx\,dx = \left[(x)\left(-\frac{1}{s}\cos sx\right)\right]_0^\pi - \int_0^\pi\left(-\frac{1}{s}\cos sx\right)dx = -\frac{\pi\cos s\pi}{s} + \frac{1}{s}\left[\frac{1}{s}\sin sx\right]_0^\pi = -\frac{\pi\cos s\pi}{s}.$$

Taking the inverse Fourier sine transform, we get

$$x = \frac{2}{\pi}\sum_{s=1}^{\infty}\left(-\frac{\pi\cos s\pi}{s}\right)\sin s\pi \qquad \text{or} \qquad \frac{2x}{\pi} = \frac{2}{\pi}\sum_{s=1}^{\infty}\left(-\frac{2\cos s\pi}{s}\right)\sin sx \qquad \ldots (9)$$

Adding (8) and (9), $$1+\frac{2x}{\pi} = \frac{2}{\pi}\sum_{s=1}^{\infty}\frac{1-3\cos s\pi}{s}\sin sx \qquad \ldots (10)$$

Using (10), (7) reduces $$\mathrm{U}(x,t) = \frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\cos s\pi}{s}e^{-s^2t}\sin sx + 1 + \frac{2x}{\pi}.$$

Physical Interpretation. Physically, U(*x, t*) represents the temperature at any point at any time *t* in a solid bounded by $x = 0$ and $x = \pi$. The conditions U(*0, t*) = 1 and U(π, *t*) = 3 express the fact that the ends $x = 0$ and $x = \pi$ and kept at temperature 1 and 3 units respectively. The expression U(*x*, 0) = 1 expresses the fact that the initial temperature of the solid is 1 unit. Equivalently, the solid can be replaced by a bar on the x – axis with end points at $x = 0$ and $x = \pi$ whose surface in insulated.

Ex. 10. *A string of density* ρ *and length* π *is stretched to a tension* ρc^2. *At time t = 0 one end (x = 0), is given a small oscillation a sin* ωt. *If the other end remains fixed,use a finite sine transform to show that the displacement of the point x at time t is*

$$a\sin\omega t\sin\frac{\omega(\pi-x)}{c}\operatorname{cosec}\left(\frac{\omega\pi}{c}\right)+\frac{2ac\omega}{\pi}\sum_{s=1}^{\infty}\frac{1}{\omega^2-s^2c^2}\sin sx\sin sct. \quad \textbf{[Meerut 2001, 02]}$$

Sol. Here the displacement *u*(*x, t*) of the string is given by the one dimensional wave equation

$$\partial^2u/\partial t^2 = c^2(\partial^2u/\partial x^2), \qquad \ldots (1)$$

with the boundary conditions: $u(0, t) = a\sin\omega t$, ... (2*a*)

and $u(\pi,t) = 0$, ... (2*b*)

and initial conditions $u(x, 0) = 0$ ... (3*a*)

$u_t(x, 0) = 0$ ... (3*b*)

Taking the finite fourier sine transform of both sides of (1), we get

$$\int_0^{\pi}\frac{\partial^2u}{\partial t^2}\sin sx\,dx = c^2\int_0^{\pi}\frac{\partial^2u}{\partial x^2}\sin sx\,dx$$

or $$\frac{d^2}{dt^2}\int_0^{\pi}u(x,t)\sin sx\,dx = c^2\left\{\left[\frac{\partial u}{\partial x}\sin sx\right]_0^{\pi} - s\int_0^{\pi}\frac{\partial u}{\partial x}\cos sx\,dx\right\}, \text{ integrating by parts}$$

or $$\frac{d^2\bar{u}_s}{dt^2} = -c^2s\int_0^{\pi}\frac{\partial u}{\partial x}\cos sx\,dx, \qquad \text{taking} \qquad \bar{u}_s = \bar{u}_s(s,t) = \int_0^{\pi}u(x,t)\sin sx\,dx$$

or $$\frac{d^2u}{dt^2} = -c^2s\left\{\left[u(x,t)\cos sx\right]_0^{\pi} + s\int_0^{\pi}u(x,t)\sin sx\,dx\right\}, \text{ integrating by parts}$$

or $$d^2\bar{u}_s/dt^2 = -c^2s\{u(\pi,t)\cos s\pi - u(0,t) + s\bar{u}_s\}$$

or $$(d^2\bar{u}_s/dt^2) + c^2s^2\bar{u}_s = c^2a\,s\sin\omega t, \text{using}\,(2a)\,\text{and}\,(2b)$$

or $$(\mathrm{D}^2 + c^2s^2)\bar{u}_s = c^2\,a\,s\sin\omega t, \quad \text{where} \quad \mathrm{D} \equiv d/dt \qquad \ldots (4)$$

Its auxiliary eqution is $\mathrm{D}^2 + c^2 s^2 = 0$ giving $\mathrm{D} = \pm ics$ and hence its

C.F = $c_1\cos c\,s\,t + \mathrm{C}_2\sin cst$, c_1 and c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 + c^2 s^2}(c^2 a\, s \sin \omega t) = c^2 as \frac{1}{-\omega^2 + c^2 s^2} \sin \omega t$$

$\therefore$ Solution of (4) is $$\bar{u}_s(s,t) = c_1 \cos cst + c_2 \sin cst - \frac{c^2 as \sin \omega t}{\omega^2 - c^2 s^2}. \quad \ldots (5)$$

Taking the finite fourier sine transform of (3*a*) and (3*b*), we get

$$\bar{u}_s(s,0) = 0 \quad \ldots (6a)$$

and $$(d\,\bar{u}_s / dt)_{t=0} = 0. \quad \ldots (6b)$$

Putting $t = 0$ in (5) and using (6*a*), we get $c_1 = 0$ and so (5) becomes

$$\bar{u}_s(s,t) = c_2 \sin cst - (c^2 as \sin \omega t)/(\omega^2 - c^2 s^2). \quad \ldots (7)$$

From (7), $$d\,\bar{u}_s / dt = c_2 cs \cos cst - (c^2 as\ \omega \cos \omega t)/(\omega^2 - c^2 s^2) \quad \ldots (8)$$

Putting $t = 0$ in (8), and using (6*b*), we get

$$0 = c_2 cs - (c^2 as\omega)/(\omega^2 - c^2 s^2) \qquad \text{or} \qquad c_2 = (c\ a\ \omega)/(\omega^2 - c^2 s^2)$$

$\therefore$ From (7), $$\bar{u}_s(s,t) = \frac{c\, a\, \omega}{\omega^2 - c^2 s^2} \sin cst - \frac{c^2 a\, s \sin \omega t}{\omega^2 - c^2 s^2}. \quad \ldots (9)$$

Taking the inverse finite fourier sine transform, we get

$$u(x,\ t) = \frac{2}{\pi} \sum_{s=1}^{\infty} \bar{u}_s(s,t) \sin sx = \frac{2}{\pi} \sum_{s=1}^{\infty} \left[\frac{c\, a\, \omega}{\omega^2 - c^2 s^2} \sin cst - \frac{c^2 a\, s \sin \omega t}{\omega^2 - c^2 s^2} \right] \sin sx$$

$\therefore$ $$u\ (x,\ t) = \frac{2ac^2 \sin \omega t}{\pi} \sum_{s=1}^{\infty} \frac{s \sin sx}{c^2 s^2 - \omega^2} + \frac{2a\omega c}{\pi} \sum_{s=1}^{\infty} \frac{\sin cst \sin sx}{\omega^2 - c^2 s^2} \quad \ldots (10)$$

To re-write the solution (10) in the required form, we note that

$$F_s\left\{ \sin \frac{\omega(\pi - x)}{c} \right\} = \int_0^{\pi} \sin \frac{\omega(\pi - x)}{c} \sin sx\, dx = \frac{1}{2} \int_0^{\pi} \left[\cos\left\{ \frac{\omega(\pi - x)}{c} - sx \right\} - \cos\left\{ \frac{\omega(\pi - x)}{c} + sx \right\} \right] dx$$

$$= \frac{1}{2} \left[\frac{\sin\{(\omega / c)(\pi - x) - sx\}}{-(\omega / c) - s} - \frac{\sin\{(\omega / c)(\pi - x) + sx\}}{-(\omega / c) - s} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{\sin(\omega\pi / c)}{s + \omega / c} + \frac{\sin(\omega\pi / c)}{s - \omega / c} \right] = \frac{\sin(\omega\pi / c)}{2} \frac{2s}{s^2 - \omega^2 / c^2} \quad \ldots (11)$$

$\therefore$ Taking the inverse finite fourier sine transform, (11) yields

$$\sin \frac{\omega(\pi - x)}{c} = \frac{2}{\pi} \sum_{s=1}^{\infty} \frac{sc^2}{s^2 c^2 - \omega^2} \sin \frac{\omega\pi}{c} \sin sx \quad \text{or} \quad \sum_{s=1}^{\infty} \frac{s \sin sx}{s^2 c^2 - \omega^2} = \frac{\pi}{2c^2} \sin \frac{\omega(\pi - x)}{c} \operatorname{cosec} \frac{\omega\pi}{c}$$

$\therefore$ $$\frac{2ac^2 \sin \omega t}{\pi} \sum_{s=1}^{\infty} \frac{s \sin sx}{s^2 c^2 - \omega^2} = a \sin \omega t \sin \frac{\omega(\pi - x)}{c} \operatorname{cosec} \frac{\omega\pi}{c} \quad \ldots (12)$$

Using (12), (10) may be re-written in the required form, namely,

$$u(x,\ t) = a \sin \omega t \sin \frac{\omega(\pi - x)}{c} \operatorname{cosec} \frac{\omega\pi}{c} + \frac{2ac\omega}{\pi} \sum_{s=1}^{\infty} \frac{1}{\omega^2 - c^2 s^2} \sin sx \sin cst.$$

Ex. 11. *Find the steady temperature u(x,y) in a long square bar of side π when one face is kept at constant temperature u_0 and the other faces at zero temperature. Assume that u(x, y) is bounded.*

Or

Determine a function u(x,y) which is harmonic in the open square o<x < π, o < y < π, takes a constant value u_0 on the edge y = π and vanishes on other edges of the square.

[I.F.S. 2003, Meerut 1995]

Sol. Steady temperature is governed by two-dimensional Laplace equation

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0 \quad \ldots (1)$$

with boundary conditions $u(0, y) = u(\pi, y) = 0$ ... (2)

$$u(x, 0) = 0 \quad \ldots (3)$$

and $u(x, \pi) = u_0$ ... (4)

Taking finite Fourier sine transform of (1) with respect to x, we get

$$\int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin sx\, dx + \int_0^\pi \frac{\partial^2 u}{\partial y^2} \sin sx\, dx = 0$$

or $$\left[\frac{\partial u}{\partial x} \sin sx\right]_0^\pi - s\int_0^\pi \frac{\partial u}{\partial x} \cos sx + \frac{d^2}{dy^2}\int_0^\pi u(x, y) \sin sx\, dx = 0, \text{ integrating by parts}$$

or $$-s\left\{\left[u(x, y) \cos sx\right]_0^\pi + s\int_0^\pi u(x, y) \sin sx\, dx\right\} + \frac{d^2 \bar{u}_s}{dy^2} = 0, \text{ taking } \bar{u}_s = \bar{u}_s(s, y) = \int_0^\pi u(x, y) \sin sx\, dx$$

or $$-s[u(\pi, y) \cos s\pi - u(0, y)] - s^2 \bar{u}_s + \frac{d^2 \bar{u}_s}{dy^2} = 0, \qquad \text{or} \qquad \frac{d^2 \bar{u}_s}{dy^2} - s^2 \bar{u}_s = 0, \text{ using (2)}$$

whose solution is $\bar{u}_s(s, y)$ = A cosh sy + B sinh sy, A and B being arbitrary constants ... (5)

Taking the finite Fourier sine transform of (3) and (4), we get

$$\int_0^\pi u(x,0) \sin sx\, dx = 0 \quad \text{and} \quad \int_0^\pi u(x, \pi) \sin sx\, dx = \int_0^\pi u_0 \sin sx\, dx = u_0 \left[-\frac{\cos sx}{s}\right]_0^\pi = \frac{u_0}{s}(-\cos s\pi + 1)$$

i.e., $$\bar{u}_s(s,0) = 0 \quad \ldots (6)$$

and $$\bar{u}_s(s, \pi) = (u_0 / s) \times [1 - (-1)^s] \quad \ldots (7)$$

Putting $y = 0$ in (5) and using (6), we get A = 0.

Putting $y = \pi$ and A = 0 in (5) and using (7), we get

$$(u_0 / s) \times [1 - (-1)^s] = \text{B sinh } s\pi \qquad \text{or} \qquad \text{B} = u_0[1 - (-1)^s] / s \sinh s\pi$$

$\therefore$ From (5), $$\bar{u}_s(s, y) = u_0[1 - (-1)^s] \sinh sy / s \sinh s\pi$$

Taking the inverse finite Fourier sine transform, we get

$$u(x, y) = \frac{2}{\pi} \sum_{s=1}^{\infty} \bar{u}_s(s, y) \sin sx \qquad \text{or} \qquad u(x, y) = \frac{2u_0}{\pi} \sum_{s=1}^{\infty} \frac{1 - (-1)^s}{s \sinh s\pi} \sinh sy \sin sx \quad \ldots (8)$$

But $$1 - (-1)^s = \begin{cases} 0, \text{if } s = 2n, n = 0, 1, 2, 3 \ldots \\ 2, \text{if } s = 2n + 1, n = 0, 1, 2, 3 \ldots \end{cases}$$

$\therefore$ (8) yields $$u(x, y) = \frac{4u_0}{\pi} \sum_{n=0}^{\infty} \frac{\sinh\{(2n+1)y\} \sin\{(2n+1)x\}}{(2n+1)\sinh\{(2n+1)\pi\}}$$

Ex. 12. *The cross-section of a long bar of diffusivity k is the square 0<x<π, 0 < y <π. If the four faces of the bar are maintained at zero temperature and the initial temperature is unity, use repeated finite transforms to show that the temperature at time t is ϕ(x) × ϕ(y)) where*

$$\phi(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} (2n+1)^{-1} \sin(2n+1)x \exp\left\{-k(2n+1)^2 t\right\}, \text{ where } \exp(a) = e^a.$$

Sol. Here the temperature $u(x, y, t)$ in the bar is given by two-dimentsional heat equation

$$(\partial^2 u / \partial x^2) + (\partial^2 u / \partial y^2) = (1/k)(\partial u / \partial t),\ 0 < x < \pi,\ 0 < x < \pi \qquad \ldots (1)$$

with the boundary conditions: $$u(0, y, t) = y(\pi, y, t) = 0 \qquad \ldots (2a)$$

$$u(x, 0, t) = u(x, \pi, t) = 0 \qquad \ldots (2b)$$

and initial condition: $$u(x, y, 0) = 1 \qquad \ldots (3)$$

Taking the finite Fourier sine transform of (1) w.r.t. 'x', we get

$$\int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin px\, dx + \int_0^\pi \frac{\partial^2 u}{\partial y^2} \sin px\, dx = \frac{1}{k} \int_0^\pi \frac{\partial u}{\partial t} \sin px\, dx$$

or $$\left[\frac{\partial u}{\partial x} \sin px\right]_0^\pi - p\int_0^\pi \frac{\partial u}{\partial x} \cos px\, dx + \frac{\partial^2}{\partial y^2} \int_0^\pi u(x, y, t) \sin px\, dx = \frac{1}{k} \frac{\partial}{\partial t} \int_0^\pi u(x, y, t) \sin px\, dx$$

or $$-p\left\{\left[u(x, y, t) \cos px\right]_0^\pi + p\int_0^\pi u(x, y, t) \sin px\, dx\right\} + \frac{\partial^2 \bar{u}_s'}{\partial y^2} = \frac{1}{k} \frac{\partial \bar{u}_s'}{\partial t}, \text{ where } \bar{u}_s' = \int_0^\pi u(x, y, t) \sin px\, dx$$

or $$-p\left\{u(\pi, y, t) \cos p\pi - u(0, y, t) - p\bar{u}_s'\right\} + \frac{\partial^2 \bar{u}_s'}{\partial y^2} = \frac{1}{k} \frac{\partial \bar{u}_s'}{\partial t}$$

or $$-p^2 \bar{u}_s' + (\partial^2 \bar{u}_s' / \partial y^2) = (1/k)(\partial \bar{u}_s' / \partial t), \text{using } (2a). \qquad \ldots (4)$$

Taking the finite Fourier sine transform of $(2b)$, we get

$$\bar{u}_s'(p, 0, t) = 0 \qquad \ldots (5)$$

$$\bar{u}_s'(p, \pi, t) = 0 \qquad \ldots (6)$$

Also, $$\bar{u}_s'(p, y, 0) = \int_0^\pi u(x, y, 0) \sin px\ dx = \int_0^\pi \sin px\ dx, \text{ by (3)}$$

$$= \left[-(1/p) \times \cos px\right]_0^\pi = [1 - (-1)^p]/p$$

$\therefore$ $$\bar{u}_s'(p, y, 0) = \begin{cases} 0, & \text{when } p \text{ is even} \\ 2/p, & \text{when } p \text{ is odd} \end{cases} \qquad \ldots (7)$$

Again, taking the Finite fourier sine transform of (4) *w.r.t.* 'y' we get

$$-p^2 \int_0^\pi \bar{u}_s'(p, y, t) \sin qy\, dy + \int_0^\pi \frac{\partial^2 \bar{u}_s'}{\partial y^2} \sin qy\, dy = \frac{1}{k} \int_0^\pi \frac{\partial \bar{u}_s'}{\partial t} \sin qy\, dy. \qquad \ldots (8)$$

Let $\bar{u}_s$ be the finite Fourier sine transform of $\bar{u}_s'$ (p, y, t) *w.r.t.* 'y', *i.e.*,

$$\bar{u}_s(p,q,t) = \bar{u}_s = \int_0^{\pi} \bar{u}_s'(p,y,t)\sin qy\,dy \qquad \ldots (9)$$

Then, $$\int_0^{\pi} \frac{\partial^2 \bar{u}_s'}{\partial y^2}\sin qy\,dy = \left[\frac{\partial \bar{u}_s'}{\partial y}\sin q\,y\right]_0^{\pi} - q\int_0^{\pi}\frac{\partial \bar{u}_s'}{\partial y}\cos qy\,dy$$

$$= -q\left\{\left[\bar{u}_s'(p,y,t)\cos qy\right]_0^{\pi} + q\int_0^{\pi}\bar{u}_s'(p,y,t)\sin qy\,dy\right\} = -q^2\bar{u}_s, \text{ using (5) and(6)}$$

and $$\int_0^{\pi}\frac{\partial \bar{u}_s'}{\partial t}\sin qy\,dy = \int_0^{\pi}\bar{u}_s'(p,y,t)\sin qy\,dy = \frac{d\bar{u}_s}{dt}.$$

Using the above results, (8) reduces to

$$-p^2\bar{u}_s - q^2\bar{u}_s = (1/k)(d\,\bar{u}_s/dt) \qquad \text{or} \qquad d\,\bar{u}_s/dt = -k(p^2+q^2)t,$$

whose general solution is

$$\bar{u}_s(p,q,t) = ce^{-k(p^2+q^2)t},\ c \text{ being an arbitrary constant} \qquad \ldots (10)$$

Putting $t = 0$ in (10), $\bar{u}_s(p,q,0) = c.$

$\therefore$ From (10), $$\bar{u}_s(p,q,t) = \bar{u}_s(p,q,0)\,e^{-k(p^2+q^2)t}. \qquad \ldots (11)$$

Now, taking finite Fourier sine transform of both sides of (7) w.r.t. 'y', we have

$$\int_0^{\pi}\bar{u}_s'(p,y,0)\sin qy\,dy = 0, \text{ if } p \text{ is even} \qquad i.e. \qquad \bar{u}_s(p,q,0) = 0, \text{ if } p \text{ is even}$$

and $$\int_0^{\pi}\bar{u}_s'(p,y,0)\sin qy\,dy = \int_0^{\pi}\frac{2}{p}\sin qy\,dy, \text{ if } p \text{ is odd}$$

or $$\bar{u}_s(p,q,0) = (2/pq)\times\left[-\cos qy\right]_0^{\pi} = (2/pq)\times[1-\cos q\pi], \text{ if } p \text{ is odd}$$

Thus, $$\bar{u}_s(p,q,0) = \begin{cases} 0, \text{ if } q \text{ is even and } p \text{ is odd} \\ 4/pq, \text{if } q \text{ is odd and } p \text{ is odd} \end{cases}$$

From the above discussion, it follows that $\bar{u}_s(p,q,0)$ is non-zero only when p and q are both odd. Since we are interested in non-zero solution $u(x, y, t)$, so $\bar{u}_s(p,q,0)$ must be non-zero in (11).

So putting $\bar{u}_s(p,q,0) = 4/pq$, (p and q both odd) in (11), we get

$$\bar{u}_s(p,q,t) = (4/pq)e^{-k(p^2+q^2)t}$$

or $$\bar{u}_s(p,q,t) = [4/(2n+1)(2m+1)]e^{-k[(2n+1)^2+(2m+1)^2]t} \qquad \ldots (12)$$

taking $p = 2n + 1$ and $q = 2m + 1$, where $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2\ldots$

Taking the inverse finite Fourier sine transform of $\bar{u}_s(p,q,t)$ *w.r.t* 'y', we have

$$\bar{u}_s'(p,y,t) = \frac{2}{\pi}\sum_{m=0}^{\infty}\bar{u}_s(p,q,t)\sin(2m+1)y. \qquad \ldots (13)$$

Taking again the inverse finitie Foruier sine transform of $\bar{u}_s'(p,y,t)$ *w.r.t.* 'x', we get

$$u(x, y, t) = \frac{2}{\pi}\sum_{n=0}^{\infty}\bar{u}_s'(p,y,t)\sin(2n+1)x = \frac{2}{\pi}\sum_{n=0}^{\infty}\left[\frac{2}{\pi}\sum_{m=0}^{\infty}\bar{u}_s(p,q,t)\sin(2m+1)y\right]\sin(2n+1)x, \text{ by (13)}$$

$$u\,(x,\,y,\,t) = \frac{16}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-k[(2n+1)^2+(2m+1)^2]t}}{(2n+1)(2m+1)}\sin(2n+1)x\sin(2m+1)y,\ \text{by (12)}$$

or $$u\,(x,\,y,\,t) = \phi(x)\times\varphi(y), \text{where}$$

$$\phi\,(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}(2n+1)^{-1}e^{-k(2n+1)^2t}\sin(2n+1)x \text{ and } \phi\,(y) = \frac{4}{\pi}\sum_{m=0}^{\infty}(2m+1)^{-1}e^{-k(2m+1)^2t}\sin(2m+1)x.$$

Ex. 13 *Solve* $\partial^2V/\partial x^2+\partial^2V/\partial y^2+\partial^2V/\partial z^2 = 0$, with the boundary conditions $V = V_0$ when $y = \pi$, and $V = 0$ when $x = 0$ and π, when $y = 0$ and also when $z = 0$ and π.

Sol. Given $$\partial^2\,\mathrm{V}/\partial x^2+\partial^2\,\mathrm{V}/\partial y^2+\partial^2\,\mathrm{V}/\partial z^2 = 0 \quad ...(1)$$

with boundary conditions: $$\mathrm{V}(0,\,y,\,z) = \mathrm{V}(\pi,\,y,\,z) = 0 \quad ...\,(2a)$$

$$\mathrm{V}(x,\,y,\,0) = \mathrm{V}(x,\,y,\,\pi) = 0 \quad ...\,(2b)$$

$$\mathrm{V}(x,\,0,\,z) = 0 \quad ...\,(2c)$$

and $$\mathrm{V}(x,\,\pi,\,z) = \mathrm{V}_0 \quad ...\,(2d)$$

Taking the finitie Fourier sine transform of both sides of (1) *w.r.t.* 'x', we have

$$\int_0^{\pi}\frac{\partial^2\,\mathrm{V}}{\partial x^2}\sin px\,dx+\int_0^{\pi}\frac{\partial^2\,\mathrm{V}}{\partial y^2}\sin px\,dx+\int_0^{\pi}\frac{\partial^2\,\mathrm{V}}{\partial z^2}\sin px\,dx = 0$$

or $$\left[\frac{\partial\,\mathrm{V}}{\partial x}\sin px\right]_0^{\pi}-p\int_0^{\pi}\frac{\partial\,\mathrm{V}}{\partial x}\cos pxdx+\frac{\partial^2}{\partial y^2}\int_0^{\pi}\mathrm{V}(x,y,z)\sin p\,x\,d\,x\ +\frac{\partial^2}{\partial z^2}\int_0^{\pi}\mathrm{V}(x,y,z)\sin pxdx = 0$$

or $$-p\left\{\left[\mathrm{V}(x,y,z)\cos px\right]_0^{\pi}+p\int_0^{\pi}\mathrm{V}(x,y,z)\sin px\,dx\right\}+\frac{\partial^2\,\overline{\mathrm{V}}_s'}{\partial y^2}+\frac{\partial^2\,\overline{\mathrm{V}}_s'}{\partial z^2} = 0,$$

$$[\text{Taking } \overline{\mathrm{V}}_s' = \overline{\mathrm{V}}_s'(p,y,z) = \int_0^{\pi}\mathrm{V}(x,y,z)\sin px\,dx]$$

or $$-p\left\{\mathrm{V}(\pi,y,z)\cos p\pi-\mathrm{V}(0,y,z)+p\,\overline{\mathrm{V}}_s'\right\}+\partial^2\,\overline{\mathrm{V}}_s'/\partial y^2+\partial^2\,\overline{\mathrm{V}}_s'/\partial z^2 = 0$$

or $$-p^2\,\overline{\mathrm{V}}_s'+\partial^2\,\overline{\mathrm{V}}_s'/\partial y^2+\partial^2\,\overline{\mathrm{V}}_s'/\partial z^2 = 0,\ \text{using (2}a\text{)} \quad ...\,(3)$$

Taking the finite Fourier sine transform of boundary conditions (2*b*), (2*c*) and (2*d*) *w.r.t.* 'x', we get

$$\int_0^{\pi}\mathrm{V}(x,y,0)\sin px\,dx\ =\ \int_0^{\pi}\mathrm{V}(x,y,\pi)\sin px\,dx\ =\ \int_0^{\pi}\mathrm{V}(x,0,z)\sin px\,dx = 0$$

and $$\int_0^{\pi}\mathrm{V}(x,\pi,z)\sin px\,dx\ =\ \int_0^{\pi}\mathrm{V}_0\sin px\,dx$$

i.e. $$\overline{\mathrm{V}}_s'(p,y,0) = \overline{\mathrm{V}}_s'(p,y,\pi) = \overline{\mathrm{V}}_s'(p,0,z) = 0 \quad ...\,(4)$$

and $$\overline{\mathrm{V}}_s'(p,\pi,z)\ =\ \mathrm{V}_0\left[-(1/p)\cos px\right]_0^{\pi} = (\mathrm{V}_0/p)[1-(-1)^p]$$

$$\therefore \quad \overline{\mathrm{V}}_s(p,\pi,z)\ =\ \begin{cases}0, \text{when } p \text{ is even}\\ 2\,\mathrm{V}_0/p, \text{when } p \text{ is odd}\end{cases} \quad ...\,(5)$$

Again, taking the finite Fourier sine transform of (3) *w.r.t.* '*z*', we get

$$-p^2\int_0^{\pi}\overline{V}_s'(p,y,z)\sin qz\,dz+\int_0^{\pi}\frac{\partial^2\overline{V}_s'}{\partial y^2}\sin qz\,dz+\int_0^{\pi}\frac{\partial^2\overline{V}_s'}{\partial z^2}\sin qz\,dz=0 \qquad \ldots(6)$$

Let $\overline{V}_s$ be finite Fouirer sine transform of $\overline{V}'_s(p, y, z)$, *w.r.t.* '*z*'.

$$\therefore \qquad \overline{V}_s=\overline{V}_s(p,y,q)=\int_0^{\pi}\overline{V}_s'(p,y,z)\sin qz\,dz \qquad \ldots(7)$$

Then,
$$\int_0^{\pi}\frac{\partial^2\overline{V}_s'}{\partial y^2}\sin qz\,dz=\frac{d^2}{dy^2}\int_0^{\pi}\overline{V}_s'(p,y,z)\sin qz\,dz=\frac{d^2\overline{V}_s}{dy^2}$$

and
$$\int_0^{\pi}\frac{\partial^2\overline{V}_s'}{\partial z^2}\sin qz\,dz=\left[\frac{\partial\overline{V}_s'}{\partial z}\sin qz\right]_0^{\pi}-q\int_0^{\pi}\frac{\partial\overline{V}_s'}{\partial z}\cos qz\,dz$$

$$=-q\left\{\left[\overline{V}_s'(p,y,z)\cos qz\right]_0^{\pi}+q\int_0^{\pi}\overline{V}_s'(p,y,z)\sin qz\,dz\right\}$$

$$=-q\left\{\overline{V}_s'(p,y,\pi)\cos q\pi-\overline{V}_s'(p,y,0)+q\,\overline{V}_s\,p,y,q)\right\}=-\,q^2\,\overline{V}_s, \text{ using (4).} \qquad \ldots(8)$$

Using the above results, (6) reduces to

$-p^2\,\overline{V}_s+(d^2\,\overline{V}_s/dy^2)-q^2\,\overline{V}_s=0$ or $(D^2-\mu^2)\,\overline{V}_s=0$, where $D\equiv d/dy$.

where
$$\mu=(p^2+q^2)^{1/2} \qquad \ldots(8)'$$

Its solution is $\overline{V}_s(p,y,q)=C_1\cosh\mu y+C_2\sinh\mu y$, C_1, C_2 being arbitrary constants ... (9)

From (4),
$$\overline{V}_s'(p,0,z)=0. \qquad \ldots(10)$$

Taking finite Fourier sine transform of (10) *w.r.t.z*, we get

$$\int_0^{\pi}\overline{V}_s'(p,0,z)\sin pz\,dz=0 \quad \text{or} \quad \overline{V}_s(p,0,q)=0. \qquad \ldots(11)$$

Putting $y = 0$ in (9) and using (11), we get $C_1 = 0$. So from (9), we get

$$\overline{V}_s(p,y,q)=C_2\sinh\mu y. \qquad \ldots(12)$$

Putting $y = \pi$ in (12), we get

$\overline{V}_s(p,\pi,q)=C_2\sinh\mu\pi$ or $C_2=(1/\sinh\mu\pi)\,\overline{V}_s(p,\pi,q)$

$\therefore$ From (12)
$$\overline{V}_s(p,y,q)=\overline{V}_s(p,\pi,q)\times(\sinh\mu y/\sinh\mu\pi) \qquad \ldots(13)$$

Taking finite Fourier sine transform of (5) *w.r.t. 'z'* we get

$$\int_0^{\pi}\overline{V}_s'(p,\pi,z)\sin qz\,dz=0, \text{ if } p \text{ is even} \qquad i.e. \qquad \overline{V}_s(p,\pi,q)=0, \text{ if } p \text{ is even}$$

and
$$\int_0^{\pi}\overline{V}_s'(p,\pi,z)\sin qz\,dz=\int_0^{\pi}(2V_0/p)\sin qz\,dz, \text{ if } p \text{ is odd}$$

$$\therefore \qquad \overline{V}_s(p,\pi,q)=(2\overline{V}_0/pq)\left[-\cos qz\right]_0^{\pi}=(2V_0/pq)\left[1-(-1)^q\right], \text{ if } p \text{ is odd}$$

$$\therefore \qquad \overline{V}_s(p,\pi,q) = \begin{cases} 0, \text{if } q \text{ is even}, p \text{ is odd} \\ 4V_0/pq, \text{if } q \text{ is } odd, p \text{ is odd} \end{cases}$$

From the above discussion, it follows that $\overline{V}_s(p,\pi,q)$ in non-zero only when p and q are both odd. Since we are interested in non-zero solution $V(x,y,z)$, so $\overline{V}_s(p,\pi,q)$ must be non-zero in (13). So putting $\overline{V}_0(p,\pi,q) = 4V_0/pq$, ($p$ and q both odd) in (13), we get

$$\overline{V}_s(p,\pi,q) = \frac{4V_0}{pq}\frac{\sinh \mu y}{\sinh \mu\pi} = \frac{4V_0}{(2n+1)(2m+1)}\frac{\sinh \mu y}{\sinh \mu\pi} \qquad \ldots (14)$$

where $$\mu = \sqrt{(p^2+q^2)} = \sqrt{(2n+1)^2+(2m+1)^2}, \text{ by } (8)' \qquad \ldots (15)$$

taking $p = 2n + 1$ and $q = 2m + 1$, where $n = 0, 1, 2,\ldots..$, $m = 0, 1, 2\ldots..$

Taking inverse finite Fourier sine transform of $\overline{V}_s(p,\pi,q)$ *w.r.t.*'z', we have

$$\overline{V}_s'(p,y,z) = \frac{2}{\pi}\sum_{m=0}^{\infty}\overline{V}_s(p,\pi,q)\sin(2m+1)z. \qquad \ldots (16)$$

Taking inverse finite Fourier sine transform of $\overline{V}_s'(p,y,z)$ *w.r.t.* x, we have

$$V(x,y,z) = \frac{2}{\pi}\sum_{n=0}^{\infty}\overline{V}_s'(p,y,z)\sin(2n+1)x = \frac{2}{\pi}\sum_{n=0}^{\infty}\left[\frac{2}{\pi}\sum_{m=0}^{\infty}\overline{V}_s(p,\pi,q)\sin(2m+1)z\right]\sin(2n+1)x, \text{ using } (16)$$

$$\therefore \qquad V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\sinh \mu y}{\sinh \mu\pi}\frac{\sin(2n+1)x}{2n+1}\frac{\sin(2m+1)z}{2m+1}, \text{ using } (14)$$

where μ is given by (15)

Ex. 14. *Find the steady temperature in a cube of side π when one face is kept at constant temperature V_0 and the other faces are at zero temperature.*

Sol. We know that the steady state temperature $V(x, y, z)$ in cube is given by Laplace's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$, $o < x < \pi$, $o < y < \pi$, $o < z < \pi$.

Assume that $V = V_0$ when $y = \pi$ and $V = 0$ when $x = 0$, $x = \pi$, when $y = 0$ also when $z = 0$ and $z = \pi$. Now proceed exactly as in Ex.13. Rest of the solution is the same.

Ex. 15. Use finite sine transoform to solve. $k(\partial^2 u/\partial x^2) = \partial u/\partial t$, $0 < x < l$, $t > 0$ subject to $u(0,t) = u(l,t) = 0$, $t > 0$, $u(x,0) = (2u_0x)/l$, $0 \le x \le l$ while $u(x,0) = 2u_0(1 - x/l)$, $l/2 \le x \le l$.

Ans. $$u(x,t) = \frac{8u_0}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}e^{-\{k(2n-1)^2\pi^2t/l^2\}}\sin\frac{(2n-1)\pi x}{l}$$

Ex.16. Using the finite Fourier transform, solve $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, $0 < x < \pi$, $0 < y < y_0$ subject to $u(0,y) = 0, u(\pi,y) = 1$, $u_y(x,0) = 0$, $u(x,y_0) = 0$.

Ans. $$u(x,y) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cosh ny}{\cosh ny_0}\sin nx$$

Calculus of Variations

Where is What

Chapters	*Pages*
1. Variational problems with fixed boundaries	1.3 – 1.84
2. Variational problems with moving (or free) boundaries. One sided variations	2.1 – 2.35
3. Sufficient conditions for an extremum	3.1 – 3.28

1

Variational Problems With Fixed Boundaries

1.1 Introduction

One of the main problem of the calculus of variation is to find that curve connecting two given points which either maximizes or minimizes some given integral. For example, suppose that two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are given in a plane. There are infinitely many curves joining these points, and suppose we wish to determine which of these curves is the shortest, *i.e.*, to determine the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that

$$I[y(x)] = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx \qquad ...(1)$$

is a minimum. The intuitive answer is of course a straight line. Again, suppose we are to determine which curve will generate the surface of revolution of smallest area when revolved about the x-axis, *i.e.*, to determine the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that

$$I[y(x)] = \int_{x_1}^{x_2} 2\pi y(1 + y'^2)^{1/2}\, dx \qquad ...(2)$$

is a minimum. But in this case the answer is far from clear. The *calculus of variation* provides an excellent uniform analytical method of dealing with situation of this kind.

The above problems (1) and (2) show that in the calculus of variation we consider some quantity (arc length, surface area etc.) that depends on an entire curve, and we wish to find the curve that minimizes the quantity under consideration. Thus, in general case, we wish to find the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for some given known function $f(x, y, y')$ of variables x, y, y' the integral

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx \qquad ...(3)$$

is either a maximum or minimum, also called an *extremum* or *stationary values*. A curve which satisfies this property is said to be an *extremal*. An integral such as (3) which assumes a numerical value for some class of functions $y(x)$ is said to be *functional.*

While formulating the basic problem of maximizing and minimizing the integral (2), we shall assume that the function $F(x, y, y')$ has continuous partial derivatives of second order with respect to x, y, y'. Again, we shall always restrict ourselves to select only unknown function $y(x)$ that have continuous second order derivatives and satisfy the given boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. Functions of this type are known as *admissible*.

To show mathematically that the function $y(x)$ obtained actually maximizes or minimizes the integral is much more difficult than in the corresponding problem of calculus. In calculus we use the second derivative to give sufficient conditions distinguishing one type of stationary value from another. Similar sufficient conditions will be developed in chapter 3. Since the application of sufficient conditions is quite complicated, we will not consider them in chapters 1 and 2. In actual practice, the geometry

or physics of the problem under consideration often makes it possible to determine whether a particular stationary function maximizes or minimizes the integral (or neither) and hence use of sufficient conditions may frequently be avoided.

1.2 Variation and its properties

As discussed in Art. 1.1, a variable quantity $I[y(x)]$ is a functional dependent on a function $y(x)$ if to each function $y(x)$ of a certain class of functions $y(x)$ there corresponds a definite value of I, *i.e.*, we have a correspondence : to the function $y(x)$ there corresponds a number I.

The increment, or variation, δy of the argument $y(x)$ of a functional $I[y(x)]$ is the difference $\delta y (= y(x) - y_1(x))$ between two functions belonging to a certain class.

A functional $I[y(x)]$ is said to be continuous if to a small change of $y(x)$ there corresponds a small change in the functional $I[y(x)]$. This definition requires some explanation, for the question naturally arises as to what changes of the function $y(x)$, which is the argument of the functional, are called small or, what in the same, what curves $y = y(x)$ and $y = y_1(x)$ are considered close or only slightly different. It may be assumed that the functions $y(x)$ and $y_1(x)$ are close if $|y(x) - y_1(x)|$ is small for all x for which $y(x)$ and $y_1(x)$ are defined; that is, we can consider as close such curves as having close-lying ordinates. But with this definition, the functional

$$I[y(x)] = \int_{x_1}^{x_2} F\left(x, y, y'\right) dx$$

which occurs frequently in applications will be continuous only in exceptional cases due to the presence of the argument y' in $F(x, y, y')$. Accordingly in many situations, it is more natural to consider as close only those curves which have close-lying ordinates and are close as regards the directions of tangents at the respective points; that is, to require that, for close curves, not only should $|y(x) - y_1(x)|$ be small, but also $|y'(x) - y_1'(x)|$. In some cases, it is necessary to consider as close only those functions for which $|y(x) - y_1(x)|$, $|y'(x) - y_1'(x)|$, ..., $|y^{(n)}(x) - y_1^{(n)}(x)|$ are all small.

Proximity of the curves. We now introduce the following definitions of proximity of the curves $y = y(x)$ and $y = y_1(x)$.

The curves $y = y(x)$ and $y = y_1(x)$ are close in the sense of zero order proximity if $|y(x) - y_1(x)|$ is small.

The curves $y = y(x)$ and $y = y_1(x)$ are close in the sense of first order proximity if $|y(x) - y_1(x)|$ and $|y'(x) - y_1'(x)|$ are small.

The curves $y = y(x)$ and $y = y_1(x)$ are close in the sense of nth order proximity if $|y(x) - y_1(x)|$, $|y'(x) - y_1'(x)|$, ..., $|y^{(n)}(x) - y_1^{(n)}(x)|$ are all small.

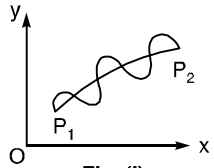


Fig. (i)

Fig. (*i*) shows two curves which are close in the sense of zero-order proximity but not in the sense of first order proximity. Fig. (*ii*) shows two curves which are close in the sense of first-order proximity.

From the above definitions it follows that if the curves are close in the sense of nth order proximity, then they are also close in the sense of any lesser order of proximity.

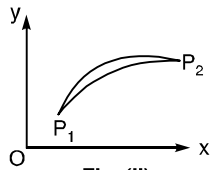


Fig. (ii)

We now proceed to refine the concept of the continuity of a functional. The functional $I[y(x)]$ is said to be continuous at $y = y_0(x)$ in the sense of nth order proximity if for any positive ε there is a $\delta > 0$ such that

$$|I[y(x)] - I[y_0(x)| < \varepsilon$$

for $\quad |y(x) - y_0(x)| < \delta, \quad |y'(x) - y'_0(x)| < \delta, \ldots\ldots\ldots\ldots, |y^{(n)}(x) - y_0^{(n)}(x)| < \delta$

It is assumed that the function $y(x)$ is taken from a class of functions on which the functional $I[y(x)]$ is defined.

Linear functional. Let us now introduce the concept of a linear functional. A linear functional is a functional $L[y(x)]$ if it satisfies that following two conditions :

(*i*) $L\,[C\,y(x)] = C\,L\,[y(x)]$, C being an arbitrary constant

(*ii*) $L\,[y_1(x) + y_2(x)] = L\,[y_1(x)] + L\,[y_2(x)]$

As an example the functional $L[y(x)] = \int_{x_1}^{x_2} \left[p(x)\, y + q(x)\, y' \right] dx$

is a linear functional.

We now proceed to define the variation of a functional $I[y(x)]$. Let the increment of a functional

$$\Delta I = I[y(x) + \delta y] - I[y(x)]$$

be re-written in the form $\Delta I = L[y(x), \delta y] + \beta(y(x), \delta y) \max |\delta y|,$

where $L[y(x), \delta y]$ is a functional linear with respect to δy, and max $|\ \delta y\ |$ is the maximum value of $|\ \delta y\ |$ and $\beta(y(x), \delta y) \to 0$ as max $|\ \delta y\ | \to 0$, Then the part of the increment of the functional that is linear with respect to δy, *i.e*. $L(y(x), \delta y)$, is called the *variation of the functional* and is denoted by δI. Thus the variation of a functional is the principal part of the increment of the functional, which part is linear in δy. In the study of functional, the variation plays the same role as the differential does in study of functions.

We now present another, almost equivalent definition of the variation δI of a functional I. Consider the functional $I[y(x) + \alpha\ \delta y]$ for fixed y and δy and different values of the parameter α. If the functional has a variation in sense of the principal linear part of the increment, then its increment will be of the form

$$\delta I = I[y(x) + \alpha\ \delta y] - I[y(x)] = L(y,\ \alpha\ \delta y) + \beta(y,\ \alpha\ \delta y)\ |\ \alpha\ |\ \max\ |\delta y|$$

Then the derivative of $I[y + \alpha\ \delta y]$ with respect to α at $\alpha = 0$ is

$$\lim_{\Delta\alpha \to 0} \frac{\Delta I}{\Delta \alpha} = \lim_{\alpha \to 0} \frac{\Delta I}{\alpha} = \lim_{\alpha \to 0} \frac{L(y, \alpha\, \delta y) + \beta(y, \alpha\, \delta y)\, |\alpha|\, \max |\delta y|}{\alpha}$$

or
$$\lim_{\Delta\alpha \to 0} \frac{\Delta I}{\Delta \alpha} = \lim_{\alpha \to 0} \frac{L(y, \alpha\, \delta y)}{\alpha} + \lim_{\alpha \to 0} \frac{\beta(y, \alpha\, \delta y)\, |\alpha|\, \max |\delta y|}{\alpha} \qquad ...(1)$$

Now, by virtue of linearity, $L(y,\ \alpha\ \delta y) = \alpha\ L(y,\ \delta y)$ ...(2)

Again, $\lim_{\alpha \to 0} \frac{\beta(y, \alpha\, \delta y)\ |\alpha|\ \max |\delta y|}{\alpha} = \lim_{\alpha \to 0} \beta(y, \alpha\, \delta y)\ \max |\delta y| = 0$ ...(3)

because $\beta(y, \alpha\, \delta y) \to 0$ as $\alpha \to 0$.

Using (2) and (3), (1) reduces to $\lim_{\Delta\alpha \to 0} \frac{\Delta I}{\Delta \alpha} = L\left(y, \delta y\right) = \delta I$

Hence the variation of a functional $I[y(x)]$ is given by

$$\delta I = [\partial I[y(x) + \alpha \delta y] / \partial \alpha]_{\alpha = 0} \qquad ...(4)$$

Definition. A functional $I[y(x)]$ attains a maximum on a curve $y = y_0(x)$ if the values of the functional $I[y(x)]$ on any curve close to $y = y_0(x)$ do not exceed $I[y_0(x)]$; that is $\Delta I = I[y(x)] - I[y_0(x)] \le 0$. If $\Delta I \le 0$ and $\Delta I = 0$ only on $y(x) = y_0(x)$, then we say that a *strict maximum* is attained on the curve $y = y_0(x)$. The curve $y = y_0(x)$, on which a minimum is attained is defined in similar manner. In this case, $\Delta I \ge 0$ for all curves close to the curve $y = y_0(x)$.

Theorem. *If a functional I[y(x)] having a variation attains a maximum or minimum on $y = y_0(x)$, where y(x) is an interior point of the domain of definition of the functional, then at $y = y_0(x)$, $\delta I = 0$.* **(Bhopal 2010)**

Proof. For fixed $y_0(x)$ and δy, $I[y_0(x) + \alpha\delta y] = \phi(\alpha)$ is a function of α, which for $\alpha = 0$, by hypothesis, attains a maximum or a minimum hence we have

$$\phi'(0) = 0 \qquad \text{so that} \qquad \left[\partial I[y_0(x) + \alpha\,\delta\alpha]/\partial\alpha\right]_{\alpha=0} = 0\,, \qquad i.e. \qquad \delta I = 0,$$

showing that the variation of a functional is zero on curves on which an extremum of the functional is achieved.

We shall now make the concept of the extremum of a functional more specific. While dealing with a maximum or minimum, more precisely, a relative maximum or minimum, we considered largest or smallest value of the functional only relative to values of the functional on close lying curves. But we have already seen that the proximity of curves may be understood in different ways. Consequently, it is necessary, in the definition of a maximum or minimum, to indicate the order of proximity.

If a functional $I[y(x)]$ attains a maximum or a minimum on a curve $y = y_0(x)$ with respect to all curves for which $|y(x) - y_0(x)|$ is small, *i.e.* with respect to curves close to $y = y_0(x)$ in the sense of zero-order proximity, then the maximum or minimum is called *strong*. However, if a functional $I[y(x)]$ attains a maximum or a minimum on a curve $y = y_0(x)$ only with respect to curves $y = y(x)$ close to $y = y_0(x)$ in the sense of first order proximity, *i.e.* with respect to curves close to $y = y_0(x)$ not only as regards ordinates but also as regards the tangent directions, then the maximum or the minimum is called *weak*.

From the above definitions, it follows that if a strong maximum (or minimum) of a functional $I[y(x)]$ is attained on a curve $y = y_0(x)$, then a weak maximum (or minimum) is also attained on the same curve. This is so because if two curves are close in the sense of first order proximity, then they are also close in the sense of zero-order proximity.

Remark. All definitions of this article and the above fundamental theorem can be readily extended to functionals dependent on several unknown functions, or dependent on one or several functions of any number of variables, *e.g.* $I\,[y_1(x), y_2(x), \ldots, y_n(x)]$ or $I[z(x_1, x_2, \ldots, x_n)]$ or $I[z_1(x_1, x_2, \ldots x_n), z_2(x_1, x_2, \ldots x_n, \ldots), \ldots, z_n(x_1, x_2, \ldots, x_n)]$.

Notations : $C[a, b]$ denotes the set of all continuous functions $\forall\, x \in [a,b]$. Again, $C^1[a,b]$ denotes the set of all functions admitting continuous first order derivatives $\forall\, x \in [a,b]$.

1.2A Solved Examples

Ex.1. *Define the following terms :*

(*i*) *Variational problems* **[Ravishankar 2004]** (*ii*) *Calculus of variation*

(*iii*) *Variational principles* (*iv*) *Functional* **[Ravishankar 2004]**

(*v*) *Admissible functions* (*vi*) *Proximity of curves* **[Ravishankar 2004]**

(*vii*) *Linear functional* **[Ravishankar 2004]** (*viii*) *Increment of a function* **[Bhopal 2004]**

(*ix*) *Variation of a functional* (*x*) Extrema of functional **[Bhopal 2010]**

Sol. (*i*) Problems which investigate extrema (maxima or minima) of functionals are known as *variational problems.*

(*ii*) The subject of variational problems of functionals is known as *calculus of variation*

(*iii*) The laws which govern functionals to attain extrema are known as variational principles

Ex.2. *Show that the functional* $I_1[y(x)] = \int_a^b \{y'(x) + y(x)\}\, dx$ *is linear in the class* $C^1[a, b]$, *but the functional* $I_2[y(x)] = \int_a^b [\, p(x)\{y'(x)\}^2 + q(x)\{y(x)\}^2\,]\, dx$ *is non-linear.* **[Bhopal 2004]**

Sol. Let k be any constant and let $y(x), \quad y_1(x), \quad y_2(x) \in C^1\ [a, b]$

Then $$I_1[k\, y(x)] = \int_a^b \left[\frac{d}{dx} \{k\, y(x)\} + k\, y(x) \right] dx \;=\; k \int_a^b \{y'(x) + y(x)\}\, dx$$

Thus, $$I_1[k\, y(x)] = k\, I_1[y(x)] \qquad ...(1)$$

Also, $$I_1\{y_1(x) + y_2(x)\} \;=\; \int_a^b \left[\frac{d}{dx} \{y_1(x) + y_2(x)\} + y_1(x) + y_2(x) \right] dx$$

$$= \int_a^b \{y_1'(x) + y_1(x)\}\, dx + \int_a^b \{y_2'(x) + y_2(x)\}\, dx$$

Thus, $$I_1\{y_1(x) + y_2(x)\} = I_1\{y_1(x)\} + I_1\{y_2(x)\} \qquad ...(2)$$

In view of (1) and (2), we see that $I_1[y(x)]$ is linear in $C^1[a, b]$.

Now, $$I_2[k\, y(x)] \;=\; \int_a^b \left[p(x) \left(\frac{d}{dx} \{ky(x)\} \right)^2 + q(x) \{k\, y(x)\}^2 \right] dx$$

$$= k^2 \int_a^b \left[p(x) \{y'(x)\}^2 + q(x) \{y(x)\}^2 \right] dx\,,$$

showing that $I_2[y(x)]$ is a non-linear functional because for a linear functional $I_2[y(x)]$, we must have

$$I_2\ [k\, y(x)] = k\, I_2[y(x)].$$

Ex.3. (*a*) *Let a functional I[y(x)] defined on the class* C^1 *[0, 1] be given by I[y(x)]* $= \int_0^1 \left[1 + \{y'(x)\}^2 \right]^{1/2} dx$, *prove that I[1] = 1, I[x]* $= \sqrt{2}$ *and I[x²]* $= \sqrt{5}/2 + (1/4)\sinh^{-1} 2$.

[Bhopal 2010; Ravishankar 2010; Jabalpur 2004]

(*b*) If a functional $I[y(x)] = \int_0^1 y(x)dx$ is defined on the class C[0,1], then prove that I $[x] = 1/2$ and $I\ [\cos(\pi x/2)] = 1/\pi$.

Sol. (*a*) Given $$I[y(x)] = \int_0^1 \left[1 + \{y'(x)\}^2 \right]^{1/2} dx \qquad ...(1)$$

To find I[1]. Here $y(x) = 1$ so that $y'(x) = 0$. Then, from (1), $$I\ [1] = \int_0^1 dx = [x]_0^1 = 1$$

To find I[x]. Here $y(x) = x$ so that $y'(x) = 1$. Then, from (1), $I(x) = \int_0^1 \sqrt{2}dx = \sqrt{2}[x]_0^1 = \sqrt{2}$

To find *I*[x^2]. Here $y(x) = x^2$ so that $y'(x) = 2x$. Then, from (1), we have

$$I\ [x^2] = \int_0^1 (1 + 4x^2)^{1/2} dx \;=\; 2 \int_0^1 \left\{ x^2 + (1/2)^2 \right\}^{1/2} dx$$

$$= 2 \left[\frac{x}{2} \left(x^2 + \frac{1}{4} \right)^{1/2} + \frac{1}{2} \times \frac{1}{4} \times \sinh^{-1} \frac{x}{(1/2)} \right] = \frac{\sqrt{5}}{2} + \frac{1}{4} \times \sinh^{-1} 2$$

(*b*) Proceed as in part (*a*)

Ex.4. *Investigate the closeness of the following curves :*

(*i*) $y(x) = (1/n) \times \sin n^2 x$, $y_1(x) = 0$ on $[0, \pi]$ **[Ravishankar 2004. Vikaram 2004]**

(*ii*) $y(x) = (1/n^2) \times \sin nx$, $y_1(x) = 0$ on $[0, \pi]$ **[Ravishankar 2004. Vikaram 2004]**

Sol. In what follows, we shall use definition of proximity given in Art.1.2. We have

(*i*) $$|y(x) - y_1(x)| = \left|\frac{\sin n^2 x}{n} - 0\right| = \frac{|\sin n^2 x|}{n} \le \frac{1}{n} \to 0 \text{ as } n \to \infty$$

But $$|y'(x) - y_1'(x)| = \left|\frac{2n^2 \cos n^2 x}{n} - 0\right| = n|\cos n^2 x| = n \text{ at } x = 2\pi/n^2$$

$$\Rightarrow |y'(x) - y_1'(x)| \to \infty \text{ as } n \to \infty \text{ at } x = 2\pi/n^2, \textit{ i.e.}, \text{ at } x = 0$$

Therefore, the curves $y(x)$ and $y_1(x)$ are close in the zeroth order of proximity but not in the first order proximity for large n.

(*ii*) $$|y(x) - y_1(x)| = \left|\frac{\sin nx}{n^2} - 0\right| = \frac{|\sin nx|}{n^2} \le \frac{1}{n^2} \to 0 \text{ as } n \to \infty$$

and $$|y'(x) - y_1'(x)| = \left|\frac{n \cos nx}{n^2} - 0\right| = \frac{|\cos nx|}{n} \le \frac{1}{n} \to 0 \text{ as } n \to \infty .$$

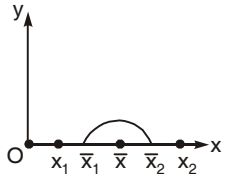
Thus, the curves $y(x)$ and $y_1(x)$ are close in the first order proximity. Proceeding as in part (*i*), we can easily verify that the given curves are not close in a higher order proximity for large n.

1.3 The fundamental lemma of the calculus of variations.

If for every continuous function $\eta(x)$ $$\int_{x_1}^{x_2} \phi(x)\eta(x)dx = 0 ,$$

where the function $\phi(x)$ *is continuous on the interval* $[x_1, x_2]$*, then* $\phi(x) \equiv 0$ *on that interval.*

Proof. Assume that $\phi(x) \neq 0$ (positive, say) at a point $x = \bar{x}$ in $x_1 \le x \le x_2$. Since $\phi(x)$ is continuous on $[x_1, x_2]$, it follows that if $\phi(x) \neq 0$, then $\phi(x)$ maintains its sign in a certain neighbourhood $\bar{x}_1 < x \le \bar{x}_2$ of the point $\bar{x}$. Since $\eta(x)$ is an arbitrary continuous function, we may choose $\eta(x)$ such that $\eta(x)$ remains positive in $\bar{x}_1 \le x \le \bar{x}_2$ while it vanishes outside this interval as shown in the adjoining figure. Hence, we obtain



$$\int_{x_1}^{x_2} \phi(x)\eta(x)dx = \int_{\bar{x}_1}^{\bar{x}_2} \phi(x)\eta(x)dx > 0 , \qquad ...(1)$$

since the product $\phi(x)\eta(x)$ remains positive everywhere in $[x_1, x_2]$ and vanishes outside this interval.

But by hypothesis, $$\int_{x_1}^{x_2} \phi(x)\eta(x)dx = 0 \qquad ...(2)$$

which contradicts (1). This contradiction shows that our assumption $\phi(x) \neq 0$ at some point $\bar{x}$ must be wrong and so $\phi(x) \equiv 0$ on $[x_1, x_2]$.

Remark. The statement of the lemma and its proof do not change if the following restrictions are imposed on the functions $\eta(x_1) = \eta(x_2) = 0$; $\eta(x)$ has continuous derivatives upto order *p,*

$$|\eta^{(n)}(x)| < \varepsilon \qquad \text{where} \qquad n = 0, 1, ..., q; \qquad q \le p.$$

1.4 Euler's equation (MDU Rohtak 2005; Kanpur 2005; K.U. Kurukshetra 2004)

Let us examine the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y(x), y'(x))dx \qquad ...(1)$$

for an extreme value, the boundary points of the admissible curves being fixed [see fig. (*i*)]

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2 \qquad ...(2)$$

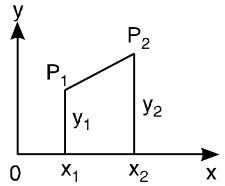


Fig. (*i*)

We assume that the given function $F(x, y, y')$ is twice differentiable with respect to any combination of its arguments.

Let $y = y(x)$ be the curve which extremizes the functional (1) such that $y(x)$ is twice differentiable and satisfies the boundary conditions (2). Assume that $y = \bar{y}(x)$ is an admissible curve close to $y = y(x)$ such that both $y(x)$ and $\bar{y}(x)$ are included in a one-parameter family of curves

$$y(x, \alpha) = y(x) + \alpha[\bar{y}(x) - y(x)]\,; \qquad ...(3)$$

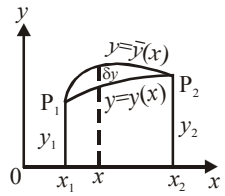


Fig. (*ii*)

for $\alpha = 0$ we get curve $y = y(x)$, for $\alpha = 1$ we get $y = \bar{y}(x)$ as shown in Fig. (*ii*).

Now, $$\bar{y}(x) - y(x) = \delta y \qquad ...(4)$$

where δy is known as the variation of the function $y(x)$

Then, (3) reduces to $$y(x, \alpha) = y(x) + \alpha\, \delta y \qquad ...(5)$$

The variation δy is a function of x. So from (4), we have

$$(\delta y)' = \bar{y}'(x) - y'(x) = \delta y'\,, \qquad ...(6)$$

showing that the derivative of the variation is equal to the variation of the derivative.

Now on the curves of the family (5), the functional (1) reduces to a function of α, say $\phi(\alpha)$, where

$$\phi(\alpha) = \int_{x_1}^{x_2} F(x, y(x, \alpha), y'(x, \alpha))\, dx \qquad ...(7)$$

A necessary condition for the extremum of the function $\phi(\alpha)$ for $\alpha = 0$ is, as is known from calculus, that its derivative for $\alpha = 0$ must vanish, *i.e.*,

$$\left(d\phi/d\alpha\right)_{\alpha=0} = 0 \qquad ...(8)$$

Using Leibnitz rule of differentiation under the integral sign, (7) gives

$$\frac{d\phi}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial F(x, y(x, \alpha), y'(x, \alpha))}{\partial \alpha} dx \qquad ...(9)$$

By the chain rule for differentiating functions of several variables, we get

$$\frac{\partial F(x, y(x, \alpha), y'(x, \alpha))}{\partial \alpha} = \frac{\partial F}{\partial x}\,\frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial y}\,\frac{\partial y(x, \alpha)}{\partial \alpha} + \frac{\partial F}{\partial y'}\,\frac{\partial y'(x, \alpha)}{\partial \alpha} \qquad ...(10)$$

From (5), $$y'(x, \alpha) = y'(x) + \alpha\, (\delta y)' = y'(x) + \alpha\, \delta y' \qquad ...(11)$$

From (5) and (11), $$\frac{\partial y(x, \alpha)}{\partial \alpha} = \delta y \quad \text{and} \quad \frac{\partial y'(x, \alpha)}{\partial \alpha} = \delta y' \qquad ...(12)$$

Using (12) and noting that $\partial x/\partial\alpha = 0$, (10) reduces to

$$\frac{\partial F(x, y(x,\alpha), y'(x,\alpha))}{\partial\alpha} = F_y(x, y(x,\alpha), y'(x,\alpha))\,\delta y + F_{y'}(x, y(x,\alpha), y'(x,\alpha))\,\delta y', \qquad ...(13)$$

where a subscript denotes partial derivative with respect to the indicated variable.

Using (13), (9) reduces to

$$\frac{d\phi}{d\alpha} = \int_{x_1}^{x_2} [F_y(x, y(x,\alpha), y'(x,\alpha))\ \delta y + F_{y'}(x, y(x,\alpha), y'(x,\alpha))\,\delta y']\,dx$$

$$\therefore \qquad \left(\frac{d\phi}{d\alpha}\right)_{\alpha=0} = \int_{x_1}^{x_2} [F_y(x, y(x), y'(x))\,\delta y + F_{y'}(x, y(x), y'(x))\,\delta y']\,dx \qquad ...(14)$$

We know that $(d\phi/d\alpha)_{\alpha=0}$ is known as the variation of the functional and is denoted by δI. A necessary condition for the extremum *I* is that its variation vanishes, *i.e.* $\delta I = 0$. Using (14), for the functional (1) the condition $\delta I = 0$ takes the form $\int_{x_1}^{x_2} (F_y\ \delta y + F_{y'}\ \delta y')\,dx = 0$

or $\int_{x_1}^{x_2} F_y\ \delta y\,dx + \int_{x_1}^{x_2} F_{y'}\,(\delta y)'\,dx = 0$, as $\delta y' = (\delta y)'$

or $$\int_{x_1}^{x_2} F_y\ \delta y\,dx + \Big[F_{y'}\,\delta y\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{dF_{y'}}{dx}\,\delta y\,dx = 0 \qquad ...(15)$$

[on integrating by parts the second term]

Since $y = y(x)$ and $y = \bar{y}(x)$ both pass through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we have

$$y(x_1) = \bar{y}(x_1) \quad \text{and} \quad y(x_2) = \bar{y}(x_2) \qquad ...(16)$$

Using (4) and (16), we have

$$\delta y\,\big|_{x=x_1} = \bar{y}(x_1) - y(x_1) = 0 \quad \text{and} \quad \delta y\,\big|_{x=x_2} = \bar{y}(x_2) - y(x_2) = 0$$

Hence (15) reduces to $$\int_{x_1}^{x_2} \left(F_y - \frac{d}{dx}F_{y'}\right)\delta y\,dx = 0 \qquad ...(17)$$

Now the first factor $F_y - \frac{d}{dx}F_{y'}$ on the extremizing curve $y = y(x)$ is a given continuous function, while the second factor δy, because of the arbitrary choice of the comparison curve $y = \bar{y}(x)$, is an arbitrary function, subject to the vanishing of δy at $x = x_1$ and $x = x_2$. Hence Equation (17) and the fundamental lemma of the calculus of variation (see Art. 1.3) imply that a necessary condition for the functional (1) to have an extreme value is that the extremizing function $y = y(x)$ satisfy the differential equation

$$F_y - \frac{d}{dx}F_{y'} = 0 \qquad \text{or} \qquad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \qquad x_1 \le x \le x_2 \qquad ...(18)$$

which is known as *Euler's equation* for the functional (1). Some authors call (18) as *Euler-Lagrange equation*.

1.4 A An alternative proof of Euler's equation.

Let us examine the functional $$I = \int_{x_1}^{x_2} F(x, y(x), y'(x))dx \qquad ...(1)$$

Suppose that F is a given function which is twice differentiable with respect to any combination of its arguments (*i.e.* x, y and y'). Again suppose that an admissible curve $y = y(x)$ exists which is twice differentiable on $[x_2, x_2]$, which satisfies the fixed boundary conditions

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2 \qquad ...(2)$$

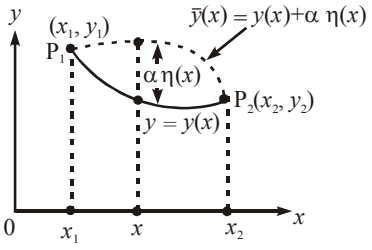
and which extremizes (*i.e.* minimizes or maximizes) the functional $I[y(x)]$.

Let $\eta(x)$ be any function with the properties that $\eta''(x)$ is continuous and

$$\eta(x_1) = \eta(x_2) = 0 \qquad ...(3)$$

If α is a small parameter, then $\bar{y}(x) = y(x) + \alpha\, \eta(x)$ ...(4)

represents a one-parameter family of curves. The vertical deviation of a curve in this family from the extremizing curve $y = y(x)$ is $\alpha\,\eta(x)$, as shown in the adjoining figure. The difference $\bar{y} - y = \alpha\, \eta(x)$ is known as the variation of the function y and is denoted by δy. This notation can be developed into a useful formalism and is the source of the name calculus of variation. From (4), we note that for each family of this type, that is for each choice of the function $\eta(x)$, the extremizing curve $y = y(x)$ belongs to the family and corresponds to the value of the parameter $\alpha = 0$.



From (4), $\bar{y}'(x) = y'(x) + \alpha\, \eta'(x)$ ...(5)

Now, with $\eta(x)$ fixed, replacing y and y' by $\bar{y}(x)$ and $\bar{y}'(x)$ respectively in the functional (1), we get a function of α,

$$I(\alpha) = \int_{x_1}^{x_2} F(x, y, \bar{y}')\, dx \qquad ...(6)$$

Since setting $\alpha = 0$ has the effect of replacing $\bar{y}(x)$ and $\bar{y}'(x)$ in equation (6) by the extremizing function $y(x)$ and its derivative $y'(x)$, $I(\alpha)$ must take on its extreme value when $\alpha = 0$. This happens no matter what particular value of function $\eta(x)$ is involved in (4) and (5). But, if $I(\alpha)$ has an extremum for $\alpha = 0$, then by elementary calculus, a necessary condition is given by

$$I'(\alpha) = 0 \qquad ...(7)$$

Using Letbnitz's rule of differentiation under the integral sign, (6) yields

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial x} F(x, \bar{y}, \bar{y}')\, dx \qquad ...(8)$$

By the chain rule for differentiating functions of several variables, we obtain

$$\frac{\partial}{\partial \alpha} F(x, \bar{y}, \bar{y}') = \frac{\partial F}{\partial x}\,\frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial \bar{y}}\,\frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial F}{\partial \bar{y}'}\,\frac{\partial \bar{y}'}{\partial \alpha} = 0 + \frac{\partial F}{\partial \bar{y}}\,\eta(x) + \frac{\partial F}{\partial \bar{y}'}\,\eta'(x), \text{ by (4) and (5)}$$

Hence (8) reduces to $$I'(\alpha) = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}}\,\eta(x) + \frac{\partial F}{\partial \bar{y}'}\,\eta'(x) \right) dx$$

which, upon setting $\alpha = 0$, and making use of (7), gives

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\,\eta(x) + \frac{\partial F}{\partial y'}\,\eta'(x) \right) dx = 0, \qquad ...(9)$$

where we have used the facts that when $\alpha = 0$, $\bar{y}(x) = y(x)$ and $\bar{y}'(x) = y'(x)$ in view of relations (4) and (5).

Re-writing (9) and integrating by parts the second term on the L.H.S of (9), we get

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y}\ \eta(x)\,dx + \left[\frac{\partial F}{\partial y'}\ \eta(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\ \eta(x)dx = 0$$

or
$$\int_{x_1}^{x_2} \eta(x)\ \left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}dx = 0\text{, using (3)} \qquad ...(10)$$

Except for the conditions (3) to be satisfied and the requirement of being twice differentiable on $[x_1, x_2]$, $\eta(x)$ is an arbitrary function. Hence, if F is any function for which $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ is continuous on $[x_1, x_2]$, equation (10) and the fundamental lemma of the calculus of variations (refer Art. 1.3) imply that a necessary condition for the functional (1) to have an extremum value is that the extrimizing function $y = y(x)$ must satisfy the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{or} \qquad F_y - \frac{d}{dx}F_{y'} = 0, \qquad x_1 < x < x_2 \qquad ...(1)$$

which is known as *Euler's equation*.

1.5 Alternative forms of Euler's Equation, namely,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \qquad ...(I)$$

Derivation of second form Euler's equation. Since F is a function of x, y, y', we have

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\ \frac{dy}{dx} + \frac{\partial F}{\partial y'}\ \frac{dy'}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\ y' + \frac{\partial F}{\partial y'}\ y'' \qquad ...(i)$$

Again
$$\frac{d}{dx}\left(y'\ \frac{\partial F}{\partial y'}\right) = y'\ \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}\ y'' \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we have

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\ \frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\ y' - y'\ \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

or
$$\frac{d}{dx}\left(F - y'\ \frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = y'\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\} = y' \times 0\text{, using (I)}$$

Thus,
$$\frac{d}{dx}\left(F - y'\ \frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0 \qquad ...(II)$$

which is an alternative second form of Euler's equation (I)

Derivation of third form of Euler's equation. Since $\partial F/\partial y'$ is a function of x, y and y', we have

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\ \frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\ \frac{dy'}{dx}$$

or
$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial^2 F}{\partial x\,\partial y'} + y'\,\frac{\partial^2 F}{\partial y\,\partial y'} + y''\,\frac{\partial^2 F}{\partial y'^2} \qquad ...(iii)$$

Using (*iii*), form (I) reduces to

$$\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial x\,\partial y'}-y'\frac{\partial^2 F}{\partial y\,\partial y'}-y''\frac{\partial^2 F}{\partial y'^2}=0 \qquad ...(III)$$

which is an alternative third form of *Euler's equation*.

1.5.A. Illustrative Solved Examples Based on Art. 1.5

Ex. 1. *Show that the Euler's equation for the functional* $\int_{x_1}^{x_2}\left\{a(x)y'^2+2b(x)yy'+c(x)y^2\right\}dx$ *is a second order linear differential equation.* **[MDU Rohtak 2004]**

Sol. Comparing the given functional with $\int_{x_1}^{x_2}F(x,y,y')dx$, we get

$$F(x, y, y') = a(x)\,y'^2 + 2b(x)\,y\,y' + c(x)\,y^2 \qquad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \qquad ...(2)$$

From (1), $\partial F/\partial y = 2\,b(x)\,y' + 2\,c(x)\,y$ and $\partial F/\partial y' = 2\,a(x)\,y' + 2\,b(x)\,y$

$\therefore$ (2) becomes $$2b(x)\,y' + 2c(x)\,y - \frac{d}{dx}\{2a(x)\,y' + 2b(x)\,y\} = 0$$

or $$2b(x)\,y' + 2c(x)\,y - \{2a'(x)\,y' + 2a(x)\,y'' + 2b'(x)\,y + 2b(x)\,y'\} = 0$$

or $$a(x)\,y'' + a'(x)\,y' - \{b'(x) - c(x)\}\,y = 0,$$

which is a second order linear differential equation

Ex.2. (*a*) *Find the extremal of the functional* $\int_a^b (y + y^3/3)dx$

(*b*) *Find the extremal of the functional* $\int_1^3 (3x-y)ydx$ *that satisfy the boundary conditions* $y(1) = 1$, $y(3) = 9/2$.

(*c*) *Find the extremals of the functional* $\int_0^1 (e^y + xy')dx$ *that satisfy the boundary conditions* $y(0) = 0$, $y(1) = 0$.

Sol. (*a*) Comparing the given functional with $\int_a^b F(x,y,y')dx$, here $F(x, y, y') = y + y^3/3$

$\therefore$ Euler's equation $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$ becomes $1 + y^2 - \frac{d}{dx}(0) = 0$ so that $y = \pm i$,

which is impossible. Hence the given functional has no extremal.

(*b*) Comparing the given functional with $\int_1^3 F(x,y,y')dx$, we get $F(x, y, y') = 3xy - y^2$. Hence the Euler's equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \quad \text{reduces to} \quad 3x - 2y\frac{d}{dx}(0) = 0 \quad \text{or} \quad 3x - 2y = 0, \qquad ...(1)$$

which is the only extremal of the given functional.

Using the given boundary condition $y(1) = 1$, *i.e.*, $y = 1$ when $x = 1$, (1) reduces to $1 = 0$, which in absurd. Hence the given problem has no solution.

(*c*) Comparing the given functional with $\int_0^1 F(x,y,y')dx$, we get $(x, y, y') = e^y + xy'$. Hence the Euler's equation

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ reduces to $e^y - \frac{d}{dx}(x) = 0$ or $e^y = 1$, *i.e.*, $y = 0$, ...(1)

which is the only extremal of the given functional. Further, (1) satisfies the given boundary conditions $y(0) = y(1) = 0$. Hence $y = 0$ is the required extremal.

Ex. 3. *Find the extremals of the functional* $\int_0^{\pi}(4y\cos x + y'^2 - y^2)dx$ *that satify the given boundary conditions* $y(0) = y(\pi) = 0$. **[KU Kurukshetra 2004]**

Sol. Comparing the given functional with $\int_0^{\pi} F(x, y, y')dx$, we get

$$F(x, y, y') = 4y\cos x + y'^2 - y^2 \quad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = 4\cos x - 2y$, $\frac{\partial F}{\partial y'} = 2y'$, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(2\frac{dy}{dx}\right) = 2y''$

$\therefore$ (2) reduces to $4\cos x - 2y - 2y'' = 0$ or $(D^2 + 1)y = 2\cos x$, $D \equiv d/dx$ ...(3)

The auxiliary equation of (3) is $D^2 + 1 = 0$ so that $D = \pm i$.

$\therefore$ C.F. of (3) = $C_1\cos x + C_2\sin x$, C_1 and C_2 being arbitrary constants

and P.I. of (3) = $\frac{1}{D^2+1}\,2\cos x = 2 \times \frac{x}{(2\times 1)} \times \sin x$, as $\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax$

$\therefore$ Solution of (3) is $y = C_1\cos x + C_2\sin x + x\sin x$ ...(4)

Using the given bourdary conditions $y(0) = y(\pi) = 0$, (4) gives $0 = C_1$ and $0 = -C_1$ so that $C_1 = 0$. Hence (4) becomes $y = C_2\sin x + x\sin x$ or $y = (C_2 + x)\sin x$, which is the required family of extremals, since C_2 is an arbitrary constant.

Ex. 4. (*a*) *Find the stationary function of the functional* $\int_0^1 (y^2 + y'^2)dx$ *subject to the boundary conditions* $y(0)=0$, $y(1) = 1$.

(*b*) *Find the extremal of the functional* $\int_0^{\pi/2}(y^2 - y'^2 - 8y\cosh x)dx$ *that satisfies the boundary conditions* $y(0) = 2$, $y(\pi/2) = 2\cosh(\pi/2)$.

Sol. (*a*) Comparing the given functional with $\int_0^1 F(x, y, y')dx$, we get

$$F(x, y, y') = y^2 + y'^2 \quad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = 2y$, $\frac{\partial F}{\partial y'} = 2y'$, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(2\frac{dy}{dx}\right) = 2y''$

$\therefore$ (2) reduces to $2y - 2y' = 0$ or $(D^2 - 1)y = 0$, $D \equiv d/dx$ ...(3)

The auxiliary equation of (3) is $D^2 - 1$ giving $D = \pm 1$.

$\therefore$ Solution of (3) is $y = C_1\cosh x + C_2\sinh x$, ... (4)

where C_1 and C_2 are arbitrary constants. Using the given boundary conditions $y(0) = 0$ and $y(1) = 1$ and noting that sinh 0 = 0 and cash 0 = 1, (4) gives

$0 = C_1$ and $1 = C_1\cosh 1 + C_2\sinh 1$ so that $C_1 = 0$ and $C_2 = 1/\sinh 1$

$\therefore$ (4) reduces to $y = (\sinh x)/(\sinh 1)$,
which is the desired equation of the stationary function.

(*b*) Comparing the given functional with $\int_0^{\pi/2} F(x, y, y')dx$, we get

$$F(x, y, y') = y^2 - y'^2 - 8y \cosh x. \quad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = 2y - 8\cosh x$, $\frac{\partial F}{\partial y'} = -2y'$, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(-2\frac{dy}{dx}\right) = -2y''$

$\therefore$ (2) reduces to $2y - 8\cosh x + 2y'' = 0$ or $(D^2 + 1)y = 4\cosh x$, $D \equiv d/dx$ ...(3)

The auxiliary equation of (3) is $D^2 + 1 = 0$ giving $D = \pm i$.

$\therefore$ C.F. of (3) = $C_1 \cos x + C_2 \sin x$, C_1 and C_2 being arbitrary constants

and P.I. of (3) $= \frac{1}{D^2+1} 4\cosh x = 4\frac{1}{D^2+1}\left(\frac{e^x + e^{-x}}{2}\right)$

$$= 2\frac{1}{D^2+1}e^x + 2\frac{1}{D^2+1}e^{-x} = 2\frac{1}{1+1}e^x + 2\frac{1}{1+1}e^{-x}$$
$$= e^x + e^{-x} = 2\cosh x, \text{ as } \cosh x = (e^x + e^{-x})/2$$

$\therefore$ Solution of (3) is $y = C_1 \cos x + C_2 \sin x + 2\cosh x$ ...(4)

Using the given boundary conditions $y(0) = 2$ and $y(\pi/2) = 2\cosh(\pi/2)$ and noting that $\cosh 0 = 1$, (4) gives $2 = C_1 + 2$ and $2\cosh(\pi/2) = C_2 + 2\cosh(\pi/2) \Rightarrow C_1 = C_2 = 0$.

$\therefore$ (4) reduces to $y = 2\cosh x$, which is the required extremal.

Ex. 5. *Find the extremals and the stationary function of the functional* $\int_0^e (xy'^2 + yy')dx$ satisfying $y(1) = 0$, $y(e) = 1$.

Sol. Comparing the given functional with $\int_0^e F(x, y, y')dx$, we get

$$F(x, y, y') = xy'^2 + yy' \quad ... (1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad ... (2)$$

From (1), $\partial F/\partial y = y'$ and $\partial F/\partial y = 2xy' + y$. Hence, (2) yields

$y' - d(2xy' + y)/dx = 0$ or $y' - (3y' + 2xy'') = 0$ or $xy'' + y' = 0$

or $x^2(d^2y/dx^2) + x(dy/dx) = 0$ or $(x^2D^2 + xD)y = 0$, ... (3)

where $D \equiv d/dx$. Let $x = e^z$ *i.e.*, $z = \log x$ and let $D_1 \equiv d/dz$. Then, we have $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$. Hence (3) reduces to

$\{D_1(D_1 - 1) + D_1\}y = 0$ or $D_1^2 = 0$ ... (4)

The auxiliary equation of (4) is $D_1^2 = 0$ giving $D_1 = 0, 0$.

$\therefore$ Solution of (4) is $y = (C_1 + C_2 z)e^{0.z} = C_1 + C_2 z = C_1 + C_2 \log x$, ... (5)

which are the extremals; C_1 and C_2 being arbitrary constants,

Using the given boundary conditions $y(1) = 0$, $y(e) = 1$ and noting that $\log 1 = 0$, $\log e = 1$, (5) gives $0 = C_1$ and $1 = C_1 + C_2$ so that $C_1 = 0$ and $C_2 = 1$. Hence, (5) yields

$y = \log x$, which is the required stationary function.

Ex. 6. (*a*) *From among the curves connecting the points* $P_1(1, 3)$ *and* $P_2(2, 5)$, *find the curve on which an extremum of the functional* $\int_1^2 y'(1+x^2y')dx$ *can be attained.*

(*b*) *By using Euler's equation, find the curves passing through (0, 3) and (4, 11) such that its length between the given points is shortest.*

Sol. (*a*) Comparing the given functional with $\int_1^2 F(x, y, y')dx$, we get

$$F(x, y, y') = y' + x^2y'^2 \qquad \ldots (1)$$

$\therefore$ Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ becomes $0 - \dfrac{d}{dx}(1 + 2x^2y') = 0$

or $d(1 + 2x^2y')/dx = 0$ giving $1 + 2x^2y' = C_1$

or $\dfrac{dy}{dx} = \dfrac{C_1 - 1}{2x^2}$ or $dy = \dfrac{C_1 - 1}{2}x^{-2}dx$

Integrating, $y = -(1/2) \times (C_1 - 1)x^{-1} + C_2$ or $y = C_2 + C_3/x$, ... (2)

where $C_3 = -(1/2) \times (C_1 - 1)$. Then, (2) is the equation of the extremals.

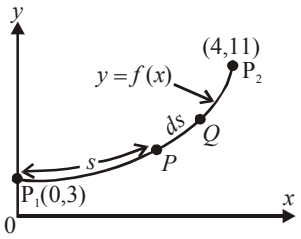
Since the desired extremal must pass through(1, 3) and (2, 5), therefore the required extremal should satisfy the boundary conditions : $y = 3$ when $x = 1$ and $y = 5$ when $x = 2$.

Using these boundary conditions, (2) gives

$3 = C_2 + C_3$ and $5 = C_2 + C_3/2$ so that $C_2 = 7$ and $C_3 = -4$.

$\therefore$ (3) yields $y = 7 - (4/x)$, which is the stationary function of the given problem. Thus, the given functional can have an extremum only on the curve $y = 7 - (4/x)$.

(*b*) Let $P_1(0, 3)$ and $P_2(4, 11)$ be two given points on xy – plane. Let $P(x,y)$ be any point on any curve joining P_1 and P_2 and let arc P_1P_2 = s. Then, we have



$$\text{arc } P_1P_2 = \int_0^4 ds = \int_0^4 (1 + y'^2)^{1/2}dx \qquad \ldots (1)$$

Let the length of the curve between the given points be minimum on the curve $y = f(x)$

Comparing the functional (1) with $\int_0^4 F(x, y, y')dx$, $F(x, y, y') = (1 + y'^2)^{1/2}$ ...(2)

$\therefore$ Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ becomes $0 - \dfrac{d}{dx}\dfrac{y'}{(1+y'^2)^{1/2}} = 0$

or $\dfrac{d}{dx}\dfrac{y'}{(1+y'^2)^{1/2}} = 0$ so that $\dfrac{y'}{(1+y'^2)^{1/2}} = C_1$

or $y'^2 = C_1(1 + y'^2)$ or $dy/dx = C_1/(1 - C_1^2)^{1/2} = C_2$ say

or $dy = C_2\,dx$ so that $y = C_2x + C_3$, ... (4)

which is the equation of the extremals, C_2, C_3 being arbitrary constants.

Since the given extremal must pass through (0, 3) and (4, 11), therefore the required extremal should satisfy the boundary conditions $y = 3$ when $x = 0$ and $y = 11$ when $x = 4$. Using these condition, (4) gives

$3 = C_3$ and $11 = 4C_2 + C_3$ so that $C_2 = 2$ and $C_3 = 3$

Hence, from (4), $y = 2x + 3$, which is a straight line passing through (0, 3) and (4, 11). Thus, the length between P_1 and P_2 is minimum along $y = 2x + 3$. Thus, the required curve is the straight line $y = 2x + 3$.

Ex. 6. (*c*) *Find the curve* $y = f(x)$ *which may correspond to the extreme value of the functional* $\int_{x_1}^{x_2} x^n (dy/dx)^2 dx$ *and passing through the fixed points* (x_1, y_1) *and* (x_2, y_2).

Sol. Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y')dx$, $F(x, y, y') = x^n y'^2$ ....(1)

$\therefore$ Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ becomes $0 - \dfrac{d}{dx}(2x^n y') = 0$

or $d(x^n y')/dx = 0$ so that $x^n y' = C_1$ or $dy/dx = C_1 x^{-n}$ or $dy = C_1 x^{-n} dx$

Integrating, $y = \{C_1/(1-n)\}x^{1-n} + C_2$, ... (2)

which is the equation of the extremals, C_1, C_2 being arbitrary constants. Since the required extremal must pass through (x_1, y_1) and (x_2, y_2), it follows that the desired extremal should satisfy the boundary conditions : $y = y_1$ when $x = x_1$ and $y = y_2$ when $x = x_2$. Hence, (2) gives

$$y_1 = \frac{C_1}{1-n} x_1^{1-n} + C_2 \quad \text{and} \quad y_2 = \frac{C_1}{1-n} x_2^{1-n} + C_2 \quad \text{... (3)}$$

From (3), $y_1 - y_2 = \dfrac{C_1(x_1^{1-n} - x_2^{1-n})}{1-n} \Rightarrow C_1 = \dfrac{(1-n)(y_1 - y_2)}{x_1^{1-n} - x_2^{1-n}}$ ... (4)

Putting the above value of C_1 in first equation of (3) yields

$$C_2 = y_1 - \frac{x_1^{1-n}(y_1 - y_2)}{x_1^{1-n} - x_2^{1-n}} \quad \text{or} \quad C_2 = \frac{y_2 x_1^{1-n} - y_1 x_2^{1-n}}{x_1^{1-n} - x_2^{1-n}} \quad \text{... (5)}$$

$\therefore$ From (2), (4) and (5), $y = \dfrac{(y_1 - y_2)x^{1-n}}{x_1^{1-n} - x_2^{1-n}} + \dfrac{y_2 x_1^{1-n} - y_1 x_2^{1-n}}{x_1^{1-n} - x_2^{1-n}}$ ... (6)

which is the equation of the stationary function of the given functional. Thus, the extreme value of the given functional correspond to the function given by (6) in the form $y = f(x)$.

Ex. 7. *On what curves can the functional* $\int_0^1 (y'^2 + 12xy)\,dx, \quad y(0) = 0, y(1) = 1$ *be extremized* ? **[Kanpur 2009**

Madras M.E. 1997, Punjab M.E. 1997; *V.T.U. 2004, CDLV 2004; KU Kurukshetra 2004]

Sol. Comparing the given functional with $\int_0^1 F(x, y, y')dx$, $F(x, y, y') = y'^2 + 12xy$ ...(1)

Euler's equation is $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ ...(2)

From (1), $\dfrac{\partial F}{\partial y} = 12x$, $\dfrac{\partial F}{\partial y'} = 2y'$ and $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = \dfrac{d(2y')}{dx} = 2y''$

With these values, (2) gives $12x - 2y'' = 0$ or $y'' = 6x$

Integrating it, $y' = 3x^2 + C_1$, C_1 being an arbitrary constant

Integrating it, $y = x^3 + C_1 x + C_2$, C_2 being an arbitrary constant ...(3)

---

*V.T.U. stand for Visveswaraiah Technological University, Belgaum

Using the given boundary conditions $y(0) = 0$, $y(1) = 1$, (3) gives

$0 = C_2$ and $1 = 1 + C_1 + C_2$ so that $C_1 = C_2 = 0$.

So, by (3), we find that an extremum can be achieved only on $y = x^3$.

Ex. 8(*a*) *On what curve the functional* $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx$ *with y(0) = 0 and* $y(\pi/2) = 0$ *be extremized* ? **(Purvanchal 2006)**

Sol. Comparing the given functional with $\int_0^{\pi/2} F(x, y, y')dx$, we get

$$F(x, y, y') = y'^2 - y^2 + 2xy \qquad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = -2y + 2x$, $\frac{\partial F}{\partial y'} = 2y'$ and $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}(2y') = 2y''$

With these values, (2) gives $-2y + 2x - 2y'' = 0$ or $y'' + y = x$

or $(D^2 + 1)y = x$, where $D \equiv d/dx$ ...(3)

Its auxiliary equation is $D^2 + 1 = 0$ so that $D = \pm i$

$\therefore$ $C.F. = C_1 \cos x + C_2 \sin x$, C_1, C_2 being arbitrary constants

$$\text{P.I. of (3)} = \frac{1}{D^2 + 1}\, x = (1 + D^2)^{-1} x = (1 - D^2 + ...)x = x$$

Hence the general solution of (3) is given by

$$y = C_1 \cos x + C_2 \sin x + x \qquad ...(4)$$

Using the given boundary conditions $y(0) = 0$, $y(\pi/2) = 0$, (4) gives

$0 = C_1$ and $0 = C_2 + \pi/2$ so that $C_1 = 0$ and $C_2 = -\pi/2$

So, by (4), we find that an extremum can be achieves only on the curve $y = x - (\pi/2) \times \sin x$.

Ex. 8(*b*) *Find the extremal for the functional*

$$I[y(x)] = \int_0^{\pi/2} (y^2 - y'^2 - 2y \sin x)dx,\ y(0) = 0, y(\pi/2) = 1$$

Sol. Comparing the given functional with $\int_0^{\pi/2} F(x, y, y')dx$, we have

$$F(x, y, y') = y^2 - y'^2 - 2y \sin x \qquad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = 2y - 2\sin x$, $\frac{\partial F}{\partial y'} = -2y'$ and $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}(-2y') = -2y''$.

With these values, (2) gives $2y - 2\sin x + 2y'' = 0$ or $y'' + y = \sin x$

$(D^2 + 1)y = \sin x$ where $D \equiv d/dx$ ...(3)

Auxiliary equating of (3) is $D^2 + 1 = 0$ so that $D = \pm i$

$\therefore$ C.F. $= C_1 \cos x + C_2 \sin x$, C_1 and C_2 being arbitrary constants.

P.I. of (3) $= \dfrac{1}{D^2+1}\sin x = -\dfrac{x}{2}\cos x$, using the result $\dfrac{1}{D^2+a^2}\sin ax = -\dfrac{x}{2a}\cos ax$

Hence the general solution of (3) is given by

$$y = C_1 \cos x + C_2 \sin x - (x/2) \times \cos x \qquad ...(4)$$

Using the given boundary conditions $y(0) = 0$ and $y(\pi/2) = 1$, (4) gives $C_1 = 0$ and $C_2 = 1$. Hence the required extremal is given by $y = \sin x - (x/2) \times \cos x$

It is the curve between two given fixed points (0, 0) and $(\pi/2, 1)$, for which the given functional will be extremum.

Ex. 9. *Test for an extremum of the functional* $\int_0^1 \left(xy + y^2 - 2y^2 y'\right)dx,\ y(0) = 1,\ y(1) = 2$

(Kanpur 2004; MDU Rohtak 2005)

Sol. Comparing the given functional with $\int_0^1 F(x, y, y')\,dx$, we get

$$F(x, y, y') = xy + y^2 - 2y^2 y' \qquad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\dfrac{\partial F}{\partial y} = x + 2y - 4y\,y'$, $\dfrac{\partial F}{\partial y'} = -2y^2$ and $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = -4y\,y'$

With these values, (2) gives $x + 2y - 4y\,y' - (-4y\,y') = 0$

or $x + 2y = 0$ or $y = -(x/2)$ ...(3)

Here the extremal (3) cannot satisfy the given boundary conditions $y(0) = 1$ and $y(1) = 2$. Hence an extremum cannot be achieved in the class of continuous functions.

An important note. In the above examples, Euler's equation was readily integrable. However, that is not always possible, since only in exceptional cases can second order differential equations be integrated in the closed form. In what follows, we shall consider some special cases, where Euler's equation admits of integration.

1.5B Case I. If F is independent of y' so that $\partial F/\partial y' = 0$ (Kanpur 2004)

Then the Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ reduces to $\dfrac{\partial F}{\partial y} = 0$, which is a finite equation, and not a differential equation. The solution of $\partial F/\partial y = 0$ does not contain any arbitrary constant and therefore, generally speaking, does not satisfy the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. Hence, in general, there does not exist a solution of this variational problem. Only in exceptional cases when the curve $\partial F/\partial y = 0$ passes through the boundary points (x_1, y_1) and (x_2, y_2) does there exist a curve on which an extremum can be attained.

Ex. 10. *Test for an extremum the functional* $I[y(x)] = \int_{x_1}^{x_2} y^2 dx,\ y(x_1) = y_1,\ y(x_2) = y_2$.

Sol. Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y')dx$, $F(x, y, y') = y^2$ ...(1)

The Euler's equation is $$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \qquad ...(2)$$

From (1), $\partial F/\partial y = 2y$ and $\partial F/\partial y' = 0$ so that (2) reduces to $y = 0$. The extremal $y = 0$ passes through the points only for $y_1 = 0$ and $y_2 = 0$. If $y_1 = 0$ and $y_2 = 0$, then, clearly, the function $y = 0$ minimizes the given functional $I[y(x)]$, since $I[y(x)] \geq 0$ and $I[y(x)] = 0$ for $y = 0$. However, if at least one of the y_1 and y_2 is not zero, then the functional is not minimized on continuous functions.

11.5C Case II The function F is linearly dependent on y' such that

$$F(x, y, y') = M(x, y) + N(x, y)\, y' \qquad \textbf{(Kanpur 2005)}$$

Then, $\dfrac{\partial F}{\partial y}=\dfrac{\partial M}{\partial y}+\dfrac{\partial N}{\partial y}y'$, $\dfrac{\partial F}{\partial y'}=N(x,y)$, $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)=\dfrac{dN(x,y)}{dx}$

Hence the Euler's equation $\dfrac{\partial F}{\partial y}-\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)=0$ becomes

$$\frac{\partial M}{\partial y}+\frac{\partial N}{\partial y}\,y'-\frac{dN(x,y)}{dx}=0 \quad \text{or} \quad \frac{\partial M}{\partial y}+\frac{\partial N}{\partial y}\,y'-\left(\frac{\partial N}{\partial x}+\frac{\partial N}{\partial y}\frac{dy}{dx}\right)=0$$

so that $$\partial M/\partial y-(\partial N/\partial x)=0\,, \qquad ...(1)$$

which is a finite equation, and not a differential equation. So the curve given by (1) does not, in general, satisfy the given boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. Hence the variational problem, in general, does not posses a solution in the class of continuous functions. In fact, when (1) holds, then the expression $Mdx + Ndy$ is an exact differential and the given functional

$$I[y(x)]=\int_{x_1}^{x_2}F(x,y,y')dx=\int_{x_1}^{x_2}(Mdx+Ndy)$$

is independent of the path of integration, the value of the functional $I[y(x)]$ is constant on admissible curves. In such cases, therefore, the variational problems becomes meaningless.

Ex. 11. *Test for an extremum the functional* $\int_0^1(y^2+x^2y')dx, \quad y(0)=0, \quad y(1)=a$

Sol. Comparing the given functional with $\int_0^1 F(x,y,y')dx\,, \quad F(x,y,y')=y^2+x^2y'$ ...(1)

Euler's equation is $$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \qquad ...(2)$$

From (1), $\dfrac{\partial F}{\partial y}=2y$, $\dfrac{\partial F}{\partial y'}=x^2$, $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)=\dfrac{dx^2}{dx}=2x$

With these values, (2) becomes $2y - 2x = 0$ or $y = x$ ...(3)

(3) satisfies the first boundary condition $y(0) = 0$. However, the second boundary condition $y(1) = a$ is satisfied by (3) only if $a = 1$. But if $a \neq 1$,, then there is no extremal that satisfy the given boundary conditions.

Ex. 12. *Test an extremum the functional* $I[y(x)]=\int_{x_1}^{x_2}(y+xy')dx$ or $I[y(x)]=\int_{x_1}^{x_2}(ydx+xdy)$, *subject to boundary conditions* $y(x_1) = y_1$ *and* $y(x_2) = y_2$.

Sol. Comparing the given functional with $\int_{x_1}^{x_2}F(x,y,y')dx\,, \quad F(x,y,y')=y+xy'$ ...(1)

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\partial F/\partial y = 1$, $\partial F/\partial y' = x$. Then (2) reduces to the identity $1 \equiv 1$. The integrand $ydx + xdy$ is an exact differential and so the integral does not depend on the path of integration. In fact, we have

$$I[y(x)] = \int_{x_1}^{x_2} (ydx + xdy) = \int_{x_1}^{x_2} d(xy) = \left[[xy]\right]_{x=x_1, y=y_1}^{x=x_2, y=y_2} = x_2 y_2 - x_1 y_1$$

This value of the functional $I[y(x)]$ is constant on admissible curves. Hence the given variational problem is meaningless.

1.5D Case III. The function F is independent of x and y, *i.e.* F is dependent only on y', so that $F = F(y')$. Euler's equation (refer third form in Art. 1.5) is given by

$$\partial F/\partial y - \partial^2 F/\partial x\,\partial y' - y'(\partial^2 F/\partial y\,\partial y') - y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(1)$$

Since $F = F(y')$, here $\partial F/\partial y = 0$ and $\partial F/\partial x = 0$

and hence $\partial^2 F/\partial x\,\partial y' = 0$ and $\partial^2 F/\partial y\,\partial y' = 0$

Hence (1) reduces to $y''(\partial^2 F/\partial y'^2) = 0$

Thus, we have either $y'' = 0$ or $\partial^2 F/\partial y'^2 = 0$

If $y'' = 0$, then integrating twice, we get $y = C_1 x + C_2$ ...(2)

which is a two parameter family of straight lines.

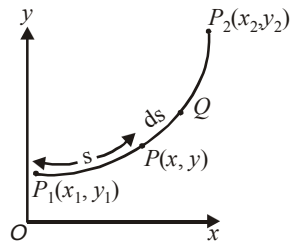
But if $\partial^2 F/\partial y'^2 = 0$ has one or several real roots $y' = K_n$, then $y = K_n\, x + C$ ...(3)

which is one parameter family of straight lines contained in the two parameter family (2). It follows, that in the present case III, extremals are all possible straight lines.

Ex. 13. *Show that the shortest distance between two fixed points in the Euclidean xy-plane is a straight line.* **(KU Kurukshetra 2005, GATE 2000, Kanpur 2004, Nagpur 2005; V.T.U. 2003) (Ravishankar 2010)**

or *Find the extremals of* $\int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx$ *and thus show that the shortest distance between two points in a plane is straight line.* **(Meerut 2008, 2010; Kanpur 2003)**

Sol. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two given points on a plane. Let $P(x, y)$ be any point on any curve joining P_1 and P_2 and let arc $P_1P = s$. Then arc length P_1P_2 is given by



$$\text{arc } P_1P_2 = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx \qquad ...(1)$$

where $y(x_1) = y$ and $y(x_2) = y_2$ ...(2)

Comparing (1) with $\int_{x_1}^{x_2} F(x, y, y')dx$, we have

$$F(x, y, y') = (1 + y'^2)^{1/2} \qquad ...(3)$$

Euler's equation is $$\partial F/\partial y - \partial^2 F/\partial x\,\partial y' - y'(\partial^2 F/\partial y\,\partial y') - y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(4)$$

From (3), $\partial F/\partial y = 0$, $\partial F/\partial x = 0$, so that $\partial^2 F/\partial x\,\partial y' = 0$ and $\partial^2 F/\partial y\,\partial y' = 0$

Hence (4) reduces to $$-y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(5)$$

Since $\partial^2 F/\partial y'^2 \neq 0$, (5) reduces to $y'' = 0$ or $d^2y/dx^2 = 0$

giving $dy/dx = C_1$ and $y = C_1x + C_2$, ...(6)

which is a straight line. Since (6) passes through (x_1, y_1) and (x_2, y_2), we get

$$y_1 = C_1x_1 + C_2 \quad ...(7)$$

and $$y_2 = C_1x_2 + C_2 \quad ...(8)$$

Subtracting (7) from (8), $y_2 - y_1 = C_1(x_2 - x_1)$ ...(9)

Subtracting (7) from (6), $y - y_1 = C_1(x - x_1)$ ...(10)

Dividing (10) by (9), we obtain

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \text{or} \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

which is a straight line joining the given two points P_1 and P_2. Hence the shortest distance between P_1 and P_2 in a plane is a straight line.

1.5E Case IV. *F* is independent of *y* so that *F* = *F*(*x*, *y*′). **(Bangalore 2005)**

Then $\dfrac{\partial F}{\partial y} = 0$ and so Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ reduces to $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

Its integration yields $\partial F/\partial y' = C$, where C is a constant. Since this relation is independent of y, it can be solved for y' as a function of x. Another integration leads to a solution involving two arbitrary constants which can be obtained by using the given boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Sometimes $\partial F/\partial y' = C$ can be solved by means of introducing a properly chosen parameter.

Ex. 14(*a*) *Prove that the extremal of* $\int_0^2 (y'^2/x)dx$ with $y(0) = 0$ and $y(2) = 1$ *is parabola.* **(Banglore 2004)**

(*b*) *Find the extremals of the functional* $\int_{x_1}^{x_2} (y'^2/x^3)dx$ **(V.T.U. 2003)**

Sol. (*a*) Comparing the given functional with $\int_0^2 F(x, y, y')dx$, $F(x, y, y') = y'^2/x$ ...(1)

Euler's equation is $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ ...(2)

From (1), $\partial F/\partial y = 0$ and $\partial F/\partial y' = 2y'/x$ ...(3)

Since $\dfrac{\partial F}{\partial y} = 0$, (2) reduces to $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

Integrating it, $\partial F/\partial y' = C_1$ or $2y'/x = C_1$, by (3)

or $dy/dx = (C_1/2) \times x$ or $dy = (C_1/2) \times x\,dx$

Integrating, $y = (C_1/4)x^2 + C_2$, C_1 and C_2 being arbitrary constants ...(4)

Using the given boundary conditions $y(0) = 0$, $y(2) = 1$, (4) gives

$0 = C_2$ and $1 = C_1 + C_2$ so that $C_1 = 1$, $C_2 = 0$

Hence by (4), an extremum can be attained only on the curve $y = x^2/4$, *i.e.* $x^2 = 4y$, which is a parabola.

(*b*) Proceed as in part (*a*). **Ans.** $y = C_1x^4 + C_2$ which is a two-parameter family of curves.

Ex. 15(a) *Find the extremal of the following functionals*

(*a*) $\int_1^2 (x^3 / y'^2)dx$ *with* $y(1) = 0$ *and* $y(2) = 3$. **(Kanpur 2011)**

(*b*) $\int_0^4 \{(y')^{-2} / x^{-3}\}\, dx, \quad y(0) = 0, \quad y(4) = 4$. **(MDU Rohtak 2006)**

Sol. (a) Comparing the given functional with $\int_1^2 F(x, y, y')dx$, $F(x, y, y') = x^3 \big/ y'^2$ ...(1)

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\partial F/\partial y = 0$ and $\partial F/\partial y' = -2x^3 \big/ y'^3$ ...(3)

Since $\dfrac{\partial F}{\partial y} = 0$, (2) reduces to $$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Integrating it, $\partial F/\partial y = -2C_1^{-3}$, C_1 being an arbitrary constant

$\therefore$ Using (3), we get $-2x^3 \big/ y'^3 = -2C_1^{-3}$ so that $y' = C_1 x$,

Integrating it, $y = (C_1/2) \times x^2 + C_2$, C_2 is an arbitrary constant ...(4)

Using the given boundary conditions $y(1) = 0$, $y(2) = 3$, (4) yields

$0 = C_1/2 + C_2$ and $3 = 2C_1 + C_2$ so that $C_1 = 2$, $C_2 = -1$

So, by (4), we see that required extremal is $y = x^2 - 1$.

(*b*) Try as in part (*a*). **Ans.** $y = x^2/4$

Ex. 16(*a*) *Find the extremum of the functional* $t[y(x)] = \int_{x_1}^{x_2} \{(1/x) \times (1 + y'^2)^{1/2}\}\, dx$

(*b*) *Show that the curve through* (1, 0) *and* (2, 1) *which minimizes* $\int_1^2 \{(1 + y'^2)/x^2\}^{1/2}\, dx$ *is a circle.* **(Madurai M.E. 2000, Bangalore 2004, Bhopal 2004, MDU Rohtak 2004, Purvanchal 2007, KU Kurukshetra 2004, Ravishankar 2004)**

(*c*) *Find the path on which a particle will take minimum time in sliding from a point* $P_1(x_1, y_1)$ *to another point* $P_2(x_2, y_2)$ *if its rate of motion* $v\,(= dx/dt)$ *is equal to x.*

[It is known as the *problem of Brachistochrone* (shortest time) in horizontal plane]

Sol. (*a*) Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y')\, dx$, we get

$$F(x, y, y') = (1/x) \times (1 + y'^2)^{1/2} \qquad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\partial F/\partial y = 0$ and $\partial F/\partial y' = (y'/x) \times (1 + y'^2)^{-1/2}$ ...(3)

Since $\partial F/\partial y = 0$, (2) reduces to $$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Integrating it, $\partial F / \partial y' = C$ or $(y'/x) \times (1 + y'^2)^{-1/2} = C$ by (3)

Thus, $y' = Cx(1 + y'^2)^{1/2}$ ...(4)

(4) will solved by introducing a parameter as follows $dy/dx = y' = \tan t$ ...(5)

Then, (4) yields $\tan t = C\,x(1 + \tan^2 t)^{1/2}$ or $\tan t = C\,x \sec t$

so that $x = C_1 \sin t$, where $C_1 = 1/C$ ...(6)

From (5), $dy = \tan t\, dx = C_1 \tan t \cos t\, dt = C_1 \sin t\, dt$, as (6) $\Rightarrow dx = C_1 \cos t\, dt$

Integrating, $y = -C_1 \cos t + C_2$, C_1, C_2 being arbitrary constants

Thus, $x = C_1 \sin t$ and $y - C_2 = -C_1 \cos t$ ...(7)

From (7), $x^2 + (y - C_2)^2 = C_1^2$ ...(8)

which is a family of circles with centre on y-axis.

(*b*) Proceed exactly as in part (*a*) upto (8). In the present problem, using the boundary conditions $x = 1$, $y = 0$ and $x = 2$, $y = 1$, (8) yields

$1 + C_2^2 = C_1^2$ and $4 + (1 - C_2)^2 = C_1^2$ giving $C_1 = \sqrt{5}$, $C_2 = 2$

Hence from (8) the required curve is the circle $x^2 + (y - 2)^2 = 5$

(*c*) From calculus, we know that $ds = (1 + y'^2)^{1/2} dx$ ...(1)

Given that $ds/dt = v = x$ and hence $ds = x\, dt$ ...(2)

From (1) and (2), $x\,dt = (1 + y'^2)^{1/2} dx$

or $dt = (1/x) \times (1 + y'^2)^{1/2} dx$ so that $t[y(x)] = \int_{x_1}^{x_2} \{(1/x) \times (1 + y'^2)^{1/2}\} dx$

where $t[y(x)]$ is the time taken in moving from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ on the curve $y = y(x)$.

Now proceed as in part (*a*).

1.5F. Case V. *F* is dependent on *y* and *y′* alone so that *F* = *F*(*y*, *y′*), *i.e.*, *F* is independent of *x* so that $\partial F/\partial x = 0$. (Meerut 2010; Kanpur 2003, 04, Nagpur 2005, 10)

Here Euler's equation (refer second form in Art. 1.5) is given by

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$$

Since $\partial F/\partial x = 0$, we have $$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

Integrating it, $F - y'(\partial F/\partial y') = C_1$, ...(1)

where C_1 is a constant. Since first order equation (1) does not contain x explicitly, it may be integrated by solving for y' and separating the variables, or by introducing a parameter.

Note : In some cases Euler's equation in its first form (see Art. 1.5) can also be used.

Ex. 17. *Obtain the Euler's equation for the extremals of the functional* $\int_{x_1}^{x_2} (y^2 - yy' + y'^2) dx$

Sol. Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y') dx$, we get

$$F(x, y, y') = y^2 - yy' + y'^2 \quad \text{...(1)}$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{...(2)}$$

From (1), $\frac{\partial F}{\partial y} = 2y - y'$, $\frac{\partial F}{\partial y'} = -y + 2y'$ and $\frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = -y' + 2y''$

Using these values, the required Euler's equation (2) becomes

$$2y - y' - (-y' + 2y'') = 0 \quad \text{or} \quad y'' - y = 0$$

Ex. 18(*a*) Test for an extremal the functional $I[y(x)] = \int_0^{\pi/2} (y'^2 - y^2)dx,\ y(0) = 0,\ y(\pi/2) = 1$ **(Kanpur 2003)**

Sol. Comparing the given functional with $\int_0^{\pi/2} F(x, y, y')\,dx$, $F(x, y, y') = y'^2 - y^2$ ...(1)

Euler's equation is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ ...(2)

From (1), $\frac{\partial F}{\partial y} = -2y$, $\frac{\partial F}{\partial y'} = 2y'$ and $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$

Using these values, (2) reduces to

$$-2y - (2y'') = 0 \quad \text{or} \quad (D^2 + 1)y = 0 \quad \text{where} \quad D \equiv d/dx \qquad ...(3)$$

The general solution of (3) is $y = C_1 \cos x + C_2 \sin x$ ...(4)

Using the given boundary conditions $y(0) = 0, y(\pi/2) = 1$, we get $C_1 = 0$ and $C_2 = 1$. Hence, from (4), an extremum can be attained only on the curve $y = \sin x$.

Ex. 18(*b*) *Find the extremal of the functional* $I[y(x)] = \int_0^1 \frac{1 + y^2}{y'} dx$, *through the origin and the point* (1, 1). **(Kanpur 2011)**

Sol. Comparing the given functional with $\int_0^1 F(x, y, y')dx$, we have

$$F(x, y, y') = (1 + y^2)/y' \qquad ...(1)$$

which is independent of x. Hence the Euler's equation yields (refer equation (1) of case V)

$$F - y'(\partial F/\partial y') = C \qquad ...(2)$$

where C_1 is an arbitrary constant. From (1), we have

$$\partial F/\partial y' = -(1 + y^2)/y'^2 \qquad ...(3)$$

Using (1) and (3), (2) reduced to

$$\frac{1 + y^2}{y'} + \frac{1 + y^2}{y'} = C_1 \quad \text{or} \quad 2(1 + y^2)\frac{dy}{dx} = C_1 \quad \text{or} \quad \frac{dy}{1 + y^2} = \frac{2}{C_1}dx$$

so that $\tan^{-1} y = (2/C_1)x + C_2$, C_1 and C_2 being arbitrary constants ...(4)

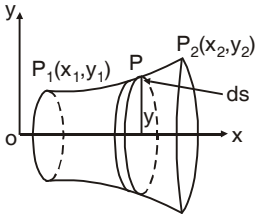
Using the given boundary conditions $y = 0$ when $x = 0$ and $y = 1$ when $x = 1$, (4) yields $C_2 = 0$ and $\pi/4 = 2/C_1$. Hence the desired extremal (*i.e.* the curve between the two fixed points (0, 0) and (1, 1) for which the functional is extremum) is given by

$$\tan^{-1} y = \pi x/4 \quad \textit{i.e.} \quad y = \tan(\pi x/4)$$

Ex. 19(*a*) *Find the curve passing through the points* (x_1, y_1) *and* (x_2, y_2) *which when rotated about the x-axis gives a minimum surface area.* **(Bilaspur 2004, Kanpur 2006, 10, 11)**

(*b*) *Prove that the extremal of* $\int_{x_1}^{x_2} y(1 + y'^2)^{1/2} dx$ *is the catenary* $y = a \cosh(ax + b)$. **(Meerut 2005, 10; Bangloare 2004)**

Sol. (*a*) When a curve joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is revolved about x-axis, the area of the surface of revolution is given by



$$S[y(x)] = 2\pi\int_{x_1}^{x_2} y(1+y'^2)^{1/2}\,dx$$

Comparing this with $\int_{x_1}^{x_2} F(x, y, y')dx$ and omitting the irrelevant factor 2π, we have $F(x, y, y') = y(1+y'^2)^{1/2}$

Since $F(x, y, y')$ is a function of y and y' only, a first integral of Euler's equation is given by (refer case V, Art. 1.5*F*)

$$F - y'(\partial F/\partial y') = \text{constant} = C$$

or $$y(1+y'^2)^{1/2} - y'\times(y/2)\times(1+y'^2)^{-1/2}\times 2y' = C$$

or $y(1+y'^2) - yy'^2 = C\,(1+y'^2)^{1/2}$ or $y = C(1+y'^2)^{1/2}$

or $y'^2 = \dfrac{y^2 - C^2}{C^2}$ or $\dfrac{dy}{dx} = \dfrac{(y^2 - C^2)^{1/2}}{C}$

Separing variables, $\dfrac{dy}{(y^2 - C^2)^{1/2}} = \dfrac{dx}{C}$

Integrating, $\cosh^{-1}(y/C) = x/C + b/C$ or $y = C\cosh\{(x+b)/C\}$ ...(1)

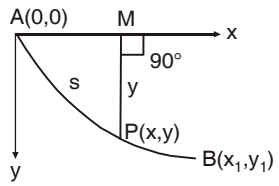
where b and C are two arbitrary constants. (1) gives a two parameter family of catenaries. The constants b and C can be obtained from the given boundary conditions, namely that $y(x_1) = y_1$ and $y(x_2) = y_2$ depending on the positions of the points P_1 and P_2. There may be one, two or no solution.

(*b*) Proceed as in part (*a*)

Note : The above problem is also important in connection with soap films which are known to have shapes with minimum surface areas.

Ex. 20 (The problem of the brachistochrone) *Find the curve connecting given points A and B which is traversed by a particle sliding from A to B in the shortest time (friction and resistance of the medium are ignored).* **(Meerut 2006, 10; Kanpur 2003, 2010, 2011)**

Sol. Let A be taken as origin and let us measure y downward as shown in the figure. Let coordinates of B be (x_1, y_1). We wish to minimize the time of descent from A to B. At any time t, let the particle be at $P(x, y)$ such that arc $OP = s$. Using the principle of conservation of energy, we have



Kinetic energy at P – kinetic energy. at A

= work done in moving the particle from A to P.

or $(1/2)\times mv^2 - 0 = mgy$ or $(ds/dt)^2 = 2gy$ or $ds/dt = \sqrt{2gy}$

Hence the time spent in moving from $A(0, 0)$ to $B(x_1, y_1)$ is given by

$$t[y(x)] = \int_A^B \frac{ds}{(2gy)^{1/2}} = \int_0^{x_1} \frac{(1+y'^2)^{1/2}}{(2gy)^{1/2}}\,dx$$

Comparing this with $\int_0^{x_1} F(x, y, y')\,dx$ and omitting the irrelevant factor $1/\sqrt{2g}$, we have

$$F(x, y, y') = (1 + y'^2)^{1/2} / \sqrt{y} \qquad ...(1)$$

Since $F(x, y, y')$ is independent of x, a first integral of Euler's equation is (refer case V, Art 1.5*F*)

$$F - y'(\partial F/\partial y') = \text{constant} = C$$

or $$\frac{1}{\sqrt{y}} \times (1 + y'^2)^{1/2} - y' \times \frac{1}{2\sqrt{y}} (1 + y'^2)^{-1/2} \times 2y' = C \text{, using (1)}$$

or $$1 + (y')^2 - (y')^2 = C\sqrt{y}(1 + y'^2)^{1/2} \quad \text{or} \quad y(1 + y'^2) = C_1 \qquad ...(2)$$

where $C_1 = 1/C^2$. Introduce the parameter t, by putting

$$y' = dy/dx = \cot t \qquad ...(3)$$

Then, from (2), $$y = \frac{C_1}{1 + \cot^2 t} = C_1 \sin^2 t = (C_1/2) \times (1 - \cos 2t)$$

From (3), $$dx = \frac{dy}{y'} = \frac{2C_1 \sin t \cos t\ dt}{\cot t} = 2C_1 \sin^2 t\ dt = C_1(1 - \cos 2t)dt$$

Integrating, $$x = C_1\left(t - \frac{\sin 2t}{2}\right) + C_2 = \frac{C_1}{2}(2t - \sin 2t) + C_2$$

Hence the equation of the given curve in parametric form is given by

$$x - C_2 = (C_1/2) \times (2t - \sin 2t), \qquad y = (C_1/2) \times (1 - \cos 2t) \qquad ...(4)$$

Taking $C_1/2 = a$ and $2t = \phi$, (4) reduces to

$$x - C_2 = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi) \qquad ...(5)$$

Now, at A (0, 0), where $x = 0$ and $y = 0$, (5) yields $\phi = 0$ and so $C_2 = 0$. Hence, from (5), we get the equation of a family of cycloids in standard parametric form

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi) \qquad ...(6)$$

where a is the radius of a rolling circle. (5) has been found from the condition of passage of the cycloid through the point $B(x_1, y_1)$. Thus, the brachistochrone is a cycloid.

Ex. 21. *If* (r_1, θ_1) *and* (r_2, θ_2) *are two points in a plane then the length of the arc of the curve joining these points is given by* $I[r(\theta)] = \int_{\theta_1}^{\theta_2} (r^2 + r'^2)^{1/2}\,d\theta$, *where* $r' = dr/d\theta$

By minimizing the above integral obtain the equation of a straight line in polar coordinates.

(Nagpur 2010; KU Kurukshetra 2006)

[Note that here, we have variables r and θ *in place of the usual variables x and y]*

Sol. Comparing the given functional with $\int_{\theta_1}^{\theta_2} F(\theta, r, r')d\theta$, we get

$$F(\theta, r, r') = (r^2 + r'^2)^{1/2} \qquad ...(1)$$

$\therefore$ Euler's equation $\frac{\partial F}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) = 0$ reduces to $$\frac{r}{(r^2 + r'^2)^{1/2}} - \frac{d}{d\theta}\frac{r'}{(r^2 + r'^2)^{1/2}} = 0$$

or $$\frac{r}{(r^2+r'^2)^{1/2}}-\frac{r''(r^2+r'^2)^{1/2}-r'\times(1/2)\times(r^2+r'^2)^{-1/2}(2rr'+2r'r'')}{(r^2+r'^2)}=0$$

or $$\frac{r}{(r^2+r'^2)^{1/2}}-\frac{r''(r^2+r'^2)-r'(rr'+r'r'')}{(r^2+r'^2)\ (r^2+r'^2)^{1/2}}=0$$

or $$r(r^2+r'^2)-r''(r^2+r'^2)+r'(rr'+r'r'')=0$$

or $$r^3+2rr'^2-r^2r''=0 \quad \text{or} \quad rr''-2r'-r^2=0 \qquad ...(2)$$

Let $$p=r'=dr/d\theta \qquad ...(3)$$

Then, $$r''=\frac{d^2r}{d\theta^2}=\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)=\frac{dp}{d\theta}=\frac{dp}{dr}\ \ \frac{dr}{d\theta}=p\ \ \frac{dp}{d\theta} \qquad ...(4)$$

Using (3) and (4), (2) reduces to

$$r\,p\,(dp/dr)-2p^2-r^2=0 \quad \text{or} \quad dp/dr=2(p/r)+(r/p) \qquad ...(5)$$

Let $p/r=v$ *i.e.*, $p=rv$ so that $dp/dr=v+r\,(dv/dr)$

Then, (5) yields $v+r\,(dv/dr)=2v+1/v$ or $r\,(dv/dr)=v+1/v$

or $$\frac{2dr}{r}=\frac{2vdv}{v^2+1} \quad \text{so that} \quad 2\log r-\log a^2=\log(v^2+1)$$

or $$v^2+1=r^2/a^2 \quad \text{or} \quad v^2=(r^2-a^2)/a^2 \quad \text{or} \quad (p/r)^2=(r^2-a^2)/a^2$$

or $$p=r(r^2-a^2)/2 \quad \text{or} \quad dr/d\theta=r\,(r^2-a^2)/a$$

or $$d\theta=a\frac{dr}{r(r^2-a^2)^{1/2}} \quad \text{so that} \quad \theta-\alpha=\sec^{-1}(r/a)$$

or $$\sec(\theta-\alpha)=r/a \quad \text{or} \quad r\cos(\theta-\alpha)=a\,, \qquad ...(6)$$

where a and α are arbitrary constants. (6) is the polar equation of a straight line. Thus, the minimum distance between two points in a plane is a straight line given by (6).

1.6 Invariance of Euler's equation under co-ordinate transformation (Kanpur 2003, 04)

When a given functional $$I[y(x)]=\int_{x_1}^{x_2}F(x,y,y')dx \qquad ...(1)$$

is transformed by replacing the independent variable or by a simultaneous replacement of the desired function and the independent variable, then the extremals of the functional can be obtained by using the Euler's equation for the transformed integrand. Here in lies the so called the "*principle of invariance of Euler equation*"

Let $x=x(u,v)$ and $y=y(u,v)$ ...(1)

such that $$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\neq 0 \qquad ...(2)$$

where $x_u=\partial x/\partial u$, $x_v=\partial x/\partial v$, $y_u=\partial y/\partial u$ and $y_v=\partial y/\partial v$

Again, let $v'=dv/du$ then, we have

$dx=(\partial x/\partial u)du+(\partial x/\partial v)dv=x_u du+x_v dv$ and $dy=(\partial y/\partial u)du+(\partial y/\partial v)dv=y_u du+y_v dv$

$\therefore \quad dx = \{x_u + x_v(dv/du)\}du = (x_u + x_v v')du$ and $dy = \{y_u + y_v(dv/du)\}du = (y_u + y_v v')du$ ...(3)

Thus, $y' = dy/dx = (y_u + y_v v')/(x_u + x_v v')$ ...(4)

Using (1), (3) and (4), we obtain

$$\therefore \quad \int_{x_1}^{x_2} F(x, y, y')dx = \int_{u_1}^{u_2} F\left\{x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right\}(x_u + x_v v')du \qquad ...(5)$$

where we have assumed that $u = u_1$ when $x = x_1$ and $u = u_2$ when $x = x_2$

Thus, (5) takes the form $\int_{x_1}^{x_2} F(x, y, y')dx = \int_{u_1}^{u_2} G(u, v, v')\,du$ ...(6)

Hence the given functional (1) takes the new form $I[v(u)] = \int_{u_1}^{u_2} G(u, v, v')du$ ...(7)

and hence the extremals of the original functional (1) can be obtained from Euler's equation for (7), namely,

$$\frac{dG}{\partial v} - \frac{d}{du}\left(\frac{\partial G}{\partial v'}\right) = 0 \qquad ...(8)$$

Note : While solving problems based on Art. 1.6, the change of variables can be made directly in the integral representing the given functional. We then write and solve the Euler's equation for the new functional containing new variables. In the extremals so obtained, the new variables are then replaced by the original variables in order to obtain the required extremals.

1.6A Solved examples based on Art. 1.6

Ex. 1. *Find the extremals of the functional* $\int_{\theta_1}^{\theta_2} (r^2 + r'^2)^{1/2}\, d\theta$ where $r = r(\theta)$ and $r' = dr/d\theta$.

(K.U. Kurukshetra 2004, 06)

***Sol.** Let $I[r(\theta)] = \int_{\theta_1}^{\theta_2} (r^2 + r'^2)^{1/2}\, d\theta$ ... (1)

We know that $x = r\cos\theta$ and $y = r\sin\theta$ ... (2)

(2) $\Rightarrow$ $dx/d\theta = -r\sin\theta + r'\cos\theta$ and $dy/d\theta = r\cos\theta + r'\sin\theta$

$\therefore \quad (dx/d\theta)^2 + (dy/d\theta)^2 = r^2 + r'^2$ ... (3)

Also, let $\theta = \theta_1 \Rightarrow x = x_1$ and $\theta = \theta_2 \Rightarrow x = x_2$ ... (4)

$$\therefore \quad \int_{\theta_1}^{\theta_2} (r^2 + r'^2)^{1/2} d\theta = \int_{\theta_1}^{\theta_2} \left\{(dx/d\theta)^2 + (dy/d\theta)^2\right\}^{1/2} d\theta \text{, using (3)}$$

$$= \int_{x_1}^{x_2} \left\{(dx)^2 + (dy)^2\right\}^{1/2} = \int_{x_1}^{x_2} (1 + y'^2)^{1/2} dx \text{, using (4)}$$

Let the transformed functional be denoted by $J[y(x)] = \int_{x_1}^{x_2} (1 + y'^2)^{1/2} dx$ ... (5)

Comparing (5) with functional $\int_{x_1}^{x_2} F(x, y, y')dx$, here $F(x, y, y') = (1 + y'^2)^{1/2}$ ... (6)

---

* If we compare (1) with I $[r(\theta)] = \int_{\theta_1}^{\theta_2} G(\theta, r, r')d\theta$, we have

$G(\theta, r, r'^1 = (r^2 + r'^2)^{1/2}$. Then, the Euler's equation

$\frac{\partial G}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial G}{\partial r'}\right) = 0$ becomes $\frac{r}{(r^2 + r'^2)^{1/2}} - \frac{d}{d\theta}\left(\frac{r'}{(r^2 + r'^2)^{1/2}}\right) = 0,$

which cannot be easily solved. Hence, we have used the technique of Art. 1.6 to solve the given problem

$\therefore$ Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ becomes $0 - \dfrac{d}{dx}\left(\dfrac{y'}{(1+y'^2)^{1/2}}\right) = 0$

or $y'/(1+y'^2)^{1/2} = C_1$ giving $y' = C_1/(1 - C_1^2)^{1/2} = C_2$, say

or $dy/dx = C_2$ so that $y = C_2 x + C_3$ ... (7)

From (2) and (7), the required extremals of the given functional (1) are given by

$$r \sin \theta = C_1 r \cos \theta + C_2.$$

Ex. 2. *Find the extremals of the functional* $\int_{\theta_1}^{\theta_2} r\sin\theta(r^2 + r'^2)^{1/2} d\theta$ *by using the transformations* $x = r\cos\theta$, $y = r\sin\theta$. **(Nagpur 2005)**

Sol. Let $I\,[\,r(\theta)] = \int_{\theta_1}^{\theta_2} r\sin\theta(r^2 + r'^2)^{1/2} d\theta$ ... (1)

Given $x = r\cos\theta$ and $y = r\sin\theta$ ... (2)

(2) $\Rightarrow$ $dx/d\theta = -r\sin\theta + r'\cos\theta$ and $dy/d\theta = r\cos\theta + r'\sin\theta$

$\therefore$ $(dx/d\theta)^2 + (dy/d\theta)^2 = r^2 + r'^2$ ... (3)

Also, let $\theta = \theta_1 \Rightarrow x = x_1$ and $\theta = \theta_2 \Rightarrow x = x_2$ ... (4)

$\therefore$ $\int_{\theta_1}^{\theta_2} r\sin\theta(r^2 + r'^2)^{1/2} d\theta = \int_{x_1}^{x_2} y\left\{(dx/d\theta)^2 + (dy/d\theta)^2\right\}^{1/2} d\theta$, by (3)

$= \int_{x_1}^{x_2} y\left\{(dx)^2 + (dy)^2\right\}^{1/2} = \int_{x_1}^{x_2} y(1+y'^2)^{1/2} dx$, by (4)

Let the transformed functional be denoted by $J[y(x)]$.

$\therefore$ $J[y(x)] = \int_{x_1}^{x_2} y(1+y'^2)^{1/2} dx$ ... (5)

Comparing (5) with the functional $\int_{x_1}^{x_2} F(x, y, y')dx$, $F(x, y, y') = y(1+y'^2)^{1/2}$ ... (6)

$\therefore$ Euler' equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ gives $(1+y'^2)^{1/2} - \dfrac{d}{dx}\left(\dfrac{yy'}{(1+y'^2)^{1/2}}\right) = 0$

or $2\left(\dfrac{yy'}{(1+y'^2)^{1/2}}\right)\dfrac{d}{dx}\left(\dfrac{yy'}{(1+y'^2)^{1/2}}\right) = 2yy'$ or $\dfrac{d}{dx}\left(\dfrac{yy'}{(1+y'^2)^{1/2}}\right)^2 = \dfrac{dy^2}{dx}$

Integrating, $(y^2y'^2)/(1+y'^2) = y^2 - c_1^2$ or $y^2/(1+y'^2) = c_1^2$

or $1 + y'^2 = \dfrac{y^2}{c_1^2}$ or $\dfrac{dy}{dx} = \left(\dfrac{y^2 - c_1^2}{c_1^2}\right)^{1/2}$ or $\dfrac{dy}{(y^2 - c_1^2)^{1/2}} = \dfrac{dx}{c_1}$.

Integrating, $\log\{y + (y^2 - c_1^2)^{1/2}\} = x/c_1 + c_2$, c_1 and c_2 being arbitrary constants

Putting $x = r\cos\theta$, $y = r\sin\theta$, the extremals of the given functional (1) are given

$$\log\{r\sin\theta + (r^2\sin^2\theta - c_1^2)^{1/2}\} = c_2 + (r\cos\theta)/c_1.$$

Ex. 3. *Find the extremals of the functional* $I[y(x)] = \int_0^{\ln 2} (e^{-x}y'^2 - e^x y^2)dx$, where $\ln 2 = \log_e 2$

Sol. Let u and v be two new variables such that

$x = \ln u = \log_e u$ and $y = v$ ...(1)

Then, $x = \ln 2 \Rightarrow \log_e u = \log_e 2 \Rightarrow u = 2$

and $x = 0 \Rightarrow \log_e u = 0 \Rightarrow u = e^0 = 1$

From (1), $x_u = \partial x/\partial u = 1/u$, $x_v = \partial x/\partial v = 0$, $y_u = \partial y/\partial u = 0$ and $y_v = \partial y/\partial v = 1$

Hence, $y' = \dfrac{y_u + y_v v'}{x_u + x_v v'} = uv'$ and $dx = (x_u + x_v v')du = (1/u)du$, where $v' = dv/du$ ...(2)

Using (1) and (2), we have

$$\int_0^{\log_e 2} (e^{-x}y'^2 + e^x y^2)dx = \int_1^2 \left\{e^{-\log_e u}(uv')^2 - e^{\log_e u}v^2\right\}\frac{du}{u} = \int_1^2 (v'^2 - v^2)du \quad ...(3)$$

Hence the given functional reduces to

$$I[v(u)] = \int_1^2 (v'^2 - v^2)du \quad ...(4)$$

Comparing (4) with $I[v(u)] = \int_1^2 G(u, v, v')du$, we have $G(u, v, v') = v'^2 - v^2$ ...(5)

Euler's equation for (5) is $\dfrac{\partial G}{\partial v} - \dfrac{d}{du}\left(\dfrac{\partial G}{\partial v'}\right) = 0$

i.e. $-v - \dfrac{d}{du}(2v') = 0$ or $v'' + v = 0$ or $(D^2 + 1)v = 0$, where $D \equiv d/dv$

whose general solution is $v = C_1 \cos u + C_2 \sin u$, C_1 and C_2 being arbitrary constants.

or $y = C_1 \cos e^x + C_2 \sin e^x$, since $(1) \Rightarrow u = e^x$

This gives us the desired extremal for the given functional.

Ex. 4. *Find the extremals through* (0, 0) *and* (1, 1) *of* $I[y(x)] = \int_{x_1}^{x_2} y^2(y'^2 - x^2)dx$

Sol. Given $I[y(x)] = \int_{x_1}^{x_2} y^2(y'^2 - x^2)dx$ ...(1)

Comparing (1) with $\int_{x_1}^{x_2} F(x, y, y')dx$, here $F(x, y, y') = y^2y'^2 - y^2x^2$ ...(2)

Then, Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ reduces to

$$2yy'^2 - 2yx^2 - \frac{d}{dx}(2y^2y') = 0 \quad \text{or} \quad yy'' + y'^2 + x^2 = 0 \quad ...(3)$$

which cannot be solved easily. Therefore, we proceed to solve the given problem by using the following transformations

$$x^2 = u \quad \text{and} \quad y^2 = v \quad \text{so that} \quad x = u^{1/2} \quad \text{and} \quad y = v^{1/2} \quad ...(4)$$

Hence, $y' = \dfrac{y_u + y_v v'}{x_u + x_v v'} = \dfrac{(1/2v^{1/2})v'}{(1/2u^{1/2})} = \left(\dfrac{u}{v}\right)^{1/2} v'$ ...(5)

and $dx = (x_u + x_v v')du = (1/2u^{1/2})\,du$, ...(6)

where $x_u = \partial x/\partial u$, $x_v = \partial x/\partial v$, $y_u = \partial y/\partial u$, $y_v = \partial y/\partial v$ and $v' = dv/du$

Using (4), (5) and (6), we have

$$\int_{x_1}^{x_2} y^2(y'^2 - x^2)dx = \int_{u_1}^{u_2} v\left(\frac{u\,v'^2}{v} - u\right)\frac{du}{2u^{1/2}} = \int_{u_1}^{u_2} \frac{1}{2}(u^{1/2}v'^2 - vu^{1/2})\,du$$

where we have assumed that $u = u_1$ when $x = x_1$ and $u = u_2$, when $x = x_2$

Hence (1) reduces to $$I[v(u)] = \int_{u_1}^{u_2} \{(1/2) \times (u^{1/2} v'^2 - v u^{1/2})\}\, du \qquad ...(7)$$

Comparing (7) with $I[v(u)] = \int_{u_1}^{u_2} G(u, v, v')du$, $G(u, v, v') = (1/2) \times (u^{1/2} v'^2 - v u^{1/2})$ ...(8)

Euler's equation for (8) is $$\frac{\partial G}{\partial v} - \frac{d}{du}\left(\frac{\partial G}{\partial v'}\right) = 0$$

i.e. $$-\frac{1}{2}u^{1/2} - \frac{d}{du}(u^{1/2} v') = 0 \quad \text{or} \quad \frac{u^{1/2}}{2} + u^{1/2} v'' + \frac{v'}{2u^{1/2}} = 0 \qquad ...(9)$$

Let $p = v' = dv/du$ so that $v'' = d^2 v/du^2 = dp/du$ ...(10)

Using (10), (9) yields $$\frac{u^{1/2}}{2} + u^{1/2}\frac{dp}{du} + \frac{p}{2u^{1/2}} = 0$$

or $$(dp/du) + (1/2u) \times p = -(1/2) \qquad ...(11)$$

which is a linear differential equation

$$\text{Integrating factor of (11)} = e^{\int (1/2u)du} = e^{(1/2)\times \log u} = e^{\log u^{1/2}} = u^{1/2}$$

and hence solution of (11) is $$p u^{1/2} = C_1 + \int \{(-1/2) \times u^{1/2}\} du$$

or $$p u^{1/2} = C_1 - (1/3) \times u^{3/2} \quad \text{or} \quad p = C_1 u^{-1/2} - u/3$$

or $$dv/du = C_1 u^{-1/2} - u/3 \quad \text{or} \quad dv = (C_1 u^{-1/2} - u/3)du$$

Integrating it, $v = C_2 + 2C_1 u^{1/2} - (u^2/6)$, C_1 and C_2 being arbitrary constants. ...(12)

Using back the transformations (4), (12) reduces to

$$y^2 = C_2 + 2C_1 x - (x^4/6) \qquad ...(13)$$

Since the required extremal must pass through (0, 0) and (1, 1), (13) yields $C_2 = 0$ and $C_1 = 7/12$. Hence from (13), the required extremal is the curve $y = (7x - x^4)/6$.

Alternative solution. We can solve the above problem without using the transformations (4). But for that purpose we have to solve (3) by using the following typical procedure :

Re-writing (3), $d(yy')/dx = -x^2$ so that $yy' = C_1 - x^3/3$ ...(*i*)

Re-writing (*i*), $dy^2/dx = 2C_1 - 2x^3/3$ so that $y^2 = 2C_1 x - x^4/6 + C_2$

which is the same as (13). Then the rest of solution is the same as above after getting (13).

1.7 Geodesics

A geodesics on a surface is a curve along which the distance between two points on the surface is a minimum. To find the geodescis on a surface is a variational problem involving the conditional extremum.

1.7A. Illustrative Examples based on Art. 1.7

Ex. 1. *Prove that the geodescis on a plane is a straight line.* **(Banglore 2004, 05)**

Sol. Refer Ex. 13 of Art. 1.5A.

Ex. 2. *Show that geodesics on a sphere of radius a are its great circles.*

(CDLU 2004, MDU Rohtak 2005, 06)

Sol. In spherical polar coordinate (r, θ, ϕ) the element of arc *ds* on the surface of the sphere of radius *a* is given by*

$$(ds)^2 = (dr)^2 + (r\, d\theta)^2 + (r \sin\theta\, d\phi)^2 \qquad ...(1)$$

But for the given sphere of radius *a*, $r = a$ so that $dr = 0$

Hence (1) reduces to $(ds)^2 = (a\, d\theta)^2 + (a \sin\theta\, d\theta)^2$

giving $ds = a(1 + \phi'^2 \sin^2\theta)^{1/2} d\theta$, where $\phi' = d\phi/d\theta$ ...(2)

Therefore arc of length *s* between two points $P_1(a, \theta_1, \phi_1)$ and $P_2(a, \theta_2, \phi_2)$ is given by

$$s = \int_{P_1}^{P_2} ds = a\int_{\theta_1}^{\theta_2} (1 + \phi'^2 \sin^2\theta)^{1/2} d\theta \text{, by (2)} \qquad ...(3)$$

Comparing (3) with $\int_{\theta_1}^{\theta_2} F(\theta, \phi, \phi')d\theta$, here $F(\theta, \phi, \phi') = (1 + \phi'^2 \sin^2\theta)^{1/2}$ ...(4)

[Note that here we have variables θ and ϕ in place of the usual variables *x* and *y* of Art. 1.4]

Euler's equation is $$\frac{\partial F}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right) = 0 \qquad ...(5)$$

From (4), $\dfrac{\partial F}{\partial \phi} = 0$ and so (5) gives $\dfrac{d}{d\theta}\left(\dfrac{\partial F}{\partial \phi'}\right) = 0$

Integrating it, $\partial F/\partial \phi' = \text{constant} = C$, say ...(6)

or $$(1/2) \times (1 + \phi'^2 \sin^2\theta)^{-1/2} \times 2\phi' \sin^2\theta = C \text{, using (4)}$$

or $\sin^4\theta \times \phi'^2 = C^2(1 + \sin^2\theta\, \phi'^2)$ or $\sin^2\theta(\sin^2\theta - C^2)\phi'^2 = C^2$

or $\dfrac{d\phi}{d\theta} = \dfrac{C}{\sin\theta(\sin^2\theta - C^2)^{1/2}}$ or $\dfrac{d\phi}{d\theta} = \dfrac{C\ \text{cosec}^2\theta}{(1 - C^2\text{cosec}^2\theta)^{1/2}}$

or $d\phi = \dfrac{C\ \text{cosec}^2\theta\, d\theta}{\{1 - C^2(1 + \cot^2\theta)\}^{1/2}} = -\dfrac{du}{\{(1 - C^2) - u^2\}^{1/2}}$, on putting $C\cot\theta = u$, $-C\ \text{cosec}^2\theta\, d\theta = du$

Integrating, $$\phi = -\sin^{-1}\frac{u}{(1 - C^2)^{1/2}} + C' = -\sin^{-1}\frac{C\cot\theta}{(1 - C^2)^{1/2}} + C'$$

or $$\{C/(1 - C^2)^{1/2}\}\cot\theta = \sin(C' - \phi) = \sin C' \cos\phi - \cos C' \sin\phi$$

or $\cot\theta = C^{-1}(1 - C^2)^{1/2} \sin C' \cos\phi - C^{-1}(1 - C^2)^{1/2} \cos C' \sin\phi$ or $\cot\theta = A\cos\phi + B\sin\phi$

where *A* and *B* are new arbitrary constants. Re-writing it, we get

or $$a\cos\theta = A\, a \sin\theta\cos\phi + B\, a \sin\theta\sin\phi \qquad ...(7)$$

If (*x*, *y*, *z*) be the cartesian coordinates of any point $P(a, \theta, \phi)$ on the surface of the given sphere of radius *a*, then we know that

$x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$

Hence (7) reduces to $z = Ax + By$ ...(8)

---

* Refer Art. 4.4C on page 156 in author's book "Vector Calculus" published by S. Chand and Co., New Delhi.

(8) is a plane through the centre (0, 0, 0) of the sphere which cuts the given sphere along a great circle. Hence the required geodesics are the arcs of the great circle.

Ex. 3. *Find the geodesics of a right circular cylinder of radius a.* **(CDLU 2004)**

Sol. In cylindrical polar coordinates (ρ, θ, ϕ) the element of arc ds on the surface of the cylinder of radius a is given by*

$$(ds)^2 = (d\rho)^2 + (\rho\, d\theta)^2 + (dz)^2 \qquad ...(1)$$

But for the given cylinder of radius a, $\rho = a$ so that $d\rho = 0$

Hence (1) rduces to $(ds)^2 = (a\, d\theta)^2 + (dz)^2$

giving $ds = (a^2 + z'^2)^{1/2}\, d\theta$, where $z' = dz/d\theta$ ...(2)

Therefore arc of length s between two points $P_1(a, \theta_1, z_1)$ and $P_2(a, \theta_2, z_2)$ is given by

$$s = \int_{P_1}^{P_2} ds = \int_{\theta_1}^{\theta_2} (a^2 + z'^2)^{1/2}\, d\theta \text{, by (2)} \qquad ...(3)$$

Comparing (3) with $\int_{\theta_1}^{\theta_2} F(\theta, z, z')\, d\theta$, here $F(\theta, z, z') = (a^2 + z'^2)^{1/2}$ ...(4)

[Note that here we have variables θ and z in place of the usual variables x and y of Art. 1.4]

Euler's equation is $$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0 \qquad ...(5)$$

From (4), $\dfrac{\partial F}{\partial z} = 0$ and so (5) gives $\dfrac{d}{d\theta}\left(\dfrac{\partial F}{\partial z'}\right) = 0$

Integrating it, $\partial F/\partial z' = \text{constant} = C$, say ...(6)

or $(1/2) \times (a^2 + z'^2)^{-1/2} \times 2z' = C$ or $z'^2 = C^2(a^2 + z'^2)$, using (4)

or $\left(\dfrac{dz}{d\theta}\right)^2 (1 - C^2) = a^2 C^2$ so that $d\theta = \dfrac{(1 - C^2)^{1/2}}{aC} dz$

Integrating, $\theta = mz + b$, where $m = (1 - C^2)^{1/2}/aC$ ...(7)

This shows that the required geodesics are circular helix given by (7).

Ex. 4. *Find the geodesics on a right circular cone of semi-vertical angle* α.

(KU Kurukshetra 2004)

Sol. In spherical polar coordinates (r, θ, ϕ) the element of arc ds on the surface of a right circular cone of semi-vertical angle α is given by

$$(ds)^2 = (dr)^2 + (r\, d\theta)^2 + (r \sin\theta\, d\phi)^2 \qquad ...(1)$$

But for the given cone of semi-vertical angle α, we have $\theta = \alpha$ so that $d\theta = 0$. Then (1) reduces to $(ds)^2 = (dr)^2 + (r \sin\alpha\, d\phi)^2$ so that

$$ds = (r^2 \sin^2\alpha + r'^2)^{1/2}\, d\phi, \quad \text{where} \quad r' = dr/d\phi \qquad ...(2)$$

Therefore arc of length s between two points $P_1(r_1, \alpha_1, \phi_1)$ and $P_2(r_2, \alpha_2, \phi_2)$ is given by

$$s = \int_{P_1}^{P_2} ds = a\int_{\phi_1}^{\phi_2} (r^2 \sin^2\alpha + r'^2)^{1/2}\, d\phi \qquad ...(3)$$

---

* Refer Art. 4.4B on page 154 in author's book "Vector Calculus" published by S. Chand and Co., New Delhi.

Comparing (3) with $\int_{\phi_1}^{\phi_2} F(\phi, r, r')d\phi$, here $F(\phi, r, r') = (r^2 \sin^2 \alpha + r'^2)^{1/2}$ ...(4)

[Note that here we have variables ϕ and r in place of the usual variables x and y of Art. 1.4]

Since $F(\phi, r, r')$ is independent of ϕ, so by case V Art. 1.5F, we have $F - r'(\partial F/\partial r') = C$

or $(r^2 \sin^2 \alpha + r'^2)^{1/2} - r' \times (1/2) \times (r^2 \sin^2 \alpha + r'^2)^{-1/2} \times 2r' = C$ by (4)

or $r^2 \sin^2 \alpha + (r')^2 - (r')^2 = C(r^2 \sin^2 \alpha + r'^2)^{1/2}$ or $r^4 \sin^4 \alpha = C^2(r^2 \sin^2 \alpha + r'^2)$

or $$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4 \sin^4 \alpha}{C^2} - r^2 \sin^2 \alpha = \frac{r^2 \sin^2 \alpha (r^2 \sin^2 \alpha - C^2)}{C^2}$$

or $$\frac{dr}{d\phi} = \frac{r \sin \alpha}{C}(r^2 \sin^2 \alpha - C^2)^{1/2}$$ or $$d\phi = \frac{C}{\sin \alpha} \frac{dr}{r(r^2 \sin^2 \alpha - C^2)^{1/2}}$$

Integrating, $\phi = \frac{C}{\sin \alpha} \int \frac{-(1/u^2)du}{(1/u)\{(1/u^2) \times \sin^2 \alpha - C^2\}^{1/2}}$, on putting $r = 1/u$ and $dr = -(1/u^2)du$

or $$\phi = -\frac{C}{\sin \alpha} \int \frac{du}{(\sin^2 \alpha - u^2 C^2)^{1/2}} = \frac{1}{\sin \alpha} \cos^{-1}\left(\frac{Cu}{\sin \alpha}\right) - \frac{b}{\sin \alpha}$$

or $\phi \sin \alpha + b = \cos^{-1}(C/r \sin \alpha)$ or $C/r \sin \alpha = \cos(\phi \sin \alpha + b)$

or $r = (C/\sin \alpha)\sec(\phi \sin \alpha + b)$ or $r = a \sec(\phi \sin \alpha + b)$

where a and b are arbitrary constants. Hence the required geodesics are given by the two parameter family of the above curves.

EXERCISE 1(A)

1. (*a*) Find the extremals of the functional $I[y(x)] = \int_{x_1}^{x_2} y'(1 + x^2 y')dx$

(*b*) Find the extremals of the functional $I[y(x)] = \int_{x_1}^{x_2} (y'^2 + 2yy' - 16y^2)dx$

Ans. (*a*) $y = C_1/x + C_2$ (*b*) $y = C_1 \sin(4x - C_2)$

2. Find the extremals of the functional $I[y(x)] = \int_{x_1}^{x_2} (xy' + y'^2)dx$ **Ans.** $y = C_1 x + C_2 - x^2/4$

3. (*a*) Find the extremals of the functional $I[y(x)] = \int_{x_1}^{x_2} \{(1/y) \times (1 + y'^2)^{1/2}\}dx$

(*b*) Find the extremals of the functional $I[y(x)] = \int_{x_1}^{x_2} \{(1/y'^2) \times (1 + y^2)\}dx$

Ans. (*a*) $(x - C_1)^2 + y^2 = C_2^2$ (*b*) $y = \sinh(C_1 x + C_2)$

4. Show that the general solution of the Euler's equation for the integral

$\int_a^b (1/y) \times \{1 + (dy/dx)^2\}^{1/2}\ dx$ *is* $(x - h)^2 + y^2 = k^2$ **(Kanpur 2010)**

5. Find extremals of functionals (*i*) $I[y(x)] = \int_{x_1}^{x_2} (y^2 + 2xy\,y')dx$, $y(x_1) = y_1$, $y(x_2) = y_2$

(*ii*) $I[(y(x)] = \int_0^1 (y'^2 + y' + 1)dx$, $y(0) = 1$, $y(1) = 2$ **[Bhopal 2010]**

Ans. (*i*) Since the integral is independent of the path of integration, hence the variational problem is meaningless;

(*ii*) $y = x + 1$

7. Find the extremals of the functional $\int_{x_1}^{x_2}(y^2+y'^2+2ye^x)dx$ **(Kanpur 2006, 07)**

Ans. $y=(x/2)\times e^x+C_1e^x+C_2e^{-x}$

8. Find the extremals of the functional $\int_{x_1}^{x_2}(y^2-y'^2-2y\sin x)dx$

Ans. $y=-(x/2)\times\cos x+C_1\cos x+C_2\sin x$

9. Find the extremals of the functional $\int_{x_1}^{x_2}(y^2+y'^2+2y\,\text{sech}\,x)dx$

Ans. $y=C_1\cosh x+C_2\sinh x+x\sinh x-\cosh x\log_e\cosh x$

10. Find the extremals and extremum value of the functional $\int_{1/2}^{1}x^2y'^2dx$ **(Kanpur 2005)**

Ans. $y=3-(1/x)$; value $=1$

11. Find the extremals and extremum value of the functional $\int_0^2(x-y')^2dx$

Ans. $y=x+x^2/2$; value $=2$

12. A particle is moving with a force perpendicular to and proportional to its distance from the line of zero velocity. Show that the path of quickest descent (brachistochrone) is a circle.

13. Find the stationary function of $\int_0^4(xy'-y'^2)dx$ which is determined by the boundary conditions $y(0)=0$ and $y(4)=3$. **Ans.** $y=(x^2-x)/4$

14. If $F(x,y,y')$ does not involve y explicitly, show that the extremals of $\int_{x_1}^{x_2}F(x,y,y')dx$ are solutions of $\partial F/\partial y'=C$, where C is an arbitrary constant.

15. Show that if $F(x,y,y')$ is a function of y' alone, then the extremals of $\int_{x_1}^{x_2}F(x,y,y')\,dx$ are straight lines.

16. Find the extremals of the functional $I=\int_{\theta_0}^{\theta_1}r\sin\theta\,(r^2+r'^2)^{1/2}d\theta$ **(Nagpur 2005)**

17. If $P_1(r_1,\theta_1)$ and $P_2(r_2,\theta_2)$ are two points in a plane, then the length of a curve from P_1 to P_2 is given by $$I=\int_{P_1}^{P_2}\sqrt{dr^2+r^2d\theta^2}$$

Find the polar equation of a straight line by minimizing this integral

(*a*) with θ as the independent variable (*b*) with *r* as the independent variable.

Ans. In both case (*a*) and (*b*), we get $r\cos(\theta-C_1)=C_2$

18. Show that the extremal of $I[y(x)]=\int_0^{\pi/2}(y'^2-y^2+4y\sin^2x)dx$, $y(0)=y(\pi/2)=1/3$ is given by $y=(2\sin x+\cos 2x)/3$.

19. Use Euler-Lagrange condition to show that the shortest distance between two points in a plane is a straight line. **(GATE 2000)**

20. *Find the extremals of functional* $I[y(x)]=\int_1^2\{(1+y'^2)^{1/2}\times(1/x)\}dx$ *subject to conditions* $y(1)=0$, $y(2)=2$. **[Purvanchal 2006]**

Ans. $x^2+y^2-(7/2)\times y-1=0$

21. (*a*) *Find the extremal of the functional* $\int_0^1 (2e^y - y^2)dx$ *that satisfy the boundary conditions* $y(0) = 1, y(1) = e$. **Ans.** The given problem has no solution.

(*b*) *Show that the extremal of the functional* $\int_0^{\pi/2} y(2x - y)dx$ *that satisfy the boundary conditions* $y(0) = 0, y(\pi/2) = \pi/2$ *is* $y = x$.

(*c*) *Show that the stationary function of the functional* $\int_0^4 (xy' - y'^2)dx$ *which is determined by the boundary conditions* $y(0) = 0$ *and* $y(4) = 3$ *is* $y = (x^2 - x)y$. **[Kurakshetra 2005]**

(*d*) *Find the extremals of the fucntional* $\int_0^{\pi} (y'^2 - y^2 + 4y\cos x)dx$ **[Kurakshetra 2005]**

Ans. $y = C_1 \cos x + C_2 \sin x + x \sin x$, C_1, C_2 *being arbitrary constants.*

22. (*a*) *Find the extremals and stationary function of the functional* $\int_0^1 (y'^2 + 4y^2)dx$ *subject to the boundary conditions* $y(0) = e^2, y(1) = 1$. **Ans.** $y = C_1 e^{2x} + C_2 e^{-2x}$; $y = e^{2(1-x)}$

(*b*) *Find the extremals and the stationary function of the functional* $\int_0^1 yy'^2 dx$ *that satisfy the boundary conditions* $y(0) = 1, y(1) = 4^{1/3}$. **Ans.** $(2/3) \times y^{3/2} = C_1 x + C_2$; $y = (x + 1)^{2/3}$

1.8 Variational problems for functionals involving several dependent variables of the form

$$\int_{x_1}^{x_2} F(x, y_1(x), y_2(x), ..., y_n(x), y_1'(x), y_2'(x), ..., y_n'(x))\, dx$$

Proof for special case of two dependent variables *y* and *z* **[Guwahati 2007]**

Let us examine the functional $I = \int_{x_1}^{x_2} F(x, y, z, y', z')dx$ ...(1)

Suppose we wish to find necessary condition to be satisfied by two functions $y(x)$ and $z(x)$ that extremizes the functional I. Suppose the boundary values

$y(x_1) = y_1$, $y(x_2) = y_2$, $z(x_1) = z_1$ and $z(x_2) = z_2$ ...(2)

be specified in advance. Let $\eta_1(x)$ and $\eta_2(x)$ be any two functions with the properties that $\eta_1''(x)$ and $\eta_2''(x)$ are continuous and $\eta_1(x_1) = \eta_1(x_2) = \eta_2(x_1) = \eta_2(x_2) = 0$ ...(3)

If α be a small parameter then we form the neighbouring functions

$\bar{y}(x) = y(x) + \alpha\, \eta_1(x)$ and $\bar{z}(x) = z(x) + \alpha\, \eta_2(x)$ ...(4)

(4) $\Rightarrow$ $\bar{y}'(x) = y'(x) + \alpha\, \eta_1'(x)$ and $\bar{z}'(x) = z'(x) + \alpha\, \eta_2'(x)$ ...(5)

Now, let us consider the function of α defined by $I(\alpha) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{z}, \bar{y}', \bar{z}')dx$ ...(6)

Since setting $\alpha = 0$ has the effect of replacing, $\bar{y}(x)$, $\bar{z}(x)$, $\bar{y}'(x)$, $\bar{z}'(x)$ in equation (6) by the extremizing functions $y(x)$ and $z(x)$ and their derivatives $y'(x)$ and $z'(x)$, $I(\alpha)$ must take on its extreme value when $\alpha = 0$. This happens no matter what particular value functions $\eta_1(x)$ and $\eta_2(x)$ are involved in (4) and (5). But, if $I(\alpha)$ has an extremum for $\alpha = 0$, then by elementary Calculus, a necessary condition is given by

$I'(\alpha) = 0$, *i.e.*, $dI(\alpha)/d\alpha = 0$ ...(7)

Using Leibnitz's rule of differentiation under the integral sign, (6) yields

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} F(x, \bar{y}, \bar{z}, \bar{y}', \bar{z}')dx \quad ...(8)$$

By the chain rule for differentiating functions of several variables, we get

$$\frac{\partial}{\partial\alpha}F(x,\bar{y},\bar{z},\bar{y}',\bar{z}') = \frac{\partial F}{\partial x}\frac{\partial x}{\partial\alpha}+\frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial\alpha}+\frac{\partial F}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial\alpha}+\frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial\alpha}+\frac{\partial x}{\partial \bar{z}'}\frac{\partial \bar{z}'}{\partial\alpha}$$

$$= \frac{\partial F}{\partial \bar{y}}\ \eta_1(x)+\frac{\partial F}{\partial \bar{z}}\ \eta_2(x)+\frac{\partial F}{\partial \bar{y}'}\ \eta_1'(x)+\frac{\partial F}{\partial \bar{z}'}\ \eta_2'(x),\ \text{by (4) and (5)}$$

$$\therefore\ (8)\Rightarrow \qquad I'(\alpha)=\int_{x_1}^{x_2}\left\{\frac{\partial F}{\partial \bar{y}}\,\eta_1(x)+\frac{\partial F}{\partial \bar{y}'}\ \eta_1'(x)+\frac{\partial F}{\partial \bar{z}}\ \eta_2(x)+\frac{\partial F}{\partial \bar{z}'}\ \eta_2'(x)\right\}dx$$

which, upon setting $\alpha=0$, and making use of (7), reduces to

$$\int_{x_1}^{x_2}\left\{\frac{\partial F}{\partial y}\ \eta_1(x)+\frac{\partial F}{\partial y'}\ \eta_1'(x)+\frac{\partial F}{\partial z}\ \eta_2(x)+\frac{\partial F}{\partial z'}\ \eta_2'(x)\right\}dx=0 \qquad ...(9)$$

where we have used the facts that when $\alpha=0$, $\bar{y}(x)=y(x)$, $\bar{z}(x)=z(x)$, $\bar{y}'(x)=y'(x)$ and $\bar{z}'(x)=z'(x)$ in view of relations (4) and (5).

Now, integrating by parts, we have $\int_{x_1}^{x_2}\frac{\partial F}{\partial y'}\ \eta_1'(x)dx=\left[\frac{\partial F}{\partial y'}\ \eta_1(x)\right]_x^{x_2}-\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\ \eta_1(x)dx$

Using (3), this $\Rightarrow$ $\int_{x_1}^{x_2}\frac{\partial F}{\partial y'}\ \eta_1'(x)dx=-\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\ \eta_1(x)dx$ ...(10)

Similarly, we have $\int_{x_1}^{x_2}\frac{\partial F}{\partial z'}\ \eta_2'(x)dx=-\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)\ \eta_2(x)dx$ ...(11)

Using (10) and (11), (9) reduces to

$$\int_{x_1}^{x_2}\left[\eta_1(x)\left\{\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}+\eta_2(x)\left\{\frac{\partial F}{\partial z}-\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)\right\}\right]dx=0 \qquad ...(12)$$

Since (12) must hold for all choices of the functions $\eta_1(x)$ and $\eta_2(x)$, we at once arrive at a pair of Eulers' equations :

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \qquad \text{and} \qquad \frac{\partial F}{\partial z}-\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)=0 \qquad ...(13)$$

Hence, to find the extremals of our problem, we solve the system (13).

Remark 1. Consider a particular case of the above functional, when it is independent of x, y and z. for example, consider the functional $I[y(x),z(x)]=\int_{x_1}^{x_2}F(y',z')dx$

Then it is easily seen that the system of Euler's equations is of the form

$$\frac{\partial^2 F}{\partial y'^2}\ y''+\frac{\partial^2 F}{\partial y'\partial z'}\ z''=0 \qquad \text{and} \qquad \frac{\partial^2 F}{\partial y'\partial z'}\ y''+\frac{\partial^2 F}{\partial z'^2}\ z''=0$$

Let $\begin{vmatrix}\partial^2F/\partial y'^2 & \partial^2F/\partial y'\partial z'\\ \partial^2F/\partial y'\partial z' & \partial^2F/\partial z'^2\end{vmatrix}\neq 0$

Then the above system yields $y''=0$ and $z''=0$

Hence $y = C_1x + C_2$ and $z = C_3x + C_4$, which gives a family of straight lines in space.

Remark 2. Following the above procedure without any essential change, the system of Euler's equations for finding extremals of the functional

$$I = \int_{x_1}^{x_2} (x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n')dx \qquad ...(14)$$

is given by $\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0$, where $i = 1, 2, ..., n$ ...(15)

1.8A Alternative proof of getting the system of Euler's equations (15) of Art. 1.8.

Consider the functional $I[y_1, y_2, ..., y_n] = \int_{x_1}^{x_2} f(x, y_1, y_2, ...y_n, y_1', y_2', ..., y_n')dx$ ...(1)

for the given boundary conditions of all functions

$$y_1(x_1) = y_{11}, \quad y_2(x_1) = y_{21}, \ ....., \quad y_n(x_1) = y_{n1} \qquad ...(2)$$

$$y_1(x_2) = y_{12}, \quad y_2(x_2) = y_{22}, \ ....., \quad y_n(x_2) = y_{n2} \qquad ...(3)$$

where $y_{11}, y_{21}, ..., y_{n1}, y_{12}, y_{22}, ..., y_{n2}$ are constants.

Let us vary only one of the functions $y_j(x)$ (j = 1, 2, ... n) while keeping the rest of the functions unchanged. Then the functional $I[y_1, y_2, ..., y_n]$ will reduce to a functional dependent only on a single valued function, for example on $y_i(x)$, $I[y_1, y_2, ..., y_n] = \bar{I}[y_i]$ of the form considered in Art. 1.4. Therefore, the extremizing function must satisfy Euler's equation

$$F_{y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0,$$

where boundary conditions on $y_i(x)$ at $x = x_1$ and $x = x_2$ are given by (2) and (3).

Since this argument is applicable to any function $y_i (i = 1, 2, ...n)$, we arrive at a system of second order differential equations

$$F_{y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0, \qquad (i = 1, 2, ..., n) \qquad ...(4)$$

which define, in general, a 2n-parameter family of curves in space $x, y_1, y_2, ..., y_n$ which is the required family of extremals of the given variational problem.

1.8B Solved Examples based on Art. 1.8 and 1.8A

Ex. 1. Find the extremals of the functional $I[y(x), z(x)] = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\,dx$, satisfying $y(0) = 0, \quad y(\pi/2) = 1, \quad z(0) = 0, \quad z(\pi/2) = -1$ **[Guwahati 2007] (Kanpur 2004)**

Sol. Comparing the given functional with $\int_0^{\pi/2} F(x, y, z, y', z')dx$, we have

$$F(x, y, z, y', z') = y'^2 + z'^2 + 2yz \qquad ...(1)$$

The system of Euler's equations is given by

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{and} \quad \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

i.e. $2z - \frac{d}{dx}(2y') = 0$ and $2y - \frac{d}{dx}(2z') = 0$

i.e. $z - y'' = 0$ and $y - z'' = 0$

Thus, we have $d^2y/dx^2 = z$ ...(2)

and $d^2z/dx^2 = y$ ...(3)

We shall now solve the system of simultaneous differential equations (2) and (3) as follows. Differentiating both side of (2) twice w.r.t. 'x', we get

$d^4y/dx^4 = d^2z/dx^2$ or $d^4y/dx^4 = y$, using (3)

or $(D^4 - 1)y = 0$, where $D \equiv d/dx$ ...(4)

Auxiliary equation of (4) is $D^4 - 1 = 0$ giving $D = 1, -1, i, -i$.

Hence the general soluton of (4) is given by $y = C_1e^x + C_2e^{-x} + C_2\cos x + C_4\sin x$, ...(5)
where C_1, C_2, C_3 and C_4 are arbitrary constants

From (5), $dy/dx = C_1e^x - C_2e^{-x} - C_3\sin x + C_4\cos x$

so that $d^2y/dx^2 = C_1e^x + C_2e^{-x} - C_3\cos x - C_4\sin x$

Substituting this value of d^2y/dx^2 in (2), $z = C_1e^x + C_2e^{-x} - C_3\cos x - C_4\sin x$ ...(6)

Given that when $x = 0$, $y = 0$ and $z = 0$. Hence (5) and (6) give

$C_1 + C_2 + C_3 = 0$ ...(7)

and $C_1 + C_2 - C_3 = 0$ ...(8)

Also, given that when $x = \pi/2$, $y = 1$ and $z = -1$. Hence, (5) and (6) give

$C_1e^{\pi/2} + C_2e^{-\pi/2} + C_4 = 1$ ...(9)

and $C_1e^{\pi/2} + C_2e^{-\pi/2} - C_4 = -1$ ...(10)

Subtracting (8) from (7), we have $2C_3 = 0$ giving $C_3 = 0$ ...(11)

With $C_3 = 0$, (7) reduces to $C_1 + C_2 = 0$ ...(12)

Subtracting (10) from (9), we have $2C_4 = 2$ so that $C_4 = 1$ ...(13)

Adding (9) and (10), we have $C_1e^{\pi/2} + C_2e^{-\pi/2} = 0$ ...(14)

Solving (12) and (14), we have $C_1 = C_2 = 0$ ...(15)

Using (11), (13) and (15) in (5) and (6), the required curves are $y = \sin x$ and $z = -\sin x$

Ex. 2. *Find the differential equations of the lines of propagation of light in a optically non-homogeneous medium in which the speed of light is* $v(x, y, z)$

Sol. According to well known Fermat's principle, light propagates from one point $P_1(x_1, y, z)$ to another $P_2(x_2, y, z_2)$ along a curve, for which the time T of passage of light will be least. If the equation of the required path of the light ray be $y = y(x)$ and $z = z(x)$, then we easily have

$$T = \int_{x_1}^{x_2} \frac{ds}{v(x, y, z)} = \int_{x_1}^{x_2} \frac{(1 + y'^2 + z'^2)^{1/2}}{v}\,dx, \quad \text{...(1)}$$

where ds is a line element of the path.

Comparing (1) with $\int_{x_1}^{x_2} F(x, y, z, y', z')dx$, here $F(x, y, z, y', z') = (1 + y'^2 + z')^{1/2} / v$ ...(2)

The system of two Euler's equation is given by

$$F_y - \frac{d}{dx}F_{y'} = 0 \quad \text{and} \quad F_z - \frac{d}{dx}F_{z'} = 0$$

i.e. $$\frac{\partial v}{\partial y}\ \frac{(1+y'^2+z'^2)^{1/2}}{v^2}+\frac{d}{dx}\ \frac{y'}{v(1+y'^2+z'^2)^{1/2}}=0 \qquad ...(3)$$

and $$\frac{\partial v}{\partial z}\ \frac{(1+y'^2+z'^2)^{1/2}}{v^2}+\frac{d}{dx}\ \frac{z'}{v(1+y'^2+z'^2)^{1/2}}=0 \qquad ...(4)$$

The system of equations (3) and (4) together determine the lines of light propagation.

Ex. 3. *Find the extremals of the functional* $I[y(x), z(x)] = \int_0^{\pi}(2yz-2y^2+y'^2-z'^2)dx$. *Also find the extremal which satisfies the boundary conditions* $y(0) = 0$, $y(\pi) = 1$, $z(0) = 0$, $z(\pi) = 1$. **[Kanpur 2006, 07]**

Sol. Comparing the given functional with $\int_0^{\pi} F(x, y, z, y', z')\,dx$, we have

$$F(x, y, z, y', z') = 2yz - 2y^2 + y'^2 - z'^2 \qquad ...(1)$$

The system of Euler's equation is given by

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \quad \text{and} \quad \frac{\partial F}{\partial z}-\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)=0$$

or $2z-4y-d(2y')/dx=0$ and $2y-d(-2z')/dx=0$, by (1)

or $y''+2y-z=0$ and $z''+y=0$ ...(2)

Let $D \equiv d/dx$. Then equations in (2) can be re-written as

$$(D^2+2)y - z = 0 \qquad ...(3)$$

and $$y + D^2 z = 0 \qquad ...(4)$$

Operating (3) by D^2 and adding to (4), we have

$$D^2(D^2+2)\,y + y = 0 \quad \text{or} \quad (D^2+1)^2\,y = 0 \qquad ...(5)$$

The auxiltary equation (5) is $(D^2+1)^2 = 0$ giving $D = \pm i$ (twice)

$\therefore$ Solution of (5) is $y = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x,$ ...(6)

where C_1, C_2, C_3 and C_4 are arbitary constants.

(6) $\Rightarrow$ $dy/dx = C_2\cos x - (C_1 + C_2x)\sin x + C_4\sin x + (C_3 + C_4x)\cos x$

or $dy/dx = (C_2 + C_3 + C_4x)\cos x + (C_4 - C_1 - C_2x)\sin x$ ...(7)

(7) $\Rightarrow$ $d^2y/dx^2 = C_4\cos x - (C_2 + C_3 + C_4x)\sin x - C_2\sin x + (C_4 - C_1 - C_2x)\cos x$

or $D^2y = (2C_4 - C_1 - C_2x)\cos x - (2C_2 + C_3 + C_4x)\sin x$ ...(8)

From (3), $z = D^2y + 2y$. Putting values of y and D^2y given by (6) and (8) in it, we get

$z = (2C_4 - C_1 - C_2x)\cos x - (2C_2 + C_3 + C_4x)\sin x + 2(C_1 + C_2x)\cos x + 2(C_3 + C_4x)\sin x,$

or $z = (2C_4 + C_1 + C_2x)\cos x + (C_3 - 2C_2 + C_4x)\sin x$ ...(9)

The required family of extremals is given by (6) and (9).

Using the given boundary conditions $y = 0$ when $x = 0$ and $y = 1$ when $x = \pi$, (6) yields.

$C_1 = 0$ and $-(C_1 + C_2\pi) = 1$ so that $C_1 = 0$, $C_2 = -(1/\pi)$ ...(10)

Using the given boundary conditions $z = 0$ when $x = 0$ and $z = 1$ when $x = \pi$, (9) yields.

$2C_4 + C_1 = 0$ and $-(2C_4 + C_1 + C_2\pi) = 1$ i.e., $C_4 = 0$ and $2C_4 - 1 = -1$, using (10)

Thus, $C_1 = C_4 = 0$, $C_3 = -(1/\pi)$ and C_3 is an is arbitrary constant.

Substituting these values in (6) and (9), we have

$y = C_3\sin x - (x/\pi) \times \cos x,$ $z = C_3\sin x + (1/\pi) \times 2(\sin x - x\cos x),$

which gives the required extremals, c_3 being an arbitrary constant.

Ex. 4. *Find the extremal of the functional* $\int_0^1 (1 + y'^2 + z'^2)^{1/2}\, dx$ *that satisfy the boundary conditions:* $y(0) = 0,\ y(1) = 2,\ z(0) = 0,\ z(1) = 4$.

Sol. Comparing the given functional with $\int_0^1 F(x, y, z, y', z')\, dx$, we have,

$$F(x, y, z, y', z') = (1 + y'^2 + z'^2)^{1/2} \qquad ...(1)$$

The system of Euler's equations is given by

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{and} \quad \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

or $$0 - \frac{d}{dx}\left(\frac{y'}{(1 + y'^2 + z'^2)^{1/2}}\right) = 0 \quad \text{and} \quad 0 - \frac{d}{dx}\left(\frac{z'}{(1 + y'^2 + z'^2)^{1/2}}\right) = 0$$, by (1)

Integrating these, $y'/(1 + y'^2 + z'^2)^{1/2} = c_1$, c_1 is an arbitrary constant ...(2)

and $z'/(1 + y'^2 + z'^2)^{1/2} = c_2$, c_2 is an arbitrary constant ...(3)

Dividing (2) by (3) yields $y'/z' = c_1/c_2 = c_3$, say

Thus, $y' = c_3 z'$ ...(4)

Substituting the value of y' given by (4) in (3), we get

$$\frac{z'}{\{1 + (1 + c_3^2)\, z'^2\}^{1/2}} = c_2 \quad \text{so that} \quad z' = \frac{c_2}{(1 - c_2^2 - c_2^2\, c_3^2)^{1/2}} = c_4 \text{ say}$$

or $dz/dx = c_4$ so that $z = c_4 x + c_5$ ...(5)

Since $z' = c_4$, (4) yields $y' = c_3 c_4 = c_6$, say

or $dy/dx = c_6$ so that $y = c_6 x + c_7$ ... (6)

The extremal are given by (5) and (6). Using the given boundary concitions $z(0) = 0$, $z(1) = 4$, $y(0)=0$ and $y(1) = 2$, (5) and (6) yield

$c_5 = 0$, $c_4 + c_5 = 4$, $c_7 = 0$, $c_6 + c_7 = 2$ so that $c_4 = 4$, $c_6 = 2$, $c_5 = c_7 = 0$

Putting these values in (5) and (6), we get $z = 4x$, $y = 2x$, which are the required extremals.

Ex. 5 *Show that the functional* $\int_0^1 \{2x + (dx/dt)^2 + (dy/dt)^2\}\, dt$ *such that x(0) = 1, y(0) = 1, x(1) = 1.5, y(1) = 1 is stationary for* $x = (2 + t^2)/2$, $y = 1$. **[CDLU 2004]**

Sol. Let $D \equiv d/dt$, $dx/dt = x'$ and $dy/dt = y'$.

Comparing the given functional with $\int_0^1 F(t, x, y, x', y')\, dt$, we have

$$F(t, x, y, x', y') = 2x + x'^2 + y'^2 \qquad ... (1)$$

The system of Euler's equation is given by

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right) = 0$$

or $2 - d(2x')/dx = 0$ and $0 - d(2y')/dt = 0$, using (1)

or $d^2x/dt^2 = 1$ and $d^2y/dt^2 = 0$ ... (2)

Integrating (2), $dx/dt = t + c_1$ and $dy/dt = c_3$ ... (3)

Integrating (3), $x = t^2/2 + c_1 t + c_2$ and $y = c_3 t + c_4$, ... (4)

which are the equations of the extremals; c_1, c_2, c_3 and c_4 being arbitrary constants.

Using the given boundary conditions $x(0) = 1$, $x(1) = 3/2$ $y(0) = 1$ and $y(1) = 1$, (4) yields $c_2 = 1$, $1/2 + c_1 + c_2 = 3/2$, $c_4 = 1$, $c_3 + c_4 = 1 \Rightarrow c_1 = c_3 = 0$, $c_2 = 1 = c_4$.

Putting these values in (4), we get $x = (2 + t^2)/2$, $y = 1$, showing that the given functional is, stationary for $x = (2 + t^2)/2$, $y = 1$.

Exercise 1(B)

1. Find the extremal of the functional $\int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\, dx$ that satisfy the boundary conditions $y(0) = 0$, $y(\pi/2) = -1$, $z(0) = 0$ and $z(\pi/2) = 1$.

Hint. Proceed an in Ex.1, Art. 1.8 B. **Ans.** $y = -\sin x$, $z = \sin x$

2. Find the extermal of the functional $\int_0^1 (y'^2 + z'^2)\, dx$ that satisfy the boundary conditions $y(0) = 0$, $z(0) = 0$, $y(1) = 1$, $z(1) = 2$. **Ans.** $y = x$, $z = 2x$

3. Find the extremal of the functional $\int_0^1 (y'^2 + z'^2 + 4z)dx$ that satisfy the boundary conditions $y(0) = 0$, $y(1) = 1$, $z(0) = 0$, $z(1) = 0$ **Ans.** $y = x$, $z = x^2 - x$

4. Show that the extremal of the functional $I = \int_0^{\lambda/2} (\dot{x}^2 + \dot{y}^2 + 2xy)dt$ subject to the boundary conditions : at $t = 0$, $x = y = 0$ and at $t = \pi/2$, $x = y = 1$, is $x = y = \text{sech}(\pi/2) \sinh t$. Here $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$.

5. Show that the extremal of the functional $I = \int_0^1 (\dot{x}\, y + \dot{y}\, x - x^2 - y^2)dt$ subject to the conditions, at $t = 0$, $x = y = 1$ and *at* $t = 1$, $x = y = 0$ is $x = y = \{\sinh(1 - t\}/\sinh 1$. Here $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$.

1.9 Functional depedent on higher order derivatives.

A particular case. *To show that a necessary condition for* $I = \int_{x_1}^{x_2} F(x, y, y', y'')dx$ ...(1)

to be extremum is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

Proof. Suppose that an admissable curve $y = y(x)$ exists which is twice differentiable on $[x_1, x_2]$ which satisfies the fixed boundaray conditions :

$$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_1) = y_1' \quad \text{and} \quad y'(x_2) = y_2' \qquad ...(2)$$

and which extremizes the functional I given by (1).

Let $\eta(x)$ be any function with the properties that $\eta'''(x)$ is continuous and

$$\eta(x_1) = 0 = \eta(x_2) \quad \text{and} \quad \eta'(x_1) = 0 = \eta'(x_2) \qquad ...(3)$$

If α is a small parameter, then $$\bar{y}(x) = y(x) + \alpha\, \eta(x) \qquad ...(4)$$

represents a one parameter family of curves. From (4), we note that for each family of this type, that is, for each choice of the function $\eta(x)$ the extremizing curve $y = y(x)$ belongs to the family and corresponds to the value of the parameter $\alpha = 0$.

From (4), $\bar{y}'(x) = y'(x) + \alpha\, \eta'(x)$ and $\bar{y}'' = y''(x) + \alpha\, \eta''(x)$ ...(5)

Now, with $\eta(x)$ fixed, replacing y, y' and y'' by $\bar{y}$, $\bar{y}'$ and $\bar{y}''$ respectively in the functional (1), we get a function of α

$$I(\alpha) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}', \bar{y}'')dx \qquad \ldots (6)$$

Since setting $\alpha = 0$ has the effect of replacing $\bar{y}$, $\bar{y}'$ and $\bar{y}''$ in (6) by the y, y', y'', $I(\alpha)$ must take on its extreme value when $\alpha = 0$. This happens no matter what particular value of function $\eta(x)$ is involved in (4) and (5). But if $I(\alpha)$ has an extrme value for $\alpha = 0$. then by elementary Calculus, a necessary condition is given by

$$I'(\alpha) = 0, \qquad i.e., \qquad dI(\alpha)/d\alpha = 0 \qquad \ldots (7)$$

Using Leibnitz's rule of differentiation under the integral sign, (6) gives

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{dF(x, \bar{y}, \bar{y}', \bar{y}'')}{d\alpha} dx \qquad \ldots (8)$$

By the chain rule for differentiating functions of several variables, we get

$$\frac{dF(x, \bar{y}, \bar{y}', \bar{y}'')}{d\alpha} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{d\alpha} + \frac{\partial F}{\partial \bar{y}''}\frac{\partial \bar{y}''}{\partial \alpha}$$

$$= 0 + \frac{\partial F}{\partial \bar{y}}\eta(x) + \frac{\partial F}{\partial \bar{y}'}\eta'(x) + \frac{\partial F}{\partial \bar{y}''}\eta''(x), \text{ by (4) and (5)}$$

$$\therefore \quad (8) \Rightarrow \qquad I'(\alpha) = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial \bar{y}}\ \eta(x) + \frac{\partial F}{\partial \bar{y}'}\ \eta'(x) + \frac{\partial F}{\partial \bar{y}''}\ \eta''(x) \right\} dx$$

which, upon setting $\alpha = 0$, and making use of (7) gives

$$\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\ \eta(x) + \frac{\partial F}{\partial y'}\ \eta'(x) + \frac{\partial F}{\partial y''}\eta''(x) \right\} dx = 0, \qquad \ldots (9)$$

where we have used the facts that when $\alpha = 0$, $\bar{y} = y$, $\bar{y}' = y'$ and $\bar{y}'' = y''$ in view by (4) and (5). Now, integrating by parts, we have

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'(x)dx = \left[\frac{\partial F}{\partial y'}\eta(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\eta(x)dx = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\ \eta(x)dx, \text{ by (3)} \ldots (10)$$

and $\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\ \eta''(x)dx = \left[\frac{\partial F}{\partial y''}\ \eta'(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''} \right)\ \eta'(x) = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''} \right)\ \eta'(x)dx$, by (3)

$$= -\left\{ \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y''} \right)\ \eta(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''} \right)\ \eta(x)dx \right\}, \text{ on integrating by parts again}$$

Thus, $$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\ \eta''(x)dx = \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''} \right)\eta(x)dx, \text{ using (3)} \qquad \ldots (11)$$

Using (10) and (11) in (9), we obtain

$$\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''} \right) \right\} \eta(x)dx = 0 \qquad \ldots (12)$$

Except for the conditions (3) to be satisfied , and the requirement of being twice diffentitable on $[x_1, x_2]$, $\eta(x)$ is an arbitrary function. Hence, if F is any function for which

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)$$

is continuous on $[x_1, x_2]$, equation (12) and the fundamental lemma of the calculus of variations (see Art. 1.3) imply that a necessary condition for the functional (1) to have an exterme value is that the extremizing function $y = y(x)$ satsify the differential equition

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

which is known as *Euler-Poisson equation*.

Note : In general, a necessary condition for the functional

$$I = \int_{x_1}^{x_2} F(x, y, y', y'', \ldots y^{(n)})dx \qquad \text{to be extremum is}$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

In next article 1.9 A, we now proceed to prove the above result by an alternative method.

1.9A General case. Functional dependant on higher-order derivatives.

(Kanpur 2006, 07, 09; MDU Rohtak 2006)

In this article, we purpose to find the extreme value of the functional

$$I[y(x)] = \int_{x_1}^{x_2} F[x, y(x), y'(x), \ldots y^{(n)}(x)]dx \qquad \ldots(1)$$

where we assume F to the be differentiable $n + 2$ times with respect to all its arguments and also assume that the boundary conditions are of the form

$$y(x_1) = y_1, \qquad y'(x_1) = y_1' \ldots\ldots\ldots, y^{(n-1)}(x_1) = y_1^{(n-1)} \qquad \ldots(2)$$

$$y(x_2) = y_2, \qquad y'(x_2) = y_2', \ldots\ldots\ldots, y^{(n-1)}(x_2) = y_2^{(n-1)} \qquad \ldots(3)$$

i.e., at the boundary points are given the values not only of the function but also of its dirivatives upto the order $n - 1$ inclusive. We also assume that an extremum is attained on the curve $y = y(x)$, which is $2n$ times differentiatable and any admissible comparison curve $y = \bar{y}(x)$ is also $2n$ times differentiable.

Let us consider one-parmameter family of funcitons

$$y(x, \alpha) = y(x) + \alpha\{\bar{y}(x) - y(x)\} \qquad \text{or} \qquad y(x, \alpha) = y(x) + \alpha\, \delta y$$

For $\alpha = 0$, $y(x, \alpha) = y(x)$ and for $\alpha = 1$, $\bar{y}(x, \alpha) = \bar{y}(x)$

If we consider the value of the functional $I[y(x)]$ only on curves of the family $y = y(x, \alpha)$ then the funcional (1) reduces to a function of the parameter α, which is extremized for $\alpha = 0$; hence $\left[dI[y(x, \alpha)]/d\alpha\right]_{\alpha=0} = 0$

According to Art. 1.2, this derivative is called the variation of the funcional I and is denoted by δI. As in Art. 1.4, we have

$$\delta I = \left[\frac{d}{d\alpha}\int_{x_1}^{x_2} F(x, y(x, \alpha), y'(x, \alpha), \ldots, y^{(n)}(x, \alpha)d\alpha\right]_{\alpha=0}$$

or $$\delta I = \int_{x_1}^{x_2} (F_y\ \delta y + F_{y'}\ \delta y' + ... + F_{y(n)}\ \delta y^{(n)})\ dx \qquad ... (4)$$

Integrating by parts the second term on the R.H.S of (4), we get

$$\int_{x_1}^{x_2} F_{y'}\ \delta y' dx = [F_{y'}\ \delta y]_{x_1}^{x_2} - \int_{x_1}^{x_2} \left(\frac{d}{dx} F_{y'}\right) \delta y\, dx$$

Integrating by parts the third term on the R.H.S of (4), we get

$$\int_{x_1}^{x_2} F_{y''}\ \delta y'' dx = [F_{y''}\ \delta y']_{x_1}^{x_2} - \left[\left(\frac{d}{dx} F_{y''}\right)\delta y\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{d^2}{dx^2} F_{y''}\right) \delta y\, dx$$

and so on. Finally, on integrating the last term on the R.H.S. of (4), we get

$$\int_{x_1}^{x_2} F_{y(n)}\ \delta y^{(n)} dx = \left[F_{y(n)}\ \delta y^{(n-1)}\right]_{x_1}^{x_2} - \left[\left(\frac{d}{dx} F_{y(n)}\right) \delta y^{(n-2)}\right]_{x_1}^{x_2} + ... + (-1)^n \int_{x_1}^{x_2} \left(\frac{d^n}{dx^n} F_{y(n)}\right) \delta y\, dx$$

Using the boundary conditions (2) and (3), we find that for $x = x_1$ and $x = x_2$, the variations $\delta y = \delta y' = \delta y'' = ... = \delta y^{(n-1)} = 0$. Hence the integrated parts in all the above expressions on the R.H.S. vanish. Therefore, (4) reduces to

$$\delta I = \int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + ... + (-1)^n \frac{d^n}{dx^n} F_{y(n)}\right) \delta y\ dx$$

Since on the extremizing curve, we have $\delta I = 0$

i.e. $$\int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - ... + (-1)^n \frac{d^n}{dx^n} F_{y(n)}\right) \delta y\ dx = 0$$

for an arbitrary choice of the function δy and since the first factor under the integral sign is a continuous function of x on the same curve $y = y(x)$, it follows that by virtue of the fundamental lemma (see Art. 1.3), the first factor is identically zero. Thus, we obtain

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - ... + (-1)^n \frac{d^n}{dx^n} F_{y(n)} = 0, \qquad ...(5)$$

which is known as *Euler-Poisson equation*.This is a differential equation of order $2n$ and its integral curves are known as extremals of the variation problems under consideration. Solution of (5) involves $2n$ arbitrary constants which may be determined by using the given $2n$ boundary conditions (2) and (3).

Remark. If the funcitonal I is of the form

$$I[y(x), z(x)] = \int_{x_1}^{x_2} F(x, y, y', ...y^{(n)}, z, z', ...z^{(n)})\, dx$$

then by varying only $y(x)$ and assuming $z(x)$ to be fixed, we can easily prove that the extremizing function $y(x)$ and $z(x)$ must satisfies the Euler-Poisson equation

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - ... + (-1)^n \frac{d^n}{dx^n} F_{y(n)} = 0 \qquad ...(i)$$

whereas by varying $z(x)$ and holding $y(x)$ fixed we find that the very same functions must satisfy the Euler-Poisson equation

$$F_z - \frac{d}{dx} F_{z'} + \frac{d^2}{dx^2} F_{z''} ... + (-1)^n \frac{d^n}{dx^n} F_{z(n)} = 0. \qquad ...(ii)$$

Thus, the functions $y(x)$ and $z(x)$ must satisfy a system of two equations (i) and (ii).

We can discuss in similar manner when investigating for the extremum of a functional dependant on any number of functions :

$$I[y_1, y_2, ..., y_m] = \int_{x_1}^{x_2} F(x, y_1, y_1', ..., y_1^{(n_1)}, y_2, y_2', ..., y_2^{(n_2)} \quad ..., y_m, y_m', ..., y_m^{(n_m)})\, dx$$

Varying some one function $y_i(x)$ and holding the others fixed, we obtain the basic necessary condition for an extremum in the form

$$F_{y_i} - \frac{d}{dx} y_i' + ... + (-1)^{n_i} \frac{d^{n_i}}{dx^{n_i}} F_{y_i^{(n_i)}} = 0, \qquad i = 1, 2, ..., m$$

1.9B. Solved examples based on Art. 1.9 and Art. 1.9A

Ex. 1. *Find the extremal of the functional* $I[y(x)] = \int_0^1 (1 + y''^2) dx, \quad y(0) = 0,$ $y'(0) = 1,\ y(1) = 1,\ y'(1) = 1$

Sol. Comparing the given functional with $\int_0^1 F(x, y, y', y'') dx,$ we get

$$F(x, y, y', y'') = 1 + y''^2 \qquad ...(1)$$

Euler-Poisson equation is $$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0 \qquad ...(2)$$

From (1), $F_y = 0,$ $F_{y'} = 0,$ $F_{y''} = 2y''$. Hence (2) reduces to

$$\frac{d^2}{dx^2}(2y'') = 0 \qquad \text{or} \qquad \frac{d^4 y}{dx^4} = 0$$

whose general solution is $$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3, \qquad ...(3)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

From (3), $$y' = C_2 + 2C_3 x + 3C_4 x^2 \qquad ...(4)$$

Since $y(0) = 0$ and $y(1) = 1$, (3) gives

$$C_1 = 0 \qquad \text{and} \qquad C_1 + C_2 + C_3 + C_4 = 1 \qquad ...(5)$$

Since $y'(0) = 1$ and $y'(1) = 1$, (4) gives

$$C_2 = 1 \qquad \text{and} \qquad C_2 + 2C_3 + 3C_4 = 1 \qquad ...(6)$$

Solving (5) and (6), $C_1 = 0,$ $C_2 = 1,$ $C_3 = 0$ and $C_4 = 0$

Hence, by (3), the required extremum can be attained only on the line $y = x$.

Ex. 2. *Find the extremal of the functional* $I[y(x)] = \int_{-1}^{0} (480y - y'''^2)\, dx,\ y(0) = y'(0) = y''(0) = 0$

(Nagpur 2005, 10)

Sol. Comparing the given functional with $\int_{-1}^{0} F(x, y, y', y'', y''') dx$, we have

$$F(x, y, y', y'', y''') = 480y - y''''^2 \qquad ...(1)$$

Euler-Poisson equation is $$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \frac{d^3}{dx^3} F_{y'''} = 0 \qquad ...(2)$$

From (1), $F_y = 480$, $F_{y'} = 0$, $F_{y''} = 0$, and $F_{y'''} = -2y'''$

$\therefore$ (2) reduces to $480 - \dfrac{d^3}{dx^3}\left(-2\dfrac{d^3y}{dx^3}\right) = 0$ or $\dfrac{d^6y}{dx^6} = -240$

Integrating it, $d^5y/dx^5 = -240x + C_1 \Rightarrow d^4y/dx^4 = -120x^2 + C_1x + C_2$

$\Rightarrow$ $d^3y/dx^3 = -40x^3 + (C_1/2)\times x^2 + C_2x + C_3$

$\Rightarrow$ $y''(x) = d^2y/dx^2 = -10x^4 + (C_1/6)\times x^3 + (C_2/2)\times x^2 + C_3x + C_4$ ...(3)

$\Rightarrow$ $y'(x) = dy/dx = -2x^5 + (C_1/18)\times x^3 + (C_2/6)\times x^3 + (C_3/2)\times x^2 + C_4x + C_5$ ...(4)

$\Rightarrow$ $y = -(1/3)\times x^6 + (C_1/72)\times x^5 + (C_2/24)\times x^4 + (C_3/6)\times x^3 + (C_4/2)\times x^2 + C_5x + C_6$ ...(5)

Putting $x = 0$ in (3), (4) and (5) and using the given conditions, we obtain $C_4 = C_5 = C_6 = 0$.

Then, (5) reduces to $y = -(x^3/3) + Ax^4 + Bx^5 + Cx^3$, ...(6)

where $A\,(= C_1/72)$, $B\,(= C_2/24)$ and $C\,(= C_3/6)$ are new arbitrary constants. The required extremals are curves given by (6).

Ex. 3(*a*) *Find the extremal of the functional* $I[y(x)] = \int_0^{\pi/2}(y''^2 - y^2 + x^2)dx$, $y(0) = 1$, $y'(0) = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = -1$ **(Kanpur 2003, Madras M.E. 2000)**

Sol. Comparing the given functional with $\int_0^{\pi/2} F(x, y, y', y'')dx$, we get

$$F(x, y, y', y'') = y''^2 - y^2 + x^2 \quad ...(1)$$

Euler-Poisson equation is $F_y - \dfrac{d}{dx}F_{y'} + \dfrac{d^2}{dx^2}F_{y''} = 0$ ...(2)

From (1), $F_y = -2y$, $F_{y'} = 0$, $F_{y''} = 2y''$. Therefore, (2) reduces to

$$-2y + \frac{d^2}{dx^2}(2y'') = 0 \quad \text{or} \quad -2y + 2\frac{d^4y}{dx^4} = 0$$

or $(D^4 - 1)y = 0$, where $D \equiv d/dx$ ...(3)

The auxiliary equation of (3) is $D^4 - 1 = 0$ *i.e.* $(D^2 - 1)(D^2 + 1) = 0$

whose roots are $D = \pm 1, \pm i$. Hence, the general solution of (3) is given by

$$y = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x, \quad ...(4)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

From (4), $y' = C_1e^x - C_2e^{-x} - C_3\sin x + C_4\cos x$ ...(5)

Since $y(0) = 1$ and $y(\pi/2) = 0$, (4) gives $C_1 + C_2 + C_3 = 0$ ...(6)

and $C_1\,e^{\pi/2} + C_2\,e^{-\pi/2} + C_4 = 0$ ...(7)

Since $y'(0) = 0$ and $y'(\pi/2) = -1$, (5) gives $C_1 - C_2 + C_4 = 0$ ...(8)

and $C_1\,e^{\pi/2} - C_2\,e^{-\pi/2} - C_3 = -1$ ... (9)

Adding (6) and (9), we get $C_1(1 + e^{\pi/2}) + C_2(1 - e^{-\pi/2}) = 0$ ...(10)

Subtracting (7) from (8), we get $C_1(1 - e^{\pi/2}) - C_2(1 + e^{-\pi/2}) = 0$ ...(11)

We now solve (10) and (11). Let D denote the determinant of coefficients of (10) and (11).

Then, $D = \begin{vmatrix} 1+e^{\pi/2} & 1-e^{-\pi/2} \\ 1-e^{\pi/2} & -\left(1+e^{-\pi/2}\right) \end{vmatrix} = -\left(1+e^{\pi/2}\right)\left(1+e^{-\pi/2}\right) - \left(1-e^{\pi/2}\right)\left(1-e^{-\pi/2}\right) = -4 \neq 0$

Since $D \neq 0$, the system of homogeneous equations (10) and (11) has only the trival solution *i.e.* $C_1 = 0$, $C_2 = 0$. With these values, (6) and (8) give $C_3 = 1$ and $C_4 = 0$. Hence, from (4), the required extremum can be attained only on the curve $y = \cos x$.

Ex. 3(*b*) *Find the extremal for the functional* $\int_0^{\pi/4} (y''^2 - y^2 + x^2)dx$, $y(0) = 0, y'(0) = 1,$ $y(\pi/4) = y'(\pi/4) = 1/\sqrt{2}$ **(Madras M.E. 2000)**

Sol. Proceed as in Ex. 3(*a*) [**Ans.** $y = \sin x$]

Ex. 4. *Find the extremal of the functional* $\int_{-l}^{l} \left(\frac{1}{2}\mu y''^2 + \rho y\right)dx$ *that satisfies the boundary conditions* $y(-l) = 0$, $y'(-l) = 0$, $y(l) = 0$, $y'(l) = 0$ **(Kanpur 2004)**

Sol. Comparing the given functional with $\int_{-l}^{l} F(x, y, y', y'')dx$, we get

$$F(x, y, y', y'') = (1/2) \times \mu y''^2 + \rho y \qquad ...(1)$$

Euler-Poisson equation is $$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0 \qquad ...(2)$$

From (1), $F_y = \rho$, $F_{y'} = 0$, $F_{y''} = \mu y''$. Therefore, (2) reduces to

$$\rho + \frac{d^2}{dx^2}(\mu y'') = 0 \quad \text{or} \quad \frac{d^4 y}{dx^4} = -\frac{\rho}{\mu} \quad \text{or} \quad D^4 y = -(\rho/\mu), \quad D \equiv d/dx \qquad ...(2)'$$

whose auxiliary equation is $D^4 = 0$, giving $D = 0, 0, 0, 0$ and so

C.F. $= C_1x^3 + C_2x^2 + C_3x + C_4$, C_1, C_2, C_3 and C_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4}\left(-\frac{\rho}{\mu}\right) = -\frac{\rho}{\mu}\frac{1}{D^3}\int dx = -\frac{\rho}{\mu}\frac{1}{D^3}x = -\frac{\rho}{\mu}\frac{1}{D^2}\int x\,dx$$

$$= -\frac{\rho}{\mu}\frac{1}{D^2}\frac{x^2}{2} = -\frac{\rho}{\mu}\frac{1}{D}\int \frac{x^2}{2}dx = -\frac{\rho}{\mu}\frac{1}{D}\frac{x^3}{6} = -\frac{\rho}{\mu}\int \frac{x^3}{6}dx = -\frac{\rho x^4}{24\mu}$$

$\therefore$ Solution of (2)′ is $$y = -(\rho/24\mu) \times x^4 + C_1x^3 + C_2x^2 + C_3x + C_4 \qquad ...(3)$$

$\therefore$ $$y' = -(\rho/6\mu) \times x^3 + 3C_1x^2 + 2C_2x + C_3 \qquad ...(4)$$

Using $y(-l) = 0$ and $y(l) = 0$, (3) gives

$$-(\rho/24\mu) \times l^4 - C_1l^3 + C_2l^2 - C_3l + C_4 = 0 \qquad ...(5)$$

and $$-(\rho/24\mu) \times l^4 + C_1l^3 + C_2l^2 + C_3l + C_4 = 0 \qquad ...(6)$$

Using $y'(-l) = 0$ and $y'(l) = 0$, (4) gives

$$(\rho/6\mu) \times l^3 + 3C_1l^2 - 2C_2l + C_3 = 0 \qquad ...(7)$$

and $$-(\rho/6\mu) \times l^3 + 3C_1l^2 + 2C_2l + C_3 = 0 \qquad ...(8)$$

Subtracting (6) from (5), we get $2C_1l^3 + 2C_3l = 0$ ...(9)

Adding (7) and (8), we get $6C_1l^2 + 2C_3 = 0$ ...(10)

Let D denote the determinant of the coefficients of the system of equation (9) and (10). Then,

$$D = \begin{vmatrix} 2l^3 & 2l \\ 6l^2 & 2 \end{vmatrix} = 4l^3 - 12l^3 = -8l^3 \neq 0$$

Since $D \neq 0$, the system of homogeneous equations (9) and (10) has only the trival solution $C_1 = C_3 = 0$. With these values (7) and (5) give $C_2 = \rho l^2/12\mu$ and $C_4 = -(\rho l^4/24\mu)$. Hence from (3), the required extremum can be achieved only, on the curve

$$y = -(\rho/24\mu)\times x^4 + (\rho l^2/12\mu)\times x^2 - (\rho/24\mu)\times l^4$$

or $y = -(\rho/24\mu)\times(x^4 - 2l^2x^2 + l^4)$ or $y = -(\rho/24\mu)\times(x^2 - l^2)^2$

Ex. 5(*a*) *Find the stationary function of the funcitonal* $\int_{x_1}^{x_2}(y'^2 + yy'')dx$, *subject to the conditions* $y(x_1) = \alpha$, $y'(x_1) = \beta$, $y(x_2) = \gamma$ and $y'(x_2) = \delta$.

(*b*) *Find the extremal of the functional* $\int_{x_1}^{x_2}(y + y'')dx$ that satisfies the boundary conditions $y(x_1) = \alpha$, $y'(x_1) = \beta$, $y(x_2) = \gamma$ and $y'(x_2) = \delta$.

Sol. (*a*) Comparing the given functional with $\int_{x_1}^{x_2}(x, y, y', y'')dx$, we get

$$F(x, y, y', y'') = y'^2 + yy'' \quad \text{... (1)}$$

Euler-Poisson equation is $F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0$ ... (2)

or $y'' - d(2y')/dx + d^2y/dx^2 = 0$, using (1)

so that $y'' - 2y'' + y'' = 0$ or $0 = 0$, which is not a differential equation. Hence there in no extremal and therefore no stationary function.

(b) Hint. As is part (*a*), $F(x, y, y', y'') = y + y''$. Hence Euler-Poisson equation yields $I - do/dx + d^2(1)/dx^2 = 0$, *i.e.*, $I = 0$, which is impossible. Hence the given problem has no solution, because it does not admit of any extremal.

Ex. 6. *Find the extremal and stationary function of the functional* $(1/2)\times\int_0^1 (y'')^2 dx$ *satisfying the boundary conditions* $y(0) = 0$, $y(1) = 1/2$, $y'(0) = 0$ *and* $y'(1) = 1$.

Sol. Comparing the given functional with $\int_0^1 F(x, y, y', y'')dx$, we get

$$F(x, y, y', y'') = (1/2)\times y'' \quad \text{... (1)}$$

Euler-Poisson equation is $F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0$ ... (2)

Using (1), (2) yields $0 - \frac{d(0)}{dx} + \frac{d^2y''}{dx^2} = 0$ or $\frac{d^4y}{dx^4} = 0$

$\therefore$ $d^3y/dx^3 = C_1 \Rightarrow d^2y/dx^2 = C_1x + C_2 \Rightarrow dy/dx = (C_1x^2)/2 + C_2x + C_3$

or $y = (C_1/6)\times x^3 + (C_2/2)\times x^2 + C_3x + C_4$ or $y = C_5x^3 + C_6x^2 + C_3x + C_4$ ... (3)

where $C_5(= C_1/6)$, $C_6(= C_2/2)$, C_3 and C_4 are arbitrary constants. (3) is the equation of the extremals

Differentiating (3) w.r.t. x gives $y' = 3C_5x^2 + 2C_6x + C_3$ ...(4)

Using the given boundary conditions $y(0) = 0$, $y(1) = 1/2$, (3) yields

$C_4 = 0$ and $C_5 + C_6 + C_3 + C_4 = 1/2$ so that $C_4 = 0$, $C_5 + C_6 + C_3 = 1/2$ ... (5)

Using the given boundary condiiton $y'(0) = 0$, $y'(1) = 1$, (4) yields

$C_3 = 0$ and $3C_5 + 2C_6 + C_3 = 1$ so that $C_3 = 0$, $3C_5 + 2C_6 = 1$ ...(6)

From (5) and (6), $C_5 + C_6 = 1/2$, $3C_5 + 2C_6 = 1$ so that $C_5 = 0$, $C_6 = 1/2$ ... (7)

Thus, $C_3 = C_4 = C_5 = 0$, $C_6 = 1/2$ and hence (3) gives $y = x^2/2$,

which is the equation of the required stationary function.

Ex. 7. *Find the extremal of the functional* $\int_{-1}^{0}(240y + y'''^2)dx$ *subject to the conditions* $y(-1) = 1$, $y(0) = 0$, $y'(-1) = -4.5$, $y'(0) = 0$, $y''(-1) = 16$ and $y''(0) = 0$.

Sol. Comparing the given functional with $\int_{-1}^{0} F(x, y, y', y'', y''')dx$, we have

$$F(x, y, y', y'', y''') = 240y + y'''^2 \quad \text{... (1)}$$

Euler-Poisson equation is $F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \frac{d^3}{dx^3}F_{y'''} = 0$

or $240 - \frac{d(0)}{dx} + \frac{d^2(0)}{dx^2} - \frac{d^3}{dx^3}(2y''') = 0$, using (1)

or $d^6y/dx^6 = 120 \Rightarrow d^5y/dx^3 = 120x + C_1$

$\Rightarrow d^4y/dx^4 = 60x^2 + C_1x + C_2 \Rightarrow d^3y/dx^3 = 20x^3 + (C_1/2) \times x^2 + C_2x + C_3$

$\Rightarrow d^2y/dx^2 = 5x^4 + (C_1/6) \times x^3 + (C_2/2) \times x^2 + C_3x + C_4$

$\Rightarrow dy/dx = x^5 + (C_1/24) \times x^4 + (C_2/6) \times x^3 + (C_3/2) \times x^2 + C_4x + C_5$

$\therefore y = x^6/6 + (C_1/120) \times x^5 + (C_2/24) \times x^4 + (C_3/6) \times x^3 + (C_4/2) \times x^2 + C_5x + C_6$

or $y = x^6/6 + C_7x^5 + C_8x^4 + C_9x^3 + C_{10}x^2 + C_5x + C_6$, ... (2)

where C_7 ($= C_1/120$), C_8 ($= C_2/24$), C_9 ($= C_3/6$) and C_{10} ($= C_4/2$) are new arbitrary constants.

(2) $\Rightarrow$ $y' = x^5 + 5C_7x^4 + 4C_8x^3 + 3C_9x^2 + 2C_{10}x + C_5$ ... (3)

and (3) $\Rightarrow$ $y'' = 5x^4 + 20C_7x^3 + 12C_8x^2 + 6C_9x + 2C_{10}$ ... (4)

Using the given boundary conditions $y(-1) = 1$, $y(0) = 0$, (2) yields

$1/6 - C_7 + C_8 - C_9 + C_{10} - C_5 + C_6 = 1$ and $C_6 = 0$ ... (5)

Using the given boundary conditions $y'(-1) = -(9/2)$, $y'(0) = 0$, (3) yields

$-1 + 5C_7 - 4C_8 + 3C_9 - 2C_{10} + C_5 = -9/2$ and $C_5 = 0$ ... (6)

Using the given boundary conditions $y''(-1) = 16$, $y''(0) = 0$, (4) yields

$5 - 20C_7 + 12C_8 - 6C_9 + 2C_{10} = 16$ and $C_{10} = 0$ ... (7)

Since $C_6 = C_5 = C_{10} = 0$, (5), (6) and (7) respectively reduce to

$C_7 - C_8 + C_9 = -5/6$, $5C_7 - 4C_8 + 3C_9 = -7/2$, $20C_7 - 12C_8 + 3C_9 = -11$

Solving these, $C_7 = 0$, $C_8 = 1$ and $C_9 = 1/6$

Thus, $C_6 = C_5 = C_{10} = C_7 = 0$, $C_8 = 1$ and $C_9 = 1/6$. Hence (2) yields $y = x^6/6 + x^4 + x^3/6$

or $y = (x^3/6) \times (x^3 + 6x + 1)$, which is the equation of the required extremal.

Ex. 4. *Find the extremals and the stationary function of the functional* $\int_0^1 (y'^2 + y''^2)dx$ *with boundary conditions* $y(0) = 0$, $y(1) = \sinh 1$, $y'(0) = 1$ and $y'(1) = \cosh 1$. **[MDU Rohtak 2004]**

Sol. Comparing the given functional with $\int_0^1 F(x, y, y', y'')dx$, we get

$$F(x, y, y', y'') = y'^2 + y''^2 \quad \text{... (1)}$$

Euler-Poisson equation is $F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0$ ... (2)

Using (1), (2) yields $0 - \frac{d(2y')}{dx} + \frac{d^2(2y'')}{dx^2} = 0$ or $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0$

or $(D^4 - D^2)y = 0$, where $D \equiv d/dx$ ... (3)

Its auxiliary equation is $D^4 - D^2 = 0$ giving $D = 0, 0, 1, -1$.

$\therefore$ solution of (3) is $y = C_1 + C_2x + C_3e^x + C_4e^{-x}$, ... (4)

where C_1, C_2, C_3 and C_4 are arbitrary constants. From (4), we have

$$y' = C_2 + C_3e^x - C_4e^{-x} \quad \text{... (5)}$$

Using the given boundary conditions $y(0) = 0$ and $y(1) = \sinh 1$, (4) yields

$C_1 + C_3 + C_4 = 0$ and $C_1 + C_2 + C_3e + C_4e^{-1} = \sinh 1$ ... (6)

Using the given boundary conditions $y'(0) = 0$ and $y'(1) = \cosh 1$, (5) yields

$C_2 + C_3 - C_4 = 1$ and $C_2 + C_3e - C_4e^{-1} = \cosh 1$ ... (7)

Solving (6) and (7), $C_1 = C_2 = 0$, $C_3 = 1/2$, $C_4 = -1/2$. Then (4) $\Rightarrow y = (e^x - e^{-x})/2 = \sinh x$ which is the desired equation of the required stationary function.

EXERCISE 1(C)

Find the extremals of the following functionals:

1. $I[y(x)] = \int_{x_1}^{x_2} (16y^2 - y''^2 + x^2)dx$ **[Kanpur 2005, 2010; KU Kurukshetra 2006]**

Ans. $y = C_1e^{2x} + C_2e^{-2x} + C_3\cos 2x + C_4\sin 2x$

2. $I[y(x)] = \int_{x_1}^{x_2} (2xy + y'''^2)dx$ **[Sagar 2004]**

Ans. $y = C_1 + C_2x + C_3x^2 + C_4x^3 + C_5x^4 + C_6x^6 + x^7/7!$

3. $I[y(x)] = \int_{x_1}^{x_2} (y''^2 - 2y'^2 + y^2 - 2y\sin x)dx$ **[Kanpur 2006]**

Ans. $y = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x - (x^2/4)\times\sin x$

4. $I[y(x)] = \int_{x_1}^{x_2} (y'''^2 + y^2 - 2yx^3)dx$.

Ans. $y = C_1e^x + C_2e^{-x} + e^{x/2}\left\{C_3\cos(x\sqrt{3}/2) + C_4\sin(x\sqrt{3}/2)\right\}$

$+ \ e^{-x/2}\left\{C_5\cos(x\sqrt{3}/2) + C_6\sin(x\sqrt{3}/2)\right\} + x^3$

5. $\int_{x_1}^{x_2} (360x^2y - y''^2)dx$. **Ans.** $y = x^6/2 + C_1x^3 + C_1x^2 + C_3x + C_4$

6. $\int_{x_1}^{x_2} (y''^2 - y^2 + x^2)dx$. **Ans.** $y = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x$

7. $\int_{x_1}^{x_2} (y^2 + 2y'^2 + y''^2)dx$. **[Kanpur 2007]** **Ans.** $y = (C_1 + C_2x)e^x + (C_3 + C_4x)e^{-x}$

8. $\int_a^b (4y^2 - y''^2 + 2x^2)dx$ **[MDU Rohtak 2006]**

Ans. $y = C_1e^{x\sqrt{2}} + C_2e^{-x\sqrt{2}} + C_3\cos(x\sqrt{2}) + C_4\sin(x\sqrt{2})$

9. Show that the extremal for functional $I[y(x)] = \int_0^{\pi} (y''^2 + 4y^2 - x^2 y')dx$, $y(0) = y(\pi/2) = y(3\pi/2) = y(\pi) = 0$ is $y = x/4 + (\pi/8 \sinh \pi) \times [2 \cos x \sinh x + \sin x \{3 \sinh (x - \pi/2) + \sinh (x - 3\pi/2)\}]$

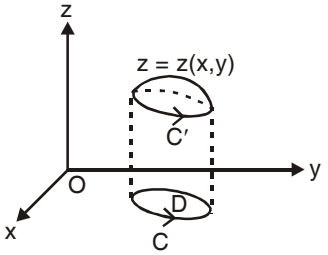
10. Show that the extremal for the functional $I[y(x)] = \int_{-1}^{1} (yy''' + y'''^2/2)dx$, $y(-1) = y(0) = y(1) = 0$, $y'(-1) = y'(0) = y'(1) = 1$ is given by $y = x + (3x^5 - 5x^3)/2$.

1.10 Functionals dependent on the functions of several independent variables

We propose to investigate the following functional for an extremum

$$I[z(x, y)] = \iint_D F(x, y, z, p, q)dx\,dy, \quad \text{where} \quad p = \partial z/\partial x \quad \text{and} \quad q = \partial z/\partial y \qquad ...(1)$$

the values of the function $z(x, y)$ are given on the boundary C of the domain D, that is, a spatial path (or contour) C' is given, through which all permissible surfaces have to pass (see the adjoining figure). We assume that F is thrice differentiable.



Take some admissible surface $z = \bar{z}(x, y)$ close to $z = z(x, y)$ and include the surfaces $z = z(x, y)$ and $z = \bar{z}(x, y)$ in a one-parameter family of surfaces

$$z(x, y, \alpha) = z(x, y) + \alpha\,\delta z \qquad \text{where} \qquad \delta z = \bar{z}(x, y) - z(x, y)$$

for $\alpha = 0$, we get the surface $z = z(x, y)$, for $\alpha = 1$, we have $z = \bar{z}(x, y)$. δz is called the variation of the function $z(x, y)$.

On functions of the family $z(x, y, \alpha)$, the functional I reduces to the function of α, which has to have an extremum for $\alpha = 0$. Hence, we have

$$[\partial I[z(x, y, \alpha)]/\partial\alpha]_{\alpha=0} = 0$$

As discussed in Art. 1.2, the derivative of $I[z(x, y, \alpha)]$ with respect to α, for $\alpha = 0$ is known as the variation of the function and is denoted by δI. Accordingly, we have

$$\delta I = \left[\frac{\partial}{\partial\alpha}\iint_D F(x, y, z(x, y, \alpha), p(x, y, \alpha), q(x, y, \alpha))\,dx\,dy\right]_{\alpha=0}$$

or
$$\delta I = \iint_D (F_z\,\delta z + F_p\,\delta p + F_q\,\delta q)dx\,dy, \qquad ...(2)$$

where
$$z(x, y, \alpha) = z(x, y) + \alpha\,\delta z$$

$$p(x, y, \alpha) = \frac{\partial z(x, y, \alpha)}{\partial x} = p(x, y) + \alpha\,\delta p \quad \text{and} \quad q(x, y, \alpha) = \frac{\partial z(x, y, \alpha)}{\partial y} = q(x, y) + \alpha\,\delta q$$

Now, we have
$$\frac{\partial(F_p\,\delta z)}{\partial x} = \frac{\partial F_p}{\partial x}\delta z + F_p\,\delta p \Rightarrow F_p\,\delta p = \frac{\partial\left(F_p\,\delta z\right)}{\partial x} - \frac{\delta F_p}{\partial x}\delta z$$

and
$$\frac{\partial(F_q\,\delta z)}{\partial y} = \frac{\partial F_q}{\partial y}\delta z + F_q\,\delta q \Rightarrow F_q\,\delta q = \frac{\partial(F_q\,\delta z)}{\partial y} - \frac{\partial F_q}{\partial y}\delta z$$

Using the above two results, we have

$$\iint_D (F_p\,\delta p + F_q\,\delta q)dx\,dy = \iint_D \left\{\frac{\partial}{\partial x}(F_p\,\delta z) + \frac{\partial}{\partial y}(F_q\,\delta z)\right\}dx\,dy - \iint_D \left(\frac{\partial F_p}{\partial x} + \frac{\partial F_q}{\partial y}\right)\delta z\,dx\,dy \qquad ...(3)$$

where $\partial F_p / \partial x$ is known as total partial derivative with respect to x. While computing it, y is assumed to be fixed, but the dependence of z, p and q upon x is taken into account. Hence, we get

$$\frac{\partial F_p}{\partial x} = F_{px} + F_{pz}\frac{\partial z}{\partial x} + F_{pp}\frac{\partial p}{\partial x} + F_{pq}\frac{\partial q}{\partial x}$$

Similarly,
$$\frac{\partial F_q}{\partial y} = F_{qy} + F_{qz}\frac{\partial z}{\partial x} + F_{qp}\frac{\partial p}{\partial y} + F_{qq}\frac{\partial q}{\partial y}$$

Using the well known Green's theorem*, we have

$$\iint_D \left\{\frac{\partial}{\partial x}(F_p \delta z) + \frac{\partial}{\partial y}(F_q \delta z)\right\} dx\, dy = \int_C \left(F_p\, dy - F_q\, dx\right)\delta z = 0 \qquad ...(4)$$

The last integral is equal to zero, since on the contour C the variation $\delta z = 0$ because all permissible surfaces pass through one and the same spatial contour C'. Using (4), (3) reduces to

$$\iint_D (F_p\ \delta p + F_q\ \delta q)dx\, dy = -\iint_D \left(\frac{\partial}{\partial x}F_p + \frac{\partial}{\partial y}F_q\right)\delta z\, dx\, dy \qquad ...(5)$$

Using (5), (2) reduces to $$\delta I = \iint_D F_z \delta z\ dx\ dy - \iint_D \left(\frac{\partial}{\partial x}F_p + \frac{\partial}{\partial y}F_q\right)\delta z\, dx\, dy$$

Hence the necessary condition $\delta I = 0$ for an extremum of the functional (1) takes the form

$$\iint_D \left(F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)\delta z\, dx\, dy = 0 \qquad ...(6)$$

Since the variation δz is arbitrary (only restrictions of a general nature are imposed on δz that have to do with continuity and differentiability, vanishing on the contour C, etc.) and the first factor is continuous, it follows from the fundamental lemma of the calculus variation (see Art. 1.3) that on the extremizing surface $z = z(x, y)$, we must have

$$F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q = 0, \qquad i.e., \qquad \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0, \qquad ...(7)$$

which is Euler's equation for an extremal of functional (1). As before, a stationary function (if one exists) is an extremal that satisfies the given boundary conditions.

Thus the required extremizing functon $z(x, y)$ is obtained from the solution of the second order partial differential equation (7) which is known as *Euler-Ostrogradsky equation.*

Remark 1. For the functional

$$I[z(x_1, x_2, ..., x_n)] = \iint ... \int_D F(x_1, x_2, ..., x_n, z, p_1, p_2, ..., p_n)dx_1 dx_2 ... dx_n$$

where $p_i = \partial z / \partial x_i$, in exactly similar way, we get from the basic necessary condition for an extremum $\delta I = 0$, the following Euler-Ostrogradsky equation

---

* **Green's theorem in plane.** If D is a closed region in the xy-plane bounded by a simple closed curve C and if $M(x,y)$ and $N(x,y)$ are continuous functions having continuous partial derivatives in D, then

$$\iint_D (\partial N / \partial x + \partial M / \partial y)\, dx\, dy = \int_C (N\, dy - M\, dx),$$

where C is traversed in the positive (anti-clockwise) direction.

$$F_z - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{p_i} = 0,$$

which the function $z = z(x_1, x_2, ..., x_n)$ extremizing the functional *I* must satisfy.

Remark 2. If the integrand of the functional *I* depends on derivatives of higher order, then, by applying several times the transformations used in driving the Euler-Ostrogradsky equation, we find, as the necessary condtion for an extremum, that the extremizing function must sastisfy an equation similar to the Euler-Poission equation (refer Art. 1.9)

For example, for the functional $I[z(x,y)] = \iint_D F(x, y, z, p, q, r, s, t)\, dx\, dy$ ...(*i*)

we get the equation $F_z - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q + \frac{\partial^2}{\partial x^2} F_r + \frac{\partial^2}{\partial x \partial y} F_s + \frac{\partial^2}{\partial y^2} F_t = 0,$ ...(*ii*)

where $p = \partial z/\partial x, \quad q = \partial z/\partial y, \quad r = \partial^2 z/\partial x^2, \quad s = \partial^2 z/\partial x \partial y, \quad t = \partial^2 z/\partial y^2$

Hence the fourth-order partial differential equation (*ii*) must be satisfied by the function extremizing the functional *I* given by (*i*).

1.10A. ILLUSTRATIVE SOLVED EXAMPLES BASED ON ART. 1.10

Ex. 1. *Obtain the Euler-Ostrogradsky equation for* $I[z(x,y)] = \iint_D \{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\} dxdy$ *where the values of z are prescribed on the boundary C of the domain D.*

Sol. Writing $\partial z/\partial x = p, \partial z/\partial y = q,$ the given functional takes the form

$$I[z(x,y)] = \iint_D (p^2 + q^2) dx\, dy \qquad ...(1)$$

Comparing (1) with $\iint_D F(x, y, z, p, q) dx\, dy$, here $F(x, y, z, p, q) = p^2 + q^2$ ...(2)

Euler-Ostrogradsky equation for extremal of (1) is given by

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0 \qquad ...(3)$$

(2) $\Rightarrow$ $\partial F/\partial z = 0, \quad \partial F/\partial p = 2p = 2(\partial z/\partial x), \quad \partial F/\partial q = 2q = 2(\partial z/\partial y)$

$\therefore$ (3) becomes $-\frac{\partial}{\partial x}\left(2\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(2\frac{\partial z}{\partial y}\right) = 0$ or $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0,$

which is well known Laplace equation. We are to find a solution, continuous in *D*, of this equation that takes on prescribed values on the boundary *C* of the domain *D*. This is one of the basic problems of mathematical physics, known as *Dirichlet problem*.

Ex. 2. *Obtain the Euler-Ostrogradsky equation for*

$I[z(x,y)] = \iint_D \left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 2z\, f(x,y)\right\} dx\, dy$, *where the values of z are prescribed on the boundary C of the domain.* **(Kanpur 2006, 07, 08)**

Sol. Writing $\partial z/\partial x = p, \partial z/\partial y = q,$ the given functional is

$$I[z(x,y)] = \iint_D \{p^2 + q^2 + 2z\, f(x,y)\} dxdy \qquad ...(1)$$

Comparing (1) with $\iint_D F(x,y,z,p,q)\,dxdy$, $\quad F(x,y,z,p,q) = p^2 + q^2 + 2z\ f(x,y)$ ...(2)

Euler-Ostrogradsky-equation for extremal of (1) is given by

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0 \qquad ...(3)$$

(2) $\Rightarrow$ $\quad \partial F/\partial z = 2f(x,y), \quad \partial F/\partial p = 2p = 2(\partial z/\partial x), \quad \partial F/\partial q = 2q = 2(\partial z/\partial y)$

$\therefore$ (3) becomes $\quad 2f(x,y) - \frac{\partial}{\partial x}\left(2\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(2\frac{\partial z}{\partial y}\right) = 0 \qquad$ or $\qquad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x,y),$

which is well known *Poisson's equation*. We are to find a solution, continuous in D of this equation that takes on prescribed values on the boundary C of the domain D.

Ex. 3. *Obtain the surface of minimum area, stretched over a given closed curve C, enclosing the domain D in the xy plane.*

Sol. From advanced calculus, we know that the required given problem reduces to find the extremal of the functional

$$S[z(x,y)] = \iint_D \left\{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2\right\}^{1/2} dx\ dy \ \ i.e.\ \ S[z(x,y)] = \iint_D (1 + p^2 + q^2)^{1/2}\,dxdy \quad ...(1)$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Comparing (1) with $\iint_D F(x,y,z,p,q)\,dx\,dy$

$$F(x,y,z,p,q) = (1 + p^2 + q^2)^{1/2} \qquad ...(2)$$

Euler-Ostrogradsky equation for extremal of (1) is given by

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0 \qquad ...(3)$$

(2) $\Rightarrow$ $\quad \partial F/\partial z = 0, \quad \partial F/\partial p = p(1 + p^2 + q^2)^{-1/2}, \quad \partial F/\partial q = q(1 + p^2 + q^2)^{-1/2}$

$\therefore$ (3) yields $\quad -\frac{\partial}{\partial x}\ \frac{p}{(1 + p^2 + q^2)^{1/2}} - \frac{\partial}{\partial y}\ \frac{q}{(1 + p^2 + q^2)^{1/2}} = 0$

or $\quad \frac{\partial}{\partial x}\ \frac{z_x}{(1 + z_x^2 + z_y^2)^{1/2}} + \frac{\partial}{\partial y}\ \frac{z_y}{(1 + z_x^2 + z_y^2)^{1/2}} = 0, \qquad ...(4)$

where $z_x = \partial z/\partial x$ and $z_y = \partial z/\partial y$. In what follows, we shall use the usual notations, namely, $z_{xx} = \partial^2 z/\partial x^2$, $z_{yy} = \partial^2 z/\partial y^2$, $z_{xy} = \partial^2 z/\partial x \partial y$ and $z_{yx} = \partial^2 z/\partial y \partial x$. Of course $z_{xy} = z_{yx}$. Then, (4) reduces to

$$z_{xx}(1 + z_x^2 + z_y^2)^{-1/2} - (1/2) \times z_x(1 + z_x^2 + z_y^2)^{-3/2} \times 2(z_x z_{xx} + z_y z_{yx})$$
$$+ z_{yy}(1 + z_x^2 + z_y^2)^{-1/2} - (1/2) \times z_y(1 + z_x^2 + z_y^2)^{-3/2} \times 2(z_x z_{xy} + 2z_y z_{yy}) = 0$$

or $\quad z_{xx}(1 + z_x^2 + z_y^2) - z_x(z_x z_{xx} + z_y z_{xy}) + z_{yy}(1 + z_x^2 + z_y^2) - z_y(z_x z_{xy} + z_y z_{yy}) = 0$, as $z_{xy} = z_{yx}$

or $\quad z_{xx}(1 + z_y^2) + z_{yy}(1 + z_x^2) - 2z_x z_y z_{xy} = 0$

i.e. $\quad \frac{\partial^2 z}{\partial x^2}\left\{1 + (\partial z/\partial y)^2\right\} + \frac{\partial^2 z}{\partial y^2}\left\{1 + (\partial z/\partial x)^2\right\} - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y} = 0,$

whose solution will yield the desired minimal surface.

Ex. 4. *Obtain the Euler-Ostrogrodsky equation for*

$$I[u(x,y,z)] = \iiint_D \left\{(\partial u/\partial x)^2 + (\partial u/\partial y)^2 + (\partial u/\partial z)^2\right\} dx\,dy\,dz$$

where the values of u are prescribed on the boundary C of the doamin D.

Sol. Let $p_1 = \partial u/\partial x$, $p_2 = \partial u/\partial y$, $p_3 = \partial u/\partial z$. Then the given functional can be re-written as

$$I[u(x,y,z)] = \iiint_D (p_1^2 + p_2^2 + p_3^2)\,dx\,dy\,dz \qquad ...(1)$$

Comparing (1) with $\iiint_D F(x,y,z,u,p_1,p_2,p_3)\,dx\,dy\,dz$, we get

$$F(x,y,z,u,p_1,p_{2,}p_3) = p_1^2 + p_2^2 + p_3^2 \qquad ...(2)$$

Euler-Ostrogradsky equation (refer remark 1 of Art. 1.10) for extremal of (1) is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial p_2}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial p_3}\right) = 0 \quad \text{or} \quad 0 - \frac{\partial}{\partial x}(2p_1) - \frac{\partial}{\partial y}(2p_2) - \frac{\partial}{\partial z}(2p_3) = 0 \text{, using (2)}$$

or $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = 0 \qquad \text{or} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Ex. 5. *Find the fourth-order partial differential equation satisfied by the function extremizing the functional* $I[z(x,y)] = \iint_D \left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 2(\partial^2 z/\partial x \partial y)^2\right\} dx\,dy$,

where the values of z are prescribed on the boundary C of the domain D.

Sol. Let $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$ and $t = \partial^2 z/\partial y^2$. Then the given funcitonal can be re-written as $I[z(x,y)] = \iint_D (r^2 + t^2 + 2s^2)\,dx\,dy$ ...(1)

Comparing (1) with $\iint_D F(x,y,z,p,q,r,s,t)\,dx\,dy$, we get

$$F(x,y,z,p,q,r,s,t) = r^2 + t^2 + 2s^2 \qquad ...(2)$$

Then, the partial differential equation satisfied by the function extremizing the functional (1) is given by $$F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q + \frac{\partial^2}{\partial x^2}F_r + \frac{\partial^2}{\partial x\,\partial y}F_s + \frac{\partial^2}{\partial y^2}F_t = 0$$

or $$0 - \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(0) + \frac{\partial^2}{\partial x^2}(2r) + \frac{\partial^2}{\partial x \partial y}(4s) + \frac{\partial}{\partial y^2}(2t) = 0 \text{, using (2)}$$

or $$2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 z}{\partial x^2}\right) + 4\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 z}{\partial x \partial y}\right) + 2\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 z}{\partial y^2}\right) = 0 \qquad \text{or} \qquad \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = 0,$$

which is well known *biharmonic equation*. It can be expressed as $\nabla^2\nabla^2 z = 0$

EXERCISE 1(D)

1. Show that the Euler-Ostrogradsky equation for the functional (which is obtained while finding a surface of minimal area stretched on a given contour)

$S[z(z,y)] = \iint_D \left\{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2\right\}^{1/2} dx\,dy$ is given by

$$\frac{\partial^2 z}{\partial x^2}\left\{1+\left(\partial z/\partial y\right)^2\right\}-2\frac{\partial z}{\partial x}\,\frac{\partial z}{\partial y}\,\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}\left\{1+\left(\partial z/\partial x\right)^2\right\}=0$$

2. Show that the fourth-order partial differential equation satisfied by the function extremizing the functional $I[z(x,y)] = \iint_D \{(\partial^2 z/\partial x^2)^2+(\partial^2 z/\partial y^2)^2+2(\partial^2 z/\partial x\partial y)^2-2z\,f(x,y)\}\,dxdy$

is given by $\partial^4 z/\partial x^4+2(\partial^4 z/\partial x^2\partial y^2)+\partial^4 z/\partial y^4 = f(x,y)$

3. Show that Ostrogradsky equation for $I[z(x,y)] = \iint_D \left\{(\partial z/\partial x)^2-(\partial z/\partial y)^2\right\} dx\,dy$

is given by $(\partial^2 z/\partial x^2)-(\partial^2 z/\partial y^2)=0$

4. Show that the Ostrogradsky equation for the functional

$$I[u(x,y,z)] = \iiint_D \left\{(\partial u/\partial x)^2+(\partial u/\partial y)^2+(\partial u/\partial z)^2+2u\ f(x,y,z)\right\} dx\,dy\,dz$$

is given by $\partial^2 u/\partial x^2+\partial^2 u/\partial y^2+\partial^2 u/\partial z^2 = f(x,y,z)$ **(Vikram 2004)**

5. Obtain the fourth-order partial differential equation satisfied by the function extremizing the following functionals: (*i*) $I[z(x,y)] = \iint_D \left\{(\partial z/\partial x)^2+(\partial z/\partial y)^2\right\} dxdy$

(*ii*) $I[z(x,y)] = \iint_D [(\partial^2 z/\partial x^2+\partial^2 z/\partial y^2)^2-2(1-\mu)\ \{(\partial^2 z/\partial x^2)(\partial^2 z/\partial y^2)$

$-(\partial^2 z/\partial x\partial y)^2\}]dxdy$, where μ is a parameter.

6. Show that the extremal for functional $I[z(x),z(y)] = \iint_D \left\{(\partial z/\partial x)^2+(\partial z/\partial y)^2\right\} dxdy$

is solution of Laplace equation $\partial^2 z/\partial x^2+\partial^2 z/\partial y^2 = 0$

7. Find the general solution of the extremal of the following functionals:

(*i*) $\iint (p^2xy/2-qx^2y^2/2)\,dx\,dy$, where $p=\partial z/\partial x, q=\partial z/\partial y$

(*ii*) $\iint (q^2xy/2-px^2y^2/2)\,dx\,dy$, where $p=\partial z/\partial x, q=\partial z/\partial y$

(*iii*) $\iint (xyz+ypq+xp^2)dxdy$, where $p=\partial z/\partial x, q=\partial z/\partial y$

Ans. (*i*) $z = C_1(y)\log x + C_2(y) + x^3/9$ (*ii*) $z = C_1(x)\log y + C_2(y) + y^3/9$ (*iii*) $z = C_1(y) - C_2(y)/(2x^2) + yx^2/15$

1.11 Variational problems in parametric form

While investigating a given functional $I[y(x)] = \int_{x_1}^{x_2} F(x,y,y')dx$ ...(1)

for an extremum, it is more convenient and sometimes advisable to obtain the solution in the parametric form $x = x(t), y = y(t)$; then the functional (1) will be reduced to the following form:

$$I[x(t), y(t)] = \int_{t_1}^{t_2} F\left(x(t), y(t), \frac{\dot{y}(t)}{\dot{x}(t)} \right) \dot{x}(t) dt \,, \qquad ...(2)$$

where the integration is taken along the line $x = x(t), \quad y = y(t), \quad$ for $\quad t_1 \le t \le t_2 \qquad ...(3)$

and a dot denotes derivative with respect to *t*.

It is to be noted that after transformation of the variables, the integrand in (2) does not contain *t* explicitly and, with respect to the variables $\dot{x}$ and $\dot{y}$, is a homogeneous function of the first degree.

It follow that the functional *I [x(t), y(t)]* is not an arbitrary functional of the form

$$\int_{t_1}^{t_2} \phi\left[t, x(t), y(t), \dot{x}(t), \dot{y}(t)\right] dt$$

that depends on two functions $x(t)$ and $y(t)$, but only an extremely particular case of such a functional, since its integrand does not contain t explicitly and is a homogeneous function of the first degree in the variables $\dot{x}$ and $\dot{y}$.

It we were to choose any other parametric representation of the desired curve $x = x(\tau)$, $y = y(\tau)$ then the functional $I[x, y]$ would be reduced to the form

$$\int_{\tau_1}^{\tau_2} F(x(\tau), y(\tau), \dot{y}_\tau / \dot{x}_\tau)\, \dot{x}_\tau \, d\tau$$

where $\dot{x}_\tau = dx/d\tau$ and $\dot{y}_\tau = dy/d\tau$. Consequently, we see that the integrand of the functional *I* does not change its form when the parametric representation of the curve is changed. Thus, the functional I depends on the type of curve and not its parametric representation.

For further discussion we now state and prove the following theorem.

Theorem. *If the integrand of the functional* $I\,[x(t), y(t)] = \int_{t_1}^{t_2} \phi(t, x(t), y(t), \dot{x}(t), \dot{y}(t)) dt$

does not contain t explicitly and is a homogeneous function of the first degree in $\dot{x}$ *and* $\dot{y}$*, the the functional I[x(t), y(t)] depends only on the type of the curve x = x(t), y = y(t), and not its parametric representation.*

Proof. Assume that $I[x(t), y(t)] = \int_{t_1}^{t_2} \phi\left(x(t), y(t), \dot{x}(t), \dot{y}(t)\right) dt \qquad ...(4)$

where $\phi(x, y, k\dot{x}, k\dot{y}) = k\,\phi(x, y, \dot{x}, \dot{y}) \qquad ...(5)$

Let us consider a new parametric representation

$$\tau = \phi(t) \qquad [\dot{\phi}(t) \ne 0], \qquad x = x(\tau), \qquad y = y(\tau) \qquad ...(6)$$

From (4) and (6), we obtain

$$\int_{t_1}^{t_2} \phi\left(x(t), y(t), \dot{x}(t), \dot{y}(t)\right) dt = \int_{\tau_1}^{\tau_2} \phi\left(x(\tau), y(\tau), \dot{x}_\tau(\tau)\dot{\phi}(t), y_\tau(\tau)\dot{\phi}(t)\right) \frac{d\tau}{\dot{\phi}(t)} \qquad ...(7)$$

Since ϕ is a homogeneous fuction of the first degree in *x* and *y*, we get

$$\phi(x, y, \dot{x}_\tau \dot{\phi}, \dot{y}_\tau \dot{\phi}) = \dot{\phi}\,\phi(x, y, \dot{x}_\tau, \dot{y}_\tau)$$

Using the above fact, (7), reduces to $\int_{t_1}^{t_2} \phi(x, y, \dot{x}_t, \dot{y}_t) dt = \int_{\tau_1}^{\tau_2} \phi(x, y, \dot{x}_\tau, \dot{y}_\tau) d\tau,$

showing that the integrand has not changed with a change in the parametric representation.

The arc length $\int_{t_1}^{t_2} (\dot{x}^2 + \dot{y}^2)^{1/2} dt$ and an area bounded by a ceratin curve $(1/2) \times \int_{t_1}^{t_2} (x\dot{y} - y\dot{x})\, dt$ are examples of such functionals.

In order to find the extremals of the functionals of this kind, $I[x(t), y(t)] = \int_{t_1}^{t_2} \phi(x, y, \dot{x}, \dot{y})\, dt$, where ϕ is a homogeneous function of the first degree in x and y, and also for functionals with an arbitrary function $\phi(t, x, y, \dot{x}, \dot{y})$, we have to solve a system of Euler's equations (see Art. 1.8)

$$\phi_x - \frac{d}{dt}\phi_{\dot{x}} = 0 \qquad \text{and} \qquad \phi_y - \frac{d}{dt}\phi_{\dot{y}} = 0 \qquad ...(8)$$

However, in the case under consideration, equations in (8) are not independent, since they must be satisfied by a certain solution $x = x(t)$, $y = y(t)$ and also by any other pairs of functions with a different parametric representation of the same curve, which in the case of Euler's equations being independent, would lead to a contradiction with the theorem of the existence and uniqueness of a solution of a system of differential equations. Hence, it follows that in (8), any one equation is a consequence of the other. Accordingly, to find the extremals, we must consider one of the Euler equations (8) and integrate it together with the equation defining the choice of parameter. For example, to the equation $\phi_{\dot{x}} - d\phi_{\dot{x}}/dt = 0$ we can adjoin the equation $\dot{x}^2 + \dot{y}^2 = 1$, which indicates that the arc length of the curve is taken as the parameter.

Remark. The Weirstrassian form of Euler's equations (8) is given by

$$\frac{1}{\rho} = \frac{\phi_{x\dot{y}} - \phi_{y\dot{x}}}{\phi_1 (\dot{x}^2 + \dot{y}^2)^{3/2}}, \qquad ...(9)$$

where ρ is the radius of curvature of the extremal and ϕ_1 is the common value of the ratios

$$\phi_1 = \phi_{\dot{x}\dot{x}} \Big/ \dot{y}^2 = \phi_{\dot{y}\dot{y}} \Big/ \dot{x}^2 = -(\phi_{\dot{x}\dot{y}} \Big/ \dot{x}\dot{y}) \qquad ...(10)$$

1.11A Euler's equation in parameter form (Alternative proof)

Consider a functional of the form

$$I[x(t), y(t)] = \int_{t_1}^{t_2} F(t, x, y, \dot{x}, \dot{y})dt, \qquad \dot{x} = dx/dt, \qquad \dot{y} = dy/dt \qquad ...(1)$$

subject to given fixed boundary conditions at t_1 and t_2, where x and y both are functions of another parameter t. Proceeding as in previous articles, introduce parameter α in (1) so that

$$I(\alpha) = \int_{t_1}^{t_2} F(t, x + \alpha\,\eta_1, y + \alpha\,\eta_2,\ \dot{x} + \alpha\,\dot{\eta}_1, \dot{y} + \alpha\,\dot{\eta}_2)dt, \qquad ...(2)$$

where η_1 and η_2 are functions of t and both vanish at t_1 and t_2 and $\dot{\eta}_1 = d\eta_1/dt$, $\dot{\eta}_2 = d\eta_2/dt$.

Now expanding the integrand is (2) in Taylor's series, we get

$$I(\alpha) = \int_{t_1}^{t_2} \{F(t, x, y, \dot{x}, \dot{y}) + \alpha\eta_1 \frac{\partial F}{\partial x} + \alpha\eta_2 \frac{\partial F}{\partial y} + \alpha\dot{\eta}_1 \frac{\partial F}{\partial \dot{x}} + \alpha\dot{\eta}_2 \frac{\partial F}{\partial \dot{y}}$$

$$+ \text{terms containing } \alpha^2 \text{ and higher powers of } \alpha\}dt$$

$$\therefore \quad \frac{dI(\alpha)}{d\alpha} = \int_{t_1}^{t_2} \{\eta_1 \frac{\partial F}{\partial x} + \eta_2 \frac{\partial F}{\partial y} + \dot{\eta}_1 \frac{\partial F}{\partial \dot{x}} + \dot{\eta}_2 \frac{\partial F}{\partial \dot{y}}$$

$$+ \text{terms containing } \alpha \text{ and its higher powers}\}dt \qquad ...(3)$$

Now, $$\int_{t_1}^{t_2} \dot{\eta}_1 \frac{\partial F}{\partial \dot{x}} dt = \left[\eta_1 \frac{\partial F}{\partial \dot{x}}\right]_{t_1}^{t_2} - \int_{t_2}^{t_2} \eta_1 \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)dt = -\int_{t_1}^{t_2} \eta_1 \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)dt$$

$$\left[\because \left[\eta_1\right]_{t_1}^{t_2} = \eta_1(t_2) - \eta_1(t_1) = 0\right]$$

and $$\int_{t_1}^{t_2} \dot{\eta}_2 \frac{\partial F}{\partial \dot{y}} dt = \left[\eta_2 \frac{\partial F}{\partial \dot{y}} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta_2 \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}} \right) dt = -\int_{t_1}^{t_2} \eta_2 \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}} \right) dt$$

$$\left[\because \ [\eta_2]_{t_1}^{t_2} = \eta_2(t_2) - \eta_2(t_1) = 0 \right]$$

Hence (3) may be re-written as

$$\frac{dI(\alpha)}{d\alpha} = \int_{t_1}^{t_2} \left[\eta_1 \left\{ \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}} \right) \right\} + \eta_2 \left\{ \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}} \right) \right\} + \text{terms containing } \alpha \text{ and its higher powers} \right] dt$$

Applying extremum condition $dI(\alpha)/d\alpha = 0$ at $\alpha = 0$, we have

$$\int_{t_1}^{t_2} \eta_1 \left\{ \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}} \right) \right\} dt + \int_{t_1}^{t_2} \eta_2 \left\{ \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}} \right) \right\} dt = 0$$

giving $$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}} \right) = 0 \qquad ...(4)$$

and $$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}} \right) = 0 \qquad ...(5)$$

Equations (4) and (5) are the required Euler's equations in parametric form. It should be noted that these two equations are not independent but one equation is a consequence of the other to obtain the extremals.

1.11B. SOLVED EXAMPLES BASED ON ART. 1.11 AND 1.11A

Ex. 1. *Find the extremal for the functional* $I\left[x(t), y(t)\right] = \int_{t_1}^{t_2} \{(\dot{x}^2 + \dot{y}^2)^{1/2} + a^2(x\dot{y} - y\dot{x})\}\, dt$, *a* being a constant

Sol. Comparing the given functional with $I[x(t), y(t)] = \int_{t_1}^{t_2} \phi(x, y, \dot{x}, \dot{y}) dt$, we get

$$\phi(x, y, \dot{x}, \dot{y}) = (\dot{x}^2 + \dot{y}^2)^{1/2} + a^2(x\dot{y} - y\dot{x}) \qquad ...(i)$$

Clearly, $\phi(x, y, \dot{x}, \dot{y})$ is homogeneous function of first degree in $\dot{x}$ and $\dot{y}$. The Weirstrassian form of Euler's equation is given by (9) (refer Art. 1.11) where ϕ_1, is given by (10). (refer Art. 1.11). From (i), we have

$\phi_x = a^2\dot{y}$ so that $\phi_{x\dot{y}} = a^2$; $\phi_y = -a^2\dot{x}$ so that $\phi_{y\dot{x}} = -a^2$ ...(*ii*)

Also, $\phi_{\dot{x}} = (1/2) \times (\dot{x}^2 + \dot{y}^2)^{-1/2} \times 2\dot{x} - a^2 y = \dot{x}\,(\dot{x}^2 + \dot{y}^2)^{-1/2} - a^2 y$

$\Rightarrow$ $\phi_{\dot{x}\dot{x}} = (\dot{x}^2 + \dot{y}^2)^{-1/2} + \dot{x}\left\{-(1/2) \times (\dot{x}^2 + \dot{y}^2)^{-3/2} \times 2\dot{x}\right\}$

Thus, $$\phi_{\dot{x}\dot{x}} = \frac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} - \frac{\dot{x}^2}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{\dot{y}^2}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \qquad ...(iii)$$

$\therefore$ Using result (10) of Art. 1.11, here $\phi_1 = \phi_{\dot{x}\dot{x}} / \dot{y}^2 = 1/(\dot{x}^2 + \dot{y}^2)^{3/2}$ ...(*iv*)

From (9) of At. 1.11, (*ii*) and (*iv*), $\frac{1}{\rho} = \frac{a^2 - (-a^2)}{1}$ so that $\rho = \frac{1}{2a^2}$,

showing that ρ is constant and hence the extremals are circles.

Alternative solution. Comparing the given functional with $\int_{t_1}^{t_2} \phi(t, x, y, \dot{x}, \dot{y}) dt$, we get

$$\phi(t, x, y, \dot{x}, \dot{y}) = (\dot{x}^2 + \dot{y}^2)^{1/2} + a^2(x\dot{y} - y\dot{x}) \quad ...(1)$$

Euler's equations in parametric form are given by

$$\frac{\partial \phi}{\partial x} - \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{x}}\right) = 0 \quad ...(2)$$

and

$$\frac{\partial \phi}{\partial y} - \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{y}}\right) = 0 \quad ...(3)$$

From (1), $\partial\phi / \partial x = a^2 \dot{y}$, $\partial\phi / \partial \dot{x} = \dot{x}(\dot{x}^2 + \dot{y}^2)^{-1/2} - a^2 y$ ...(4)

and

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{x}}\right) = \ddot{x}(\dot{x}^2 + \dot{y}^2)^{-1/2} - \dot{x}(\dot{x}^2 + \dot{y}^2)^{-3/2}(\dot{x}\ddot{x} + \dot{y}\ddot{y}) - a^2\dot{y}, \quad ...(5)$$

where $\ddot{x} = d^2x/dt^2$ and $\ddot{y} = d^2y/dt^2$. Using (4) and (5), (2) reduces to

$$2a^2\dot{y} - (\dot{x}^2 + \dot{y}^2)^{-3/2}\left\{\ddot{x}(\dot{x}^2 + \dot{y}^2 - \dot{x}^2\ddot{x} - \dot{x}\dot{y}\ddot{y}\right\} = 0$$

or $2a^2 = (\dot{x}^2 + \dot{y}^2)^{-3/2}(\dot{y}\ddot{x} - \dot{x}\ddot{y})$ or $(\dot{x}^2 + \dot{y}^2)^{3/2} / (\dot{y}\ddot{x} - \dot{x}\ddot{y}) = 1/2a^2$ ...(6)

Again, from (1) $\partial\phi / \partial y = -a^2\dot{x}$, $\partial\phi / \partial \dot{y} = \dot{y}(\dot{x}^2 + \dot{y}^2)^{1/2} + a^2 x$ ...(7)

and

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{y}}\right) = \ddot{y}(\dot{x}^2 + \dot{y}^2)^{-1/2} - \dot{y}(\dot{x}^2 + \dot{y}^2)^{-3/2}(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + a^2\dot{x} \quad ...(8)$$

Using (7) and (8) , (3) reduces to $-2a^2\dot{x} - (\dot{x}^2 + \dot{y}^2)^{-3/2}\{\ddot{y}(\dot{x}^2 + \dot{y}^2) - \dot{y}\dot{x}\ddot{x} - \dot{y}^2\ddot{y}\} = 0$

or $2a^2 = (\dot{x}^2 + \dot{y}^2)^{-3/2} / (\dot{y}\ddot{x} - \dot{x}\ddot{y})$ or $(\dot{x}^2 + \dot{y}^2)^{-3/2} / (\dot{y}\ddot{x} - \dot{x}\ddot{y}) = 1/2a^2$ ...(9)

Using the value of radius of curvature ρ for a parametric curve $x = x(t)$, $y = y(t)$ as found in calculus, (6) and (9) both imply that $\rho = 1/2a^2 =$ constant,
which shows that the extremals are circles.

Ex. 2. *Find the extremal of the functional* $I[x(t), y(t)] = \int_{t=0}^{\pi/4} (\dot{x}\dot{y} + 2x^2 + 2y^2) dt$,

$\dot{x} = dx/dt$, $y = dy/dt$, *subject to the initial conditions at* $t = 0$, $x = y = 0$; at $t = \pi/4, x = y = 1$.

Sol. Comparing the given functional with $\int_0^{\pi/4} \phi(t, x, y, \dot{x}, \dot{y}) dt$, we get

$$\phi(t, x, y, \dot{x}, \dot{y}) = \dot{x}\dot{y} + 2x^2 + 2y^2. \quad ...(1)$$

Euler's equation in parametric form are given by

$$\frac{\partial \phi}{\partial x} - \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{x}}\right) = 0 \quad ...(2)$$

and

$$\frac{\partial \phi}{\partial y} - \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{y}}\right) = 0 \quad ...(3)$$

From (I), $\frac{\partial \phi}{\partial x} = 4x$, $\frac{\partial \phi}{\partial \dot{x}} = \dot{y}$ and $\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{x}}\right) = \ddot{y}$.

Hence (2) reduces to $4x - \ddot{y} = 0$ or $\ddot{y} - 4x = 0$ ...(4)

Again, from (1), $\frac{\partial \varphi}{\partial y} = 4y$, $\frac{\partial \varphi}{\partial \dot{y}} = \dot{x}$ and $\frac{d}{dt}\left(\frac{\partial \varphi}{\partial \dot{y}}\right) = \ddot{x}$

Hence (3) reduces to $4y - \ddot{x} = 0$ or $\ddot{x} - 4y = 0$ ...(5)

In order to find the required extermal curve, we shall solve the system of simultaneous equations (4) and (5). Differentiating (5) twice with respect to 't', we have

$(d^4x/dt^4) - 4\ddot{y} = 0$ or $(d^4x/dt^2) - 16x = 0$, by (4) ...(6)

Re-writing (6), $(D^4 - 16)x = 0$ where $D \equiv d/dt$ ...(7)

The auxiliary equation of (7) is $D^4 - 16 = 0$ or $(D^2 - 4)(D^2 + 4) = 0$.

So $D = \pm 2, \pm 2i$. Hence the general solution of (7) is

$$x = c_1 \cosh 2t + c_2 \sinh 2t + c_3 \cos 2t + c_4 \sin 2t \quad ...(8)$$

Now differentiating (8) twice w.r.t. 't', we have

$$\ddot{x} = 4(c_1 \cosh 2t + c_2 \sinh 2t - c_3 \cos 2t - c_4 \sin 2t) \quad ...(9)$$

Substituting this value of $\ddot{x}$ in (5), we have

$$y = c_1 \cosh 2t + c_2 \sinh 2t - c_3 \cos 2t - c_4 \sin 2t \quad ...(10)$$

Given that $x = y = 0$ when $t = 0$. Hence (8) and (10) yield

$c_1 + c_3 = 0$ and $c_1 - c_3 = 0$ so that $c_1 = c_3 = 0$ ...(11)

Also, given that $x = y = 1$ when $t = \pi/4$, Hence (8) and (10) yield

$c_2 \sinh(\pi/2) + c_4 = 1$ and $c_2 \sinh(\pi/2) - c_4 = 1$

Solving these, we get $c_4 = 0$ and $c_2 = 1/\sinh(\pi/2)$.

Hence from (8) and (9) the required extremal in parametric form is given by

$$x = y = (\sinh 2t)/\sin(\pi/2)$$

Ex. 3. *Find the extremals of the functional* $\int_{t_1}^{t_2} y(\dot{x}^2 + \dot{y}^2)^{1/2} dt$. **[KU Kurukshetra 2004]**

Sol. Comparing the given functional with $\int_{t_1}^{t_2} \phi(x, y, \dot{x}, \dot{y}) dt$, we have

$$\phi(x, y, \dot{x}, \dot{y}) = y(\dot{x}^2 + \dot{y}^2)^{1/2} \quad ...(1)$$

Then, $\phi(x, y, c\dot{x}, c\dot{y}) = y\{(c\dot{x})^2 + (c\dot{y})^2\}^{1/2} = c\ \phi\ (x, y, \dot{x}, \dot{y})$,

showing that ϕ is a positive homogeneous function of degree one in x and y. Hence ϕ must satisy the following system of Euler's equations

$$\phi_x - \frac{d}{dt}\phi_{\dot{x}} = 0 \quad ...(2)$$

and

$$\phi_y - \frac{d}{dt}\phi_{\dot{y}} = 0 \quad ...(3)$$

We shall use (2) to find the required extremals.

Using (1), (2) reduces to $0-\frac{d}{dt}\left(\frac{y\dot{x}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\right)=0$

Integrating, $y\dot{x}/(\dot{x}^2+\dot{y}^2)^{1/2}=c_1$ ...(4)

Case I. Let $c_1=0$, Then, (4) $\Rightarrow$ $y\,\dot{x}=0$ $\Rightarrow$ $y=0$ or $\dot{x}=0$ $\Rightarrow$ $y=0, x=c_2$

Case II. Let $c_1\neq 0$.Then, $(4)\Rightarrow y^2\dot{x}^2=c_1^2(\dot{x}^2+\dot{y}^2)$

or $y^2=c_1^2\{1+(\dot{y}/\dot{x})^2\}$ or $y^2=c_1^2\{1+(dy/dx)^2\}$

or $\left(\frac{dy}{dx}\right)^2=\frac{y^2-c_1^2}{c_1^2}$ or $\frac{c_1dy}{(y^2-c_1^2)^{1/2}}=dx$

Integrating, $c_1\cosh^{-1}(y/c_1)=x+c_3$ or $y=c_1\cosh(x/c_1+c_3/c_1)$.

Thus, the required extremals are $y=0$, $x=c_2$ and $y=c_1\cosh(x/c_1+c_3/c_1)$, c_1,c_2 and c_3 being arbitrary constants

Note: In what follows, we show that equation (3) also leads to the same extremals. Using (1), (3) reduces to

$(\dot{x}^2+\dot{y}^2)^{1/2}-\frac{d}{dt}\left(\frac{y\dot{y}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\right)=0$ or $\frac{2y\dot{y}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\frac{d}{dt}\left(\frac{y\dot{y}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\right)=2y\dot{y}$

or $\frac{d}{dt}\left[\frac{y\dot{y}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\right]^2=\frac{dy^2}{dt^2}$ so that $\frac{y^2\dot{y}^2}{\dot{x}^2+\dot{y}^2}=y^2+c_1^2$

or $(y^2\dot{x}^2)/(\dot{x}^2+\dot{y}^2)=c_1^2$ or $(y\,\dot{x})/(\dot{x}^2+\dot{y}^2)^{1/2}=c_1$,

which is exactly the same as equation (4). Thus, (2) and (3) both lead to the same solution.

Ex. 4. *Find the extremal of the functional* $\int_{t_1}^{t_2}(\dot{x}^2+\dot{y}^2)^{1/2}dt$, *where* $\dot{x}=dx/dt, \dot{y}=dy/dt$.

[MDU Rohtak 2004, 05]

Sol. Comparing the given functional with $\int_{t_1}^{t_2}\phi(x,y,\dot{x},\dot{y})dt$, we have

$$\phi(x,y,\dot{x},\dot{y})=(\dot{x}^2+\dot{y}^2)^{1/2} \quad \text{... (1)}$$

Then, $\phi(x,y,\ c\,\dot{x},\ c\,\dot{y})=\{(c\dot{x})^2+(c\dot{y})^2\}^{1/2}=c\ \phi\ (x,y,\dot{x},\dot{y})$,

showing that ϕ is a positive homogeneous function of degree one in x and y. Hence ϕ must satisfy the following system of Euler's equations (refer. Art 1.11)

$$\phi_x-\frac{d}{dt}\phi_{\dot{x}}=0 \quad \text{... (2)}$$

and

$$\phi_y-\frac{d}{dt}\phi_{\dot{y}}=0 \quad \text{... (3)}$$

We shall use (2) only to get the extremals. From (1) and (2), we have

$$0-\frac{d}{dt}\left(\frac{\dot{x}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\right)=0 \quad \text{so that} \quad \frac{\dot{x}}{(\dot{x}^2+\dot{y}^2)^{1/2}}=c_1 \quad \text{... (4)}$$

Two cases arise :

Case I. Let $c_1=0$. Then, $(4)\Rightarrow dx/dt=0\Rightarrow x=c_2$, where c_2 is an arbitrary constant.

Case II. Let $c_1\neq 0$. Then, $(4)\Rightarrow \dot{x}^2=c_1^2(\dot{x}^2+\dot{y}^2)$

or $1 = c_1^2(1+\dot{y}^2/\dot{x}^2)$ or $1 = c_1\{1+(dy/dx)^2\}$

or $dy/dx = (1-c_1^2)^{1/2}/c_1 = c_3$, say or $dy = c_3 dx$

Integrating, $y = c_3 x + c_4$, c_3, c_4 being arbitrary constants.

Therefore, the required extremals are straight lines $x = c_2$ and $y = c_3 x + c_4$.

Note. It can be shown that equation (3) also yields the same extremals.

Ex. 5. *Find the stationary function of the functional* $\int_{(-1,0)}^{(1,0)} \{k(\dot{x}^2+\dot{y}^2)^{1/2} - \dot{x}y\}\, dt, k > 0$.

Sol. Comparing the given functional with $\int_{(-1,0)}^{(1,0)} \phi(x, y, \dot{x}, \dot{y})dt$, we have

$$\phi(x, y, \dot{x}, \dot{y}) = k(\dot{x}^2+\dot{y}^2)^{1/2} - \dot{x}y \qquad \ldots (1)$$

Then, $\phi(x, y, c\,\dot{x}, c\dot{y}) = k\{(c\dot{x})^2 + (c\dot{y})^2\}^{1/2} - c\,\dot{x}\,\dot{y} = c\,\phi\,(x, y, \dot{x}, \dot{y})$,

showing that ϕ is a positive homogeneous function of degree one in x and y. Hence ϕ must satisfy the Weirstrassian form of the Euler's equation (refer remark of Art. 1.11)

$$1/\rho = (\phi_{x\dot{y}} - \phi_{y\dot{x}})/\{\phi_1(\dot{x}^2+\dot{y}^2)^{1/2}\}, \qquad \ldots (2)$$

where $\phi_1 = \phi_{\dot{x}\dot{x}}/\dot{y}^2 = \phi_{\dot{y}\dot{y}}/\dot{x}^2 = -\{\phi_{\dot{x}\dot{y}}/(\dot{x}\dot{y})\}$ ... (3)

Using (1), $\phi_x = 0$, $\phi_y = -\dot{x}$, $\phi_{x\dot{y}} = 0$ and $\phi_{y\dot{x}} = -1$ ... (4)

$$\phi_{\dot{x}} = \frac{k\,\dot{x}}{(\dot{x}^2+\dot{y}^2)^{1/2}}, \phi_{\dot{x}\dot{x}} = k\,\frac{(\dot{x}^2+\dot{y}^2)^{1/2} - \dot{x}\times\{\dot{x}/(\dot{x}^2+\dot{y}^2)^{1/2}\}}{\dot{x}^2+\dot{y}^2} = \frac{k\,\dot{y}^2}{(\dot{x}^2+\dot{y}^2)^{1/2}}$$

$\therefore$ (3) $\Rightarrow$ $$\phi_1 = \frac{\phi_{\dot{x}\dot{x}}}{\dot{y}^2} = \frac{k\,\dot{y}^2}{(\dot{x}^2+\dot{y}^2)^{3/2}} \times \frac{1}{\dot{y}^2} = \frac{k}{(\dot{x}^2+\dot{y}^2)^{3/2}} \qquad \ldots (5)$$

Using (3) and (5), (2) reduces to $$\frac{1}{\rho} = \frac{0-(-1)}{[k/(\dot{x}^2+\dot{y}^2)^{1/2}\}\times(\dot{x}^2+\dot{y}^2)^{1/2}} = \frac{1}{k}$$

$\Rightarrow$ The curvature $1/\rho$ of the extremals is $1/k$, a constant.

$\Rightarrow$ The extremals are arcs of circles of radius = $\rho = k$.

Therefore, the required stationary function is the arc of the circle passing through (–1, 0) and (1, 0) and of radius k.

1.12 Isoperimetric Problems. (Bangalore 2005)

The ancient Greeks proposed the problem of finding a closed curve of given length which encloses maximum area. They called this the *isoperimetric problem*. The term isoperimetric problem is extended to include the general case of finding extremals of one integral subject to any constraint requiring a second integral to take on a prescribed value.

Method of solving isoperimetric problems. The simplest kind of isoperimetric problem consists of finding a curve $y = y(x)$ that extremizes an integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx \qquad \ldots(1)$$

relative to twice-differentiable function on $[x_1, x_2]$ which satisfy an integral constraint

$$J = \int_{x_1}^{x_2} g(x, y, y')dx = c \text{ , } c \text{ being a fixed constant} \qquad \ldots(2)$$

as well as a pair of prescribe boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ ...(3)

We also assume that both *f* and *g* are twice differentiable with respect to the arguments. As in Art.1.4 A, we assume that $y = y(x)$ is actual stationary curve and disturb it slightly to find the desired analytical condition. However, we cannot proceed as in Art.1.4 A and use only one - parameter families of comparison curves, because any change in value of a single parameter would in general change the value of the integral in equation (2) whose value must remain fixed. Therefore, instead, we use two-parameter families of curves

$$\bar{y}(x) = y(x) + \alpha_1 \eta_1(x) + \alpha_2 \eta_2(x), \quad ...(4)$$

where $\eta_1(x)$ and $\eta_2(x)$ are arbitrany twice-differentiable functions on $[x_1, x_2]$ such that

$$\eta_1(x_1) = \eta_1(x_2) = 0 \quad \text{and} \quad \eta_2(x_1) = \eta_2(x_2) = 0 \quad ...(5)$$

These conditions ensure that all comparison curves $y = \bar{y}(x)$ satisfy the same boundary conditions as the extremizing curve $y = y(x)$. Again, for every $\eta_1(x)$ and $\eta_2(x), \bar{y}(x) = y(x)$ if $\alpha_1 = \alpha_2 = 0$. Differentiating (4) w.r.t. '*x*', we have

$$\bar{y}'(x) = y'(x) + \alpha_1 \eta_1'(x) + \alpha_2 \eta_2'(x) \quad ...(6)$$

Replacing *y* and y' by $\bar{y}(x)$ and $\bar{y}'(x)$ respectively in (1) and (2), we get

$$I(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\, dx \quad ...(7)$$

and

$$J(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} g(x, \bar{y}, \bar{y}')dx \quad ...(8)$$

Since a functional relation $J(\alpha_1, \alpha_2) = c$, *c* being a constant ...(9)

exists between α_1 and α_2 , these parameters are not independent.

Now, by definition, *y(x)* extremizes I subject to the constraint (2). Hence, when $\alpha_1 = \alpha_2 = 0, I(\alpha_1, \alpha_2)$ has an extremum relative to values of α_1 and α_2 which satisfy relation (9). Our given isoperimetric problem thus reduces to find out the condition(*s*) *y(x)* must satisfy when it is known that, for $\alpha_1 = \alpha_2 = 0$, the ordinary function $I(\alpha_1, \alpha_2)$ of two variables has an extremum subject to condition (9). Such problem is generally solved by the method of Lagrange multipliers. Accordingly, we multiply (2) by λ and then add it to (1). Thus we get the following setting.

$$H(\alpha_1, \alpha_2) = I(\alpha_1, \alpha_2) + \lambda J(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')dx\,, \quad ...(10)$$

where the Lagrange multiplier λ is an arbitrary constant and the function $F(x, \bar{y}, \bar{y}')$ is defined by

$$F(x, \bar{y}, \bar{y}') = f(x, \bar{y}, \bar{y}') + \lambda\, g(x, \bar{y}, \bar{y}') \quad ...(11)$$

Now, if $I(\alpha_1, \alpha_2)$ has an extremum for $\alpha_1 = \alpha_2 = 0$, then by calculus, necessary conditions are given by

$$\partial H/\partial\alpha_1 = \partial H/\partial\alpha_2 = 0, \quad \text{when} \quad \alpha_1 = \alpha_2 = 0 \quad ...(12)$$

Computing these derivatives with help of (4), (6) and (10), we get

$$\frac{\partial H}{\partial \alpha_i} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \alpha_i} + \frac{\partial F}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \alpha_i} \right) dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}}\, \eta_i + \frac{\partial F}{\partial \bar{y}'}\, \eta_i' \right) dx, \text{ where } i = 1, 2 \quad ...(13)$$

Since $\bar{y}$ and $\bar{y}'$ reduce to the extremizing function *y* and its derivative y' , when $\alpha_1 = \alpha_2 = 0$, we have from (12) and (13)

$$\left[\frac{\partial H}{\partial \alpha_i}\right]_{\alpha_1=0,\,\alpha_2=0} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\ \eta_i + \frac{\partial F}{\partial y'}\ \eta_i'\right)dx = 0, \text{ where } i = 1,2 \qquad ...(14)$$

Integrating by parts the second term on the R.H.S. of (14), we obtain

$$\int_{x_1}^{x_2}\frac{\partial F}{\partial y}\ \eta_i(x)\,dx + \left[\frac{\partial F}{\partial y'}\ \eta_i(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\ \eta_i(x)dx = 0, \text{ where } i = 1,2$$

or
$$\int_{x_1}^{x_2}\eta_i(x)\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}dx = 0, \text{ using (5), where } i = 1, 2 \qquad ...(15)$$

Except for the conditions (5) to be satisfied, and the reqirement of being twice-differentiable on $[x_1, x_2], \eta_1(x)$ and $\eta_2(x)$ are arbitrary functions. Hence, if H is any function for which $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ is continuous on $[x_1, x_2]$, from equation (15) and the fundamental lemma of the calculus of variations (see Art. 1.3), we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \qquad ...(16)$$

as the Euler equation which a function y must satisfy if it is to extremize I subject to an integral constraint (2).

Remark 1. Extensions can be made to cases where there are more constraint conditions.

Remark 2. In the case of functionals that depend on two or more functions, this result can be extended in the same manner as explained in Art.1.8. For example, if

$$I = \int_{x_1}^{x_2} g(x, y, z, y', z')dx$$

has a stationary value subject to the condition

$$J = \int_{x_1}^{x_2} g(x, y, z, y', z')\,dx = \text{constant} = c, \text{ say}$$

then the stationary functions $y(x)$and $z(x)$ must satisfy the system of equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

where, $$F(x, y, z, y', z') = f(x, y, z, y', z') + \lambda\, g(x, y, z, y', z')$$

1.13 Working rule for solving isoperimetric problems

Step1. Suppose we wish to find a curve $y = y(x)$ that gives extreme value of the functional

$$\int_{x_1}^{x_2} f(x, y, y')dx \qquad ...(1)$$

while keeping another integral $$\int_{x_1}^{x_2} g(x, y, y')dx = \text{constant} = c, \text{ say} \qquad ...(2)$$

Step 2. Let $y = y(x)$ satisfy the boundary conditions

$$y(x_1) = y_1 \qquad \text{and} \qquad y(x_2) = y_2 \qquad ...(3)$$

Step 3. Let $$F = f(x, y, y') + \lambda g(x, y, y'),$$

where λ is called the Lagrange multiplier.

Step 4. The required extreme value of $y(x)$ must satisfy Euler's equation

$$F_y - \frac{d}{dx} F_{y'} = 0 \qquad \text{or} \qquad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(4)$$

i.e.,
$$f_y - \frac{d}{dx} f_{y'} + \lambda\left(g_y - \frac{d}{dx} g_{y'}\right) = 0 \qquad ... (4)'$$

Step 5. The solutions of (4) (*i.e.* extremals of our problem) will involve three undetermined parameters: two constants of integration, and the Lagrange multilier λ. If an extremizing function exists, definite values for these parameters can be found by imposing the two boundary conditions (3) and the requirement that the value of the integral in (2) be as prescribed.

1.13A. ILLUSTRATIVE EXAMPLES BASED ON ART. 1.13

Ex. 1. (*a*) *Find the extremal of the functional* $\int_0^2 y'^2\, dx$ *under the constraint* $\int_0^2 y dx = 1$ given $y(0) = 0$ and $y(2) = 1$. **(Bangalore 2004)**

Sol. Given functional is $I = \int_0^2 y'^2\, dx$, ...(1)

Also given that $J = \int_0^2 y\, dx = 1$ ...(2)

The given boundary conditions are $y(0) = 0$ and $y(2) = 1$ ...(3)

Let $F(x, y, y') = y'^2 + \lambda y,$ ...(4)

where λ is Lagrange multiplier. Then the required extremal satsfies the Euler's equation,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{or} \qquad \lambda - \frac{d}{dx}(2y') = 0, \text{ using (4)}$$

or $$\frac{\lambda}{2} - \frac{d^2 y}{dx^2} = 0 \qquad \text{or} \qquad \frac{d^2 y}{dx^2} = \frac{\lambda}{2}$$

Integrating it twice, we get $y = (\lambda/4) \times x^2 + c_1 x + c_2$, ...(5)

where c_1 and c_2 are arbitrary constants.

Using (3), (5) yields $0 = c_2$ and $1 = \lambda + 2c_1$ *i.e.* $c_1 = (1 - \lambda)/2$.

$\therefore$ (5) becames $y = (\lambda/4) \times x^2 + (1/2) \times (1 - \lambda) x$ ...(6)

Substituting this value of y in (2), we have

$$\int_0^2 \{(\lambda/4) \times x^2 + (1 - \lambda) \times (x/2)\} dx = 1 \qquad \text{or} \qquad [(\lambda/12) \times x^3 + (1 - \lambda) \times (x^2/4)]_0^2 = 1$$

or $(2\lambda/3) + 1 - \lambda = 1$ so that $\lambda = 0$. Hence $c_1 = 1/2$.

From (5), the required extremal is given by $y = x/2$.

Ex. 1(*b*) Find the extremal for which the functional $I = \int_0^1 y'^2 dx$, $y(0) = 0, y(1) = 1$ *is extremum, subject to the condition* $J = \int_0^1 y\, dx = 2$.

Sol. Do as in Ex.1(*a*) **Ans.** $y = 1 - 6x - 6x^2$

Ex. 2. *Find the extremal of the functional* $\int_0^{\pi}(y'^2 - y^2)dx$ *under the conditions* $y(0) = 0, y(\pi) = 1$ *and subject to the constraint* $\int_0^{\pi} y\,dx = 1$.

Sol. Given functional is $I = \int_0^{\pi}(y'^2 - y^2)\,dx$ ...(1)

Also given that $J = \int_0^{\pi} y\,dx = 1$ ...(2)

The given boundary conditions are $y(0) = 0$ and $y(\pi) = 1$ ...(3)

Let $F(x, y, y') = y'^2 - y^2 + \lambda y$ ...(4)

where λ is the Lagrange multiplier. Then the required extremal satifies the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

i.e. $-2y + \lambda - \frac{d}{dx}(2y') = 0$ or $y'' + y = \frac{\lambda}{2}$, using (4)

or $(D^2 + 1)y = \lambda/2,$ where $D \equiv d/dx$ ...(5)

Auxiliary equation of (5) is $D^2 + 1 = 0$ so that $D = \pm i$

$\therefore$ C.F. of (5) = $c_1 \cos x + c_2 \sin x$, c_1 and c_2 being arbitrary constants

P.I.of (5) $= \frac{1}{D^2+1}\ \frac{\lambda}{2} = \frac{\lambda}{2}(1 + D^2)^{-1}1 = \frac{\lambda}{2}(1 - D^2 + ...)1 = \frac{\lambda}{2}$

Hence the general solution of (5) is $y = c_1 \cos x + c_2 \sin x + \lambda/2$ ...(6)

Using (3), (6) gives $0 = c_1 + \lambda/2$ and $1 = -c_1 + \lambda/2$

Solving these, $\lambda = 1$ and $c_1 = -(1/2)$. Hence (6) reduces to

$$y = -(1/2) \times \cos x + c_2 \sin x + 1/2 \quad ...(7)$$

Using (7), (2) reduces to

$$\int_0^{\pi}\left(-\frac{1}{2}\cos x + c_2 \sin x + \frac{1}{2}\right)dx = 1 \quad \text{or} \quad \left[-\frac{1}{2}\sin x - c_2 \cos x + \frac{x}{2}\right]_0^{\pi} = 1$$

Thus, $2c_2 + \pi/2 = 1$ so that $c_2 = (2 - \pi)/4$. So from (7) the required extremal is

$$y = -\frac{1}{2}\cos x + \frac{2-\pi}{4}\sin x + \frac{1}{2} \quad \text{or} \quad y = \frac{1}{2}(1 - \cos x) + \frac{2-\pi}{4}\sin x$$

Ex. 3. *Find a function y(x) for which* $\int_0^1 (x^2 + y'^2)dx$ *is stationary, given that* $\int_0^1 y^2 dx = 2; y(0) = 0, y(1) = 0$.

Sol. Given functional is $I = \int_0^1 (x^2 + y'^2)\,dx$ ...(1)

Also given that $J = \int_0^1 y^2 dx = 2$ ...(2)

The given boundary conditions are $y(0) = 0$ and $y(1) = 0$ ...(3)

Let $F(x, y, y') = x^2 + y'^2 + \lambda y^2,$ ...(4)

where λ is the Lagrange multiplier. Then the required extremal satisfies the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad i.e. \quad 2\lambda y - \frac{d}{dx}(2y') = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} - \lambda y = 0, \quad \text{using (4)}$$

or $(D^2 - \lambda)y = 0, \quad$ where $D \equiv d/dx$ ...(5)

Three cases arise:

Case 1. Let λ = 0. Then solution of (5) is given by $y = c_1 x + c_2$ ...(6)

Using (3), (6) gives $0 = c_2$ and $0 = c_1 + c_2$ so that $c_1 = c_2 = 0$. Then (6) reduces to $y = 0$, which cannot satisfy (2), Hence $\lambda = 0$ leads to no solution.

Case II. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (5) becomes $(D^2 - \mu^2)y = 0$, whose solution is

$y = c_1 e^{\mu x} + c_2 e^{-\mu x}$, c_1 and c_2 being arbitrary constants ...(7)

Using (3), (7) gives $c_1 + c_2 = 0$ and $c_1 e^{\mu} + c_2 e^{-\mu} = 0$ so that $c_1 = c_2 = 0$. Hence as before, we get $y = 0$ which is not the required solution.

Case III. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (5) becomes $(D^2 + \mu^2)y = 0$ whose solution is

$y = c_1 \cos \mu x + c_2 \sin \mu x$, c_1 and c_2 being arbitrary constants ...(8)

Using (3), (8) given $c_1 = 0$ and $c_2 \sin \mu = 0$. For non-zero solutions of (8), we must take $c_2 \neq 0$. Hence, we get

$$c_2 \sin \mu = 0 \quad \Rightarrow \quad \sin \mu = 0 \quad \Rightarrow \quad \mu = m\pi, \text{ where } m \text{ is an integer}$$

Hence, (8) reduces to $y = c_2 \sin m\pi x$ ...(9)

Using (9), (2) reduces to $\int_0^1 c_2^2 \sin^2 m\pi x \, dx = 2$ or $c_2^2 \int_0^1 \{(1 - \cos 2m\pi x)/2\} dx = 2$

or $(c_2^2/2) \times \left[x - (1/2m\pi) \times \sin 2m\pi x\right]_0^1 = 2$ so that $c_2^2 = 4$ or $c_2 = \pm 2$.

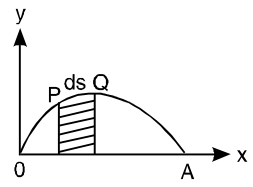
Hence, from (9), the required function is given by

$y = \pm 2 \sin m\pi x$, where m is an integer.

Ex. 4. (*a*) *Determine the curve of length l which passes through the points* (0, 0) *and* (1, 0) *and for which the area between the curve and the x-axis is a maximum.*

(*b*) *Find the curve of fixed length l that joins the points* $P_1(x_1, 0)$ *and* $P_2(x_2, 0)$ *lies above the x-axix, and encloses the maximum area between itself and the x-axis.* **(Kanpur 2003)**

Sol. (*a*) Here *l* is the fixed perimeter of a plane curve passing through two given points *O* (0,0) and *A*(1, 0). Let S be area enclosed by that plane curve and *x*-axis. Then, we are to maximize



$$S = \int_0^1 y \, dx \quad ...(1)$$

subject to the constraint $\int_0^1 (1 + y'^2)^{1/2} \, dx = l$ ...(2)

and the boundary conditions $y(0) = 0$ and $y(1) = 0.$ ...(3)

Let $F(x, y, y') = y + \lambda(1 + y'^2)^{1/2},$ ...(4)

where λ is the Lagrange multiplier. Then the required extremal satisfies the Euler's equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \qquad \text{or} \qquad 1-\frac{d}{dx}\left\{\frac{\lambda y'}{(1+y'^2)^{1/2}}\right\}=0, \text{ by (4)} \quad ...(5)$$

Integrating (5), $x-\dfrac{\lambda y'}{(1+y'^2)^{1/2}}=c_1$ or $(x-c_1)^2=\dfrac{\lambda^2 y'^2}{1+y'^2}$

or $\dfrac{1+y'^2}{y'^2}=\dfrac{\lambda^2}{(x-c_1)^2}$ so that $y'^2=\dfrac{(x-c_1)^2}{\lambda^2-(x-c_1)^2}$

or $\dfrac{dy}{dx}=\pm\dfrac{x-c_1}{\{\lambda^2-(x-c_1)^2\}^{1/2}}$ or $dy=\pm\dfrac{(x-c_1)dx}{\{\lambda^2-(x-c_1)^2\}^{1/2}}$

Integrating, $y=c_2\pm\{\lambda^2-(x-c_1)^2\}^{1/2}$ so that $(y-c_2)^2=\lambda^2-(x-c_1)^2$

or $(x-c_1)^2+(y-c_2)^2=\lambda^2,$ which is the equation of a circle with radius λ.

Alternative solution. First proceed upto equation (5). Then, we get

$$1-\lambda\ \frac{y''(1+y'^2)^{1/2}-y'\times(1/2)\times(1+y'^2)^{-1/2}\times 2y'y''}{(1+y'^2)}=0$$

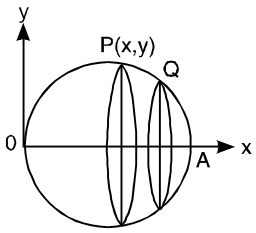
or $\dfrac{1}{\lambda}-\dfrac{y''}{(1+y'^2)^{3/2}}=0$ or $\dfrac{y''}{(1+y'^2)^{3/2}}=\dfrac{1}{\lambda},$

showing that the curvature of the required curve is constant and equals $1/\lambda$. It follows that the required curve is an arc of a circle of radius λ.

(*b*) Proceed an in part (*a*) **Ans.** circle.

Ex. 5. *Show that the sphere is the solid figure of revolution which, for a given surface, has maximum volume.* **(CDLU 2004)**

Sol. Let us consider the arc OPA of the curve which rotates about the *x*-axis. Let *S* and *V* be the surface and volume of the solid generated by arc OPA about *x*-axis where $OA = a$. Our problem is to maximize



$$V=\int_0^a \pi\, y^2 dx \qquad ...(1)$$

subject to the constraint $S=\int_0^a 2\pi y(1+y'^2)^{1/2}dx=\text{constant}$ ...(2)

and the boundary conditions $y(0)=0$ and $y(a)=0$ ...(3)

Let $F(x,y,y')=\pi y^2+\lambda\times 2\pi y(1+y'^2)^{1/2},$ ...(4)

where λ is the Lagrange multiplier. Then the required extremal satisfies the Euler's equation in the following form (note that here *F* is independent of *x* and so we use Case *V*, Art. 1.5F)

$$F-y'(\partial F/\partial y')=\text{constant}=c_1, \text{ say}$$

i.e. $$\pi y^2+2\pi\lambda y(1+y'^2)^{1/2}-y'\times 2\pi y\times\frac{y'}{(1+y'^2)^{1/2}}=c_1$$

or $$\pi y^2+(2\pi\lambda y)/(1+y'^2)^{1/2}=c_1 \qquad ...(5)$$

Since the curve OPA passes through O and A for which $y = 0$, hence (5) gives $c_1 = 0$. Then (5) yields

$$\pi y^2 + \frac{2\pi\lambda y}{(1+y'^2)^{1/2}} = 0 \qquad \text{giving} \qquad y' = \frac{dy}{dx} = \frac{(4\lambda^2 - y^2)^{1/2}}{y}$$

Seperating variables and integrating, we have

$$\int dx = \int \frac{y\,dy}{(4\lambda^2 - y^2)^{1/2}} + c_2 \qquad \text{or} \qquad x = c_2 - (4\lambda^2 - y^2)^{1/2} \qquad ...(6)$$

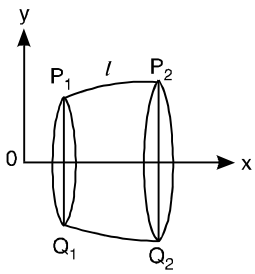
Since the curve OPA passes through (0, 0), (6) gives $0 = c_2 - 2\lambda$ so that $c_2 = 2\lambda$. Then (6) reduces to

$$x = 2\lambda - (4\lambda^2 - y^2)^{1/2} \qquad \text{or} \qquad (x - 2\lambda)^2 + y^2 = (2\lambda)^2,$$

which is a circle with centre $(2\lambda, 0)$ and radius 2λ. It follows that the figure formed by the revolution of the given arc of curve (*i.e.* circle) is a sphere.

Ex. 6. *Show that the curve C of given length l which minimizes the curved surface area of solid generated by the revolution of C about x-axis is a catenary.* **(V.T.U. 2000)**

Sol. Let us consider the arc P_1P_2 of the curve which rotates about the *x*-axis. Let *l* and *S* be the length of arc P_1P_2 and surface of the solid generated by arc P_1P_2 about the *x*-axis. Let coordinates of P_1 and P_2 be (x_1, y_1) and (x_2, y_2) respectively. Our problem is to minimize



$$S = \int_{x_1}^{x_2} 2\pi y(1+y'^2)^{1/2}\,dx \qquad ...(1)$$

subject to the constraint $\int_{x_1}^{x_2} (1+y'^2)^{1/2}\,dx = \text{constant} = l$ ...(2)

and the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ ...(3)

Let $F(x, y, y') = 2\pi y(1+y'^2)^{1/2} + \lambda(1+y'^2)^{1/2},$

where λ is the Lagrange multiplier Since *F* does not contain *x*, Euler's equation takes the form (refer case V, Art. 1.5F)

$$F - y'(\partial F/\partial y') = \text{constant} = c_1 \times 2\pi \text{, where } c_1 \text{ is constant}$$

or $$(2\pi y + \lambda)(1+y'^2)^{1/2} - y'(2\pi y + \lambda) \times \left\{ y'/(1+y'^2)^{1/2} \right\} = 2\pi c_1 \qquad ...(4)$$

Let use choose $\lambda = -2\pi\mu,$ where μ is new Lagrange multiplier Then (4) may be re-written as

$$(1+y'^2)^{1/2} - \frac{y'^2}{(1+y'^2)^{1/2}} = \frac{c_1}{y-\mu} \qquad \text{or} \qquad \frac{1}{(1+y'^2)^{1/2}} = \frac{c_1}{y-\mu}$$

or $$1 + \left(\frac{dy}{dx}\right)^2 = \frac{(y-\mu)^2}{c_1^2} \qquad \text{or} \qquad \frac{dy}{dx} = (1/c_1) \times \{(y-\mu)^2 - c_1^2\}^{1/2}$$

Seperating variables and integrating, we have

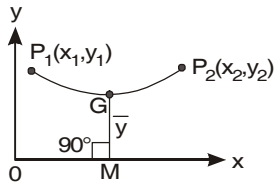
$$\int \frac{dx}{c_1} = \int \frac{dy}{\{(y-\mu)^2 - c_1^2\}^{1/2}} \qquad \text{or} \qquad \frac{x}{c_1} + c_2 = \cosh^{-1}\frac{y-\mu}{c_1}$$

or $$y = \mu + c_1 \cosh\left(\frac{x}{c_1} + c_2\right) \quad \text{or} \quad y = \mu + a\cosh\left(\frac{x-b}{a}\right) \text{ [choosing } c_1 = a \text{ and } c_2 = -b/a\text{]}$$

which are catenaries with base lines parallel to the *x*-axis.

Ex. 7 (*a*) *Find the shape of an absolutely flexible, inextensible homogeneous and heavy string of given length l suspended between two points, not necessarily in the same horizontal line.*

Sol. suppose that the given string hangs between two fixed points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, not necessarily in the same same horizontal line as shown in the adjoining figure. Let G be its centre of gravity. Then string in equilibrium takes a shape such that G occupies the lowest position. Hence we must find the minimum value of the y-coordinate $\bar{y}$ of G given by



$$\bar{y} = I[y(x)] = \frac{\int_{x_1}^{x_2} y(1+y'^2)^{1/2}\,dx}{\int_{x_1}^{x_2} (1+y'^2)^{1/2}\,dx} \qquad ...(1)$$

subject to the constraint $$\int_{x_1}^{x_2} (1+y'^2)^{1/2}\,dx = l \qquad ...(2)$$

In view of (2) , (1) shows that we have to minimize the numerator of the right side of (1) subject to constraint (2). we, therefore, write.

$$F(x, y, y') = y(1+y'^2)^{1/2} + \lambda(1+y'^2)^{1/2} = (y+\lambda)(1+y'^2)^{1/2}, \qquad ...(3)$$

where λ is the Lagrange multiplier. Since F does not contain x, Euler's equation takes the form (refer case V, Art. 1.5F)

$$F - y'(\partial F / \partial y') = \text{constant} = c_1,\ c_1 \text{ being an arbitrary constant}$$

or $$(y+\lambda)(1+y'^2)^{1/2} - y'^2(y+\lambda)(1+y'^2)^{-1/2} = c_1 \quad \text{or} \quad y+\lambda = c_1(1+y'^2)^{1/2}$$

or $$1+\left(\frac{dy}{dx}\right)^2 = \frac{(y+\lambda)^2}{c_1^{\,2}} \quad \text{or} \quad \frac{dy}{dx} = \frac{\{(y+\lambda)^2 - c_1^{\,2}\}^{1/2}}{c_1}$$

Seperating variables and then integrating, we have

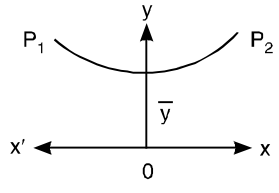
$$x = c_1 \int \frac{dy}{\{(y+\lambda)^2 - c_1^2\}^{1/2}} + c_2 \quad \text{or} \quad x = c_1 \cosh^{-1}\left(\frac{y+\lambda}{c_1}\right) + c_2$$

or $$\cosh^{-1}\frac{y+\lambda}{c_1} = \frac{x-c_2}{c_1} \quad \text{or} \quad y+\lambda = c_1 \cosh\frac{x-c_2}{c_1}, \qquad ...(4)$$

which is the required curve representing a catenary. The three constants λ, c_1 and c_2 can be obtained with help of (2) and the given boundary conditions that the string passes through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Ex. 7 (*b*) *Determine the curve of prescribed length 2l which joins the points (-a,b) and (a,b) and has its centre of gravity as low as possible.*

Sol. Let P_1P_2 be an arc joining the given points $P_1(-a, b)$ and $P_2(a, b)$. The y-coordinate of the centre of gravity of the required curve is given by



$$I = \frac{\int_{-a}^{a} y\,ds}{\int_{-a}^{a} ds} = \frac{1}{2l}\int_{-a}^{a} y(1+y'^2)^{1/2}\,dx, \qquad ...(1)$$

where we have used the given constraint, namely

$$\int_{-a}^{a} ds = \int_{-a}^{a} (1+y'^2)^{1/2}\,dx = 2l\,, \qquad i.e., \qquad (1/2l)\times\int_{-a}^{a}(1+y'^2)^{1/2}dx = 1 \qquad ...(2)$$

The boundary conditions are $y(-a) = b$ and $y(a) = b$ ...(3)

Let $$F(x, y, y') = \frac{y}{2l}(1+y'^2)^{1/2} + \frac{\lambda}{2l}(1+y'^2)^{1/2} = \frac{(y+\lambda)(1+y'^2)^{1/2}}{2l}, \qquad ...(4)$$

where λ is the Lagrange multiplier. Since F does not contain x, Euler's equation takes the form (refer case V, Art. 1.5F). $F - y'(\partial F/\partial y') = \text{constant} = C$

or $$\frac{(y+\lambda)(1+y'^2)^{1/2}}{2l} - \frac{y+\lambda}{2l}\times\frac{y'^2}{(1+y'^2)^{1/2}} = C \qquad \text{or} \qquad \frac{y+\lambda}{(1+y'^2)^{1/2}} = C_1$$

where $C_1 = 2Cl$. Re-writting the above equation, we have

$$1+y'^2 = \frac{(y+\lambda)^2}{C_1^2} \qquad \text{or} \qquad \frac{dy}{dx} = \left\{\frac{(y+\lambda)^2 - C_1^2}{C_1^2}\right\}^{1/2}$$

Separating variable and then integrating, we get

$$x = C_1\int \frac{dy}{\{(y+\lambda)^2 - C_1^2\}^{1/2}} + C_2 \qquad \text{or} \qquad x = C_1\cosh^{-1}\frac{y+\lambda}{C_1} + C_2$$

or $$\cosh^{-1}\frac{y+\lambda}{C_1} = \frac{x-C_2}{C_1} \qquad \text{so that} \qquad y = C_1\cosh\frac{x-C_2}{C_1} - \lambda \qquad ...(5)$$

which is a complete solution of the Euler's equation on [–a, a] and the boundary conditions (3) will be satisfied by this solution if and only if

$$\frac{b+\lambda}{C_1} = \cosh\frac{-a-C_2}{C_1} \qquad \text{and} \qquad \frac{b+\lambda}{C_1} = \cosh\frac{a-C_2}{C_1}$$

that is to say if and only if $(a + C_2)/C_1 = (a - C_2)/C_1$, Hence $C_2 = 0$ and so the required curve must be symmetric with respect to the y-axis. Thus, we get

$$\lambda = C_1\cosh(a/C_1) - b \qquad ...(6)$$

Also, (5) reduces to $$y = C_1\cosh(x/C_1) - \lambda \qquad ...(7)$$

Using (7), (2) reduces to

$$(1/2l)\times\int_{-a}^{a}\{1+\sinh^2(x/C_1)\}^{1/2}dx = 1 \qquad \text{or} \qquad \int_{-a}^{a}\cosh(x/C_1)\,dx = 2l$$

or $$\left[C_1\sinh(x/C_1)\right]_{-a}^{a} = 2l \qquad \text{or} \qquad 2C_1\sinh(a/C_1) = 2l \qquad ...(8)$$

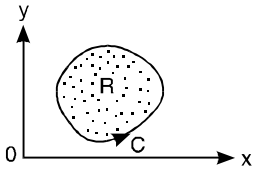
When C_1 has been found from (8) (by some approximate numerical procedure), the value of λ can be found and the equation of the required curve can be obtained from (7). In particular, from (6), we have

$$\lambda = C_1\cosh(a/c) - b = C_1\{1+\sinh^2(a/C_1)\}^{1/2} - b = (C_1^2 + l^2)^{1/2} - b$$

and then (7) reduces to $$y = C_1\cosh(x/a) + b - (C_1^2 + l^2)^{1/2}.$$

Ex. 8. (*The Dido problem*) *Find the closed plane curve of given length that encloses the largest area.* (*Kanpur 2010*)

Sol. Let A be the area of the region R bounded by closed plane curve C of the given length l. Then, by calculus, we have



$$A = \frac{1}{2}\oint_C (xdy - ydx) = \frac{1}{2}\oint_C (xy' - y)dx \qquad ...(1)$$

and $$\oint_C (1 + y'^2)^{1/2} dx = \text{constant} = l\,, \qquad ...(2)$$

where $\oint_C$ indicates that the curve C is closed and that it is described in the positive direction. Here we are to maximize A subject to the constraint (2)

Let $$F = (1/2) \times (xy' - y) + \lambda(1 + y'^2)^{1/2}, \qquad ...(3)$$

where λ is the Lagrange multiplier. The required function $y(x)$ satisfies the Euler's equation namely,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$-\frac{1}{2} - \frac{d}{dx}\left\{\frac{x}{2} + \frac{\lambda y'}{(1 + y'^2)^{1/2}}\right\} = 0 \quad \text{or} \quad \frac{d}{dx}\left\{\frac{\lambda y'}{(1 + y'^2)^{1/2}}\right\} = -1$$

Integrating, $$\frac{\lambda y'}{(1 + y'^2)^{1/2}} = -x + C_1 \quad \text{or} \quad \frac{y'^2}{1 + y'^2} = \frac{(x - C_1)^2}{\lambda^2}$$

or $$y'^2 = \left(\frac{dy}{dx}\right)^2 = \frac{(x - C_1)^2}{\lambda^2 - (x - C_1)^2} \quad \text{or} \quad dy = \pm\frac{(x - C_1)dx}{\{\lambda^2 - (x - C_1)^2\}^{1/2}}$$

Integrating, $$y - C_2 = \pm\{\lambda^2 - (x - C_1)^2\} \quad \text{or} \quad (x - C_1)^2 + (y - C_2)^2 = \lambda^2,$$

which is a circle of radius λ.

Alternative Solution. Let us find the equation of the plane curve $r = f(\theta)$ in polar coordinates (r, θ). Let A be the area bounded by it and let l be the length of this closed curve. Then, we are maximize A given by

$$A = (1/2) \times \int_0^{2\pi} r^2 d\theta$$

subject to the condition $$\int_0^{2\pi} \{r^2 + (dr/d\theta)^2\}^{1/2} d\theta = \text{constant} = l$$

Let λ be a parameter. Then, we consider the functional

$$I = \int_0^{2\pi} F(\theta, r, r')d\theta, \text{ taking } r' = dr/d\theta$$

where $$F(\theta, r, r') = (r^2/2) + \lambda(r^2 + r'^2)^{1/2} \qquad ...(1)$$

The required function $r(\theta)$ satisfies the Euler's equation

$$\frac{\partial F}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) = 0 \qquad ...(2)$$

$(1) \Rightarrow \quad \partial F/\partial r = r + \lambda r(r^2 + r'^2)^{-1/2}, \qquad \partial F/\partial r' = \lambda r'(r^2 + r'^2)^{-1/2}$ and so

$$\frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) = \lambda r''(r^2 + r'^2)^{-1/2} - (1/2) \times \lambda r'(r^2 + r'^2)^{-1/2}(2rr' + 2r'r'')$$

$$= \lambda r(rr'' - r'^2)(r^2 + r'^2)^{-3/2}, \text{ on simplification}$$

where $r'' = d^2r/d\theta^2$. Substituting the above values in (2), we get

$$r + \lambda r(r^2 + r'^2)^{-1/2} - \lambda r(rr'' + r'^2)(r^2 + r'^2)^{-3/2} = 0$$

or $r + \lambda r(r^2 + r'^2)^{-3/2}\{r^2 + r'^2 - (rr'' - r'^2)\} = 0$ or $r + \lambda r(r^2 + r'^2)^{-3/2}(r^2 + 2r'^2 - rr') = 0$

Thus, $$-\lambda = (r^2 + r'^2)^{3/2}/(r^2 + 2r'^2 - rr') \quad ...(3)$$

From calculus, we know that radius of curvature for $r = f(\theta)$ is given by

$$\rho = (r^2 + r'^2)^{3/2}/(r^2 + 2r'^2 - rr') \quad ...(4)$$

From (3) and (4), it follows that the radius of curvature of the required curve is constatnt. Hence the curve of given length enclosing a maximum area is a circle.

Ex. 9. *Find the extremal of the functional* $\int_{x_1}^{x_2} y'^2 dx$ *subject to the condition* $\int_{x_1}^{x_2} dx = c$, *a constant.* **[KU kurukshetra 2006]**

Sol. Given functional is $$I = \int_{x_1}^{x_2} y'^2 dx \quad ... (1)$$

Also, given that $$J = \int_{x_1}^{x_2} y\, dx = c \quad ... (2)$$

Let $$F(x, y, y') = y'^2 + \lambda y, \quad ... (3)$$

where λ is the Lagrange multipliler. Then, the required extremal satisfies the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{or} \quad \lambda - \frac{d(2y')}{dx} = 0 \text{ , Using (3)}$$

or $2(dy^2 dx^2) = \lambda$ or $dy^2/dx^2 = \lambda/2$ ... (4)

Integrating (4), $$dy/dx = \lambda x/2 + c_1 \quad ... (5)$$

Integrating (5), $$y = \lambda x^2/4 + c_1 x + c_2 \quad ... (6)$$

Substituting the above value of y in (2), we have

$$\int_{x_1}^{x_2}\left(\frac{\lambda x^2}{2} + c_1 x + c_2\right)dx = c \quad \text{or} \quad \left[\frac{\lambda x^3}{12} + \frac{c_1 x^2}{2} + c_2 x\right]_{x_1}^{x_2} = c$$

or $$(\lambda/12)\times(x_2^3 - x_1^3) + (c_1/2)\times(x_2^2 - x_1^2) + c_2(x_2 - x_1) = c$$

or $$(\lambda/12)\times(x_2^3 - x_1^3) = c - (c_1/2)\times(x_2^2 - x_1^2) - c_2(x_2 - x_1)$$

or $$\lambda = 6\{2c - c_1(x_2^2 - x_1^2) - 2c_2(x_2 - x_1)\}/(x_2^3 - x_1^3)$$

Substituting the above value of λ in (6), we have

$$y = 3x^2\{2c - c_1(x_2^2 - x_1^2) - 2c_2(x_2 - x_1)\}/2(x_2^3 - x_1^3) + c_1 x + c_2,$$

which is the required equation of the extremals, c_1 and c_2 being orbitrary constants

Ex. 10. *Find the extremal of the functional* $\int_0^1 y'^2 dx$, $y(0) = 1,\ y(1) = 6$ *subject to the condition* $\int_0^1 y\, dx = 3$.

Sol. Given functional is $$I = \int_0^1 y'^2 dx \quad ... (1)$$

Also, given that $J = \int_0^1 y\,dx = 3$ ... (2)

The given boundary condition are $y(0) = 1$ and $y(1) = 6$ ... (3)

Let $F(x, y, y') = y'^2 + \lambda y,$ ... (4)

where λ is Lagrange multiplier. Then the required extremal satisfies the Euler's equation

$$\frac{\partial F}{dy} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{or} \quad \lambda - \frac{d(2y')}{dx} = 0, \text{ using (4)}$$

or $2(d^2y/dx^2) = \lambda$ or $d^2y/dx^2 = \lambda/2$

$\Rightarrow \quad dy/dx = \lambda x/2 + c_1$ so that $y = \lambda x^2/4 + c_1 x + c_2$ ... (5)

Using the boundary conditions (3), (5) yields

$c_2 = 1$ and $\lambda/4 + c_1 + c_2 = 6$ so that $c_2 = 1$ and $c_1 = 5 - (\lambda(4)$ ... (6)

Using (6), (5) reduces to $y = (\lambda/4) \times x^2 + (5 - \lambda/4)x + 1.$ ... (7)

Substituting the above of y in (2), we have

$$\int_0^1 \left\{\frac{\lambda x^2}{4} + \left(5 - \frac{\lambda}{4}\right)x + 1\right\} dx = 3 \quad \text{or} \quad \left[\frac{\lambda x^3}{12} + \left(5 - \frac{\lambda}{4}\right) \times \frac{x^2}{2} + x\right]_0^1 = 3$$

or $\lambda/12 + (5/2 - \lambda/8) + 1 = 3$ so that $\lambda = 12$.

With $\lambda = 12$, (5) reduces to $y = 3x^2 + 2x + 1$, which is the equation of the desired extremal.

Ex. 11. *Among all curves of length l(>2a) in the upper half plane passing through the points (– a, 0) and (a, 0), find the one which together with the segment – a ≤ x ≤ a encloses maximum area.*

Sol. Here l is the fixed perimeter of a plane curve passing through two given points $P_1(-a, 0)$ and $P_2(a, 0)$ Let S be the area enclosed by that plane curve and x-axis. Then, we are to maximize

$$S = \int_{-a}^{a} y\,dx \quad \text{... (1)}$$

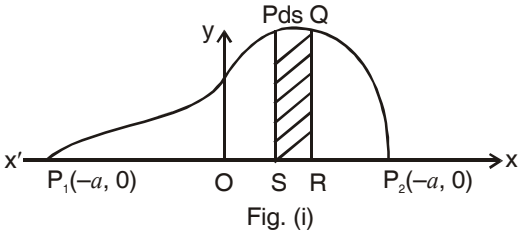


Fig. (i)

subject to the constraint $\int_{-a}^{a} (1 + y'^2)^{1/2} = l$ ... (2)

and the boundary condition $y(-a) = y(a) = 0$ ... (3)

Let $F(x, y, y') = y + \lambda(1 + y'^2)^{1/2},$ ... (4)

where λ is the Lagrange multiplier. Then the required extremal satisfies the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{or} \quad 1 - \frac{d}{dx}\left\{\frac{\lambda y'}{(1 + y'^2)^{1/2}}\right\} = 0, \text{ using (4)} \quad \text{... (5)}$$

Integrating (5), $\frac{\lambda y'}{(1 + y'^2)^{1/2}} = x + c_1$ or $(x + c_1)^2 = \frac{\lambda^2 y'^2}{1 + y'^2}$

or $\frac{1 + y'^2}{y'^2} = \frac{\lambda^2}{(x + c_1)^2}$ so that $y'^2 = \frac{(x + c_1)^2}{\lambda^2 - (x + c_1)^1}$ ... (6)

(6) $\Rightarrow \quad \frac{dy}{dx} = \pm \frac{x + c_1}{\{\lambda^2 - (x + c_1)^2\}^{1/2}}$ or $dy = \pm \frac{(x + c_1)dx}{\{\lambda^2 - (x + c_1)^2\}^{1/2}}$

Integrating, $y + c_2 = \pm \{\lambda^2 - (x + c_1)^2\}^{1/2}$ or $(x + c_1)^2 (y + c_2)^2 = \lambda^2$ ... (7)

Using the given boundary condition (3), (7) yields

$(-a + c_1)^2 + c_2^2 = \lambda^2$ *i.e.*, $a^2 - 2ac_1 + c_1^2 + c_2^2 = \lambda^2$ ... (8)

and $(a + c_1)^2 + c_2^2 = \lambda$ *i.e.*, $a^2 + 2a\,c_1 + c_1^2 + c_2^2 = \lambda^2$ ... (9)

Substracting (8) from (9), we get $4ac_1 = 0$ so that $c_1 = 0$ ... (10)

Using (10), (8) yields, $a^2 + c_2^2 = \lambda^2$ so that $c_2 = (\lambda^2 - a^2)^{1/2}$ ... (11)

Using (10) and (11), (7) yields, $x^2 + \{y + (\lambda^2 - a^2)^{1/2}\}^2 = \lambda^2$ ... (12)

From (6) and (10), $y'^2 = x^2/(\lambda^2 - x^2)$. Hence (2) reduces to

$$l = \int_{-a}^{a} \left(1 + \frac{x^2}{\lambda^2 - x^2}\right)^{1/2} dx = 2\lambda \int_0^a \frac{dx}{(\lambda^2 - x^2)^{1/2}} = 2\lambda \left[\sin^{-1}\frac{x}{\lambda}\right]_0^a$$

or $l = 2\lambda \sin^{-1}(a/\lambda)$ or $\sin(l/2\lambda) = a/\lambda$ ... (13)

Let $\lambda = \lambda_0$ be a solution of the transcendental equation (13). Then (12) yields

$$x^2 + \{y + (\lambda_0^2 - a^2)^{1/2}\}^2 = \lambda_0^2 \quad \text{... (14)}$$

Therefore the required arc is an arc of the circle (14) whose centre in at $C(0, -(\lambda_0^2 - a^2)^{1/2}/2)$ and radius λ_0 as shown in figure (ii).

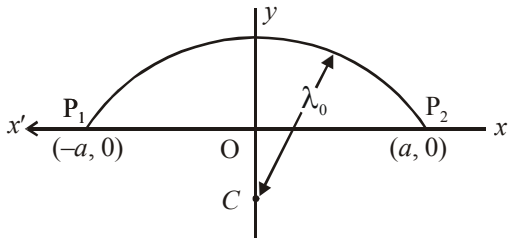


Fig. (ii)

Note: In particular case take $l = \pi a$ so that $l > 2a$. Then, from (13), we have

$$\sin\frac{\pi}{2}\left(\frac{a}{\lambda}\right) = \frac{a}{\lambda} \text{ giving } \lambda = \lambda_0 = a$$

Hence (14) reduces to circle $x^2 + y^2 = a^2$ as shown in figure (iii)

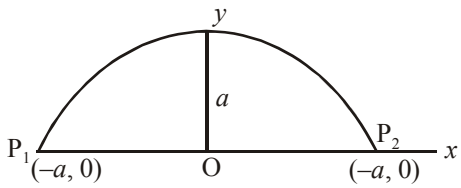


Fig. (iii)

1.14 Reciprocity principle

It can be easily verified that the extremals of the functional

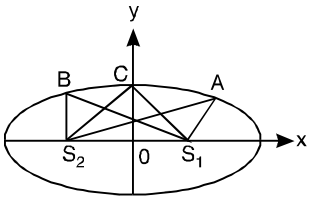
$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')dx \quad \text{...(1)}$$

subject to the constraint $J[y(x)] = \int_{x_1}^{x_2} g(x, y, y')dx = \text{constant}$ ...(2)

coincide with the extremals of $J[y(x)]$ under the constraint $I[y(x)]$ = constant. This is so because Euler's equations of the two variational problems are the same. This property is called the *reciprocity principle*. For example, the problem of maximum area bounded by a closed curve of given length, and the problem of the minimum length of a closed curve bounded by a given area are reciprocal and have common extremals (which are known to be circles, as shown in solved example 8, Art. 1.13A).

Example. *Show that an isosceles triangle has the smallest perimeter for a given area and a given base.*

Sol. As shown in the adjoining figure, let the distance S_1S_2 between two foci of an ellipse be the common base of the triangles AS_1S_2, BS_1S_2 and CS_1S_2.



For an ellipse, we know that sum of focal distances of any point on an ellipse is constant and equal to length $2a$ of its major axis. Thus, we have

$$AS_1 + AS_2 = BS_1 + BS_2 = CS_1 + \mathrm{CS}_2 = 2a$$

$$\Rightarrow \quad AS_1 + AS_2 + S_1S_2 = BS_1 + BS_2 + S_1S_2 = CS_1 + CS_2 + S_1S_2 = 2a + S_1S_2,$$

showing that the perimeter of each triangle under consideration is the same. But the isosceles triangle CS_1S_2 has the maximum area since it has the maximum height. So, according to the reciprocity principle, an isosceles triangle has the smallest perimeter for a given area and a given base.

EXERCISE 1(E)

1. A curve C of given length l connects (x_1, y_1) and (x_2, y_2). Find the shape of C so that the area bounded by C, the lines $x = x_1, x = x_2$ and the x-axis will be a maximum. **Ans.** Circle

2. Find the extremals of the functional $\int_0^1 y'^2\,dx$, $y(0) = 0, y(1) = 1/4$ subject to the condition $\int_0^1 (y - y'^2)\,dx = 1/12$. **Ans.** $y = (2x - x^2)/4$

3(*a*) Prove that the extremal of the isoperimetric problem $I[y(x)] = \int_1^4 y'^2 dx$, $y(1) = 3,\ y(4) = 24,$ subject to the condition $\int_1^4 y dx = 36$ is a parabola. **(Madras M.E. 2000)**

(*b*) Find all functions y, that satisfy the conditions $y(0) = y(\pi) = 0$ and which render the integral $I = \int_0^\pi y'^2 dx$ stationery subject to the constraint $\int_0^\pi y^2 dx = 1$.

4. Find the maximum value of $(1/2) \times \int_{t_1}^{t_2} (x\dot{y} - y\dot{x})\,dt$ given that $\int_{t_1}^{t_2} (\dot{x}^2 + \dot{y}^2)dt = \text{constant} = l,$ where $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$. **Ans.** $(x - C_1)^2 + (y - C_2)^2 = \lambda^2$, which is a circle

5. A curve in the first quadrant joins (0, 0) and (1, 0) and has a given area beneath it. Show that the shortest such curve is an arc of a circle.

6. Find the closed plane curve of given length that encloses the maximum area by using polar coordinates.

Hint : Refer alternative solution of solved Ex. 8 of Art. 1.13A.

7. A rope of length l suspended from two fixed points hangs so that its centre of gravity is as low as possible. Prove that the curve in which the rope hangs is a caterary.

Hint : Refer Ex. 7(*a*) of Art. 1.13A.

8. Find a function $y(x)$ such that $\int_0^\pi y^2 dx = 1$ which makes $\int_0^\pi y''^2 dx$ a minimum if $y(0) = 0,$ $y(\pi) = 0,\quad y''(0) = 0,\quad y''(\pi) = 0$. **Ans.** $y = B_n \sin nx,\ n = 1, 2, 3, \ldots$

9. Show that the extremum of the functional $I[y(x)] = \int_0^\pi (y'^2 - y^2)dx, y(0) = 0, y(\pi) = 1,$ subject to the constraint $I[y(x)] = \int_0^\pi y dx = 1,$ is given by the family of curves $y = (1/2) \times (1 - \cos x + c \sin x)$, where c is an arbitrary constant.

10. Show that the curve, revolving about x-axis, for a given surface of revolution, containing maximum area, is a circle.

1.15 Hamilton's principle. Variational principle of least action *(Kanpur 2011, Meerut 2009)*

Consider first a single particle of mass m moving in a force field. Let the position vector from a fixed origin to the particle at time t be denoted by $\boldsymbol{r}$. Then, according to Newton's laws of motion, the actual path followed is governed by the vector equation

$$m(d^2\boldsymbol{r}/dt^2) - \boldsymbol{F} = \boldsymbol{0}, \qquad ...(1)$$

where $\boldsymbol{F}$ is the force acting on the particle. Now consider any other path $\boldsymbol{r} + \delta\boldsymbol{r}$. In what follows, we assume only that the true and the varied path coincide at two distinct instants $t = t_1$ and $t = t_2$, that is, that the variation $\delta\boldsymbol{r}$ vanish at $t = t_1$ and $t = t_2$

$$\therefore \qquad [\delta\boldsymbol{r}]_{t=t_1} = [\delta\boldsymbol{r}]_{t=t_2} = \boldsymbol{0} \qquad ...(2)$$

At any intermediate time t we then have to consider the true path and the varied path $\boldsymbol{r} + \delta\boldsymbol{r}$. Taking the scalar product of the variation into (1) and then integrating the result with respect to time over the interval (t_1, t_2), we get

$$\int_{t_1}^{t_2} \left(m\frac{d^2\boldsymbol{r}}{dt^2} \cdot \delta\boldsymbol{r} - \boldsymbol{F} \cdot \delta\boldsymbol{r} \right) dt = 0$$

or $$m\{[(d\boldsymbol{r}/dt)\cdot\delta\boldsymbol{r}]_{t_1}^{t_2} - \int_{t_1}^{t_2} (d\boldsymbol{r}/dt)\cdot\delta(d\boldsymbol{r}/dt)dt\} - \int_{t_1}^{t_2} \boldsymbol{F}\cdot\delta\boldsymbol{r}\, dt = 0$$

[on integrating the first integral by parts]

or $$\int_{t_1}^{t_2} \delta\{(m/2)\times(d\boldsymbol{r}/dt)^2\}dt + \int_{t_1}^{t_2} \mathbf{F}\cdot\delta\boldsymbol{r}\ dt = 0 \qquad ...(3)$$

[using (2) and the relation $(d\boldsymbol{r}/dt)\cdot\delta(d\boldsymbol{r}/dt) = (1/2)\,\delta(d\boldsymbol{r}/dt)^2$]

Now, T = the kinetic energy of the particle = $(m/2)\times(d\boldsymbol{r}/dt)^2$

$$\therefore \qquad \delta[(m/2)\times(d\boldsymbol{r}/dt)^2] = \delta T \qquad ...(4)$$

Using (4), (3) reduces to $$\int_{t_1}^{t_2} (\delta T + \boldsymbol{F}\cdot\delta\boldsymbol{r})\, dt = 0 \qquad ...(5)$$

This is *Hamilton's principle* in its most general form, as applied to the motion of a single particle. However, if the force is conservative it can be re-written in a more concise form.

Let $\boldsymbol{F} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$. We recall that a force field $\boldsymbol{F}$ is *conservative* if

$$\boldsymbol{F}\cdot d\boldsymbol{r} = Xdx + Ydy + Zdz$$

is the differential $d\phi$ of a single-valued function ϕ. Also, then $\boldsymbol{F}$ = grad ϕ. The function ϕ is called the *force potential* and its negative, say V, is called the *potential energy*. It is to be noted that ϕ and V each involve an irrelevant arbitrary additive constant.

It follows that $\boldsymbol{F}$ is conservative if there exists a single valued function such that

$$\boldsymbol{F}\cdot\delta\boldsymbol{r} = -\delta V \qquad ...(6)$$

Using (6), Hamilton's principle takes the form $$\delta\int_{t_1}^{t_2} (T - V)dt = 0 \qquad ...(7)$$

which holds when a potential function V exists *i.e.* when the forces acting are conservative. For such a problem Hamilton's principle (7) states that the motion is such that the integral of the difference between the kinetic and potential energies is stationary for the true path.

The *action* (or *Hamilton's integral*) is defined as $A = \int_{t_1}^{t_2} (T - V)dt$ ...(8)

and in general its value depends on the path along which the particle moves in passing from P_1 and P_2 where P_1 and P_2 are the positions of the particle at times t_1 and t_2 respectively.

It is quite easy to give simple examples in which the actual path of a particle maximizes the action. However, if the time interval is sufficiently short, then it can be proved that the action is necessarily a minimum . In this form, Hamilton's principle is known as the principle of *least action* and can be loosely interpreted as saying that nature tends to equalize the kinetic and potntial energies throughout the motion.

The energy difference = $L = (T - V)$ is known as the *kinetic potential* or the *Lagarangian function*. Using this function, (7) may be re-written as

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \qquad ...(9)$$

When the force field is non-conservative, then the potential energy function does not exist, but (5) still holds and $\boldsymbol{F} \cdot \delta \boldsymbol{r}$ is the work done by the force $\boldsymbol{F}$ in a small displacement $\delta \boldsymbol{r}$. In particular, when the force is conservative this element of work is equivalent to $\delta\phi = -\delta V$.

The above derivation can easily be extended to a system of *n* particles by summation, and to a continous system by integration. For example, if the position vector of the *i th* particle of mass m_i be $\boldsymbol{r}_i$ and if m_i is subject to a force $\boldsymbol{F}_i$, then the total kintic energy is given by

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \left(\frac{d\boldsymbol{r}_i}{dt} \right)^2 \qquad ...(10)$$

while the total work done by the forces acting is given by $\sum_{i=1}^{n} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i$ ...(11)

Finally, The principle applies equally well to a general dynamical system consisting of particles and rigid bodies subject to inter-connections and constraints. It is to noted that the derivation is independent of the coordinates specifying the system.

OBJECTIVE PROBLEMS ON CHEPTER 1

Write (a), (b), (c) or (d) whichever is correct:

1. The extremum of the functional $I = \int_0^1 (y'^2 + 12xy)dx$ satisfying the conditions $y(0) = 0$ and $y(1) = 1$ is attained on the curve (a) $y = \sin^2(\pi x/2)$

(b) $y = \sin(\pi x/2)$ (c) $y = x^3$ (d) $y = (1/2) \times \{x^3 + \sin(\pi x/2)\}$. **[GATE 2005]**

Ans. (c). Refer Ex. 7 of Art 1.5A

2. (*i*) Extremals $y = y(x)$ for the variational problem $I[y(x)] = \int_0^1 (y + y')^2 dx$ satisfies the differential equation. (a) $y'' + y = 0$ (b) $y'' - y = 0$ (c) $y'' + y' = 0$ (d) $y' + y = 0$ **[GATE 2003]**

(*ii*) The functional $I[y(x)] = \int_0^2 (y'^2 + 6xy + x^3)dx = 0, y(0) = 0,\ y(2) = 2$ can be extremized on the curve (a) $y = x$ (b) $2y = x^3$ (c) $y = x^3 - 6x$ (d) $2y = x^3 - 2x$. **[GATE 2003]**

Ans. (*i*) (*b*); (*ii*) (*d*)

3. (*i*) The Euler's equation for a functional of the form $\int_a^b F(x, y)dx$ is

(a) $F_{y'} = C_1$ (b) $F_y - y'F_{y'} = C_1$ (c) $F_y = C_1$ (d) none of these **[Kanpur 2003, 2004]**

(*ii*) The curve of shortest distance between two fixed points is
(*a*) straight line (*b*) a circle (*c*) a cycloid (*d*) none of these. **[Kanpur 2004]**
(*iii*) The extremizing curve of the Brachistochrone problem is
(*a*) circle (*b*) a catenary (*c*) a cyloid (*d*) a straight line. **[Kanpur 2003]**
Ans. (*i*) (*d*) ; (*ii*) (*a*) ; (*iii*) (*a*)

4. *An extremal of the functional* $I[y(x)] = \int_a^b F(x, y, y')dx$, $y(a) = y_1$, $y(b) = y_2$ *satisfies Euler's equation which in general*
(*a*) *is a second order linear ordinary differential equation (ODE)*
(*b*) *is a non-linear ODE of order greater than two*
(*c*) *admits a unique solution satisfying the conditions* $y(a) = y_1$, $y(b) = y_2$
(*d*) *may not admit a solution satisfying the conditions* $y(a) = y_1$, $y(b) = y_2$ **[GATE 2002]**
Ans (*d*). As an example see Ex. 9(a) of Art. 1.5A, Art. 1.5B and 1.5C.

5. *The extremals for the functional* $I[y(x)] = \int_{x_1}^{x_2} (xy' + y'^2)dx$, *are given by the following family of curves.* (*a*) $y = C_1 + C_2x + x^2/4$ (*b*) $y = 1 + C_1x + (C_2/4) \times x^2$
(*c*) $y = C_1 + x + (C_2/4) \times x^2$ (*d*) $y = C_1 + C_2x - (x^2/4)$ **[GATE 2004]**
Ans (*d*) Proceed as in Ex. 14 of Art 1.5A.

6. The curve of given length that encloses maximum area between itself and the x-axis is a
(*a*) circle (*b*) cycloid (*c*) catenary (*d*) hyperbola. **[Kanpur 2003]**
Sol. Ans. (a) Refer Ex. 4 or Art 1.13 A.

7. Let I be the functional defined by $I[y(x)] = \int_0^{\pi/2} (y'^2 - y^2)dx$, $y(0) = 0$, $y(\pi/2) = 1$ where the unknown function $y(x)$ possesses two derivatives everywhere in $(0, \pi/2)$. Then (*a*) The functional has an extremum which cannot be acieved in the class of continous functions (*b*) The corresponding Euler's equation does not have a unique solution satisfying the given boundary conditions (*c*) I is not linear (d) I is linear. **[GATE 2006]**
Sol. Ans (b) Refer Ex. 18(a) of Art. 1.5 A.

8. Let $I[y(x)] = \int_0^1 F(x, y, y')dx$, satisfying $y(0) = 0$, $y(1) = 1$, where F has continuous second order derivatives w.r.t its arguments and the unknown function $y(x)$ possesses two derivatives everywhere in (0, 1). If the function F depends only on x and y', then the Euler's equation is an ordinary differential equation in y, which in general is (*a*) first order linear
(b) first order non-linear (*c*) second order linear (*d*) second order non-linear. **[GATE 2006]**
Sol. Ans. (a). Refer Art. I.5.E.

9. The functional $I[y(x)] = \int_0^1 (y + y'')dx$ defined on the set of functions $C^2([0, 1])$ satisfying $y(0) = 1$, $y(1) = 1$, $(dy/dx)_{x=0} = 0$ and $(dy/dx)_{x=1} = -1$ admits. (*a*) only one extremal (*b*) exactly two extremals (*c*) Infinite number of extremals (*d*) no extremals. **[GATE 2006]**

Sol. Ans. (*d*) Comparing the given functional with $\int_0^1 F(x, y, y', y'')dx$, we have

$$F(x, y, y', y'') = y + y'' \quad \ldots (1)$$

$\therefore$ Euler's –Poisson equation $F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0$ reduces to $1 - 0 - d^2(1)/dx^2 = 0$, *i.e.*, $1 = 0$, which is absurd. So the given functional can admit no extremal.

10. The extremum of the variational problem $\int_0^{\pi/8} (y'^2 + 24y' - 16y^2)dx$, $y(0) = 0$, $y(\pi/8) = 1$ occurs for the curve
(*a*) $y = \sin 4x$ (*b*) $y = \sqrt{2}\sin 2x$ (*c*) $y = 1 - \cos 4x$ (*d*) $y = (1 - \cos 8x)/2$ **(GATE 2007)**
Sol. Ans. (*a*). Proceed as in Ex. 18(*a*), Art. 1.5 F.

MISCELLANEOUS EXAMPLES ON CHAPTER 1

1. *Explain the following terms associated with Euler's equation : (i) Stationary value (ii) stationary point (iii) stationary function (iv) Extremals (v) Extremum* **[Guwahati 2007]**

2. *Find the extremals and the extremum of the functional*

$I[y] = \int_1^8 \left\{ 9(dy/dx)^2 - (2y^2/x^2) \right\} dx$ for which y (1) = 1 and y (8) = 4 . **[Guwahati 2007]**

Sol. Given $I[y] = \int_1^8 \left\{ 9y'^2 - (2y^2/x^2) \right\} dx$ ... (1)

Comparing the given functional with $\int_1^8 F(x, y, y')dx$, we get

$$F(x, y, y') = 9y'^2 - (2y^2/x^2) \quad \text{... (2)}$$

From (2), $\frac{\partial F}{\partial y} = -\frac{4y}{x^2}$, $\frac{\partial F}{\partial y'} = 18y'$ and so $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 18y''$

$\therefore$ Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ gives $-\frac{4y}{x^2} - 18y'' = 0$

or $(9x^2D^2 + 2)\, y = 0,$ where $D \equiv d/dx$ ... (3)

To solve (3), let $x = e^z$ and $D_1 \equiv xD \equiv d/dz$ so that $x^2D^2 = D_1(D_1 - 1)$. Hence (3) reduces to

$\{9D_1(D_1 - 1) + 2\}\, y = 0$ or $(9D_1^2 - 9D_1 + 2)\, y = 0$ ... (4)

Auxiliary equation of (4) is $9D_1^2 - 9D_1 + 2 = 0$ giving $D_1 = 1/3, 2/3$.

$\therefore$ Solution of (4) is $y = c_1e^{z/3} + c_2e^{2z/3} = c_1(e^z)^{1/3} + c_2(e^z)^{2/3}$

or $y = c_1 x^{1/3} + c_2 x^{2/3},$ as $x = e^z$. ... (5)

(1) gives the required extremals.

Using the given boundary conditions $y(1) = 1$ and $y(8) = 4$, (15) gives

$$c_1 + c_2 = 1 \quad \text{and} \quad c_1(8)^{1/3} + c_2(8)^{2/3} = 4$$

$$\Rightarrow \quad c_1 + c_2 = 1 \quad \text{and} \quad 2c_1 + 4c_2 = 4 \quad \Rightarrow \quad c_1 = 0 \text{ and } c_2 = 1$$

Hence, from (5), $y = x^{2/3}$ where gives the required extremum.

Ex. 3. The possible values of α for which the variational problem

$I[y(x)] = \int_0^1 (3y^2 + 2x^3y')\, dx,\ y(\alpha) = 1$ has extremals are

(*a*) –1, 0 (*b*) 0, 1 (*c*) –1, 1 (*d*) –1, 0, 1 **[GATE 2008]**

Sol. Ans. (c). Here $F(x, y, y') = 3y^2 + 2x^3y'$. So Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ yields

$6y - d(2x^3)/dx = 0$ or $y = x^2$. Since $y(\alpha) = 1$, we get $1 = \alpha^2$ Hence, $\alpha = 1, -1$

Ex. 4. (*i*) The extremal of the functional $\int_0^1 (y + x^2 + y'^2/4)dx,\ y(0) = 0,\ y(1) = 0$ is

(*a*) $4(x^2 - x)$ (*b*) $3(x^2 - x)$ (*c*) $2(x^2 - x)$ (*d*) $x^2 - x$ **[GATE 2009]**

(*ii*) The Euler's equation for the variational problem

minimize $I\,[y\,(x)] = \int_0^1 (2x - xy - y')\,y'dx$ is

(*a*) $2y'' - y = 2$ (*b*) $2y'' + y = 2$ (*c*) $y'' + 2y = 0$ (*d*) $2y'' - y = 0$ **[GATE 2010]**

[**Ans.** (*i*). (*d*); (*ii*). (*b*)]

Ex.5. Show that the shortest curve joining two fixed points is a straight line by using variational method. **[Meerut 2008, 10]**

Ex.6. Show that the area of the surface of revolution of the curve $y = f(x)$ is $2\pi\int_{x_1}^{x_2} y\sqrt{1 + y'^2}\,dx.$ Hence show that for this to be minimum, the curve must be a catenary.

[Meerut 2005, 2010, 2011]

Ex. 7. Show that the path followed by a particle in sliding from one point to another in the absence of friction in the shortest time is a cycloid. **[Meerut 2006, 10]**

Ex. 8. A functional $I[y(x)]$ has maxima or minima at the curve $y = y_0(x)$, if

(*a*) $\Delta I = 1$ (*b*) $\Delta I = -1$ (*c*) $\Delta I = 0$ (*d*) $\Delta I > 0$ **[Bhopal 2010]**

Ex.9. Find the shortest distance between two points A and B in a plane using polar coordinates. **[Nagpur 2010]**

Hint. Refer Ex. 21, page 1.27.

Ex.10. Find the extremals of the functional $\int_0^{\pi/2} \{2xy + (dx/dt)^2 + (dy/dt)^2\}dt$ with conditions $x\,(0) = 0,\ x(\pi/2) = -1,\ y\,(0) = 0,\ y(\pi/2) = 1$ **(Kanpur 2011)** **Ans.** $x = -\sin t,\ y = \sin t$

2

Variational Problems with Moving (Or Free) Boundaries. One side Variations

2.1 Variational problems with moving (or free) boundaries

While investigating the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx \qquad ...(1)$$

in Art 1.4 of chapter 1, we assumed that the boundary points (x_1, y_1) and (x_2, y_2) are fixed. In what follows, we shall assume that one or both of the boundary points can move. This implies that the class of admissible curves is extended because, in addition to the comparison curves with fixed boundary points, we can now also admit curves with variable boundary points From this, it follows that if on a curve $y = y(x)$, an extremum is attained in a problem with moving boundary point, then surely enough the extremum is all the more attained on a restricted class of curves with fixed boundary points with the curve $y = y(x)$ and hence, the basic condition for attaining an extremum in a problem with fixed boundaries must be satisfied. Thus the curves $y = y(x)$ on which extremum of the functional (1) is attained in a moving boundary problem must be solutions of the Euler's equation

$$F_y - \frac{d}{dx}F_{y'} = 0, \qquad i.e., \qquad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

Thus, the curves $y = y(x)$ on which an extremum is attained in the moving boundary problem must be extremals.

The general solution of (2) contains two arbitrary constants, for the determination of which we require two conditions. In the problem with fixed boundary points, these conditions were (see Art 1.4, chapter 1)

$$y(x_1) = y_1 \qquad \text{and} \qquad y(x_2) = y_2 \qquad ...(3)$$

On the other hand, in the moving boundary problems, one or both these conditions (3) are absent and, therefore, the missing condition (or conditions) for a determination of the arbitrary constants of the general solution of Euler's equation have to obtained from the basic necessary condition for an extremum which is that the variation of δI must be equal to zero.

From the above discussion, it follows that in the moving boundary problem an extremum is attained only on solutions $y = y(x, c_1, c_2)$ of the Euler's equation, c_1 and c_2 being arbitrary constants involved in the general solution of (2). Therefore, in what follows, we can consider the value of the functional only on functions of this family. Thus the functional $I[y(x, c_1, c_2)]$ is reduced to a function of the parameters c_1 and c_2 and of the limits of integration x_1 and x_2 , while the variation of the functional coincides with the differential of this function. For the sake of simplicity, let us assume that one of the boundary point (x_1, y_1) is fixed while the other boundary point (x_2, y_2) can move and passes to point $(x_2 + \delta x_2, y_2 + \delta y_2)$.

We shall call the permissible curves $y = y(x)$ and $y = y(x) + \delta y$ neighbouring if the absolute values of the variations δy and $\delta y'$ are small and the absolute values of the increments δx_2 and δy_2 are also small (the increments δx_2 and δy_2 are also known as variations of the limit values x_2 and y_2).

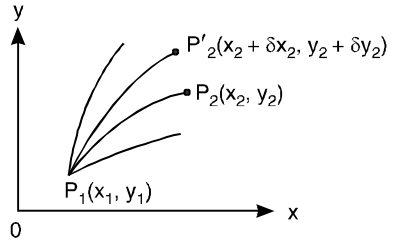


Fig. (i). A pencil of extremals through a given point

The extremals passing through the point (x_1, y_1) form a pencil of extremals $y = y(x, c_1)$ as shown in figure (*i*). The functional $I[y(x, c_1)]$ on the curves of this pencil becomes a function of c_1 and x_2. If the curves of the pencil $y = y(x, c_1)$ in the neighbourhood of the extrema under consideration do not intersect, then $I[y(x, c_1)]$ may be considered as a single valued function of x_2 and y_2. This is so because the specification of x_2 and y_2 determines the extremal of the pencil uniquely and hence determines the value of the functional.

We now find the variation of the functional $I[y(x, c_1)]$ on the extremals of the pencil $y = y(x, c_1)$ when the boundary point moves from (x_2, y_2) to $(x_2 + \delta x_2, y_2 + \delta y_2)$. Since the functional $I[y(x, c_1)]$ on the curves of the pencil is reduced to a function of x_2 and y_2, its variation concide with the differential of this function. From the increment ΔI, we now remove the principal part that is linear in δx_2 and δy_2.

Now, the increment ΔI is given by

$$\Delta I = \int_{x_1}^{x_2 + \delta x_2} F(x, y + \delta y, y' + \delta y')\, dx - \int_{x_1}^{x_2} F(x, y, y')\, dx$$

or
$$\Delta I = \int_{x_1}^{x_2} \{F(x, y + \delta y, y' + \delta y') - F(x, y, y')\} dx + \int_{x_2}^{x_2 + \delta x_2} F(x, y + \delta y, y' + \delta y')\, dx \qquad ...(4)$$

Using the mean-value theorem, the second term on R.H.S. of (4) can be re-written as

$$\int_{x_2}^{x_2 + \delta x_2} F(x, y + \delta y, y' + \delta y') dx = [F]_{x = x_2 + \theta \cdot \delta x_2} \delta x_2, \text{ where } 0 < \theta < 1 \qquad ...(7)$$

Again, by virtue of the continuity of F, we have

$$[F]_{x = x_2 + \theta \cdot \delta x_2} = [F(x, y, y')]_{x = x_2} + \varepsilon, \qquad ...(8)$$

where $\varepsilon \to 0$ as $\delta x_2 \to 0$ and $\delta y_2 \to 0$

From (7) and (8), $$\int_{x_2}^{x_2 + \delta x_2} F(x, y + \delta y, y' + \delta y') dx = \left[F(x, y, y')\right]_{x = x_2} \delta x_2 + \varepsilon\ \delta x_2 \qquad ...(9)$$

Using Taylor's expansion, first term on R.H.S. of (4) can be re-written as

$$\int_{x_1}^{x_2} \left\{F(x, y + \delta y, y' + \delta y') - F(x, y, y')\right\} dx = \int_{x_1}^{x_2} \left\{F_y(x, y, y')\ \delta y + F_{y'}(x, y, y')\ \delta y'\right\} dx + R, \qquad ...(10)$$

where R is an infinitesimal of the order higher than δy or $\delta y'$

On integrating by parts, we have

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y'\right) dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y\, dx + \left[\frac{\partial F}{\partial y'} \delta y\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \delta y\, dx$$

$$= \left[F_{y'}\ \delta y\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'}\right) \delta y\, dx \qquad ...(11)$$

Since the values of the functional I are taken only on the extremals, therefore

$$F_y - \frac{d}{dx} F_{y'} = 0 \qquad ...(12)$$

Again, since the boundary point (x_1, y_1) is fixed, so $[\delta y]_{x=x_1} = 0$ ...(13)

Using (12) and (13), (11) reduces to

$$\int_{x_1}^{x_2} (F_y \ \delta y + F_{y'} \ \delta y')dx = [F_{y'} \ \delta y]_{x=x_2} \qquad ...(14)$$

Note that $[\delta y]_{x=x_2} \neq \delta y_2$, since δy_2 is the increment of y_2 when the boundary point is displaced from from $P_2(x_2, y_2)$ to $P_2'(x_2 + \delta x_2, y_2 + \delta y_2)$. In fact, $[\delta y_2]_{x=x_2}$ is the increment of the ordinate at the point x_2 when passing from the extremal joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ to the one joining $P_1(x_1, y_1)$ and $P_2'(x_2 + \delta x_2, y_2 + \delta y_2)$ as shown is Fig. (*ii*).

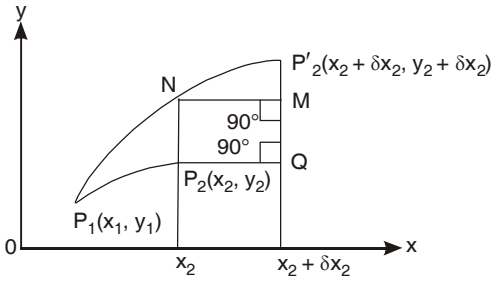


Fig. (ii) Extremizing curve and a neighbouring curve through a given point.

From figure (*ii*), it is clear that $P_2N = [\delta y]_{x=x_2}$, $QP_2' = \delta y_2$ and $MP_2' \approx y'(x_2)\delta x_2$, where ≈ stands for approximately equal to. Then, we get

$P_2'N = QP_2' - MP_2'$ so that $[\delta y]_{x=x_2} \approx \delta y_2 - y'(x_2)\,\delta x_2$

The above equality is valid within the infinitesimals of higher order. Thus, finally, we have

$$\int_{x_2}^{x_2+\delta x_2} F \ dx \approx [F]_{x=x_2} \ \delta x_2$$

and $$\int_{x_1}^{x_2} \{F(x, y+\delta y, y'+\delta y') - F(x, y, y')\}dx \approx [F_{y'}]_{x=x_2} (\delta y_2 - y'(x_2)\,\delta x_2),$$

where the approximate equalities likewise hold true to within terms higher than the first in δx_2 and δy_2. Hence, from (4), we have

$$\delta I = [F]_{x=x_2} \delta x_2 + [F_{y'}]_{x=x_2} \ (\delta y_2 - y'(x_2) \ \delta x_2) \quad \text{or} \quad \delta I = [F - y' \ F_{y'}]_{x=x_2} \ \delta x_2 + [F_{y'}]_{x=x_2} \ \delta y_2$$

or $$d\,\bar{I}\ (x_2, y_2) = [F - y' \ F_{y'}]_{x=x_2} \ dx_2 + \left[F_{y'}\right]_{x=x_2} dy_2,$$

where $\bar{I}(x_2, y_2)$ is a function into which the functional I was transformed on the extermals $y = y(x_1, c_1)$, and $dx_2 = \Delta x_2 = \delta x_2$, $dy_2 = \Delta y_2 = \delta y_2$ are increment of the coordinates of the boundary points.

The basic necessary condition for an extermum $\delta I = 0$ takes the form

$$[F - y' \ F_{y'}]_{x=x_2} \ \delta x_2 + [F_{y'}]_{x=x_2} \ \delta y_2 = 0 \qquad ...(15)$$

The following four cases arise :

Case I. When the variations δx_2 and δy_2 are independent

Then from (15), the necessary conditions for the extermum of (1) are given by

$$[F - y' F_{y'}]_{x=x_2} = 0 \qquad ...(16)$$

and $$[F_{y'}]_{x=x_2} = 0 \qquad ...(17)$$

Case II. When variations δx_2 and δy_2 are dependent

Suppose the boundary point (x_2, y_2) moves along the curve $y_2 = \psi(x_2)$, ...(18)

Then $\delta y_2 \approx \psi'(x_2)\,\delta x_2$ and, hence the condition (15) takes the following form

$$[F + (\psi' - y')F_{y'}]_{x=x_2}\,\delta x_2 = 0 \qquad ...(19)$$

Since δx_2 varies arbitrarily, $(19) \Rightarrow$ $[F + (\psi' - y')F_{y'}]_{x=x_2} = 0$ ...(20)

The above condition (20) establishes a relationship between the slopes ψ' and y' at the moving boundary point. It is known as the *transversality condition*. The conditions (18) and (20) are sufficient to determine the extremals of the pencil $y = y(x_1, c_1)$ on which as extermum may be attained.

If the boundary point (x_1, y_1) moves along the curve $y_1 = \phi(x_1)$, then proceeding as above we will find that the transversality condition $[F - (\phi' - y')\,F_{y'}]_{x=x_1} = 0$ ...(21)

must also be satisfied at the point (x_1, y_1).

Particular Case. There is a particular situation when the transversality conditon (20) reduces to the *orthogonality condition*. Suppose that function $F(x, y, y')$ in (1) is given by

$$F(x, y, y') = A(x, y)\cdot(1 + y'^2)^{1/2}, \qquad ...(22)$$

where $A(x, y) \neq 0$ at the movable boundary point (x_2, y_2). In this case the transversality condition takes the form

$$A(x,y)\cdot(1+y'^2)^{1/2} + \frac{A(x,y)\,y'}{(1+y'^2)^{1/2}}(\psi' - y') = 0 \quad \text{or} \quad \{A(x,y)\,(1+\psi' y')\}\big/(1+y'^2)^{1/2} = 0$$

Since by hypothesis, $A(x, y) \neq 0$, we get $1 + \psi' y' = 0$ so that $y' = -1/\psi'$ at $x = x_2$, showing that the transversality condition in this case is reduced to orthogonality condition. Thus the tangents on the extremal and $y = \psi(x)$ at $x = x_2$ are perpendicular to each other.

Case III. When the moving boundary point (x_2, y_2) can move only along a vertical straight line $x = x_2$ [see Fig. (*iii*)]

Then $\delta x_2 = 0$ and hence the condition (15) takes the form $[F_{y'}]_{x=x_2} = 0$ ...(23)

The condition (23) is known as a *natural boundary condition*.

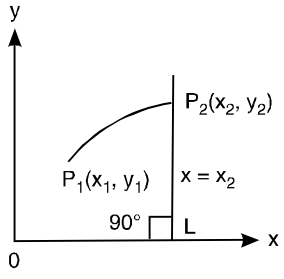


Fig. (iii)

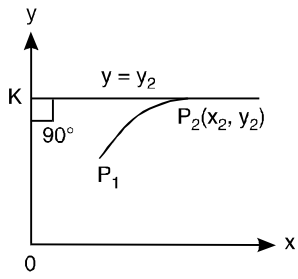


Fig. (iv)

Case IV. When the moving boundary point (x_2, y_2) can move only along a horizontal line $y = y_2$ (see Fig.iv)

Then $\delta y_2 = 0$ and hence the condition (15) takes the form $[F - y'F_{y'}]_{x=x_2} = 0$ ...(24)

Remark. In the above discussion we have studied the variational problem when the boundary points (x_1, y_1) and (x_2, y_2) moved on two curves in explicit form $y = \phi(x)$ and $y = \psi(x)$. However, if the two curve are in implicit form

$\phi(x, y) = 0$ and $\psi(x, y) = 0$

then the two transversality conditions (20) and (21) take the follwing forms:

$$\left[\frac{\partial F}{\partial y'}\left(\frac{\partial \phi}{\partial x} + y'\,\frac{\partial \phi}{\partial y}\right) - F\,\frac{\partial \phi}{\partial y}\right]_{x=x_1,\, y=y_1} = 0 \qquad ...(20)'$$

and

$$\left[\frac{\partial F}{\partial y'}\left(\frac{\partial \psi}{\partial x} + y'\,\frac{\partial \psi}{\partial y}\right) - F\,\frac{\partial \psi}{\partial y}\right]_{x=x_2,\, y=y_2} = 0 \qquad ...(21)'$$

Hence we shall use (20)′ and (21)′ in place of (20) and (21) while dealing with implicit forms of the given curves on which the end points (x_1, y_1) and (x_2, y_2) of the desired curves (for which the functional is extermum) move.

2.1A Some important results for direct applications

(*a*) Prove that for a functional of the form $[I(y(x)] = \int_{x_1}^{x_2} A(x, y)\,(1 + y'^2)^{1/2}\,dx$, where $A(x,y) \neq 0$ on the boundary curves, the extremals are orthogonal to the curves.

(*b*) Prove that for a functional of the form $I[y(x)] = \int_{x_1}^{x_2} A(x, y)(1 + y'^2)^{1/2}\,dx$, where $A(x, y) \neq 0$ at the moving boundary points $P_1\,(x_1,\ \phi(x_1))$ and $P_2\,(x_2, \psi(x_2))$, the transversality conditions are of the form

$y'(x) = -\{1/\phi'(x)\}$ *at* $x = x_1$ and $y'(x) = -\{1/\psi'(x)\}$ *at* $x = x_2$

Proof. (*a*) Refer particular case of case II of Art 2.1

(*b*) comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y')dx$, we have

$$F(x, y, y') = A(x, y)\ \times\ (1 + y'^2)^{1/2} \qquad ... (1)$$

The transversality conditions at $x = x_1$ and $x = x_2$ are given by (refer Art 2.1)

$$[F - (\phi' - y')F_{y'}]_{x=x_1} = 0 \qquad ... (2)$$

and

$$[F - (\psi' - y')F_{y'}]_{x=x_2} = 0 \qquad ...(3)$$

From (1), $F_{y'} = \partial F / \partial y' = A(x, y) \times y'(1 + y'^2)^{-1/2}$ ... (4)

Subtituting the values of F and $F_{y'}$ given by (1) and (4) in (2) yields

$$[A(x, y)(1 + y'^2)^{1/2} + (\phi' - y')A(x, y) \times y'(1 + y'^2)^{-1/2}]_{x=x_1} = 0$$

or $[A(x, y)\{1 + y'^2 + (\phi' - y')y'\}]_{x=x_1} = 0$ or $[1 + \phi' y']_{x=x_1} = 0$

Since $A(x, y) \neq 0$ at $x = x_1$, we have $y' = -\{1/\phi'(x)]$ at $x = x_1$

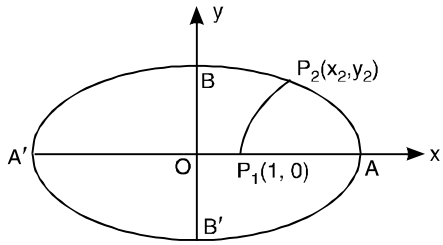
Likewise, from (3), we get $y' = -\,[1/\psi'\ (x)\}$ at $x = x_2$

2.2 ILLUSTRATIVE SOLVED EXAMPLES BASED ON ART. 2.1 AND ART. 2.1A

Ex. 1. (*a*) *Find the shortest distance between the point* $P_1(1,0)$ *and the ellipse* $4x^2+9y^2=36$.

[Bhopal 2004; MDU Rohtak 2004; Ravishankar 2004, 05]

Sol. We are to find the shortest between $P_1(1,0)$ and $P_2(x_2,y_2)$ where P_2 lies on the ellipse $ABA'B'$ given by $4x^2+9y^2=36$ ...(1)



The arc length P_1P_2 of the minimizing curve $y=y(x)$ is given by

$$l[y(x)]=\int_0^{x_2}(1+y'^2)^{1/2}\,dx\,, \qquad ...(2)$$

where the end point $P_1(1,0)$ is fixed while the end $P_2(x_2,y_2)$ lies on (1).

Comparing the given functional with $\int_0^{x_2}F(x,y,y')dx$, we have

$$F(x,y,y')=(1+y'^2)^{1/2} \qquad ...(3)$$

Since $F(x,y,y')$ is independent of x and y, we use the following form of Euler's equation (refer case III in Art, 1.5D, chapter 1)

$$\partial F/\partial y-\partial^2F/\partial x\partial y'-y'(\partial^2F/\partial y\partial y')-y''(\partial^2F/\partial y'^2)=0 \qquad ...(4)$$

From (3), $\partial F/\partial y=\partial F/\partial x=0$ so that $\partial^2F/\partial x\partial y'=\partial^2F/\partial y\partial y'=0$

Hence (4) reduces to $-y''(\partial^2F/\partial y'^2)=0$ ...(5)

Since $\partial^2F/\partial y'^2\neq 0$, (5) reduces to $y''=0$ or $d^2y/dx^2=0$

giving $dy/dx=C_1$ so that $y=C_1x+C_2$ ...(6)

Since (6) must pass through $P_1(1,0)$, we have

$$0=C_1+C_2 \quad \text{so that} \quad C_2=-C_1 \qquad ...(7)$$

Since (6) must pass through $P_2(x_2,y_2)$, we have

$$y_2=C_1x_2+C_2 \quad \text{or} \quad y_2=C_1x_2-C_1\text{, using (7)} \qquad ...(8)$$

Here the end $P_2(x_2,y_2)$ lies on (1) *i.e.* $y=(2/3)\times(9-x^2)^{1/2}=\psi(x)$, say ...(9)

Therefore, $4x_2^2+9y_2^2=36$ ...(10)

Hence the transversality condition (refer equation (20) of case II of Art. 2.1) is given by

$$[F+(\psi'-y')F_{y'}]_{x=x_2}=0$$

i.e.
$$\left[(1+y'^2)^{1/2}+\left\{\frac{2}{3}\times\frac{1}{2}\times\frac{(-2x)}{(9-x^2)^{1/2}}-y'\right\}\times\frac{1}{2}\times\frac{2y'}{(1+y'^2)^{1/2}}\right]_{x=x_2}=0$$

or
$$(1+C_1^2)^{1/2}+\left(\frac{-2x_2}{3(9-x_2^2)^{1/2}}-C_1\right)\times\frac{C_1}{(1+C_1^2)^{1/2}}=0 \qquad [\because \text{ from (6), we have } y'=C_1]$$

or
$$1+C_1^2-\frac{2x_2C_1}{3(9-x_2^2)^{1/2}}-C_1^2=0 \quad \text{or} \quad 3(9-x_2^2)^{1/2}=2x_2C_1 \qquad ...(11)$$

Using (8), (10) reduces to $4x_2^2 + 9C_1^2(x_2 - 1)^2 = 36$ ...(12)

From (11) and (12), we have $9(9 - x_2^2) = 4x_2^2 C_1^2$ ...(13)

and $4(9 - x_2^2) = 9C_1^2(x_2 - 1)^2$ ...(14)

Dividing (13) by (14), we have

$9/4 = (4x_2^2)/9(x_2 - 1)^2$ or $9(x_2 - 1) = 4x_2$ or $x_2 = 9/5$.

Substituting this value of x_2 in (11), we have

$3(9 - 81/25)^{1/2} = 2 \times (9/5) \times C_1$ so that $C_1 = 2$.

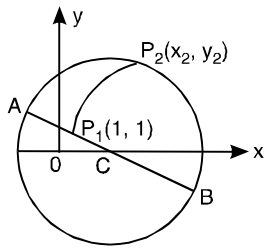
Substituting the values of x_2 and C_1 in (8), we have $y_2 = C_1(x_2 - 1) = 2(9/5 - 1) = 8/5$

From (8), the required shortest distance is attained along the line $y = 2(x - 1)$ and the required shortest distance l between $P_1(1,0)$ and the given ellipse (1) is given by the distance between $P_1(1,0)$ and $P_2(9/5, 8/5)$ and so

the shortest distance $= l = \{(9/5 - 1)^2 + (8/5 - 0)^2\} = (16/25 + 64/25)^{1/2} = 4\sqrt{5}/5$

Ex. 1 (*b*) *Find the extreme distances of an interior point (*1, 1*) from the curve* $(x - 2)^2 + y^2 = 9$.

Sol. We are to find the shortest distance between $P_1(1,1)$ and $P_2(x_2, y_2)$ where P_2 lies on the given curve (circle)



$$(x - 2)^2 + y^2 = 9 \quad ...(1)$$

The are length P_1P_2 of the minimizing curve $y = y(x)$ is given by $l[y(x)] = \int_0^{x_2} (1 + y'^2)^{1/2} dx$ ...(2)

where the end $P_1(1,1)$ is fixed while $P_2(x_2, y_2)$ lies on (1).

Comparing the given funcitional with $\int_0^{x_2} F(x, y, y')\, dx$, we have

$$F(x, y, y') = (1 + y'^2)^{1/2} \quad ...(3)$$

Since $F(x, y, y')$ is independent of x and y, we shall use the following form of Euler's equation (refer form III in Art. 1.5D of chapter 1).

$$\partial F/\partial y - \partial^2 F/\partial x \partial y' - y'(\partial^2 F/\partial y \partial y') - y''(\partial^2 F/\partial y'^2) = 0 \quad ...(4)$$

(3) $\Rightarrow$ $\partial F/\partial y = 0$, $\partial F/\partial x = 0$ so that $\partial^2 F/\partial x \partial y' = 0$ and $\partial^2 F/\partial y \partial y' = 0$

Hence (4) reduces to $-y''(\partial^2 F/\partial^2 y'^2) = 0$ ...(5)

Since $\partial^2 F/\partial y'^2 \neq 0$, (5) reduces to $y'' = 0$ or $d^2y/dx^2 = 0$,

so that $dy/dx = C_1$ and hence $y = C_1 x + C_2$ ...(6)

Since (6) must pass through $P_1(1,1)$, we have

$1 = C_1 + C_2$ so that $C_2 = 1 - C_1$ ...(7)

Again, since (6) must pass through $P_2(x_2, y_2)$ we have

$y_2 = C_1 x_2 + C_2$ so that $y_2 = C_1(x_2 - 1) + 1$, using (7) ...(8)

Now, the end $P_2(x_2, y_2)$ lies on (1), *i.e.*, $y = \{9-(x-2)^2\}^{1/2} = \psi(x)$, say ...(9)

Hence $(x_2-2)^2 + y_2^2 = 9$ ...(10)

Here the transversality condition (refer result (20) of case II of Art. 2.1) is

$$[F + (\psi' - y')F_{y'}]_{x=x_2} = 0 \qquad ...(11)$$

From (6) and (9), $y' = C_1$ and $\psi' = -(x-2)\{9-(x-2)^2\}^{-1/2}$ ...(12)

From (3), $F_{y'} = \partial F / \partial y' = y'(1+y'^2)^{-1/2} = C_1(1+C_1^2)^{-1/2}$

Also, since $y' = C_1$, from (3), we get $F = (1+C_1^2)^{1/2}$ ...(13)

Using (12) and (13), (11) reduces to

$$\left[(1+C_1^2)^{1/2} + \left\{(x-2)\Big/\sqrt{9-(x-2)^2} - C_1\right\} \times C_1(1+C_1^2)^{-1/2}\right]_{x=x_2} = 0$$

or $$(1+C_1^2)^{1/2} + \left\{(x_2-2)\Big/\sqrt{9-(x_2-2)^2} - C_1\right\} \times C_1(1+C_1^2)^{-1/2} = 0$$

or $$1 + C_1^2 - \{C_1(x_2-2)\}/\{9-(x_2-2)^2\}^{1/2} - C_1^2 = 0$$

or $$C_1(x_2-2) = \left\{9-(x_2-2)^2\right\}^{1/2} \qquad ...(14)$$

From (9), $$y_2 = \left\{9-(x_2-2)^2\right\}^{1/2}$$

or $$C_1(x_2-1) + 1 = \left\{9-(x_2-2)^2\right\}^{1/2}, \text{ using (8)} \qquad ...(15)$$

From (14) and (15), $C_1(x_2-2) = C_1(x_2-1) + 1$ and so $C_1 = -1$

Substituting the above value of C_1 in (7), we get $C_2 = 2$.

Hence, from (6), the line along which extremum (shortest and longest) distances will occurs is given by $y = -x + 2$ or $x + y = 2$, ...(16)

which also passes through the centre $C(2,0)$ of circle (1).

Since $C_1 = -1$, (15) reduces to $-(x_2-1)+1 = \{9-(x_2-2)^2\}^{1/2}$ or $2(x_2-2)^2 = 9$

or $x_2 - 2 = \pm 3/\sqrt{2}$ so that $x_2 = 2 \pm 3/\sqrt{2}$

Since $(x_2-2)^2 = 9/2$, (10) yeilds $9/2 + y_2^2 = 9$ so that $y_2 = \pm 3/\sqrt{2}$

Thus there are two points on the circle (1), namely, $A(2-3/\sqrt{2}, 3/\sqrt{2})$ and $B(2+3/\sqrt{2}, -3/\sqrt{2})$ the distances of which from $P_1(1,1)$ will be shortest and longest respectively (as shown in figure) and will be respectively as

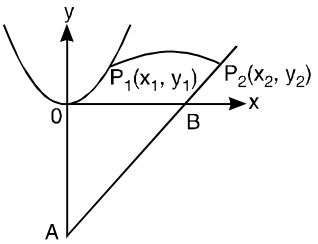
$$AP_1 = \int_{2-3/\sqrt{2}}^{1} (1+y'^2)^{1/2}\,dx = \int_{2-3/\sqrt{2}}^{1} (1+C_1^2)\,dx = \int_{2-3/\sqrt{2}}^{1} \sqrt{2}\,dx, \text{ as } C_1 = -1$$

$$= \sqrt{2}\,[x]_{2-3/\sqrt{2}}^{1} = \sqrt{2}\,[1-(2-3/\sqrt{2})] = 3-\sqrt{2}$$

and $$BP_1 = \int_{1}^{2+3/\sqrt{2}} (1+y'^2)^{1/2}\,dx = \int_{1}^{2+3/\sqrt{2}} \sqrt{2}\,dx = 3+\sqrt{2}$$

Ex. 2 (*a*) *Find the shortest distance between the parabola* $y = x^2$ *and the straight line* $y = x - 5$.
[Kanpur 2011; CDLU 2004; Jabalpur 2004; Ravishankar 2004]

Sol. Here we are to find the shortest distance between the point $P_1(x_1, y_1)$ lying on the parabola



$$y = x^2 = \phi(x), \text{ say} \qquad ...(1)$$

and $P_2(x_2, y_2)$ lying on the straight line

$$y = x - 5 = \psi(x), \text{ say} \qquad ...(2)$$

The arc length P_1P_2 of the minimizing curve $y = y(x)$ is given by

$$I[y(x)] = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx, \qquad ...(3)$$

Here both the boundary points P_1 and P_2 move on curves (1) and (2) respectively.

Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y')dx$, we get

$$F(x, y, y') = (1 + y'^2)^{1/2} \qquad ...(4)$$

Since $F(x, y, y')$ is independent of x and y, we use the following form of Euler's equation (refer form III, in Art 1.5D chapter 1)

$$\partial F/\partial y - \partial^2 F/\partial x \partial y' - y'(\partial^2 F/\partial y \partial y') - y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(5)$$

From (4), $\partial F/\partial y = \partial F/\partial x = 0$ so that $\partial^2 F/\partial x \partial y' = \partial^2 F/\partial y \partial y' = 0$

Hence (4) reduces to $-y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(6)$

Since $\partial^2 F/\partial y'^2 \neq 0$, (6) reduces to $y'' = 0$ or $d^2y/dx^2 = 0$

giving $dy/dx = C_1$ so that $y = C_1 x + C_2 \qquad ...(7)$

Since P_1 and P_2 lie on (7), $y_1 = C_1 x_1 + C_2$ and $y_2 = C_1 x_2 + C_2 \qquad ...(8)$

Again, since P_1 lies on (1) and P_2 lies on (2), we get

$$y_1 = x_1^2 \quad \text{and} \quad y_2 = x_2 - 5 \qquad ...(9)$$

From (8) and (9), we have $C_1 x_1 + C_2 = x_1^2 \qquad ...(10)$

and $C_1 x_2 + C_2 = x_2 - 5 \qquad ...(11)$

Using the transversality conditions at P_1 and P_1 (refer equations (20) and (21) of case II of Art. 2.1), we get

$$[F + (\phi' - y')F_{y'}]_{x=x_1} = 0 \quad \text{and} \quad [F + (\psi' - y')F_{y'}]_{x=x_2} = 0$$

i.e. $\left[(1 + y'^2)^{1/2} + (2x - y')\, y'(1 + y'^2)^{-1/2}\right]_{x=x_1} = 0 \qquad ...(12)$

and $\left[(1 + y'^2)^{1/2} + (1 - y')\, y'(1 + y'^2)^{-1/2}\right]_{x=x_2} = 0 \qquad ...(13)$

From (7), it follows that $y' = C_1$. Hence (12) and (13) give

$$(1 + C_1^2)^{1/2} + (2x_1 - C_1)\, C_1(1 + C_1^2)^{-1/2} = 0 \quad \text{or} \quad 1 + 2x_1 C_1 = 0 \qquad ...(14)$$

and $(1+C_1^2)^{1/2} + (1-C_1)\, C_1(1+C_1^2)^{-1/2} = 0$ or $1 + C_1 = 0$ ...(15)

From (14) and (15), $C_1 = -1$ and $x_1 = 1/2$. ...(16)

Using (16), (10) gives $-(1/2) + C_2 = 1/4$ so that $C_2 = 3/4$ ...(17)

Using (16) and (17), (11) gives $-x_2 + 3/4 = x_2 - 5$ so that $x_2 = 23/8$

Therefore, from (7) the required extremal is given by $y = -x + 3/4$ and the required shortest distance l along this extremal between the given parabola (1) and the straight line (2) is given by

$$l = \int_{1/2}^{23/8} (1+y'^2)^{1/2}\, dx = \int_{1/2}^{23/8} (1+1)^{1/2}\, dx, \quad \text{as} \quad y' = C_1 = -1$$

$$= \sqrt{2}\ [x]_{1/2}^{23/8} = \sqrt{2}\ (23/8 - 1/2) = 19\sqrt{2}/8.$$

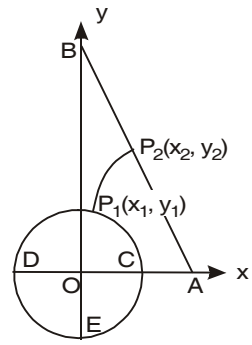
Ex. 2(*b*) *Find the shortest distance between circle* $x^2 + y^2 = 4$ *and straight line* $2x + y = 6$.

Sol. We are to find the shortest distance between the point $P_1(x_1, y_1)$ lying on the upper semi-circle DP_1C

$$x^2 + y^2 = 4 \quad \text{or} \quad y = (4 - x^2)^{1/2} = \phi(x), \text{ say} \qquad ...(1)$$

and $P_2(x_2, y_2)$ lying on the straight line

$$y = 6 - 2x = \psi(x), \text{ say} \qquad ...(2)$$



The arc length P_1P_2 of the extremizing curve $y = y(x)$ is given by

$$I[y(x)] = \int_{x_1}^{x_2} (1+y'^2)^{1/2}\, dx \qquad ...(3)$$

Here both the boundary points P_1 and P_2 move on the curves (1) and (2) respectively. Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y')dx$, we get

$$F(x, y, y') = (1+y'^2)^{1/2} \qquad ...(4)$$

Since $F(x, y, y')$ is independent of x and y, we use the following form of Euler's equation (refer form III in Art. 1.5D of chapter 1).

$$\partial F/\partial y - \partial^2 F/\partial x \partial y' - y'(\partial^2 F/\partial y \partial y') - y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(5)$$

(4) $\Rightarrow$ $\partial F/\partial y = 0$, $\partial F/\partial x = 0$ so that $\partial^2 F/\partial x \partial y' = 0$ and $\partial^2 F/\partial y \partial y' = 0$

Hence (5) reduces to $-y''(\partial F/\partial y'^2) = 0$ ...(6)

Since $\partial^2 F/\partial y'^2 \neq 0$ (6) reduces to $y'' = 0$ or $d^2y/dx^2 = 0$

giving $dy/dx = C_1$ so that $y = C_1 x + C_2$ ...(7)

Since P_1 and P_2 lie on (7), $y_1 = C_1 x_1 + C_2$ and $y_2 = C_1 x_2 + C_2$ ...(8)

Again, since P_1 and P_2 lie on (1) and (2) respectively, we get

$$y_1 = (4 - x_1^2)^{1/2} \quad \text{and} \quad y_2 = 6 - 2x_2 \qquad ...(9)$$

From (8) and (9), $C_1 x_1 + C_2 = (4 - x_1^2)^{1/2}$ ...(10)

and $C_1 x_2 + C_2 = 6 - 2x_2$ ...(11)

Using the transversality conditions at P_1 and P_2 (refer equations (20) and (21) of case II of Art. 2.1), we have

$$\left[F+(\phi'-y')F_{y'}\right]_{x=x_1}=0 \qquad \text{and} \qquad \left[F+(\psi'-y')F_{y'}\right]_{x=x_2}=0.$$

Using (1), (2) and (4), the above relations reduce to

$$\left[(1+y'^2)^{1/2}+\left\{-x(4-x^2)^{-1/2}-y'\right\}\times y'\ (1+y'^2)^{-1/2}\right]_{x=x_1}=0$$

and

$$\left[(1+y'^2)^{1/2}+(-2-y')\times y'\ (1+y'^2)^{-1/2}\right]_{x=x_2}=0$$

From (7), $y'=C_1$. Hence the above relations take the forms

$$(1+C_1^2)^{1/2}-\left\{x_1(4-x_1^2)^{-1/2}+C_1\right\}\times C_1\ (1+C_1^2)^{-1/2}=0 \qquad ...(12)$$

and

$$(1+C_1^2)^{1/2}-(2+C_1)\times C_1\ (1+C_1^2)^{-1/2}=0 \qquad ...(13)$$

From (13), $1+C_1^2-C_1(2+C_1)=0$ so that $C_1=1/2$

Re-writing (12), $1+C_1^2-C_1\left\{x_1(4-x_1^2)^{-1/2}+C_1\right\}=0$

or $1=C_1x_1(4-x_1^2)^{-1/2}$ or $1=(x_1/2)\times(4-x_1^2)^{-1/2}$, as $C_1=1/2$

or $4(4-x_1^2)=x_1^2$ or $5x_1^2=16$ or $x_1=4/\sqrt{5}$

Since $x_1=4/\sqrt{5}$, $(9)\Rightarrow y_1=(4-16/5)^{1/2}=2/\sqrt{5}$

Hence coordinates of P_1 are $(4/\sqrt{5},2/\sqrt{5})$. Putting $x_1=4/\sqrt{5}$ and $C_1=1/2$ in (10) gives

$$(1/2)\times(4/\sqrt{5})+C_2=(4-16/5)^{1/2} \qquad \text{so that} \qquad C_2=0$$

Since $C_1=1/2$ and $C_2=0$, (11) reduces to $(1/2)\times x_2+0=6-2x_2$ so that $x_2=12/5$.

Hence, from (9), $y_2=6-2\times(12/5)=6/5$.

Thus, coordinates of P_2 are $(12/5,6/5)$ Therefore, from (7), the required extremal is $y=x/2$ and the required shortest distance l between the given circle and the straight line is given by

$$l=\int_{4/\sqrt{5}}^{12/\sqrt{5}}(1+y'^2)^{1/2}dx=\int_{4/\sqrt{5}}^{12/\sqrt{5}}(1+1/4)^{1/2}dx, \quad \text{as} \quad y'=C_1=1/2$$

$$=\frac{\sqrt{5}}{2}\left[x\right]_{4/\sqrt{5}}^{12/\sqrt{5}}=\frac{\sqrt{5}}{2}\left(\frac{12}{\sqrt{5}}-\frac{4}{\sqrt{5}}\right)=2\left(\frac{3\sqrt{5}}{5}-1\right).$$

Ex. 2(*c*) *Find the shortest distance between* $y^2=4x$ *and* $(x-9)^2+y^2=4$.

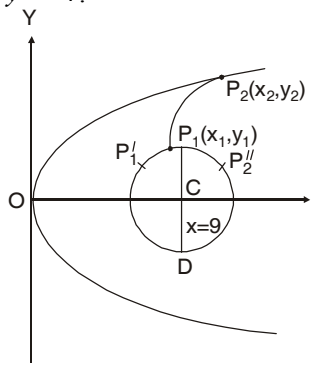
Sol. We are to find the shortest distance between the point $P_1(x_1,y_1)$ lying on the circle

$(x-9)^2+y^2=4$ or $y=\{4-(x-9^2)\}^{1/2}=\phi(x)$, say ...(1)

and $P_2(x_2,y_2)$ lyning on the paralola

$y^2=4x$ or $y=2\sqrt{x}=\psi(x)$, say ...(2)

The arc length P_1P_2 of the extremizing curve $y=y(x)$ is given by



$$l[y(x)] = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx \qquad ...(3)$$

Here both the boundary points P_1 and P_2 move on the curves (1) and (2) respectively.

Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, y')\, dx,$ we get

$$F(x, y, y') = (1 + y'^2)^{1/2} \qquad ...(4)$$

Since $F(x, y, y')$ is independent of x and y, we use the following form of Euler's equation (refer form of III in Art.1.5D, chapter 1)

$$\partial F/\partial y - \partial^2 F/\partial x \partial y' - y'(\partial^2 F/\partial y \partial y') - y'^2 (\partial^2 F/\partial y'^2) = 0 \qquad ...(5)$$

From (4), $\partial F/\partial y = \partial F/\partial x = 0$ so that $\partial^2 F/\partial x \partial y' = \partial^2 F/\partial y \partial y' = 0$

Hence (5) reduces to $-y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(6)$

Since $\partial^2 F/\partial y'^2 \neq 0,$ (6) rduces to $y'' = 0$ or $d^2 y/dx^2 = 0$

giving $dy/dx = c_1$ so that $y = c_1 x + c_2 \qquad ...(7)$

Since P_1 and P_2 lie on (7), we have $y_1 = c_1 x_1 + c_2$ and $y_2 = c_1 x_2 + c_2 \qquad ...(8)$

Again since P_1 and P_2 lie on (1) and (2) respectively, we get

$$y_1 = \{4 - (x_1 - 9)^2\}^{1/2} \quad \text{and} \quad y_2 = 2\sqrt{x_2} \qquad ...(9)$$

From (8) and (9), $c_1 x_1 + c_2 = \{4 - (x_1 - 9)^2\}^{1/2} \qquad ...(10)$

and $c_1 x_2 + c_2 = 2\sqrt{x_2} \qquad ...(11)$

Using the transversality conditions at P_1 and P_2 (refer equations (20) and (21) of case II of Art 2.1), we get

$$\left[F + (\phi' - y')F_{y'}\right]_{x=x_1} = 0 \quad \text{and} \quad \left[F + (\psi' - y')F_{y'}\right]_{x=x_2} = 0$$

Using (1), (2) and (4), the above relations reduce to

$$\left[(1 + y'^2)^{1/2} + \left\{-(x-9)/\sqrt{4 - (x-9)^2} - y'\right\} \times y' (1 + y'^2)^{-1/2}\right]_{x=x_1} = 0$$

and $$\left[(1 + y'^2)^{1/2} + \left\{(1/\sqrt{x} - y') \times y' (1 + y'^2)^{-1/2}\right\}\right]_{x=x_2} = 0$$

From (7), $y' = c_1$. Hence the above relations take the form

$$(1 + c_1^2)^{1/2} + \left[-(x_1 - 9)/\left\{4 - (x_1 - 9)^2\right\}^{1/2} - c_1\right] \times c_1 (1 + c_1^2)^{-1/2} = 0 \qquad ...(12)$$

and $$(1 + c_1^2)^{1/2} + (1/\sqrt{x_2} - c_1) \times c_1 (1 + c_1^2)^{-1/2} = 0 \qquad ...(13)$$

Re-writing (13), $1 + c_1^2 + c_1(1/\sqrt{x_2} - c_1) = 0$ so that $c_1 = -\sqrt{x_2} \qquad ...(14)$

Re-writing (12), $1 + c_1^2 - c_1\left[(x_1 - 9)/\{4 - (x_1 - 9)^2\}^{1/2} + c_1\right] = 0$

or $$c_1 (x_1 - 9) = \{4 - (x_1 - 9)^2\}^{1/2} \qquad ...(15)$$

From (10) and (15), $c_1 (x_1 - 9) = c_1 x_1 + c_2$ so that $c_2 = -9c_1 \qquad ...(16)$

From (14) and (16), $c_2 = 9\sqrt{x_2}$ ...(17)

Substituting the values of c_1 and c_2 as given by (14) and (17) in (11), we have

$-x_2\sqrt{x_2} + 9\sqrt{x_2} = 2\sqrt{x_2}$ so that $x_2 = 7$

Hence, from (9), $y_2 = 2\sqrt{7}$. Thus, coordinates of P_2 are $(7, 2\sqrt{7})$.

Since $x_2 = 7$, $(14) \Rightarrow c_1 = -\sqrt{7}$ and so $(16) \Rightarrow$ $c_2 = 9\sqrt{7}$

Since $c_1 = -\sqrt{7}$, (15) reduces to

$-\sqrt{7}(x_1 - 9) = \{4 - (x_1 - 9)^2\}^{1/2}$ or $7(x_1 - 9)^2 = 4 - (x_1 - 9)^2$

or $8(x_1 - 9)^2 = 4$ or $x_1 - 9 = \pm 1/\sqrt{2}$ or $x_1 = 9 \pm 1/\sqrt{2}$

Then, from (9), we get $y_1 = (4 - 1/2)^{1/2} = \pm\sqrt{7}/\sqrt{2}$.

Thus, as shown in the figure, there will be two points P_1' and P_1'' on both sides of the line $x = 9$, on the circle (1); $P_1'(9 - 1/\sqrt{2}, \sqrt{7/2})$ will lie on left and $P_1''(9 + 1/\sqrt{2}, \sqrt{7/2})$ will be on the right of the line $x = 9$. Clearly P_1' will have the shortest distance from $P_2(7, 2\sqrt{7})$.

Therefore, from (7) the required extremal is $y = -x\sqrt{7} + 7\sqrt{7}$ and the desired shortest distance l between the given circle and the given paralola is given by

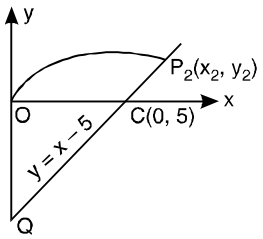
$$l = \int_7^{9-1/\sqrt{2}} (1 + y'^2)^{1/2}\, dx = \int_7^{9-1/\sqrt{2}} (1+7)^{1/2}\, dx, \quad \text{as} \quad y' = c_1 = -\sqrt{7}$$

$$= \sqrt{8}\,[x]_7^{9-1/\sqrt{2}} = \sqrt{8}\,(9 - 1/\sqrt{2} - 7) = \sqrt{8}\,(2 - 1/\sqrt{2})$$

Ex. 3. *Test for the extremum the functional* $\int_0^{x_2} \{(1 + y'^2)^{1/2}/y\}dx$ *given that y(0) = 0 and the second boundary moves on the line y = x – 5.*

Sol. Comparing the given functional with $\int_0^{x_2} F(x, y, y')\, dx$,

we get $F(x, y, y') = (1 + y'^2)^{1/2}/y$ ...(1)



Here one boundary point $(x_1, y_1)[= (0, 0)]$ is fixed while the other moving boundary point (x_2, y_2) lies on $y = x - 5$ so that

$y_2 = x_2 - 5$ ...(2)

Since F is independent of x, we use the following form of Euler's equation (refer case V, on Art. 1.5F of chapter 1)

$F - y'(\partial F/\partial y') = c$, c being an arbitrary constant

i.e. $(1/2) \times (1 + y'^2)^{1/2} - (y'/2y) \times (1 + y'^2)^{-1/2} \times 2y' = c$, using (1)

or $\dfrac{1}{y(1 + y'^2)^{1/2}} = c$ so that $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{c_1^2}{y^2}$, taking $c = \dfrac{1}{c_1}$

Thus $\dfrac{dy}{dx} = \pm\dfrac{(c_1^2 - y^2)^{1/2}}{y}$ or $dx = \pm\dfrac{y\, dy}{(c_1^2 - y^2)^{1/2}}$

Integrating, $x - c_2 = \pm(c_1^2 - y^2)^{1/2}$ or $(x - c_2)^2 = c_1^2 - y^2$

Thus, $(x-c_2)^2 + y_2 = c_1^2$, c_1 and c_2 being arbitrary constants. ...(3)

Using the given boudary condition $y(0)=0$, (3) gives $c_2 = c_1$ and hence (3) reduces to

$$(x-c_1)^2 + y^2 = c_1^2 \qquad ...(4)$$

Since $F(x, y, y')$ given by (1) is of the special form $A(x, y)\cdot(1+y'^2)^{1/2}$, it follows that for the given functional the tranversality condition reduces to the orthogonality condition (see remark of case II in Art. 2.1). It follows that the straight line $y = x-5$ must be a diameter of the circle (4). Therefore, the centre $(c_1, 0)$ of the required circle (4) must lie at the point (5, 0) where the straight line $y = x-5$ meets x-axis. Thus $c_1 = 5$ and so the required circle is

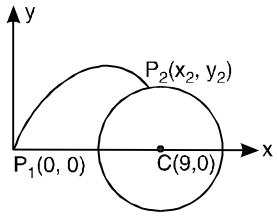
$$(x-5)^2 + y^2 = 5^2 \qquad \text{so that} \qquad y = \pm(10x - x^2)^{1/2}.$$

It follows that the extremum can be attained only on arcs of the circles

$$y = (10x - x^2)^{1/2} \qquad \text{and} \qquad y = -(10x - x^2)^{1/2}$$

Ex. 4. *Using only the basic necessary condition* $\delta I = 0$, *find the curve on which an extremum of the functional* $I[y(x)] = \int_0^{x_2} \{(1+y'^2)^{1/2}/y\}\, dx; \qquad y(0) = 0$ *can be achieved if the second boundary point* (x_2, y_2) *can move along the circumference* $(x-9)^2 + y^2 = 9$. **(Bilaspur 2004; Kanpur 2005)**

Sol. Comparing the given functional with $\int_0^{x_2} F(x, y, y')dx$, we have $F(x, y, y') = (1+y'^2)^{1/2}/y$ ...(1)



Here one boundary point $P_1(x_1, y_1)$ $[=(0,0)]$ is fixed while the other moving boundary point $P_2(x_2, y_2)$ lies on the circle

$$(x-9)^2 + y^2 = 9 \qquad ...(2)$$

$$\therefore \qquad (x_2 - 9)^2 + y_2^2 = 9 \qquad \text{so that} \qquad x_2^2 - 18x_2 + y_2^2 = -72 \qquad ...(3)$$

Since F is free from x, in this case a first integral of Euler's equation takes the form (see case V, Art 1.5F of chapter 1) $F - y'(\partial F/\partial y') = C$

or $(1/y)\times(1+y'^2)^{1/2} - (y'/2y)\times(1+y'^2)^{-1/2}\times 2y' = C$, using (1)

or $\dfrac{1}{y(1+y'^2)^{1/2}} = C$ or $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{C_1^2}{y^2}$, where $C_1 = \dfrac{1}{C}$

Thus, $dy/dx = \pm(C_1^2 - y^2)^{1/2}/y$

Separating variables, we have

$$dx = \pm\frac{y\,dy}{(C_1^2 - y^2)^{1/2}} \qquad \text{or} \qquad dx = \pm\frac{1}{2}(C_1^2 - y^2)^{-1/2}\times(-2y)dy$$

Integrating, $x - C_2 = \pm(C_1^2 - y^2)^{1/2}$ or $(x-C_2)^2 + y^2 = C_1^2$ ...(4)

Since $y(0) = 0$, (4) must pass through $P_1(0,0)$ and so $C_2 = C_1$.

Thus, (3) reduces to $(x-C_1)^2+y^2=C_1^2$ or $x^2-2C_1x+y^2=0$ ...(5)

Since $P_2(x_2,y_2)$ lie on (5), $(x_2-C_1)^2+y_2^2=C_1^2$ so that $x_2^2-2C_1x_2+y_2^2=0$ ...(6)

Subtracting (6) from (3), we get $-18x_2+2C_1x_2=-72$ or $x_2(9-C_1)=36$ ...(7)

Since $F(x,y,y')$ given by (1) is of the form $A(x,y)\cdot(1+y'^2)^{1/2}$ the transversality condition of the moving point $P_2(x_2,y_2)$ reduces to the orthogonality condition. It follows that the required extremal must be the arc of a circle belonging to (5) which is orthogonal to the circle (2) as shown in the figure. Now in view of orthogonality of circles (2) and (5) at $P_2(x_2,y_2)$, the tangent to (5) at P_2 must pass throught he center (9, 0) of circle (2).

Equation of the tangent to (5) at $P_2(x_2,y_2)$ is given by $xx_2-C_1(x+x_2)+yy_2=0$ ...(8)

As discussed above, (8) must pass through (9, 0).

$\therefore$ $9x_2-C_1(9+x_2)=0$ or $(9-C_1)x_2=9C_1$ ...(9)

Solving (7) and (9) for x_2 and C_1, we get $C_1=4$ and $x_2=36/5$. Hence from (5), we have

$x^2-8x+y^2=0$ giving $y=\pm(8x-x^2)^{1/2}$

Hence an extremum can be achieved only on the arcs of the circles

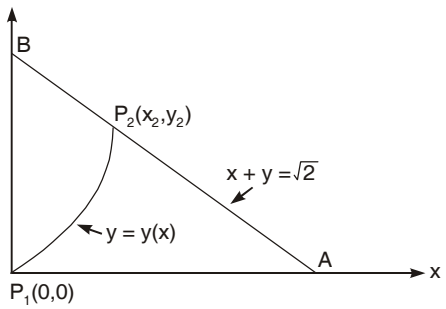
$y=(8x-x^2)^{1/2}$ and $y=-(8x-x^2)^{1/2}$

Ex. 5. *Find the shortest distance between the point*

(*a*) P_1 *(0,0) and the straight line* $x+y=\sqrt{2}$ **(MDU Rohtak 2005)**

(*b*) P_1 *(– 1,3) and the straight line* $y+3x=1$

Sol. (*a*) We are to find the shortest distance between P_1 (0,0) and P_2 (x_2, y_2) where P_2 lies on straight line $x+y=\sqrt{2}$ ...(1)



The are length P_1P_2 of the minimizing curve $y = y(x)$ is given by $I\,[y(x)] = \int_0^{x_2}(1+y'^2)^{1/2}dx$ ...(2)

where the end point P_1 (0,0) is fixed while P_2 (x_2, y_2) is a variable point lying on (1). Comparing the functional (2) with $\int_0^{x_2}F(x,y,y')dx$, we have

$$F(x,y,y')=(1+y'^2)^{1/2} \qquad ...(3)$$

Since $F(x,y,y')$ is independent of *x* and *y*, we use the following form of Euler's equation (refer form III of Art. 1.5 D of chapter 1)

$$\partial F/\partial y-\partial^2F/\partial x\partial y'-y'(\partial^2F/\partial y\partial y')-y''(\partial^2F/\partial y'^2)=0 \qquad ...(4)$$

From (3), $\partial F/\partial y=0=\partial F/\partial x$ so that $\partial^2F/\partial x\,\partial y=0=\partial^2F/\partial y\partial y'$

$\therefore$ (4) reduces to $-y''(\partial^2F/\partial y'^2)=0$ ...(5)

Since $\partial^2F/\partial y'^2\neq 0$, $(5)\Rightarrow$ $y''=0$ or $d^2y/dx^2=0$

Integrating, $dydx=c_1$ so that $y=c_1x+c_2$ ...(6)

Since (6) passes through (0,0), we have $c_2=0$

Again, since (6) must pass through $P_2\ (x_2, y_2)$, we have

$$y_2 = c_1 x_2 + c_2 \quad \text{or} \quad y_2 = c_1 x_2, \quad \text{as} \quad c_2 = 0 \qquad ...(7)$$

Now, the end point $P_2\ (x_2, y_2)$ lies on (1), *i.e.*, $y = \sqrt{2} - x = \psi(x)$, say ...(8)

$$\therefore \qquad y_2 = \sqrt{2} - x_2 \qquad ...(9)$$

Here the transversality condition at at $x = x_2$ is given by (refer Art. 2.1)

$$[F + (\psi' - y')F_{y'}]_{x=x_2} = 0 \qquad ...(10)$$

From (6) and (8), $y' = c_1$ and $\psi' = -1$ ...(11)

From (3), $F_{y'} = \partial F / \partial y' = y'(1 + y'^2)^{-1/2} = c_1(1 + c_1^2)^{-1/2}$ ...(12)

Since $y' = c_1$, (3) $\Rightarrow$ $F(x, y, y') = (1 + c_1^2)^{1/2}$ ...(13)

Using (11), (12) and (13), (10) reduces to

$$[(1 + c_1^2)^{1/2} + (-1 - c_1)c_1(1 + c_1^2)^{-1/2}]_{x=x_2} = 0 \quad \text{or} \quad (1 + c_1^2) - (1 + c_1)c_1 = 0 \quad \text{so that} \quad c_1 = 1$$

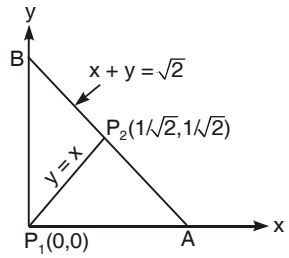
Since $c_1 = 1$ and $c_2 = 0$, from (6), the line along which extremum (shortest and longest) distances will occur is given by $y = x$ ...(14)

Since $P_2\ (x_2, y_2)$ lies on (14), we have $y_2 = x_2$ ...(15)

Solving (9) and (15), we get $x_2 = y_2 = 1/\sqrt{2}$. Thus, the coordinates of P_2 are $(1/\sqrt{2}, 1/\sqrt{2})$. The required shortest distance l along the extremal $y = x$ between $P_1\ (0,0)$ and straight line (1) is given by

$$l = \int_0^{1/\sqrt{2}} (1 + y'^2)^{1/2}\, dx = \int_0^{1/\sqrt{2}} (1 + c_1^2)^{1/2}\, dx$$

$$= \int_0^{1/\sqrt{2}} \sqrt{2}\, dx = 1, \text{ as } c_1 = 1$$



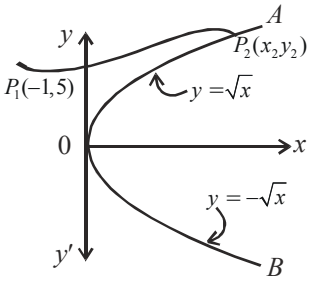
Note. The actual position of the minimizing curve ($y = x$) is shown in adjoining figure.

(*b*) Proceed as in part (*a*). Equation of extremal is $3y = x + 10$ and the shortest distance is $\sqrt{10}/10$ units.

Ex. 6. *Find the shortest distance from the point P_1 (– 1, 5) to the parabola $y^2 = x$.*

[KU Kurukshetra 2006, MDU Rohtak 2005]

Sol. Here equation of the given parabola AOB is $y^2 = x$. Then, clearly the equation of the upper half OA of the same parabola is given by $y = \sqrt{x}$. We are to find the shortest distance between P_1 (–1, 5) and $P_2\ (x_2, y_2,)$, where (x_2, y_2) lies on the parabola $y = \sqrt{x} = \psi\ (x)$, say ...(1)



The arc length P_1P_2 of the minimizing curve $y = y\ (x)$ is given by $I[y(x)] = \int_{-1}^{x_2} (1 + y'^2)^{1/2}\, dx,$ ...(2)

where the end point P_1 is fixed while P_2 is a variable point lying on (1).

Comparing the functional (2) with $\int_{-1}^{x_2} F(x, y, y')dx$, $F(x, y, y') = (1 + y'^2)^{1/2}$ ...(3)

Since $F(x, y, y')$ is independent of x and y, we use the following form of Euler's equation

$$\partial F/\partial y - \partial^2 F/\partial x \partial y' - y'(\partial^2 F/\partial y \partial y') - y''(\partial^2 F/\partial y'^2) = 0 \quad ...(4)$$

From (3), $\partial F/\partial y = 0 = \partial F/\partial x$ so that $\partial^2 F/\partial x \partial y' = 0 = \partial^2 F/\partial y \partial y'$

Hence (4) reduces to $-y''(\partial^2 F/\partial y'^2) = 0$ ...(5)

Since $\partial^2 F/\partial y'^2 \neq 0$ (5) yields $y'' = 0$ or $d^2y/dx^2 = 0$

so that $dy/dx = c_1$ and hence $y = c_1 x + c_2$ ...(6)

Since (6) must pass through $P_1(-1,5)$, $5 = -c_1 + c_2$ so that $c_2 = c_1 + 5$ ...(7)

Again, since (6) must pass through $P_2(x_2, y_2)$, we have

$y_2 = c_1 x_2 + c_2$ or $y_2 = c_1 x_2 + c_1 + 5$, by (7)

Thus, $y_2 = c_1 (x_2 + 1) + 5$ ...(8)

Now, the end point $P_2 (x_2, y_2)$ lies on (1) and hence

$y_2 = \sqrt{x_2}$ and hence $\sqrt{x_2} = c_1 (x_2 + 1) + 5$, by (8) ...(9)

Here the transversality condition is given by $[F + (\psi' - y')F_{y'}]_{x=x_2} = 0$ ...(10)

From (1) and (6), $\psi' = 1/2\sqrt{x}$ and $y' = c_1$ ...(11)

From (3), $F_{y'} = \partial F/\partial y' = y'(1 + y'^2)^{-1/2} = c_1(1 + c_1^2)^{-1/2}$ ...(12)

Also, since $y' = c_1$, (3) $\Rightarrow$ $F(x, y, y') = (1 + c_1^2)^{1/2}$ ...(13)

Using (11), (12) and (13), (10) reduces to

$$[(1 + c_1^2)^{1/2} + \{(1/2\sqrt{x}) - c_1\}\, c_1(1 + c_1^2)^{-1/2}]_{x=x_2} = 0$$

or $1 + c_1^2 + c_1\{(1/2\sqrt{x_2}) - c_1\} = 0$ or $c_1 = -2\sqrt{x_2}$ ...(14)

Substituting the above value of C_1 in (9), we have

$\sqrt{x_2} = -2\sqrt{x_2}(x_2 + 1) + 5$ or $5 = \sqrt{x_2}(1 + 2x_2 + 2)$

or $25 = x_2 (2x_2 + 3)^2$ or $4x_2^3 + 12x_2^2 + 9x_2 - 25 = 0$

or $4x_2^2(x_2 - 1) + 16\, x_2\, (x_2 - 1) + 25\, (x_2 - 1) = 0$ or $(x_2 - 1)\,(4x_2^2 + 16x_2 + 25) = 0$

$\therefore$ $x_2 = 1,\ \{-16 \pm (256 - 4 \times 4 \times 25)^{1/2}\}/2,$

showing that $x_2 = 1$ is the only real root.

From (9), $y_2 = \sqrt{x_2}$. Since $x_2 = 1$, $y_2 = 1$. Hence (14) $\Rightarrow$ $c_1 = -2$.

Since $c_1 = -2$, (7) $\Rightarrow c_2 = -2 + 5 = 3$. Thus, the coordinates of P_2 are (1, 1) and the equation of the extramizing curve, from (6), is given by $y = 3 - 2x$.

The required shortest distance l along the extremal $y = 3 - 2x$ between P_1 (-1, 5) and parabola (1) is given by

$$l = \int_{-1}^{1} (1 + y'^2)^{1/2}\, dx = \int_{-1}^{1} (1 + c_1^2)^{1/2}\, dx = \int_{-1}^{1} \{1 + (-2)^2\}^{1/2}\, dx = 2\sqrt{5} \text{ units}$$

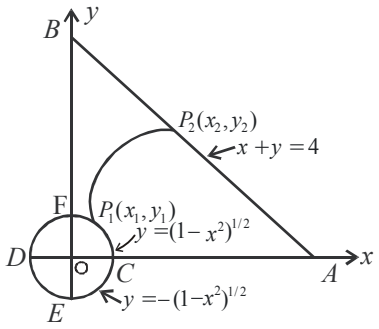
Ex. 7. *Find the shortest distance between the circle $x^2 + y^2 = 1$ and the straight line $x + y = 4$.*

[Kanpur 2011; Bilaspur 2004; Ravishankar 2004; Sagar 2004]

Sol. Here equation of the given circle CFDE is $x^2 + y^2 = 1$. Then, clearly the equation of the upper half CFD of the same circle is given by $y = (1 - x^2)^{1/2}$. We are to find the shortest distance between the point $P_1 (x_1, y_1)$ lying on $y = (1 - x^2)^{1/2} = \phi(x)$, say ...(1)

and $P_2 (x_2, y_2)$ lying on the straight line $y = 4 - x = \psi(x)$, say ...(2)

Here P_1 and P_2 are both variable points. The arc length P_1P_2 of the extremizing curve $y = y(x)$ is given



by $$I[y(x)] = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx \qquad ...(3)$$

Comparing the functional (3) with $\int_{x_1}^{x_2} F(x, y, y')dx$,

we get $$F(x, y, y') = (1 + y'^2)^{1/2}\, dx \qquad ...(4)$$

Since $F(x, y, y')$ independent of x and y, we use the following form of Euler's equation

$$\partial F/\partial y - \partial^2 F/\partial x \partial y' - y'(\partial^2 F/\partial y \partial y') - y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(5)$$

From (4), $\partial F/\partial y = 0 = \partial F/\partial x$ so that $\partial^2 F/\partial x \partial y' = 0 = \partial^2 F/\partial y \partial y'$

Hence (5) reduces to $$-y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(6)$$

Since $\partial^2 F/\partial y'^2 \neq 0$, (6) $\Rightarrow$ $y'' = 0$, *i.e.*, $d^2y/dx^2 = 0$

Integrating, $dy/dx = c_1$ so that $y = c_1 x + c_2$ ...(7)

Since $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie on (7), we have

$$y_1 = c_1 x + c_2 \quad \text{and} \quad y_2 = c_1\, x_2 + c_2 \qquad ...(8)$$

Again, since P_1 ad P_2 lie on (1) and (2) respectively, we get

$$y_1 = (1 - x_1^2)^{1/2} \quad \text{and} \quad y_2 = 4 - x_2 \qquad ...(9)$$

From (8) and (9), $$c_1 x + c_2 = (1 - x_1^2)^{1/2} \qquad ...(10)$$

and $$c_1 x_2 + c_2 = 4 - x_2 \qquad ...(11)$$

Using the transversality conditions at P_1 and P_2, we have

$$[F + (\phi' - y')F_{y'}]_{x=x_1} = 0 \quad \text{and} \quad [F + (\psi' - y')F_{y'}]_{x=x_2} = 0$$

Using (1), (2) and (4), the above conditions take the following forms:

$$[(1 + y'^2)^{1/2} + \{-x(1 - x^2)^{-1/2} - y'\} \times y'(1 + y'^2)^{-1/2}]_{x=x_1} = 0$$

and $$[(1 + y'^2)^{1/2} + (-1 - y') \times y'(1 + y'^2)^{-1/2}]_{x=x_2} = 0$$

Since $y' = c_1$, by (7), hence the above relations reduce to

$$(1 + c_1^2)^{1/2} - \{x_1(1 - x_1^2)^{-1/2} + c_1\} \times c_1(1 + c_1^2)^{-1/2} = 0 \qquad ...(12)$$

and $$(1 + c_1^2)^{1/2} - (1 + c_1) \times c_1(1 + c_1^2)^{-1/2} = 0 \qquad ...(13)$$

From (13), $1 + c_1^2 = c_1(1 + c_1)$ so that $c_1 = 1$

Re-writing (12), $$(1 + c_1^2) - c_1\{x_1(1 - x_1^2)^{-1/2} + c_i\} = 0$$

or $1 = c_1 x_1 (1 - x_1^2)^{-1/2}$ or $1 = x_1(1 - x_1^2)^{-1/2}$, as $c_1 = 1$

or $(1 - x_1^2)^{1/2} = x_1$ or $1 - x_1^2 = x_1^2$ so that $x_1 = 1/\sqrt{2}$, as $x_1 > 0$

Since $x_1 = 1/\sqrt{2}$, (9) $\Rightarrow$ $y_1 = (1 - 1/2)^{1/2} = 1/\sqrt{2}$

Hence the coordinates of P_1 are $(1/\sqrt{2}, 1/\sqrt{2})$. Putting $x_1 = 1/\sqrt{2}$ and $c_1 = 1$ in (10), we have

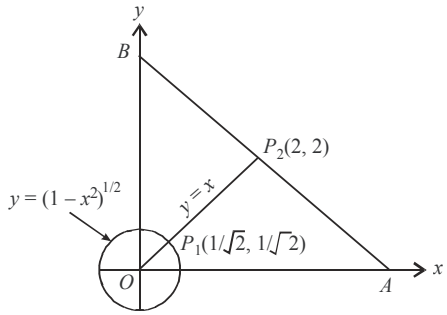
$1/\sqrt{2} + c_2 = (1 - 1/2)^{1/2}$ so that $c_2 = 0$

Putting $c_1 = 1$ and $c_2 = 0$ is (11), we get $x_2 = 2$.

Since $x_2 = 2$, (9) $\Rightarrow y_2 = 4 - x_2 = 2$. Thus the coordinates of P_2 are (2,2). Since $c_1 = 1$ and $c_2 = 0$, (7) yields $y = x$. Thus, $y = x$ minimizes the functional (3), *i.e.*, the distance between

$$P_1\ (1\sqrt{2}, 1/\sqrt{2}) \text{ and } P_2\ (2,2).$$

The required shortest distance l along the extremal $y = x$ between P_1 and P_2 is given by



$$l = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx = \int_{1/\sqrt{2}}^{2} (1 + c_1^2)^{1/2}\, dx, \text{ as } x_1 = 1/\sqrt{2}\,,\ x_2 = 2 \text{ and } y' = c_1$$

$$= \int_{1/\sqrt{2}}^{2} (1+1)^{1/2}\, dx = (2\sqrt{2} - 1) \text{ units, as } c_1 = 1$$

Note: The actual position of the minimizing curve namely, straight line $y = x$ is shown in the above figure.

Ex. 8. ***The problem of the brachistochrone.*** *Find the curve connecting given points $P_1(0,0)$ and $P_2(x_2, y_2)$ which is traversed by a particle sliding from P_1 to P_2 in the shortest time when P_1 is fixed while P_2 is allowed to move along the vertical line $x = x_2$. (Friction and resistance of the medium are ignored)*

Sol. Let $P_1(0,0)$ be taken as origin and let us measure y downward, as shown in the figure (*i*). Let the moving boundary point $P_2(x_2, y_2)$ move on the line $x = x_2$, where x_2 is constant. We wish to minimize the time of descent from P_1 to P_2. At any time t, let the particle be at $P(x, y)$ such that arc $OP = s$. Using the principle of conservation of energy, we have

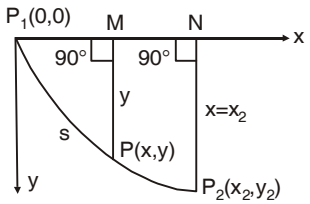


Figure (i)

Kinetic energy at P – kinetic energy at P_1

= work done in moving the particle from P_1 to P

or $(1/2) \times m v^2 - 0 = m g y$ or $(ds/dt)^2 = 2gy$ or $ds/dt = \sqrt{2g\,y}$

Hence the time spent in moving from P_1 to P_2 is given by

$$t[y(x)] = \int_{P_1}^{P_2} \frac{ds}{(2gy)^{1/2}} = \int_0^{x_2} \{(1 + y'^2)^{1/2} \big/ \sqrt{2gy}\}\, dx \qquad ...(1)$$

Comparing the functional (1) with $\int_0^{x_2} F(x, y, y')$ and omitting the irrelevent factor $1/\sqrt{2g}$,

$$F(x, y, y') = (1 + y'^2) \big/ \sqrt{y} \qquad ...(2)$$

Since $F(x, y, y')$ is independent of x, a first integral of Euler's equation is (refer case V, Art. 1.5F of chapter 1) $F - y'(\partial F / \partial y') = \text{constant} = C$

or $(1/\sqrt{y}) \times (1 + y'^2)^{1/2} - (y'/2\sqrt{y}) \times (1 + y'^2)^{-1/2} \times 2y' = C$, using (2)

or $1 + y'^2 - y'^2 = C\sqrt{y}\,(1 + y'^2)^{1/2}$ or $y\,(1 + y'^2) = C_1$ ...(3)

where $C_1 = 1/C^2$. Introduce the parameter t, by putting $y' = dy/dx = \cot t$ ...(4)

Then, from (3), $$y = \frac{C_1}{1 + \cot^2 t} = C_1 \sin^2 t = \frac{C_1}{2} \times (1 - \cos 2t) \qquad ...(5)$$

(4) and (5) $\Rightarrow dx = \dfrac{dy}{y'} = \dfrac{2C_1 \sin t \cos t\, dt}{\cos t} = 2C_1 \sin^2 t\, dt$ or $dx = C_1(1-\cos 2t)dt$

Integrating, $x = C_1\left(t - \dfrac{\sin 2t}{2}\right) + C_2 = \dfrac{C_1}{2}(2t - \sin 2t) + C_2$, ...(6)

where C_1 and C_2 are arbitrary constants.

From (5) and (6), the equation of the desired curve is

$$x - C_2 = (C_1/2)\times(2t - \sin 2t), \qquad y = (C_1/2)\times(1-\cos 2t) \qquad ...(7)$$

Taking $C_1/2 = a$ and $2t = \phi$, (7) reduces to $x - C_2 = a(\phi - \sin\phi), \quad y = a(1-\cos\phi)$ ...(8)

Now, at $P_1(0,0)$, when $x = 0, y = 0$ and so (8) gives $\phi = 0$ and $C_2 = 0$. Hence the extremals of (1) are cycloids of the following form: $x = a(\phi - \sin\phi), \quad y = a(1-\cos\phi)$ ...(9)

Since the moving boundary point can move only a vertical line $x = x_2$, so the basic necessary condition for extremum (1) is (refer case III, Art.2.1)

$$[F_{y'}]_{x=x_2} = 0 \qquad \text{or} \qquad \left[(1/2\sqrt{y})\times(1+y'^2)^{-1/2}\times 2y'\right]_{x=x_2} = 0$$

so that $y'(x_2) = 0$ *i.e.*, $(dy/dx)_{x=x_2} = 0$ ...(10)

(10) shows that $dy/dx = 0$ at $x = x_2$ so that the tangent to the cycloid at P_2 must be parallel to the x-axis or in other words, that the cycloid must be perpendicular to the line $x = x_2$ at P_2. This implies that P_2 must at a vertex of the cycloid (refer figure (*ii*)). From the tracing of the cycloid (9), we know that the cusp P_2 corresponds to the value $\phi = \pi$. Hence (9) yields $x_2 = C_1\pi$ so that $C_1 = x_2/\pi$. So, by (9) an extermum can be attained only on the cycloid $x = (x_2/\pi)\times(\phi - \sin\phi), \quad y = (x_2/\pi)\times(1-\cos\phi)$.

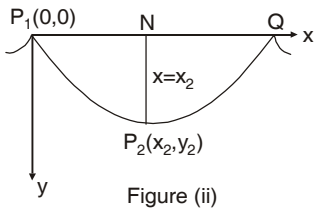


Figure (ii)

Ex. 9(*a*) *Find the shortest distance between the point* (1, 1) *and a curve in implicit form* $x^2 + y^2 + xy = 1$.

(*b*) *Show that the shortest distance between the point (3,3) and a curve in implicit form* $(x^2+y^2)^2 - (x^2+y^2+4xy) = 0$ *is* $\sqrt{2}(3-\sqrt{3/2})$ *and is along the line* $y = x$.

Sol. (*a*) We are to find the shortest distance between the given fixed point $P_1(1,1)$ and a variable point $P_2(x_2, y_2)$ lying on the given curve

$$\phi(x,y) = x^2 + y^2 + xy - 1 = 0 \qquad ...(1)$$

so that $$x_2^2 + y_2^2 + x_2 y_2 - 1 = 0 \qquad ...(2)$$

The arc length P_1P_2 of the extremizing curve $y = y(x)$ is given by

$$l[y(x)] = \int_0^{x_2} (1+y'^2)^{1/2}\, dx \qquad ...(3)$$

Comparing (3) with $\int_0^{x_2} F(x,y,y')dx$, here $F(x,y,y') = (1+y'^2)^{1/2}$ ...(4)

Since $F(x,y,y')$ is independent of x and y, we use the following form of Euler's equation (refer form III, in Art. 1.5D of chapter 1)

$$\partial F/\partial y - \partial^2 F/\partial x \partial y' - y'(\partial^2 F/\partial y \partial y') - y''(\partial^2 F/\partial y'^2) = 0 \qquad ...(5)$$

From (3), $\partial F/\partial y = 0 = \partial F/\partial x$ and so $\partial^2 F/\partial x \partial y' = \partial^2 F/\partial y \partial y = 0$.

Using these facts, (5) yields $-y''(\partial^2 F/\partial y'^2) = 0$

Since $\partial^2 F/\partial y'^2 \neq 0$, we get $y'' = 0$ or $d^2y/dx^2 = 0$

Integrating it twice, we get $y = C_1 x + C_2$ ...(6)

Since (1,1) and (x_2, y_2) lie on (6), we have

$C_1 + C_2 = 1$ so that $C_2 = 1 - C_1$ ...(7)

and $y_2 = C_1 x_2 + C_2$ ...(8)

Substituting the value of C_2 given by (7) in (8), we get $y_2 - 1 = C_1(x_2 - 1)$ ...(9)

Also, from (6), $y' = C_1$ ...(10)

Since (1) is in implicit form, we use the following form of transversality condition (refer remark of Art. 2.1)

$$\left[\frac{\partial F}{\partial y'}\left(\frac{\partial \phi}{\partial x} + y'\frac{\partial \phi}{\partial y}\right) - F\left(\frac{\partial \phi}{\partial y}\right)\right]_{x=x_2, y=y_2} = 0$$

or $$\left[y'(1+y'^2)^{-1/2}\left(\frac{\partial \phi}{\partial x} + y'\frac{\partial \phi}{\partial y}\right) - (1+y'^2)^{1/2}\left(\frac{\partial \phi}{\partial y}\right)\right]_{x=x_2, y=y_2} = 0$$

or $$\left[C_1(1+C_1^2)^{-1/2}\left(\frac{\partial \phi}{\partial x} + C_1\frac{\partial \phi}{\partial y}\right) - (1+C_1^2)^{1/2}\left(\frac{\partial \phi}{\partial y}\right)\right]_{x=x_2, y=y_2} = 0$$, using (10)

or $$\left[C_1\left(\frac{\partial \phi}{\partial x} + C_1\frac{\partial \phi}{\partial y}\right) - \left(1+C_1^2\right)\left(\frac{\partial \phi}{\partial y}\right)\right]_{x=x_2, y=y_2} = 0$$

or $$\left[C_1(\partial\phi/\partial x) - \partial\phi/\partial y\right]_{x=x_2, y=y_2} = 0$$

or $$C_1 = \left(\frac{\partial \phi/\partial y}{\partial \phi/\partial x}\right)_{x=x_2, y=y_2} = \left(\frac{2y+x}{2x+y}\right)_{x=x_2, y=y_2} = \frac{2y_2 + x_2}{2x_2 + y_2}$$, by (1) ...(11)

or $$\frac{y_2 - 1}{x_2 - 1} = \frac{2y_2 + x_2}{2x_2 + y_2}$$, since by (9), $C_1 = \frac{y_2 - 1}{x_2 - 1}$

or $(y_2 - 1)(2x_2 + y_2) = (2y_2 + x_2)(x_2 - 1)$ or $(y_2 - x_2)(y_2 + x_2 + 1) = 0$

Thus, either $y_2 = x_2$ or $y_2 + x_2 + 1 = 0$ ...(12)

When $y_2 = x_2$, (11) $\Rightarrow C_1 = 1$ and then (7) $\Rightarrow C_2 = 0$

Hence from (6), the line of the shortest distance between (1,1) and a point on the given curve (1) is $y = x$.

When $y_2 = x_2$, (1) gives $3x_2^2 - 1 = 0$ so that $x_2 = 1/\sqrt{3}$. Thus, the coordinates of $P_2(x_2, y_2)$ are $(1/\sqrt{3}, 1/\sqrt{3})$.

Hence the required extremum is attained along the line $y = x$ and the desired extremum value of distance between $P_1(1,1)$ and $P_2(1/\sqrt{3}, 1/\sqrt{3})$ is

$$= \int_{1/\sqrt{3}}^{1} (1+y'^2)^{1/2}\,dx = \int_{1/\sqrt{3}}^{1} \sqrt{2}\,dx = \sqrt{2}(1-1/\sqrt{3}), \text{ as } y' = C_1 = 1 \qquad ...(13)$$

Finally, from (12), when $y_2 + x_2 + 1 = 0$ then $y_2 = -x_2 - 1$ ...(14)

Substituting the value of y_2 as given by (14) in (2) we get

$$x_2^2 + (x_2+1)^2 + x_2(-x_2-1) - 1 = 0 \quad \text{giving} \quad x_2 = 0 \quad \text{or} \quad x_2 = -1$$

When $x_2 = 0$ then (14) $\Rightarrow y_2 = -1$. Then, by (11), $C_1 = 2$ and so (7) $\Rightarrow C_2 = -1$ and (6) reduces to $y = 2x - 1$. So the required extremum distance is attained along the line $y = 2x - 1$ and the extremum distance between $P_1(1,1)$ and $P_2(0,-1)$ is

$$\int_0^1 (1+y'^2)^{1/2}\,dx = \int_0^1 (1+4)^{1/2}\,dx = \sqrt{5}, \qquad \text{as} \qquad y' = C_1 = 2 \qquad ...(15)$$

Next, if $x_2 = -1$, then (14) $\Rightarrow y_2 = 0$. Then, by (11), $C_1 = 1/2$ and so (7) $\Rightarrow C_2 = 1/2$. Hence, (6) reduces to $y = (1/2)\times(x+1)$. So the required extreme distance is attained along the line $y = (x+1)/2$ and extreme distance between $P_1(1,1)$ and $P_2(-1,0)$ is

$$= \int_{-1}^{1} (1+y'^2)^{1/2}\,dx = \int_{-1}^{1} (1+1/4)^{1/2}\,dx = (\sqrt{5}/2)\times 2 = \sqrt{5}, \text{ as } y' = C_1 = 1/2 \qquad ...(16)$$

From (13),(15) and (16), we find that the shortest distance is attained along $y = x$ and the value of the shortest distance is $\sqrt{2}(1-1/\sqrt{3})$.

(*b*) Try yourself.

EXERCISE 2(A)

1. Find the extremum distances of an interior point (1, 1) from the circle $x^2 + y^2 = 4$.

Ans. Shortest and longest distances occur along the extemizing curve $y = x$. The shortest distance $= 2 - \sqrt{2}$ and the longest distance $= 2 + \sqrt{2}$.

2. Find the extremum distances of an interior point (1,1) from the curve $x^2/9 + y^2/4 = 1$.

Ans. The extermizing curve is $2y = x + 1$ and the shortest distance is nearly 1.1085

3. Find the shortest distance between $y^2 = 4x$ and the straight line $x + y = -5$.

Ans. Shortest distance $= 2\sqrt{2}$ extremizing curve is $y = x - 3$

4. Find the curve between $y^2 = 4x$ and $x + y = -5$ for which the functional $I = \int_{x_1}^{x_2} y'^2 dx$ is extremum.

Ans. The required curve is the straight line joining the point (1, –2) lying on the given parabola $y^2 = 4x$ and the point (5, –10) lying on the given straight line $x + y = -5$.

5. Find a function on which an extremum of the functional $I[y(x)] = \int_0^{\pi/4} (y^2 - y'^2)dx$, $y(0) = 0$ can be achieved if the other boundary point is permitted to slide along the straight line $x = \pi/4$. **Ans.** $y = 0$

6. Find the transversality condition for the functional

$$I[y(x)] = \int_{x_1}^{x_2} A(x,y)\, e^{\tan^{-1} y'} (1+y'^2)^{1/2}\,dx, \qquad A(x,y) \neq 0$$

Hint. Use case II of Art.2.1 and get $(y' - \phi')/(1 + y'\phi') = 1$

showing that the externals must intersect the curve $y = \phi(x)$ along which the boundary point slides at an angle $\pi/4$.

7. Using the basic necessary condition for an extremum $\delta I = 0$, find the function on which the following functional can be extremized.

$$I[y(x)] = \int_0^1 (y''^2 - 2xy)dx, y(0) = y'(0); y(1) = 1/120, y'(1) \text{ is not given}$$

Ans. $y = x^5/120 + (x^2 - x^3)/24$

2.3 Variational problem with a moving boundary for a funcitonal dependent on two functions

While investigating the functional $$I[y(x), z(x)] = \int_{x_1}^{x_2} F(x, y, z, y', z')dx \qquad ...(1)$$

in Art. 1.8 of chapter 1, we assumed that the boundary points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are fixed. In what follows, we shall assume that one or both of the boundary points can move.This implies that the class of admissible curves is extended because, in addition to the comparison curves with fixed boundary points, we can now admit curves with variable boundary points. From this, it follows that if on a curve, an extremum is attained in a problem with moving boundary points, then surely enough the extremum is all the more attained on a restricted class of curves with fixed boundary points. Therefore, the basic condition for attaining an extremum in a problem with fixed boundaries must be satisfied, *i.e.* an extremum is attained only on the integral curves of the system of Euler's equations

$$F_y - \frac{d}{dx}F_{y'} = 0 \qquad \text{and} \qquad F_z - \frac{d}{dx}F_{z'} = 0 \qquad ...(2)$$

The general solution of (2) contains four arbitrary constants. Let us assume that the boundary point P_1 is fixed. Since P_1 is fixed, it is possible to eliminate two arbitrary constants from the solutions of (2). The other two constants have to be determined from the necessary condition $\delta I = 0$ for extremum, where δI is the variation of I. While computing the variation we shall now assume that the functional is specified on solutions of the system of Euler's equations, because an extremum is achieved only on them. Then the functional I is reduced to the function $G(x_2, y_2, z_2)$ of the coordinates x_2, y_2, z_2 of the variable point $P_2(x_2, y_2, z_2)$ and the variation of the functional is reduced to the differential of this function*.

The variation of $I, \delta I$ can now be easily computed by proceeding exactly as in Art. 2.1 We thus obtain

$$\Delta I = \int_{x_1}^{x_2+\delta x_2} F(x, y+\delta y, z+\delta z, y'+\delta y', z'+\delta z')dx - \int_{x_1}^{x_2} F(x, y, z, y', z')dx$$

or

$$\Delta I = \int_{x_1}^{x_2} \{F(x, y+\delta y, z+\delta z, y'+\delta y', z'+\delta z') - F(x, y, z, y', z')\}dx$$

$$+ \int_{x_2}^{x_2+\delta x_2} F(x, y+\delta y, z+\delta z, y'+\delta y', z'+\delta z')\,dx \qquad ... (3)$$

As in Art. 2.1, in the first integral on R.H.S. of (3), isolate the principal part by using Taylor's expansion whereas in the second integral on R.H.S. of (3) use mean value theorem and take advantage of he continuity of the function $F(x, y, z, y', z')$. After using these transformations, (3) reduces to

---

* The function G will be single-valued if the extremals of the pencil centred at $P_1(x_1, y_1, z_1)$ do not intersect, because then the point $P_2(x_2, y_2, z_2)$ uniquely defines an extremal.

$$\delta I = \left[F\right]_{x=x_2} \delta x_2 + \int_{x_1}^{x_2} [F_y\ \delta y + F_z\ \delta z + F_{y'}\ \delta y' + F_{z'}\ \delta z']\, dx \qquad ...(4)$$

Now,integrating by parts the last two terms under the integral sign in (4), we get

$$\delta I = \left[F\right]_{x=x_2} \delta x_2 + \left[F_{y'}\ \delta y\right]_{x_1}^{x_2} + \left[F_{z'}\ \delta z\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\left(F_y - \frac{d}{dx} F_{y'}\right)\delta y + \left(F_z - \frac{d}{dx} F_{z'}\right)\delta z\right] dx \qquad ...(4)$$

Since the values of the functional I are taken only on the extremals, so we have

$$F_y - \frac{d}{dx} F_{y'} \equiv 0 \qquad \text{and} \qquad F_z - \frac{d}{dx} F_{z'} \equiv 0 \qquad ...(5)$$

Again, since the boundary point (x_1, y_1, z_1) is fixed, so we have

$$\left[\delta y\right]_{x=x_1} = 0 \qquad \text{and} \qquad \left[\delta z\right]_{x=x_1} = 0 \qquad ...(6)$$

Using (5) and (6), (4) reduces $\delta I = \left[F\right]_{x=x_2} \delta x_2 + \left[F_{y'}\ \delta y\right]_{x=x_2} + \left[F_{z'}\ \delta z\right]_{x=x_2}$ ...(7)

Discussing in the same manner as in Art. 2.1 we get

$$\left[\delta y\right]_{x=x_1} \approx \delta y_2 - y'(x_2)\ \delta x_2 \qquad \text{and} \qquad \left[\delta z\right]_{x=x_1} \approx \delta z_2 - z'(x_2)\ \delta x_2$$

Then, (7) reduces to $\delta I = \left[F - y'F_{y'} - z'F_{z'}\right]_{x=x_2} \delta x_2 + \left[F_{y'}\right]_{x=x_2} \delta y_2 + [F_{y'}]_{x=x_2} \delta z_2$

The basic necessary condition for an extremum $\delta I = 0$ takes the form

$$[F - y'F_{y'} - z'F_{z'}]_{x=x_2} \delta x_2 + [F_{y'}]_{x=x_2} \delta y_2 + [F_{z'}]_{x=x_2} \delta z_2 = 0 \qquad ...(8)$$

The following three case arise :

Case I. When the variations δx_2, δy_2 and δz_2 are independent.

Then from (8), the necessary conditions for the extremum of (1) are

$$[F - y'F_{y'} - z'F_{z'}]_{x=x_2}, \qquad [F_{y'}]_{x=x_2} = 0 \qquad \text{and} \qquad [F_{z'}]_{x=x_2} = 0 \qquad ...(9)$$

Case II. When the boundary point $P_2(x_2, y_2, z_2)$ can move along some given curve specified by the equations $y = \phi(x)$ and $z = \psi(x)$ so that $y_2 = \phi(x_2)$, $z_2 = \psi(x_2)$

Then $\delta y_2 = \phi'(x_2)\, \delta x_2$ and $\delta z_2 = \psi'(x_2)\, \delta x_2$ and so (8) reduces to

$$[F + (\phi' - y')F_{y'} + (\psi' - z')F_{z'}]_{x=x_2} \delta x_2 = 0 \qquad ...(10)$$

Since δx_2 varies arbitrarily, (10) leads to $[F + (\phi' - y')F_{y'} + (\psi' - z')F_{z'}]_{x=x_2} = 0$ ...(11)

Then above condition (11) is known as the *transversality condition* in the problem of investigating the functional (1) for an extremum. Along with the equations $y_2 = \psi(x_2)$ and $z_2 = \psi(x_2)$, the condition (11) gives the equations necessary for getting the two arbitrary constants in the general solution of Euler's equations (5).

Case III. When the boundary point $P_2(x_2, y_2, z_2)$ can move over some given surface $z = \phi(x, y)$ so that $z_2 = \phi(x_2, y_2)$

Then $\delta z_2 = \phi_{x_2} \delta x_2 + \phi_{y_2}\ \delta y_2$ and the variations δx_2 and δy_2 are arbitrary. Therefore, (8) reduces to $[F - y'F_{y'} + (\phi_x - z')F_{z'}]_{x=x_2} \delta x_2 + [F_{y'} + \phi_y F_{z'}]_{x=x_2} \delta y_2 = 0$ ...(12)

Since δx_2 and δy_2 are independent, (12) yields

$$[F - y'F_{y'} + (\phi_x - z')F_{z'}]_{x=x_2} = 0 \qquad \text{and} \qquad [F_{y'} + \phi_y F_{z'}]_{x=x_2} = 0 \qquad ...(13)$$

There two conditions, together with the equation $z_2 = \phi(x_2, y_2)$ will enable us to get two arbitrary constants in the general solution of the system of Euler's equation (5).

Remark 1. If the boundary point $P_1(x_1, y_1, z_1)$ is moving , then proceeding as above, we arrive at similar conditions in all the cases I, II and III.

Remark 2. The above arguments can be easily extended to a functional of the form

$$I[y_1(x), y_2(x), .., y_n(x)] = \int_{x_1}^{x_2} F\left((x, y_1(x), y_2(x), .., y_n(x), y_1'(x), y_2'(x), .., y_n'(x)\right) dx$$

Without changing the method of proof, we find that in the case the moving point $P_2(x_2, y_{12}, y_{22}, .., y_{n2})$ the condition is

$$\left[F - \sum_{i=1}^{n} y_i' \; F_{y_i'}\right]_{x=x_2} \delta x_2 + \sum_{i=1}^{n} \left[F_{y_i'}\right]_{x=x_2} \delta y_{i2} = 0$$

Remark 3. *We now find nature of the transversality condition for the functional*

$$I = \int_{x_1}^{x_2} A(x, y, z)\,(1 + y'^2 + z'^2)^{1/2}\, dx\,, \qquad \text{if} \qquad z_2 = \phi\left(x_2, y_2\right).$$

Comparing the given functional with $I = \int_{x_1}^{x_2} F(x, y, z, y', z')dx$, we have

$$F(x, y, z, y', z') = A(x, y, z)\,(1 + y'^2 + z'^2) \qquad ...(1)$$

The transversality conditions [refer equation (13) of case III of Art. 2.3] are given by

$$\left[F_y - y'F_{y'} + (\phi_x - z')F_{z'}\right]_{x=x_2} = 0 \qquad \text{and} \qquad \left[F_{y'} + \phi_y F_{z'}\right]_{x=x_2} = 0 \qquad ...(2)$$

Using (1), (2) reduces to $1 + \phi_x z' = 0$ and $y' + \phi_y z' = 0$ for $x = x_2$

so that $1/\phi_x = y'/\phi_y = z'/(-1)$ for $x = x_2$ ...(3)

Relations (3) are the conditions for parallelism of the vector of the tangent $\boldsymbol{t}(1, y', z')$ to the desired extremal at the point (x_2, y_2, z_2) and the vector of the normal $\boldsymbol{n}(\phi_x, \phi_y, -1)$ to the surface $z = \phi(x, y)$ at the same point. It follows that, for functional (1), the transversality condition reduces to the orthogonality condition of the extremal to the surface $z = \phi(x, y)$.

2.3A. Working rule for solving variational problems with a moving boundary for a functional dependent on two functions (for details and proof, refer Art. 2.3)

Let $$I[y(z), z(x)] = \int_{x_1}^{x_2} F(x, y, z, y', z')dx \qquad ...(1)$$

be a functional defined on the set of functions $y = y(x)$ and $z = z(x)$ which admit of continuous first order derivatives. Let the point $P_1(x_1, y_1, z_1)$ corresponding to the lower limit in the integral (1) be fixed while the other boundary point $P_2(x_2, y_2, z_2)$ be variable point moving along a certain curve specified by the equations $y = \phi(x)$, $z = \psi(x)$ so that $y_2 = \phi(x_2)$ and $z_2 = \psi(x_2)$. Also assume that $F(x, y, z, y', z')$ be differentiable three times with respect to all its arguments.

If $I[y(x), z(x)]$ has an extremum on a curve $y = y(x), z = (x),$ then $y(x)$ and $z(x)$ satisfy the following system of Euler's equations:

$$F_y - \frac{d}{dx}F_{y'} = 0 \qquad \text{and} \qquad F_z - \frac{d}{dx}F_{z'} = 0 \qquad ...(2)$$

Also, the following transversality condition must be satisfied:

$$[F + (\phi' - y')F_{y'} + (\psi' - z')F_{z'}]_{x=x_2} = 0 \qquad ...(3)$$

As an illustration of the above working rule, refer Ex.1 of the next 2.4.

Another form of the above variational problem

In place of moving point P_2 along the curve given by $y = \phi(x)$, $z = \psi(x)$ as discussed above, now suppose we are given that the moving poing P_2 moves along the surface $z = \phi(x, y)$. Then, if functional $I[y(x), z(x)]$ has an extremum on a curve $y = y(x)$, $z = z(x)$, then $y(x)$ and $z(x)$ satisty the following system of Euler's equations:

$$F_y - \frac{d}{dx}F_{y'} = 0 \qquad \text{and} \qquad F_z - \frac{d}{dx}F_{z'} = 0 \qquad ...(2)$$

Now, the following two modified transversality conditions must be satisfied:

$$[F - y'F_{y'} + (\phi_x - z')F_{z'}]_{x=x_2} = 0 \qquad \text{and} \qquad [F_{y'} + \phi_y F_{z'}]_{x=x_2} = 0 \qquad ...(3)$$

where $\phi_x = \partial\phi / dx$ and $\phi_y = \partial\phi / dy$

As an illustration of the above another form, refer Ex.2 of the next Art. 2.4.

2.4 Solved examples based on Art 2.3 and 2.3A

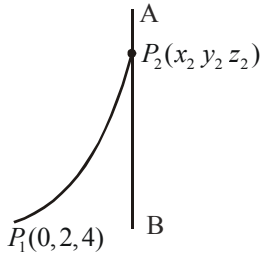
Ex.1 *Find the shortest distance of the point (0, 2, 4) to the stroight line (x–1)/1 = y/3 = z/4.*

[KU kurukshetra 2005]

Sol. We are to find the shortest distance between $P_1(0, 2, 4)$ and $P_2(x_2, y_2 z_2)$ where P_2 moves on the given line, AB, namely,

$$(x–1)/1 = y/3 = z/4 \qquad ...(1)$$

i.e., $y = 3x - 3 = \phi(x)$, $z = 4x - 4 = y(x)$, say ...(2)



Let $y = y(x)$, $z = z(x)$ be any curve passing through P_1 and P_2, then, the arc P_1P_2 is given by

$$I[y(x), z(x)] = \int_0^{x_2} (1 + y'^2 + z'^2)^{1/2}\, dx \qquad ...(3)$$

Comparing the functional (3) with $\int_0^{x_2} F(x, y, z, y', z')dx$,

here $$F(x, y, z, y', z') = (1 + y'^2 + z'^2)^{1/2} \qquad ...(4)$$

Let the length of the curve between P_1 and P_2 be minimum on the curve given by $y = y(x)$, $z = z(x)$. Then, the functional (3) has a minimum value and hence $y(x)$ and $z(x)$ must satisfy the following system of Euler's equations:

$$F_y - \frac{d}{dx}F_{y'} = 0 \qquad \text{and} \qquad F_z - \frac{d}{dx}F_{z'} = 0 \qquad ...(5)$$

Using (4), equations in (5) become

$$0 - \frac{d}{dx}\left(\frac{y'}{(1 + y'^2 + z'^2)^{1/2}}\right) = 0 \qquad \text{so that} \qquad \frac{y'}{(1 + y'^2 + z'^2)^{1/2}} = C_1 \qquad ...(6)$$

and $$0 - \frac{d}{dx}\left(\frac{z'}{(1 + y'^2 + z'^2)^{1/2}}\right) = 0 \qquad \text{so that} \qquad \frac{z'}{(1 + y'^2 + z'^2)^{1/2}} = C_2 \qquad ...(7)$$

Dividing (6) by (7), we have $y' / z' = C_1 / C_2 = C_3$, say

Thus, we have $y' = C_3 z'$ ...(8)

Now, (7) and (8) yield $z' / \{1 + (1 + c_3^{\,2})z'^2\}^{1/2} = c_2$

or $z'^2 = C_2^2\{1 + (1 + C_3^2)z'^2\}$ so that $z' = C_2 / (1 - C_2^2 - C_2^2C_3^2)^{1/2} = C_4$, say

Thus, $dz/dx = C_4$ so that $z = C_4 x + C_5$ ...(9)

Since $z' = C_4$, (8) $\Rightarrow$ $y' = C_3C_4 = C_6$, say

or $dy/dx = C_6$ so that $y = C_6x + C_7$ ...(10)

Thus, extremals are given by (9) and (10). Since the required extremal has to pass through $P_1(0, 2, 4)$ and $P_2(x_2, y_2, z_3)$, the boundary conditions are given by

$y(0) = 2$, $z(0) = 4$, $y(x_2) = 3x_2 - 3$, $z(x_2) = 4x_2 - 4$ ...(11)

Now, (9) and $z(0) = 4 \Rightarrow C_5 = 4$; Also, (10) and $y(0) = 2 \Rightarrow C_7 = 2$ ...(12)

Thus, $C_4 = (4x_2 - 8)/x_2$ ...(13)

Again (10) and $y(x_2) = 3x_2 - 3 \Rightarrow C_6 x_2 + C_7 = 3x_2 - 3 \Rightarrow C_6 x_2 + 2 = 3x_2 - 3$ by (12)

Thus, $C_6 = (3x_2 - 5)/x_2$ ...(14)

Here the following transversality condition must be satisfied

$$[F + (\phi' - y')F_{y'} + (\psi' - z')F_{z'}]_{x=x_2} = 0 \quad ...(15)$$

Using (2) and (4), (15) reduces to

$$[(1+y'^2+z'^2)^{1/2} + (3-y')y'(1+y'^2+z'^2)^{-1/2} + (4-z')z'(1+y'^2+z'^2)^{-1/2}]_{x=x_2} = 0$$

or $$[1+y'^2+z'^2+(3-y')y'+(4-z')z']_{x=x_2} = 0$$

or $[1+3y'+4z']_{x=x_2} = 0$ or $1+3C_6+4C_4 = 0$, using (9) and (10)

or $1+(3/x_2)\times(3x_2-5)+(4/x_2)\times(4x_2-8) = 0$, using (13) and (14)

This yields $x_2 = 47/26$ ...(16)

Substituting the above value of x_2 in (13) and (14), we get

$C_4 = 4 - 8 \times (26/47) = -(20/47)$ and $C_6 = 3 - 5 \times (26/47) = 11/47$

Putting $C_5 = 4$, $C_7 = 2$, $C_4 = -(20/47)$ and $C_6 = 11/47$ in (9) and (10), we have

$y = 2 + (11x/47)$, $z = 4 - (20x/47)$, ...(17)

which gives the straight line along which shortest distance will occur.

(17) can be re-written as $(x - 0)/47 = (y - 2)/11 = (z - 4)/(-20)$.

Therefore, the required shortest distance l along the extremal (17) between the given point $P_1(0, 2, 4)$ and the given straight line (1) is given by

$$l = \int_0^{x_2} (1+y'^2+z'^2)^{1/2}\,dx = \int_0^{47/26} (1+C_6^2+C_4^2)^{1/2}\,dx, \text{ using (9) and (10)}$$

$$= \int_0^{47/26} \{1+(11/47)^2+(-20/47)^2\}^{1/2}\,dx = (105/26)^{1/2} \text{ units}$$

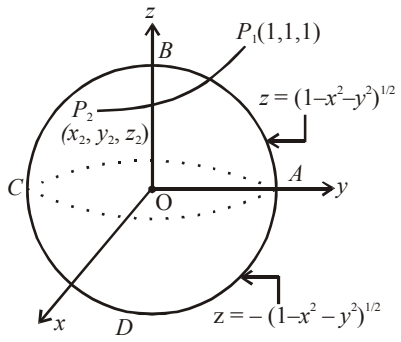
Ex. 2. *Find the shortest distance from the point $P_1(1, 1, 1)$ to the sphere $x^2 + y^2 + z^2 = 1$.*

[KU Kurukshetra 2004]

Sol. Given sphere is $x^2 + y^2 + z^2 = 1$ so that $z = \pm (1 - x^2 - y^2)^{1/2}$. Cearly, $z = (1 - x^2 - y^2)^{1/2}$ will represent the upper half sphere ABC. We are to find the shortest distance between $P_1(1,1,1)$ and $P_2(x_2, y_2, z_2)$ where P_2 is a variable point lying on the upper half sphere ABC, given by

$$z = (1-x^2-y^2)^{1\backslash 2} = \phi(x, y), \text{ say} \quad ...(1)$$

Let $y = y(x)$, $z = z(x)$ be any curve passing through P_1 and P_2. Then, the arc length P_1P_2 is given by



$$I[y(x), z(x)] = \int_1^{x_2} (1 + y'^2 + z'^2)^{1/2} dx \qquad ...(2)$$

Comparing the functional (2) with $\int_1^{x_2} F(x, y, z, y', z')dx$, we have

$$F(x, y, z, y', z') = (1 + y'^2 + z'^2)^{1/2} \qquad ...(3)$$

Let the length of the curve between P_1 and P_2 be minimum on the curve given by $y = y(x)$, $z = z(x)$. Then, the functional (2) has a minimum value and hence $y(x)$ and $z(x)$ must satists the following system of Euler's equations.

$$F_y - \frac{d}{dx} F_{y'} = 0 \qquad ...(4)$$

and

$$F_z - \frac{d}{dx} F_{z'} = 0 \qquad ...(5)$$

Using (3), (4) and (5) reduce to

$$0 - \frac{d}{dx}\left(\frac{y'}{(1 + y'^2 + z'^2)^{1/2}} \right) = 0 \quad \text{so that} \quad \frac{y'}{(1 + y'^2 + z'^2)^{1/2}} = C_1 \qquad ...(6)$$

and

$$0 - \frac{d}{dx}\left(\frac{z'}{(1 + y'^2 + z'^2)^{1/2}} \right) = 0 \quad \text{so that} \quad \frac{z'}{(1 + y'^2 + z'^2)^{1/2}} = C_2 \qquad ...(7)$$

Dividing (6) by (7), we get $y'/z' = C_1/C_2 = C_3$, say

Thus, we have $y' = C_3 z'$ ...(8)

Now, (7) and (8) $\Rightarrow$ $z'/\{1 + (1 + C_3^2)z'^2\}^{1/2} = C_2$

or $z'^2 = C_2^2\{1 + (1 + C_3^2)z'^2\}$ so that $z' = C_2/(1 - C_2^{\,2} - C_2^2 C_3^2)^{1/2} = C_4$, say

or $dz/dx = C_4$ so that $z = C_4 x + C_5$ ...(9)

Since $z' = C_4$, (8) $\Rightarrow$ $y' = C_3 C_4 = C_6$, say

or $dy/dx = C_6$ so that $y = C_6 x + C_7$ ...(10)

Thus, the required extremals are given by (9) and (10). Since the required extemal has to pass through $P_1(1, 1, 1)$ and $P_2(x_2, y_2, z_2)$, the boundary conditions of the given problem are:

$y(1) = 1$, $z(1) = 1$, $y(x_2) = y_2$ and $z(x_2) = z_2$, where $z_2 = (1 - x_2^2 - y_2^2)^{1/2}$ ...(11)

Now, (9) and $z(1) = 1$ $\Rightarrow$ $C_4 + C_5 = 1$ ...(12)

(10) and $y(1) = 1$ $\Rightarrow$ $C_6 + C_7 = 1$ ...(13)

(9) and $z(x_2) = z_2$ $\Rightarrow$ $C_4 x_2 + C_5 = z_2$ ...(14)

(10) and $y(x_2) = y_2$ $\Rightarrow$ $C_6 x_2 + C_7 = y_2$ ...(15)

For the present problem transversality conditions are

$$[F - y'F_{y'} + (\phi_x - z')F_{z'}]_{x=x_2} = 0 \qquad ...(16)$$

and

$$[F_{y'} + \phi_y F_{z'}]_{x=x_2} = 0 \qquad ...(17)$$

Now, (1) $\Rightarrow$ $\phi_x = \partial\phi/\partial x = -x(1 - x^2 - y^2)^{-1/2}$, $\phi_y = \partial\phi/\partial y = -y(1 - x'^2 - y^2)^{1/2}$ ...(18)

Using (3) and (18), (16) reduces to

$$\left[(1 + y'^2 + z'^2)^{1/2} - \frac{y'^2}{(1 + y'^2 + z'^2)^{1/2}} + \left(-\frac{x}{(1 - x^2 - y^2)^{1/2}} - z' \right) \times \frac{z'}{(1 + y'^2 + z'^2)^{1/2}} \right]_{x=x_2} = 0$$

or $\quad 1+y'^2+z'^2-y'^2+\{-x_2(1-x_2^2-y_2^2)^{-1/2}-z'\}z'=0, \qquad$ as $x=x_2 \Rightarrow y=y_2$

or $\quad 1+z'^2-x_2z_2^{-1}z'-z'^2=0,$ since $z_2=(1-x_2^2-y_2^2)^{1/2}$

or $\quad 1-\dfrac{x_2C_4}{z_2}=0, \qquad$ as $\qquad z'=C_4$ by (9)

Thus, we have $\qquad z_2=C_4\,x_2 \qquad$ ...(19)

Using (3) and (18), (17) reduces to

$$\left[\frac{y'}{(1+y'^2+z'^2)^{1/2}}+\left(-\frac{y}{(1-x^2-y^2)^{1/2}}\right)\times\frac{z'}{(1+y'^2+z'^2)^{1/2}}\right]_{x=x_2}=0$$

or $\quad y'-y_2z'(1-x_2^2-y_2^2)^{1/2}=0, \qquad$ as $x=x_2 \Rightarrow y=y_2$

or $\quad C_6-y_2\,C_4\,z_2^{-1}=0$, as $y'=C_6$ by (10) and $z'=C_4$ by (9)

Thus, we have $\qquad C_6\,z_2=C_4\,y_2 \qquad$ ...(20)

From (14) and (19), $\qquad z_2+C_5=z_2 \qquad$ so that $\qquad C_5=0 \qquad$ ...(21)

Since $C_5=0$, $\qquad (12) \Rightarrow C_4=1 \qquad$ ...(22)

Using (22), $\qquad (20) \Rightarrow y_2=C_6\,z_2 \qquad$ ...(23)

(19) and (23) $\Rightarrow \quad y_2=C_6\,C_4\,x_2 \Rightarrow y_2=C_6\,x_2$, as $C_4=1 \qquad$ ...(24)

(15) and (24) $\Rightarrow \quad y_2+C_7=y_2 \Rightarrow C_7=0 \qquad$ ...(25)

Using (25), $\qquad (13) \Rightarrow C_6=1 \qquad$ ...(26)

Since $C_4=1$, $\quad (19) \Rightarrow z_2=x_2.$ $\qquad$ Also, since $C_6=1$, $\quad (23) \Rightarrow y_2=z_2$

Thus, $y_2=z_2=x_2$. Hence (11) reduces to

$x_2=(1-x_2^2-x_2^2)^{1/2} \qquad$ so that $\qquad x_2^2=1-2x_2^2 \qquad$ or $\qquad x_2=1/\sqrt{3}$

Thus, $x_2=y_2=z_2=1/\sqrt{3}$ and hence the coordinates or P_2 are $(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$.

Since $C_4=1$, $C_5=0$, $C_6=1$, $C_7=0$, (9) and (10) reduce to $y=x$, $z=x$, which is the straight line along which the required shortest distance will occur.

Therefore, the required shortest distance l along the extremal $y=x$, $z=x$ between $P_1(1,1,1)$ and the upper half sphere is given by

$$l=\int_0^{x_2}(1+y'^2+z'^2)^{1/2}\,dx=\int_0^{1/\sqrt{3}}(1+C_6^2+C_4^2)^{1/2}\,dx\,,\ \text{using (9) and (10)}$$

$$=\int_0^{1/\sqrt{3}}(1+1+1)^{1/2}\,dx\ =\sqrt{3}(1/\sqrt{3}-1)=(\sqrt{3}-1)\ \text{units, as } C_6=C_4=1$$

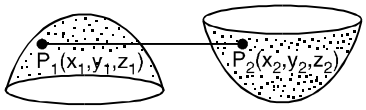
Ex. 3. *Find the shortest distance between two given surfaces* $z=\phi(x,y)$ *and* $z=\psi(x,y)$.

Sol. Here we are to compute minimum value of the functional

$$I\left[y(x),z(x)\right]=\int_{x_1}^{x_2}(1+y'^2+z'^2)dx \qquad ...(1)$$

Since the integrand of (1) is independent of x, y, and z the extremals are straight lines in space (see remark 1 of Art.1.8). Further, since (1) is particular case of the functional

$$\int_{x_1}^{x_2}A(x,y,z)\ (1+y'^2+z'^2)^{1/2}\,dx\,,$$



hence as discussed in remark 3 of Art. 2.3, the transversality conditions both at the point $P_1(x_1,y_1,z_1)$ and at the point $P_2(x_2,y_2,z_2)$ reduces to the orthogonality conditions of the

extremal to both the surfaces $z = \phi(x, y)$ and $z = \psi(x, y)$. Hence the shortest distance can be achieved only along the straight lines that are orthogonal both to the surface $z = \phi(x, y)$ at the point $P_1 = (x_1, y_1, z_1)$ and to the surface $z = \psi(x, y)$ at the point $P_2 = (x_2, y_2, z_2)$.

Ex. 4. *Test for an extermum the functional* $I = \int_{x_1}^{x_2} (y'^2 + z'^2 + 2yz)dx$ *with* $y(0) = 0,\ z(0) = 0$ *and the point* (x_2, y_2, z_2) *moves over the fixed plane* $x = x_2$.

Sol. Comparing the given functional with $\int_{x_1}^{x_2} F(x, y, z, y', z')dx$, we have

$$F(x, y, z, y', z') = y'^2 + z'^2 + 2yz \qquad ...(1)$$

We know that the required extermum can be attained only on the integral curves of Euler's equations $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ and $\dfrac{\partial F}{\partial z} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial z'}\right) = 0$

i.e. $2z - d(2y')/dx = 0$ and $2y - d(2z')/dx = 0$, using (1)

i.e. $$d^2y/dx^2 - z = 0 \qquad ...(2)$$

and $$d^2z/dx^2 - y = 0 \qquad ...(3)$$

Substituting the value of z given by (2) in (3), we get

$$d^4y/dx^4 - y = 0 \quad \text{or} \quad (D^4 - 1)y = 0, \quad \text{where} \quad D \equiv d/dx \qquad ...(4)$$

Its auxiliary equation is $D^4 - 1 = 0$ so that $D = \pm 1, \pm i$

Hence, the general solution of (4) is given by

$$y = C_1 \cosh x + C_2 \sinh x + C_3 \cos x + C_4 \sin x, \qquad ...(5)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

From (5), $dy/dx = C_1 \sinh x + C_2 \cosh x - C_3 \sin x + C_4 \cos x$

and $$d^2y/dx^2 = C_1 \cosh x + C_2 \sinh x - C_3 \cos x - C_4 \sin x \qquad ...(6)$$

From (2) and (6), $$z = C_1 \cosh x + C_2 \sinh x - C_3 \cos x - C_4 \sin x \qquad ...(7)$$

Using the given boundary condition $y(0) = 0$, (5) $\Rightarrow$ $C_1 + C_3 = 0$ ...(8)

Using the given boundary condition $z(0) = 0$, (7) $\Rightarrow$ $C_1 - C_3 = 0$ ...(9)

From (8) and (9), we get $C_1 = C_3 = 0$. Hence (5) and (7) yield

$$y = C_2 \sinh x + C_4 \sin x \qquad ...(10)$$

and $$z = C_2 \sinh x - C_4 \sin x \qquad ...(11)$$

The necessary condition for the extermum of I is given by (refer condition (8) of Art.2.3)

$$[F - y'F_{y'} - z'F_{z'}]_{x=x_2}\ \delta x_2 + [F_{y'}]_{x=x_2}\ \delta y_2 + [F_{z'}]_{x=x_2}\ \delta z_2 = 0.$$

Since the moving boundary point (x_2, y_2, z_2) always moves on the fixed plane $x = x_2$, where x_2 is fixed, hence $\delta x_2 = 0$. Therefore the above equation reduces to

$$[F_{y'}]_{x=x_2}\ \delta y_2 + [F_{z'}]_{x=x_2}\ \delta z_2 = 0 \qquad ...(12)$$

Since δy_2 and δz_2 are arbitrary, (12) imply that $[F_{y'}]_{x=x_2} = 0$ and $[F_{z'}]_{x=x_2} = 0$

$\Rightarrow$ $[2y']_{x=x_2} = 0$ and $[2z']_{x=x_2} = 0$ $\Rightarrow$ $y'(x_2) = 0$ and $z'(x_2) = 0$ ...(13)

From (10) and (11), $y'(x) = C_2 \cosh x + C_4 \cos x$, and $z'(x) = C_2 \cosh x - C_4 \cos x$

Replacing x by x_2 in the above relations and using (13), we get

$$C_2 \cosh x_2 + C_4 \cos x_2 = 0 \quad \text{and} \quad C_2 \cosh x_2 - C_4 \cos x_2 = 0 \qquad ...(14)$$

Two cases arise:

Case (*i*). If $\cos x_2 \neq 0$ then (14) $\Rightarrow C_2 = C_4 = 0$. Hence from (10) and (11), it follows that the required extremum may be attained only on the straight line $y = 0, z = 0$.

Case (*ii*). If $\cos x_2 = 0$, *i.e.*, $x_2 = n\pi + \pi/2$, where n is a integer, then (14) $\Rightarrow C_2 = 0$ and C_4 is an arbitrary constant. So, from (10) and (11), it follows that the required extermum is given by $y = C_4 \sin x$ and $z = -C_4 \sin x$. In this case, it can be easily verified that the functional $I = 0$ for any arbitrary value of C_4.

EXERCIS 2(B)

Ex. 1. *Find the extremal of* $I[y(x), z(x)] = \int_{x_1}^{x_2} (y'z' + 2y^2 + 2z^2)\, dx$ with $y(0) = 0$, $z(0) = 0$ *and the point* (x_2, y_2, z_2) *moves over the fixed plane* $x = x_2$.

Sol. Proceed as in Ex. 4 of Art. 2.4.

Ans. If $\cos 2x_2 \neq 0$, then extermum is attained only on the line $y = 0$, $z = 0$. Next, if $\cos 2x_2 = 0$ *i.e.* $x_2 = n\pi/2 + \pi/4$, n being and integer, the extremal is given by $y = C_4 \sin 2x$ and $z = -C_4 \sin 2x$, C_4 being an arbitrary constant.

Ex. 2(*a*) *Find the shortest distance from the point* (*0, 2, 0*) *to the straight line* $x = y = z$.

(KU Kurukshetra 2004)

(*b*) *Find the shortest distance from the point* $P_1(0, 2, 1)$ *to the straight line* $x/1 = y/2 = z/3$.

(KU Kurukshetra 2005)

Hint. Proceed as in Ex. 1. **Ans.** (*i*) $(2\sqrt{6})/3$ (*ii*) $\sqrt{6}/2$ units

2.5 One sided variations

While discussing some variational problem involving an extermum of a functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx \qquad ...(1)$$

a restriction is imposed on the class of permissible curves in such a manner that the curves cannot pass through points of a certain region R bounded by the curve $\psi(x, y) = 0$ as shown in figure (*i*). In such problems one of the following two situations may arise :

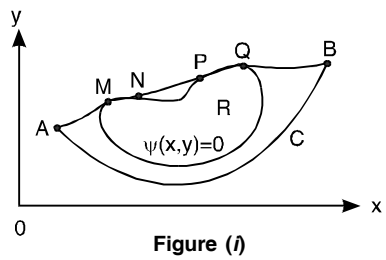


Figure (*i*)

Situation (*i*). When the extremizing curve *C* passes completely outside the boundaries of the region *R*. Then it must be an extremal, since in this case the presence of the prohibited region R does not at all affect the properties of the functional and its variations in the neighbourhood of the curve C, and the arguments of chapter1 hold true.

Situation (*ii*). When the extremizing curve *C* consists of arcs lying outside the boundary of *R* and also consists of parts of the boundary of the region *R*. Then a new situation arises; only one-sided variations of the curve C are possible on parts of the boundary of the region R, since

permissible curves are prohibited from entering the region *R*. Parts of the curve *C* that lie outside the boundary of *R* must, as before, be extremals, since if we vary the curve *C* only on such a segment that permits two-sided variations, the presence of the region *R* will not affect the variations of *y*, and the conclusions of chapter 1 continue to hold true.

From the above discussion, it follows that an extremum can be attained only on curves consisting of arcs of extremals and parts of the boundary of the region *R*. Therefore,in order to construct the required extremizing curve we must get conditions, at the points of transition of the extremal to the boundary of the region *R*, which permit determining these points. In the case shown in figure (*ii*), it is necessary to derive conditions at the points of transition *M*, *N*, *P* and *Q*. In what follows, we shall derive condition at the point *M*. In exactly analogous manner, the necessary conditions at other points *N*, *P* and *Q* can be derived.

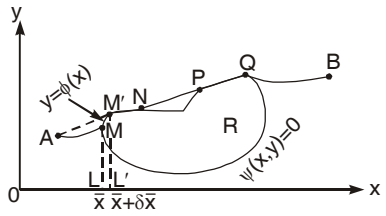


Figure (ii)

While calculating the variation δI of the functional

$$I = \int_{x_1}^{x_2} F(x, y, y')dx = \int_{x_1}^{x'} F(x, y, y')dx + \int_{x'}^{x_2} F(x, y, y')dx = I_1 + I_2, \text{ say} \qquad ...(2)$$

we assume that the variation is caused solely by the displacement of the point $M(\bar{x}, \bar{y})$ on the curve $\psi(x, y) = 0$, that is, it may be assumed that for any position of the point *M* on the curve $\psi(x, y) = 0$, the arc AM is already an extremal, and the segment *MNPQB* does not vary. The functional I_1 has a boundary point moving along the boundary of the region R, whose equation is $\psi(x, y) = 0$;or in the form solved for y in the neighbourhood of the point: $y = \phi(x)$. Hence proceeding as in case II of Art.2.1, we have

$$\delta I_1 = [F + (\phi' - y')F_{y'}]_{x=\bar{x}}\ \delta\bar{x} \qquad ...(3)$$

The functional I_2 also has a moving boundary point $(\bar{x}, \bar{y})$. But in the neighbourhood of this point, the curve $y = \phi(x)$ on which an extremum can be attained does not vary. It follows that the variation of the functional I_2 in the translation of the point $M(\bar{x}, \bar{y})$ to the position $M'(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y})$ only reduces to a change in the lower limit of integration and

$$\Delta I_2 = \int_{\bar{x}+\delta\bar{x}}^{x_2} F(x, y, y')\,dx - \int_{\bar{x}}^{x_2} F(x, y, y')\,dx = -\int_{\bar{x}}^{\bar{x}+\delta\bar{x}} F\left(x, y, y'\right)dx = -\int_{\bar{x}}^{\bar{x}+\delta\bar{x}} F(x, \phi(x), \phi'(x))\,dx$$

$[\because\ y = \phi(x)$ on the interval $(\bar{x}, \bar{x} + \delta\bar{x})\ ]$

Using the mean value theorem and continuity of *F*, we obtain

$$\Delta I_2 = -\left[F(x, \phi(x), \phi'(x))\right]_{x=\bar{x}}\ \delta\bar{x} + \varepsilon\delta\bar{x}\text{, where } \varepsilon \to 0 \text{ as } \delta\bar{x} \to 0.$$

Hence $$\delta I_2 = -\left[F(x, \phi(x), \phi'(x))\right]_{x=\bar{x}}\ \delta\bar{x} \qquad ...(4)$$

From (2), (3) and (4), we obtain $\delta I = \delta I_1 + \delta I_2$.

or $$\delta I = \left[F(x, y, y') + (\phi' - y')F_{y'}(x, y, y')\right]_{x=\bar{x}}\ \delta\bar{x} - \left[F(x, y, \phi')\right]_{x=\bar{x}}\ \delta\bar{x}$$

or $$\delta I = \left[F(x, y, y') - F(x, y, \phi') - (y' - \phi')F_{y'}(x, y, y')\right]_{x=\bar{x}}\ \delta\bar{x} \qquad [\because\ y(\bar{x}) = \phi(\bar{x})]$$

Since $\delta\bar{x}$ is arbitrary, the necessary condition for an extremum $\delta I = 0$, reduces to

$$\left[F(x, y, y') - F(x, y, \phi') - (y' - \phi')F_{y'}(x, y, y')\right]_{x=\bar{x}} = 0 \qquad ...(5)$$

Applying the mean value theorem, (5) yields

$$(y'-\phi')\left[F_{y'}(x,y,q)-F_{y'}(x,y,y')\right]_{x=\bar{x}}=0, \qquad ...(6)$$

where q lies between $y'(\bar{x})$ and $\phi'(\bar{x})$. Again applying the mean value theorem, (6) yields

$$[(y'-\phi')\,(q-y')F_{y'y'}(x,y,q')]_{x=\bar{x}}=0, \qquad ...(7)$$

where q' lies between q and $y'(\bar{x})$.

In what follows, we further assume that $F_{y'y'}(x,y,q')\neq 0$,

which is a valid assumption for many variational problems. In this case the required condition at M is of the form $y'(\bar{x})=\phi'(\bar{x})$

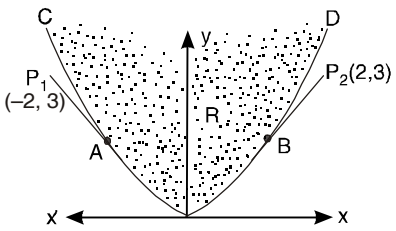
because $q=y'$ only when $y'(\bar{x})=\phi'(\bar{x})$, since q is a value lying between $y'(\bar{x})$ and $\phi'(\bar{x})$.

It follows that at the point M the extremal AM and the boundary curve MN have a common tangent [the left tangent for the curve $y=y(x)$ and the right tangent for the curve $y=\phi(x)$] *Thus, the extremal is tangent to the boundary of the region R at the point M.*

2.6. ILLUSTRATIVE SOLVED EXAMPLES BASED ON ART. 2.5

Ex. 1. *Find the shortest path from the point* $P_1(-2,3)$ *to the point* $P_2(2,3)$ *located in the region* $y\le x^2$. **(Ravishekhar 2005)**

Sol. Let the parabola $x^2=y$ be denoted by the curve CAOBD. Then region $y\le x^2$ is the region R (shown by dots), which is the interior of the parabola $x^2=y$. Here we arc to determine the shortest path joining $P_1(-2,3)$ and $P_2(2,3)$ in such a way that the curve cannot pass through points of the region R.



The arc length of the required minimizing curve $y=y(x)$ is given by $I[y(x)]=\int_{-2}^{2}(1+y'^2)^{1/2}\,dx,$ ...(1)

subject to the conditions $y\le x^2,$ $y(-2)=3$ and $y(2)=3$ ...(2)

Comparing (1) with $I[y(x)]=\int_{-2}^{2}F(x,y,y')dx$, here $F(x,y,y')=(1+y'^2)^{1/2}$ ...(3)

Since $F(x,y,y')$ is independent of x and y, we use the following form of Euler's equation (refer Case III of Art. 1.5D of chapter 1)

$$\partial F/\partial y-\partial^2F/\partial x\partial y'-y'(\partial^2F/\partial y\partial y')-y''(\partial^2F/\partial y'^2)=0 \qquad ...(4)$$

From (3), $\partial F/\partial y=\partial F/\partial x=0$ and so $\partial^2F/\partial x\partial y'=0=\partial^2F/\partial y\partial y'$

Hence (4) reduces to $-y''(\partial^2F/\partial y'^2)=0$ ...(5)

From (3), $\partial F/\partial y'=(1/2)\times(1+y'^2)^{-1/2}\times 2y'=y'(1+y'^2)^{-1/2}$

and hence $\partial^2F/\partial y'^2=(1+y'^2)^{-1/2}+y'\times(-1/2)\times(1+y'^2)^{-3/2}\times(2y')$

Thus, $\dfrac{\partial^2F}{\partial y'^2}=\dfrac{1}{(1+y'^2)^{1/2}}-\dfrac{y'^2}{(1+y'^2)^{3/2}}=\dfrac{1}{(1+y'^2)^{3/2}}\neq 0$ ...(6)

From (5) and (6), we get $y''=0$ or $d^2y/dx^2=0$

giving $dy/dx=C_1,$ so that $y=C_1x+C_2$ ...(7)

Again, since $F_{y'y'} = \partial^2 F / \partial y'^2 \neq 0$, hence by result of Art. 2.5, the required extremal will consist of portions of the straight lines AP_1 and BP_2 which are tangents to the parabola $x^2 = y$ together with the portion AOB of the parabola.

Let the coordinates of A and B be $(-x', y')$ and (x', y') respectively. Since A and B lie on $x^2 = y$, we get $x'^2 = y'$ and $x'^2 = -y'$.

Thus, the coordinates of A and B can be taken as $(-x', x'^2)$ and (x', x'^2) respectively. From coordinate geometry, the equation of tangent to the parabola $x^2 = y$ at $B(x', x'^2)$ is given by

$$xx' = (y + x'^2)/2 \quad \text{or} \quad 2xx' = y + x'^2 \qquad ...(8)$$

Since (8) passes through $P_2(2,3)$ we must have

$4x' = 3 + x'^2$ or $x'^2 - 4x' + 3 = 0$ so that $x' = 3$ or 1.

From figure it is clear that x' cannot be greater than 2. So we reject $x' = 3$ and take $x' = 1$. Hence the coordinates of B are (1,1). Again, from (8), the equation of the tangent BP_2 is given by $y = 2x - 1$. Proceeding similarly, we can show that the equation tangent AP_1 is $y = -2x - 1$.

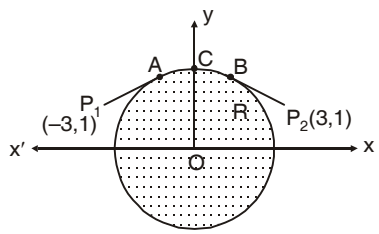
Therefore, the required extremal is given by $$y = \begin{cases} -2x - 1, & \text{if } -2 \le x \le -1 \\ x^2, & \text{if } -1 \le x \le 1 \\ 2x - 1, & \text{if } 1 \le x \le 2 \end{cases} \qquad ...(9)$$

This extremal clearly minimizes the functional.

From (9), we see than the extremal consists of a segment of straight line AP_1 that is tangent to the parabola $x^2 = y$, an arc AOB of the parabola $x^2 = y$, and again a segment of a straight line BP_2 that is tangent to the parabola $x^2 = y$.

Ex. 2. *Find the curves on which the following functional attain extremum* $I = \int_{-3}^{3} y'^2 dx$, $y(-3) = 1,\ y(3) = 1$ *subject to the condition that the admissible curve cannot pass inside the area bounded by the circle* $x^2 + y^2 = 5$.

Sol. Let the circle $x^2 + y^2 = 5$ ...(1) be denoted by the curve ACB. Let R be the region interior to the circle. Here we are to determine extremal of the functional $I[y(x)] = \int_{-3}^{3} y'^2 dx$, ...(2) between two points $P_1(-3,1)$ and $P_2(3,1)$ in such way that the curve cannot pass inside the region R.



Comparing (2) with $I[y(x)] = \int_{-3}^{3} F(x, y, y') dx$, we have $F(x, y, y') = y'^2$ ...(3)

Since $F(x, y, y')$ is independent of x and y, we use the following form of Euler's equation (refer Case III, Art 1.5D of chapter 1).

$$\partial F / \partial y - \partial^2 F / \partial x \partial y' - y'(\partial^2 F / \partial y \partial y') - y''(\partial^2 F / \partial y'^2) = 0 \qquad ...(4)$$

From (3), $\partial F / \partial y = \partial F / \partial x = 0$ so that $\partial^2 F / \partial x \partial y = \partial^2 F / \partial y \partial y' = 0$ ...(5)

Again, from (3), $\partial F / \partial y' = 2y'$ so that $\partial^2 F / \partial y'^2 = 2 \neq 0$ ...(6)

Using (5) and (6), (4) yields $-2y'' = 0$ or $d^2 y / dx^2 = 0$

Integrating it twice, $y = C_1x + C_2$, C_1 and C_2 being arbitrary constants. ...(7)
showing that the extremals of the functional (2) are straight lines.

From (6), since $F_{y'y'} = \partial^2 F / \partial y'^2 \neq 0$, hence (refer a result of Art. 2.5) the required extremal will consist of portions of straight lines P_1A and P_2B which are tangents to the given circle (1) and of the portion ACB of the circle (1) as shown in the figure.

Let the coordinates of A and B be $(-x', y')$ and (x', y') respectively. Since A and B lie on (1), we have $x'^2 + y'^2 = 5$ ...(8)

From coordinate geometry, the equations of the tangents to (1) at $A(-x', y')$ and $B(x', y')$ are given by $-xx' + yy' = 5$ ...(9)
and $xx' + yy' = 5$ ...(10)

Since (10) passes through (3,1) , we have
$3x' + y' = 5$ so that $y' = 5 - 3x'$ ...(11)

Substituting the value of y' as given by (11) in (8), we get

$x'^2 + (5 - 3x')^2 = 5$ or $x'^2 - 3x' + 2 = 0$ so that $x' = 2$ or 1.

Using (11), we find that $y' = -1$ when $x' = 2$ and $y' = 2$ when $x' = 1$.

Hence coordinates of B are (2, –1) or (1, 2). But from figure we see that coordinates (2, –1) of B are not admissible. Hence we take (1, 2) as coordinates of B. Then corresponding coordinates of A are (–1, 2).

Putting $x' = 1$ and $y' = 2$ in (9) and (10), we see that the equations of tangents at A and B are $-x + 2y = 5$ and $x + 2y = 5$ respectively.

Thus, the required extremal is given by

$$y = \begin{cases} x/2 + 5/2, & \text{(tangent } P_1A\text{)}, \quad \text{for} \quad -3 \le x \le -1 \\ (5 - x^2)^{1/2} & \text{(arc } ACB\text{)}, \quad \text{for} \quad -1 \le x \le 1 \\ -x/2 + 5/2 & \text{(tangent } P_2B\text{)}, \quad \text{for} \quad 1 \le x \le 3 \end{cases}$$

EXERCISE 2 (C)

1. Find the curves on which an extremum of the functional $I[y(x)] = \int_0^{10} y'^2 dx, y(0) = 0 = y(10)$ can be achieved provided that the permissible curves cannot pass inside a circle bounded by the circumference $(x-5)^2 + y^2 = 9$.

Ans. $$y = \begin{cases} \pm 3x/4, & \text{if} \quad 0 \le x \le 16/5 \\ \pm\{9 - (x-5)^2\}^{1/2}, & \text{if} \quad 16/5 \le x \le 34/5 \\ \pm 3(x-10)/4, & \text{if} \quad 34/5 \le x \le 10 \end{cases}$$

2. Find the extremal of shortest distance between two fixed points (2,3) and (2, –3), located in the region $y^2 \ge 4x$

Hint. Do like solved example 1 by using all the well known results after interchanging x and y. The required extremal is given by

$$x = \begin{cases} -y - 1, & \text{for} \quad -3 \le y \le -2 \\ y^2/4, & \text{for} \quad -2 \le y \le 2 \\ y - 1, & \text{for} \quad 2 \le y \le 3 \end{cases}$$

3

Sufficient conditions for an Extremum

3.1 Proper field, central field and field of extremals

Proper field.

A family of curves $y = y(x, c)$ is said to form a *proper field* in a given region D of the xy plane if one and only one curve of the family passes through every point of the region D. For example, inside the circle $x^2 + y^2 = 1$, the family of parallel lines $y = x + c$ (c being an arbitrary constant) forms a proper field since through any point of the above circle there passes one and only one straight line of the family as shown in figure (*i*). On the other hand, the family of parabolas $y = (x - c)^2 - 1$ inside the same circle $x^2 + y^2 = 1$ does not form a proper field since the parabolas of this family intersect inside the circle as shown in figure (*ii*).

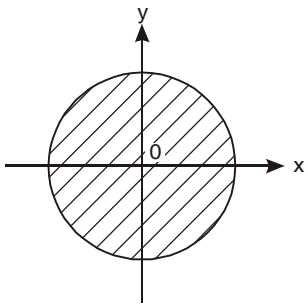


Fig. (i)

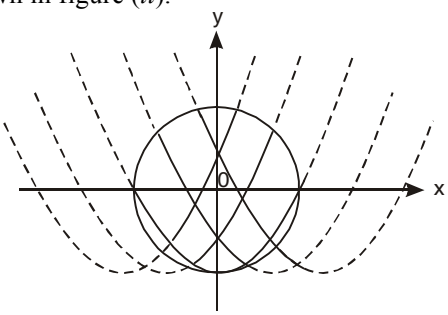


Fig. (ii)

Central field.

If all the curves of the family $y = y(x, c)$ pass through a certain point (x_0, y_0), *i.e.*, if they form a pencil of curves, then they do not form a proper field in the region D, if the centre of the pencil (x_0, y_0) belongs to D. However, if the curves of the pencil cover the entire region D and do not intersect anywhere in this region, with the exception of the centre of the pencil (x_0, y_0), then the family $y = y(x, c)$ is said to form a *central field* as shown in figure (*iii*).

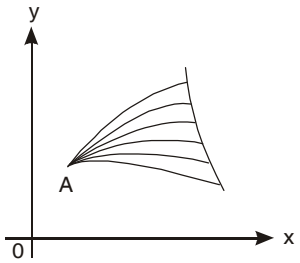


Fig. (iii)

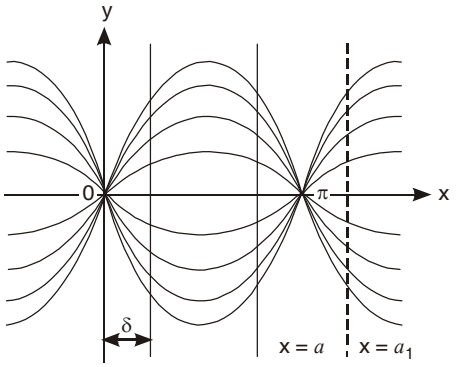


Fig. (iv)

For example, the pencil of sinusoids $y = c \sin x$ for $0 \le x \le a, a < \pi$ forms a central field [see Fig (*iv*)]. But the above mentioned pencil of sinusoids forms a proper field in a sufficiently small neighbourhood of the segment of *x*-axis for $\delta \le x \le a$, where $\delta > 0, a < \pi$ as shown in figure (*iv*). Again the above mentioned pencil of sinusoids does not form a proper field in a neighbourhood of the segment of the *x*-axis, for $0 \le x \le a_1, a_1 > \pi$ as shown in figure (*iv*).

Extremal field.

If a proper field or a central field is formed by a family of extremals of a given variational problem, then it is known as an *extremal field*.

Let the curve $y = y(x)$ be an extremal of a variational problem concerning the extremum of an elementary functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx \qquad ...(1)$$

and let the boundary points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be fixed. We say that the extremal $y = y(x)$ is included in the extremal field if a family of extremals $y = y(x, c)$ can be found that forms a field with the curve $y = y(x)$ as a member of this field for some value of $c = c_0$, say, and the extremal $y = y(x)$ does not lie on the boundary of the region D in which the family $y = y(x, c)$ forms the proper field as shown in figure (*v*).

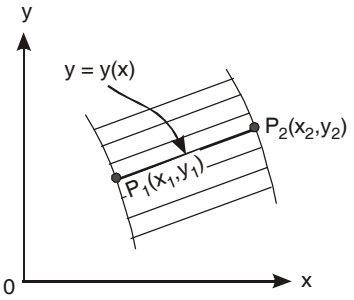


Fig. (v). An extremal embedded in an extremal field

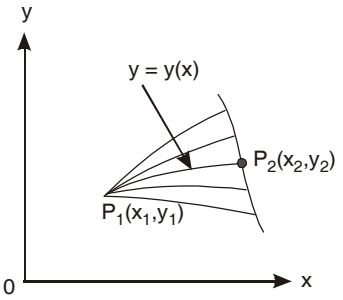


Fig. (vi). An extremal embedded in a central field

Similarly, if a pencil of extremals centered at the point $P_1(x_1, y_1)$ forms a field in the neighbourhood of the extremal $y = y(x)$ that passes through this point, then the central field including the given extremal $y = y(x)$ is said to have been obtained. In the given case, for the parameter of the family of extremals we can take the slope of the tangent line to the curves of the pencil at the point $P_1(x_1, y_1)$ as shown in the figure (*vi*).

3.1A Solved examples based on Art. 3.1

Ex. 1. *Consider the functional* $I[y(x)] = \int_0^a (y'^2 - y^2)dx.$

Let it be required to include the arc of the extremal y = 0 that connects the points (0, 0) and (a, 0) where $0 < a < \pi$ *in the central field of extremals.*

Solution. Comparing the given functional with $\int_0^a F(x, y, y')\,dx$, we have

$$F(x, y, y') = y'^2 - y^2 \qquad ...(1)$$

Euler's equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\dfrac{\partial F}{\partial y} = -2y$, $\dfrac{\partial F}{\partial y'} = 2y'$ and $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = \dfrac{d(2y')}{dx} = 2y''$

Using these values, (2) reduces to

$-2y-2y''=0$ or $(D^2+1)y=0,$ where $D=d/dx$ ...(3)

The general solution of (3) is $y=c_1\cos x+c_2\sin x$ ...(4)

Since the required extremal passes through (0, 0), (4) yields, $c_1=0$ and hence (4) yields.

$$y=c_2\sin x \qquad ...(5)$$

and the curves of this pencil form as central field on the interval $0\le x\le a, a<\pi$ including, for $c_2=0,$ the extremal is $y=0$. (refer figure (*iv*) of Art. 3.1)

From (5), $dy/dx=c_2\cos x$ so that $c_2=(dy/dx)_{(0,0)}$, showing that the parameter of the family of extremals (5) can be taken as the value of dy/dx at $(0,0)$. However, in the above problem, if $a\ge\pi$, then the family of extremals (5) does not form an extremal field as discussed in Art 3.1.

Ex. 2. *Find the proper and central fields of extremals for the functional* $\int_0^{\pi/4}(y'^2-y^2+2x^2+4)dx$.

Sol. Comparing the given functional with $\int_0^{\pi/4}F(x,y,y')\,dx$, $F(x,y,y')=y^2-y^2+2x^2+4$

$\therefore$ Euler's equation $\dfrac{\partial F}{\partial y}-\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)=0 \quad\Rightarrow\quad -2y-\dfrac{d(2y')}{dx}=0$

$d^2y/dx^2+y=0$ or $(D^2+1)\,y=0,$ where $D\equiv d/dx$ ...(1)

Auxiliary equation of (1) is $D^2+1=0$ so that $D=\pm i$ and hence solution of (1) is given by

$y=c_1\cos x+c_2\sin x$, c_1 and c_2 being arbitrary constants ...(2)

(2) in the equation of the extremals.

For $c_2=0$, (2) yields $y=c_1\cos x$, which is a proper field of extremals in the domain $0\le x\le\pi/4$. Again, for $c_1=0$, (2) yields $y=c_2\sin x$, which is a central field of extremals in the domain $0\le x\le\pi/4$.

Ex. 3. *Show that the extremal of the variational problem* $\int_0^2(y'^3+\sin^2 x)dx,\ y(0)=0,\ y(2)=6$ *is included in a central field of extremals of the given functional.*

Sol. Comparing the given functional with $\int_0^2F(x,y,y')\,dx,$ $F(x,y,y')=y'^3+\sin^2 x$

$\therefore$ Euler's equation $\dfrac{\partial F}{\partial y}-\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)=0 \quad\Rightarrow\quad 0-\dfrac{d(3y'^2)}{dx}=0$

Integrating it, $y'^2=c_1^2$ or $y'=dy/dx=c_1$ so that $y=c_1x+c_2$ ...(1)

where c_1 and c_2 are arbitrary constants.

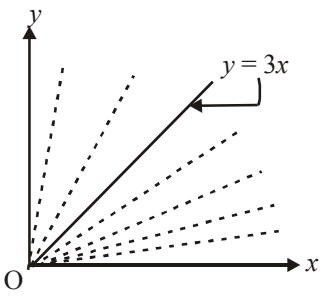
(1) is the equation of the extremals. Using the given boundary conditions $y(0)=$ and $y(2)=6$, (1) gives

$0=c_2$ and $6=2c_1+c_2$ so that $c_1=3$ and $c_2=0$.

Therefore, $(2)\Rightarrow$ $y=3x,$

which is the extremal of the given variational problem.

For $c_2=0$, (1) yields $y=c_1x$, which is a central field of extremals in the domain $0<x\le 2$ with centre at (0, 0). For $c_1=3$, $y=c_1x$ reduces, to $y=3x$, showing that the extremal $y=3x$ is included in the central field of extremals $y=c_1x$.



Ex. 4. *Show that the extremal of the variational problem* $\int_0^2 (y'^2 + x^2)\, dx,\ y(0) = 1,\ y(2) = 3$ *is included in a proper field of extremals of the given functional.*

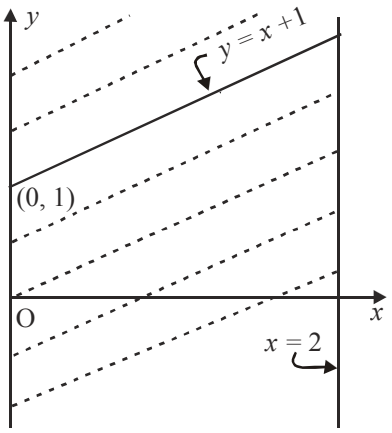
Sol. Comparing the given functional with $\int_0^2 F(x, y, y')\, dx$, here

$$F(x, y, y') = y'^2 + x^2$$

$\therefore$ Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad 0 - \frac{d(2y')}{dx} = 0$$

or $dy'/dx = 0 \Rightarrow y' = c_1 \Rightarrow y = c_1 x + c_2$ ...(1)



(1) is the equation of the extremals. Using the given boundary conditions $y(0) = 1$ and $y(2) = 3$, (1) yields $c_2 = 1$ and $2c_1 + c_2 = 3$ so that $c_1 = c_2 = 1$.

Hence (1) $\Rightarrow y = x + 1$, which is the extremal of the given variational problem.

For $c_1 = 1$, (1) reduces to $y = x + c_2$, which is a proper field of the extremals in the domain $0 \le x \le 2$. For $c_2 = 1$, $y = x + c_2$, yields $y = x + 1$, showing that the extremal $y = x + 1$ is included in the proper field of extremals $y = x + c_2$.

Exercise 3 (A)

1. Find the proper and central fields for the following functionals :

(i) $\int_0^4 y'^2 dx$ (ii) $\int_0^1 (y^2 - y'^2)\, dx$

Ans. (i). Proper field is $y = c_2,\ 0 \le x \le 4$ and central field is $y = c_1 x,\ 0 < x \le 4$ with centre at (0, 0).

(ii) Proper field is $y = c_1 \cos x,\ 0 \le x \le 1$ and central field in $y = c_2 \sin x,\ 0 < x \le 1$ with centre at (0, 0).

2. Show that the extremal of the variational problem $\int_0^1 (y'^2 - 2xy)\, dx,\ y(0) = y(1) = 0$ is included in a central field of extremals of the given functional.

3. Show that the extremal of the variational problem $\int_0^2 (y'^3 + \cos^2 x)\, dx,\ y(0) = y(2) = 2$ is included in a proper field of the given functional.

3.2 Jacobi condition

We know that two infinitely close curves of the family $F(x, y, c) = 0$ intersect at points of the c-discriminant curve defined by the equations $F(x, y, c) = 0$ and $\partial F/\partial c = 0$

Note that the c-discriminant curve includes, among other things, the envelope of the family and locus of the nodal points and the locus of cusps. If we have a pencil of curves with centre at $P_1(x_1, y_1)$, then P_1 belongs to the c-discriminant curve.

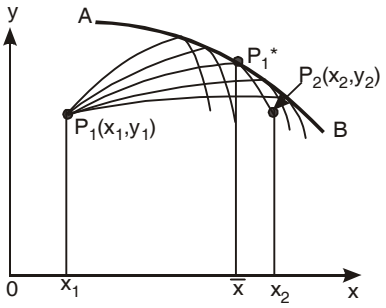


Fig. (i)

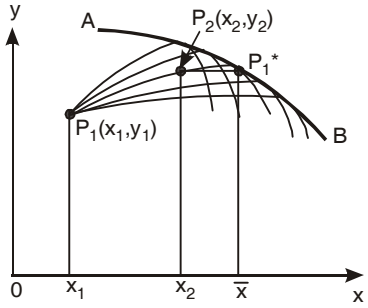


Fig. (ii)

Consider a pencil of extremals $y = y(x,c)$ passing through P_1 and get its c-discriminant curve $\phi\ (x,\ y) = 0$. Now, by definition, the envelope AB of this pencil of curves belongs to $\phi(x,\ y) = 0$. The point P_1* where the extremal (given by $y = y(x)$) touches AB is known as the conjugate point of P_1. As shown in figures (*i*) and (*ii*), we see that the following two situations may arise:

Situation (*i*). When the conjugate point P_1* of P_1 lies between P_1 and P_2 as shown in figure (*i*). In this case the extremals of the pencil close to P_1P_2 intersect and so the extremal P_1P_2 cannot be included in a central field.

Situation (*ii*). When the point $P_2(x_2, y_2)$ lies between P_1 and P_1* as shown in figure (*ii*). In this case the extremals of the pencil close to P_1P_2 do not intersect. Hence the extremals close to P_1P_2 form a central field which includes the arc P_1P_2.

The above results may be sum up as follows:

Jacobi condition. *In order to be able to include an arc P_1P_2 of the extremal in a central field of extremals, it is sufficient that the conjugate point of P_1 does not lie on the arc P_1P_2.*

We now proceed to state Jacobi condition analytically. Let $y = y(x,c)$ be the equation of a pencil of extremals with centre at the point P_1 such that the parameter c being, for definiteness, considered as coinciding with the slope y' of the extremals of the pencil at the point P_1. We know that the c-discriminant is defined by equations $y = y(x,c)$ and $\partial\, y(x,c)/\partial c = 0$

Clearly along every fixed curve of the family, the derivative $\partial y(x,c)/\partial c$ is a function of x *alone*.

Let $\partial y(x,c)/\partial c = u,$ where c is given ...(1)

From (1), $u' = du/dx = \partial^2 y(x,c)/\partial c\ \partial x$ ...(2)

Since $y = y(x,\ c)$ is a solution of Euler's equation for the extremum of $I[y(x)]$ given by

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx\,, \qquad ...(3)$$

we get $$F_y(x, y\ (x,c), y'_x(x,c)) - \frac{d}{dx} F_{y'}(x, y(x,c), y'_x(x,c)) = 0\,, \qquad ...(4)$$

where $y'_x(x, c) = \partial y(x, c)/\partial x$

Differentiating the identity (4) w.r.t. 'c' and using (1), we get

$$F_{yy}\ u + F_{yy'}\ u' - \frac{d}{dx}(F_{yy'}\ u + F_{y'y'}\ u') = 0 \quad \text{or} \quad \left(F_{yy} - \frac{d}{dx} F_{yy'}\right)u - \frac{d}{dx}\left(F_{y'y'}\ u'\right) = 0, \qquad ...(5)$$

where $F_{yy}(x, y, y'), F_{yy'}(x, y, y')$ and $F_{y'y'}(x, y, y')$ are known functions of x, since the second argument y is equal to a solution of Euler's equation $y = y(x, c)$ taken for the value $c = c_0$ that corresponds to the extremal P_1P_2. The above homogeneous linear equation (5) of the second order in u is known as *Jacobi's equation*.

If the solution of (5), which vanishes at the centre of the pencil for $x = x_1$, (the centre of pencil always belong to the c-discriminant curve), also vanishes at some point of the interval $x_1 < x < x_2$, then the point conjugate to P_1 defined by the equation

$$y(x, c) = 0 \quad \text{and} \quad \partial y(x, c)/\partial c = 0 \quad \text{or} \quad u = 0$$

lies on the arc P_1P_2 of the extremal.

On the other hand, if there exists a solution of (5), which vanishes at the centre of the pencil for $x = x_1$ and does not further vanish at any point of the interval $x_1 \le x \le x_2$, then there are no points conjugate to P_1 lying on the arc P_1P_2. Hence the Jacobi condition is fulfilled, and the arc P_1P_2 of the extremal may be included in the central field of extremals centred at the point P_1.

Remark. It has also been proved that the Jacobi condition is necessary for attaining an extremum; *i.e.*, for the extremizing curve P_1P_2 the point conjugate to P_1 cannot lie in the interval $x_1 \le x \le x_2$.

3.2A Solved examples based on Art. 3.1 and 3.2

Ex 1. *Find the c-discriminant of the following families of curves :*

(i) $y\,(c - x) - c^2 = 0$. **[K.U. Kurukshetra 2006]**

(ii) $y = cx + c^2$ *(iii)* $c^2\,(x^2 - a^2) - 2cxy + y^2 + a^2 = 0$

Sol. (*i*) Given $y(c - x) - c^2 = 0$, c being parameter ...(1)

Let $F(x, y, c) = y(c - x) - c^2$ ...(2)

From (2), $\partial F / \partial c = y - 2c$ ...(3)

Then, c-discriminant is obtained by eliminating c between the equations

$F(x, y, c) = 0$ *i.e.*, $y\,(c - x) - c^2 = 0$ ...(4)

and $\partial F / \partial c = 0$ *i.e.*, $y - 2c = 0$ ...(5)

From (5), $c = y/x$. Substituting this value of c in (4), the equation of the c-discriminant is

$$y(y/2 - x) - (y^2/2) = 0 \quad \text{or} \quad (y^2/4) - xy = 0 \quad \text{or} \quad y(y - 4x) = 0$$

Parts (ii) and (iii). **Ans.** (ii) $y = -(x^2/y)$ (iii) $x^2 - y^2 = a^2$

Ex 2. *Show that the Jacobi condition is fulfilled for the functional* $\int_{-1}^{1} (12xy + y'^2 + x^2)\,dx$ *with fixed boundaries* $P_1(-1, -2)$ *and* $P_2(1, 0)$. **[KU Kurukshetra 2006]**

Sol. Comparing the given functional with $\int_{-1}^{1} F(x, y, y')\,dx$, we get

$$F(x, y, y') = 12xy + y'^2 + x^2 \qquad ...(1)$$

$\therefore$ Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad 12x - \dfrac{d(2y')}{dx} = 0$

or $d^2y/dx^2 = 6x$ so that $dy/dx = 3x^2 + c_1$

Intergrating it again, $y = x^3 + c_1x + c_2$, c_1 and c_2 being arbitrary constants. ...(2)
which is the equation of the extremals. Now, using the given boundary conditions, namely, $y = -2$ when $x = -1$ and $y = 0$ when $x = 1$, we have

$$-1 - c_1 + c_2 = -2 \quad \text{and} \quad 1 + c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = 0 \quad \text{and} \quad c_2 = -1 \qquad ...(3)$$

Substituting these values in (2), we get $y = x^3 - 1$,
which is the extremal of the given variational problem.

Using (1), Jacobi's equation $\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)u - \frac{d}{dx}(F_{y'y'}u) = 0 \Rightarrow \left(0 - \frac{d0}{dx}\right)u - \frac{d(2u')}{dx} = 0$

or $du'/dx = 0$ so that $u' = du/dx = c_3$

Integrating, $u = c_3x + c_4$, c_3 and c_4 being arbitrary constants. ...(4)

Using the condition $u(-1) = 0$, $(4) \Rightarrow 0 = -c_3 + c_4$ so that $c_4 = c_3$

Then, (4) yields $u(x) = c_3(x + 1)$, where $c_3 \neq 0$ ...(5)

Now, $(5) \Rightarrow$ $u(x) \neq 0$, for all $x \in]-1, 1]$ ...(6)

From (6), it follows that no conjugate point of P_1 lie on the arc P_1P_2, *i.e.*, on the curve of $y = x^3 - 1$. Therefore, the Jacobi condition is fulfilled and hence the curve of the extremal $y = x^3 - 1$ is included in a central field of extremals with centre at P_1.

Verification. Setting $c_2 = c_1 - 1$ is (2), the family of extremals (2) of the given variational problem reduces to $y = x^3 + c_1x + c_1 - 1$ *i.e.*, $y = x^3 - 1 + c_1(x + 1)$, ...(7)
which is a central field of extremals with centre at P_1 in the domain $-1 < x \leq 1$. Setting $c_1 = 0$ in (7), we get $y = x^3 - 1$, showing that the extremal (3) is included in the central field of extremals given by (7) with centre at $P_1(-1, -2)$.

Ex. 3. *Show that the Jacobi condition is fulfilled for the functional* $\int_0^5 (9y^2 + y'^2 - 3x)\,dx$ *with fixed boundaries* P_1 *(0, 0) and* P_2 *(5, 0).*

Sol. Comparing the given functional with $\int_0^5 F(x, y, y')\,dx$, we get

$$F(x, y, y') = 9y^2 + y'^2 - 3x \qquad ...(1)$$

$\therefore$ Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad 18y - \frac{d(2y')}{dx} = 0$

or $d^2y/dx^2 - 9y = 0$ or $(D^2 - 9)y = 0$, where $D \equiv d/dx$

whose auxiliary equation is $D^2 - 9 = 0$ giving $D = 3, -3$

and so its solution is

$y = C_1 \cosh 3x + C_2 \sinh 3x$, where C_1 and C_2 are arbitrary constants. ...(2)

which is the equation of the extremals. Now, using the given boundary conditions, namely, $y = 0$ when $x = 0$ and $y = 0$ when $x = 5$, (2) yields (noting that $\sinh 0 = 0$ and $\cosh 0 = 1$)

$C_1 = 0$ and $C_1 \cosh 15 + C_2 \sinh 15 = 0$ so that $C_1 = 0$ and $C_2 = 0$

Substituting these values in (2), we have $y = 0$,
which is the extremal of the given variational problem.

Using (1), Jacobi's equation $\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)u - \frac{d}{dx}(F_{y'y'}u') = 0 \Rightarrow \left(18 - \frac{d0}{dx}\right)u - \frac{d(2u')}{dx} = 0$

or $d^2u/dx^2 - 9u = 0$ or $(D^2 - 9)u = 0, D \equiv d/dx$

and its solution, as before, is $u = C_3 \cosh 3x + C_4 \sinh 3x$, ...(4)
where C_3 and C_4 are arbitrary constants.

Using the condition $u(0) = 0$, (4) yields $C_3 = 0$

$\therefore$ (4) reduces to $u(x) = C_4 \sinh x, \quad c_4 \neq 0$ ...(5)

Now, $(5) \Rightarrow u(x) \neq 0$ for all $x \in]0, 5]$ ...(6)

From (6), it follows that no conjugate point of P_1 lie on the arc P_1P_2, *i.e.*, on the curve of $y = 0$. Therefore, the Jacobi condition is fulfilled and hence the curve of the extremal $y = 0$ is included in a central field of extremals with centre at P_1.

Verification. Setting $C_1 = 0$ in (2), the family of extremals of the given variational problem reduces to $y = C_2 \sinh 3x$ ...(7)

which is a central field of extremals with centre at P_1 in the domain $0 < x \leq 5$. Setting $C_2 = 0$ in (7), we get $y = 0$, showing that the extremal (3) is included in the central field of extremal given by (7) with centre at P_1 (0, 0).

Ex 4. *Show that the Jacobi condition in fulfilled for the extremal* $\int_0^a (y'^2 - 4y^2 + e^{-x^2})dx, a \neq n\pi/2$ *with fixed boundaries* P_1 *(0, 0) and* P_2 *(a, 0).*

Sol. Comparing the given functional with $\int_0^a F(x, y, y')\, dx$, we get

$$F(x, y, y') = y'^2 - 4y^2 + e^{-x^2} \quad ...(1)$$

$\therefore$ Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad -8y - \dfrac{d(2y')}{dx} = 0$

or $d^2y/dx^2 + 4y = 0$ or $(D^2 + 4)y = 0$, where $D \equiv d/dx$

whose auxiliary equation is $D^2 + 4 = 0$ giving $D = \pm 2i$ and hence

$y = c_1 \cos 2x + c_2 \sin 2x$, c_1 and c_2 being arbitrary constants. ...(2)

which is the equation of the extremals.

Now, using the given boundary conditions, namely, $y = 0$ when $x = 0$ and $y = 0$ when $x = a$, (2) yields $c_1 = 0$ and $c_1 \cos 2a + c_2 \sin 2a = 0$ so that $c_1 = c_2 = 0$, if $a \neq n\pi/2$

Substituting these values in (2), $y = 0$, ...(3)

which is the extremal of the given variational problem.

Using (1), Jacobi's equation $\left(F_{yy} - \dfrac{d}{dx}F_{yy'}\right)u - \dfrac{d}{dx}(F_{y'y'}u') = 0 \Rightarrow \left(-8 - \dfrac{d0}{dx}\right)u - \dfrac{d(2u')}{dx} = 0$

or $d^2u/dx^2 + 4u = 0$ or $(D^2 + 4)u = 0$, whose solution, as before, is

$u = c_3 \cos 2x + c_4 \sin 2x$, c_3 and c_4 being arbitrary constants. ...(4)

Using the condition $u(0) = 0$, (4) yields $c_3 = 0$.

Hence, (4) reduces to $u(x) = c_4 \sin 2x, \; c_4 \neq 0$ ...(5)

The following two situations arise depending on values of a.

Situation I : Suppose that $a < \pi/2$. Then $0 < a < \pi/2 \Rightarrow 0 < 2a < \pi \Rightarrow \sin 2x \neq 0$

$\therefore$ $(5) \Rightarrow$ $u(x) = c_4 \sin 2x \neq 0$, for all $x \in]0, a]$ ...(6)

From (6), it follows that no conjugate point of P_1 lie on the arc P_1P_2, *i.e.*, on $y = 0$.

Therefore, the Jacobi condition is fulfilled and hence the curve of the extremals $y = 0$ is included in a central field of extremals with centre at P_1 (0, 0).

Verification. Setting $c_1 = 0$ in (2), we get $y = c_2 \sin 2x,$ ...(7)
which is a central field of extremals in the domain $0 < x \leq a$ with centre at P_1. Setting $c_2 = 0$ in (7), we get $y = 0$, showing that the extremal (3) is included in the central field of extremals given by (7) with centre at P_1.

Situation II. Suppose that $a > \pi/2$. Then $\pi/2 \in]0, a[$ such that $\sin 2x = 0$ when $x = \pi/2$. Thus, we have $u(x) = c_4 \sin 2x = 0$ for $x = \pi/2 \in]0, a[$,
showing that a conjugate point of P_1 lies on the arc $P_1 P_2$, *i.e.*, on the curve of $y = 0$. So by the Jacobi condition, the curve of the extremal $y = 0$ is not included in a central field of the extremals with centre at P_1.

Verification. Setting $c_1 = 0$ in (2), we get $y = c_2 \sin 2x,$ ...(8)
which is not a central field of extremals because each curve in the family passes through (0, 0) and $(\pi/2, 0)$ and so (8) does not pass through a unique point of domain $0 < x \leq a$.

Again, setting $c_2 = 0$ in (2), we get $y = c_1, \cos 2x,$ ...(9)
which is not a central field of extremals because there is no centre of this family of curves. Hence the extremal $y = 0$ is not included in a central field.

Ex. 5. *Is the Jacobi condition fulfilled for extremal of the functional* $I[y(x)] = \int_0^a (y'^2 + y^2 + x^2)dx$, *that passes through the points* $P_1(0,0)$ *and* $P_2(a,0)$*?*

[Ravishankar 2004, 10; Kurukshetra 2004]

Sol. Comparing the given functional with $\int_0^a F(x, y, y')dx$, $F(x, y, y') = y'^2 + y^2 + x^2$ ...(1)

The Jacobi's equation is given by $\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)u - \frac{d}{dx}\left(F_{y'y'}\, u'\right) = 0$ ...(2)

where $u = \partial y(x,c)/\partial c$ and $y = y(x, c)$ is the equation of a pencil of extremals with centre at $P_1(0,0)$.

From (1), $F_{yy} = 2,$ $F_{y'y'} = 2,$ $F_{yy'} = 0$ and so (2) yeilds

$(2-0)u - d(2u')/dx = 0$ or $u'' - u = 0$ or $(D^2 - 1)u = 0$ ...(3)

where $D \equiv d/dx$. Auxiliary equation of (3) is $D^2 - 1 = 0$ so that $D = \pm 1$. Therefore, the general solution of (3) is given by

$u = C_1 \cosh x + C_2 \sinh x$, where C_1 and C_2 are arbitrary constants. ...(4)

Using the condition $u(0) = 0$,(4) gives $C_1 = 0$. Then (4) reduces to $u = C_2 \sinh x$

Clearly the curves of the pencil $u = C_2 \sinh x$ intersect the x-axis only at the point $x = 0$. Hence, the Jacobi condition is fulfilled for any value of a.

Ex. 6. *Is the Jacobi condition fulfilled for the extremal of the functional* $I[y(x)] = \int_0^a (y'^2 - y^2)dx$, *that passes through the points* $P_1(0,0)$ *and* $P_2(a,0)$*?*

Sol. Comparing the given functional with $\int_0^a (y'^2 - y^2)dx$, $F(x, y, x') = y'^2 - y^2$ ...(1)

The Jacobi's equation is given by $\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)u - \frac{d}{dx}(F_{y'y'}\, u') = 0,$ ...(2)

where $u = \partial y(x,c)/\partial c$ and $y = y(x, c)$ is the equation of a pencil of extremals with centre at $P_1(0,0)$.

From (1), $F_{yy} = -2,$ $F_{y'y'} = 2,$ $F_{yy'} = 0$ and so (2) yeilds

$(-2-0)u - d(2u')/dx = 0$ or $u'' + u = 0$ or $(D^2 + 1)u = 0$ ...(3)

where $D \equiv d/dx$. Auxiliary equation of (3) is $D^2 + 1 = 0$ so that $D = \pm i$. Therefore, the general solution of (3) is given by

$$u = C_1 \cos x + C_2 \sin x, \text{ where } C_1 \text{ and } C_2 \text{ are arbitrary constants.} \quad ...(4)$$

Using the condition $u(0) = 0$, (4) gives $C_1 = 0$. Then (4) reduces to $u = C_2 \sin x$

Clearly the curves of the pencil $u = C_2 \sin x$ intersect the x-axis when $\sin x = 0$, *i.e.* at the points $x = n\pi$, where n is an integer. Therefore, if $0 < a < \pi$, then on the interval $0 \le x \le a$ the curves of the pencil $u = C_2 \sin x$ intersect the x-axis only at the point $x = 0$ and the Jacobi condition is fulfilled. On the other hand, if $a \ge \pi$, then on the interval $0 \le x \le a$, the curves of the pencil $u = C_2 \sin x$ intersect the x-axis in at least one more point $x = \pi$ and the Jacobi condition is not fulfilled.

Note. Please compare this example with solved example 1of Art. 3.1 A

Ex. 7. *Show that the Jacobi condition is fulfilled for the following variational problems:*

(*i*) $\int_0^1 (1 + y'^2)^{1/2}\, dx$ with fixed boundaries $P_1(0, 0)$ and $P_2(1,0)$

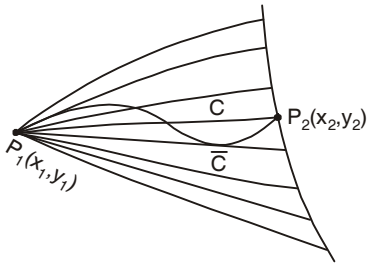
(*ii*) $\int_0^a (x^2 + y'^2)\, dx$, $a > 0$ with fixed boundaries $P_1(0, 0)$ and $P_2(a, 2)$. **[Kurukshetra 2005]**

3.3 Weierstrass function

Consider the extremum of the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx, \qquad y(x_1) = y_1, \qquad y(x_2) = y_2 \quad ...(1)$$

Assume that the extremal C that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ satisfies the Jacobi's condition. Hence, the extremal C can be included in the central field* whose slope is $p(x, y)$ as shown in the figure. In what follows the symbols $\int_{\bar{C}} F(x, y, y')dx$ and $\int_C F(x, y, y')dx$ will represent the values of the functional (1) taken along the arcs of the curves $\bar{C}$ and C, where $\bar{C}$ is permissible curve close to extremal curve C.



In order to examine the sign of the increment ΔI of the functional I when passing from the extremal C to $\bar{C}$, first of all we shall re-write the increment

$$\Delta I = \int_{\bar{C}} F(x, y, y')dx - \int_C F(x, y, y')dx \quad ...(2)$$

in a suitable form as follows. Consider the auxiliary functional

$$\int_{\bar{C}} \{F(x, y, p) + (y' - p)F_P(x, y, p)\}\, dx, \qquad \text{where} \qquad y' = dy/dx \quad ...(3)$$

which reduces to $\int_C F(x, y, y')dx$ on the extremal C, since $dy/dx = p$ on extremals of the field. Now, re-writing the functional (3), we have

$$\int_{\bar{C}} [\{F(x, y, p) - pF_p(x, y, p)\}dx + F_p(x, y, p)\, dy] \quad ...(4)$$

which is the integral of an exact differential. In fact, the differential of the function $\bar{I}(x, y)$, into which the functional $I[y(x)]$ is transformed on the extremals of the field has the form (refer Art 2.1 for understanding)

---

* It might be assumed that the extremal is included in the proper field and not in a central field.

$$d\bar{I} = \{F(x, y, y') - y'F_{y'}(x, y, y')\}dx + F_{y'}(x, y, y')dy$$

which differs from the integrand in (4) with y' replaced by p, the slope of the tangent line to the extremals of the field. Thus, on the extremal C the integral of (3) coincides with the integral $\int_C F(x, y, y')\,dx$, and since the functional (3) is the integral of an exact differential and, hence, does not depend on the path of integration, it follows that

$$\int_C F(x, y, y')dx = \int_{\bar{C}} \{F(x, y, p) + (y' - p)F_p(x, y, p)\}dx \quad ...(5)$$

not only for $\bar{C} = C$ but for any choice of $\bar{C}$ as well.

Using (5), (2) may be re-written as

$$\Delta I = \int_{\bar{C}} F(x, y, y')dx - \int_{\bar{C}} \{F(x, y, p) + (y' - p)F_p(x, y, p)\}dx$$

or $$\Delta I = \int_{\bar{C}} \{F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p)\}dx \quad \text{or} \quad \Delta I = \int_{\bar{C}} E(x, y, p, y')dx \quad ...(6)$$

where $$E(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p), \quad ...(7)$$

where $E(x, y, p, y')$ is known as the *Weierstrass function*.

From (6), it clearly follows that a sufficient condition for the functional I to attain a *minimum* on the extremal C is $E \geq 0$ since it implies $\Delta I \geq 0$. Similarly, a sufficient condition for a *maximum* is $E \leq 0$ since it implies $\Delta I \leq 0$. Again, a sufficient condition for a *weak minimum* is $E \geq 0$ (or for a *weak maximum* a sufficient conditions is $E \leq 0$) is satisfied for values of x, y, close to the values of x, y on the extremal C under consideration, and for the values of y' close to $p(x, y)$ on the same extremal. However, for a *strong minimum* (or *strong maximum*), the same inequality must hold for the same x, y but now for arbitrary y', since in the case of a *strong extremum*, close lying curves may have arbitrary directions of tangent lines and in the case of a weak extremum the values of y' on close lying curves are close to the values of $y' = p$ on the extremal C.

In the following table we have given sufficient conditions for the functional I to attain an extremum on the curve C.

S.No.	For a weak extremum	For a strong extremum
1.	The curve C is an extremal satisfying the boundary conditions (1).	The curve C is an extremal satisfying the boundary conditions (1).
2.	The extremal C must be included in the field of extremals or Jacobi condition must be fulfilled.	The extremal C must be included in the field of extremals or Jacobi condition must be fulfilled.
3.	The Weierstrass function E does not change sign at a point (x, y) close to the curve C or for values of y' close to $p(x, y)$ on the extremal. For a minimum, $E \geq 0$, and for a maximum $E \leq 0$.	The Weierstrass function E does not change sign at any point (x, y) close to the curve C and for arbitrary values of y'. For a minimum $E \geq 0$, and for a maximum $E \leq 0$.

Remark : *We can show that the Weierstrass condition is necessary. More precisely, if in a central field including the extremal C, the Weierstrass function E has opposite signs at points of the extremal for certain y', then a strong extremum is not attained. If this property occurs for values of y' arbitrary close to p, then a weak extremum is not attained.*

3.3A. SOLVED EXAMPLES BASED ON ART. 3.3

Ex. 1. *Investigate for an extremum the functional* $I[y(x)] = \int_0^1 \left(x + 2y + y'^2/2\right)dx$, $y(0) = 0,\ y(1) = 0$

Sol. Comparing the given functional with $\int_0^1 F(x, y, y')dx$, we have

$$F(x, y, y') = x + 2y + (1/2) \times y'^2 \qquad ...(1)$$

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = 2$, $\frac{\partial F}{\partial y'} = y'$ so that $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{dy'}{dx} = y''$

With these values, (2) gives $2 - (d^2y/dx^2) = 0$ or $d^2y/dx^2 = 2$

Integrating, $dy/dx = 2x + C_1$ so that $y = x^2 + C_1x + C_2$ ...(3)

Using the given boundary conditions $y(0) = 0$ and $y(1) = 0$, (3) yields

$0 = C_2$ and $0 = 1 + C_1 + C_2$ so that $C_2 = 0,\quad C_1 = -1$

Then, from (3), we have $$y = x^2 - x \qquad ...(4)$$

Thus an extremal of the given functional is attained only on the curve given by (4). Now, the Jacobi's equation is given by

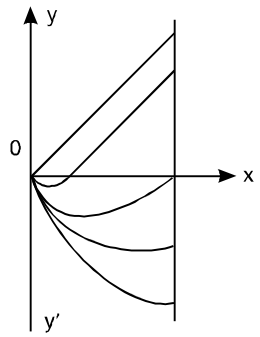
$$\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)u - \frac{d}{dx}\left(F_{y'y'}\,u'\right) = 0 \qquad ...(5)$$

From (1), $F_{yy} = 0,\quad F_{yy'} = 0$ and $F_{y'y'} = 1$ and so (5) yeilds

$-(du'/dx) = 0$ or $d^2u/dx^2 = 0$ giving $u = C_3x + C_4$ ...(6)

Since $u(0) = 0$, (6) yeilds $C_4 = 0$ and so $u = C_3x$

Since $u(x) = C_3x$ does not vanish (for $C_3 \neq 0$) anywhere in [0, 1] except at $x = 0$, we find that the Jacobi condition is satisfied and the extremal (4) can be included in the central field of extremals given by $y = x^2 + C_1x$ with centre at the origin (0, 0) as shown in figure.



Now, the Weierstrass function is given by

$$E(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p) \qquad ...(7)$$

From (1), $$F(x, y, p) = x + 2y + p^2/2 \qquad ...(8)$$

From(8), $$F_p(x, y, p) = p \qquad ...(9)$$

Using (1), (8) and (9), (7) reduces to

$$E(x, y, p, y') = x + 2y + y'^2/2 - (x + 2y + p^2/2) - (y' - p) \times p = y'^2/2 - py' + p^2/2 = (y' - p)^2/2,$$

showing that $E \geq 0$ for arbitrary y'. Hence, a strong minimum is attained on the external (4).

Using (4), we have

$$I[y(x)] = \int_0^1 \left(x + 2y + y'^2/2\right) dx = \int_0^1 \{x + 2(x^2 - x) + (1/2) \times (2x-1)^2\}\, dx = \int_0^1 (4x^2 - 3x + 1/2)\, dx$$

$$[\because y = x^2 - x \Rightarrow y' = 2x - 1]$$

$$= [4 \times (x^3/3) - 3 \times (x^2/2) + x/2]_0^1 = 4/3 - 3/2 + 1/2 = 1/3$$

Thus the value of the strong minimum on the extremal (4) is 1/3.

Ex. 2. *Test for an extremal of* $I[y(x)] = \int_0^a y'^3\, dx, \quad y(0) = 0, \quad y(a) = b, \quad a > 0, \quad b > 0$

Sol. Comparing the given functional with $\int_0^a F(x, y, y')dx$, here $F(x, y, y') = y'^3$ ...(1)

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial y'} = 3y'^2$ so that $\frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = \frac{d(3y'^2)}{dx} = 6y''$

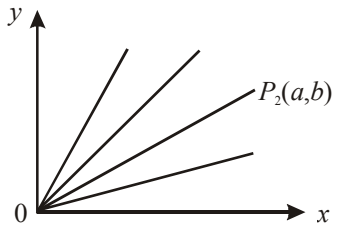
With these values, (2) gives $0 - 6y'' = 0$ or $y'' = 0$

Integrating, $y' = C_1$ so that $y = C_1 x + C_2$ ...(3)

Using the boundary conditions $y(0) = 0$ and $y(a) = b$, (3) yeilds

$0 = C_2$ and $b = C_1 a + C_2$ so that $C_2 = 0$ and $C_1 = b/a$

Then from (3), $y = (b/a) \times x$ ...(4)



Thus an extremal of the given functional is attained only on the straight lines given by (4).

Here the pencil of straight lines $y = C_1 x$ centred at the point (0, 0) forms a central field that includes the extremals $y = (b/a) \times x$ as shown in the figure by straight line OP_2.

From (1), $F(x, y, p) = p^3$ and so $F_p(x, y, p) = 3p^2$ ...(5)

Now the Weierstrass function is given by

$$E(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p) F_p(x, y, p) \qquad ...(6)$$

Using (1) and (5), (6) reduces to

$$E(x, y, p, y') = y'^3 - p^3 - (y' - p) \times (3p^2) = (y' - p)^2 (y' + 2p) \qquad ...(7)$$

On the extremal $y = (b/a) \times x$, the slope of the field $p = b/a > 0$ and if y' assumes values close to $p = b/a$, then $E \geq 0$, and, hence all the conditions that are sufficient for attaining a weak minimum are satisfied. Therefore a weak minimum is attained on the extremal $y = (b/a) \times x$.

On the other hand , if y' takes on arbitrary values, then $(y' + 2p)$ may have any sign and, so the function *E* changes sign, and the conditions sufficient for attaining a strong minimum are not satisfied. Further, using result of remark of Art. 3.3, we see that a strong minimum is not attained on the straight line $y = (b/a) \times x$.

Ex. 3. *Test for an extremum the functional* $\int_0^a (6y'^2 - y'^4 + yy')dx;\ y(0) = 0;$ $y(a) = b;\ a > 0, b > 0$ *in the class of continuous functions with continuous first derivative.*

Sol. Left as an exercise for the reader.

Remark : From the above examples, we have seen that the testing the sign of the Weierstrass function $E(x, y, p, y')$ creates some difficulties. It is ,therefore, desirable to replace the condition of retaining the sign by the function $E(x, y, p, y')$ by a more easily verifiable condition, which is obtained in the next article.

3.4 Legendre condition (Kanpur 2006, 07)

First of all proceed as in Art 3.3 upto equation (7). Then, we proceed as follows to arrive at Legendre condition. Let the function $F(x, y, y')$ be thrice differentiable with respect to y'. Then by Taylor's formula, we have

$$F(x, y, y') = F(x, y, p) + (y' - p)F_p(x, y, p) + (1/2!) \times (y' - p)^2 F_{y'y'}(x, y, q), \quad ...(8)$$

where q lies between y' and p.

Substituting the value of $F(x, y, y')$ given by (8) in equation (7) of Art. 3.3, we get

$$E(x, y, p, y') = (1/2) \times (y' - p)^2 F_{y'y'}(x, y, q), \quad ...(9)$$

showing that the function E does not change sign if $F_{y'y'}(x, y, q)$ does not. It follows that while investigating for a weak extremum the function $F_{y'y'}(x, y, q)$ must retain sign for values of x and y at points close to the points of the extremal under consideration, and for values of q close to $p(x, y)$. If $F_{y'y'}(x, y, y') \neq 0$ at points of the external C, then by virtue of continuity this second derivative maintains sign both at points close to the curve C and for values of y' close to the values of y' on the curve C.

Thus for testing for a weak minimum, the condition $E \geq 0$ *may be replaced by* $F_{y'y'} > 0$ *on the extremal C and for testing for a weak maximum, the condition* $E \leq 0$ *may be replaced by* $F_{y'y'} < 0$ *on C. This is known as the Legendre condition.*

On the other hand, while testing for a strong minimum the condition $E \geq 0$ *may be replaced by the condition* $F_{y'y'}(x, y, q) \geq 0$ *at points (x, y) close to points of the curve C for arbitrary values of q. In this case, of course, we assume that the formula (8) holds good for any* y'. *Similarly, while testing for a strong maximum the condition* $E \leq 0$ *may be replaced by the condtion* $F_{y'y'}(x, y, q) \leq 0$ *at points (x, y) close to points of the curve C for arbitrary values of q. Here, again we assume that (8) holds good for any* y'.

Remark. The above results can be easily extended, without substantial modifications, to functional of the form

$$I[y_1, y_2, ..., y_n] = \int_{x_1}^{x_2} F(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n')\, dx$$

subject to $y_i(x_1) = y_{i1}$ and $y_i(x_2) = y_{i2}, \quad i = 1, 2, ..., n$

In this case, the Weierstrass function is given by

$$E = F(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n') - F(x, y_1, y_2, ..., y_n, p_1, p_2, ..., p_n)$$
$$- \sum_{i=1}^{n} (y_i' - p_i)\, F_{p_i}(x, y_1, y_2, ..., y_n, p_1, p_2, ..., p_n),$$

where p_i are the slope functions of the field, on which certain restriction are imposed (under these restrictions it is known as a *special field*).

The legendre condition $F_{y'y'} \geq 0$ is replaced by the following conditions:

$$F_{y_1'y_1'} \geq 0, \quad \begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} \\ F_{y_2'y_1'} & F_{y_2'y_2'} \end{vmatrix} \geq 0 \ , ..., \begin{vmatrix} F_{y_1'y_1'} & F_{y_1'y_2'} & ... & F_{y_1'y_n'} \\ F_{y_2'y_1'} & F_{y_2'y_2'} & ... & F_{y_2'y_n'} \\ ... & ... & ... & ... \\ F_{y_n'y_1'} & F_{y_n'y_2'} & ... & F_{y_n'y_n'} \end{vmatrix}$$

3.4A Working rule for testing for a weak minimum / weak maximum and strong minimum/ strong maximum of an extremal of a given functional with help of Legendre condition.

Legendre condition. Let $$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx \qquad ...(1)$$

be a functional with fixed boundaries $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let $F(x, y, y')$ possess a continuous partial *derivative* $F_{y'y'}$, that is, $\partial^2 F / \partial y' \partial y'$. Let C be the curve of the extremal of the given functional which passes thorough P_1 and P_2. Also, assume that the extremal of the curve C is included in a field of extremals. Then the following table provides the sufficient conditions for the nature of the extremal of the given functional.

S.No.	Nature of the extremal of the given functional	Sufficient conditions
1.	For a weak minimum to be attained on the extremal on the curve C	$F_{y'y'} > 0$ on C
2.	For a weak maximum to be attained on the extremal on the curve C	$F_{y'y'} < 0$ on C
3.	For a strong minimum to be attained on the extremal on the curve C	$F_{y'y'} \geq 0$ at points close to C and also for arbitrary values of y'
4.	For a strong maximum to be attained on the extremal on the curve C	$F_{y'y'} \leq 0$ at points close to C and also for arbitrary values of y'

3.4B SOLVED EXAMPLES BASED ON ART. 3.4 AND 3.4 A

Ex. 1. *Test for an extermum the functional* $I[y(x)] = \int_0^2 (e^{y'} + 3)dx, \quad y(0) = 0, \quad y(2) = 1$

[KU Kurukshetra 2005; Ravishankar 2004]

Sol. Comparing the given functional with $\int_0^2 F(x, y, y')dx$, here $F(x, y, y') = e^{y'} + 3$ ...(1)

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial y'} = e^{y'}$ so that $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{de^{y'}}{dx} = e^{y'} y''$.

With these values, (2) gives $0 - e^{y'} y'' = 0$ or $y'' = 0$

Integrating it, $y' = C_1$ so that $y = C_1 x + C_2$ ...(3)

showing that an extremal of the given functional is attained only on the straight lines (3).

Since $y(0) = 0$ and $y(2) = 1$, (3) $\Rightarrow$ $0 = C_2$ and $1 = 2C_1$ $\Rightarrow$ $C_1 = 1/2, \ C_2 = 0$

Hence the extremal satisfying the boundary condition is $y = x/2$ which is included in the central field of extremals $y = C_1 x$.

Since $F(x, y, y')$ is three times differentiable with respect to y' for any y' and $F_{y'y'} = e^{y'} > 0$ for any value of y', it follows by (Legendre condition) that on the straight line $y = x/2$ a strong minimum is attained.

Ex. 2. *Test for an extermum the functional* $I[y(x)] = \int_0^a (y'^2 - y^2)dx,\ a > 0;\ y(0) = 0,\ y(a) = 0$

Sol. Comparing the given functional with $\int_0^a F(x, y, y')dx$, $F(x, y, y') = y'^2 - y^2$ ...(1)

Euler's equation is $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad ...(2)$$

From (1), $\frac{\partial F}{\partial y} = -2y$, $\frac{\partial F}{\partial y'} = 2y'$ and $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d(2y')}{dx} = 2y''$

Using these values, (2) reduces to

$-2y - 2y'' = 0$ or $(D^2 + 1)y = 0$, where $D \equiv d/dx$ ...(3)

The general solution of (3) is $y = C_1 \cos x + C_2 \sin x$, ...(4)

where C_1 and C_2 are arbitrary constants.

Since $y(0) = 0$ and $y(a) = 0$, (4) gives $0 = C_1$ and $0 = C_2 \sin a$, If $a \neq n\pi$ where n is an integer, then $C_1 = 0$ and $C_2 = 0$. Thus, for $a \neq n\pi$ an extremum may be attained only on the straight line $y = 0$. If $a < \pi$, then the pencil of extremals $y = C_2 \sin x$ with centre at (0, 0) forms a central field. For $a > \pi$, the Jocobi condition is not satisfied (see figure (*iv*) and discussion of Art. 3.1)

Since $F(x, y, y')$ is three times differentiable with respect to y' for any value of y' and $F_{y'y'} = 2 > 0$ for any values of y', it follows (by Legendre condition) that on the straight line $y = 0$ a strong minimum is attained for $a < \pi$. Again, by virtue of remark of Art. 3.2 we see that for $a > \pi$ a minimum is not attained on $y = 0$.

Ex. 3. *Test for an extremum the functional* $I[y(x)] = \int_0^a y'^3 dx,\ y(0) = 0,\ y(a) = b,\ a > 0,\ b > 0$

[We have already solved this example in Ex. 2 of Art. 3.3A. However, we again solve this problem by using Legendre condition to show that how investigation with respect to weak extermum can be simplified]

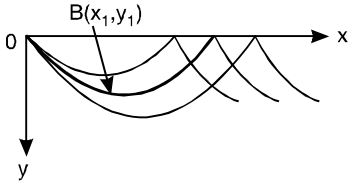
Refer solution of Ex. 2 of Art. 3.3A. Then, as before, the pencil of straight lines $y = C_1 x$ centred at the point (0, 0) forms a central field that includes the extremal $y = (b/a) \times x$. On the extremal $y = (b/a) \times x$, $F_{y'y'} = 6y' = 6 \times (b/a) > 0$. So on the straight line $y = (b/a)x$, a weak minimum is attained. For arbitrary y', $F_{y'y'} = 6y'$ changes sign; hence the above mentioned sufficient conditions for achieving a strong minimum are not satisfied. However, we cannot say that a strong extremum is not attained.

Ex. 4. *Test for an extermum the functional* $I[y(x)] = \int_0^{x_1} \{(1+y'^2)/y\}^{1/2}\,dx,\ y(0)=0,\ y(x_1)=y_1$

Sol. Comparing the given functional with $\int_0^{x_1} F(x,y,y')dx$, we have

$$F(x,y,y') = (1+y'^2)^{1/2}\big/\sqrt{y} \qquad ...(1)$$

Proceed exactly as in Ex. 20, of Art. 1.5F of chapter 1 to get $x = a(\phi - \sin\phi),\quad y = a(1-\cos\phi)$ ...(*i*) where a is determined from the condition that (*i*) must pass through the second boundary point $B(x_1, y_1)$, if $x_1 < 2\pi a$ as shown in the figure. Now, from (1) we have



$$F_{y'} = y'/\sqrt{y}\sqrt{1+y'^2} \qquad \text{and therefore} \qquad F_{y'y'} = 1/y^{1/2}(1+y'^2)^{3/2} > 0 \text{ ,for any } y'$$

Hence, by Legendre condition, for $x_1 < 2\pi a$, a strong minimum is attained on the cycloid (i)

Ex. 5. *Test for an extermum the functional* $I[y(x)] = \int_0^a (y/y'^2)\,dx,\ y(0)=1,$ $y(a)=b,\ a>0,\ 0<b<1$

Sol. Comparing the given functional with $\int_0^a F(x,y,y')dx$, here $F(x,y,y') = y/y'^2$ ...(1)

Since $F(x,y,y')$ is a function of y and y' only, a first integral of Euler's equation is (refer case V, Art. 1.5F of Chapter 1) $F - y'(\partial F/\partial y') = \text{constant} = C$

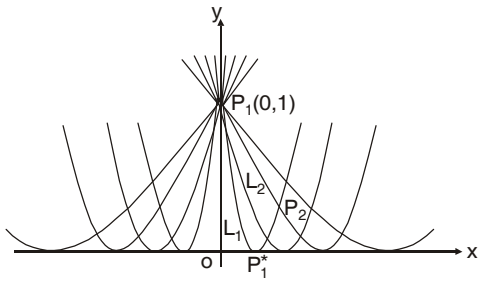
or $\dfrac{y}{y'^2} - y' \times \left(-\dfrac{2y}{y'^3}\right) = C_1$ or $\dfrac{3y}{y'^2} = \dfrac{3}{4C^2}$, where $C_1 = \dfrac{3}{4C^2}$

or $y'^2 = 4C^2\, y$ or $dy/dx = \pm 2C\, y^{1/2}$

Separating variables, $\pm(1/2) \times y^{-1/2} dy = C\,dx$

Integrating, $\pm y^{1/2} = Cx + C'$ or $y = (Cx + C')^2$, ...(2)

which is a family of parabolas. Using condition $y(0) = 1$, (2) gives $C' = 1$. Now, the pencil of parabolas $y = (Cx + 1)^2$ with centre at $P_1(0, 1)$ has a c-discriminant curve $y = 0$ as shown in the figure. Two parabolas of this pencil pass through the point $P_2(a, b)$. On the arc P_1P_2 of one of them (say L_1) lies the point P_1^* which is conjugate to P_1. On the other hand, on the other arc (say, L_2) there is no conjugate point. It follows that the Jacobi condition is fulfilled on the arc L_2, and an extremum can be attained on this arc of the parabola.



In the neighbourhood of the extremal under consideration, $F_{y'y'} = 6y/y'^4 > 0$ for arbitrary y'. But due to this fact we cannot say that a strong minimum can be attained on the arc L_2, since the function $F(x,y,y') = y/y'^2$ cannot be represented in the form

$$F(x,y,y') = F(x,y,p) + (y'-p)F_p(x,y,p) + (1/2)\times(y'^2-p)^2 F_{y'y'}(x,y,q)$$

for arbitrary values of y' due to the presence of a discontinuity of the function $F(x, y, y')$ when $y' = 0$. However, we can only say that a weak minimum is achieved on L_2, since for values of y' close to the slope of the field on the curve L_2 we have an expansion of the function $F(x, y, y')$ by Taylor's formula. A complete discussion of this function for an extermum involves the consideration of Weierstrass function given by

$$E(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p)$$

i.e. $$E(x, y, p, y') = \frac{y}{y'^2} - \frac{y}{p^2} + \frac{2y}{p^3}(y' - p) = \frac{y(y' - p)^2(2y' + p)}{y'^2 p^3}$$

Since the factor $(2y' + p)$ changes sign for arbitrary y', hence with help of remark of Art. 3.3, we conclude that a strong minimum cannot be attained on the arc L_2.

Ex. 6. Using Legendre condition, test for an extremum of the functional $I[y(x)] = \int_0^1 yy'^2 dx$, $y(0) = 4, y(1) = 4$.

Sol. Comparing the given functional with $\int_0^1 F(x, y, y')dx$, $F(x, y, y') = yy'^2$ ... (1)

$\therefore$ Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow y'^2 - \frac{d(2yy')}{dx} = 0$

or $y'^2 - 2(yy'' + y'^2) = 0$ or $2yy'' + y'^2 = 0$

Multiplying both sides by y', we get $y'^3 + 2yy'y'' = 0$

or $y' \times y'^2 + y \times (2yy'') = 0$ or $d(yy'^2)/dx = 0$

Integrating, $yy'^2 = C_1^2$ so that $y' = dy/dx = C_1 / y^{1/2}$

or $y^{1/2}\, dy = C_1 dx$ so that $(2/3) \times y^{3/2} = C_1 x + C_2$, ... (2)

which is the equation of the extremals.

Now, using the given boundary conditions $y(0) = 4$ and $y(1) = 4$, (2) yields

$(2/3) \times (4)^{3/2} = C_2$, $(2/3) \times 4^{3/2} = C_1 + C_2 \Rightarrow C_2 = 16/3$, $C_1 + C_2 = 16/3, \Rightarrow C_2 = 16/3$, $C_1 = 0$.

Substituting these values in (2), we obtain

$(2/3) \times y^{3/2} = 10/3$ or $y^{3/2} = 8$ or $y = 4$, ...(3)

which is the extremal of the given variational problem.

Setting $C_1 = 0$ in (2) yields $y = (3C_2/2)^{2/3}$, ...(4)

which is the proper field of extremals.

Setting $C_2 = 16/3$ in (4), we have $y = 4$, showing that the extremal $y = 4$ is included in the proper field of extremals given by (4).

From (1), $F_{y'} = \partial F / dy' = 2yy'$ so that $F_{y'y'} = \partial^2 F / dy' \partial y' = 2y$.

Clearly, the function $2y$ is continuous and $F_{y'y'} = 2y \geq 0$ at points close to the extremal $y = 4$ and also for arbitrary values of y'. Hence by Legendre condition, the given functional has a strong minimum on the extremal $y = 4$ of the given functional.

Ex. 7. *Using Legendre condition, test for an extremum of the functional* $I[y(x)] = \int_2^3 (x^3 / y'^2)dx$, $y(2) = 4, y(3) = 9$.

Sol. Comparing the given functional with $\int_2^3 F(x, y, y')dx$, here $F(x, y, y') = x^3 / y'^2$ ...(1)

Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad 0 - \frac{d}{dx}\left(-\frac{2x^3}{y'^3}\right) = 0$

$\frac{d}{dx}\left(\frac{x^3}{y'^3}\right) = 0$ so that $\frac{x^3}{y'^3} = \frac{1}{8C_1^3}$

$\therefore \quad y' = dy/dx = 2C_1 x \quad$ or $\quad dy = 2C_1 x\,dx.$

Integrating, $y = C_1 x^2 + C_2,$ where C_1 and C_2 are arbitrary constants. ...(2)

which is the equation of the extremals. Now, using the given boundary conditions $y(2) = 4$ and $y(3) = 9$, (2) yields $4C_1 + C_2 = 4$ and $9C_1` + C_2 = 9$ so that $C_1 = 1$ and $C_2 = 0$.

Substituting these values in (2), we get $y = x^2,$ ...(3)

which is the extremal of the given variabtional problem.

Setting $C_2 = 0$ in (2), we have $y = C_1 x^2,$ ...(4)

which is a central field of extremals with centre at (0, 0) and domain $2 \le x \le 3$.

Setting $C_1 = 1$ in (4), we get $y = x^2$, showing that the extremal $y = x^2$ is included in the central field given by (4).

Now, $(1) \Rightarrow F_{y'} = \partial F/\partial y' = -(2x^3/y'^3) \Rightarrow F_{y'y'} = \partial^2 F/\partial y'\partial y' = 6x^3/y'^4$ ..(5)

Since for $y' = 0$, $F_{y'y'}$ is not defined, it follows that on the curve $y = x^2$ neither a strong maximum nor a strong minimum is attained for the given functional

Now, from (2) $y' = 2x$. Hence on the extremal $y = x^2$

$$F_{y'y'} = (6x^3)/(16x^4) = 3/(8x) > 0, \text{ for } 2 \le x \le 3.$$

Thus, $F_{y'y'} > 0$ on the extremal $y = x^2$. Therefore, by Legendre condition, it follows that on the curve $y = x^2$ a weak minimum is attained for the given functional.

Ex. 8. *Using Legendre condition, test for the functional* $I[y(x)] = \int_0^2 (4 - e^{y'})dx, y(0) = 0, y(2) = 1$ *for an extremum.* **[M.D.U Rohtak 2005]**

Sol. Comparing the given functional with $\int_0^2 F(x, y, y')dx, \quad F(x, y, y') = 4 - e^{y'}$ ...(1)

$\therefore$ Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad 0 - \frac{d}{dx}(-e^{y'}) = 0$

or $de^{y'}/dx = 0$ so that $e^{y'} = e^{c_1}$

$\therefore \quad y' = dy/dx = C_1$ so that $y = C_1 x + C_2,$ ...(2)

which in the equation of the extramals.

Now, using the given boundary conditions $y(0) = 0$ and $y(2) = 1$, (2) yields

$C_2 = 0$ and $2C_1 + C_2 = 1$ so that $C_1 = 1/2$ and $C_2 = 0$

Substituting these values in (2), we get $y = x/2,$ ...(3)

which is the extremal of the given variational problem.

For $C_2 = 0$, (2) yields $y = C_1 x,$ ..(4)

which is a central field of extremals with centre (0, 0) and domain $0 \le x \le 2$.

For $C_1 = 1/2$, (4) reduces to $y = x/2$, showing that the extremal $y = x/2$ is included in the central field of extremals given by (4).

Now, from (1), $F_{y'} = -e^{y'}$ and so $F_{y'y'} = -e^{y'}$, which is continuous.

Clearly, $F_{y'y'} = -e^{y'} \le 0$, for all values of y'.

Therefore, by Legendre condition, it follows that a strong maximum is attained on the extremal $y = x/2$ of the given functional.

Ex. 9. *Using Legendre condition, test for an extremum of the following functionals:*

i) $\int_0^2 (4 + e^{y'})dx, y(0) = 0, y(2) = 1$ **[KU Kurukshetra 2004]**

Ans. Strong minimum on the extremal $y = x/2$.

(ii) $\int_0^1 (x^2 + y'^2)dx, y(0) = -1, y(1) = 1$ **Ans.** Strong minimum on the extremal $y = 2x - 1$.

(iii) $\int_1^2 (xy'^4 - 2yy'^3)dx, y(1) = 0, y(2) = 1$ **Ans.** Weak minimum on the extremal $y = x - 1$.

3.4C An alternative form of Jacobi equation. Jacobi condition.

Consider a functional $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx$ ... (1)

subject to the boundary conditions : $y(x_1) = y_1$ and $y(x_2) = y_2$. ... (2)

The second variation $\delta^2 I$ of the functional I is given by

$$\delta^2 I = \int_{x_1}^{x_2} \left\{ F_{yy} (\delta y)^2 + 2F_{yy'} \delta y \delta y' + F_{y'y'} (\delta y')^2 \right\} dx \quad \text{... (3)}$$

Let $z = \delta y$. Then, we get $z' = \frac{dz}{dx} = \frac{d(\delta y)}{dx} = \delta\left(\frac{dy}{dx}\right) = \delta y'$ ... (4)

Let G denote the integrard in (3). Then, using (4), we get

$$G(x, z, z') = F_{yy} z^2 + 2F_{yy'} z z' + F_{y'y'} z'^2$$

The Euler's equation for the function G is given by $\frac{\partial G}{\partial z} - \frac{d}{dx}\left(\frac{\partial G}{\partial z'}\right) = 0$

or $$2zF_{yy} + 2F_{yy'} z' - \frac{d}{dx}(2F_{yy'} z + 2F_{y'y'} z') = 0$$

or $$\left(F_{yy} - \frac{d}{dx} F_{yy'}\right) z - \frac{d}{dx}(F_{y'y'} z') = 0, \quad \text{... (5)}$$

which is Jacobi equation. It is a second order, homogemeous and linear differential equation in $z(x)$, *i.e.*, $\delta y(x)$.

Now, at either boundary point $P_1(x_1, y_1)$ or $P_2(x_2, y_2)$, $\delta y = 0$ because there is no variation in y at any of these two points, *i.e.*, $[\delta y]_{x=x_1} = 0$ and $[\delta y]_{x=x_2} = 0$, ... (6) showing that the solution of (5), *i.e.*, δy vanishes at P_1 and it again vanishes at P_2.

A point $\bar{x} \neq x_1$, at which the solution again vanishes, is known as *conjugate point* of P_1 and is denoted by P_1^*. If $\bar{x}$ does not belong to the interval $[x_1, x_2]$, then it would imply that the solution of (5) which vanishes at $P_1[x = x_1]$, does not further vanish at any point in $x_1 \le x \le x_2$, *i.e.*, no point conjugate to P_1 lies on the curve P_1P_2. In this situation the other fixed end of the extremal may occur beyond the point $P_2(x = x_2)$. This means that the curve $P_1 P_2$ will only be an arc of the extremal.

However, if $\bar{x}$ belongs to the interval $[x_1, x_2]$, then it would imply that the solution of (5), *i.e.*, δy vanishes again at some point P_1^* ($x = \bar{x}$) before x_2. This also implies that neighbouring extremals intersect at P_1^* because only then variation in y vanishes at P_1^*. It follows that all the curves of the central field will terminate at P_1^* ($x = \bar{x}$) occuring at left of x_2. This fact contradicts the assumption (6).

Thus if the solution of Jacobi equation, *i.e.*, δy vanishes at one end $P_1(x = x_1)$ then to ensure that curve $P_1 P_2$ is either extremal or an arc of the extremal, δy must vanish (*i.e.*, conjugate point P_1^* of P_1 must occur) before P_2 $(x = x_2)$. Clearly, it may vanish beyond P_2 $(x = x_2)$. This sufficient condition is known as *Jacobi condition.*

Remark. Suppose the solution of Jacobi equation vanishes at $x = x_2$. Then the first fixed boundary may occur at $x = x_1$ or on its left.

3.4D Solved examples based on Art. 3.4C

Ex. 1. *Show that Jacobi condition for the central field of extremals for* $I[y(x)] = \int_0^{\pi/2} (xyy' + y^2 - y'^2/2)\,dx$; $z(0) = 0$; *where* $z = \delta y$ *is fulfuilled. Further show that for extremals, the functional has weak maximum.*

Sol. Comparing the given functional with $\int_0^{\pi/2} (x, y, y')dx$, we get

$$F(x, y, y') = xyy' + y^2 - y'^2/2 \quad \ldots (1)$$

The Jacobi equation is $$(F_{yy} - \frac{d}{dx}F_{yy'})z - \frac{d}{dx}\left(F_{y'y'}z'\right) = 0 \quad \ldots (2)$$

From (1), $F_y = xy' + 2y$, $F_{yy} = 2$, $F_{yy'} = x$, $F_{y'} = xy - y'$, $F_{y'y'} = -1$ ... (3)

Using (3), (2) yields $$\left(2 - \frac{dx}{dx}\right)z - \frac{d(-z')}{dx} = 0$$

or $z'' + z = 0$ or $(D^2 + 1)\,z = 0$, where $D \equiv d/dx$ ... (4)

Its auxiliary equation is $D^2 + 1 = 0$ so that $D = \pm i$ and so its solution is

$z = C_1 \cos x + C_2 \sin x$, where C_1 and C_2 are arbitrary constants ... (5)

Using the given condition $z(0) = 0$, (5) yields $C_1 = 0$

$\therefore$ (5) reduces to $z = C_2 \sin x$ or $\delta y = C_2 \sin x$ ... (6)

which is the central field of extremals for the given functional.

From (6), we see that z is zero at $x = 0$ and again vanishes at $x = \pi$. Hence $x = \pi$ is conjugate point of $x = 0$ for every family of (5). Since $x = \pi$ lies beyond $x = \pi/2$ (*i.e.*, the conjugate point does not lie in the interval $0 \le x \le \pi/2$), therefore, a curve $z = \sin x$ (which satisfies an additional condition $z(\pi/2) = 1$) represents only part of the extremal. Hence Jacobi condition is satisfied.

From (3), $F_{y'y'} = -1 < 0$. Hence, by Legendre condition, a weak maximum is attained for $z = C_2 \sin x$.

Ex. 2. *Discuss the Jacobi and Legendre conditions for the functional* $\int_0^1 \{(x^2 y'^2)/2 - 2xyy' + y\}dx$, $z(0) = 0$ *and* $z = \delta y$. *Further derive the extremal satisfying* $z(1) = 1/2$ *and emanating from* $(0, 1)$.

Sol. Comparing the given functional with $\int_0^1 F(x, y, y')dx$, we get

$$(x, y, y') = (x^2 y'^2)/2 - 2xyy' + y \quad \ldots (1)$$

The Jacobi equation is $$\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)z - \frac{d}{dx}(F_{y'y'}z') = 0 \quad \ldots (2)$$

From (1), $F_y = -2xy' + 1$, $F_{yy} = 0$, $F_{yy'} = -2x$, $F_{y'} = x^2y' - 2xy$ and $F_{y'y'} = x^2$ ... (3)

Using (3), (2) yields $$\left(0 - \frac{d(-2x)}{dx}\right)z - \frac{d(x^2 z')}{dx} = 0$$

or $2z-(2xz'+x^2z'')=0$ or $(x^2D^2+2xD-2)z=0, \quad D\equiv d/dx$ ... (4)

Let $x=e^t$ or $\log x=t$ and let $D_1\equiv d/dt$. Then, $xD=D_1$ and $x^2D^2=D_1(D_1-1)$. Hence, (4) reduces to $\{D_1(D_1-1)+2D_1-2\}\,z=0$ or $(D_1^2+D_1-2)z=0$

Its auxiliary equation is $D_1^2+D_1-2=0$ so that $D_1=1,-2$ and so its solution is

$z=C_1e^t+C_1e^{-2t}=C_1e^t+C_2(e^t)^{-2}$, C_1, and C_2 being arbitrary constants

or $z=C_1x+C_2x^{-2}$, as $e^t=x$ or $x^2z=C_1x^3+C_2$. ... (5)

Using the boundary condition $z(0)=0$, (5) yields $C_2=0$

Hence (5) reduces to $z=C_1x$, ... (6)

which is the equation of the central field of extremals.

Using the boundary condiiton $z(1)=1/2$, (6) yields $C_1=1/2$. Hence, the required extremal is given by $z=x/2$

or $\delta y=x/2$ ... (7)

From (7), we see $\delta y=0$ at $x=0$ and δy is never zero beyond $x=0$. Hence Jacobi condition is fulfilled.

Now, from (3), $F_{y'y'}=x^2>0$ for any $x>0$. Hence by Legendre condition, the given functional attains weak minimum for the extremal $z=x/2$.

Integrating (7), $y=x^2/4+C_3$. C_3 being an arbitrary constant. ... (8)

For the required extremal emanating from the point (0, 1), we have y = 1 when $x=0$. Hence (8) yields $C_3=1$. Therefore the required extremal emanating from (0, 1) is given by $y=x^2/4+1$.

Ex. 3. *Show that the functional* $\int_0^a (x^2+y^2+y'^2)\,dx$, *z(0) = 0 and* $z=\delta y$, *the Jacobi condition is fulfilled and for the extremals* $z=C\sinh x$, *the functional has a weak minimum.*

Hints. Proved as in Ex. 2.

4. *(a) Show that the extremal through origin for which the functional* $\int_1^2(-y^2+x^2y'^2/2)dx$ *attains weak minimum at every point of the domain, subject to the conditions that its first order variations at the two ends of the domain are one and zero respectively, is* $y=-(8/x)-(x^2/14)$.

(b) Show that the extremal y(x) through origin, whose first order variation at two ends of the domain (1, 2) are 1 and 2 respectively, and for which the functional of integrand (1 + y + $xy'^2/2$*) attains weak minimum at every point of the domain, is* $y=x\left\{1+(\log x-1)/\log_e^2\right\}$.

Sol. *Left an exercises for the reader.*

3.4E Second Variation and determination of conditions for weak minimum (or maximum)

In the present article we shall examine how the sufficient conditions for a weak minimum (or maximum)may be derived by a different technique based on the study of the sign of the second variation of the functional

$$I[y(x)]=\int_{x_1}^{x_2}F(x,y,y')dx, \qquad \text{if } y(x_1)=y_1, \quad y(x_2)=y_2 \qquad ...(1)$$

Using the Taylor's formula, we transform the increment of the functional in the elementary problem (1) in the following form:

$$\Delta I=\int_{x_1}^{x_2}\left\{F(x,y+\delta y,y'+\delta y')-F(x,y,y')\right\}dx$$

$$=\int_{x_1}^{x_2}(F_y\delta y+F_{y'}\delta y')dx+(1/2)\times\int_{x_1}^{x_2}\{F_{yy}(\delta y)^2+2F_{yy'}\delta y\delta y'+F_{y'y'}(\delta y')^2\}dx+R, \qquad ...(2)$$

where R is of order higher than second in δy and $\delta y'$.

On the extremal, the first variation represented by the first term on the R.H.S. of (2) vanishes,

i.e. $$\int_{x_1}^{x_2} (F_y\ \delta y + F_{y'}\ \delta y')\, dx = 0 \qquad ...(3)$$

Using (3), (2) reduces to

$$\Delta I = (1/2) \times \int_{x_1}^{x_2} \{F_{yy}\ (\delta y)^2 + 2F_{yy'}\ \delta y\ \delta y' + F_{y'y'}\ (\delta y')^2\}\, dx + R \qquad ...(4)$$

While investigating for a weak extermum, δy and $\delta y'$ are sufficiently small, and in this case the sign of the increment ΔI is determined by the sign of the terms on the R.H.S. of (4) involving the lowest degrees of δy and $\delta y'$. Hence from (4), it follows that the sign of ΔI, in general, coincides with the sign of the second variation $\delta^2 I$, being given by

$$\delta^2 I = \int_{x_1}^{x_2} \{F_{yy} (\delta y)^2 + 2F_{yy'}\ \delta y\ \delta y' + F_{y'y'}\ (\delta y')^2\} dx \qquad ...(5)$$

The Legendre condition and the Jacobi condition together are the conditions that ensure constancy of sign of the second variation, and thus also constancy of the increment ΔI in the problem involving a weak extermum.

Consider the integral $$\int_{x_1}^{x_2} \{G'(x) \times (\delta y)^2 + 2G(x)\ \delta y\ \delta y'\} dx \qquad ...(6)$$

where $G(x)$ is an arbitrary differentiable function. We now show that this integral is equal to zero by using the given boundary conditions in (1) so that

$$[\delta y]_{x=x_1} = 0 \qquad \text{and} \qquad [\delta y]_{x=x_2} = 0 \qquad ...(7)$$

$$\therefore \int_{x_1}^{x_2} \{G(x) \times (\delta y)^2 + 2G(x)\ \delta y\ \delta y'\}\, dx = \int_{x_1}^{x_2} d\left((G(x) \times (\delta y)^2\right) dx = \Big[G(x) \times (\delta y)^2 \Big]_{x_1}^{x_2} = 0 \text{, by (7)}$$

Thus, $$0 = \int_{x_1}^{x_2} \left\{ G'(x) \times (\delta y)^2 + 2G(x)\ \delta y\ \delta y' \right\} dx \qquad ...(8)$$

Adding the corresponding sides of (5) and (8), we have

$$\delta^2 I = \int_{x_1}^{x_2} \left\{ (F_{yy} + G') \times (\delta y)^2 + 2(F_{yy'} + G) \delta y\ \delta y' + F_{y'y'} \times (\delta\, y')^2 \right\} dx \qquad ...(9)$$

Now, let us choose the function $G(x)$so that the integrand in (9) is a perfect square. Then the function $G(x)$ must satisfy the equation $$F_{y'y'}(F_{yy} + G') - (F_{yy'} + G)^2 = 0 \qquad ...(10)$$

For such a choice of $G(x)$, (9) takes the form: $$\delta^2 I = \int_{x_1}^{x_2} F_{y'y'} \left(\delta y' + \frac{F_{yy'} + G}{F_{y'y'}}\, \delta y \right)^2 dx\,,$$

showing that the sign of $\delta^2 I$ is the same as that of $F_{y'y'}$. However, such a transformation is possible only on the assumption that (10) can yield differentiable solution $G(x)$ on the interval (x_1, x_2).

Transforming (10) to new variables by the substitution

$$G(x) = -F_{yy'} - F_{y'y'} \times (u'/u)\,, \qquad ...(11)$$

where u is new unknown function, we have

$$\left(F_{yy} - \frac{d}{dx} F_{yy'} \right) u - \frac{d}{dx} \left(F_{y'y'}\ u' \right) = 0\,, \qquad ...(12)$$

which is Jacobi's equation derived already in Art. 3.2.

If a solution of (12) exists that does not varnish for $x_1 < x \le x_2$, *i.e.* if the Jacobi condition is satisfied, then for the same values of x there exists a continuous and differentiable solution $G(x)$ (given by (11)) of the equation (9). Thus fulfillment of both Legendre and Jacobi conditions ensures that the sign of second variation $\delta^2 I$ does not change and hence, they are sufficient conditions for a weak minimum $(F_{y'y'} > 0)$ or weak maximum $(F_{y'y'} < 0)$.

EXERCISE 3 (B)

Test the following functionals for extrema:

1. $I[y(x)] = \int_0^2 (xy' + y'^2)dx, \quad y(0) = 1, \quad y(2) = 0$

2. $I[y(x)] = \int_1^2 y'(1 + x^2 y')dx, \quad y(1) = 3, \quad y(2) = 5$

3. $I[y(x)] = \int_{-1}^2 y'(1 + x^2 y')dx, \quad y(-1) = y(2) = 1$

4. $I[y(x)] = \int_1^2 (x^2 y'^2 + 12y^2)dx, \quad y(1) = 1, \quad y(2) = 8$

5. $I[y(x)] = \int_0^{x_1} (1/y')dx, \quad y(0) = 0, \quad y(x_1) = y_1; \quad x_1 > 0, \quad y_1 > 0$

6. $I[y(x)] = \int_0^{x_1} (1/y'^2)\,dx, \quad y(0) = 0, \quad y(x_1) = y_1; \quad x_1 > 0, \quad y_1 > 0$

7. $I[y(x)] = \int_1^2 (x^3/y'^2)\,dx\,dx, \quad y(1) = 1, \quad y(2) = 4$ **(Vikram 2004)**

8. $I[y(x)] = \int_1^3 (12xy + y'^2)dx, \quad y(1) = 0, \quad y(3) = 26$

9. $I[y(x)] = \int_{-1}^2 y'(1 + x^2 y')dx, \quad y(-1) = 1, \quad y(2) = 4$

10. $I[y(x)] = \int_0^a (y'^2 + 2yy' - 16y^2)dx, \quad a > 0, \quad y(0) = 0, \quad y(a) = 0$

11. $I[y(x)] = \int_0^1 (y'^2 + y^2 + 2ye^{2x})dx, \quad y(0) = 1/3, \quad y(1) = (1/3) \times e^2$

12. $I[y(x)] = \int_0^{\pi/4} (y^2 - y'^2 + 6y \sin 2x)dx, \quad y(0) = 0, \quad y(\pi/4) = 1$

13. $I[y(x)] = \int_0^{\pi/4} (4y^2 - y'^2 + 8y)dx, \quad y(0) = -1, \quad y(\pi/4) = 0$

14. $I[y(x)] = \int_0^2 (y^2 + y'^2 - 2xy)dx, \quad y(0) = 0, \quad y(2) = 3$

15. Find the minimum of the functional $I[y(x)] = \int_0^1 (y'^2/2 + yy' + y' + y)\,dx$ if the values at the ends of the interval are not given. **(Sagar 2004, Indore 2004,)**

16. The functional $\int_0^1 (1 + x)\,(y')^2\,dx,\ y(0) = 0,\ y(1) = 0$ possesses

(*a*) strong maximum (*b*) strong minimum (*c*) weak maximum but not strong maximum
(*d*) weak minimum but not strong minimum **(GATE 2007)**

ANSWERS

1. A strong minimum is attained for $y = 1 - (x^2/4)$

2. A strong minimum is attained for $y = 7 - (4/x)$

3. A strong minimum is attained for $y = 1$ **4.** A strong minimum is attained for $y = x^3$

5. A weak minimum is attained for $y = (xy_1)/x_1$

6. A weak minimum is attained for $y = (xy_1)/x_1$

7. A weak minimum is attained for $y = x^2$ **8.** A strong maximum is attained for $y = x^2 - 1$

9. An extremum is not attained on continuous curves.

10. A strong minimum is attained for $y = 0$ if $0 < a < \pi/4$ but if $a > \pi/4$, there is no minimum.

11. A strong minimum is attained for $y = (1/3) \times e^{2x}$

12. A strong maximum is attained for $y = \sin 2x$

13. A strong maximum is attained for $y = \sin 2x - 1$

14. A strong minimum is attained for $y = x + (\sinh x / \sinh 2)$

15. A strong minimum is attained on $y = (x^2 - x - 1)/2$

3.5 Determination of the second variation of a functional

Definition : Let $I[y(x)]$ be a functional defined on a certain class C of functions $y(x)$. The second variation of the functional $I[y(x)]$ is defined as the value of the second derivative of the functional $I[y(x) + \alpha\, \delta y(x)]$ with respect to the parameter α when $\alpha = 0$.

The second variation of $I[y(x)]$ is denoted by $\delta^2 I$ and is given by

$$\delta^2 1 = \left[\frac{\delta^2}{\delta \alpha^2} [\{y(x) + \alpha\, \delta\, y(x)] \right]_{\alpha = 0}$$

Note 1 : The variation of the functional $I[y(x)]$ is denoted and defined as (refer equation (4) in Art. 1.2 of chapter 1)

$$\delta I = \left[\frac{\partial}{\partial \alpha} I\{y(x) + \alpha\, \delta\, y(x)\}] \right]_{\alpha = 0}$$

Note 2 : In what follows, we shall use the following Leibnit'z rule of differentiation under the sign of integration

$$\frac{d}{d\alpha} \int_{x_1}^{x_2} F(x, \alpha) dx = \int_{x_1}^{x_2} \frac{\partial F(x, \alpha)}{\partial \alpha} dx, \text{ where } x_1 \text{ and } x_2 \text{ are constants.}$$

3.5A Solved examples based on Art. 3.5

Ex. 1. *Find the variation of the following functionals:*

(*i*) $\int_{x_1}^{x_2} [6y(x) + 7] dx$ (*ii*) $\int_a^b (y^2 - y'^2) dx$ (*iii*) $\int_a^{\pi} y' \sin y\, dx$

(*iv*) $I[y(x)] = y^2(0) + \int_0^1 [x\, y(x) + \{y'(x)\}^2] dx$

(*v*) $\int_{x_1}^{x_2} y dx$ (*vii*) $\int_{x_1}^{x_2} \{y(x)\}^2 dx$ (*viii*) $\int_{x_1}^{x_2} (x + y) dx$

Sol. For the sake of simplicity, we shall write y in place of writing $y(x)$.

(*i*) Let $$\mathrm{I}[y] = \int_{x_1}^{x_2} (6y + 7) dx \quad \ldots (1)$$

Let δy be an increment of y. Then, from (1), we have

$$I[y+\alpha\delta y]=\int_{x_1}^{x_2}\{6(y+\alpha\,\delta y)+7\}dx \qquad \ldots (2)$$

Differentiating both sides of (2) partially w. r. t. 'α' and using Leibnitz's rule of differentiating under the sign of integration, we have

$$\frac{\partial}{\partial\alpha}I[y+\alpha\,\delta\,y]=\int_{x_1}^{x_2}\frac{\partial}{\partial\alpha}\{6(y+\alpha\,\delta y)+7\}dx=6\int_{x_1}^{x_2}\delta y\,dx \qquad \ldots (3)$$

$\therefore$ Variation of $I[y(x)]=\delta[1(y)]=\left(\dfrac{\partial}{\partial\alpha}\mathrm{I}[y+\alpha\,\delta\,y\right)_{\alpha=0}=6\int_{x_1}^{x_2}\delta y\,dx,$ by (3)

(*ii*) Let $$I[y]=\int_{x_1}^{x_2}(y^2-y'^2)dx \qquad \ldots (1)$$

Let δy be an increment of y. Then, from (1), we have

$$I[y+\delta y]=\int_{x_1}^{x_2}\{(y+\alpha\,\delta y)^2-(y'+\alpha\ (\delta y)')^2\}dx$$

$\therefore$ $\dfrac{\partial}{\partial\alpha}I[y+\delta y]=\int_{x_1}^{x_2}\dfrac{\partial}{\partial\alpha}\{(y+\alpha\,\delta y)^2-(y'+\alpha(\delta y)')^2\}dx$, using Leibnitz's rule

$$=\int_{x_1}^{x_2}\{2(y+\alpha\,\delta y)\,\delta y-2(y'+\alpha(\delta y)')\ (\delta y)'\}dx$$

$\therefore$ $$\delta I[y]=\left(\frac{\partial}{\partial\alpha}I[y+\delta y]\right)_{\alpha=0}=2\int_{x_1}^{x_2}(y\,\delta y-y'(\delta y)')\,dx$$

(*iii*) Let $$I[y]=\int_0^{\pi}y'\sin y\,dx \qquad \ldots (1)$$

Let δy be an increment of y. Then, from (1), we have

$$I[y+\delta y]=\int_0^{\pi}(y+\alpha\,\delta y)'\sin(y+\alpha\,\delta y)\,dx$$

$\therefore$ $\dfrac{\partial}{\partial\alpha}I[y+\delta y]=\int_0^{\pi}\dfrac{\partial}{\partial\alpha}(y'+\alpha(\delta y)')\sin(y+\alpha\,\delta y)\,dx,$ using Leibnitz's rule

$$=\int_0^{\pi}\{(\delta y)'\sin(y+\alpha\,\delta y)+(y'+\alpha\,(\delta y)')\cos(y+\alpha\,\delta y)\delta y\}dx$$

$\therefore$ $$\delta I[y]=\left[\frac{\partial}{\partial\alpha}I[y+\delta y]\right]_{\alpha=0}=\int_0^{\pi}\{(\delta y)'\sin y+y'\cos y\,\delta y\}\,dx$$

(*iv*) Here $$I[y]=y^2(0)+\int_0^1(xy+y'^2)dx \qquad \ldots (1)$$

Let δy be an increment in y. Then, (1) yields

$$I[y+\alpha\,\delta y]=\{y(0)+\alpha\,\delta y(0)\}^2+\int_0^1[x(y+\alpha\,\delta y)+\{(y'+\alpha(\delta y)'\}^2]dx$$

$\therefore$ $\dfrac{\partial}{\partial\alpha}I[y+\alpha\,\delta y]=2\{y(0)+\alpha\,\delta y(o)\}\delta y(0)+\int_0^1\dfrac{\partial}{\partial\alpha}\left[xy+x\alpha\,\delta y+\{y'+\alpha(\delta y)'\}^2\right]dx$

$$=2y(0)\,\delta y(0)+2\alpha\{\delta y(0)\}^2+\int_0^1\left[x\,\delta y+2\{y'+\alpha(\delta y)'\}(\delta y)'\right]dx$$

$\therefore$ $$I[y]=\left[\frac{\partial}{\partial\alpha}I[y+\delta y]\right]_{\alpha=0}=2y(0)\,\delta y(0)+\int_0^1\{x\,\delta y+2y'\ (\delta y)'\}dx$$

(*v*) **Ans.** $\int_{x_1}^{x_2}\delta y\,dx$ (*vi*) **Ans.** $2\int_{x_1}^{x_2}y\,\delta y\,dx$ (*vi*) **Ans.** $\int_{x_1}^{x_2}\delta y\,dx$

Ex. 2. *Find the second variation of the following functionals* :

(*i*) $\int_0^1 (xy^2 + y'^3)\,dx$ (*ii*) $\int_{x_1}^{x_2} y\,dx$ (*iii*) $\int_{x_1}^{x_2} y^2(x)\,dx$

Sol. Let $I[y] = \int_0^1 (xy^2 + y'^3)\,dx$ ... (1)

Let δy be an increment in y. Then, from (1), we have

$$I[y+\delta y] = \int_0^1 \left[x(y+\alpha\,\delta y)^2 + \{(y+\alpha\,\delta y)'\}^3 \right] dx$$

$$\therefore \quad \frac{\partial}{\partial \alpha} I[y+\delta y] = \int_0^1 \frac{\partial}{\partial \alpha} \left[x(y+\alpha\delta y)^2 + \{y' + \alpha(\delta y)'\}^3 \right] dx \text{, by Leibnitz's rule}$$

or $$\frac{\partial}{\partial \alpha} I[y+\delta y] = \int_0^1 \left[2x(y+\alpha\,\delta y)\delta y + 3\{y' + \alpha(\delta y)'\}^2 (\delta y)' \right] dx$$

$$\therefore \quad \frac{\partial^2}{\partial \alpha^2} I[y+\delta y] = \int_0^1 \frac{d}{\partial \alpha} \left[2x\delta y(y+\alpha\delta y) + 3(\delta y)'\{y' + \alpha(\delta y)'\}^2 \right] dx$$

[Using Leibnitz's rule again]

$$= \int_0^1 \left[2x(\delta y)^2 + 3(\delta y)' \times 2\{y' + \alpha(\delta y)'\}(\delta y)' \right] dx$$

$$= \int_0^1 \left[2x(\delta y)^2 + 6\{(\delta y)'\}^2 \times \{y' + \alpha(\delta y)'\} \right] dx$$

$$\therefore \text{Required second variation} = \delta^2 I[y] = \left[\frac{\partial^2}{\partial \alpha^2} I[y+\delta y] \right]_{\alpha=0} = 2\int_0^1 [x(\delta y)^2 + 3y'\{(\delta y)'\}^2]\,dx$$

(*ii*) **Ans.** 0 (*iii*) **Ans.** $2\int_{x_1}^{x_2} (\delta y)^2\,dx$

Ex. 3. *Evaluate the second variation of the functional* $e^{F(y)}$*, where F(y) is a twice differentiabel funciton.*

Sol. Let $I[y] = e^{F(y)}$ ... (1)

Let δ*y* be an increment in *y*. Then, from (1), $I[y+\alpha\,\delta y] = e^{F(y+\alpha\,\delta y)}$ ... (2)

From (2), $$\frac{\partial}{\partial \alpha} I[y+\alpha\,\delta y] = e^{F(y+\alpha\,\delta y)} F'(y+\alpha\,\delta y) \times \delta y$$

$$\therefore \quad \frac{\partial^2}{\partial \alpha^2} I[y+\alpha\,\delta y] = \delta y[e^{F(y+\alpha\,\delta y)} F'(y+\alpha\,\delta y)\delta y.F'(y+\alpha\,\delta y) + e^{F(y+\alpha\delta y)} F''(y+\alpha\delta y)\delta y]$$

$$= (\delta y)^2 e^{F(y+\alpha\,\delta y)} \left[\{F'(y+\alpha\,\delta y)\}^2 + F''(y+\alpha\,\delta y) \right]$$

$$\Rightarrow \quad \left[\frac{\partial^2}{\partial \alpha^2} I[y+\alpha\,\delta y] \right]_{\alpha=0} = (\delta y)^2 e^{F(y)} \left[\{F'(y)\}^2 + F''(y) \right]$$

$$\Rightarrow \quad \delta^2 I = e^{F(y)} \left[\{F'(y)\delta y\}^2 + F''(y)(\delta y)^2 \right] \quad \text{... (3)}$$

Now, let $J[y] = F(y)$ so that $J[y + \alpha\,\delta y] = F(y+\alpha\,\delta y)$

$$\therefore \quad \frac{\partial}{\partial \alpha} \mathrm{J}[y+\alpha\,\delta y] = F'(y+\alpha\,\delta y)\delta y \quad \text{... (4)}$$

$$(4) \Rightarrow \frac{\partial^2}{\partial \alpha^2} \mathrm{J}[y+\alpha\,\delta y] = F''(y+\alpha\,\delta y)(\delta y)^2 \quad \text{... (5)}$$

$$\therefore \quad \left[\frac{\partial}{\partial \alpha} J[y+\alpha\delta y] \right]_{\alpha=0} = F'(y)\delta y, \quad \text{and} \quad \left[\frac{\partial^2}{\partial \alpha^2} J[y+\alpha\delta y] \right]_{\alpha=0} = F''(y)(\delta y)^2$$

$$\Rightarrow \quad \delta F = F'(y)\,\delta y \quad \text{and} \quad \delta^2 F = F''(y)\,(\delta y)^2 \quad \text{... (6)}$$

Now, (3) and (6) $\Rightarrow$ $\delta^2 I = e^{F(y)} \{(\delta F)^2 + \delta^2 F\}$

Miscellaneous problems on chapter 3

Ex. 1. *The functional* $\int_0^1 (y'^2 + x^3)dx$ *, given y (1) = 1 achieves its*

(a) weak maximum on all its extremals *(b) weak minimum on all its extremals*
(c) weak maximum on some, but not on all its extremals
(d) weak minimum on some, but not on all its extremals **(GATE 2008)**

MISCELLANEOUS PROBLEMS ON PART V OF THIS BOOK

Ex. 1. *Find the variation of the following functional :*

$I[y(x)] = \int_a^b \int_a^b K(x,t)\ y(x)\, y(t)\, dx\, dt,\ a \le x \le b,\ a \le t \le b,\ K(x,t) \in L^2.$ **[Bhopal 2010]**

Sol. Let $\psi(\alpha) = I[y(x) + \alpha\, \delta y]$, where α is a parameter. Then, we have

$$\psi(\alpha) = \int_a^b \int_a^b K(x,t)\ \{y(x) + \alpha\, \delta y(x)\}\{y(t) + \alpha\, \delta y(t)\} dx\, dt$$

$$\Rightarrow \frac{\partial \psi}{\partial \alpha} = \int_a^b \int_a^b K(x,t)\ \{y(x) + \alpha \delta y(x)\}\, \delta y(t) dx dt + \int_a^b \int_a^b K(x,t)\{y(t) + \alpha\, \delta y(t)\}\ \delta y(x)\, dx dt$$

Hence, $$\delta I = \left(\frac{\partial \psi}{\partial \alpha} \right)_{\alpha = 0} = \int_a^b \int_a^b K(x,t)\ y(x) \delta y(t) dx dt + \int_a^b \int_a^b K(x,t) y(t)\, \delta y(x)\, dx dt \quad \ldots (1)$$

Interchanging x and t, we get $\int_a^b \int_a^b K(x,t)\ y(x) \delta y(t) dx dt = \int_a^b \int_a^b K(t,x) y(t)\, \delta y(x)\, dt dx$ ... (2)

Since $K(x, t)$ is symmetric, so $K(t, x) = K(x, t)$. Therefore, (2) reduces to

$$\int_a^b \int_a^b K(x,t)\ y(x) \delta y(t) dx dt = \int_a^b \int_a^b K(x,t) y(t)\, \delta y(x)\, dt dx \quad \ldots (3)$$

Interchanging the order of integration on R.H.S. of (3), we obtain

$$\int_a^b \int_a^b K(x,t)\ y(x)\, \delta y(t) dx dt = \int_a^b \int_a^b K(x,t) y(t)\, \delta y(x)\, dx dt \quad \ldots (4)$$

Using (4) in (1), the required variation δI of the given functional is given by

$$\delta I = 2 \int_a^b \int_a^b K(x,t)\ y(t)\, \delta y(x) dx dt$$

Ex. 2. *Show that the functional* $I[y(x)] = \int_0^1 x^3 \{1 + y^2(x)\}^{1/2} dx$ defined on the set of functions $y(x) \in C[0,1]$ is continuous on the function $y_0(x) = x^2$ in the sense of zeroth order proximity. **(Nagpur 2010)**

Sol. Given $$I[y(x)] = \int_0^1 x^3 \{1 + y^2(x)\}^{1/2} dx \quad \ldots (1)$$

Suppose that $y(x) = x^2 + \varepsilon \phi(x)$, where $\phi(x) \in C\ [0,1]$ and ε is arbitrarily small. Then, using (1), we have

$$I[y(x)] = I[x^2 + \varepsilon \phi(x)\ ] = \int_0^1 x^3 [1 + \{x^2 + \varepsilon\, \phi(x)\}^2\ ]^{1/2} dx \quad \ldots (2)$$

Letting $\varepsilon \to 0$, (2) yields $\lim_{\varepsilon \to 0} I[y(x)] = \int_0^1 x^3 \{1 + (x^2)^2\}^{1/2} dx = I[x^2]$, using (1) showing that the functional (1) is continuous on $y_0(x)$ in the sense of zeroth order proximity.

Ex.3. *Show that the functional* $I[y(x) = \int_0^1 \{y(x) + 2y'(x)\} dx,\ y(x) \in C^1\ [0,1]$ *is continuous on the curve* $y_0(x) = x$ in the sense of first order proximity.

Sol. Given $I[y(x)] = \int_0^1 \{y(x) + 2y'(x)\} dx$, and $y_0(x) = x$ ... (1)

$$\therefore |\ I[y(x)] - I[y_0(x)]\ | = \left| \int_0^1 \{y(x) + 2y'(x)\}\, dx, - \int_0^1 (x + 2) dx \right|, \text{ using (1)}$$

$$= \left| \int_0^1 \left[\{y(x) - x\} + 2\{y'(x) - 1\} \right] dx \right|$$

Hence, $\left|I\left[y(x)\right]-I[y_0(x)]\right| \le \int_0^1 [|\, y(x)-x\,| + 2\,|\,y'(x)-1\,|]dx$ ... (2)

Since $y(x) \in C^1[0,1]$, by definition $y(x)$ and $y'(x)$ are continuous on [0, 1]. Therefore, for given $\varepsilon > 0$, it is possible to choose $\delta > 0$ such that

$$|\, y(x)-x\,| < \delta \qquad \text{and} \qquad |\, y'(x)-1| < \delta,\ x \in [0,1] \qquad \text{... (3)}$$

From (2) and (3), $|\, I[y(x)-I[y_0(x)]\,| < \int_0^1 (\delta + 2\delta)dx = 3\delta = \varepsilon$, if $\delta = \varepsilon/3$

Thus, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\, I[y(x)]-I[y_o(x)]\,| < \varepsilon,\ x \in [0,1]$$

showing that the given functional is continuous on the function $y_0(x) = x$ in the sense of first order proximity.

Ex. 4. Find the extremals of the functional $I[y(x)] = \int_{x_1}^{x_2} x\ \{(dx)^2 + (dy)^2\}^{1/2} dz$

[Ravishankar 2005]

Sol. Given $I[y(x)] = \int_{x_1}^{x_2} x\ \{1 + (dy/dx)^2\}^{1/2} dx = \int_{x_1}^{x_2} x(1+y'^2)^{1/2} dx$ ... (1)

Comparing the given functional with $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx$, we have

$$F(x, y, y') = x(1+y'^2)^{1/2} \qquad \text{... (2)}$$

From (2), $\partial F / \partial y = 0$ and $\partial F / \partial y' = xy'(1+y'^2)^{-1/2}$ ... (3)

Hence, Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ yields $-\dfrac{d}{dx}\left\{xy'(1+y'^2)^{-1/2}\right\} = 0$ ... (4)

Integrating (4), $(xy')/(1+y'^2)^{1/2} = c_1, c_1$ being an arbitrary constant ... (5)

Equation (5) is solved by introducing a parameter t as follows : $dy/dx = y' = \tan t$ ... (6)

Using (6), (5) reduces to $(x \tan t)/\sec t = c_1$ giving $\sin t = c_1/x$ ... (7)

From (7), $\tan t = \dfrac{c_1}{(x^2 - c_1^2)^{1/2}}$ and hence (6) yields $dy = c_1 \dfrac{dx}{(x^2 - c_1^2)^{1/2}}$

Integrating, $y = c_1 \cosh^{-1}(x/c_1) + c_2$ or $x = c_1 \cosh\{(y - c_2)/c_2\}$

where c_2 in another arbitrary constant. Here (8) represents a catenary

Ex. 5. *On the interval [0,1], let y be a twice continuously differentiable function which is an extremal of the functional* $J(y) = \int_0^1 \{(1/x) \times (1+y'^2)^{1/2}\}dx$ *with* $y(0) = 0,\ y(1) = 2$. *Then for some arbitrary constant c, y satisfies.*

(*a*) $y'^2(2 - c^2x^2) = c^2x^2$ (*b*) $y'^2(2 + c^2x^2) = c^2x^2$

(*c*) $y'^2(1 - c^2x^2) = c^2x^2$, (*c*) $y'^2(1 + c^2x^2) = c^2x^2$ **(GATE 2011]**

Sol. Ans. (c). Proceed upto equation (4) of Ex. 16 (b), page 1.23. On squaring both sides of (4) we obtain $y'^2 = c^2x^2(1+y^2)$ so that $y'^2(1 - c^2x^2) = c^2x^2$

Ex. 6. *Assume F to be on twice continuously differentiable. Let J (y) be functional of the form* $\int_0^1 F(x, y')\ dx,\ 0 \le x \le 1$, *defined on the set of all continuously differentiable function y on [0, 1] satisfying* $y(0) = a,\ y(1) = b$. *For some arbitrary constant c, a necessary condition for y to be an extremum of J is*

(a) $\partial F/\partial x = c$ (*b*) $\partial F/\partial y' = c$ (*c*) $\partial F/\partial y = c$ (d) $\partial F/\partial x = 0$ **(GATE 2011)**

Sol. Ans. (*b*). Refer result (18) on page 1.10. Since the given function $F(x, y')$ is independent of y, hence $\partial F/\partial y = 0$. Hence, (18) reduces to

$$0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{giving} \qquad \frac{\partial F}{\partial y'} = C\,.$$

Ex. 7. *The functional* $\int_0^1 \{y'^2 + (y + 2y')\,y'' + kxyy' + y^2\}\,dx$, $y(0) = 0$, $y\,(1) = 1$, $y''(0) = 2$ **,** $y'(1) = 3$ is path independent if *k* equals

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4 **[GATE 2012]**

Sol. Ans. (*b*). Comparing the given functional with $\int_0^1 F\,(x, y,\, y',\, y'')\,dx$, we get

$$F(x, y, y',\, y'') = y'^2 + yy'' + 2y'y'' + kxyy' + y^2 \qquad \text{... (1)}$$

Euler-Poisson equation (refer page 1.45) is given by

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0 \qquad \text{... (2)}$$

From (1), $F_y = \partial F / \partial y = y'' + kxy' + 2y, \quad F_{y''} = \partial F / \partial y' = 2y' + 2y'' + kxy$

and $F_{y''} = \partial F / \partial y'' = y + 2y'$

Hence, (2) reduces to $y'' + kxy' + 2y - (2y'' + 2y''' + ky + kxy') + (y'' + 2y''') = 0$

giving $2y - ky = 0$ so that $k = 2$

Ex. 8. *The functional* $\int_0^1 (y'^2 + 4y^2 + 8ye^x)\,dx$, $y(0) = -(4/3)$, $y(1) = -(4e/3)$ possesses.

(*a*) strong minima on $y = -(e^x/3)$ (*b*) strong minima on $y = -(4e^x/3)$

(*c*) weak maxima on $y = -(e^x/3)$ (d) strong maxima on $y = -(4e^x/3)$ **[GATE 2012]**

Sol. Ans. (*b*). Comparing the given functional with $\int_0^1 F(x, y, y')dx$, we have

$$F(x, y, y') = y'^2 + 4y^2 + 8ye^x \qquad \text{... (1)}$$

From (1), $\frac{\partial F}{\partial y} = 8y + 8e^x, \qquad \frac{\partial F}{\partial y'} = 2y' \qquad$ so that $\frac{d}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 2y''$

Hence Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ yields $8y + 8e^x - 2y'' = 0$

giving $(D^2 - 4)y = 4e^x$, where $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 - 4 =$ giving D = 2, –2

Hence, C.F. of (2) $= c_1 e^{2x} + c_2 e^{-2x}$, c_1 and c_2 being arbitrary constants

P.I. of (2) = $\frac{1}{D^2-4}4e^x = 4\times\frac{1}{1-4}e^x = -\frac{4}{3}e^x$

Hence, solution of (2) is $y = c_1e^{2x} + c_2e^{-2x} - (4/3)\times e^x$ ... (3)

Putting $x = 0$ and $x = 1$ by turn is (3) and using the given boundary conditions namely $y = -(4/3)$ when $x = 0$ and $y = -(4e/3)$ when x = 1, we obtain

$-(4/3) = c_1 + c_2 = -(4/3)$ and $-(4e/3) = c_1e^2 + c_2e^{-2} - (4e/3)$. These give $c_1 = c_2 = 0$

Hence the extremal satisfying the given boundary conditions is $y = -(4/3)e^x$

Since $F(x, y, y')$ is three times differentiable with respect to y' for any y' and $F_{y'y'} = \partial^2F/\partial y'\partial y' = 2 > 0$ for any value of y', it follows (by Legendre condition) that a strong minima is attained on $y = -(4e^x/3)$

Ex.9. *Find the value of the curve* $y = f(x)$ *which corresponds to the extremum of the functional I [y (x)] =* $\int_{x_1}^{x_2} x^n(dy/dx)^n\,dx$ *and passing through two fixed points x =* x_1 *and x =* x_2.

Ans. $y = \{c_1/(1-n)\}x^{1-n} + c_2$

Ex.10. *If f(y) is a given function and* $u = \int_{x_0}^{x_1} f(y)(1+y'^2)^{1/2}\,dx$ *where the values of y at* $x = x_0$ *and* $x = x_1$ *are fixed, then show that the extremal value of u is given by the function* $y = \phi(x)$ *where* $\phi(x)$ *is defined by* $\phi(x) = \int \frac{dy}{[a\{f(y)\}^2 - 1]^{1/2}} = x - b$, *where a and b are constants which are determined by the boundary conditions.*

Ex.11. If $\phi(x, y)$ is a continuous function on the domain $-1 \le x \le 1, -\infty < y < \infty$ then $\phi(x, y) \in C[-1, 1]$. We define the functional $I[y(x)]$ as follows : $I[y(x)] = \int_{-1}^{1} \phi\,(x, y(x))\,dx$. If $\phi(x, y) = x^2/(1+y^2)$, then show that $I[1] = 1/3$ *and* $I[x] = 2 - (\pi/2)$

ADDITIONAL PROBLEMS ON THE ENTIRE BOOK

Ex. 1. *Solve the following boundary value problem by the use of Fourier transform:* $\partial u/\partial t = 2(\partial^2 u/\partial x^2)$, $0 < x < 4$, $t > 0$ *with boundary conditions* $u(0,t) = 0$, $u(4,t) = 0$, $u(x,0) = 3\sin \pi x - 2\sin 5\pi x$ ***(Kanpur 2009)***

Sol. Given $$(1/2) \times (\partial u/\partial t) = \partial^2 u/\partial x^2, \ 0 < x < 4, \ t > 0 \quad \text{... (1)}$$

subject to the boundary conditions : $$u(0,t) = u(4,t) = 0 \quad \text{... (2)}$$

and initial condition: $$u(x,0) = 3\sin \pi x - 2\sin 5\pi x \quad \text{... (3)}$$

Since $u(x,t)$ and $u(4,t)$ are given, taking the finite sine transform of both sides of (1) we have [Refer Art. 2.8 of Chapter 2 in Part IVB of this book.]

$$\frac{1}{2}\int_0^4 \frac{\partial u}{\partial t} \sin \frac{s\pi x}{4}\, dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{s\pi x}{4} dx$$

or $$\frac{1}{2}\frac{d}{dt}\int_0^4 u(x,t) \sin \frac{s\pi x}{4}\, dx = \left[\frac{\partial u}{\partial x} \sin \frac{s\pi x}{4}\right]_0^4 - \frac{s\pi}{4}\int_0^4 \frac{\partial u}{\partial x} \cos \frac{s\pi x}{4} dx \quad \text{... (4)}$$

[Integrating by parts]

Let $$\bar{u}_s = \bar{u}_s\ (s,t) = \int_0^4 u(x,t) \sin \frac{s\pi x}{4} dx \quad \text{...(5)}$$

Then, the equation (4) may be re-rewritten as

$$\frac{1}{2}\frac{d\bar{u}_s}{dt} = -\frac{s\pi}{4}\left\{\left[u(x,t)\cos \frac{s\pi x}{4}\right]_0^4 + \frac{s\pi}{4}\int_0^4 u(x,t)\sin \frac{s\pi x}{4} dx\right\}, \text{ integrating by parts}$$

or $$\frac{1}{2}\frac{d\bar{u}_s}{dt} = -\frac{s\pi}{4}\left[u(4,t)\cos s\pi - u(0,t)\right] - \frac{s^2\pi^2}{16}\bar{u}_s = -\frac{s^2\pi^2}{16}\bar{u}_s, \text{ using (2) and (4)}$$

Re-writting the above equation, $$(1/\bar{u}_s)\, d\bar{u}_s = -(s^2\pi^2/8)dt$$

Integrating, $$\bar{u}_s(s,t) = A_s\ e^{-(s^2\pi^2 t)/8}, \ A_s \text{ being an arbitary constant} \quad \text{... (6)}$$

Now, taking the inverse Fourier sine transform, we obtain

$$u(x,t) = \frac{2}{l}\sum_{s=1}^{\infty} A_s\ e^{-(s^2\pi^2 t)/8} \sin \frac{s\pi x}{l}, \text{ where } l = 4 \text{ for the given problem}$$

Thus, $$u(x,t) = \frac{1}{2}\sum_{s=1}^{\infty} A_s\ e^{-(s^2\pi^2 t)/8} \sin \frac{s\pi x}{4} \quad \text{... (7)}$$

Now, taking finite sine transform of the initial condition (3), we get

$$\bar{u}_s(s,0) = \int_0^4 (3\sin \pi x - 2\sin 5\pi x) \sin \frac{s\pi x}{4} dx \quad \text{... (8)}$$

Putting $\pi x/4 = y$ *and* $(\pi/4)\,dx = dy$, (8) reduces to

$$\bar{u}_s(s,0) = \frac{4}{\pi}\int_0^{\pi} (3\sin 4y - 2\sin 20y)\sin sy\ dy \qquad \ldots (9)$$

From Integral Caalculus, we know that

$$\int_0^{\pi} \sin my\ \sin ny\ dy = \begin{cases} 0, \text{ when } m \neq n \\ \pi/2, \text{ when } m = n \end{cases} \qquad \ldots (10)$$

Using (10), (9) $\Rightarrow \bar{u}_s(s,0) = 0$ when $s \neq 4$ or when $s \neq 20$. Further more,

(9) $\Rightarrow \bar{u}_s(s,0) = 3\times(4/\pi)\times(\pi/2) = 6,$ when $s = 4, \bar{u}_s(s,0) = -2\times(4/\pi)\times 2 = -4,$ when $s = 20$

From (6) and the above results, it follows that when t =0, $A_s = 0$, for all s except when s =4 and s = 20. Also we have $A_4 = 6$ and $A_{20} = -4$. Hence, from (7) the required given by

$$u(x,t) = (1/2)\times\{A_4\ e^{-(16\pi^2 t)/8}\ \sin\ (4\pi x/4) + A_{20}e^{-(400\pi^2 t)/8}\ \sin(20\pi x/4)$$

i.e. $$u(x,t) = 3e^{-2\pi^2 t}\ \sin\ \pi x - 2\ e^{-50\pi^2 t}\ \sin\ 5\pi x.$$

2. Solve the vibrating string problem described by $u_{tt} = c^2 u_{xx}$, $0 < x < l,\ t > 0,$ boundary conditions are $u(0, t) = u(l, t) = 0$ and initial conditions are $u(x, 0) = f(x)$, $u_t(x, 0) = 0$.

[**Hint.** Refer particular case 1 on page 2.38 of part III of this book.] **(Himanchal 2007)**

3. Determine the solution of the one-dimensional wave equation $\partial^2\phi/\partial x^2 - (1/c^2)\times(\partial^2\phi/\partial t^2) = 0,\ < x < a, t > 0$ with c as a constant under the following conditions :

$$\phi(x,0) = \begin{cases} x/b,\ 0 \leq\ x \leq b \\ (a-x)/(a-b),\ b < x \leq a \end{cases}$$

$\phi_t(x,0) = 0,\ 0 < x < a,$ and $\phi(0,t) = \phi(a,t) = 0,\ t > 0$.

Ans. $$\phi(x,t) = \sum_{n=1}^{\infty} \frac{2a^2}{n^2\pi^2 b^2 (a-b)} \sin\ \frac{n\pi b}{a}\ \sin\ \frac{n\pi x}{a}\ \cos\ \frac{c\pi n t}{a}$$

[**Hint.** Proceed as in Art. 2.10C, page 2.39 of part III of this book]

Ex. 4. *Show that u(x, y) = (1/n²) × sinh ny sin nx is a solution of the Laplace equation* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ subject to $u(x, 0) = 0$, $(\partial u/\partial y)_{y=0} = (1/n) \times \sin nx$, where n is a positive integer. *(Himanchal 2007, 09)*

Sol. Given $$\partial^2 u/\partial x^2 + \partial^2/\partial y^2 = 0 \quad \text{... (1)}$$

subject to $$u(x, 0) = 0 \quad \text{...(2}a\text{)}$$

and $$(\partial u/\partial y)_{y=0} = (1/n) \times \sin nx \quad \text{...(2}b\text{)}$$

Suppose that (1) has solutions of the form $$u(x, y) = X(x)\, Y(y), \quad \text{...(3)}$$

where X is a function x alone and Y that of y alone.

Substituting the values of u given by (3) in (1), we have

$$X''Y + X\,Y'' = 0 \quad \text{so that} \quad X''/X = -(Y''/Y) \quad \text{...(4)}$$

Since x and y are independent variables, (4) is true only if its each side is a constant ($= -\mu^2$, say). Then, (4) yields

$X'' + \mu^2 X = 0$ and $Y'' - \mu^2 y = 0$, whose solutions are

$$X(x) = A \cos \mu x + B \sin x\mu, \quad A \text{ and } B \text{ being arbitrary constants} \quad \text{...(5)}$$

and $$Y(y) = C \cosh \mu y + D \sinh \mu y,\ C \text{ and } D \text{ being arbitrary constants} \quad \text{...(6)}$$

Putting $y = 0$ in (3) and using (2a), we get $0 = X(x)\, Y(0)$ so that $Y(0) = 0$.

Putting $y = 0$ in (6) and using the fact that $Y(0) = 0$, (6) yields $C = 0$.

Hence, (6) reduces to $$Y(y) = D \sinh \mu y \quad \text{...(7)}$$

From (3), (5) and (7), $$u(x, y) = (E \cos \mu x + F \sin \mu x) \sinh \mu y, \quad \text{...(8)}$$

where $E(= AD)$ and $F(= BD)$ are new arbitrary constants.

From (8), $$\partial u/\partial y = \mu(E \cos \mu x + F \sin \mu x) \cosh \mu y \quad \text{...(9)}$$

Putting $y = 0$ in (9) and using (2b), we obtain

$(1/n) \times \sin nx = E\mu \cos \mu x + F\mu \sin \mu x \Rightarrow E = 0,\ \mu = n$ and $F = 1/n^2$.

Hence, from (8), $$u(x, y) = (1/n^2) \times \sinh ny \sin nx.$$

5. A thin rectangular homogenous thermally conducting plate lies in the xy-plane defined by $0 \le x \le a$, $0 \le y \le b$. The edge $y = 0$ is held at the temperature $Tx\,(x - a)$, where T is a constant, while the remaining edges are held at 0°. The other faces are insulated and no internal sources and sinks are present. Show that the steady state temperature $u\,(x, y)$ inside the plate is given by

$$u(x, y) = \frac{8Ta^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \operatorname{cosech}\left\{\frac{(2m-1)\,\pi b}{a}\right\} \sin\left\{\frac{(2m-1)\,\pi x}{a}\right\} \sinh\left\{\frac{(2m-1)\,(y-b)\pi}{a}\right\}$$

6. Solve $\nabla^2 u = 0$, $0 \le x \le a$, $0 \le y \le b$ satisfying the boundary conditions: $u\,(0, y) = u\,(x, 0) = u\,(x, b) = 0$, $u_x\,(a, y) = T \sin^3(\pi y/a)$, T being a constant.

Ans. $$u(x, y) = \frac{bT}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^3 \frac{\pi y}{a} \operatorname{sech} \frac{n\pi a}{b} \sinh \frac{n\pi x}{a}$$

7. Show that $T(x, t) = (1/\sqrt{4\pi\alpha t}) \times \exp\{(x - \xi)^2/(4\alpha t)\}$ is a solution of $\partial^2 T/\partial x^2 = (1/\alpha) \times (\partial T/\partial t)$, $-\infty < x < \infty, t > 0$ and ξ is an arbitrary constant. *(Himachal 2007)*

8. The ends A and B of a rod, 10 cm in length, are kept at temperature 0°C and 100°C until the steady state condition prevails. Suddenly the temperature at the end A is increased to 20° C, and the end B is decreased to 60°C. Show that the temperature distribution $u\,(x, t)$ in the rod at time t is given by

$$u(x, t) = 4x + 20 - \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left\{4(-1)^n - 1\right\} \exp\left\{-(kn^2\pi^2/100)t\right\} \sin \frac{n\pi x}{10}$$

9. A conducting bar of uniform cross-section lies along the x– axis with ends at $x = 0$ and $x = l$. It is kept initially at temperature 0° and its lateral surface is insulated. There are no heat sources in the bar. The end $x = 0$ is kept at 0°, and heat is suddenly applied at the end $x = l$ so that there is a constant flux Q_o at $x = l$. Show that the temperature distribution $u(x, t)$ in the bar for $t > 0$ is given by $$u(x,t) = Q_o x + \frac{8lQ_o}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)^2} \exp\left\{-\frac{k(2m-1)^2\pi^2 t}{4l^2}\right\} \sin\frac{(2m-1)\pi x}{2l}$$

10. A uniform rod of length l whose surface is thermally insulated is initially at temperature u_o. At time $t = 0$, one end is suddenly cooled to $u = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Show that the temperature distribution $u(x,t)$ is given by $$u(x,t) = \frac{4u_o}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \exp\left\{-\frac{k(2m-1)^2\pi^2 t}{4l^2}\right\} \sin\frac{(2m-1)\pi x}{2l}$$

11. Solve $\partial u/\partial t = \partial^2 u/\partial x^2$ satisfying the conditions : (*i*) $u = 0$ when $x = 0$ and I for all t (*ii*) $u = 2x$, if $0 \le x \le 1/2$ and $u = 2(1 - x)$, if $1/2 \le x \le 1$ when $t = 0$. *(Himachal 2007)*

Ans. $u(x,t) = (8/\pi^2) \times \sum_{n=1}^{\infty} \left\{(1/n^2) \times \sin(n\pi/2) \sin(n\pi x)\, e^{-n^2\pi^2 t}\right\}$

12. Show that the solution of the wave equation $u_{tt} = c^2\, u_{xx}$ under the following conditions (*i*) $u(0, t) = u(2, t) = 0$ (*ii*) $u(x, 0) = \sin^3(\pi x/2)$ (*iii*) $u_t(x, 0) = 0$ is given by $$u(x, t) = (3/4) \times \sin(\pi x/2) \cos(\pi ct/2) - (1/4) \times \sin(3\pi x/2) \cos(3\pi ct/2).$$

13. A uniform string of line density ρ is stretched to tension ρc^2 and executes a small transverse vibration in a plane through the undisturbed line of string. The ends $x = 0, l$ of the string are fixed. The string is at rest, with the point $x = b$ drawn aside through a small distance ε and released at time $t = 0$. Show that the expression for the displacement $u(x, t)$ is given by $$u(x,t) = \frac{2\varepsilon l^2}{b\pi^2(l-b)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi b}{l} \sin\frac{n\pi x}{l} \cos\frac{nc\pi t}{l}$$

14. Show that the solution of the boundary value problem described by $u_{tt} - c^2\, u_{xx} = 0$, $0 \le x \le l$, $t \ge 0$, $u(0, t) = u(l, t) = 0$, $t \ge 0$, $u(x, 0) = 10 \sin(\pi x/l)$, $0 \le x \le l$ and $u_t(x, 0) = 0$, $0 \le x \le l$ is given by $10 \cos(\pi c t/l) \sin(\pi x/l)$ *(Himachal 2007)*

15. Show that solution of one-dimensional wave equation $u_{tt} = c^2\, u_{xx}$, $0 \le x \le \pi$, $t \ge 0$ subject to $u = 0$ when $x = 0$ and $x = \pi$. while $u_t = 0$ when $t = 0$ and $u(x, 0) = x$, $0 < x < \pi$ is given by $$u(x, t) = 2\sum_{n=1}^{\infty} \{(-1)^{n+1}/n\} \sin(nx) \cos(nct)$$

16. Show that the solution of $u_{tt} = c^2\, u_{xx}$, $0 \le x \le l$, $t \ge 0$ subject to $u(0, t) = u(l, t) = 0$ for all t, $u(x,0) = 0$ and $u_t(x, 0) = b \sin^3(\pi x/l)$, $0 \le x \le l$ is given by

$u(x, t) = (bl/12c\pi) \times \{9 \sin(\pi x/l) \sin(c\pi t/l) - \sin(3\pi x/l) \sin(3c\pi t/l)\}$

17. *Derive Laplace equation $\nabla^2 V = 0$, where V is the potential function.* *(Himachal 2007)*

Sol. Let two particles of masses, m and m_1 be situated at points Q and P at a distance r apart. According to Newton's law of gravitation, the magnitude of the force F, proportional to the product of their masses and inversely proportional to the square of the distance, between them is given by $$F = G \times (m\, m_1/r^2), \qquad ...(1)$$ where G is the gravitational constant. Let $\overrightarrow{PQ} = \boldsymbol{r}$. Suppose that $m = 1$ and $G = 1$. Then the force at Q due to the mass at P is given by

$$\boldsymbol{F} = -(m_1 \boldsymbol{r}) / r^3 = \nabla(m_1 / r), \qquad ...(2)$$

which is known as the intensity of the gravitational force.

Now, if a particle of unit mass moves under the attraction of a particle of man m_1 at P from infinity up to Q, then the work done by the force $\boldsymbol{F}$ is given by

$$\int_{\infty}^{r} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\sigma}^{r} \nabla (m_1 / r) \cdot dr = \frac{m_1}{r} \qquad ...(3)$$

This is defined as the potential V at Q due to particle of mass m_1 at P and is denoted by

$$V = -(m_1/r) \qquad ...(4)$$

From (2) and (4), we have $$\boldsymbol{F} = -\nabla V \qquad ... (5)$$

Again, consider a system of particles of masses $m_1, m_2, \ldots, m_n$ which are situated at points $P_1, P_2, P_3, ..., P_n$ whose distances, measured from a point Q, are $r_1, r_2, \ldots, r_n$ respectively, Then the force of attraction $\boldsymbol{F}$ per unit mass at Q due to the system of masses is given by

$$\boldsymbol{F} = \sum_{i=1}^{n} \nabla (m_i / r_i) = \nabla \sum_{I=1}^{n} (m_i / r_i) \qquad ... (6)$$

Now, as before, the work done by the force **F** given by (6) is given by

$$\int_{\infty}^{r} \boldsymbol{F} \cdot d\boldsymbol{r} = \sum_{i=1}^{n} (m_i / r_i) = -\mathbf{V}$$

Hence, $$\nabla^2 V = -\nabla^2 \sum_{i=1}^{n} (m_i / r_i) = \sum_{i=1}^{n} \nabla^2 (m_i / r_i) = 0, \text{ since } r_i \neq 0 \qquad ...(7)$$

where $\nabla^2 V = \text{div grad} = \partial^2 / x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$, which is known as the *Laplacian operator*.

In case the matter of density ρ is continuously described in a volume τ, we have

$$V(x, y, z) = \iiint_{\tau} \frac{1}{r} \rho(\xi, \eta, \zeta)\, d\tau,$$

where $r = \{(x-\xi)^2 + (y-\eta)^2 + (z-\varsigma)^2\}^{1/2}$ and Q is outside the body. Then, it can be shown that $\nabla^2 V = 0$, which is the *Laplace equation.*

18. *Derive Poisson's equation* $\nabla^2 V = -4\pi G\rho$, *where the symbols have their usual meaning.* *(Himachal 2007, 08, 09)*

Sol. Let S be a closed surface consisting of particles of masses $m_1, m_2, ..., m_n$. Let P be an arbitrary point on S and let $\sum_{i=1}^{n} m_i = M$ be the total mass inside S. Again, let $g_1, g_2, ..., g_n$ be the gravitational fields at P due to the presence of masses $m_1, m_2, ..., m_n$ respectively within S. Also, assume that $\sum_{i=1}^{n} g_i = g$, the entire gravity field at P. Then, by Gauss law, we obtain

$$\iint_{S} \boldsymbol{g} \cdot d\boldsymbol{S} = -4\pi GM, \qquad ... (1)$$

where $M = \iint_{\tau} \rho\, d\tau$, ρ being the mass density function and τ being the volume in which the masses are distributed throughout. Since the gravity field is conservative, we have

$$\boldsymbol{g} = \nabla V, \qquad ... (2)$$

where V is a scalar potential. Now, by Gauss divergence theorem, we have

$$\iiint_{S} \boldsymbol{g} \cdot d\boldsymbol{S} = \iiint_{\tau} \nabla \cdot \boldsymbol{g}\, d\tau \qquad ...(3)$$

Again, rewriting (1) we have $\iint_S \boldsymbol{g}\cdot d\boldsymbol{S} = -4\pi G\iiint_\tau \rho\, d\tau$ ... (4)

From (3) and (4), $\iiint_\tau(\nabla\cdot\boldsymbol{g} + 4\pi G\rho)\, d\tau = 0 \Rightarrow \nabla\cdot\boldsymbol{g} = -4\pi G\rho \Rightarrow \nabla\cdot\nabla V = -4\pi G\rho$ by (2)

Thus, $\nabla^2 V = -4\pi G\rho$, which is known as *Poisson's equation.*

19. (*a*). *If $\phi(x)$ is bounded for all real values of x, show that*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\phi(\xi)\, e^{-\{(x-\xi)^2/4kt\}}\, d\xi$$

is a solution of $u_t = k\, u_{xx}$ such that $u(x, 0) = \phi(x)$. *(Himachal 2007, 08)*

(*b*) *In a one-dimensional infinite solid , $-\infty < x < \infty$, the surface $a < x < b$ is initially maintained at temperature u_o and at zero temperature everywhere outside the surface. show* that

$$u(x,t) = (u_o/2)\times\left[\text{erf}\left\{(b-x)/\sqrt{4kt}\right\} - erf\left\{(a-x)/\sqrt{4kt}\right\}\right]$$

where erf is an error function. *(Himachal 2009)*

Sol. Part (*a*). Given $\partial u/\partial t = k(\partial^2 x/\partial x^2),\quad -\infty < x < \infty,\quad t > 0$ ... (1)

with initial condition $u(x,0) = \phi(x),\quad -\infty < x < \infty$ ... (2)

Suppose that (1) has solutions of the form $u(x, t) = X(x)\ T(t)$ ... (3)

Substituting (4) into (1), we have $X''/X = T'/kT = \mu$, say

giving $X'' - \mu X = 0$ and $T' = k\mu T$ ... (4)

Solving $T' = k\mu T$, we have $T = C\, e^{k\mu t}$, C being an arbitrary constant ...(5)

If $\mu > 0$, then from (3) and (5), we see that $u(x, t)$ increases exponentially with time. In view of actual physical considerations, it is reasonable to assume that $\phi(x) \to 0$ as $|x| \to \infty$,while $1\ u\ (x, t)\ | < M$ as $|x| \to \infty$. It follows that in order that $u(x, t)$ remains bounded we must take $\mu = -\lambda^2$, where $\lambda > 0$. Then , from (4), we have $X'' + \lambda^2 X = 0$ whose general solution is given by

$X = C_1 \cos\lambda x + C_2 \sin\lambda x$, C_1 and C_2 being arbitrary constants ... (6)

Again, setting $\mu = -\lambda^2$ in (5) yields $T = Ce^{-\lambda^2 kt}$ ... (7)

From (3), (6) and (7), $u(x, t, \lambda) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\lambda^2 kt}$, ... (8)

where A (= C C_1) and B (= C C_2) are new arbitrary constants. (8) is a solution of (1) Now, $f(x)$ need not always be periodic and hence we shall use Fourier integral instead of Fourtier series in the given problem. Since A and B are arbitrary, we assume them to be functions of λ and so we write A = A (λ) and B = B (λ). Using the superposition principle, this summation of all product solutions will lead us to the relation

$$u(x,t) = \int_0^\infty u(x,t,\lambda)\, d\lambda = \int_0^\infty \left\{A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right\} e^{-\lambda^2 kt}\, d\lambda, \quad \text{... (9)}$$

which is the solution of (1). Putting $t = 0$ in (9) and using the initial condition (2), we have

$$\phi(x) = \int_o^\infty \left\{A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right\} d\lambda \quad \text{... (10)}$$

Using Fourier integral theorem, we have

$$\phi(x) = \frac{1}{\pi}\int_o^\infty\left[\int_{-\infty}^\infty \phi(\xi)\cos\lambda(x-\xi)\, d\xi\right] d\lambda = \frac{1}{\pi}\int_o^\infty\left[\int_{-\infty}^\infty \phi(\xi)\,(\cos\lambda x\cos\lambda\xi + \sin\lambda x\sin\lambda\xi)\, d\xi\right] d\lambda$$

or $$\phi(x) = \frac{1}{\pi}\int_o^\infty\left[\cos\lambda x\int_{-\infty}^\infty \phi(\xi)\cos\lambda\xi\, d\xi + \sin\lambda x\int_{-\infty}^\infty \phi(\xi)\sin\lambda\xi\, d\xi\right] d\lambda \quad \text{... (11)}$$

Let $A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(\xi)\cos\lambda\xi\, d\xi$ and $B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(\xi)\sin\lambda\xi\, d\xi$ ... (12)

Using (12), (11) yields $\phi(x) = \int_0^{\infty} \{A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\}\, d\lambda$ ... (13)

Comparing (10) and (13), it follows that (10) may be re-written as

$$\phi(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}\phi(\xi)\cos\lambda(x-\xi)\,d\xi\right]d\lambda \quad \text{... (14)}$$

Hence, (9) reduce to $u(x,t) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}\phi(\xi)\cos\lambda(x-\xi)e^{-\lambda^2 kt}d\xi\right]d\lambda$ ... (15)

Suppose the conditions for the interchange of orders of integration are satisfied.

Then, (15) yields $u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\phi(\xi)\left[\int_0^{\infty}\cos\lambda(x-\xi)e^{-\lambda^2 kt}d\lambda\right]d\xi$ ... (16)

We have a standard integral: $\int_0^{\infty}\cos(2by)e^{-y^2}dy = \frac{\sqrt{\pi}}{2}e^{-b^2}$ ... (17)

Put $y = \lambda\sqrt{kt}$, $dy = \sqrt{kt}\,d\lambda$ and $b = (x-\xi)/\sqrt{4kt}$ ... (18)

Using (18), (17) yields $\int_0^{\infty}\cos\lambda(x-\xi)e^{-\lambda^2 kt}d\lambda = \left(\sqrt{\pi}/\sqrt{4kt}\right)\times e^{-\{(x-\xi)^2/4kt\}}$ ...(19)

Using (19), (16) yields $u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty}\phi(\xi)\, e^{-\{(x-\xi)^2\}/4kt\}}\, d\xi$

Part (*b*). Here we are to solve $\partial u/\partial t = k(\partial^2 u/\partial x^2)$, $-\infty < x < \infty,\ t > 0$ ...(1)

with initial conditions: $u(x,0) = u_o$ for $a < x < b$ while $u(x,0) = 0$ for $x \le a$ or $x \ge b$ ...(2)

Let $u(x, 0) = \phi(x)$, $-\infty < x < \infty$ ...(3)

Comparing (2) and (3), here $\phi(x) = u_o$ for $a < x < b$ while $\phi(x) = 0$ for $x \le a$ or $x \ge b$. ... (4)

Proceeding as in part (*a*), the general solution of (1) and (2) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\phi(\xi)\, e^{-\{(x-\xi)^2/4kt\}}d\xi \quad \text{...(5)}$$

Using (4), (5) yields $u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_a^b e^{-\{(x-\xi)^2/4kt\}}d\xi$ ...(6)

Let $\eta = -(x-\xi)/\sqrt{4kt}$ so that $d\xi = \sqrt{4kt}\, d\eta$. Then, (6) reduces to

$$u(x,t) = \frac{u_o}{\sqrt{\pi}}\int_{(a-x)/\sqrt{4kt}}^{(b-x)/\sqrt{4kt}} e^{-\eta^2}d\eta = \frac{u_o}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^{(b-x)/\sqrt{4kt}} e^{-\eta^2}d\eta - \frac{2}{\sqrt{\pi}}\int_0^{(a-x)/\sqrt{4ht}} e^{-\eta^2}d\eta\right]$$

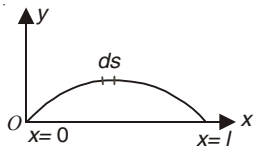
or $u(x,t) = \frac{u_0}{2}\left[erf\left(\frac{b-x}{\sqrt{4kt}}\right) - erf\left(\frac{a-x}{\sqrt{4kt}}\right)\right]$, as $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}d\eta$

20. *Prove that the total energy of a string, which is fixed at the points x = 0, x = l and executing small transverse vibrations, is given by (T/2)* $\times \int_0^l \{(\partial y/\partial x)^2 + (1/c^2)\times(\partial y/\partial t)^2\}dx$, *where*

$c^2 = T/\rho$, ρ is the uniform linear density and T is the tension. Show that if $y = f(x - ct)$, $0 \le x \le l$, then the energy of wave is equally divided between potential energy and kinetic energy.

(Himanchal 2007)

Sol. The kinetic energy (K.E) of an element *dx* of the string executing small transverse vibrations is given by $(1/2)\times(\rho\, dx)\times(\partial y/\partial t)^2$ as shown in the following figure.



Hence, the total K.E. $= \dfrac{T}{2}\int_0^l \dfrac{1}{c^2}\left(\dfrac{\partial y}{\partial t}\right)^2 dx$, as $\rho = \dfrac{T}{c^2}$ ....(1)

From calculus, $ds = \left\{1+\left(dy/dx\right)^2\right\}^{1/2} dx = \left\{1+(1/2)\times\left(dy/dx\right)^2\right\} dx$

[On expanding by Binomial Theorem]

Therefore, the stretch in the string is given by $ds - dx = (1/2)\times(\partial y/\partial x)^2$

Then, the potential energy (P.E.) of this element is given by $(T/2)\times(\partial y/\partial x)^2 dx$.

Hence, we have total $P.E. = \dfrac{T}{2}\int_0^l \left(\dfrac{\partial y}{\partial x}\right)^2 dx$ ... (2)

Adding (1) and (2), the required total energy $= \dfrac{T}{2}\int_0^l \left\{\dfrac{1}{c^2}\left(\dfrac{\partial y}{\partial t}\right)^2 + \left(\dfrac{\partial y}{\partial x}\right)^2\right\} dx$ ...(3)

Particular case. Given $y = f(x - ct)$ ...(4)

From (4), $\partial y/\partial t = -c\, f'(x - ct)$ and $\partial y/\partial x = f'(x - ct)$ ... (5)

Using (5), from (1) and (2), we have

total K.E. $= \dfrac{1}{2}T\int_0^l [f'(x-ct)]^2 dx$ and total $P.E. = \dfrac{1}{2}T\int_0^l \left[f'(x-ct)\right]^2 dx$,

showing that the total kinetic energy = total potential energy, as required.

21. *A string of length l is released from rest in the position $y = f(x)$. Show that the total energy of the string is $(\pi^2 T/4l)\times\sum_{s=1}^{\infty} s^2 k_s^2$ where $k_s = (2/l)\times\int_0^l f(x)\sin(s\pi x/l)\, dx$, T being tension in the string. If the mid-point of a string is pulled aside through a small distance and then released, show that in the subsequent motion the fundamental mode contributes $8/\pi^2$ of the total energy.*

(Himachal 2007.08)

22. *(a) Slove the following Poisson equation: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 2$ subject to the boundary conditions $u(0, y) = u(5, y) = u(x, 0) = u(x, 4) = 0$.*

(b) Find the stress function ψ which satisfies the Poisson equation $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = -2$, $0 \le x \le 1$, $0 \le y \le 1$ with the boundary conditions $\psi = 0$ on the sides $x = 0$, $x = 1$, $y = 0$ and on $y = 1$. (*Himachal 2007, 08*)

Sol. (*a*) Given $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 2, \quad 0 \le x \le 5, \quad 0 \le y \le 4$ ... (1)

with conditions $u(0, y) = u(5, y) = u(x, 0) = u(x, 4) = 0$ ... (2)

Let the required solution be of the form $u = v + w$, ... (3)

where v in a particular integral of (1) and w is the solution of the corresponding homogeneous Laplace equation. Accordingly, let

$$\nabla^2 v = \partial^2 v / \partial x^2 + \partial^2 v / \partial y^2 = 2 \quad \text{... (4)}$$

and $$\nabla^2 w = \partial^2 w / \partial x^2 + \partial^2 w / \partial y^2 = 0 \quad \text{...(5)}$$

Let v be of the form $v(x, y) = a + bx + cy + dx^2 + exy + fy^2$ ...(6)

Using (6), (4) yields $2d + 2f = 2$ or $d + f = 1$ ... (7)

Let $f = 0$ so that $d = 1$. Choosing the remaining coefficients arbitrarily, we take

$$v(x, y) = -5x + x^2 \quad \text{... (8)}$$

in order that v satisfies the boundary conditions: $v(o, y) = v(5, y) = 0$ ...(9)

Now, we proceed to find w from the following boundary value problem:

$$\partial^2 w / \partial x^2 + \partial^2 w / \partial y^2 = 0, \quad 0 \le x \le 5, \quad 0 \le y \le 4 \quad \text{...(10)}$$

satisfying the boundary conditions (using (3) and (8) and (9))

$$w(0, y) = -v(0, y) = 0, \quad w(5, y) = -v(5, y) = 0 \quad \text{... (11)}$$

$$w(x, 0) = -v(x, 0) = -(-5x + x^2) = 5x - x^2, \quad w(x, 4) = -v(x, 4) = -(-5x + x^2) = 5x - x^2 \quad \text{...(12)}$$

Suppose that (10) has solutions of the form $w(x, y) = X(x)\, Y(y)$ ...(13)

Then (10) reduces to $X''/X = -(Y''/Y) = -\lambda^2$, say

so that $$X'' + \lambda^2 X = 0 \quad \text{... (14)}$$

$$Y'' - \lambda^2 Y = 0 \quad \text{... (15)}$$

Using boundary conditions (10), (13) yields $X(0) = X(5) = 0$ ... (16)

Now, the general solutions of (14) is given by

$X(x) = A \cos \lambda x + B \sin \lambda x$, A and B being orbitrary constants. ...(17)

Putting $x = 0$ and $x = 5$ in (17) and using (16), we get $A = 0$ and $0 = B \sin 5\lambda$.

Since $B \neq 0$ for a non-trivial solution, we have

$\sin 5\lambda = 0 \Rightarrow 5\lambda = n\pi \Rightarrow \lambda = n\pi/5, \quad n = 1, 2, 3, \ldots$ ... (18)

Hence, solutions of (14) are given by $X(x) = B_n \sin(n\pi x/5), \quad n = 1, 2, 3, \ldots$ ... (19)

Using (18), (15) yields $Y'' - (n^2 \pi^2 / 25) Y = 0$ whose general solution is given by

$Y(y) = C_n \cosh(n\pi y/5) + D_n \sinh(n\pi y/5)$, C_n and D_n being arbitrary constants ...(20)

From (13), (19) and (20), $w(x, y) = \sin(n\pi x/5)\{E_n \cosh(n\pi y/5) + F_n \sinh(n\pi y/5)\}$ is a solution of (10). By the superposition principle, the general solution is given by

$$w(x, y) = \sum_{n=1}^{\infty} \sin(n\pi x/5)\{E_n \cosh(n\pi y/5) + F_n \sinh(n\pi y/5)\} \quad \text{... (21)}$$

Putting $y = 0$ in (21) and using boundary condition (11), we have

$5x - x^2 = \sum_{n=1}^{\infty} E_n \sin(n\pi x/5)$, which is Fourier sine series and hence, we have

$$E_n = \frac{2}{5}\int_0^5 (5x - x^2)\sin(n\pi x/5)\, dx \quad \text{...(22)}$$

$$= \frac{2}{5}\left[(5x - x^2)\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right) - (5 - 2x)\left(-\frac{25}{n^2\pi^2}\sin\frac{n\pi x}{5}\right) + (-2)\left(\frac{125}{n^3\pi^3}\cos\frac{n\pi x}{5}\right)\right]_0^5$$

[On using chain rule of integration by parts]

$$= \frac{2}{5}\left[-\frac{2\times 125}{\pi^3 n^3}\cos n\pi + \frac{2}{n^3}\times\frac{125}{\pi^3}\right] = \frac{100}{\pi^3}\left[\frac{1}{n^3} - \frac{\cos n\pi}{n^3}\right] = \frac{100}{\pi^3}\times\frac{1-(-1)^n}{n^3}$$

Thus, $$E_n = \begin{cases} 200/\pi^3(2m-1)^3, \text{if } n = 2m-1, \text{ where } m = 1,2,3,..... \\ 0, \text{if } n = 2m, \text{ where } m = 1,2,3,..... \end{cases} \quad \text{... (22)}'$$

Next, putting $y = 4$ in (21) and using boundary condition (11), we have

$$5x - x^2 = \sum_{n=1}^{\infty}[E_n \cosh(4n\pi/5) + F_n \sinh(4n\pi/5)]\sin(n\pi x/5), \text{ which is Fourier sine series.}$$

Hence, $$E_n \cosh\frac{4n\pi}{5} + F_n \sinh\frac{4n\pi}{5} = \frac{2}{5}\int_0^5 (5x - x^2)\sin\frac{n\pi x}{5}\,dx = E_n \text{, using (22)}$$

giving $$F_n = E_n[1 - \cosh(4n\pi/5)]/\sinh(4n\pi/5) \quad \text{... (23)}$$

Substituting the above values of F_n in (21), we have

$$w(x,y) = \sum_{n=1}^{\infty}\left[E_n \cosh\frac{n\pi y}{5} + \frac{E_n\{1-\cosh(4n\pi/5)\}}{\sinh(4n\pi/5)}\sinh\frac{n\pi y}{5}\right]\sin\frac{n\pi x}{5}$$

$$= \sum_{n=1}^{\infty} E_n \frac{\cosh(n\pi y/5)\sinh(4n\pi/5) + \sinh(n\pi y/5) - \cosh(4n\pi/5)\sinh(n\pi y/5)}{\sinh(4n\pi/5)}\sin\frac{n\pi x}{5}$$

$$= \sum_{n=1}^{\infty} E_n \frac{\sinh\{(n\pi(4-y)/5\} - \sinh(n\pi y/5)}{\sinh(4n\pi/5)}\sin\frac{n\pi x}{5}$$

$$= \sum_{m=1}^{\infty}\frac{200}{\pi^2(2m-1)^3}\frac{\sinh\{(2m-1)(4-y)\pi/5\} + \sinh\{(2m-1)\pi y/5\}}{\sinh\{4\pi(2m-1)/5\}}\sin\frac{(2m-1)\pi x}{5} \quad \text{... (24)}$$

[on putting value of E_n given by (22)]

From (3), (8) and (24), the required solution of (1) is given by

$$u(x,y) = x^2 - 5x + \frac{200}{\pi^3}\sum_{m=1}^{\infty}\frac{\sinh\{(2m-1)(4-y)/5\} + \sinh\{(2m-1)\pi y/5\}}{(2m-1)^3\sinh\{4\pi(2m-1)/5\}}\sin\frac{(2m-1)\pi x}{5}$$

Part (*b*). Do like part (*a*). Take $\psi = v + w$. Then $v(x,y) = x - x^2$ satisfies $\nabla^2 v = -2$ and boundary conditions $v = 0$ when $x = 0$ and $v = 0$ when $x = 1$. Then , find w satisfying $\nabla^2 w = 0$ and boundary conditions : $w(0, y) = w(1, y) = 0$, $0 \le y \le 1$ and $w(x, 0) = w(x, 1) = x^2 - 1$ for $0 \le x \le 1$. Now, proceed as before and show that the required solution is given by

$$\psi(x,y) = x - x^2 - \frac{8}{\pi^3}\sum_{m=1}^{\infty}\frac{\sin\{(2m-1)\pi x\}}{(2m-1)^3}\left[\frac{\sinh\{(2m-1)\pi y\} + \sinh\{(2m-1)(1-y)\pi\}}{\sinh\{(2m-1)\pi\}}\right]$$

23. *Solve* $\nabla^2 u = 0, 0 \le r \le 10, 0 \le \theta \le \pi$ *satisfying the boundary conditions:*

$u(10,\theta) = (400/\pi)\times(\pi\theta - \theta^2)$, $u(r,0) = u(r,\pi) = 0$ *and* $u(0, \theta)$ *is finite.*

Ans. $$u(r,\theta) = \frac{3200}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\left(\frac{r}{10}\right)^{2m-1}\sin\{(2m-1)\theta\}$$

24. (*a*) *Discuss interior Dirichlet problem for a circle.*

(*b*) *Discuss exterior Dirichlet problem for a circle.*

(*c*) *Discuss interior Neumann problem for a circle.*

Sol. (*a*). The interior Dirichlet problem for a circle can be described as follows:

$$\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial\theta^2) = 0, \quad 0 \le r < a, \quad 0 \le \theta \le 2\pi \qquad \ldots (1)$$

subject to the boundary condition: $u(a,\theta) = f(\theta), \qquad 0 \le \theta \le 2\pi.$ ... (2)

Then we wish to find the values of u at any in the interior of the circle $r = a$. Proceeding as in Ex. 9 (a) , page 3.16 upto the equation (16) of part III of this book, the required solution is

$$u(r,\theta) = (E_0/2) + \sum_{n=1}^{\infty} r^n \ (E_n \cos n\theta + F_n \sin n\theta). \qquad \ldots (16)$$

Putting $r = a$ in (16) and using boundary condition (2), we have

$$f(\theta) = (E_0/2) + \sum_{n=1}^{\infty} (E_n\ a^n \cos n\theta + F_n\ a^n \sin n\theta), \qquad \ldots(17)$$

which is usual expansion of $f(\theta)$ as Fourier series in $(0, 2\pi)$. Hence, we have

and

$$\left.\begin{aligned} E_0 &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, d\theta = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\, d\phi \\ E_n &= \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta = \frac{1}{a^n\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi \\ F_n &= \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta = \frac{1}{a^n\pi}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi \end{aligned}\right\} \qquad \ldots (18)$$

Hence the required solution is given by (16) and (18).

An alternative expression for u (r, θ) in closed integral form:

Substituting the values of E_0, E_n and F_n given by (18) in (16), we have

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi + \sum_{n=1}^{\infty}\left[\frac{r^n \cos n\theta}{a^n\pi}\int_0^{2\pi}\cos n\phi\ f(\phi)\, d\phi + \frac{r^n \sin n\theta}{a^n\pi}\int_0^{2\pi}\sin n\phi\, f(\phi)\, d\phi\right]$$

On interchanging the order of summation and integration, we have

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi + \frac{1}{\pi}\int_0^{2\pi} f(\phi)\ \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\cos n\phi\ \cos n\theta + \sin n\phi\ \sin n\theta\right] d\phi$$

or

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)\right] d\phi \qquad \ldots (19)$$

*Let $\quad C = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta) \quad$ and $\quad S = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \sin n(\phi-\theta) \qquad \ldots (20)$

$$(20) \Rightarrow C + iS = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \{\cos n(\phi-\theta) + i\sin n(\phi-\theta)\} = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{\text{in}(\phi-\theta)} = \sum_{n=1}^{\infty}\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}^n \qquad \ldots (21)$$

Since for the given problem $r < a$, so $r/a < 1$. Also, $|\, e^{i(\phi-\theta)}\,| = 1$. Hence, the infinite series on the R.H.S of (21) is a geometric series with common ratio less than 1 and so, (21) reduces to

$$C + iS = \frac{(r/a)e^{i(\phi-\theta)}}{1-(r/a)e^{i(\phi-\theta)}} = \frac{(r/a)\ e^{i(\phi-\theta)}\ \{1-(r/a)e^{-i(\phi-\theta)}\}}{\left\{1-(r/a)e^{i(\phi-\theta)}\right\}\times\left\{1-(r/a)e^{-i(\phi-\theta)}\right\}}$$

or

$$C + iS = \frac{(r/a)e^{i(\phi-\theta)} - (r^2/a^2)}{1-(2r/a)\cos(\phi-\theta) + (r^2/a^2)} = \frac{(r/a)\{\cos(\phi-\theta) + i\sin(\phi-\theta)\} - (r^2/a^2)}{1-(2r/a)\ \cos\ (\phi-\theta) + (r^2/a^2)}$$

---

*In order to find the sum of the infinite series occuring in (19), use $C + iS$ method. Refer author's ''Trigonometry'' published by S.Chand & Co., New Delhi.

Equating real parts on both sides and using the value of C given by (20), we get

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)=\frac{(r/a)\cos(\phi-\theta)-(r^2/a^2)}{1-(2r/a)\cos(\phi-\theta)+(r^2/a^2)}$$

$$\Rightarrow \frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)=\frac{1}{2}+\frac{(r/a)\cos(\phi-\theta)-(r^2/a^2)}{1-(2r/a)\cos(\phi-\theta)+(r^2/a^2)}=\frac{a^2-r^2}{2\{a^2-2ar\cos(\phi-\theta)+r^2\}}$$

Using the above relation, (19) reduces to

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-r^2)f(\phi)}{\{a^2-2ar\cos(\phi-\theta)+r^2\}}d\phi \qquad \ldots(22)$$

(22) is known as *Poisson's integral formula for a circle*, which gives a unique solution for the given Dirichlet problem.

Part (*b*). The exterior Dirichlet problem for a circle can be described as

$$\partial^2u/\partial r^2+(1/r)\times(\partial u/\partial r)+(1/r^2)\times(\partial^2u/\partial\theta^2)=0, \qquad r>a, \qquad 0\le\theta\le 2\pi \qquad \ldots(1)$$

subject to the boundary condition: $u(a,\theta)=f(\theta), \qquad 0\le\theta\le 2\pi \qquad \ldots(2)$

Then we wish to find the value of u at any point for which $r > a$. First proceed upto equation (14) as in Ex 9 (*a*) , page 3.16 of part III of this book. Thus, we get

$$u(r,\theta)=(A_2\,r^n+B_2\,r^{-n})\,(C_2\cos n\theta+D_2\sin n\theta), \quad n=1,2,3\ldots \qquad \ldots(14)$$

Since $u(r,\theta)$ must be bounded when $r\to\infty$, we take $A_2 = 0$ and hence (14) yields

$$u(r,\theta)=B_2r^{-n}(C_2\cos n\theta+D_2\sin n\theta) \quad \text{or} \quad u(r,\theta)=r^{-n}(E_n\cos n\theta+F_n\sin n\theta), \qquad \ldots(15)$$

where $E_n\,(=B_2\,C_2)$ and $F_n\,(=B_2\,D_2)$ are new arbitrary constants.

Using (11) and (15) the most general solution of (1) is given by

$$u(r,\theta)=E_0/2+\sum_{n=1}^{\infty}r^{-n}\,(E_n\cos n\theta+F_n\sin n\theta) \qquad \ldots(16)$$

Putting $r = a$ is (16) and using boundary condition (2), we get

$$f(\theta)=E_0/2+\sum_{n=1}^{\infty}(E_n\,a^{-n}\cos n\theta+F_n\,a^{-n}\sin n\theta) \qquad \ldots(17)$$

which is usual expansion of $f(\theta)$ as Fourier series in $(0, 2\pi)$. Hence, we obtain

$$\left.\begin{aligned}
E_0&=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\,d\theta=\frac{1}{\pi}\int_0^{2\pi}f(\phi)\,d\phi\\
E_n&=\frac{1}{a^{-n}\pi}\int_0^{2\pi}f(\theta)\cos n\theta\,d\theta=\frac{1}{a^{-n}\pi}\int_0^{2\pi}f(\phi)\cos n\phi\,d\phi\\
F_n&=\frac{1}{a^{-n}\pi}\int_0^{2\pi}f(\theta)\sin n\theta\,d\theta=\frac{1}{a^{-n}\pi}\int_0^{2\pi}f(\phi)\sin n\phi\;\,d\phi
\end{aligned}\right\} \qquad \ldots(18)$$

Hence the required solution is given by (16) and (18)

An alternative expression for *u (r,* θ) in closed form:

Substituting the values of E_o, E_n and F_n given by (18) in (16), we have

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}f(\phi)\,d\phi+\sum_{n=1}^{\infty}\left[\frac{r^{-n}\cos n\theta}{a^{-n}\pi}\int_0^{2\pi}\cos n\phi\,f(\phi)\,d\phi+\frac{r^{-n}\sin n\theta}{a^{-n}\pi}\int_0^{2\pi}\sin n\phi\,f(\phi)\,d\phi\right]$$

$$=\frac{1}{2\pi}\int_0^{2\pi}f(\phi)\,d\phi+\frac{1}{\pi}\int_0^{2\pi}f(\phi)\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\left[\cos n\phi\cos n\theta+\sin n\phi\sin n\theta\right]d\phi$$

Thus,

$$u(r,\theta)=\frac{1}{\pi}\int_0^{2\pi}f(\phi)\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\cos n(\phi-\theta)\right]d\phi \qquad \ldots(19)$$

Let $C=\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cos n(\phi-\theta)$ and $S=\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sin n(\phi-\theta)$ ... (20)

$$\Rightarrow \quad C+iS=\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\{\cos n(\phi-\theta)+i\sin n(\phi-\theta)\}=\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n e^{\text{in}(\phi-\theta)}=\sum_{n=1}^{\infty}\left\{\frac{a}{r}e^{i(\phi-\theta)}\right\}^n \quad \text{... (21)}$$

Since for the present problem, $a < r$ so $r/a < 1$, Also, $|e^{i(\phi-\theta)}|=1$, Hence, (21) yields

$$C+iS=\frac{(a/r)e^{i(\phi-\theta)}}{1-(a/r)e^{i(\phi-\theta)}}=\frac{(a/r)\,e^{i(\phi-\phi)}\{1-(a/r)e^{-i\,(\phi-\theta)}\}}{\{1-(a/r)\,e^{i(\phi-\theta}\}\times\{1-(a/r)e^{-i(\phi-\theta)}\}}$$

or $$C+iS=\frac{(a/r)e^{i(\phi-\theta)}-(a^2/r^2)}{1-(2a/r)\,\cos(\phi-\theta)+(a^2/r^2)}=\frac{(a/r)\{\cos(\phi-\theta)+i\sin(\phi-\theta)\}-(a^2/r^2)}{1-(2a/r)\cos(\phi-\theta)+(a^2/r^2)}$$

Equating real parts on both sides and using the value of C given by (20), we get

$$\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cos n(\phi-\theta)=\frac{(a/r)\,\cos(\phi-\theta)-(a^2/r^2)}{1-(2a/r)\cos(\phi-\theta)+(a^2/r^2)}$$

$$\Rightarrow \frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cos n\left(\phi-\theta\right)=\frac{1}{2}+\frac{(a/r)\cos\left(\phi-\theta\right)-(a^2/r^2)}{1-(2a/r)\cos(\phi-\theta)+(a^2/r^2)}=\frac{r^2-a^2}{2\{r^2-2ar\cos(\phi-\theta)+a^2\}}$$

Using the above relation, (19) reduces to

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-a^2)f(\phi)}{\{r^2-2ar\cos(\phi-\theta)+a^2\}}d\phi \quad \text{... (22)}$$

which gives a unique solution of the given problem.

Part (c). The interior Neumann problem for a circle is given by

$$\partial^2u/\partial r^2+(1/r)\times(\partial u/\partial r)+(1/r^2)\times(\partial^2u/\partial\theta^2)=0, \qquad 0\le r<a, \qquad 0\le\theta\le 2\pi \quad \text{...(1)}$$

subject to the boundary condition $\left[\partial u(r,\theta)/\partial r\right]_{r=a}=f(\theta)$, for $r = a$ ... (2)

Proceeding as usual, the required solution is given by

$$u(r,\theta)=A_0/2+\sum_{n=0}^{\infty}r^n\,(A_n\cos n\,\theta+B_n\sin n\,\theta),$$

where $A_n=\frac{1}{n\pi a^{n-1}}\int_0^{2\pi}f(\theta)\cos n\theta\,d\theta$ and $B_n=\frac{1}{n\pi a^{n-1}}\int_0^{2\pi}f(\theta)\sin n\,\theta\,d\theta$

The reader can easily verify that this solution can be re-written in the following integral form

$$u(r,\theta)=\frac{A_0}{2}-\frac{a}{2\pi}\int_0^{2\pi}\log_e\left[\{a^2-2ar\cos(\phi-\theta)+r^2\}/a^2\right]f(\phi)\,d\phi$$

Ex. 24 (*d*). *A thermally conducting disc with insulated faces occupies the region* $0\le r\le a$ *in the xy- plane. The temperature u on the rim r = a, is given by* $u=c, 0<\theta<\alpha$ *and* $u=0, \alpha<\theta<2\pi$, *where* $o<\alpha<2\pi$. *Show that the series expression for the temperature at interior points of the disc given by*

$$u(r,\theta)=(c/\pi)\times\left[\alpha/2+\sum_{n=1}^{\infty}(r/a)^n\times(1/n)\times\{\sin n(\alpha-\theta)+\sin n\theta\}\right]$$

Hint. First proceed as in part (*a*) of Ex. 24 upto equation (18). Now, use the fact that here $u(a,\theta)=f(\theta)=c$, $0<\theta<\alpha$ and $u(a,\theta)=f(\theta)=0$, $\alpha<\theta<2\pi$ and obtain E_0, E_n and F_n. Substitute these values in equation (16) to get the desired result.

Ex. 24 (*e*). *An infinitely long conducting cylinder of radius a, with its axis coincident with*

z- axis, the voltage u(r , θ) obeys the Laplace equation $\nabla^2 u = 0,\ a < r < \infty,\ 0 \le \theta \le 2\pi$. *Find the voltage u(r, θ) for* $r \ge a$ *if u (r , θ)* $\to 0$ *as* $r \to \infty$, *subject to the condition* $(\partial u/\partial r)_{r=a} = (u_0/a) \times \sin 3\theta$.

Hint. Proceed as in Example 24 (*b*). Then the solution of the Laplace equation satisfying $u(r, \theta) \to 0$ as $r \to \infty$, is given by $u(r,\theta) = \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta)$ ...(*i*)

From (*i*), $\partial u/\partial r = -\sum_{n=1}^{\infty} n r^{-n-1}\left(A_n \cos n\theta + B_n \sin n\theta\right)$ ...(*ii*)

Since $(\partial u/\partial r)_{r=a} = (u_o/a) \times \sin 3\theta$, (*ii*) $\Rightarrow$ $(u_o/a) \times \sin 3\theta = -\sum_{n=1}^{\infty} n\, a^{-n-1}(A_n \cos n\theta + B_n \sin n\theta)$, which is satisfied only if $A_n = 0$ for all *n*. Hence, the above equation shows that $B_n = 0$, if $n \ne 3$ and B_3 is given by $(u_o/a) \times \sin 3\theta = -3a^{-4} B_3 \sin 3\theta$ so that $B_3 = -(u_o a^3)/3$.

Hence, from (*i*), $u(r,\theta) = -(u_o/3) \times (a/r)^3 \sin 3\theta$, where $a \le r < \infty$

Example 25 (*a*). *A thin annulus occupies the region* $0 < a \le r \le b,\ 0 \le \theta \le 2\pi$. *The faces are insulated and along the inner edge, the temperature is maintained at 0°, while along the outer edge, the temperature in held at 100°. Obtain the temperature distribution in the annulus.*

(*b*) *A thin annulus occupies the region* $0 < a \le r \le b,\ 0 \le \theta \le 2\pi$. *The faces are insulated. Along the inner edge the temperature is maintained at 0°, while along the outer edge the temperature is held at k cos (θ/2) where k is a constant. Find the temperature distribution in the annulus.*

Ans. (*a*) $u(r, \theta) = 100\dfrac{\log(r/a)}{\log(b/a)}$ (*b*) $u(r,\theta) = \dfrac{8k}{\pi}\sum_{n=1}^{\infty} \dfrac{n}{4n^2-1}\left[\dfrac{(r/a)^n - (a/r)^n}{(b/a)^n - (a/b)^n}\right]\sin n\theta$

26. *Find the electrostatic potential u in the annular region bounded by the concentric spheres r = a, r = b where 0 < a < b, if the inner and outer surfaces are kept at constant potentials* u_1, *and* u_2, $u_1 \ne u_2$.

Sol. We know that the electrostatic potential satisfies the Laplace equation. Clearly, we are to find a solution with spherical symmetry which is independent of θ and ϕ. Hence, here $u = u(r)$ and hence the Laplace equation in spherical polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \quad \text{reduces to} \quad \frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0$$

Re-writing it, we get $\dfrac{d}{dr}\left(r^2 \dfrac{du}{dr}\right) = 0$ ...(1)

We are solve (1) under boundary conditions: $u = u_1$, at $r = a$ and $u = u_2$ at $r = b$ ... (2)

Integrating (1), $r^2(du/dr) = A$ or $du = (A/r^2)\,dr$ ... (3)

Integrating (3), $u = B - (A/r)$, A and B being arbitrary constants ... (4)

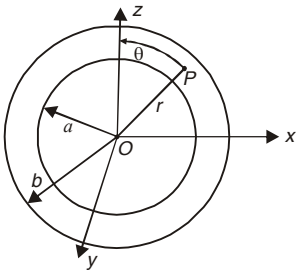
Putting $r = a$ and $r = b$ by turn and using boundary conditions (2), we get

$$B - \frac{A}{a} = u_1 \quad \text{and} \quad B - \frac{A}{b} = u_2 \ \text{giving} \quad A = \frac{u_2 - u_1}{(1/a) - (1/b)} \quad \text{and} \quad B = \frac{(u_1/b) - (u_2/a)}{(1/b) - (1/a)}$$

Substituting these values in (4), the required potential is given by

$$u = \left\{u_1(1/r - 1/b) + u_2(1/a - 1/r)\right\}/\{(1/a) - (1/b)\}$$

27 (*a*). *A thermally conducting solid bounded by two concentric spheres of radii a and b, a < b, is such that the internal boundary is kept at $f_1(\theta)$ and the outer boundary at $f_2(\theta)$. Find the steady state temperature in the solid.*



Sol. We know that the steady state temperature u satisfies the Laplace equation. In the present problem we are looking for a solution for axial symmetry which is independent of ϕ. Hence $u = u(r, \theta)$. Thus, we are solve

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0 \qquad ...(1)$$

subject to the boundary conditions; $u(a, \theta) = f_1(\theta)$ and $u(b, \theta) = f_2(\theta)$ ...(2)

Proceeding as in Ex. 2, page 5.4 of part III of this book, solution of (1) is given by

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(A_n r^n + B_n / r^{n+1}\right)P_n(\cos\theta) \qquad ...(3)$$

Putting $r = a$ and $r = b$ by turn in (3) and using the boundary conditions (2), we get

$$f_1(\theta)=\sum_{n=0}^{\infty}\left(A_n a^n + B_n / a^{n+1}\right)P_n(\cos\theta) \qquad ...(4)$$

and

$$f_2(\theta)=\sum_{n=0}^{\infty}\left(A_n b^n + B_n / b^{n+1}\right)P_n(\cos\theta) \qquad ...(5)$$

In order to get A_n and B_n, we shall use the following orthogonality relations:

$$\int_0^{\pi} P_m(\cos\theta)\, P_n(\cos\theta)\sin\theta\, d\theta=\begin{cases}0, & \text{if } m\neq n\\ 2/(2n+1), & \text{if } m=n\end{cases} \qquad ... (6)$$

Now, multiplying both sides of (4) by $P_m(\cos\theta)\sin\theta$ and integrating, we have

$$\int_0^{\pi} f_1(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta=\sum_{n=0}^{\infty}(A_n a^n + B_n / a^{n+1})\int_0^{\pi} P_n(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta$$

or

$$\int_0^{\pi} f_1(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta=(A_m a^m + B_m / a^{m+1})\frac{2}{2m+1}, \text{using (6)} \quad ... (7)$$

Similarly, (5) yields $\int_0^{\pi} f_2(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta=(A_m b^m + B_m / b^{m+1})\frac{2}{2m+1}$ ... (8)

Let $C_m=\frac{2m+1}{2}\int_0^{\pi} f_1(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta$, $D_m=\frac{2m+1}{2}\int_0^{\pi} f_2(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta$ ...(9)

Using (9), equations (7) and (8) may be rewritten as

$A_m a^m + B_m / a^{m+1} = C_m$ and $A_m b^m + B_m / b^{m+1} = D_m$

Solving the above equations for A_m and B_m we obtain

$$A_m=(C_m a^{m+1}-D_m b^{m+1})/(a^{2m+1}-b^{2m+1}),\ B_m=a^{m+1}b^{m+1}(C_m b^m - D_m a^m)/(b^{2m+1}-a^{2m+1}) \qquad ...(10)$$

The required solution is given by $u(r,\theta)=\sum_{m=0}^{\infty}(A_m r^m + B_m / r^{m+1})P_m(\cos\theta)$,

where A_m and B_m are given by (10).

27. (*b*) *A thermally conducting solid bounded by two concentric spheres a and b, a < b, is such that the inner boundary in kept at T_1 and the outer boundary at T_2 (1 – cos θ). Find the steady state temperature in the solid.*

28. *In a solid sphere of radius a, the surface is maintained at the temperature given by* $u(a,\theta) = k\cos\theta, 0 \le \theta \le \pi/2$ *and* $u(a,\theta) = 0$, $\pi/2 < \theta < \pi$. *Prove that the steady state temperature within the solid is given by*

$$u(r,\theta) = k\left\{\frac{1}{4}P_o(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)P_1(\cos\theta) + \frac{5}{16}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{3}{32}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \ldots.\right\}$$

29. *Find the potential at all points of space inside and outside of a sphere of unit radius which is maintained at a constant distribution of electric potential u (a, θ) = f (θ) = cos 2 θ*

Ans. $u(r,\theta) = -(1/3) + (4r^2/3) \times P_2(\cos\theta), r < 1; \qquad u(r,\theta) = -(1/3r) + (4/3r^3) \times P_2(\cos\theta), r > 1$

30. *Choose the correct answer in the following problems :*

(*i*) *Let* $u = \psi(x,t)$ *be the solution to the initial value problem* $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$ *for* $-\infty < x < \infty$, $t > 0$ *with* $u(x,0) = \sin x$ and $u_t(x,0) = \cos x$, *then the value of* $\psi(\pi/2, \pi/6)$ *is*

(*a*) $\sqrt{3}/2$ (*b*) 1/2 (*c*) $1/\sqrt{2}$ (*d*) 1 **(GATE 2003)**

(*ii*) *The solution of the initial value problem* $\partial^2 u/\partial t^2 = 4\left(\partial^2 u/\partial x^2\right)$, $t > 0, -\infty < x < \infty$, *satisfying the conditions* $u(x,0) = x$, $u_t(x,0) = 0$ *is*

(*a*) x (*b*) $x^2/2$ (*c*) $2x$ (*d*) $2t$ **(GATE 2001)**

(*iii*) *The vertical displacement u (x, t)* of an infinitely long elastic string is governed by the initial value problem $\partial^2 u/\partial t^2 = 4\ (\partial^2 u/\partial x^2)$, $t > 0$, $-\infty < x < \infty$, $u(x,0) = -x$ and $u_t(x,0) = 0$. The value of $u(x,t)$ at $x = 2$ and $t = 2$ is equal to

(*a*) 2 (*b*) 4 (*c*) –2 (*d*) –4 **(GATE 2011)**

So. (*i*) **Ans. (*a*).** Refer part (b) of Ex. 2 on page 2.57 of part III of this book. We know that the D'Alembert's solution for the initial value problem $\partial^2 u/\partial t^2 = c^2\ (\partial^2 u/\partial x^2)$, $t > 0, -\infty < x < \infty$, subject to the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ is given by

$$u(x,t) = (1/2) \times \{f(x+ct) + f(x-ct)\} + (1/2c) \times \int_{x-ct}^{x+ct} g(\theta)\, d\theta \qquad \ldots (1)$$

Comparing the given problem with the above problem, for the given problem we have $c = 1, f(x) = \sin x$ and $g(x) = \cos x$. Hence (1) reduces to

$$u(x,t) = (1/2) \times \{\sin(x+t) + \sin(x-t)\} + (1/2) \times \int_{x-t}^{x+t} \cos\theta\, d\theta$$

or $$u(x,t) = (1/2) \times \{\sin(x+t) + \sin(x-t)\} + (1/2) \times \{\sin(x+t) - \sin(x-t)\} = \sin(x+t)$$

Thus, $\psi(x,t) = u(x,t) = \sin(x+t)$ and so $\psi(\pi/2, \pi/6) = \sin(\pi/2 + \pi/6) = \cos(\pi/6) = \sqrt{3}/2$

(*ii*) **Ans. (*a*).** Refer the particular case of part (b) of Ex.2 on page 2.57 of part III of this book. Accordingly, D' Alembert's solution for the initial value probelm $\partial^2 u/\partial t^2 = c^2\ (\partial^2 u/\partial x^2)$, $t > 0$, $-\infty < x < \infty$, subject to the initial conditious $u(x,0) = f(x)$ and $u_t(x,0) = 0$ is given by

$$u(x,t) = (1/2) \times \{f(x+ct) + f(x-ct)\} \qquad \ldots(1)$$

Comparing the given problem with the above problem, for the given problem we have $c = 2$ and $f(x) = x$. Hence (1) reduces to

$$u(x,t) = (1/2)\times\{(x+2t)+(x-2t)\} = x$$

(*iii*) Proceed exactly as in part (ii). Note that for the present problem, we have c = 2 and $f(x) = -x$. Hence, as in part (*ii*), we have

$$u(x,t) = (1/2)\times\{-(x+2t)-(x-2t)\} = -x$$

Thus, when $x = 2$ and $t = 2$, the required value of $u(x, t)$ is –2.

Ex. 31. *The solution of the diffusion equation* $\partial^2 u/\partial x^2 = \partial u/\partial t$, $u = u(x,t)$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \cos x \sin 5x$ is

(*a*) $(e^{-36t}/2)\times(\sin 6x + e^{20t}\sin 4x)$ (*b*) $(e^{-36t}/2)\times(\sin 4x + e^{20t}\sin 6x)$

(*c*) $(e^{-20t}/2)\times(\sin 3x + e^{15t}\sin 5x)$ (*d*) $(e^{-36t}/2)\times(\sin 5x + e^{20t}\sin x)$ **[GATE 2012]**

Sol. Ans. (*a*). From Working rule 2.3B, we know that the solution of

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \text{... (1)}$$

subject to the boundary conditions $u(0,t) = u(a,t) = 0$ for all t (2)

and the iniutial condition $u(x,0) = f(x), 0 < x < a$ ... (3)

is given by $$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a} e^{-(n^2\pi^2 kt/a^2)} \quad \text{... (4)}$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = 1$, $a = \pi$ and $f(x) = \cos x \sin 5x$. Hence, from (5), the required solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin nx\, e^{-n^2 t} \quad \text{... (5)}$$

Putting t = 0 is (5) and using the given fact that $u(x,0) = \cos x \sin 5x$, we have

$$\sum_{n=1}^{\infty} E_n \sin nx = \cos x \sin 5x = (1/2)\times(\sin 6x + \sin 4x) \quad \text{... (6)}$$

Equating the coefficients of sin 6x and sin 4x on both sides of (6), we have $E_6 = E_4 = 1/2$. We also note that $E_n = 0$ for all n except when $n = 4$ or $n = 6$.

Substituting the above values of E_4 and E_6 in (6), the required solution is given by

$$u(x,t) = E_4 \sin 4x\, e^{-16t} + E_6 \sin 6x\, e^{-36t} = (e^{-36t}/2)\times(\sin 6x + e^{20t}\sin 4x)$$

Ex.32. *The function* $u(r,\theta)$ *satisfying the Laplace's equation* $\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/\partial\theta^2) = 0$, $e < r < e^2$ *subject ot the conditions* $u(e,\theta) = 1$, $u(e^2,\theta) = 0$ *is*

(*a*) $ln\left(\frac{e}{r}\right)$ (*b*) $ln\left(\frac{e}{r^2}\right)$ (*c*) $ln\left(\frac{e^2}{r}\right)$ (*d*) $\sum_{n=1}^{\infty}\frac{r-e^2}{e-e^2}\sin n\theta$ **[GATE 2012]**

Sol. Ans. (*c*). From solution of Ex. 10(*a*), page 3.19 of part III, we know that the solution of the boundary voclue problem

$$\partial^2 u/\partial r^2 + (1/r)\times(\partial u/\partial r) + (1/r^2)\times(\partial^2 u/d\theta^2) = 0,\ r_1 < r < r_2 \quad \ldots(1)$$

subject to the boundary conditions $u(r_1, \theta) = f_1(\theta)$ and $u(r_2, \theta) = f_2(\theta)$ ...(2)

is $u(r, \theta) = \dfrac{a_0 \log r + b_0}{2} + \sum_{n=1}^{\infty}\left[\left(a_n r^n + b_n r^{-n}\right)\cos n\theta + (c_n r^n + d_n r^{-n})\sin n\theta\right]$ ... (3)

where $a_0 \log r_1 + b_0 = \dfrac{1}{\pi}\int_0^{2\pi} f_1(\theta)d\theta$, $a_0 \log r_2 + b_0 = \dfrac{1}{\pi}\int_0^{2\pi} f_2(\theta)d\theta$ ... (4)

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\cos n\theta\, d\theta \qquad a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\cos n\theta\, d\theta \quad \ldots(5)$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\sin n\theta\, d\theta \qquad c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\sin n\theta\, d\theta \quad \ldots(6)$$

Comparing the given boundary value problem with the above boundary value problem, we have $r_1 = e,\quad r_2 = e^2,\quad f_1(\theta) = 1 \quad and \quad f_2(\theta) = 0$.

Substituting there values in (4), (5) and (6), we obtain

$$a_0 \log e + b_0 = \frac{1}{\pi}\int_0^{2\pi} d\theta = 2, \qquad a_0 \log e^2 + b_0 = \frac{1}{\pi}\int_0^{2\pi}(0)d\theta = 0 \quad \ldots(7)$$

$$a_n e^n + b_n e^{-n} = \frac{1}{\pi}\int_0^{2\pi}\cos n\theta\, d\theta = \left[\frac{\sin n\theta}{n\pi}\right]_0^{2\pi} = 0, \quad a_n e^{2n} + b_x e^{-2n} = \frac{1}{\pi}\int_0^{2\pi}(o)\ d\theta = 0 \quad \ldots(8)$$

$$c_n e^n + d_n e^{-n} = \frac{1}{\pi}\int_0^{2\pi}\sin n\theta d\theta = \left[-\frac{\cos n\theta}{n\pi}\right]_0^{2\pi} = 0, \quad c_n e^{2n} + d_n e^{-2n} = \frac{1}{\pi}\int_0^{2\pi}(0)d\theta = 0 \quad \ldots(9)$$

$ln\, x = \log_e x = \log x$

From (7), $a_0 + b_0 = 2$ and giving $a_0 = -2$ and $b_0 = 4$ (10)

From (8) and (9), $a_n = b_n = c_n = d_n = 0 \ \forall n = 1, 2, 3, 4, \ldots\ldots$ ... (11)

Substituting the above values of a_0, b_0, a_n, b_n, c_n and d_n in (3), the required solution is

$$u(r,\theta) = (-2\log r + 4)/2 = 2 - \log r = 2\log e - \log r = \log(e^2/r) = ln(e^2/r).$$

where $ln\, x = \log_e x = \log x$

Ex.33. (Uniqueness of the solution for the heat equation (i.e. heat conduction problem)) *Let u (x, t) be a continuously differentiable function. If u (x, t) satisfies the heat equation $u_t = k\, u_{xx}$, $0 < x < l$, $t > 0$, the initial condition u (x, 0) = f(x), $0 \le x \le l$ and the boundary conditions u (0, t) = u (l, 1) = 0, $t \ge 0$, then show that the solution is unique.*

Sol. If possible, let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of the given initial boundary valve problem and suppose that $v(x, t) = u_1(x, t) - u_2(x, t)$. It can readily be seen that $v(x, t)$ is the solution of the following initial boundary value problem:

$$v_t = k\, v_{xx},\ 0 < x < l,\ t > 0 \quad \ldots(1)$$

satisfying the initial condition: $v(x, 0) = 0,\ 0 \le x \le l$ …(2)

and the boundary conditions: $v(0, t) = v(l, t) = 0,\ t \ge 0$ …(3)

In order to prove the required result, we shall prove that the function $v(x, t)$ is identically zero. To this end, consider the integral

$$I(t) = \frac{1}{2k}\int_0^l v^2 dx \qquad \ldots(4)$$

Since the function $v(x, t)$ is continuously differentiable, we differentiate both sides of (4) with respect to t. Thus, we obtain

$$\frac{dI}{dt} = \frac{1}{k}\int_0^l v v_t \, dx = \int_0^l v v_{xx}, \text{ using (1)} \qquad \ldots(5)$$

or
$$dI/dt = [v v_x]_0^l - \int_0^l v_x^2 \, dx, \text{ integrating by parts} \qquad \ldots(6)$$

But from (3), $v(0, t) = v(l, t) = 0$. Therefore, the expression in the square breakets occurring on the right hand side of (6) vanishes and hence (6) reduces to

$$\frac{dI}{dt} = -\int_0^l v_x^2 \, dx \le 0 \qquad \ldots(7)$$

Now, from (4)
$$I(0) = \frac{1}{2k}\int_0^l [v^2]_{t=0} dx = \frac{1}{2}\int_0^l [v(x,0)]^2 dx = 0, \text{ using (2)}$$

Since $I(0) = 0$ and $dI/dt \le 0$, it follows that $I(t)$ is a non-increasing function of t, Therefore, we obtain

$$I(t) \le 0 \qquad \ldots(8)$$

Again, from (4), we have
$$I(t) \ge 0 \qquad \ldots(9)$$

From (8) and (9), we have
$$I(t) = 0 \text{ for } t \ge 0$$

Since $v(x, t)$ is continuous, $I(t) = 0 \Rightarrow v(x, 0) = 0$ for $0 \le x \le l$, $t \ge 0$. Since $v(x, t) = u_1(x, t) - u_2(x, t)$, it follows that $u_1(x, t) = u_2(x)$. Therefore, the solution of the given problem is unique.

Ex. 34. (Uniqueness of the solution for the wave equation)

Show that there exists at most one solution of the wave equation $u_{tt} = c^2 u_{xx}$, $0 < x < l$, $t > 0$ satisfying the initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $0 \le x \le l$ and the boundary conditions $u(0, t) = 0$, $u(l, t) = 0$, $t \ge 0$, where $u(x, t)$ is twice continuously differentiable function with respect to both x and t. *[Delhi B.A. [B.Sc. (Hons.) III 2012]*

Sol. If possible, let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of the given initial-boundary-value problem and suppose that $v(x, t) = u_1(x, t) - u_2(x, t)$. It can readily be seen that $v(x, t)$ is the solution of the following initial-boundary value problem:

$$v_{tt} = c^2 v_{xx}, \quad 0 \le x \le l, \quad t > 0 \qquad \ldots(1)$$

satisfying the initial conditions:
$$v(x, 0) = 0, \; v_t(x, 0) = 0, \; 0 \le x \le l \qquad \ldots(2)$$

and the boundary conditions:
$$v(0, t) = v(l, t) = 0, \; t \ge 0 \qquad \ldots(3)$$

In order to prove the required result, we shall prove that the function $v(x, t)$ is identically zero. To this end, let us consider the energy integral

$$E(t) = \frac{1}{2}\int_0^l (c^2 v_x^2 + v_t^2) \, dx, \qquad \ldots(4)$$

which physically represents the total energy of the vibrating string at time t.

Since the function $v(x, t)$ is twice continuously differentiable, we differentiate both sides of (4) with respect to t. Thus, we obtain

$$dE/dt = \int_0^l (c^2 v_x v_{xt} + v_t v_{tt}) \, dx \qquad \ldots(5)$$

Integrating the first integral in (5), we get

$$\int_0^l c^2 v_x v_{xt}\, dx = [c^2 v_x v_t]_0^l - \int_0^l c^2 v_t v_{xx}\, dx \quad \ldots(6)$$

From (3), $v(0, t) = v(l, t) = 0$ for $t \geq 0$ which implies that $v_t(0, t) = v_t(l, t) = 0$, for $t \geq 0$. Therefore, the expression in the square breakets occurring on the R.H.S. of (6) vanishes and hence (6) reduces to

$$\int_0^l c^2 v_x v_{xt}\, dx = -\int_0^l c^2 v_t v_{xx}\, dx \quad \ldots(7)$$

Using (7), (5) reduces to $$dE/dt = \int_0^l v_t (v_{tt} - c^2 v_{xx})\, dx \quad \ldots(8)$$

Now, from (1), we have $v_{tt} - c^2 v_{xx} = 0$. Therefore, (8) reduces to

$$dE/dt = 0 \quad \text{giving} \quad E(t) = \text{constant} = C, \text{ say} \quad \ldots(9)$$

We now proceed to evaluate constant C. Putting $t = 0$ in (4) and (9), we get

$$C = \frac{1}{2}\int_0^l [c^2 v_x^2 + v_t^2]_{t=0}\, dx \quad \ldots(10)$$

From (2), $v(x, 0) = 0$ which implies that $v_x(x, 0) = 0$. Again, from (2), $v_t(x, 0) = 0$. Using the results $v_x(x, 0) = 0$ and $v_t(x, 0) = 0$,(10) reduces to $C = 0$.

Since $C = 0$, (9) yields $E(t) = 0$, which can happen only $v_x = 0$ and $v_t = 0$ for $t > 0$. In order to satisfy both of these conditions, we must have $v(x, t) =$ constant. However, from (2) $v(x, 0) = 0$ and h e n c e $v(x, t) \equiv 0$. Since $v(x, t) = u_1(x, t) - u_2(x, t)$, it follows that $u_1(x, t) = u_2(x, t)$. Therefore, the solution of the given problem is unique. In other words, there exists only one solution of the given problem.

Ex. 35. *Solve* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, *for* $0 < x < \pi$, $0 < y < \pi$ *given that* $u(x, 0) = x^2$, $u(x, \pi)$ *exists only one solution of* $(\partial u/\partial x)_{(0,y)} = (\partial u/\partial x)_{(\pi,y)} = 0$. *[Meerut 2012]*

Sol. Given $$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \text{ for } 0 < x < \pi,\ 0 < y < \pi \quad \ldots(1)$$

subject boundary conditions: $$u(x, 0) = x^2 \quad \ldots(2a)$$

$$u(x, \pi) = 0 \quad \ldots(2b)$$

$$(\partial u/\partial x)_{(0,y)} = u_x(0, y) = 0 \quad \ldots(2c)$$

$$(\partial u/\partial x)_{(\pi,y)} = u_x(\pi, y) = 0 \quad \ldots(2d)$$

Suppose (1) has solution of the form $$u(x, y) = X(x)\, Y(y), \quad \ldots(3)$$

Substituting the above value of $u(x, y)$ in (1), we have

$$X''Y + XY'' = 0 \quad \text{so that} \quad X''/X = -(Y''/Y) \quad \ldots(4)$$

Since x and y are independent variables, (4) can only be true if each side is equal to the same constant, say μ. Hence, (4) yields

$$X''/X = \mu \quad \text{i.e.,} \quad d^2X/dx^2 - \mu X = 0 \quad \ldots(5)$$

$$-(Y''/Y) = \mu, \quad \text{i.e.,} \quad d^2Y/dy^2 + \mu Y = 0 \quad \ldots(6)$$

The following three cases arise:

Case I. Let $\mu = 0$**.** Then, the general solutions of (5) and (6) are given by

$X(x) = Ax + B$ and $Y(y) = Cy + D$, A, B, C and D being arbitrary constants …(7)

From (3) and (7), $u(x, y) = (Ax + B)(Cy + D)$ …(8)

From (8), $\partial u/\partial x = u_x(x, y) = A(Cy + D)$ …(9)

Putting $x = 0$ in (9) and using condition (2c), we get $0 = A(Cy + D)$. For non-trivial solution of the given problem $(Cy + D) \neq 0$ and hence we have $A = 0$. Then, (8) reduces to

$$u(x, y) = B(Cy + D) \quad \ldots(10)$$

Putting $x = \pi$ in (10) and using condition (2b), we get $0 = B(C\pi + D)$. For non-trivial solution of the given problem, $B \neq 0$ and hence $C\pi + D = 0$ giving $D = -C\pi$,

Since $A = 0$ and $D = -C\pi$, (8) reduces to

$$u(x, y) = B(Cy - C\pi) = E(y - \pi), \text{ where } E = BC \quad \ldots(11)$$

Clearly, solution (11) also satisfies the condition (2d). Then, (11) gives a solution of (1) satisfying boundary conditions (2 b), (2 c) and (2d).

Case II. Let $\mu = \lambda^2$,where $\lambda \neq 0$. Then, the general solutions of (5) and (6) are given by

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ and $Y(y) = C\cosh \lambda y + D\sinh \lambda y$, A, B, C and D being constants …(12)

From (3) and (12), $u(x, y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cosh \lambda y + D\sinh \lambda y)$ …(13)

From (13), $u_x(x, y) = \lambda(Ae^{\lambda x} - Be^{-\lambda x})(C\cosh \lambda y + D\sinh \lambda y)$ …(14)

Putting $x = 0$ in (14) and using condition (2c), we get $0 = \lambda(A - B)(C\cosh \lambda y + D\sinh \lambda y)$.

For non-trivial solution of the given problem, $(C\cosh \lambda y + D\sinh \lambda Y) \neq 0$. Also, $\lambda \neq 0$ for our case II. Hence, we obtain A – B = 0 giving B = A. Them, (13); reduces to

$$u(x, y) = A(e^{\lambda x} + e^{-\lambda x})(C\cosh \lambda y + D\sinh \lambda y) \quad \ldots(15)$$

Putting $x = \pi$ in (15) and using condition (2b), we get $0 = A(e^{\lambda \pi} + e^{-\lambda \pi})(C\cosh \lambda y) + D\sinh \lambda y)$ giving $A = 0$. Now, $A = 0 \Rightarrow B = 0$. Hence, (13) reduces to trivial solution. Hence, $\mu = \lambda^2$ is not possible.

Case III. Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then, the general solutions of (5) and (6) are given by

$X(x) = A\cos \lambda x + B\sin \lambda x$ and $Y(y) = C\cosh \lambda y + D\sinh \lambda y$, A, B, C, D and being constants …(16)

From (3) and (16), $u(x, y) = (A\cos \lambda x + B\sin \lambda x)(C\cosh \lambda y + D\sinh \lambda y)$ …(17)

From (17), $u_x(x, y) = \lambda(-A\sin \lambda x + B\cos \lambda x)(C\cosh \lambda y + D\sinh \lambda y)$ …(18)

Putting $y = \pi$ in (17) and using condition (2b), we get

$$0 = (A\cos \lambda x + B\sin \lambda x)(C\cosh \lambda \pi + D\sinh \lambda \pi) \quad \ldots(19)$$

For non-trivial solution of the given problem, $(A\cos \lambda x + B\sin \lambda x) \neq 0$. Hence,

$$(19) \Rightarrow C\cosh \lambda \pi + D\sinh \lambda \pi = 0 \text{ giving } D = -\{(C\cosh \lambda \pi) / \sinh \lambda \pi\} \quad \ldots(20)$$

Putting $x = 0$ in (18) and using condition (2c), we get $0 = \lambda B(C\cosh \lambda y + D\sinh \lambda y)$ …(21)

Since $\lambda \neq 0$ and for non-rivial solution $(C\cosh \lambda y + D\sinh \lambda y) \neq 0$, so $(21) \Rightarrow B = 0$ … (22)

Substituting values of B and D given by (22) and (20) in (17) we get

$$u(x, y) = \text{AC}\cos \lambda x \left(\cosh \lambda y - \frac{\cosh \lambda \pi}{\sinh \lambda \pi}\sin \lambda y\right) = \text{AC}\ \frac{\cos \lambda x \sinh \lambda(\pi - y)}{\sinh \lambda \pi} \quad \ldots(23)$$

From (23), $u_x(x, y) = -\lambda AC\ \dfrac{\sin \lambda x \sinh \lambda(\pi - y)}{\sinh \lambda \pi}$ …(24)

Putting $x = \pi$ in (24) and using condition (2d) yields $\lambda AC\sin \lambda \pi \sinh \lambda(\pi - y) = 0$…(25)

From (23), it follows that for existence of nn-trivial solution $u(x, y)$, $AC \neq 0$ and $\sinh \lambda (\pi - y) \neq 0$. Also we have assumed that $\lambda \neq 0$. Hence, from (25), it follows that

$$\sin \lambda\pi = 0 \quad \Rightarrow \quad \lambda\pi = n\pi \quad \Rightarrow \quad \lambda = n, \quad \text{where} \quad n = 1, 2, 3, \ldots$$

Hence from (23), non-zero solutions $u_n(x, y)$ of (1) satisfying the boundary condition (2*b*), (2*c*) and (2*d*) are given by

$$u_n(x, y) = E_n [\{\cos n x \sinh n (\pi - y)\} / \sinh n\pi\}], n = 1, 2, 3, \ldots \quad \ldots(26)$$

In order to satisfy condition (2*a*), we consider more general solution of (1) given by

$$u(x, y) = E(y - \pi) + \sum_{n=1}^{\infty} u_n(x, y) = E(y - \pi) + \sum_{n=1}^{\infty} E_n \frac{\cos nx \sinh n(\pi - y)}{\sinh n\pi} \quad \ldots(27)$$

which satisfies condition (2*b*), (2*c*) and (2*d*). Putting $y = 0$ in (27) and using condition (2*a*), we set

$$x^2 = -E\pi + \sum_{n=1}^{\infty} E_n \cos n x \quad \ldots(28)$$

Determination of E Integrating both sides of (28) w.r.t. x from $x = 0$ to $x = \pi$, we get

$$\frac{\pi^3}{3} = -E\pi^2 + \sum_{n=1}^{\infty} E_n \left[\frac{\sin nx}{n}\right]_0^{\pi} \Rightarrow -\frac{\pi^3}{3} = E\pi^3 \Rightarrow E = -\frac{\pi}{3} \quad \ldots(29)$$

Determination of E_n. Multiplying both sides of (28) by $\cos nx$ and integrating both sides of the resulting equation with respect to x from $x = 0$ to $x = \pi$, we get

$$\int_0^{\pi} x^2 \cos nx \, dx = -E\pi \int_0^{\pi} \cos nx \, dx + E_n \sum_{n=1}^{\infty} \int_0^{\pi} \cos^2 nx \, dx$$

or
$$\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi} = 0 + E_n \sum_{n=1}^{\infty} \frac{1 + \cos 2nx}{2} dx$$

[Using chain rule of integrating by parts on R.H.S.]

or
$$(2\pi \cos n\pi) / n^2 = (\pi E_n)/2 \quad \Rightarrow \quad E_n = [4(-1)^n] / n^2 \quad \ldots(30)$$

Substituting the values of E and E_n given by (29) and (30) in (27), the required solution is

$$u(x, y) = \frac{\pi(\pi - y)}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx \sinh n(\pi - y)}{n^2 \sinh n\pi}$$

Ex. 36. *Using Parseval's identity calculate:*

$$(a) \int_0^{\infty} \frac{dx}{(a^2 + x^2)^2} \qquad (b) \int_0^{\infty} \frac{x^2}{(a^2 + x^2)} \text{ if } a > 0 \qquad (\textit{Madras 2012})$$

Sol. By definition,
$$F_c\left\{\frac{1}{a^2 + x^2}\right\} = \int_0^{\infty} \frac{\cos sx}{a^2 + x^2} dx = \text{I, say} \quad \ldots(1)$$

From (1), $$\frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{a^2+x^2}dx = -\int \frac{x\sin sx}{a^2+x^2}dx \quad \ldots(2)$$

or $$\frac{dI}{ds} = -\int \frac{x^2\sin sx}{x(a^2+x^2)}dx = -\int_0^\infty \frac{\{(a^2+x^2)-a^2\}\sin sx}{x(a^2+x^2)}dx = -\int_0^\infty \frac{\sin sx}{x}dx + a^2\int_0^\infty \frac{\sin sx}{x(a^2+x^2)}dx$$

Hence $$\frac{dI}{ds} = -\frac{\pi}{2} + a^2\int_0^\infty \frac{\sin sx}{x(a^2+x^2)}dx, \qquad \text{as} \int_0^\infty \frac{\sin sx}{x}dx = \frac{\pi}{2} \quad \ldots(3)$$

From (3), $$\frac{d^2I}{ds^2} = a^2\int_0^\infty \frac{x\cos sx}{x(a^2+x^2)}dx = a^2\int_0^\infty \frac{\cos sx}{a^2+x^2}dx = a^2\text{I, using (1)}$$

or $d^2I/dx^2 - a^2I = 0$ or $(D^2 - a^2)\,\text{I} = 0$, where $D \equiv d/ds$

whose general solution in $I = Ae^{-as} + Be^{as}$, A and B being arbitrary constants …(4)

From (4), $$dI/ds = -Aae^{-as} + Bae^{as} \quad \ldots(5)$$

For $s = 0$, (1) and (4) give $$A + B = \int_0^\infty \frac{dx}{a^2+x^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty = \frac{\pi}{2a} \quad \ldots(6)$$

For $s = 0$, (3) and (5) give $$-Aa + Ba = -\frac{\pi}{2} \quad \text{or} \quad A - B = \frac{\pi}{2a} \quad \ldots(7)$$

Solving (6) and (7), $A = \pi / 2a$ and B = 0, Hence, from (1) and (4), we have

$I = (\pi/2a) \times e^{-as}$ or $\text{F}_\text{c}\{1/(a^2+x^2)\} = (\pi/2a) \times e^{-as}$ …(8)

Again, by definition, $$F_s\left\{\frac{x}{a^2+x^2}\right\} = \int_0^\infty \frac{x\sin sx}{a^2+x^2}dx = -\frac{dI}{ds}, \text{ using (2)} \quad \ldots(9)$$

Since $A = \pi / 2a$ and $B = 0$, (5) yields $dI/ds = -(\pi/2) \times e^{-as}$ …(10)

From (9) and (10), $F_s\{x/(a^2+x^2)\} = (\pi/2) \times e^{-as}$ …(11)

We now proceed to calculate the values of required integrals with help of Parseval's identities.

Part (a) : Parseval's identity for Fourier cosine transform is given by (refer note on page 1.34 of part IV B of this book)

$$\int_0^\infty \{F(x)\}^2dx = \frac{2}{\pi}\int_0^\infty [F_c\{F(x)\}]^2ds$$

Hence, $$\int_0^\infty \frac{1}{(a^2+x^2)}dx = \frac{2}{\pi}\int_0^\infty \left[F_C\left\{\frac{1}{a^2+x^2}\right\}\right]^2 ds = \frac{2}{\pi}\int_0^\infty \left(\frac{\pi}{2a}e^{-as}\right)^2 ds, \text{ using (8)}$$

$$= \frac{\pi}{2a^2}\int_0^a e^{-2as}ds = \frac{\pi}{2a^2}\left[-\frac{1}{2a}e^{-2as}\right]_0^\infty = \frac{\pi}{2a^2}\times\frac{1}{2a} = \frac{\pi}{4a^3}$$

Part (b) Parseval's identity for Fourier sine transform is given by (refer note on page 1.34 of part IV B of this book)

$$\int_0^a \{F(x)\}^2 dx = \frac{2}{\pi}\int_0^\infty [F_s\{F(x)\}]^2 ds$$

Hence $$\int_0^\infty \left(\frac{x}{a^2+x^2}\right)^2 dx = \frac{2}{\pi}\int_0^\infty \left[F_s\left\{\frac{x}{a^2+x^2}\right\}\right]^2 ds = \frac{2}{\pi}\int_0^\infty \left(\frac{\pi}{2}e^{-as}\right)^2 ds,$$ using (11)

$$= \frac{\pi}{2}\int_0^a e^{-2as} ds = \frac{\pi}{2}\left[-\frac{1}{2a}e^{-2as}\right]_0^\infty = \frac{\pi}{2}\times\frac{1}{2a} = \frac{\pi}{4a}$$